Automata and Reactive Systems

Lecture No. 7

Prof. Dr. Wolfgang Thomas

thomas@informatik.rwth-aachen.de

Lehrstuhl für Informatik VII

RWTH Aachen

Non-counting Property

$L \subseteq \Sigma^\omega$ ia called non-counting if

$$\exists n_0 \, \forall n \geq n_0 \, \forall u, v \in \Sigma^* \, \forall \beta \in \Sigma^\omega : uv^n\beta \in L \Leftrightarrow uv^{n+1}\beta \in L$$

Two claims are shown:

1. $L = (00)^*1^\omega$ does not have this property
2. Each LTL-definable ω-language has this property

Hence $L = (00)^*1^\omega$ is Büchi recognizable but not LTL-definable.

3.4 Proposition: Each LTL-definable ω-language L is non-counting:

$$\exists n_0 \; \forall n \geq n_0 \; \forall u, v \in \Sigma^* \; \forall \beta \in \Sigma^\omega : uv^n\beta \in L \Leftrightarrow uv^{n+1}\beta \in L$$

Proof: by induction on LTL-formulas φ

$\varphi = p_i$: Take $n_0 = 1$.

Whether $uv^n\beta \in L$ only depends only on first letter.
This is the same letter as in $uv^{n+1}\beta$.

$\varphi = \neg\psi$: The claim is trivial.

$\varphi = \psi_1 \wedge \psi_2$:
ψ_1, ψ_2 define non-counting L_1, L_2 (with n_1, n_2) by ind.hyp.

Take $n_0 = \max(n_1, n_2)$.
Then the claim is true for $L_1 \cap L_2$, defined by $\psi_1 \wedge \psi_2$.

$\varphi = \mathrm{X}\psi$:

By ind.hyp. assume ψ defines non-counting L with n_1.

Take $n_0 := n_1 + 1$, at least $n_0 \geq 2$.

For $n \geq n_0$ we have to show: $uv^n\beta \models \mathrm{X}\psi$ iff $uv^{n+1}\beta \models \mathrm{X}\psi$

If $u \neq \varepsilon$, say $u = au'$, then use ind.hyp

$$u'v^n\beta \models \psi \text{ iff } u'v^{n+1}\beta \models \psi$$

If $u = \varepsilon$ and $v = av'$ then use (for $n \geq n_0$)

$$v^n\beta \models \mathrm{X}\psi \text{ iff } v'v^{n-1}\beta \models \psi \text{ iff } v'v^n\beta \models \psi \text{ iff } v^{n+1}\beta \models \mathrm{X}\psi$$

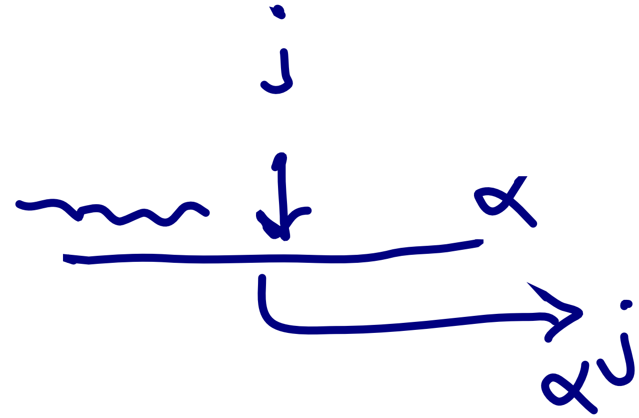
$\varphi = \psi_1 \mathrm{U} \psi_2$:

ψ_1, ψ_2 **defining non-counting** L_1, L_2 **(with** n_1, n_2**) by ind.hyp.**

Take $n_0 = 2 \cdot \max(n_1, n_2)$**.**

Show : For $n \geq n_0$**:**

$$uv^n\beta \models \psi_1 \mathrm{U} \psi_2 \;\; \text{iff} \;\; uv^{n+1}\beta \models \psi_1 \mathrm{U} \psi_2$$



More precisely:

for some j**:** $(uv^n\beta)^j \models \psi_2$ **and for** $i < j$**:** $(uv^n\beta)^i \models \psi_1$

iff

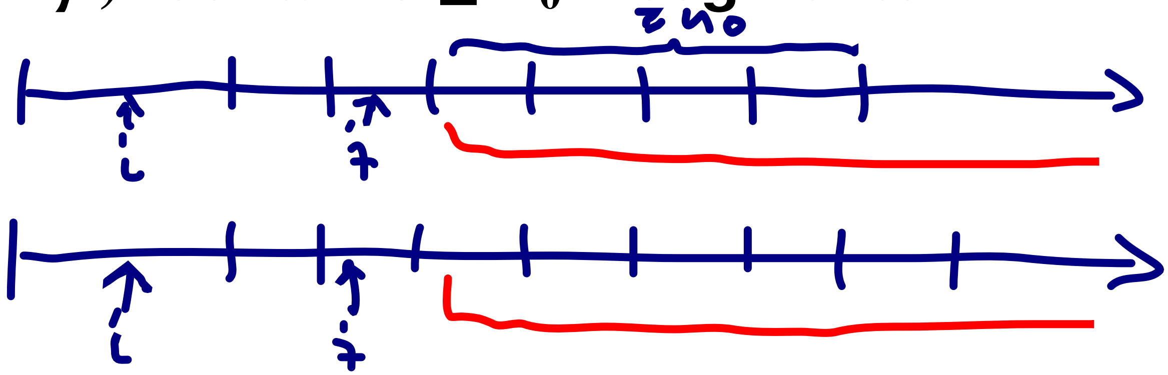
for some j**:** $(uv^{n+1}\beta)^j \models \psi_2$ **and for** $i < j$**:** $(uv^{n+1}\beta)^i \models \psi_1$

Look at the direction from left to right:

if for some j: $(uv^n\beta)^j \models \psi_2$ and for $i < j$: $(uv^n\beta)^i \models \psi_1$

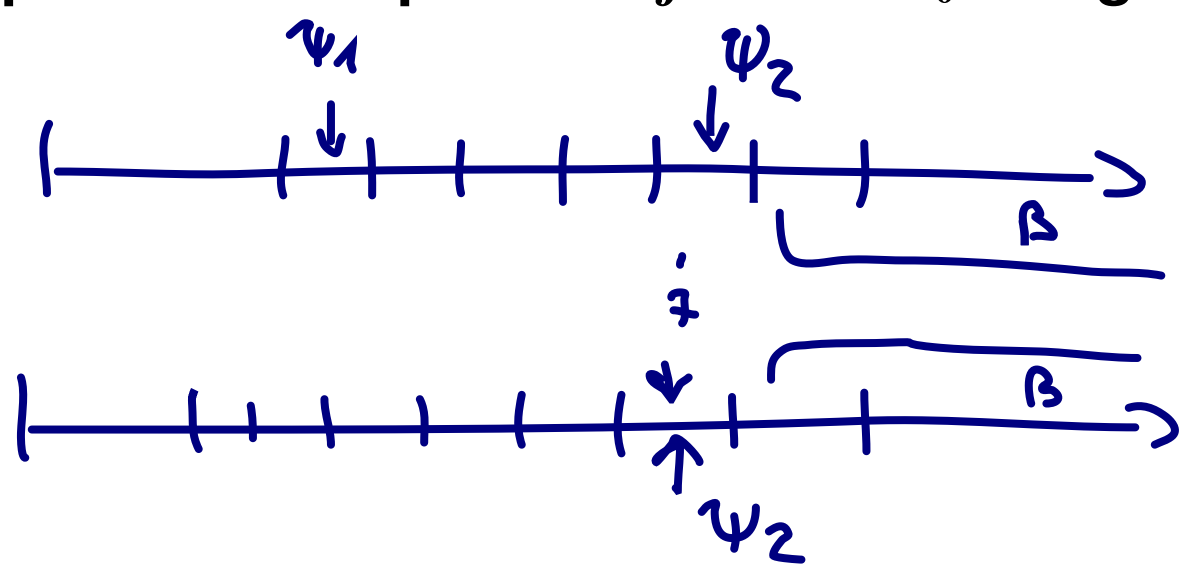
then for some j: $(uv^{n+1}\beta)^j \models \psi_2$ and for $i < j$: $(uv^{n+1}\beta)^i \models \psi_1$

Case 1: $(uv^n\beta)^j$ contains $\geq n_0$ v-segments



Case 2: $(uv^n\beta)^j$ contains $< n_0$ v-segments

Then the prefix before position j has $\geq n_0$ v-segments.



S1S

We introduce a more flexible logic than LTL:

S1S (second-order theory of one successor)

Idea: Use

- variables $s, t, \ldots$ for time-points (positions in ω-words)
- variables $X, Y, \ldots$ for sets of positions
- the constant 0 for position 0, the successor function $'$, equality $=$ and less-than relation $<$
- the usual boolean connectives and the quantifiers $\exists$, $\forall$

Example formulas

LTL-formulas and their translation to S1S:

$$\mathbf{GF}p_1 \quad : \quad \forall s \exists t(s \leq t \wedge X_1(t))$$

$$\mathbf{XX}(p_2 \rightarrow \mathbf{F}p_1) \quad : \quad X_2(0'') \rightarrow \exists t(0'' \leq t \wedge X_1(t))$$

$$\mathbf{F}(p_1 \wedge \mathbf{X}(\neg p_2 \mathbf{U} p_1)) \quad : \quad \exists t_1(X_1(t_1) \wedge \exists t_2(t_1' \leq t_2 \wedge X_1(t_2) \wedge \forall t((t_1' \leq t \wedge t < t_2) \rightarrow \neg X_2(t))))$$

A definition of $L = (00)^*1^\omega$:

$$\exists X\, \exists t(X(0) \wedge \forall s(X(s) \leftrightarrow \neg X(s')) \wedge X(t) \wedge \forall s(s < t \rightarrow \neg X_1(s)) \wedge \forall s(t \leq s \rightarrow X_1(s)))$$

0 1 2 3 4

X * _ * _ * _

t

Plan

1. **Syntax and semantics of S1S**
2. **Expressive power: Büchi recognizable ω-languages are S1S-definable**
3. **S1S-definable ω-languages are Büchi recognizable (Preparation)**

Syntax of S1S

Variables:

- **first-order variables $s, t, \ldots, x, y, \ldots$ (ranging over natural numbers, i.e. positions in ω-words)**
- **second-order variables $X, X_1, X_2, Y, Y_1, \ldots$ (ranging over sets of natural numbers)**

Terms are

- **the constant 0 and the first-order variables**
- **with any term τ also τ' (the successor of τ)**

Examples of terms: $t, t', t'', 0, 0', 0''$

Syntax of S1S

- **Atomic formulas:** $X(\tau)$, $\sigma < \tau$, $\sigma = \tau$ for terms σ, τ

  $\tau \in X$

- **First-order formulas (S1S$_1$-formulas)** are built up from atomic formulas using boolean connectives and quantifiers $\exists, \forall$ over first-order variables

- **S1S-formulas** are built up from atomic formulas using boolean connectives and quantifiers $\exists, \forall$ over first-order variables and second-order variables,

- **Existential S1S formulas** are S1S$_1$-formulas preceded by a block $\exists Y_1 \ldots \exists Y_m$ of existential second-order quantifiers.

$$\exists Y_1 \ldots \exists Y_m \underbrace{\varphi(\quad)}_{S1S_1}$$

Examples

First-order formulas:

$$\varphi_1(X) \quad : \quad \forall s \exists t(s < t \wedge X(t))$$

$$\varphi_2(X_1, X_2) \quad : \quad X_2(0'') \rightarrow \exists t(0'' \leq t \wedge X_1(t))$$

$$\begin{aligned}\varphi_3(X_1, X_2) \quad : \quad & \exists t_1(X_1(t_1) \wedge \exists t_2(t_1' \leq t_2 \wedge X_1(t_2) \wedge \\ & \forall t((t_1' \leq t \wedge t < t_2) \rightarrow \neg X_2(t))))\end{aligned}$$

An existential second-order formula:

$$\begin{aligned}\varphi_4(X_1) \quad : \quad & \exists X\, \exists t(X(0) \wedge \forall s(X(s) \leftrightarrow \neg X(s')) \wedge X(t) \\ & \wedge \forall s(s < t \rightarrow \neg X_1(s)) \wedge \forall s(t \leq s \rightarrow X_1(s)))\end{aligned}$$

Notation: $\varphi(X_1, \ldots, X_n)$ **indicates that at most the variables** $X_1, \ldots, X_n$ **occur free in** φ**, i.e. not in the scope of a quantifier.**

Semantics of S1S

- Use $\mathbb{N}$ as universe for the first-order variables
- Use $2^{\mathbb{N}}$ (the powerset of $\mathbb{N}$) as universe for the second-order variables
- Apply the standard semantics for boolean connectives and quantifiers

Write $(\mathbb{N}, 0, +1, <, P_1, \ldots, P_n) \models \varphi(X_1, \ldots, X_n)$ if φ is true in this semantics with $P_1 \subseteq \mathbb{N}, \ldots, P_n \subseteq \mathbb{N}$ as interpretations of $X_1, \ldots, X_n$.

We need only specify $\overline{P} = P_1, \ldots, P_n$:

This can be coded by the ω-word $\alpha(\overline{P}) \in ((\mathbb{B}^n)^\omega$ defined by

$$i \in P_k \iff (\alpha(i))_k = 1$$

We write simply: $\alpha(\overline{P}) \models \varphi(X_1, \ldots, X_n)$

Example

$$\varphi_3(X_1, X_2) \quad : \quad \exists t_1(X_1(t_1) \wedge \exists t_2(t'_1 \leq t_2 \wedge X_1(t_2) \wedge \forall t((\underbrace{t'_1 \leq t \wedge t < t_2}_{*}) \rightarrow \neg X_2(t))))$$

$*$ $t_1 < t < t_2$

Let P_1 be the set of even numbers, P_2 be the set of prime numbers.

$\alpha(P_1, P_2)$:

t	0	1	2	3	4	5	6	...	
	1	0	1	0	1	0	1	...	$\leftarrow P_1$
	0	0	1	1	0	1	0	...	$\leftarrow P_2$
	t_1		t_2						

We have $\alpha \models \varphi_3(X_1, X_2)$

An ω-language $L \subseteq (\mathbb{N}^n)^\omega$ is **S1S-definable** if for some S1S-formula $\varphi(X_1, \ldots, X_n)$ we have

$$L = \{\alpha \in (\mathbb{B}^n)^\omega \mid \alpha \models \varphi(X_1, \ldots, X_n)\}$$

Similarly: first-order definable, existential second-order definable

Examples:

1. $L = \{\alpha \in \mathbb{B}^\omega \mid \alpha \text{ has infinitely many } 1\}$
   is first-order definable by
   $\forall s \exists t (s < t \wedge X_1(t))$

2. $(00)^* 1^\omega$ is existential second-order definable by

$$\begin{aligned} \varphi_4(X_1) \quad : \quad & \exists X\, \exists t (X(0) \wedge \forall s (X(s) \leftrightarrow \neg X(s')) \wedge X(t) \\ & \wedge \forall s (s < t \rightarrow \neg X_1(s)) \wedge \forall s (t \leq s \rightarrow X_1(s))) \end{aligned}$$

3.5 Theorem: An LTL-definable ω-language is $S1S_1$-definable.

Illustration:

$$
\begin{array}{lcl}
\mathrm{GF}p_1 & : & \forall s \exists t (s \leq t \wedge X_1(t)) \\
\mathrm{XX}(p_2 \rightarrow \mathrm{F}p_1) & : & X_2(0'') \rightarrow \exists t (0'' \leq t \wedge X_1(t)) \\
\mathrm{F}(p_1 \wedge \mathrm{X}(\neg p_2 \mathrm{U} p_1)) & : & \exists t_1 (X_1(t_1) \wedge \exists t_2 (t'_1 \leq t_2 \wedge X_1(t_2) \wedge \\
 & & \forall t ((t'_1 \leq t \wedge t < t_2) \rightarrow \neg X_2(t))))
\end{array}
$$

Idea for general translation:

Describe the semantics of the temporal operators in S1S.

Inductive proof in the Exercises.

From Büchi automata to S1S

3.6 Theorem: A Büchi-recognizable ω-language is S1S-definable.

Idea: For Büchi automaton $\mathcal{A}$ over the input alphabet $\mathbb{B}^n$ find an S1S-formula $\varphi(X_1, \ldots, X_n)$ such that

$$\mathcal{A} \text{ accepts } \alpha \quad \text{iff} \quad \alpha \models \varphi(X_1, \ldots, X_n)$$

We express in $\varphi(X_1, \ldots, X_n)$: "There is a successful run of $\mathcal{A}$ on the input given by $X_1, \ldots, X_n$"

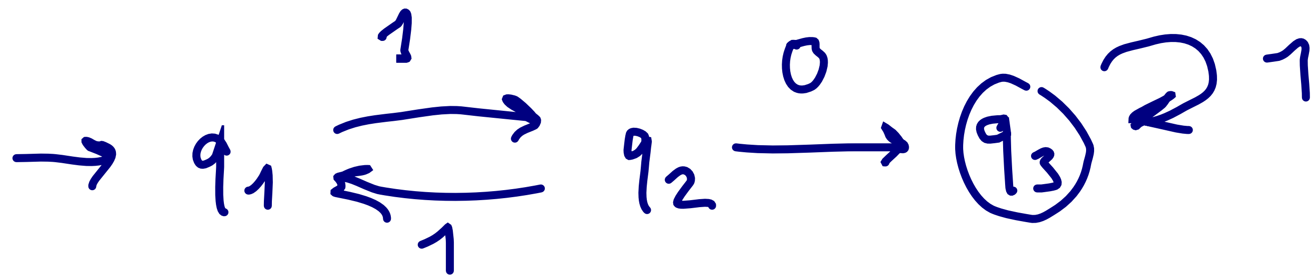
How to express the existence of a run?

Assume $\mathcal{A}$ has m states $q_1, \ldots, q_m$ (q_1 initial)

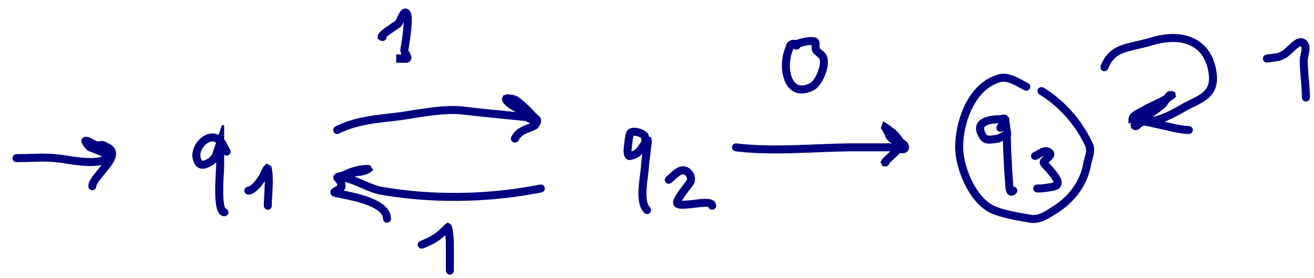
Then a run $\rho(0)\rho(1)\ldots$ is coded by m sets $Y_1, \ldots, Y_m$ with

$$i \in Y_k \iff \rho(i) = q_k$$

Example



Input:		1	1	1	0	1	1	1
Run:	q_1	1*	0	1*				
	q_2	0	1*	0	*			
	q_3	0	0	0		*	*	*



Formula $\varphi(X_1)$:

$$\exists Y_1 Y_2 Y_3\ (\mathrm{Partition}(Y_1,\ldots,Y_m) \wedge Y_1(0) \wedge \forall t($$

$$(Y_1(t) \wedge X_1(t) \wedge Y_2(t')) \vee (Y_2(t) \wedge X_1(t) \wedge Y_1(t'))$$

$$\vee (Y_2(t) \wedge \neg X_1(t) \wedge Y_3(t')) \vee (Y_3(t) \wedge X_1(t) \wedge Y_3(t')))$$

$$\wedge \forall s \exists t (s < t \wedge Y_3(t)))$$

Translation in the general case

Preparation 1:

$$\text{Partition}(Y_1, \ldots, Y_m) :=$$
$$\forall t \left(\bigvee_{i=1}^{m} Y_i(t) \right) \;\wedge\; \forall t \left(\neg \bigvee_{i \neq j} (Y_i(t) \wedge Y_j(t)) \right)$$

Preparation 2:

For $a \in \mathbb{B}^n$, say $a = (b_1, \ldots, b_n)$

we write $X_a(t)$ as an abbreviation for

$$(b_1)X_1(t) \wedge (b_2)X_2(t) \wedge \ldots (b_n)X_n(t)$$

where $(b_i) = \neg$ for $b_i = 0$, and b_i is empty for $b_i = 1$

$a = (1\ 0\ 1)$: $X_1(t) \wedge \neg X_2(t) \wedge X_3(t)$

Given the Büchi automaton $\mathcal{A} = (Q, \mathbb{B}^n, 1, \Delta, F)$

with $Q = \{1, \ldots, m\}$, define

$$
\begin{aligned}
&\varphi(X_1, \ldots, X_n) = \\
&\quad \exists Y_1 \ldots Y_m \, (\mathrm{Partition}(Y_1, \ldots, Y_m) \wedge Y_1(0) \\
&\qquad \wedge \forall t \left(\bigvee_{(i,a,j)\in\Delta} (Y_i(t) \wedge X_a(t) \wedge Y_j(t')) \right) \\
&\qquad \wedge \forall s \exists t (s < t \wedge \bigvee_{i\in F} Y_i(t))
\end{aligned}
$$

We conclude: A Büchi recognizable ω-language is existential second-order definable (within S1S).