# Carnegie Mellon University
# MSCF Program
# 46-956 Introduction to Fixed Income

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Mini 1

Lecture Notes for Week 3

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# *Page 3.1:* Comment on Yield to Maturity

If the spot-rate curve is not flat, then the yield to maturity **cannot** be used to discount individual cash flows of a bond (except in the case of a ZCB where there is only one payment).

The yield to maturity **can always** be used to discount all of the bond's cash flows, add the results together, and obtain a price for the bond. In other words, the YTM applies to the bond *as a whole* rather than to *individual payments*.

Even if bonds have the same maturity (and the yield curve is not flat), you cannot use the YTM of one bond to price another unless they have the same coupon.

# *Page 3.2:* DV01, Duration, & Convexity

Single interest rate factor (variable): $y$

Security with current price a function of the interest rate factor: $P = f(y)$

DV01: $DV01 = -\frac{f'(y)}{10{,}000}$

Duration: $D = -\frac{f'(y)}{P}$

Convexity: $C = \frac{f''(y)}{P}$

# *Page 3.3:* Use of Difference Quotients

If we don't have an analytical expression giving the price in terms of $y$, we can use difference quotients instead of derivatives. In particular, we can use

$$DV01 = -\frac{\Delta P}{10,000 \Delta y},$$

$$D = -\frac{\Delta P}{P \Delta y}.$$

The reason for the minus sign in the definitions of DV01 and duration is that for traditional bonds with fixed cash flows, an increase in the interest rate factor leads to a decrease in price. With the sign convention as above, DV01 and duration will be positive for ZCBs, Coupon Bonds, and Annuities. For these securities, the interest rate factor is usually taken to be the yield to maturity.

# *Page 3.4:* First- and Second-Order Approximations

**First-Order Approximation**

$$\frac{\Delta P}{P} = -D\Delta y$$

**Second-Order Approximation**

$$\frac{\Delta P}{P} = -D\Delta y + \frac{1}{2}C(\Delta y)^2$$

**Remark 3.1**: Some authors define convexity as

$$\frac{f''(y)}{2P}.$$

In this case, there is no factor of $\frac{1}{2}$ in the second-order approximation.

# *Page 3.5:* Sign Consequences for $D$ and $C$

**Remark 3.2**: It is very important to notice that

(i) Positive duration leads to price increases when rates fall and price decreases when rates rise.

(ii) Negative duration leads to price decreases when rates fall and price increases when rates rise.

(iii) Positive convexity makes a positive contribution to the price both when rates rise and when they fall.

(iv) Negative convexity makes a negative contribution to the price both when rates rise and when they fall.

Some terminology used in relation to convexity:

- Positive Convexity – Long on Volatility
- Negative Convexity – Short on Volatility

# *Page 3.6:* Portfolios of Fixed-Income Securities

Although DV01 is frequently quoted per some amount of face, here we shall regard prices as *dollar prices* and the DV01 of a portfolio will be based on the total price of the portfolio (rather than the price per some amount of face). Since DV01 reflects total price change, it scales with the amount invested. Since duration represents a relative price change, it does not scale with the amount invested.

In other words, suppose that two investors $A$ and $B$ each invest all of their capital in one and the same bond and that investor $A$ invests twice as much as investor $B$. Investor $A$'s DV01 will be twice as much as investors $B$'s DV01. However, their durations will be the same (as will their convexities).

# *Page 3.7:* Portfolios (Cont.)

Suppose that a portfolio is being built using fixed-income securities $S^{(1)}, S^{(2)}, \cdots, S^{(N)}$. Here we think of each of the securities as having dollar prices $P^{(1)}, P^{(2)}, \cdots, P^{(N)}$ (not expressed as a percentage of face). An investor builds a portfolio with total initial capital $X > 0$. The investor buys $\alpha^{(i)}$ shares of $S^{(i)}$, so that

$$X = \alpha^{(1)} P^{(1)} + \alpha^{(2)} P^{(2)} + \cdots + \alpha^{(N)} P^{(N)}.$$

(Some of the $\alpha^{(i)}$ can be negative here.)

For the entire portfolio we have

 Portfolios (Cont.)

$$DV01 = \sum_{i=1}^{N} \alpha^{(i)} DV01^{(i)},$$

$$D = \sum_{i=1}^{N} \left( \frac{\alpha^{(i)} P^{(i)}}{X} \right) D^{(i)},$$

$$C = \sum_{i=1}^{N} \left( \frac{\alpha^{(i)} P^{(i)}}{X} \right) C^{(i)},$$

where $DV01^{(i)}$, $D^{(i)}$, and $C^{(i)}$ are the DV01, duration, and convexity of the $i^{th}$ security.

**Remark 3.3**: The duration and convexity of each piece is weighted by the percentage of capital associated with that piece.

# *Page 3.9:* Portfolios (Cont.)

**Remark 3.4**: In this class I will use the terminology *duration* and *convexity* for portfolios *only* when the net position is *long*, i.e. for portfolios with a positive price. The meanings of duration and convexity can become confusing for a portfolio with negative price. Some of the statements in the book make sense only for portfolios with a positive price. If we want to measure the interest-rate exposure of a portfolio with a negative (or zero) net price, we can use duration and convexity to analyze components of the portfolio and combine the results. Talking about the DV01 of a portfolio with zero or negative net price is perfectly ok.

 Hedging Based on Duration and Convexity

**Remark 3.5**: If two portfolios have the same price and the same duration, then for small changes in $y$, the price changes in the two portfolios will be approximately the same (both in magnitude and sign), i.e. they have the same DV01s. This very simple idea is frequently used to construct hedges. A better hedge might be obtained by matching prices, durations, and convexities.

**Remark 3.6**: Given any three numbers $P, \delta, \gamma$ with $P > 0$ and any three maturities $T_3 > T_2 > T_1 > 0$, it is possible to construct a portfolio consisting only of ZCBs with maturities $T_1$, $T_2$, $T_3$ such that the portfolio has price $P$, duration $\delta$ and convexity $\gamma$. You will be asked to prove this result for homework.

# *Page 3.11:* Yield Based DV01 for Coupon Bonds

$$P = f(y) = F\left[(1+\frac{y}{2})^{-2T} + \frac{q}{2}\sum_{i=1}^{2T}(1+\frac{y}{2})^{-i}\right]$$

$$DV01 = -\frac{f'(y)}{10,000}$$

$$DV01 = \frac{F}{10,000} \times \frac{1}{1+\frac{y}{2}}\left[\frac{T}{(1+\frac{y}{2})^{2T}} + \frac{q}{2}\sum_{i=1}^{2T}\frac{\frac{i}{2}}{(1+\frac{y}{2})^{i}}\right]$$

# *Page 3.12:* DV01 for Coupon Bonds (Cont.)

**Remark 3.7**: A more compact expression for the DV01 can be obtained by summing the geometric series in the formula for $P$ to obtain

$$P = f(y) = F\left[(1+\frac{y}{2})^{-2T} + \frac{q}{y}(1-(1+\frac{y}{2})^{-2T})\right],$$

and then differentiating to obtain

$$DV01 = \frac{F}{10,000}\left[\frac{q}{y^2}\left(1-\frac{1}{(1+\frac{y}{2})^{2T}}\right) + \left(1-\frac{q}{y}\right)\frac{T}{(1+\frac{y}{2})^{2T+1}}\right].$$

# *Page 3.13:* Macaulay Duration

For securities with deterministic cash flows, a notion of duration based on yield to maturity was introduced by F.R. Macaulay in 1938. It is closely related to (but not identical to) the duration obtained by differentiating the price with respect to the yield.

Let $T_i = i/2$, for $i = 1, 2, \cdots, 2T$ and consider a security (with maturity $T$) that makes payments $F_i \geq 0$ at the times $T_i$ and has yield to maturity $y$, so that the price is given by

$$P = \sum_{i=1}^{2T} \frac{F_i}{(1 + \frac{y}{2})^i}.$$

Observe that

$$D = -\frac{f'(y)}{P} = \frac{1}{P\left(1 + \frac{y}{2}\right)} \sum_{i=1}^{2T} \frac{\frac{i}{2} F_i}{(1 + \frac{y}{2})^i}.$$

# *Page 3.14:* Macaulay Duration (Cont.)

The Macaulay duration is defined by

$$D_{Mac} = \frac{1}{P}\sum_{i=1}^{2T}\frac{\frac{i}{2}F_i}{(1+\frac{y}{2})^i} = \frac{1}{P}\sum_{i=1}^{2T}\frac{T_iF_i}{(1+\frac{y}{2})^i}.$$

The Macaulay duration is simply a weighted average of the payment times with weights being the present values of the payment amounts with discount factors computed using the yield to maturity.

Observe that $D_{Mac}$, can be expressed in terms of duration $D$ via the formula

$$D_{Mac} = \left(1+\frac{y}{2}\right)D.$$

# *Page 3.15:* Macaulay Duration (Cont.)

The numerical value of the Macaulay duration is fairly close to the numerical value of the duration. The Macaulay duration has the very nice property that it is exactly equal to the maturity for a zero coupon bond.

Notice that in terms of $D_{Mac}$, the first-order approximation becomes

$$\frac{\Delta P}{P} = -\frac{D_{Mac}}{(1 + \frac{y}{2})}\Delta y.$$

# *Page 3.16:* Duration for Coupon Bonds

Unless stated otherwise, when we talk about the duration of a coupon bond, it is understood that the interest rate factor is the yield to maturity. It follows from the expressions for DV01 that

$$D = \frac{F}{P} \times \frac{1}{1+\frac{y}{2}} \left[ \frac{T}{(1+\frac{y}{2})^{2T}} + \frac{q}{2} \sum_{i=1}^{2T} \frac{\frac{i}{2}}{(1+\frac{y}{2})^{i}} \right],$$

and

$$D = \frac{F}{P} \left[ \frac{q}{y^2} \left( 1 - \frac{1}{(1+\frac{y}{2})^{2T}} \right) + \left( 1 - \frac{q}{y} \right) \frac{T}{(1+\frac{y}{2})^{2T+1}} \right].$$

# *Page 3.17:* Duration for Coupon Bonds (Cont.)

For coupon bonds, the Macaulay duration is given by

$$D_{Mac} = \frac{F}{P}\left[\frac{T}{(1+\frac{y}{2})^{2T}} + \frac{q}{2}\sum_{i=1}^{2T}\frac{\frac{i}{2}}{(1+\frac{y}{2})^{i}}\right],$$

and

$$D_{Mac} = \frac{F}{P}\left(1+\frac{y}{2}\right)\left[\frac{q}{y^2}\left(1-\frac{1}{(1+\frac{y}{2})^{2T}}\right) + \left(1-\frac{q}{y}\right)\frac{T}{(1+\frac{y}{2})^{2T+1}}\right].$$

# *Page 3.18:* Example 3.1

**Example 3.1**: Consider a coupon bond with $q = .04$ and $T = 20$, and yield to maturity $y = .04231$. Let us find the duration and Macaulay duration. Using the relation between price and yield, we find that with $\lambda = (1 + .04231/2)^{-1}$:

$$P = 100\lambda^{40} + 2\frac{\lambda}{1-\lambda}(1 - \lambda^{40}) = 96.90351$$

per 100 face. Substituting into the formulas, we find that

$$D = 13.555$$

$$D_{Mac} = \left(1 + \frac{.04231}{2}\right) 13.555 = 13.842$$

 
# Example 3.2

**Example 3.2**: Consider a coupon bond with $F = 100,000$, $q = .04$, $T = 20$, and yield to maturity $y = .04231$. Find the DV01 of the bond and use the DV01 to estimate the price of the bond if $y$ is decreased by 17 basis points.

From Example 3.1, we know that $P = 96,903.51$ and $D = 13.555.$ It follows that

$$DV01 = \frac{PD}{10,000} = \frac{96,903.51(13.555)}{10,000} = 131.353.$$

If the yield decreases by 17 basis points the price of the bond will increase by approximately

$$131.353(17) = 2,233.00.$$

If we use the exact relation between price and yield, we find that when $y = .04061$ (a decrease of 17 bp), the price of the bond is 99,170.10. The exact change in bond price is an increase of 2266.59.

# *Page 3.20:* Duration and DV01 for Par Coupon Bonds

For par coupon bonds, we have

$$\frac{F}{P} = 1, \quad \frac{q}{y^2} = \frac{1}{y}, \quad \left(1 - \frac{q}{y}\right) = 0,$$

and the formulas for duration and DV01 simplify considerably:

$$D = \frac{1}{y}\left[1 - \frac{1}{(1+\frac{y}{2})^{2T}}\right],$$

$$D_{Mac} = \left(\frac{1}{y} + \frac{1}{2}\right)\left[1 - \frac{1}{(1+\frac{y}{2})^{2T}}\right],$$

$$DV01 = \frac{F}{10,000y}\left[1 - \frac{1}{(1+\frac{y}{2})^{2T}}\right].$$

# *Page 3.21:* 10-Year Treasuries

**Example 3.3**: The current value of the 10-year par-coupon yield is approximately $y = 1.34\%$. The duration of the 10-year par bond is therefore

$$D = \frac{1}{.0134}\left[1 - \frac{1}{(1+\frac{.0134}{2})^{20}}\right] \approx 9.3298.$$

This means that the DV01 of the 10-year par bond is approximately \$9.33 per \$10,000 face (i.e. 9.33 cents per \$100 face).

For purposes of illustration, let's use a value of $D = 9.3$ for the 10-year par bond. Suppose that we own a 10-year par bond with face value \$10,000. If the yield increases by 4 basis points to 1.38%, then the value of the bond will decrease by approximately \$37.20 ($4 \times 9.3$). If we have a 10-year par bond with face value \$1,000,000 and the yield increases from 1.34% to 1.38% the value of the bond will drop by approximately \$3,720 ($100 \times 9.3 \times 4$).

# *Page 3.22:* 10-Year Treasuries (Cont.)

**Remark**: With a value of .667% (September 2020) for the par-coupon yield we would get $D = 9.658$. With a value of 1.63% (September 10, 2019) we would get $D = 9.133$. With a value of 2.98% (September 12, 2018) for the 10-year par-coupon yield we would get $D = 8.593$. With a yield of $3.59\%$ (April 4, 2011), we would get $D = 8.33979$. With a yield of $6.01\%$ (July 17, 2000) we would get $D = 7.43530$. With a yield of $1\%$, we would get $D = 9.49371$.

**Remark**: The 10-year on the run bond has a yield of 1.34% and a coupon of 1.25%. This leads to $D \approx 9.365$ instead of $9.3298$, not a very significant difference.

**Exercise**: What happens to the duration of a par-coupon bond as $y \to 0$? What happens what $y < 0$?

I will talk about this in the discussion session on Friday if you are interested.

# *Page 3.23:* Annuities

For an annuity that pays the same fixed amount every 6 months for $T$ years, it can be shown that

$$D = \frac{1}{y} - \left(\frac{1}{1+\frac{y}{2}}\right)\left[\frac{T}{(1+\frac{y}{2})^{2T} - 1}\right],$$

$$D_{Mac} = \frac{1}{y} + \frac{1}{2} - \frac{T}{(1+\frac{y}{2})^{2T} - 1}.$$

(These formulas do not seem to be in Tuckman & Serrat.)

# *Page 3.24:* Perpetuities

Recall that a perpetuity pays the same amount $A$ every 6 months indefinitely. Recall also that for a perpetuity, we have

$$P = \frac{2A}{y},$$

from which we compute

$$\frac{dP}{dy} = -\frac{2A}{y^2}.$$

It follows that

$$D = \frac{1}{y}, \quad D_{Mac} = \frac{1}{y} + \frac{1}{2}.$$

# *Page 3.25:* Example 3.4

Assume that the spot-rate curve is flat at .08. Portfolio A holds a par-coupon bond with price $P^{(1)} = \$500,000$ and maturity $T^{(1)} = 10$ and a ZCB with price $P^{(2)} = \$1,000,000$ and maturity $T^{(2)} = 20$. Let us find the Macaulay duration and DV01 for this portfolio. Observe that

$$D_{Mac}^{(1)} = 1.04\left[\frac{1}{.08}\left(1 - \frac{1}{(1.04)^{20}}\right)\right] = 7.06697,$$

and that

$$D_{Mac}^{(2)} = 20.$$

## *Page 3.26:* Example 3.4 (Cont.)

It follows that

$$D_{Mac}^{(A)} = \frac{500,000(7.06697) + 1,000,000(20)}{1,500,000} = 15.689,$$

and

$$\begin{aligned} DV01^{(A)} &= \tfrac{1{,}500{,}000}{10{,}000} D^{(A)} \\ &= \tfrac{1{,}500{,}000}{10{,}000} \left(\tfrac{15.689}{1.04}\right) = 2,262.84. \end{aligned}$$

# *Page 3.27:* Example 3.4 (Cont.)

Portfolio B is to consist of a single par-coupon bond having maturity $T^{(3)} = 15$ and face value $F^{(3)}$. Find a value for $F^{(3)}$ so that for small parallel shifts in the spot rate curve, the price change in Portfolio A will be the same as the price change in Portfolio B (to a first-order approximation).
Observe that

$$DV01^{(B)} = \frac{F^{(3)}}{10,000}\left[\frac{1}{.08}\left(1 - \frac{1}{(1.04)^{30}}\right)\right].$$

Setting $DV01^{(A)} = DV01^{(B)}$ and solving for $F^{(3)}$, we find that

$$F^{(3)} = 2,617,200.$$

# *Page 3.28:* Example 3.4 (Cont.)

By approximately how much will the price of Portfolio A change if the there is a parallel shift downward of 24 basis points in the spot rate curve?

$$\Delta P \approx (2,262.84)24 = 54,308.$$

Notice that price should increase.

# *Page 3.29:* Convexity for Coupon Bonds

Of course, it would be possible to introduce several different types of convexity. We shall not do so here. We shall make the convention that when we talk about convexity for securities with deterministic cash flows, that we mean convexity computed by differentiating the price twice with respect to the yield and dividing the result by the price.

# *Page 3.30:* Convexity for Coupon Bonds (Cont.)

Let $y$ denote the yield to maturity. Differentiating earlier expressions for $f'(y)$, we get

$$C = \frac{F}{P(1+\frac{y}{2})^2}\left[\frac{T(T+.5)}{(1+\frac{y}{2})^{2T}} + \frac{q}{2}\sum_{i=1}^{2T}\frac{i(i+1)}{4(1+\frac{y}{2})^i}\right].$$

$$C = \tfrac{F}{P}\left[\tfrac{2q}{y^3}\left(1-\tfrac{1}{(1+\frac{y}{2})^{2T}}\right) - \tfrac{2q}{y^2}\tfrac{T}{(1+\frac{y}{2})^{2T+1}} + \left(1-\tfrac{q}{y}\right)\tfrac{T^2+.5T}{(1+\frac{y}{2})^{2T+2}}\right]$$

## *Page 3.31:* Dependence of DV01, Duration, and Convexity on Maturity and Yield for ZCBs

$$P = f(y) = F(1 + \frac{y}{2})^{-2T}$$

$$f'(y) = -TF \cdot \left(1 + \frac{y}{2}\right)^{-2T-1} = -\left(\frac{T}{1 + \frac{y}{2}}\right) f(y)$$

# *Page 3.32:* ZCBs and Yield-Based Sensitivity (Cont.)

$$
\begin{aligned}
f''(y) &= -\left(\frac{T}{1+\frac{y}{2}}\right) f'(y) + \frac{T}{2}\left(1+\frac{y}{2}\right)^{-2} f(y) \\
&= \left(\frac{T}{1+\frac{y}{2}}\right)^2 f(y) + \frac{1}{2}\frac{T}{(1+\frac{y}{2})^2} f(y) \\
&= \left[\frac{T^2+.5T}{(1+\frac{y}{2})^2}\right] f(y).
\end{aligned}
$$

# *Page 3.33:* ZCBs and Yield-Based Sensitivity (Cont.)

$$DV01 = \frac{1}{10,000}\left[\frac{FT}{(1+\frac{y}{2})^{2T+1}}\right]$$

$$D = \frac{T}{1+\frac{y}{2}}$$

$$D_{Mac} = T$$

$$C = \frac{T(T+\frac{1}{2})}{(1+\frac{y}{2})^2}$$

# *Page 3.34:* Zero Coupon Bonds: DV01

Using Calculus, it is straightforward to show that (when the yield is held fixed at some value $y > 0$) the DV01 of a ZCB increases with increasing maturity until $T$ reaches the critical value

$$T = \frac{1}{2\ln\left(1+\frac{y}{2}\right)} \approx \frac{1}{y},$$

and decreases for $T$ larger than this value, tending to zero as $T \to \infty$.

The phenomenon of DV01 decreasing as maturity increases for a ZCB can certainly occur in the actual markets.

# *Page 3.35:* Zero Coupon Bonds: Duration and Convexity

- The Macaulay duration of a ZCB always equals the maturity.
- For fixed $y > 0$, the convexity of a ZCB increases (quadratically) with increasing $T$.
- For fixed $T > 0$, the convexity of a ZCB decreases with increasing $y$.

# *Page 3.36:* Coupon Bonds

$$DV01 = \frac{F}{10,000}\left[\frac{q}{y^2}\left(1 - \frac{1}{(1+\frac{y}{2})^{2T}}\right) + \left(1 - \frac{q}{y}\right)\frac{T}{(1+\frac{y}{2})^{2T+1}}\right]$$

$$D_{Mac} = \frac{F}{P}\left[\frac{T}{(1+\frac{y}{2})^{2T}} + \frac{q}{2}\sum_{i=1}^{2T}\frac{\frac{i}{2}}{(1+\frac{y}{2})^{i}}\right]$$

$$D_{Mac} = \frac{F}{P}\times\left(1+\frac{y}{2}\right)\left[\frac{q}{y^2}\left(1 - \frac{1}{(1+\frac{y}{2})^{2T}}\right) + \left(1 - \frac{q}{y}\right)\frac{T}{(1+\frac{y}{2})^{2T+1}}\right]$$

# *Page 3.37:* Limiting Duration for Coupon Bonds as Maturity Increases

Using the explicit formula for $D_{Mac}$ in terms of $y$ for a coupon bond, it can be shown that for fixed $q, y > 0$, we have

$$D_{Mac} \to \frac{1}{2} + \frac{1}{y} \ \text{as}\ T \to \infty.$$

# *Page 3.38:* Par Coupon Bonds

$$DV01 = \frac{F}{10,000y}\left[1 - \frac{1}{(1+\frac{y}{2})^{2T}}\right]$$

$$D = \frac{1}{y}\left[1 - \frac{1}{(1+\frac{y}{2})^{2T}}\right]$$

$$D_{Mac} = \frac{1+\frac{y}{2}}{y}\left[1 - \frac{1}{(1+\frac{y}{2})^{2T}}\right]$$

# Page 3.39: Perpetuities

Consider a perpetuity that pays $A$ twice per year.

$$DV01 = \frac{2A}{10,000y^2}$$

$$D = \frac{1}{y}$$

$$D_{Mac} = \frac{1}{2} + \frac{1}{y}$$

$$C = \frac{2}{y^2}$$

# *Page 3.40:* Rules of Thumb for Duration, DV01, and Convexity for Coupon Bonds

**Remark 3.8**: There are a number of useful “rules of thumb” for duration, DV01, and convexity for coupon bonds. However, some caution must be exercised in applying these rules in practice because, unless the yield curve is flat, changing the coupon rate or maturity of a bond will generally change the yield, so it may not be realistic to think of changing bond parameters “one at a time”. Moreover, some of these rules (such as *duration increases with maturity*) are valid under typical market conditions, but could be violated under *extreme conditions*.

# *Page 3.41:* Duration: Rules of Thumb for Coupon Bonds

Roughly speaking:

- Duration increases with increasing maturity. (Could be violated under extreme conditions.)
- Duration decreases with increasing coupon rate.
- Duration decreases with increasing yield to maturity.

To state these rules more precisely: For $q, y, T > 0$, let $D_{Mac}(q, y, T)$ denote the Macaulay duration of a bond with coupon rate $q$, yield to maturity $y$, and maturity $T$. Then we have the following result.

# *Page 3.42:* Duration: Rules of Thumb (Cont.)

**Proposition**:

(i) For fixed $q, y > 0$ $\lim_{T \to \infty} D_{Mac}(q, y, T) = \frac{1}{2} + \frac{1}{y}$.

(ii) For fixed $q, y > 0$ with $q \geq y$, $D_{Mac}(q, y, T)$ increases with increasing $T$.

(iii) For fixed $q, y > 0$ with $q < y$, there is a critical maturity $T^*$ (depending on $q$ and $y$) such that $D_{Mac}(q, y, T)$ increases with $T$ until $T$ reaches $T^*$ and then decrease as $T$ is increased beyond $T^*$. (The value of $T^*$ is typically much larger than the maturities of traded bonds.)

(iv) For fixed $y, T > 0$, $D_{Mac}(q, y, T)$ decreases with increasing $q$.

(v) For fixed $q, T > 0$, $D_{Mac}(q, y, T)$ decreases with increasing $y$.

 Duration:Rules of Thumb (Cont.)

**Remark 3.9**: The phenomenon in item (iii) of the proposition is mostly a "mathematical curiosity" because for typical $q$ and $y$ the corresponding values of $T^*$ are very large. There is no simple closed-form analytical expression for the "critical maturity" $T^*$. Pianca (2006) gives a formula for $T^*$ in terms of the so-called Lambert function. I do not know of any actual bonds that have had their value of $T^*$ be less than or equal to their maturity. However, it could conceivably happen. With $y = .05$ and $q = .01$, $T^* \approx 53$ years. However, with $y = .10$ and $q = .02$, $T^* \approx 27$.

# *Page 3.44:* DV01: Rules of Thumb for Coupon Bonds

Roughly speaking:

- DV01 increases with increasing maturity (exceptions can occur for discount bonds with large maturities).
- DV01 increases with increasing coupon.
- DV01 decreases with increasing yield.

To state these rules a bit more carefully: For $F, q, y, T > 0$, let $DV01(F, q, y, T)$ denote the DV01 of a coupon bond with face value $F$, coupon rate $q$, yield to maturity $y$ and maturity $T$. Then we have the following result.

# *Page 3.45:* DV01: Rules of Thumb (Cont.)

**Proposition**:

(i) For fixed $F, q, y > 0$, $\lim_{T \to \infty} DV01(F, q, y, T) = \frac{Fq}{10{,}000y^2}$.

(ii) For fixed $F, q, y > 0$ with $q \geq y$, $DV01(F, q, y, T)$ increases with increasing $T$.

(iii) For fixed $F, q, y > 0$ with $q < y$, there is a critical maturity $T^{**}$ such $DV01(F, q, y, T)$ increases with $T$ until $T$ reaches $T^{**}$ and then decreases for $T$ larger than $T^{**}$.

(iv) For fixed $F, y, T > 0$, $DV01(F, q, y, T)$ increases with increasing $q$.

(v) For fixed $F, q, T > 0$, $DV01(F, q, y, T)$ decreases with increasing $y$.

# *Page 3.46:* Convexity for Coupon Bonds: Rules of Thumb

Roughly speaking:

- Convexity increases with increasing maturity. (This can be violated for deeply discounted bonds.)
- Convexity decreases with increasing coupon.
- Convexity decreases with increasing yield to maturity.

# *Page 3.47:* Barbells Versus Bullets

In the asset-liability context, *barbelling* refers to the use of a portfolio of short-term and long-term bonds rather than intermediate term bonds.

Suppose that an asset manager has a portfolio of liabilities with Macaulay duration 9 years. The proceeds gained from those liabilities could be used to purchase several assets with duration 9 years, or alternatively, to purchase, say 2-year and 30-year securities that as a portfolio have a duration of 9 years. Let's look at an example.

 Example 3.5

**Example 3.5**: Let us assume that the spot-rate curve is flat at 6%. For a 9-year ZCB (a *bullet*) we have

$$D_{Mac} = 9, \quad C = \frac{9(9.5)}{(1 + \frac{.06}{2})^2} = 80.59195.$$

For a portfolio with 75% of its value in 2-year zeros and 25% of its value in 30-year zeros we have

$$D_{Mac}^{Barbell} = 9, \quad C^{Barbell} = \frac{.75(2)(2.5)}{(1 + \frac{.06}{2})^2} + . \frac{.25(30)(30.5)}{(1 + \frac{.06}{2})^2} = 219.15355.$$

# *Page 3.49:* Example 3.5 (Cont.)

Notice that the convexity of the barbell is substantially greater than the convexity of the bullet.

**Question**: What does this mean? **Answer**: Assuming parallel

shifts, if the yield moves away from 6%, the price of the barbell portfolio will be above the price of the bullet portfolio, whether rates rise or fall.

Let's look at some numbers (assuming 100 is invested in each portfolio) and that there is a parallel shift in the yield curve.

# *Page 3.50:* Example 3.5 (Cont.)

Exact Price Changes

| **Shift** | **Bullet** $\Delta P$ | **Barbell** $\Delta P$ |
|---|---|---|
| 10 bp | -.8697698 | -.8629362 |
| 50 bp | -4.269802 | -4.108030 |
| -10 bp | .8778291 | .8848532 |
| -50 bp | 4.471323 | 4.656949 |

$1^{st}$ Order Approximations

| **Shift** | **Bullet** $\Delta P$ | **Barbell** $\Delta P$ |
|---|---|---|
| 10 bp | -.8737864 | -.8737864 |
| 50 bp | -4.368932 | -4.368932 |
| -10 bp | .8737864 | .8737864 |
| -50 bp | 4.368932 | 4.368932 |

# *Page 3.51:* Example 3.5 (Cont.)

Exact Price Changes

| **Shift** | **Bullet** $\Delta P$ | **Barbell** $\Delta P$ |
|---|---|---|
| 10 bp | -.8697698 | -.8629362 |
| 50 bp | -4.269802 | -4.108030 |
| -10 bp | .8778291 | .8848532 |
| -50 bp | 4.471323 | 4.656949 |

$2^{nd}$ Order Approximations

| **Shift** | **Bullet** $\Delta P$ | **Barbell** $\Delta P$ |
|---|---|---|
| 10 bp | -.869756 | -.8628288 |
| 50 bp | -4.268192 | -4.09499 |
| -10 bp | .877816 | .8847441 |
| -50 bp | 4.46967 | 4.6428740 |

# *Page 3.52:* Example 3.5 (Cont.)

**Question**: Does this mean that there are real arbitrage opportunities trading with bullets and barbells in practice?

**Answer**: No. In the real world, the term structure will not be flat, so the barbell and bullet will have different yields. Also, we cannot be sure that rates will move in parallel. As we shall see on Assignment 4, an assumption of parallel shifts in the yield curve leads to “difficulties” when the term structure is flat.

 
# A Nonparallel Shift

Let's look at a nonparallel shift. Suppose that $\hat{r}(2) = .062$, $\hat{r}(9) = .064$, and $\hat{r}(30) = .066$. Then we have

$$P^{bullet} = 96.5685 \qquad P^{barbell} = 95.7063$$

In this case the bullet outperforms the barbell.

# *Page 3.54:* General Barbell vs Bullet (flat yield curve)

For fixed $y > 0$ define the function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(t) = \frac{t(t + .5)}{(1 + \frac{y}{2})^2}, \quad t \in \mathbb{R},$$

and assume that the term structure is flat at $y$. Notice that $g$ is a convex function because $g''(t) \geq 0$. Let $T_1, T_2$ with $0 < T_1 < T_2$ be given. Let $\alpha$ with $0 < \alpha < 1$ be given. Suppose that we build a barbell portfolio by investing $\alpha$ in a zero with maturity $T_1$ and $1 - \alpha$ in a zero with maturity $T_2$.

# *Page 3.55:* General Barbell Versus Bullet (Cont.)

The duration and convexity of the barbell are given by

$$D^{barbell} = \frac{\alpha T_1 + (1-\alpha) T_2}{1 + \frac{y}{2}},$$

$$D^{barbell}_{Mac} = \alpha T_1 + (1-\alpha) T_2,$$

$$C^{barbell} = \alpha g(T_1) + (1-\alpha) g(T_2).$$

 
# General Barbell Versus Bullet (Cont.)

The bullet portfolio having the same price (namely 1) and the same duration is obtained by investing \$1 in a zero with maturity

$$T = \alpha T_1 + (1 - \alpha) T_2.$$

The convexity of the bullet portfolio is

$$C^{bullet} = g(T) = g(\alpha T_1 + (1 - \alpha) T_2).$$

# *Page 3.57:* General Barbell Versus Bullet (Cont.)

By the definition of a convex function, we have

$$g(\alpha T_1 + (1-\alpha)T_2) \leq \alpha g(T_1) + (1-\alpha)g(T_2),$$

which says that

$$C^{bullet} \leq C^{barbell}.$$

In fact, the inequality is strict because $g$ is *strictly convex*.

 General Barbell Versus Bullet (Cont.)

**Remark 3.10**: Trades in which an intermediate-maturity security is purchased (or sold) and two securities whose maturities straddle the intermediate maturity are sold (or purchased) are referred to as *butterfly trades*.

# Duration and Convexity for a Callable Bond

**Example 3.6** A callable bond is a bond that the issuer has the right to buy back at some fixed set of prices on some fixed dates before maturity. Suppose that there is a bond with $q = .05$ and maturity $T = 10$ callable in one year at par. (Here, for simplicity, we assume that there is only one possible call date.) The value of the callable bond should be the difference in value between the underlying bond and a European call option on the underlying bond struck at par with exercise date $T_E = 1$ year. (Here the underlying is a non-callable coupon bond with $q = .05$ and $T = 10$; we shall refer to the underlying here as the "bond".)

Suppose we know that the prices, durations, and convexities of the (underlying) bond and the call option are as is in the table below. Here $y$ represents the yield to maturity of the (underlying) bond. In all tables below, the prices are for $100 face of the (underlying) bond.

# *Page 3.60:* Example 3.6 (Cont.)

| Rate Level | Bond Price | Bond D | Bond C |
|---|---|---|---|
| 4.00% | 108.1757 | 7.9263 | 75.4725 |
| 5.00% | 100.0000 | 7.7983 | 73.6287 |
| 6.00% | 92.5613 | 7.6686 | 71.7854 |

| Rate Level | Option Price | Option D | Option C |
|---|---|---|---|
| 4.00% | 8.1506 | 78.7438 | 2,800.9970 |
| 5.00% | 3.0501 | 121.2927 | 9,503.3302 |
| 6.00% | .6879 | 180.9833 | 25,627.6335 |

# *Page 3.61:* Example 3.6 (Cont.)

$$P^{Callable} = P^{Bond} - P^{Option}$$

Let us compute the duration of the callable bond when $y = .04$. At this level for $y$ the price of the long position on the bond is 108.1757/100.0251 times the price of the callable and the price of the short position on the option is -8.1506/100.0251 times the price of the callable. This leads to

$$D^{Callable} = \frac{108.1757(7.9263)}{100.0251} - \frac{8.1506(78.7438)}{100.0251} = 2.155692.$$

 
# Example 3.6 (Cont.)

The calculation of convexity for the callable bond is similar. The results are summarized in a table.

| **Rate Level** | **Callable Price** | **Callable D** | **Callable C** |
|---|---|---|---|
| 4.00% | 100.0251 | 2.1557 | -146.618 |
| 5.00% | 96.9499 | 4.2277 | -223.035 |
| 6.00% | 91.8734 | 6.3709 | -119.563 |

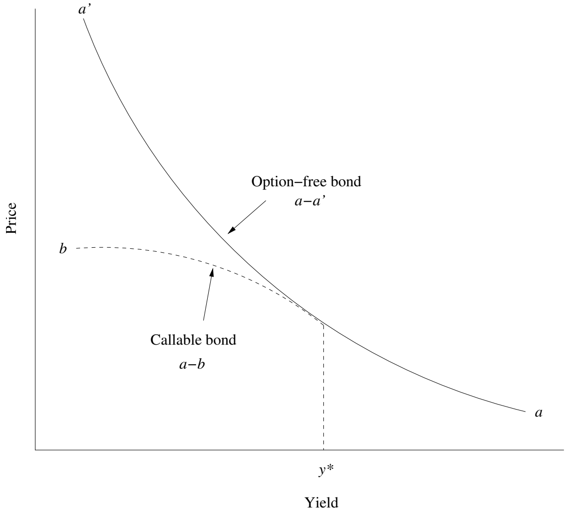
The callable bond exhibits *negative convexity*.

 Callable Bonds (Cont.)

**Remark 3.11**: In general callable bonds exhibit positive convexity when market rates are high (relative to the coupon rate) because there is little likelihood of the bond being called. However, when current market rates are low, the bond is subject to "price compression" and the price-yield curve lies below its tangent lines. Also for callable bonds a *yield to call* is frequently quoted (especially when current conditions suggest that the bond may be called). The yield to call is simply the yield to maturity computed under the assumption that the bond will be called at the earliest future call date. We shall have several homework exercises concerning callable (and putable) bonds after we introduce some term structure models.

# *Page 3.64:* Callable Bonds (Cont.)

The graph below is taken from *Fixed Income Mathematics* by Frank J. Fabozzi.



# *Page 3.65:* Parallel Shifts of a Nonflat Yield Curve

One consider parallel shifts of a nonflat spot-rate curve. However, a couple of caveats are in order.

- Different securities will have different yields, so we must be careful.
- For a parallel shift of amount $\alpha$, the yields of typical securities will change by an amount close to (but not exactly equal to) $\alpha$.

# *Page 3.66:* Swap Payments & Forward Rates

Let $r_{t-.5,t}$ denote the spot rate that will prevail at time $t - .5$ for loans to be settled with a single payment at time $t$. The floating payment at time $t$ in a standard interest rate swap (with semiannual payments) is given by

$$\frac{F}{2} r_{t-.5,t}.$$

It is very useful to observe that the time-0 price of a single payment of this amount at time $t$ is

$$\frac{F}{2} f(t) d(t),$$

where $f(t)$ is the forward (agreed upon at time 0) for a loan initiated at time $t - .5$ and settled at time $t$.

## *Page 3.67:* Swap Payments & Forward Rates

To verify the claim, we use replication. At time 0, we purchase a ZCB with face value $F$ and maturity $t - .5$ and also short a ZCB with face value $F$ and maturity $t$. At time $t - .5$, we invest $F$ until time $t$ at the rate $r_{t-.5,t}$ and then at time $t$, we pay $F$ to close out the short bond position. The time-0 price of this portfolio is

$$F[d(t-.5) - d(t)] = Fd(t)\left[\frac{d(t-.5)}{d(t)} - 1\right] = Fd(t)\frac{f(t)}{2}.$$

**Warning**: This does not mean that $r_{t-.5,t}$ is the same as $f(t)$ when $t > .5$. Remember that $f(t)$ is know at time 0, but $r_{t-.5,t}$ is not known until time $t - .5$.