Automata and Reactive Systems

Lecture No. 12

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Deciding E- and Büchi-Recognizability

Question:

Given a Muller automaton $\mathcal{A} = (Q, \Sigma, q_0, \delta, \mathcal{F})$,

can one decide algorithmically whether the language $L(\mathcal{A})$ is deterministic Büchi recognizable or even E-recognizable?

Recall: A deterministic automaton $\mathcal{A} = (Q, \Sigma, q_0, \delta, F)$ is called

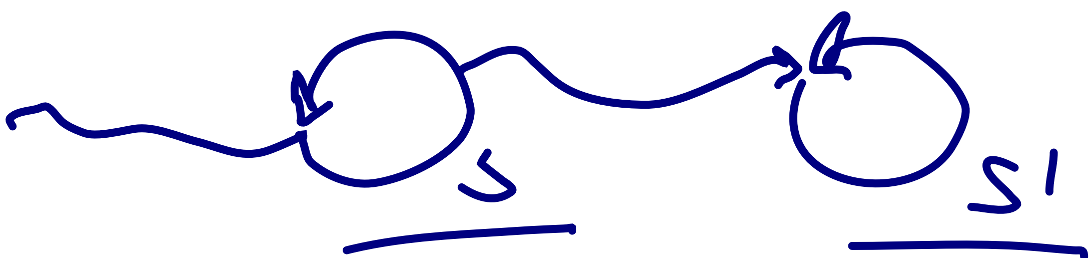
- **E-automaton** if it is used with the acceptance condition
  "run ρ is successful iff $\rho(i) \in F$ for some i"
- **Büchi automaton** if it is used with the acceptance condition
  "run ρ is successful iff $\rho(i) \in F$ for infinitely many i"

Loop Structure

By a loop we mean a nonempty subset $S \subseteq Q$ such that for all $s, s' \in S$ there is a word $w \in \Sigma^+$ with $\delta(s, w) = s'$

As sets in $\mathcal{F}$ we only take loops.

Call $\mathcal{F}$ closed under reachable loops iff each loop S' reachable from a loop $S \in \mathcal{F}$ also belongs to $\mathcal{F}$



Call $\mathcal{F}$ closed under superloops iff each loop $S' \supseteq S$ for a loop $S \in \mathcal{F}$ also belongs to $\mathcal{F}$

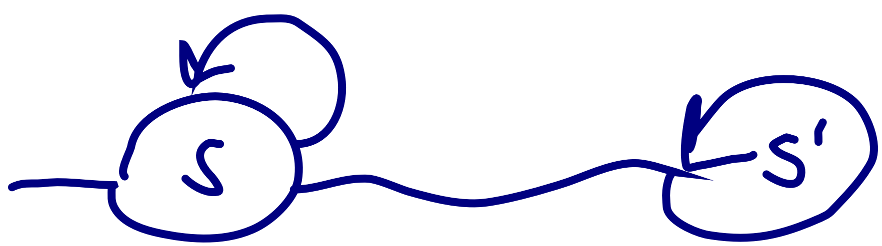
Remarks

Let $\mathcal{F}_1$ be the set of loops reachable from the loops of $\mathcal{F}$

Let $\mathcal{F}_2$ be the set of loops $S' \supseteq S$ with $S \in \mathcal{F}$

Remark:

1. $\mathcal{F}$ is closed under reachable loops iff $\mathcal{F} = \mathcal{F}_1$
2. $\mathcal{F}$ is closed superloops iff $\mathcal{F} = \mathcal{F}_2$
3. Each superloop of an $\mathcal{F}$-loop is also reachable from an $\mathcal{F}$-loop; so:
   If $\mathcal{F}$ is closed under reachable loops then also under superloops.



Landweber's Theorem

6.3 Theorem: Let $\mathcal{A} = (Q, \Sigma, q_0, \delta, \mathcal{F})$ be a Muller automaton.

1. $L(\mathcal{A})$ is E-recognizable iff $\mathcal{F}$ is closed under reachable loops.
2. $L(\mathcal{A})$ is deterministic Büchi recognizable iff $\mathcal{F}$ is closed under superloops.

Both closure conditions on the loop structure can be tested effectively.

6.4 Corollary: It is decidable whether the ω-language specified by a Muller automaton is E-recognizable, respectively deterministic Büchi recognizable.

Proof on E-Recognizability

Consider a Muller automaton $\mathcal{A}$ where $\mathcal{F}$ is closed under reachable loops.

Show that $L(\mathcal{A})$ is E-recognizable.

Define the E-automaton $\mathcal{A}' = (Q, \Sigma, q_0, \delta, \bigcup \mathcal{F})$. Then

$\mathcal{A}'$ E-accepts α

iff $\mathcal{A}'$ on α reaches a state from a loop $S \in \mathcal{F}$

iff $\mathcal{A}$ on α finally assumes a loop reachable from a loop $S \in \mathcal{F}$

iff (since $\mathcal{F}$ is closed under reachable loops)
$\mathcal{A}$ on α finally assumes a loop in $\mathcal{F}$

iff $\mathcal{A}$ accepts α

E-Recognizability: The Converse

Assume $L(\mathcal{A})$ is recognized by the E-automaton $\mathcal{B}$

Let $q \in S \in \mathcal{F}$. Show: Loop S' reachable from $q \Rightarrow S' \in \mathcal{F}$

Pick $u \in \Sigma^*$ with $\delta_{\mathcal{A}}(q_0, u) = q$

Continue u by $\gamma \in \Sigma^\omega$, to cause $\mathcal{A}$ to loop through S; so $u\gamma \in L(\mathcal{A})$

The E-automaton $\mathcal{B}$ on $u\gamma$ somewhere reaches a final state, say after prefix uv this has happened.

We extend uv by w, causing $\mathcal{A}$ in loop S to return to q

Consider any loop S' reachable from q: $\mathcal{A}$ will finally assume S' on input $uvw\gamma'$ for suitable γ'

Due to the prefix uv, $\mathcal{B}$ E-accepts $uvw\gamma'$

So also $\mathcal{A}$ accepts $uvw\gamma'$, hence $S' \in \mathcal{F}$

Proof on Büchi Recognizability

Consider a Muller automaton $\mathcal{A} = (Q, \Sigma, q_0, \delta, \mathcal{F})$ where $\mathcal{F}$ is closed under superloops.

Construct a Büchi automaton $\mathcal{A}'$ as follows, over the state set $Q \times 2^Q$

In the first component, $\mathcal{A}'$ simulates $\mathcal{A}$

In the second component the visited states are accumulated until a set from $\mathcal{F}$ is reached or surpassed,

then the second component is reset to $\emptyset$ (final state).

So the Büchi automaton $\mathcal{A}'$ accepts α iff $\mathcal{A}$ on α infinitely often passes through loops $S' \supseteq S$ for loops $S \in \mathcal{F}$

Show: The Büchi automaton $\mathcal{A}'$ is equivalent to $\mathcal{A}$

$\mathcal{A}'$ is equivalent to $\mathcal{A}$

$\mathcal{A}'$ accepts α

iff on input α, $\mathcal{A}$ infinitely often passes through loops $S' \supseteq S$ where $S \in \mathcal{F}$

iff (since only finitely many such S' exist) for some $S' \supseteq S$ with $S \in \mathcal{F}$, precisely the states of S' are visited infinitely often

iff (since $\mathcal{F}$ is closed under superloops) for some $S \in \mathcal{F}$, precisely the states of S are visited infinitely often

iff $\mathcal{A}$ accepts α

Büchi Recognizability: The Converse

Assume that the Büchi automaton $\mathcal{B}$ with final state set F recognizes $L(\mathcal{A})$

Show for a superloop S' of loop $S \in \mathcal{F}$ that also $S' \in \mathcal{F}$

Task: Find $\alpha \in L(\mathcal{A})$ which finally lets $\mathcal{A}$ cycle through S'

Pick $q \in S$, reached by $\mathcal{A}$ via w. Continue w by γ such that $\mathcal{A}$ loops through S and hence accepts.

So $\mathcal{B}$ on $w\gamma$ infinitely often visits F, say first after wu_1

Continuation via v_1 through S leads $\mathcal{A}$ back to q, then a travel through the superloop S' via x_1 again back to q

$\underbrace{u\ v}_{S}\ \underbrace{x}_{S'}$

Repetition yields $wu_1v_1x_1u_2v_2x_2\ldots$ such that $\mathcal{B}$ assumes a final state after each u_i; so $\mathcal{A}$ accepts,

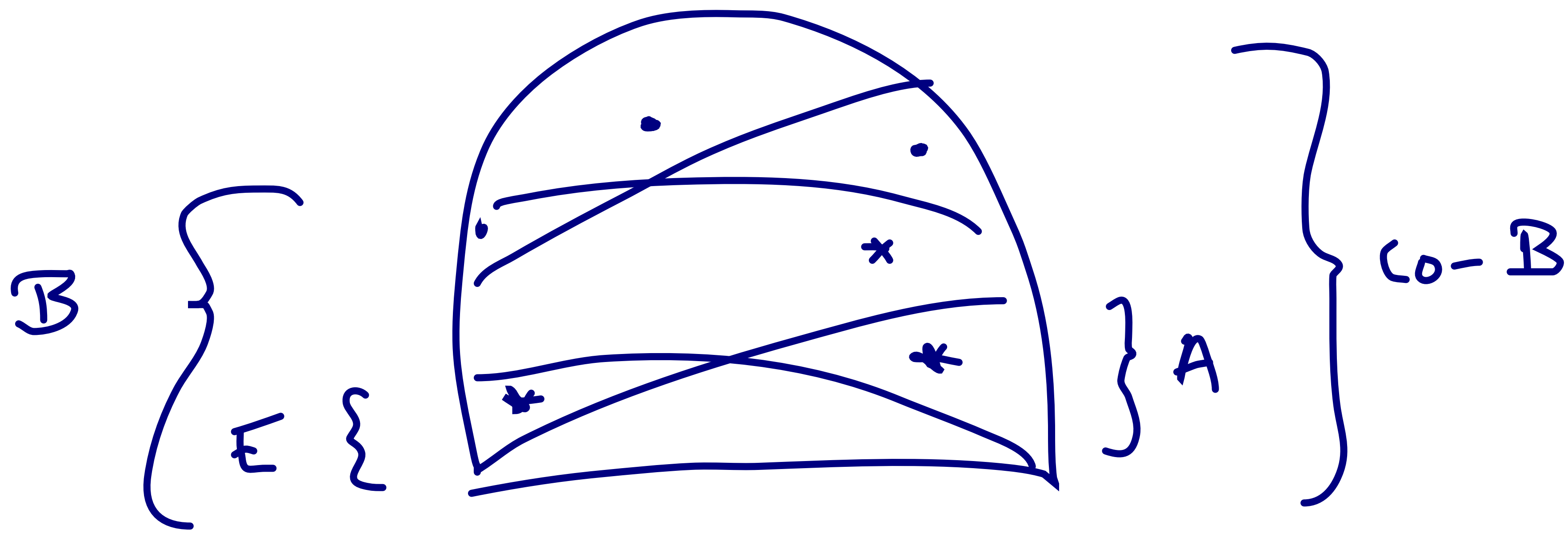
and due to the x_i, $\mathcal{A}$ visits the S'-states again and again.

Dual version of Landweber's Theorem on Büchi recognizability

6.5 Theorem: Let $\mathcal{A} = (Q, \Sigma, q_0, \delta, \mathcal{F})$ be a Muller automaton.

$L(\mathcal{A})$ is deterministic co-Büchi recognizable iff $\mathcal{F}$ is closed under subloops.

Proof: Exercise.



Boolean combinations

Recall:

An ω-language is regular (say recognized by a deterministic Muller automaton) iff it is a boolean combination of deterministic Büchi recognizable ω-languages.

We analyze the boolean combinations of E-recognizable ω-languages.

First aim: Characterization by an analogue of Muller automata: Staiger-Wagner automata.

Second aim: Relation to deterministic Büchi and co-Büchi recognizable ω-languages.

Staiger-Wagner automata

For a run $\rho \in Q^\omega$ let

$$\mathrm{Occ}(\rho) := \{q \in Q \mid \exists i : \rho(i) \in F\}$$

Remark: Given $F \subseteq Q$, a run ρ is

- E-accepting iff $\mathrm{Occ}(\rho) \cap F \neq \emptyset$
- A-accepting iff $\mathrm{Occ}(\rho) \subseteq F$

An ω-automaton $\mathcal{A} = (Q, \Sigma, q_0, \delta, \mathcal{F})$ with $\mathcal{F} \subseteq 2^Q$ is called **Staiger-Wagner automaton** if it is used with the following "Staiger-Wagner acceptance condition":

ρ is successful if $\mathrm{Occ}(\rho) \in \mathcal{F}$,

i.e the states which occur in ρ form a set in $\mathcal{F}$