Automata and Reactive Systems

Lecture No. 11

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6 Classification of Regular ω-Languages

We have introduced properties of system runs:

- **guarantee condition ("Sometime p_1 becomes true")**
- **safety condition ("Always p_1 is true")**
- **recurrence condition ("Again and again, p_1 is true")**

Plan:

1. **Definition of a natural classification scheme based on deterministic automata**
2. **Comparison of the levels of this classification**
3. **Decision to which level a given property belongs**

The four basic types of sequence properties

Intuition:

- **Guarantee condition requires that some finite prefix has a certain property**
- **Safety condition requires that all finite prefixes have a certain property**
- **Recurrence condition requires that infinitely many finite prefixes have a certain property**
- **Persistence condition requires that almost all (i.e. from a certain point onwards all) finite prefixes have a certain property**

We shall describe the prefix properties by deterministic automata

The four basic acceptance conditions

Let $\mathcal{A} = (Q, \Sigma, q_0, \delta, F)$ be a deterministic automaton.

We introduce four acceptance conditions for runs of $\mathcal{A}$.

Call a run ρ

- E-accepting if for some i, the state $\rho(i)$ belongs to F
- A-accepting if for all i, the state $\rho(i)$ belongs to F
- Büchi accepting if for infinitely many i, $\rho(i) \in F$
- co-Büchi accepting if for almost all i, $\rho(i) \in F$

Formally, the acceptance conditions are

- $\exists i\ \rho(i) \in F, \quad \forall i\ \rho(i) \in F$
- $\forall j\ \exists i \geq j\ \rho(i) \in F, \quad \exists j\ \forall i \geq j\ \rho(i) \in F$

Recognizability

We speak of a (deterministic)

E-automaton, A-automaton, Büchi automaton, co-Büchi automaton

if the E-, A-, Büchi, co-Büchi acceptance condition is used

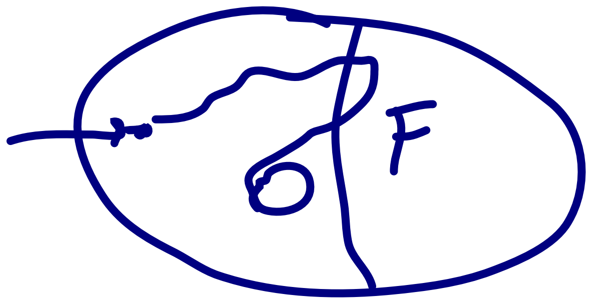
The corresponding ω-languages are called

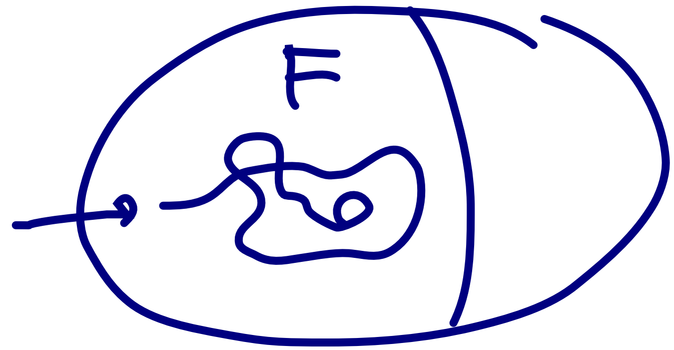
E-recognizable, A-recognizable, deterministic Büchi recognizable, deterministic co-Büchi recognizable.

In the following we always consider deterministic automata (and sometimes skip the term “deterministic”).

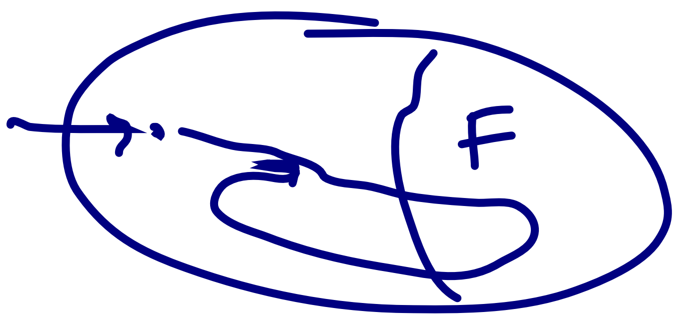
Pictoral illustration of accepting paths:

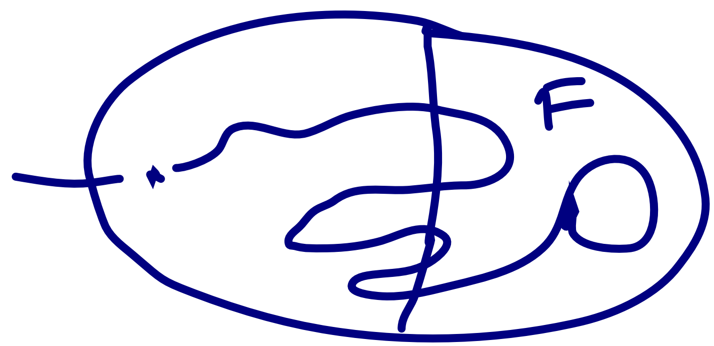
E-acceptance and A-acceptance





Büchi acceptance and co-Büchi acceptance





Characterization of E- and Büchi-recognizability

Remark:

(a) An ω-language $L \subseteq \Sigma^\omega$ is E-recognizable iff it is of the form $L = U \cdot \Sigma^\omega$ for some regular $*$-language U.

(b) An ω-language $L \subseteq \Sigma^\omega$ is det. Büchi-recognizable iff it is of the form $\lim(U)$ for some regular $*$-language U.

Proof of (a): [(b) was shown earlier]:

Let L be E-recognized by $\mathcal{A} = (Q, \Sigma, q_0, \delta, F)$

Let U be the $*$-language recognized by $\mathcal{A}$

$\mathcal{A}$ accepts α

iff the unique run of $\mathcal{A}$ reaches F after a finite prefix of α

iff some prefix of α belongs to U

iff $\alpha \in U \cdot \Sigma^\omega$

Complementation and Dual Acceptance

6.1 Lemma (Complementation Lemma):

(a) An ω-language $L \subseteq \Sigma^\omega$ is E-recognizable iff the complement language $\Sigma^\omega \setminus L$ is A-recognizable.

(b) An ω-language $L \subseteq \Sigma^\omega$ is Büchi-recognizable iff the complement language $\Sigma^\omega \setminus L$ is co-Büchi-recognizable.

Proof: Assume L is E-recognized by $\mathcal{A} = (Q, \Sigma, q_0, \delta, F)$.

$\alpha \in \Sigma^\omega \setminus L$

iff α has no prefix leading $\mathcal{A}$ into F

iff all prefixes of α lead $\mathcal{A}$ into states of $Q \setminus F$

iff α is A-accepted by $\mathcal{A}' = (Q, \Sigma, q_0, \delta, Q \setminus F)$

The other cases are analogous.

E versus A, Büchi versus co-Büchi

Remark:

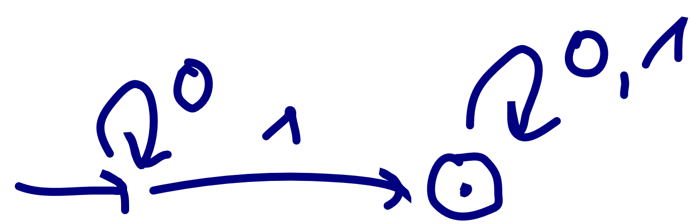
1. $\mathbb{B}^* \cdot 1 \cdot \mathbb{B}^\omega$ is E-recognizable, but not A-recognizable
2. $\{0^\omega\}$ is A-recognizable but not E-recognizable
3. $(0^*1)^\omega$ is Büchi recognizable but not co-Büchi recognizable
4. $\mathbb{B}^*0^\omega$ is co-Büchi recognizable but not Büchi recognizable.

Note:

$$\{0^\omega\} = \mathbb{B}^\omega \setminus (\mathbb{B}^* \cdot 1 \cdot \mathbb{B}^\omega), \quad \mathbb{B}^*0^\omega = \mathbb{B}^\omega \setminus (0^*1)^\omega$$

Proof:

ad 1.: E-recognizability is clear.

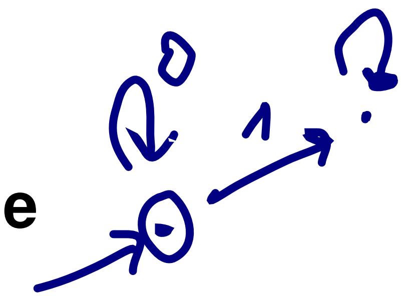


Assume $\mathbb{B}^* \cdot 1 \cdot \mathbb{B}^\omega$ is A-recognizable, say by $\mathcal{A}$ with n states.

Consider $\mathcal{A}$ on $0^n 1 0^\omega$; all states of the run are final.

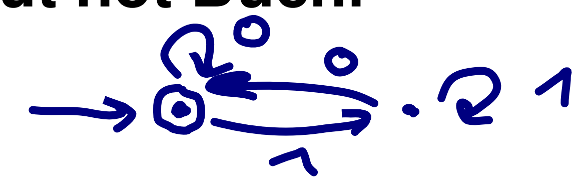
Before input letter 1 there is a state repetition (loop of final states).

So with this loop $\mathcal{A}$ accepts also the input word 0^ω, contradiction.



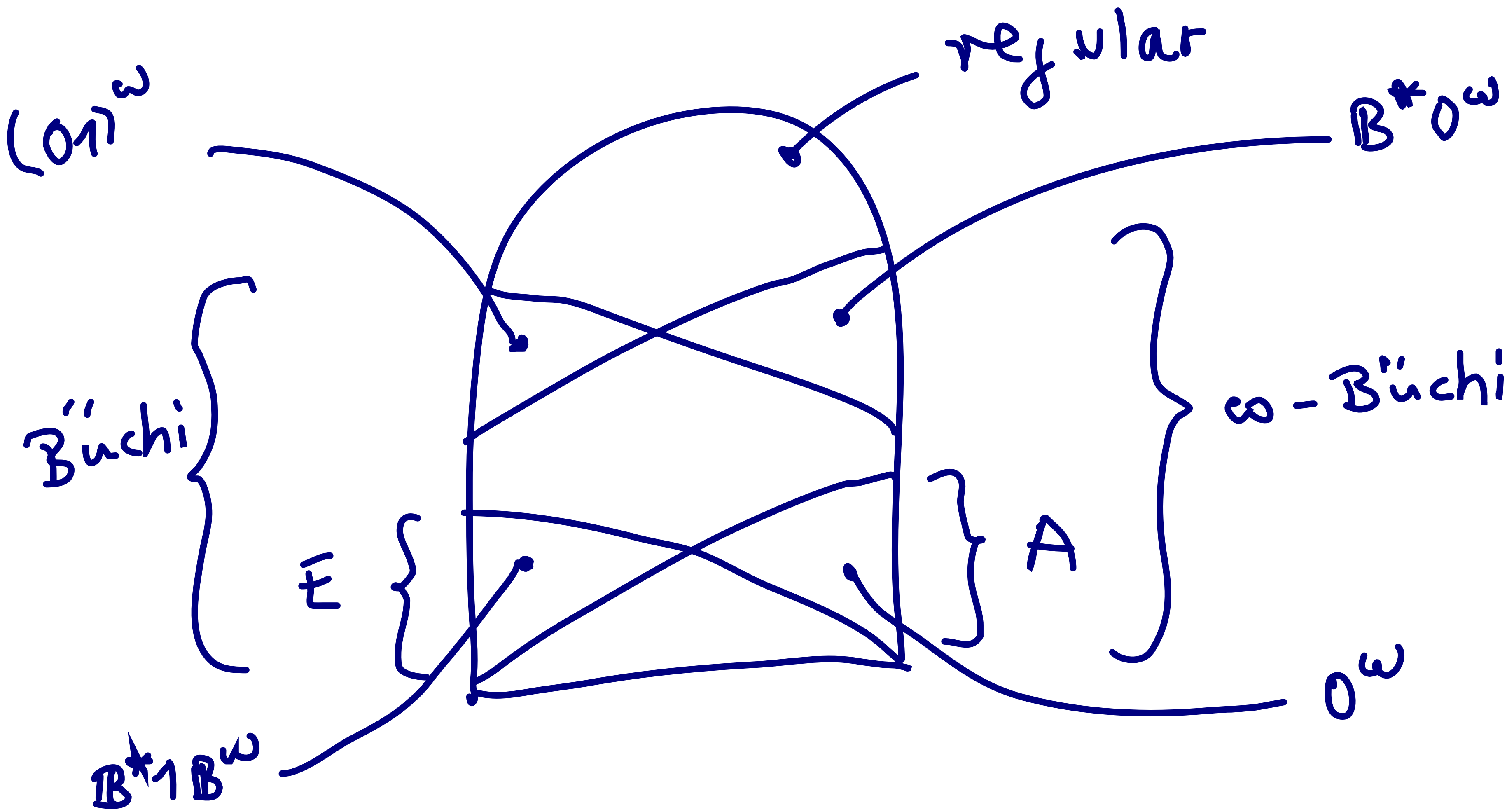
ad **2.**: $\{0^\omega\}$ is A-recognizable but not E-recognizable follows by the Complement Lemma.

ad **4.**: $\mathbb{B}^* 0^\omega$ is co-Büchi recognizable but not Büchi recognizable was shown earlier.



ad **3.**: $(0^* 1)^\omega$ Büchi recognizable but not co-Büchi recognizable follows by the Complement Lemma.

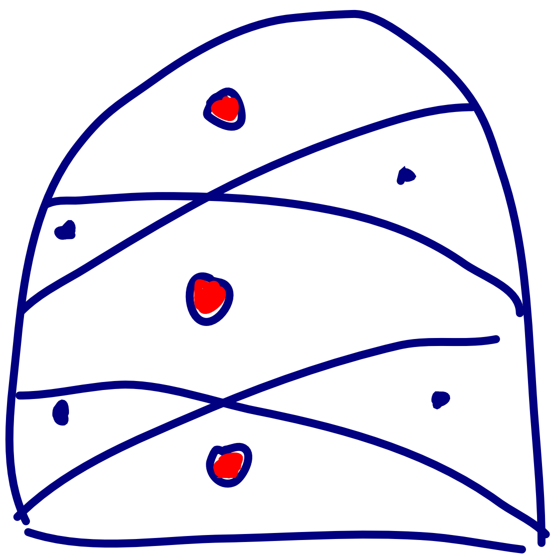
For the classes of E-, A-, deterministic Büchi-, and deterministic co-Büchi recognizable ω-languages, the following inclusion diagram holds:



Proof strategy

(a) **For the inclusion claims show: An E-recognizable ω-language is both deterministic Büchi and co-Büchi recognizable. (The other claims are clear.)**

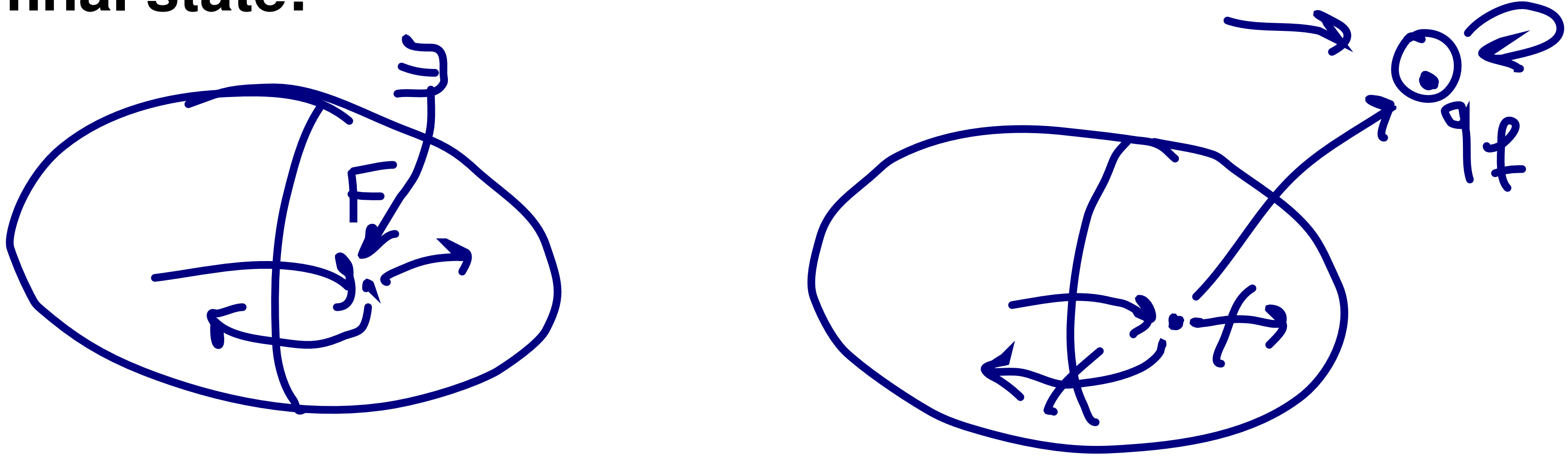
(b) **For the properness of the inclusions we have to exhibit seven ω-languages. We have exhibited already four of them:**



ad **(a)**:

An E-recognizable ω-language is both deterministic Büchi and co-Büchi recognizable.

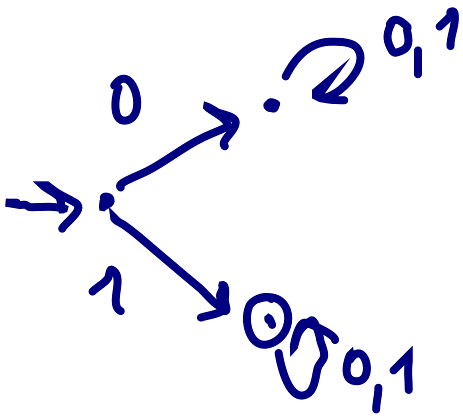
Modify a given E-automaton by changing all transitions from final states into transitions to a new state q_f which is now the only final state:

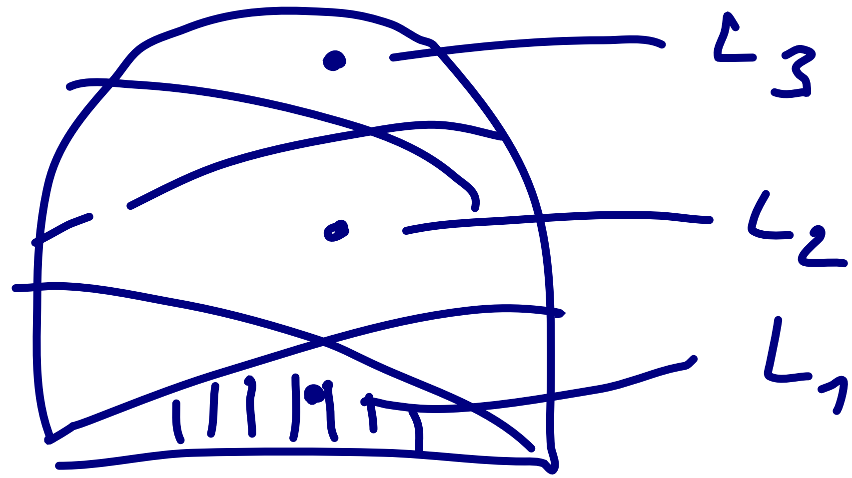


Then the E-automaton reaches a final state iff the new automaton eventually stays in q_f (i.e. Büchi accepts and co-Büchi accepts).

Three example languages

ad **(b)**:





1. $L_1 := 1 \cdot \mathbb{B}^\omega$ is E- and A-recognizable.

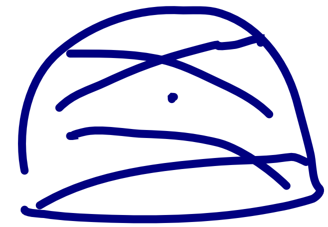
2. $L_2 := \{\alpha \in \mathbb{B}^\omega \mid$ 00 occurs, but 11 never occurs in $\alpha\}$

3. $L_3 := \{\alpha \in \mathbb{B}^\omega \mid$ 00 occurs infinitely often,
   but 11 occurs only finitely often in $\alpha\}$

L_1: clear. L_3: Exercise

$L_2 := \{\alpha \in \mathbb{B}^\omega \mid$ 00 occurs, but 11 never occurs in $\alpha\}$

is neither E- nor A-recognizable, but both deterministic Büchi and co-Büchi recognizable.



Assume L is E-recognizable, say by $\mathcal{A}$ with n states.

On input 0^ω a final state occurs within prefix 0^n.

Then $\mathcal{A}$ E-accepts also 0^n001^ω, contradiction.

Assume L is A-recognizable, say by $\mathcal{A}$ with n states.

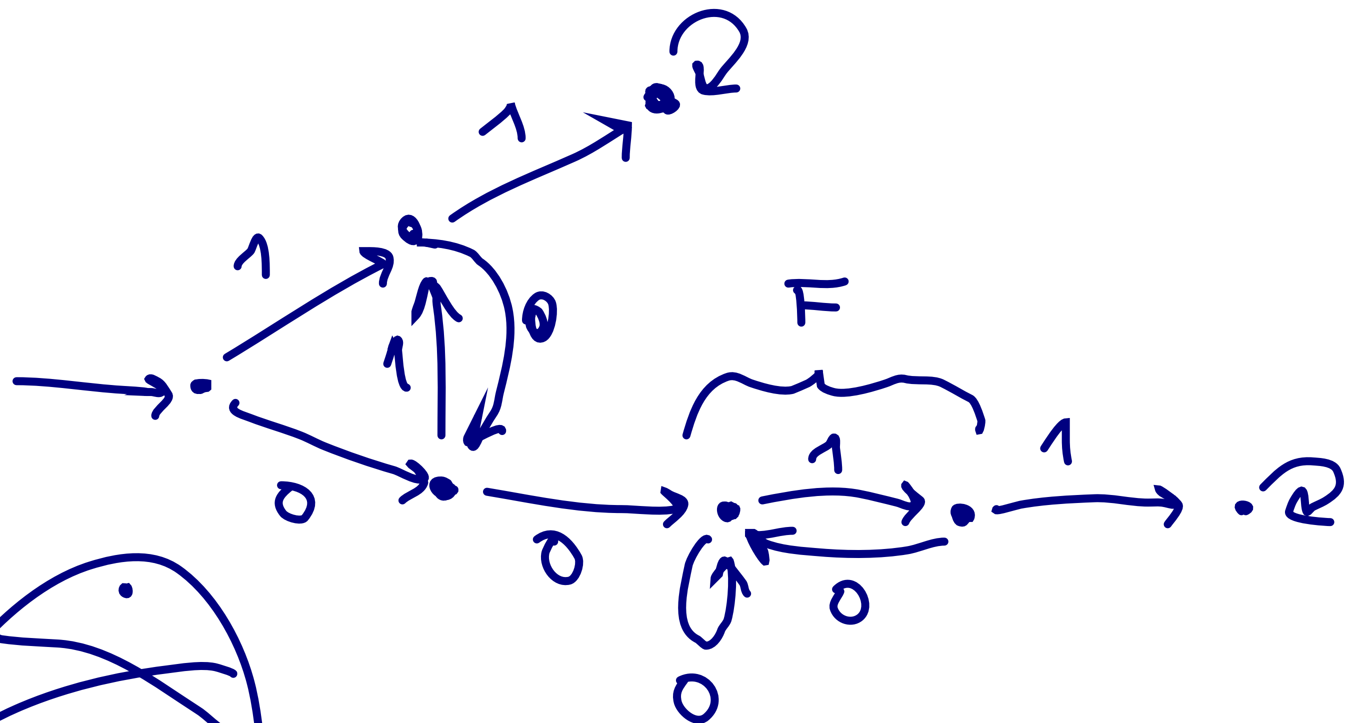
On input $(01)^{n+1}0^\omega$ only final states are visited.

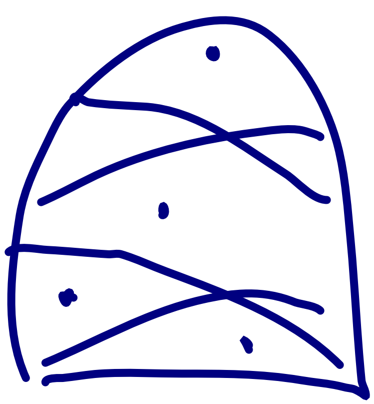
Before the 0-suffix a state repetition occurs after letters 1.

So also $(01)^\omega$ is accepted, a contradiction.

$L_2 := \{\alpha \in \mathbb{B}^\omega \mid$ **00 occurs, but 11 never occurs in** $\alpha\}$

is both deterministic Büchi and co-Büchi recognizable.





Deciding the levels

Question: Given a Muller automaton $\mathcal{A} = (Q, \Sigma, q_0, \delta, \mathcal{F})$,

can one decide algorithmically whether the language $L(\mathcal{A})$ is already det. Büchi recognizable or even E-recognizable?

We shall formulate criteria which can be checked effectively.

So the hierarchy of regular ω-languages is “effective”.

We assume that in a given Muller automaton each state is reachable from the initial state via some finite input word.

By a loop we mean a nonempty subset $S \subseteq Q$ such that for all $s, s' \in S$ there is a word $w \in \Sigma^+$ with $\delta(s, w) = s'$

Remark: Any set $\mathrm{Inf}(\rho)$ is a loop. We may assume that $\mathcal{F}$ only contains loops.

Deciding E-recognizability

Given $\mathcal{A} = (Q, \Sigma, q_0, \delta, \mathcal{F})$ define

$$\mathcal{F}_1 = \{S \subseteq Q \mid S \text{ is loop and reachable from a loop of } \mathcal{F}\}$$

Remark: $\mathcal{F} \subseteq \mathcal{F}_1$

$\{q\} \in \mathcal{F}$

6.2 Theorem (E-Recognizability):

Given a Muller automaton $\mathcal{A}$ as above, $L(\mathcal{A})$ is E-recognizable iff $\mathcal{F} = \mathcal{F}_1$

(in other words: each loop reachable from a loop in $\mathcal{F}$ already is in $\mathcal{F}$ itself).

Note: This closure property of the system of accepting loops can be checked effectively.