Carnegie Mellon University
MSCF Program
46-956 Introduction to Fixed Income

Fall 2021

Mini 1

Lecture Notes for Week 4

William Hrusa

Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh, PA 15213

September 21 & 23, 2021

Page 4.1: Multiple Interest Rate Factors

Although a single-factor approach provides a valuable quick approximation, it is based on a premise that rates of all maturities are perfectly correlated, i.e. that movements of the entire term structure can be adequately described by a single random variable. There is little reason to believe that the change in the 6-month rate will perfectly predict the change in the 30-year rate, for example.

At the other extreme is complete *immunization*, which uses one ZCB for each individual cash flow being hedged, requires no assumptions about how rates move, but is almost certainly very costly and usually impractical.

Very significant improvements over a one-factor hedge can be obtained by using 3 or 4 interest rate factors.

Page 4.2: Key Rate Shifts

We shall concentrate here on an approach known as *key rate shifts*. The same kind of reasoning is applicable to other choices of interest-rate factors. When we implement key rate shifts, we can use spot rates or par-coupon yields (or other types of rates). If we will be hedging using zero coupon bonds, then spot rates are more natural. If we plan to hedge with par-coupon bonds, then it is more natural to use par-coupon yields.

We must first choose a set of key rates. For purposes of illustration, let us use par-coupon yields. Suppose that we decide to use 2, 5, 10, and 30 year bonds. (The details are quite a bit simpler with spot rates. You will use spot rates for homework.) The approach described here is appropriate for securities with deterministic cash flows.

Page 4.3: Key Rates (Cont.)

Once we have decided what type of rates, how many key rates, and the terms of the key rates, there is still a very important decision to be made. We need to decide on how a change in each key rate will influence the rates of other maturities. Of course, we want set things up so that a change in any individual key rate does not force a change in any of the other chosen key rates. In other words, a change in the 10-year rate, should not force changes in the 2-year, 5-year, or 30-year rates, but it should force a change in the rates for terms near 10 years. We also want a change in a given key rate to have more of an influence on nearby rates. There are many ways to accomplish this. We focus on a particularly simple one here. Before describing the precise nature of the perturbations that we shall use, let's take a look at the form of the first- and second-order approximations when there are multiple factors. Let us denote by y_1, y_2, y_3, y_4, the 2-year,5-year, 10-year, and 30-year par coupon rates, respectively.

Page 4.4: Key Rate DV01s

We shall assume that

$$P = f(y_1, y_2, y_3, y_4)$$

for some security with deterministic cash flows. The first-order approximation can be expressed as

$$\Delta P = \frac{\partial f}{\partial y_1}\Delta y_1 + \frac{\partial f}{\partial y_2}\Delta y_2 + \frac{\partial f}{\partial y_3}\Delta y_3 + \frac{\partial f}{\partial y_4}\Delta y_4.$$

The quantities

$$-\frac{\frac{\partial f}{\partial y_i}}{10,000}$$

can be thought of as the *DV01s* corresponding to the i^{th} key rate.

Page 4.5: Key Rate Durations and Convexities

The quantities

$$-\frac{\frac{\partial f}{\partial y_i}}{P}$$

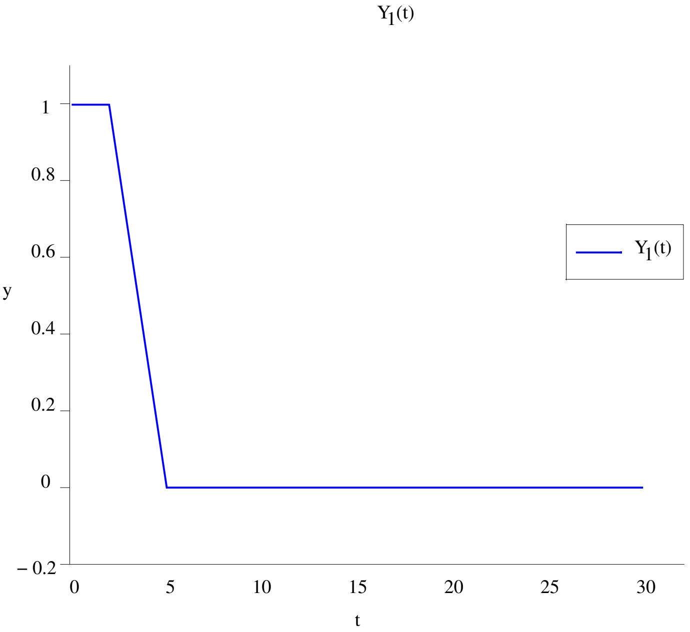
can be thought of as the *durations* corresponding to the i^{th} key rate. The second-order approximation takes the form

$$\Delta P = \sum_{i=1}^{4} \frac{\partial f}{\partial y_i} \Delta y_i + \frac{1}{2} \sum_{i,j=1}^{4} \frac{\partial^2 f}{\partial y_i \partial y_j} \Delta y_i \Delta y_j.$$

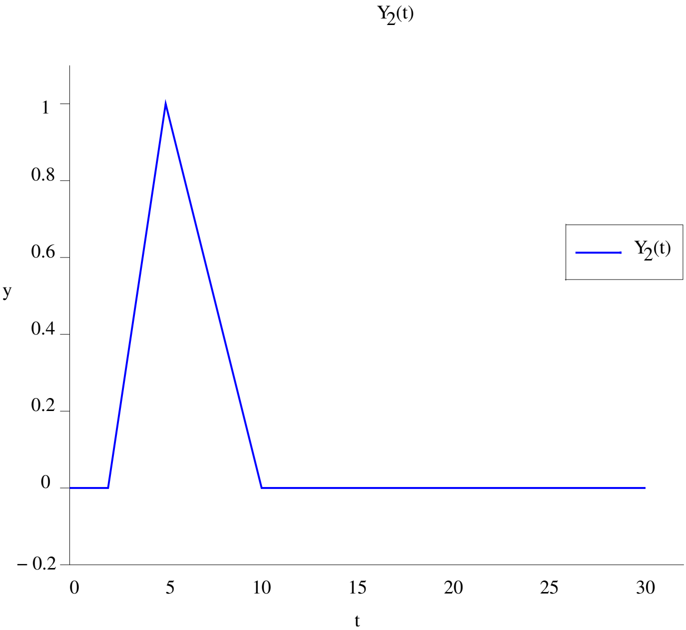
With multiple factors, duration is described by a vector and convexity is described by a matrix.

Page 4.6: Key Rate Perturbations

Let us assume that a change of one basis point in each individual key rate y_i leads to a change in the par-coupon yield curve of $Y_i(t)$ basis points where the functions Y_i are as shown in the 5 graphs that follow.

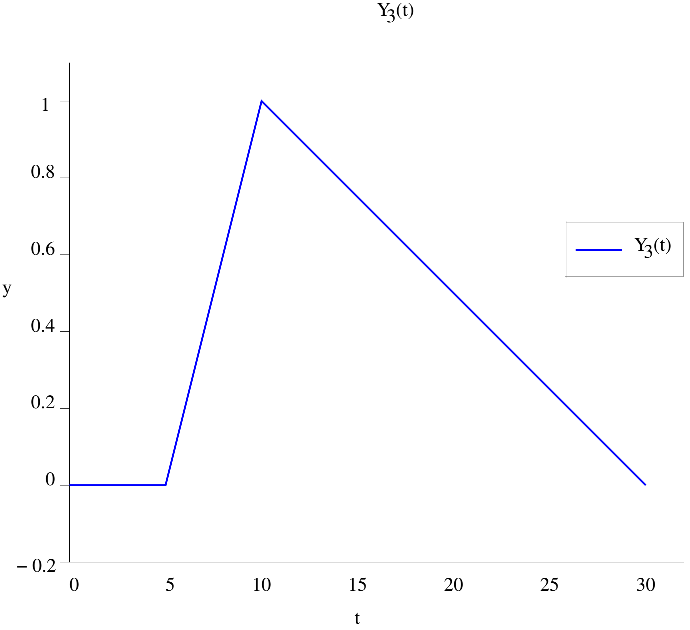


$Y_1(t)$
1
0.8
0.6
0.4
0.2
0
− 0.2
y
0
5
10
15
20
25
30
t
$Y_1(t)$

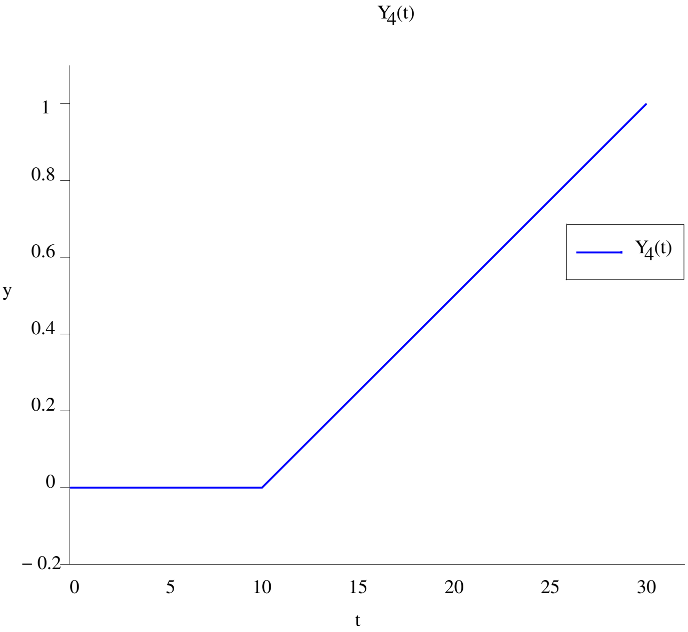


$Y_2(t)$
y
t
1
0.8
0.6
0.4
0.2
0
– 0.2
0
5
10
15
20
25
30
$Y_2(t)$

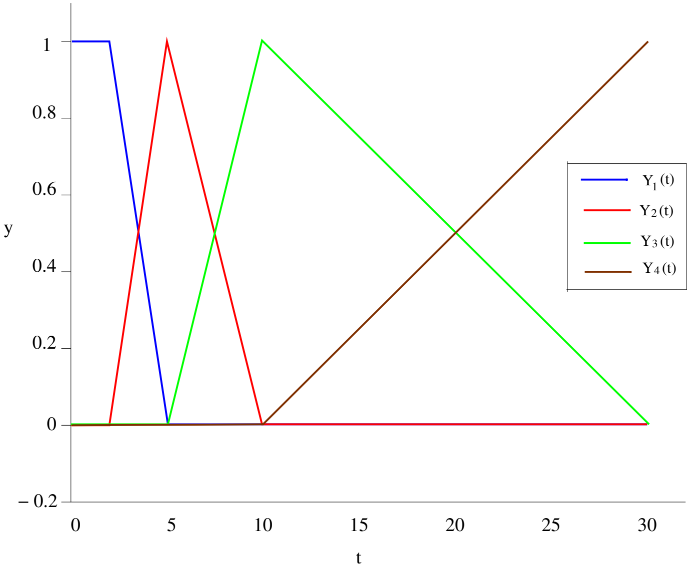
Page 4.9



Page 4.10



Page 4.11



Page 4.12: Perturbed Yield Curve

Let us denote the current par-coupon yield for maturity t by $y(t)$. We shall denote the observed values of the 2-, 5-, 10-, and 30-year yields by y_1, y_2, y_3, y_4. We also put

$$y_1^* = y(2), \quad y_2^* = y(5), \quad y_3^* = y(10), \quad y_4^* = y(30);$$

these are the *reference yield values*. Suppose that a change in the term structure occurs and we observe new values y_1, y_2, y_3, y_4 for the key rates. Then our "perturbed" par-coupon yield curve will be given by

$$y(t) \;+\; (y_1 - y_1^*)Y_1(t) \;+\; (y_2 - y_2^*)Y_2(t)$$

$$+(y_3 - y_3^*)Y_3(t) \;+\; (y_4 - y_4^*)Y_4(t).$$

Page 4.13: Key Rates (Cont.)

Since a yield curve completely determines the price of our security, the price can be expressed as

$$P = f(y_1, y_2, y_3, y_4).$$

Of course, this is not the exact price, because it is highly unlikely that the entire yield curve will have moved to fit the "perturbed" yield curve exactly, but with 4 key rates this should be a pretty good approximation.

The formula above uses the perturbed yield curve to compute the new price. Notice that

$$P^{original} = f(y_1^*, y_2^*, y_3^*, y_4^*).$$

Page 4.14: First-Order Approximation

Recall that the first-order approximation can be expressed as

$$\Delta P = \frac{\partial f}{\partial y_1}\Delta y_1 + \frac{\partial f}{\partial y_2}\Delta y_2 + \frac{\partial f}{\partial y_3}\Delta y_3 + \frac{\partial f}{\partial y_4}\Delta y_4.$$

In practice we would probably not compute the partial derivatives explicitly, because an explicit formula for the price would likely be too complicated. Instead, we would compute prices numerically and use difference quotients.

Page 4.15: Par Coupon yields and Discount Factors

The calculation of prices from par coupon yields is a bit involved, but easy to do on a spread sheet. Recall that the par coupon yield $y(\frac{n+1}{2})$ corresponding to maturity $\frac{n+1}{2}$ is given by

$$y\left(\frac{n+1}{2}\right) = \frac{2[1 - d(\frac{n+1}{2})]}{\sum_{i=1}^{n+1} d(\frac{i}{2})}.$$

Starting with a par-coupon yield curve $y(t)$, the discount factors can be computed recursively by

$$d\left(\frac{1}{2}\right) = \frac{1}{1 + \frac{y(\frac{1}{2})}{2}}; \quad d\left(\frac{n+1}{2}\right) = \frac{2 - y(\frac{n+1}{2})\sum_{i=1}^{n} d(\frac{i}{2})}{2 + y(\frac{n+1}{2})}.$$

Page 4.16: Prices from Par Coupon Yields

For securities with deterministic cash flows, the prices are completely determined by the discount factors. We assume that a reference (or starting) yield curve $y(t)$ is given. Let us agree to use the key rates and perturbation functions (or "shock functions") described above. For any set of inputs, y_1, y_2, y_3, y_4, we can determine a new yield curve, which determines new discount factors. The new discount factors can be used to determine a new price.

Page 4.17: Flow Chart

reference yield + perturbations $\rightarrow$ new yield curve

new yield curve $\rightarrow$ new discount factors

new discount factors $\rightarrow$ new price

$$P = f(y_1, y_2, y_3, y_4).$$

Page 4.18: A 30-year Annuity

For purposes of illustration, let us analyze an annuity making payments of \$3,250 every 6 months for the next 30 years. Also, for purposes of illustration, let us assume that the par-coupon yield curve is currently flat at 5%; this implies that the spot-rate curve is flat at 5%. The assumption that the yield curve is initially flat is not significant here; it simply allows us to give the data in a simple way.

There is a spread sheet on Canvas (in a module called "Spreadsheets') that allows you to change the data in this example.

Page 4.19: Annuity (Cont.)

To determine the initial price of the annuity, we put

$$\lambda = \frac{1}{1 + \frac{.05}{2}} = .975609756.$$

Then, we have

$$P = 3,250 \sum_{i=1}^{60} \lambda^i = 3,250 \frac{\lambda - \lambda^{61}}{1 - \lambda} = 100,453.13.$$

Page 4.20: Annuity (Cont.)

Using the procedure outlined above, we compute the prices corresponding to a 1 BP change in each of the key rates individually:

$$f(.0501, .05, .05, .05) = 100,452.15,$$

$$f(.05, .0501, .05, .05) = 100,449.36,$$

$$f(.05, .05, .0501, .05) = 100,410.77,$$

$$f(.05, .05, .05, .0501) = 100,385.88$$

(There is no simple formula to compute these values. They are obtained by the procedure discussed above: compute new yields, then discount factors, then prices.)

Page 4.21: Annuity (Cont.)

We can now compute the DV01s and durations corresponding to each key rate. We compute these for the 10-year rate and summarize all results in a table.

$$DV01_3 = -\frac{f(.05, .05, .0501, .05) - f(.05, .05, .05, .05)}{10,000(.0001)} = 42.36$$

$$D_3 = \frac{10,000 DV01_3}{f(.05, .05, .05, .05)} = 4.217.$$

Paage 4.22: Annuity (Cont.)

Key Rate	DV01	Duration
2 year	.98	.098
5 year	3.77	.375
10 year	42.36	4.217
30 year	67.25	6.695

We can perform a nice and simple check at this point. We can compute exactly the price of the annuity if the entire yield curve shifts by 1 BP and compare the answer to what we get from the first-order approximation if all key rates move up by 1 BP.

Page 4.23: Annuity (Cont.)

1^{st}-Order Approximation

$$\Delta P = -(.98 + 3.77 + 42.36 + 67.25) = -114.36,$$

$$P = 100,453.13 - 114.36 = 100,338.77.$$

Computing the exact price if the par-coupon yield is flat at 5.01% gives

$$100,338.81,$$

which agrees very well with the price we computed using the 1^{st} order approximation.

 Approximating Price Changes

Example 4.1 Compute the approximate price of the annuity assuming that the 2-year rate increases by 5 BP, the 5-year rate increases by 3 BP, the 10-year rate increases by 2 BP, and the 30-year rate drops by 1 BP. Using the key-rate DV01s we get

$$\Delta P = -(.98 \times 5 + 3.77 \times 3 + 42.36 \times 2 + 67.25 \times (-1)) = -33.68,$$

$$P = 100,453.13 - 33.68 = 100,419.45.$$

Page 4.25: Using the Key Rates to Construct a Hedge

Suppose that an agent has sold the annuity and wishes to hedge his short position using 2-, 5-, 10-, and 30-year par-coupon bonds. Here we assume that there are liquid par-coupon bonds of all 4 desired maturities. Let F_1, F_2, F_3, F_4 denote the face amounts of the 2-, 5-, 10-, and 30-year bonds used in the hedging portfolio. We choose the F's so that DV01s match for each of the key rates. Therefore, we need to compute the DV01s of the 2-, 5-, 10-, and 30-year par-coupon bonds, assuming that the yields to maturity are 5%. These are summarized in the table below.

Page 4.26: Key Rate DV01s & Durations

Bond Maturity	DV01 (per 100 face)	Duration
2 years	.018810	1.8810
5 years	.043760	4.3760
10 years	.077946	7.7946
30 years	.154543	15.4543

We can compute the DV01s of the par-coupon bonds using the formula that expresses the DV01 of a par bond in terms of face value, maturity, and yield to maturity. We can also compute them by using difference quotients.

Page 4.27: Calculating Face Amounts

Matching the DV01s for each of the key rates, we get

$$\frac{F_1(.018810)}{100} = .98, \quad \frac{F_2(.043670)}{100} = 3.77,$$

$$\frac{F_3(.077946)}{100} = 42.36, \quad \frac{F_4(.154543)}{100} = 67.25.$$

Solving for the face amounts gives $F_1 = \$5,209.99$, $F_2 = \$8,632.93$, $F_3 = \$54,345.32$, and $F_4 = \$43,515.40$.

page 4.28: Generalization of the Method

The above ideas can be used with other types of “shift functions” (or perturbations to the yield curve). Suppose we have a yield curve $y(t)$, $0 \le t \le T$ and we also have shift functions $X_i(t)$, $0 \le t \le T$, $i = 1, 2, \cdots, N$.

We postulate that the yield curve will move to something of the form

$$y(t) + \sum_{i=1}^{N} x_i X_i(t),$$

where the x_i are small “loading factors”.

Page 4.29: Constructing a General Hedge

Suppose we have sold a security S whose price is given by

$$P^{(S)} = f(x_1, x_2, \cdots, x_N),$$

and we wish to hedge the sale of this security using a portfolio of basic securities. (Typically we will want N securities in the hedging portfolio.) If the price of the hedging portfolio is given by

$$P^{(H)} = g(x_1, x_2, \cdots, x_N)$$

then we choose the amounts of each of the securities in the hedging portfolio so that

$$\frac{\partial}{\partial x_i} f(0, 0, \cdots, 0) = \frac{\partial}{\partial x_i} g(0, 0, \cdots, 0), \quad i = 1, 2, \cdots, N.$$

This is analogous to matching all of the key rate DV01s. (Note that $x_i = 0$ for $i = 1, 2, \cdots, N$ corresponds to the original yield curve – the perturbation is zero.)

Page 4.30: Some Comments on Hedging

Remark 4.1: The method of *bucket shifts* uses parallel shifts of a large number of "buckets" of forward rates. This technique is frequently used to hedge the risk in swaps.

Remark 4.2: A one-factor approach is very simple, but cannot protect against all possible moves of the spot rates. At the other extreme is complete *immunization* in which all cash flows are matched exactly. (Of course, this is not always feasible.) The multifactor approach is between these two extremes and can give excellent results.

Remark 4.3: For your benefit, I suggest that you read the material in Tuckman & Serrat on regression based hedging.

Page 4.31: Volatility-Weighted Hedging

We conclude our discussion of hedging for now with a simple, but useful idea, known as Volatility-Weighted Hedging. Suppose that an agent wishes to hedge the sale of a 20-year par coupon bond by purchasing a 30-year par coupon bond. The agent knows from experience that a change of 1 BP in the 30-year par coupon yield typically corresponds to a change of 1.1 BP in the 20-year par coupon yield.

If we denote by $DV01_{20}$ and $DV01_{30}$ the DV01s (per 100 face) of the 20- and 30-year par coupon bonds, then we want

$$F_{20} \times 1.1 \times \frac{DV01_{20}}{100} = F_{30} \times \frac{DV01_{30}}{100}.$$

Page 4.32: Example 4.2

Suppose that an agent who is short $10,000,000 face of a 20-year par coupon bond wishes to hedge her interest rate risk by purchasing a 30-year par coupon bond. Let us assume that the 20-year par coupon yield is 5.8%, the 30-year par coupon yield is 6%, and the agent knows from experience that a 1 BP change in the 30-year par coupon rate typically corresponds to a 1.1 BP change in the 20-year par coupon rate. The DV01s per 100 face are given by

$$DV01_{20} = .117465, \quad DV01_{30} = .138378.$$

With $F_{20} = 10,000,000$, we find that

$$F_{30} = 10,000,000 \times 1.1 \times \frac{.117465}{.138378} = 9,337,575.$$

(A hedge based on assumption of parallel shifts in the par coupon rate would suggest the purchase of $8,488,704 face of the 30-year bond.)

Page 4.33: Example 4.3

Suppose that an agent who is short \$10,000,000 face of a 20-year par coupon bond wishes to hedge his interest rate risk by investing equal amounts in a 10-year par coupon bond and a 30-year par coupon. Let us denote the par coupon yield for maturity t by $y_{pc}(t)$. Assume that

$$y_{pc}(10) = 5.2\%, \quad y_{pc}(20) = 5.8\%, \quad y_{pc}(30) = 6\%.$$

Assume also that the agent knows from experience that every change of 1 BP in $y_{pc}(30)$ corresponds to a change of 1.1 BP in $y_{pc}(20)$ and a change of 1.2 BP in $y_{pc}(10)$. The DV01's per 100 face are given by

$$DV01_{10} = .077215, \quad DV01_{20} = .117465, \quad DV01_{30} = .138378.$$

 Example 4.3 (Cont.)

Let us denote the unknown face amount of the 10- and 30-year bonds by F. (These face amounts are equal because the prices of the two bonds are the same and they are both par bonds.)

Matching the DV01 of the portfolio of the 10- and 30-year bond with the DV01 of the 20-year bond and solving for F we find that

$$F = \frac{10,000,000(1.1)(.117465)}{.077215(1.2) + .138378} = 5,592,700.$$

Page 4.35: Principal Component Analysis (PCA)

An extremely important emperical approach to hedging is based on the idea of *principal components*. We will discuss the method in more detail during the last lecture and you will study it thoroughly in Financial Data Science II.

The idea is to identify uncorrelated random variables that are statistically most significant in observed movements of the term structure, and use these random variables to construct hedges. It is interesting to note that the first principal component looks very much like a parallel shift and captures about 80% of the variance in the term structure. The first 3 principal components together typically capture about 95% of the variance in the term structure.

Page 4.36: Term Structure Models

In order to analyze fixed-income securities whose cash flows are not deterministic, we need to make a model for the evolution of interest rates.

There are two types of models: *short-rate models* and *whole-yield models*. We shall discuss the distinctions later.

We shall do a rigorous mathematical analysis in the context of binomial trees. If time permits, I will write down and briefly discuss continuous time versions of several of the most important models.

Page 4.37: A One-Period Binomial Pricing Model

We begin by discussing a simple one-period binomial pricing model. This should serve two purposes: (i) It should provide motivation for the way the multi-period model is formulated. (ii) Analysis of multiperiod problems can often be carried out by breaking things down into a sequence of one-period problems, so the one-period case will help solve multiperiod problems.

Suppose that there are two trading times (or dates) $t = 0$ and $t = 1$. We assume that there is a zero-coupon bond that will pay \$1 at time 1 (i.e., the bond has face value $F = 1$ and maturity $T = 1$.) Denote the price of this bond at time 0 by $B_{0,1}$. We assume that there is another basic (traded) security that has initial price $S_0 > 0$ and whose price $S_1 > 0$ at time 1 takes one of two possible different values, each with strictly positive probability.

Page 4.38: Sample Space & Probability Measure

Since there are two possibilities for the price of the risky asset at time 1, it is natural to take

$$\Omega = \{H, T\}$$

as the sample space. It is helpful to think of a random experiment in which a single coin is tossed and think of H as corresponding to a head and T as corresponding to a tail.

We write $\mathbb{P}(H)$ for the probability of a head and $\mathbb{P}(T)$ for the probability of a tail, i.e. we use the symbol $\mathbb{P}$ for the probability measure corresponding to the coin toss. We assume

$$\mathbb{P}(H) > 0, \quad \mathbb{P}(T) > 0,$$

but we do not necessarily require the coin to be "fair" (i.e., have equal probability for head and tail).

Page 4.39: Risky Asset

We are assuming that $S_1(H) \neq S_1(T)$. Without loss of generality, we assume that

$$S_1(H) > S_1(T).$$

(Indeed, if $S_1(T) > S_1(H)$, we could relabel the sides of the coin.)

The risky asset could, for example, be a zero-coupon bond that matures at some future date (such as time 2) that we are not considering in our model.

 Interest Rate

Let us denote the yield to maturity of the zero-coupon bond that matures at time 1 by R_0. Here we use the one-period compounding convention, so that

$$B_{0,1} = \frac{1}{1 + R_0}.$$

We can solve for R_0 to obtain

$$R_0 = \frac{1}{B_{0,1}} - 1.$$

A very simple, but important, observation is that we can describe the bond by giving the time-0 price, or by giving the interest rate (bond yield).

Page 4.41: Description of a Trading Strategy

In order to describe a trading strategy (or portfolio) in this model, we simply need to specify how much money is invested in the maturity-1 bond at time 0 and how much money is invested in the risky asset at time 0.

If we specify the total initial capital X_0 and the number of shares Δ_0 of the risky asset purchased at time 0, then we can easily compute how much money is invested in the bond and how much money is invested in the risky asset. For consistency with what you will do in Stochastic Calculus, I will describe trading strategies by giving the total initial capital and the number of shares of the risky asset purchased at time 0.

Page 4.42: Terminal Capital of a Strategy

Consider a strategy described by (X_0, Δ_0). (This means that the total initial capital is X_0 and that the portfolio will hold Δ_0 shares of the risky asset.) Let us denote the capital of the strategy at time 1 by X_1. Notice that X_1 is a *random variable* on Ω. The initial capital invested in the risky asset is $\Delta_0 S_0$ and the initial capital invested in the bond is $X_0 - \Delta_0 S_0$. The terminal capital is given by

$$\begin{aligned} X_1(\omega) &= (X_0 - \Delta_0 S_0)\tfrac{1}{B_{0,1}} + \Delta_0 S_1(\omega) \\ &= (X_0 - \Delta_0 S_0)(1 + R_0) + \Delta_0 S_1(\omega) \end{aligned}$$

for all $\omega \in \{H, T\}$.

Page 4.43: Arbitrage Strategies

By an *arbitrage strategy*, we mean a strategy with

(i) $X_0 = 0$,

(ii) $\mathbb{P}[X_1 \geq 0] = 1$,

(iii) $\mathbb{P}[X_1 > 0] > 0$.

It is clear that a strategy is an arbitrage if and only if

(i) $X_0 = 0$,

(ii') $X_1(T) \geq 0, \quad X_1(H) \geq 0$,

(iii') $X_1(T) + X_1(H) > 0$.

Page 4.44: Absence of Arbitrage

The model is free of arbitrage if and only if

$$(*) \qquad \frac{S_1(T)}{S_0} < \frac{1}{B_{0,1}} < \frac{S_1(H)}{S_0}.$$

If we put

$$u = \frac{S_1(H)}{S_0}, \quad d = \frac{S_1(T)}{S_0},$$

and recall that

$$R_0 = \frac{1}{B_{0,1}} - 1$$

then (*) is equivalent to

$$(**) \qquad d < 1 + R_0 < u.$$

Page 4.45: No-Arbitrage Inequalities

Unless stated otherwise, we assume that (*) holds. Notice that (*) is also equivalent to

$$(***) \qquad \frac{S_1(T)}{S_0} < 1 + R_0 < \frac{S_1(H)}{S_0}.$$

Page 4.46: Replication and Pricing

Consider a derivative security that pays the amount $V_1(\omega)$ at time 1. Let us try to replicate the payoff of the security by a portfolio with initial capital X_0 and Δ_0 shares of the risky asset. We need to find X_0 and Δ_0 satisfying

$$(X_0 - \Delta_0 S_0)(1 + R_0) + \Delta_0 S_1(H) = V_1(H), \tag{1}$$

$$(X_0 - \Delta_0 S_0)(1 + R_0) + \Delta_0 S_1(T) = V_1(T). \tag{2}$$

Page 4.47: Replication and Pricing (Cont.)

Subtracting (2) from (1), we obtain

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}. \tag{3}$$

Rearranging (1), we find that

$$X_0(1 + R_0) = V_1(H) + \Delta_0[S_0(1 + R_0) - S_1(H)]. \tag{4}$$

Substituting from (3) into (4), we obtain

$$X_0(1 + R_0) = \tilde{p} V_1(H) + \tilde{q} V_1(T), \tag{5}$$

Page 4.48: Risk-Neutral Probabilities

where

$$(6) \qquad \tilde{p} = \frac{S_0(1+R_0) - S_1(T)}{S_1(H) - S_1(T)}, \quad \tilde{q} = \frac{S_1(H) - S_0(1+R_0)}{S_1(H) - S_1(T)}$$

Observe that

$$(7) \qquad \tilde{p} > 0, \quad \tilde{q} > 0,$$

by virtue of (***). Moreover, we have

$$(8) \qquad \tilde{p} + \tilde{q} = 1.$$

We can define another probability measure $\tilde{\mathbb{P}}$ on Ω by

$$\tilde{\mathbb{P}}(H) = \tilde{p}, \quad \tilde{\mathbb{P}}(T) = \tilde{q}.$$

Page 4.49: Risk Neutral Probabilities (Cont.)

The measure $\tilde{\mathbb{P}}$ is called a *risk-neutral measure* or *pricing measure*.

We conclude that V is replicable and that the initial capital X_0 required to replicate V is satisfies

$$X_0(1 + R_0) = \tilde{\mathbb{E}}[V_1].$$

(Here, $\tilde{\mathbb{E}}$ denotes expectation under the risk-neutral measure.)

The arbitrage-free time-0 price of V is given by

$$V_0 = \tilde{\mathbb{E}}\left[\frac{V_1}{1 + R_0}\right].$$

Page 4.50: Example 4.4

Consider a one-period binomial model with $R_0 = .06$, $S_0 = 100$, $S_1(H) = 130$, and $S_1(T) = 94$.

Let V be a put option on S with strike price $K = 100$. Find the time-0 price V_0 of V.

We begin by finding the risk-neutral probabilities:

$$\tilde{p} = \frac{100(1.06) - 94}{130 - 94} = \frac{1}{3}, \quad \tilde{q} = \frac{130 - 100(1.06)}{130 - 94} = \frac{2}{3}.$$

Page 4.50: Example 4.4 (Cont.)

The option payoff is given by

$$V_1(H) = 0, \quad V_1(T) = 6.$$

The time-0 price of the option is given by

$$V_0 = \frac{\frac{1}{3}(0) + \frac{2}{3}(6)}{1.06} = 3.77.$$

Page 4.52: Some Remarks

Remark 4.4: Notice that if instead of being given R_0 and price information for the risky asset, we were given R_0 and the risk-neutral measure, we could still compute the arbitrage-free price of any given payoff at time 1.

Remark 4.5: It is customary to define

$$u = \frac{S_1(H)}{S_0}, \quad d = \frac{S_1(T)}{S_0}.$$

In this case, the no arbitrage condition becomes

$$d < 1 + R_0 < u,$$

and the risk-neutral probabilities are given by

$$\tilde{p} = \frac{1 + R_0 - d}{u - d}, \quad \tilde{q} = \frac{u - 1 - R_0}{u - d}.$$

Page 4.53: N-Period Binomial Models

In the N-period binomial model, there will be $N+1$ trading dates $0, 1, 2, \cdots, N$. We shall write $B_{n,m}$ for the price at time n of a ZCB that pays \$1 at time $m \geq n$. We shall write R_n for the spot rate that will prevail at time n for loans to be settled at time $n+1$ (one-period compounding). We can either model the bond prices $B_{n,n+1}$ or the interest rates R_n, because we have

$$B_{n,n+1} = \frac{1}{1+R_n}.$$

The interest rates R_n are known at time n, but not earlier.

We shall describe the evolution of the interest rates in this course.

Page 4.54: Negative Interest Rates

Remark 4.6: Negative interest rates, although mathematically possible, have been considered questionable up until recently. However, the possibility of negative interest rates in and of itself does not imply arbitrage in the sense that the term is used in the MSCF Program. This is because we do not allow an agent to hold cash in a self-financing strategy – all capital must be invested in the market. However, if you are allowed to hold cash without risk, then negative interest rates imply arbitrage. You should be aware of this point when you communicate with others, especially on interviews. Now, because of the economic situation in Europe and Japan, many banks are interested in models that allow negative interest rates.

Page 4.55: Short Rate Models

A risk-neutral measure (or pricing measure) is prescribed and the evolution of the short rates is described under the risk-neutral measure. Such models have no arbitrage (although they may give rise to negative interest rates). The spot rate curve is an output of the model, rather than an input. Typically there are parameters in the model that are chosen to try to match the spot-rate curve as closely as possible. One must be very careful in interpreting prices computed by the model – they are internally consistent in the sense that no arbitrage can occur by trading securities within the model. However, we need to worry about whether we can hedge a position in terms of securities that are traded in the actual market.

Whole-Yield Models

The current spot-rate curve is an input to the model, and the random evolution of the spot-rate curve is modeled. Modeling the evolution of the spot-rate curve in such a way that arbitrage is not possible is a very complex issue. (Heath-Jarrow-Morton). A key idea is to use forward rates. Prices are again computed using a risk-neutral measure.

Remark 4.7: What I am calling short-rate models are often called *equilibrium models* and what I am calling whole-yield models are often called *arbitrage-free models*.

 Stochastic Analysis on Coin-Toss Space

Let N be a fixed positive integer. There are $N+1$ times or dates, namely $0, 1, 2, \cdots, N$.

Sample Space: $\Omega = \{H, T\}^N$ – the set of all lists or strings of length N, with each entry in the list being H or T. We imagine that a coin is being tossed N times. "H" represents a head and "T" represents a tail. A typical element of Ω is denoted by $\omega = (\omega_1, \omega_2, \cdots, \omega_N)$. If $\omega_i = H$ then the i^{th} toss is a head, while $\omega_i = T$ indicates that the i^{th} toss is a tail. Sometimes the elements of Ω are referred to as *paths*. There are 2^N paths in Ω, a fact that has important consequences concerning actual computations when N is large.

Page 4.58: Probability Measures

Probability Measures: We shall equip Ω with one or more probability measures. Unless stated otherwise, we assume that each path has strictly positive probability.

Binomial Product Measures: A probability measure on Ω is said to be a *binomial product measure* (BPM) if there exist numbers $a, b > 0$ with $a + b = 1$ such that

$$\mathbb{P}(\omega) = a^{\#H(\omega)} b^{\#T(\omega)} \text{ for all } \omega \in \Omega.$$

Here $\#H(\omega)$ and $\#T(\omega)$ represent the number of heads and tails, respectively, in the string ω. We refer to a as the *probability of heads* and to b as the *probability of tails*.

Remark 4.8: The term "binomial product measure" is not standard, but I think that is very useful. Binomial product measures correspond to independent coin tosses. If $\mathbb{P}$ is a BPM, then for each $n \in \{1, 2, \cdots, N\}$ we have

$$\mathbb{P}[\{\omega \in \Omega : \omega_n = H\}] = a,$$

$$\mathbb{P}[\{\omega \in \Omega : \omega_n = T\}] = b.$$

Page 4.60: Adapted Processes & Information

In financial markets, more information becomes available as time evolves. When we build models, it is crucial to ensure that prices of securities at time n and decisions made at time n (such as the number of shares to hold in a portfolio) are based solely on the information available at time n. This leads to the notion of an *adapted process*. In coin-toss space, the information available at time n is represented by the partial string $(\omega_1, \omega_2, \cdots, \omega_n)$.

Page 4.61: Measurability

Def: Let $n \in \{0, 1, \cdots, N\}$ be given. A random variable Y on Ω is said to be time-n measurable provided that

$$Y(\omega) = Y(\omega_1, \cdots, \omega_n, \mu_{n+1}, \cdots, \mu_N)$$

for all $\omega \in \Omega$ and all $\mu_{n+1}, \cdots, \mu_N \in \{H, T\}$. (In other words, the value of $Y(\omega)$ depends only on the first n entries in the string ω. Y is completely determined by the information available at time n.)

For a time-n measurable random variable Y, we write $Y(\omega_1, \cdots, \omega_n)$ in place of $Y(\omega)$ when convenient.

Def: Let $k, m \in \{0, 1, \cdots, N\}$ with $k \leq m$ be given. A list $(Y_n)_{k \leq n \leq m}$ of random variables on Ω is said to be an adapted process provided that Y_n is time-n measurable for every n with $k \leq n \leq m$.

Page 4.62: Random Walk

For each $j \in \{1, 2, \cdots, N\}$, define the random variable X_j on Ω by

$$X_j(\omega) = \begin{cases} 1 & \text{if } \omega_j = H \\ \\ -1 & \text{if } \omega_j = T, \end{cases}$$

and define the adapted process $(M_n)_{0 \leq n \leq N}$ by

$$M_0 = 0, \quad M_n = \sum_{j=1}^{n} X_j, \quad n = 1, \cdots, N$$

Remark 4.9: It is very useful to observe that

$$M_n(\omega_1, \cdots, \omega_n) = \#H(\omega_1, \cdots, \omega_n) - \#T(\omega_1, \cdots, \omega_n).$$

Here $\#H(\omega_1, \cdots, \omega_n)$ and $\#T(\omega_1, \cdots, \omega_n)$ represent the number of heads and tails seen up to time n. If the measure is a binomial product measure with probability of heads equal to $\frac{1}{2}$, then $(M_n)_{0 \le n \le N}$ is called a *symmetric random walk* and can be thought of as a discrete version of Brownian motion. In fact, Brownian motion can be obtained as a limit of appropriately scaled random walks.

Page 4.64: Binomial Short-Rate Models

A binomial short-rate model is characterized by an adapted process $(R_n)_{0 \le n \le N-1}$ with

$$R_n(\omega) > -1 \quad \text{for all } \omega \in \Omega,$$

called an *interest rate process*, and a probability measure $\tilde{\mathbb{P}}$ on Ω, called a *pricing measure*. (Most people call $\tilde{\mathbb{P}}$ a *risk-neutral measure*. However, for now, I want to use the term "pricing measure" because the term "risk-neutral measure" seems to lead to confusion for some people.)

If \$1 is deposited in the bank at time n, the value of the account at time $n+1$ will be $\$1 \times (1 + R_n)$. The interest rate for borrowing is the same as the interest rate for investing.

Page 4.65: Computing Prices

The measure $\tilde{\mathbb{P}}$ is used to compute prices in the following way. Let Y be a time-m measurable random variable. The price at time 0 to receive a payment of amount $Y(\omega_1, \cdots, \omega_m)$ at time m is given by

$$\tilde{\mathbb{E}}[D_m Y],$$

where

$$D_m = \frac{1}{(1+R_0)(1+R_1)\cdots(1+R_{m-1})}.$$

We make the convention that $D_0 = 1$.

Page 4.66: Computing Prices (Cont.)

The process $(D_n)_{0 \le n \le N}$ is called the *discount process*. It is a *predictable* process, because the value of D_n can be determined from the information available at time $n-1$.

For $0 \le n \le m \le N$, the price at time n to receive a payment of amount $Y(\omega_1, \cdots, \omega_m)$ at time m is computed via the formula

$$\frac{1}{D_n}\tilde{\mathbb{E}}_n[D_m Y].$$

Here $\tilde{\mathbb{E}}_n$ denotes the conditional expectation under the pricing measure given the information at time n.

Page 4.67: Conditional Expectations

If the pricing measure is a binomial product measure with probability of heads $\tilde{p}$ and probability of tails $\tilde{q}$ then the conditional expectation at time n, $\tilde{\mathbb{E}}_n[W](\omega_1, \omega_2, \cdots, \omega_n)$ of a random variable W is given by

$$\sum_{\omega_{n+1}, \cdots, \omega_N \in \{H, T\}} W(\omega) \tilde{p}^{\# H(\omega_{n+1}, \cdots, \omega_N)} \tilde{q}^{\# T(\omega_{n+1}, \cdots, \omega_N)}.$$

Observe that

$$\tilde{\mathbb{E}}_0[W] = \tilde{\mathbb{E}}[W],$$

and

$$\tilde{\mathbb{E}}_N[W] = W.$$

Page 4.68: Some Properties of Conditional Expectations

Assume that $\tilde{\mathbb{P}}$ is a binomial product measure with probablity of heads equal to $\tilde{p}$ and probability of tails equal to $\tilde{q}$. Let X and Y be random variables on $\Omega = \{H, T\}^N$, c_1 and c_2 be constants, and n be an integer with $0 \leq n \leq N$. Then we have

(i) (Linearity) $\tilde{\mathbb{E}}_n[c_1 X + c_2 Y] = c_1 \tilde{\mathbb{E}}_n[X] + c_2 \tilde{\mathbb{E}}_n[Y]$

(ii) $\tilde{\mathbb{E}}_n[X]$ is time-n measuarble.

(iii) (One-Step Ahead Property) If X is time-$n+1$ measurable, then

$$\tilde{\mathbb{E}}_n[X](\omega_1, \cdots, \omega_n) = \tilde{p} X(\omega_1, \cdots, \omega_n, H) + \tilde{q} X(\omega_1, \cdots, \omega_n, T)$$

Page 4.69: Properties of Conditional Expectation (Cont.)

(iv) (Taking Out a Known Quantity) If X is time-n measurable then

$$\tilde{\mathbb{E}}_n[XY] = X\tilde{\mathbb{E}}_n[Y]$$

(v) (Iterated Conditioning) If $n \leq m \leq N$ then

$$\tilde{\mathbb{E}}_n[Y] = \tilde{\mathbb{E}}_n[\tilde{\mathbb{E}}_m[Y]]$$

Page 4.70: Some Remarks

Remark 4.10: Notice that the discount factor for time n is given by

$$d(n) = \tilde{\mathbb{E}}[D_n].$$

Given an interest rate process and a pricing measure we can compute the zero-coupon yield curve (spot-rate curve).

Remark 4.11: By the 1^{st} fundamental theorem of asset pricing (to be treated in detail in Stochastic Calculus), a binomial pricing model will be arbitrage free if and only if there is a probability measure $\tilde{\mathbb{P}}$ on Ω such that for all integers m, n with $0 \leq n \leq m$ and every time-m measurable random variable Y, the price at time n to receive a single payment of amount Y at time m is given by

$$\frac{1}{D_n}\tilde{\mathbb{E}}_n[D_m Y].$$

The measure $\tilde{\mathbb{P}}$ need not be a binomial product measure. (We need only the easy direction of the theorem here.)

A Quick Look at Four Important Binomial Models

We put

$$\Delta R_n = R_{n+1} - R_n.$$

Ho-Lee Model:

$$\Delta R_n = \lambda_{n+1} + \sigma X_{n+1}$$

$$R_n = R_0 + \sum_{i=1}^{n} \lambda_i + \sigma M_n.$$

Remark 4.12: The Ho-Lee Model can be written in the form

$$R_n(\omega_1, \cdots, \omega_n) = a_n + b \cdot \#H(\omega_1, \cdots, \omega_n),$$

where

$$a_n = R_0 - \sigma n + \sum_{i=1}^{n} \lambda_i, \quad b = 2\sigma.$$

We shall frequently use the more general form

$$R_n(\omega_1, \cdots, \omega_n) = a_n + b_n \cdot \#H(\omega_1, \cdots, \omega_n).$$

Notice that both of these lead to recombining trees for the short rates. This model provides a great deal of flexibility in fitting the current term structure.

Binomial Version of Vasicek Model: $0 < k < 1$

$$\Delta R_n = k(\theta - R_n) + \sigma X_{n+1}$$

The solution of this difference equation is given by

$$R_n = (1-k)^n (R_0 - \theta) + \theta + \sigma \sum_{j=1}^{n} (1-k)^{n-j} X_j.$$

Binomial Version of Hull-White Model: $0 < k < 1$

$$\Delta R_n = k(\theta_{n+1} - R_n) + \sigma_{n+1} X_{n+1}$$

One can also let k depend on n.

Black-Derman-Toy Model:

$$\Delta(\ln R_n) = -\frac{\Delta\sigma_n}{\sigma_n}(\ln\theta_{n+1} - \ln R_n) + \sigma_{n+1}X_{n+1}.$$

After some work, this can be rewritten as

$$R_n(\omega_1, \cdots, \omega_n) = a_n b_n^{\#H(\omega_1, \cdots, \omega_n)}.$$

Notice that this leads to a recombining tree for the short rates.