Fixed-Income Securities
Lecture 4: Hedging Interest Rate Risk Exposure
Traditional Methods

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- Matching maturities

- Duration

- Effective duration

- Multiple duration measures

Interest rate risk exposure

Everyone who trades in interest-sensitive securities is (or should be) concerned about their interest rate risk exposure. This is given different names in different contexts, but in modern terminology it all falls under the umbrella of risk management.

- Investment policy or guidelines

- Market neutral

- Immunization

- Value at risk

- Duration and effective duration

- Hedge ratio (beta or delta)

Some stylized properties of interest rate risk

- Rates tend to move together

- A single factor model is a good first approximation

- Short rates move more than long rates

- Interest rate volatility moves around

- Volatility is on average higher when rates are higher

- Rates are on average higher when inflation is higher

Matching cash flows

Advantages:

- Simple

- Reliable

Disadvantages:

- Cannot hedge complex bonds or derivatives

- No obvious corresponding risk measure

- May over-hedge and incur high costs

In-class exercise: matching cash flows

You are managing the final years of a pension fund. There are three remaining dates at which lump-sum payments will be made to beneficiaries: $317 million 6 months from now, $208 million 12 months from now, and $104 million 18 months from now.

- What is the market value of the pension liability?

- What portfolio of the three Treasury bonds below would immunize the liability? (Match the cash flows.)
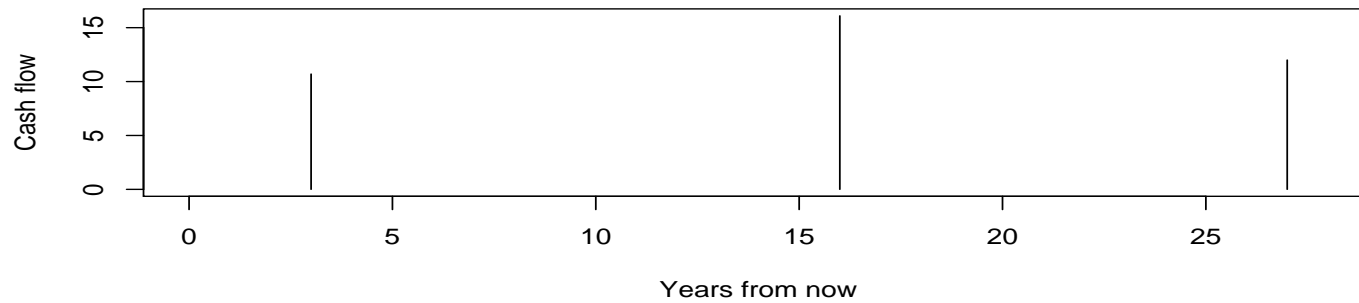
time (months out)	0	6	12	18
T-Bond 1	-100	103	0	0
T-Bond 2	-98	2	102	0
T-Bond 3	-103	4	4	104

Nearly matching cash flows: is this a good hedge?

**Liability we want to hedge**
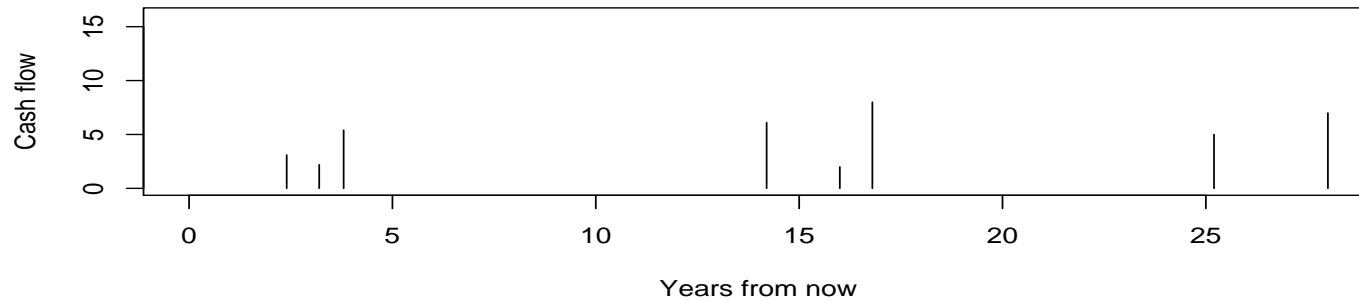
Proposed hedge

Nearly matching cash flows: a diagnostic tool

Intuitively, it seems like the proposed hedge might do well, but how can we get an objective measure of how well it will do? We can use the same tool as we used for intuition, namely, plotting the net amount at risk at each future date as a measure of our exposure to the forward rate.

For example, consider a liability of $100 a year from now with a present value of $95, and $200 two years from now with a present value of $190. Then the exposure is the full present value $285 from now until a year from now, the present value $190 of the second cash flow only from one year from now until two years from now, and $0 beyond two years.

Consider funding the liability with a portfolio that pays $298 one and one-half years from now, with present value of $285. Then this investment has a risk exposure of $285 from now until one and one-half years out and $0 thereafter. The net exposure is the difference of the two, which is $0 from now until one year out, -95 from one year out until one and one-half years out, $190 from one and one-half years out until two years out, and $0 thereafter.

Nearly matching cash flows: analysis

Net difference in remaining PV
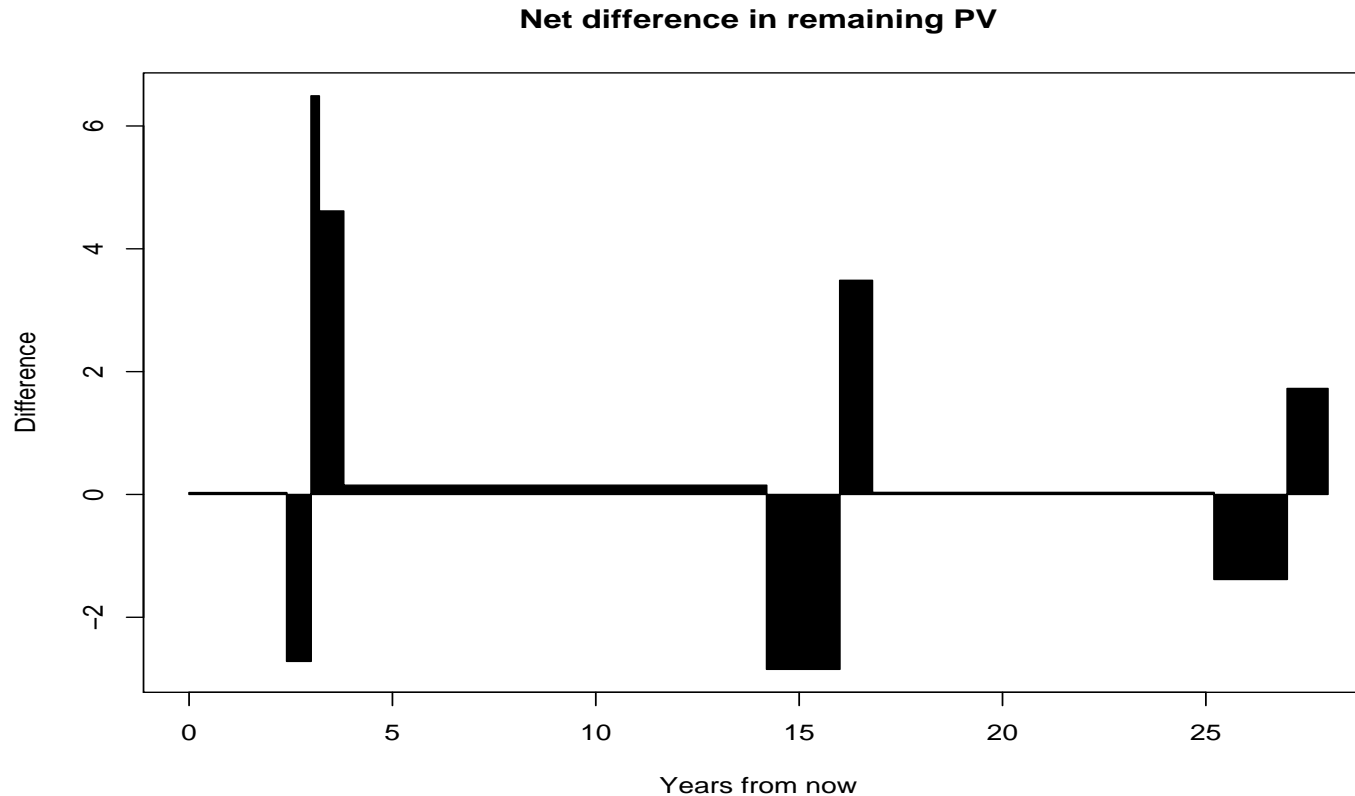
This looks at the net exposure to different forward rates from the liability plus candidate hedge. This plot gives the difference between the present value of the subsequent cash flows between the liability and the candidate hedge.

- Strict matching of cash flows: *difference* $= 0$

- Effective approximate hedge

  - total area between the difference and the axis is small
  - areas above and below the axis are nearly matched in time

- Matched duration: total *signed* area is zero

Caution: fixed claims only

The exposure analysis only works (in this form) for nonrandom claims. For example, a bond fully indexed to the short rate has *no* exposure to shocks in interest rates. The same is true of the Macauley duration defined below: the formula for duration assumes nonrandom claims and does not work for floating-rate bonds and more complex claims.

Duration

When there is a single source of interest rate risk, it is useful to think of our measure of interest rate risk being the equivalent investment in a zero-coupon bond with the same risk exposure. The traditional (Macauley) measure of duration can be derived in a world in which there is a flat term structure that can move up or down. With a flat term structure, a small change δ in the interest rate gives an approximate proportional change in the value of a zero-coupon bond with time-to-maturity T.

$$\frac{1/(1+r+\delta)^T - 1/(1+r)^T}{1/(1+r)^T} \approx -\frac{\delta}{1+r}T$$

For a bond promising cash flows c_s at each future time s,

$$P(r) = \sum_{t=1}^{T} \frac{c_t}{(1+r)^t}$$

is the value of the bond. Then for a small change δ in the interest rate r, the proportionate change in value is approximately

$$\frac{P(r+\delta) - P(r)}{P(r)} \approx -\frac{\delta}{1+r} \frac{\Sigma_{s=1}^{T} s \frac{c_s}{(1+r)^s}}{\Sigma_{s=1}^{T} \frac{c_s}{(1+r)^s}}$$

$$\approx -\frac{\delta}{1+r} \; duration$$

The bond's $duration$ is the value-weighted average time-to-maturity of the claim:

$$duration = \sum_{t=1}^{T} \frac{\frac{c_t}{(1+r)^t}}{\Sigma_{s=1}^{T} \frac{c_s}{(1+r)^s}} \; t$$

Matching risk exposures for small changes in r for the zero-coupon bond and the general bond, we see that the time-to-maturity of the zero-coupon bond is equal to the duration of the general bond.

Duration: some observations

- The duration of a coupon bond or self-amortizing bond falls as rates rise.

- For near maturity coupon bonds, duration is close to time-to-maturity.

- For long maturity coupon bonds, duration is much less than time-to-maturity.

- As maturity approaches infinity, the duration of a par coupon bond approaches $1/r$ (exact with continuous coupon and compounding, $(1+r/2)/r$ with semi-annual coupon and compounding)
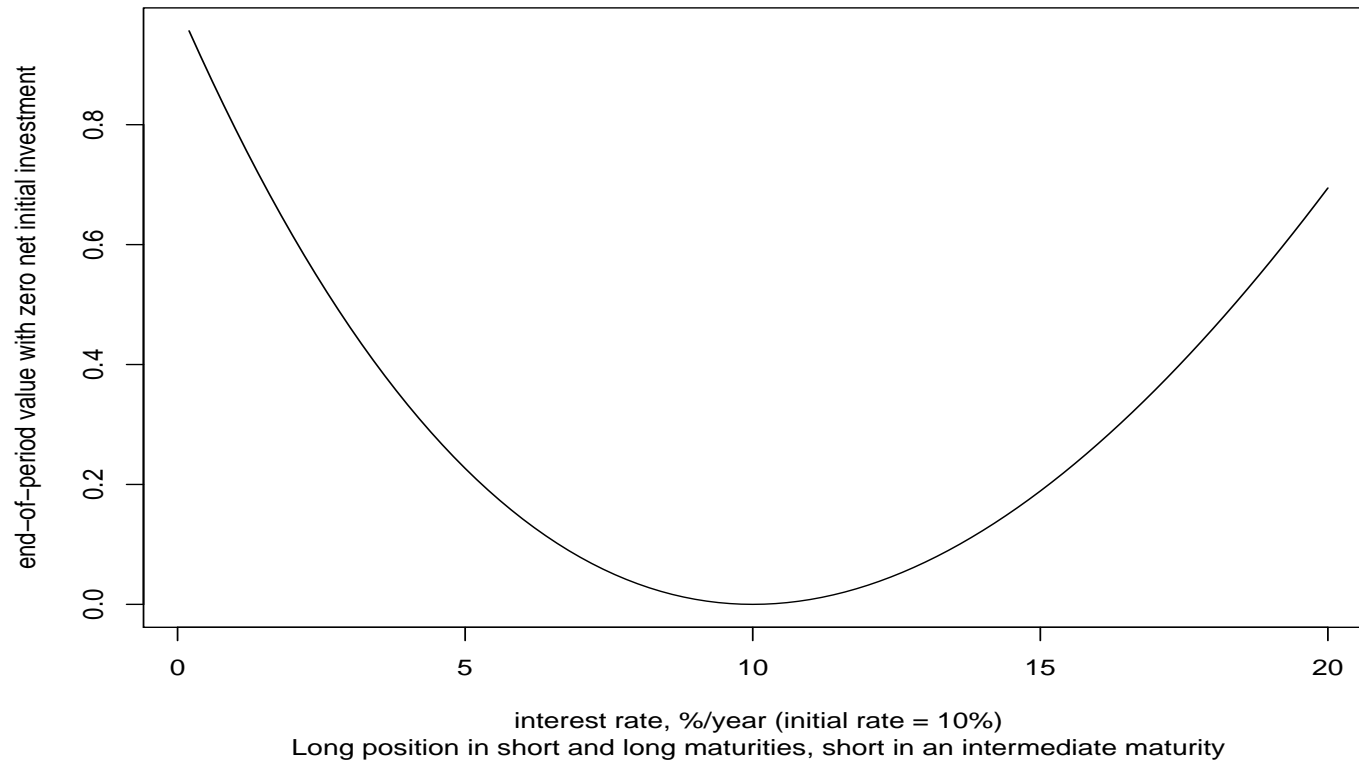
Duration for par coupon bonds

Duration of par coupon bonds for different maturities and coupon rates

Legend:
- 0.0%
- 2.5%
- 5.0%
- 7.5%
- 10.0%
- 12.5%
- 15.0%
- 17.5%
- 20.0%

Years to maturity

Duration in years
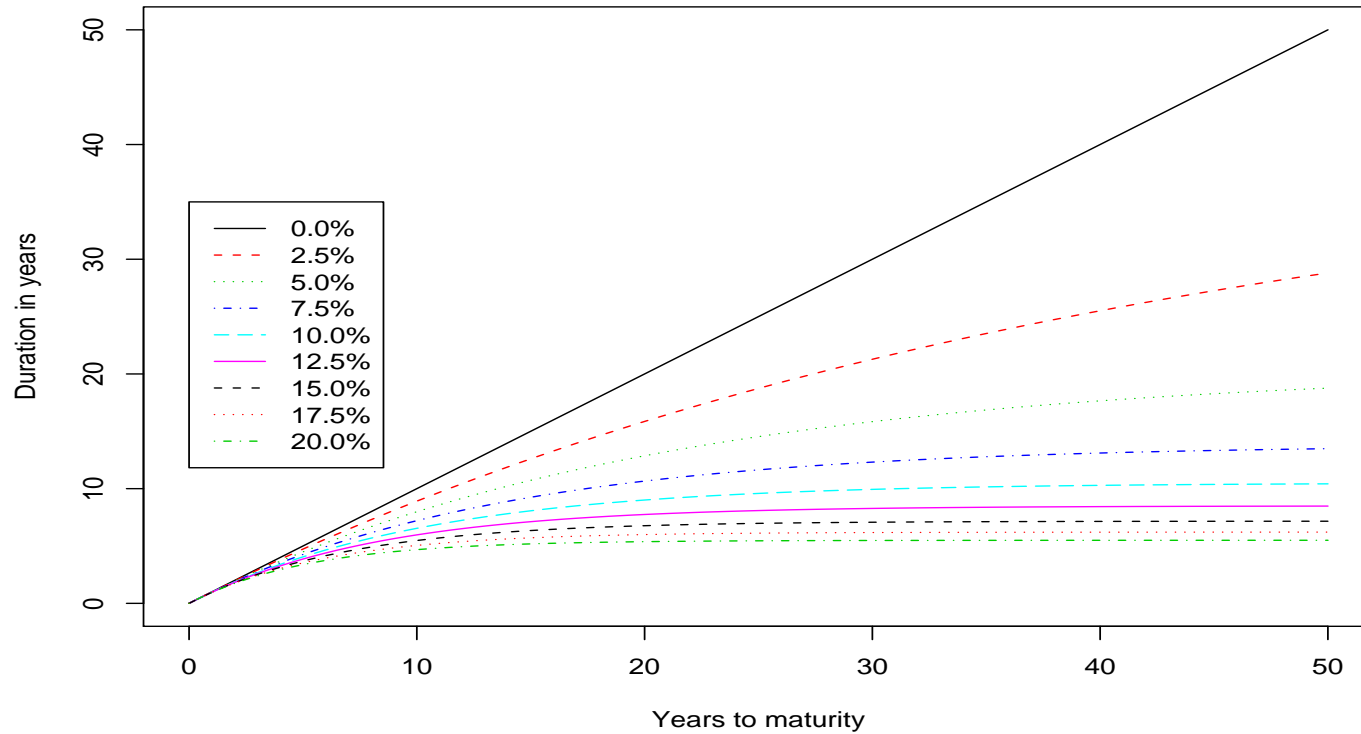
Parallel moves in a flat yield curve: the arb!

Arbitrage (!) if the yield curve is always flat but can move

Long position in short and long maturities, short in an intermediate maturity

Strangely enough, having a flat yield curve that moves up and down implies there is arbitrage! This is related to convexity of the bond price in the interest rate.

Effective duration

Effective duration captures the good features of duration while addressing its lack of flexibility. The effective duration of an interest-sensitive security is the time-to-maturity of the zero-coupon bond with the same interest sensitivity. If we are looking at nonrandom claims, the effective duration is equal to Macauley duration. Effective duration can also be computed given different assumptions about interest rate shocks that do not hit all yields equally (which is good because short rates move around more than long rates). Also, effective duration can be computed for a variety of interest derivatives if we know how their prices depend interest rates. Option pricing theory is an ideal tool for performing this analysis; in the next lecture we will consider the use of option pricing tools in pricing interest derivatives. The rest of this lecture is devoted to a more traditional approach.

Effective duration of riskless bonds

In the traditional approach to defining effective duration, we need to make an assumption about the shape of the impact of interest rate shocks on the yield curve. Suppose we start with the forward rate curve $f(s, t)$ at time s for different future times t and we think of an interest rate shock moving us to a nearby curve $f^*(0, t)$ depending on the shock δ as

$$f^*(s, t) = f(s, t) + \delta x(t - s)$$

where $x(t$-$s)$ is the sensitivity of the forward rate t-s periods out to this sort of shock. Each different function $x(.)$ will give us a different duration measure.

Effective duration: shape of the shock

Factor analysis of errors in predicting next-period bond yields suggests using factors corresponding coarsely to the level of the yield curve, the slope of the yield curve, and curvature of the yield curve. The factor corresponding to levels explains the lion's share of the variance and has a sensitivity of the forward price to the shock that declines as time-to-maturity increases. According to the estimates in one paper of mine,[1] the function $x(t - s) = exp(-.125(t - s))$ is a good fit for this dominant factor. The corresponding shock to zero-coupon yields is $sens(t - s) = (1 - exp(-.125(t - s)))/.125$, that is, the effective duration of a bond with cash flows c_t at times $t = 1, 2, ..., T$ will have an effective duration that solves

$$sens(duration) = \left(\frac{\Sigma_{s=1}^{T} sens(s)c_s D(0, s)}{\Sigma_{s=1}^{T} c_s D(0, s)} \right)$$

which is the same as the formula for Macauley duration except substituting $y(s)$ for the impact s on both sides and using the general formula $D(0, s)$ for the discount factor. To solve for duration, it is useful to note that the $\log()$ (base e) is the inverse of $\exp()$, and therefore $duration = -log(1 - .125 * sens)/.125$.
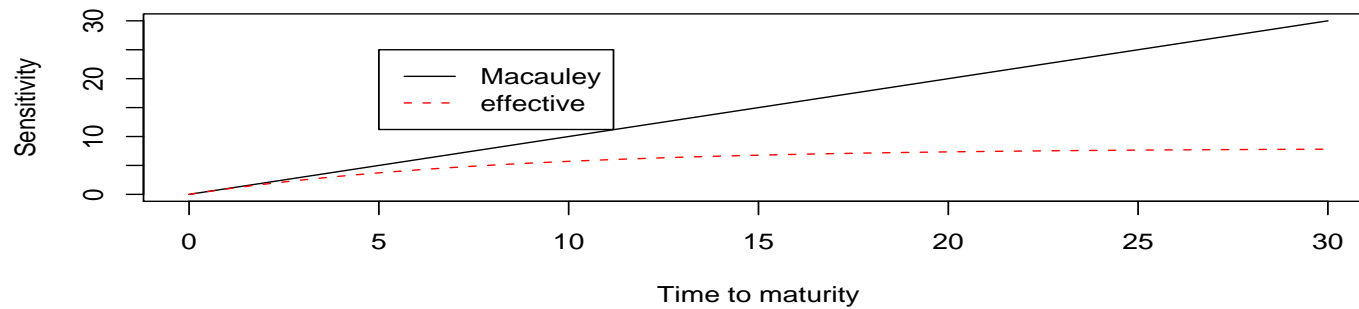
[1]Dybvig, Philip H., Bond and Bond Option Pricing Based on the Current Term Structure, 1997, *Mathematics of Derivative Securities*, Michael A. H. Dempster and Stanley Pliska, eds., Cambridge University Press.

Shape of the assumed shock
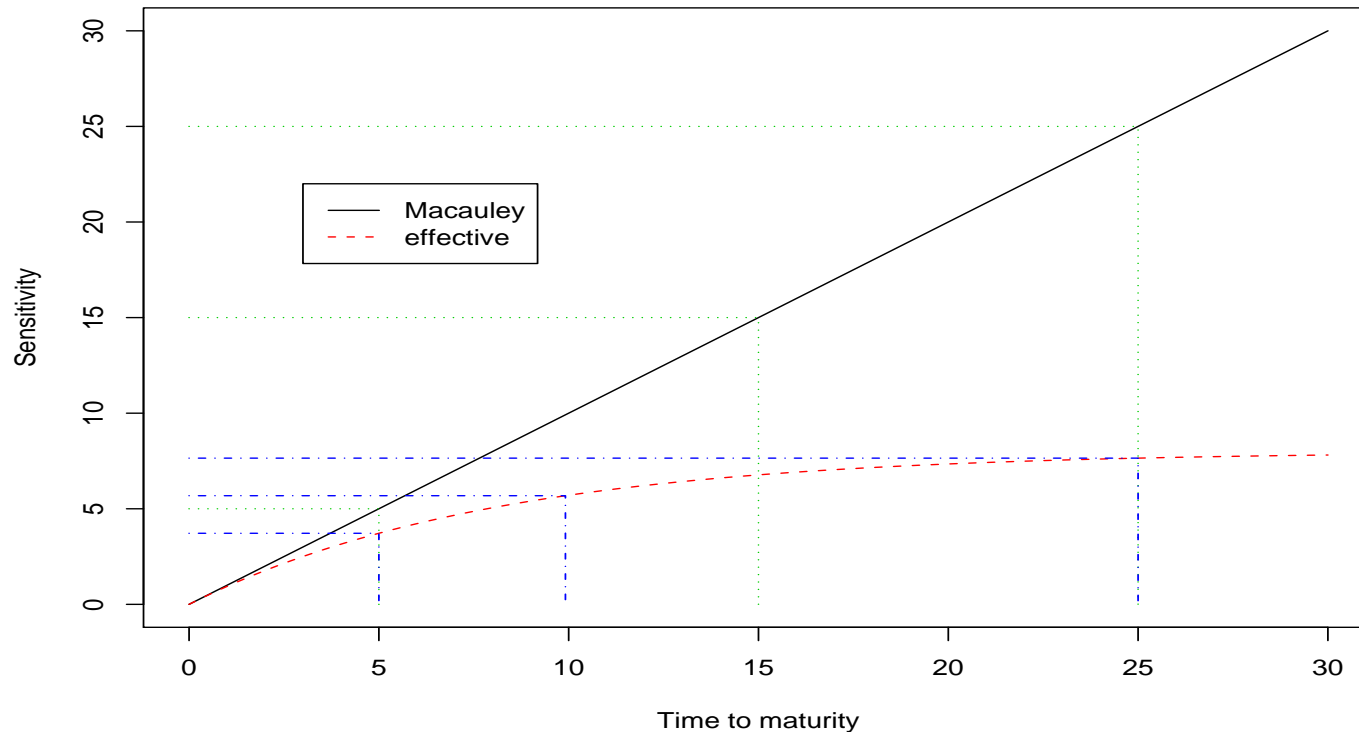
Sensitivity of forward rates to an interest shock
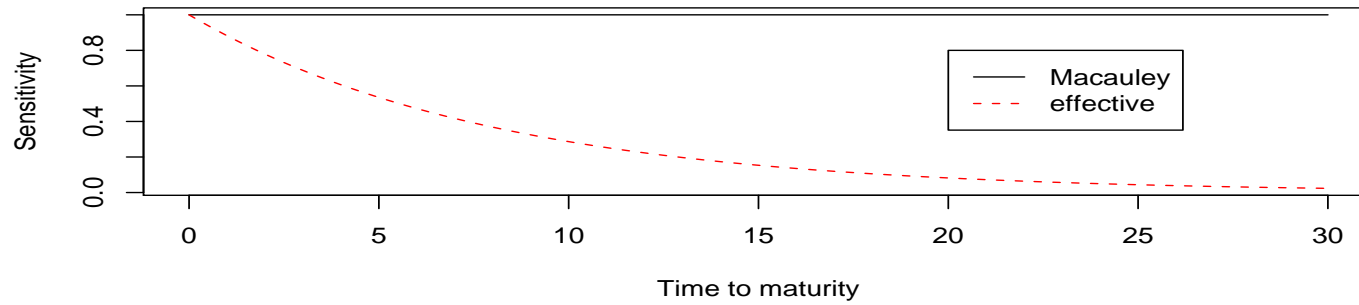
Sensitivity of zero−coupon prices to an interest shock

Effective duration computation

Sensitivity of zero–coupon prices to an interest shock

What is the effective duration of a portfolio half invested in 5-year zero-coupon bonds and half invested in 25-year zero-coupon bonds? The Macauley duration is 15, while the effective duration is about 10.

In-class exercise: effective duration

Assuming the sensitivity of discount bond prices to a shock is given by $sens(t - s) = (1 - exp(-.125(t - s)))/.125$ (as we have been assuming), compute the Macauley duration and the effective duration of a bond which pays $3/4$ of its value at 30 years out and $1/4$ of the value 10 years out. Either use the graph on the previous slide to obtain an approximate value, or use the formulas from a couple of slides back to perform a more exact computation.

Multiple duration measures

One interesting traditional extension to using duration is to use multiple duration measures and to assert that the risk exposure is matched if each of the duration measures is matched. Thus, we may have a separate duration measure for up-and-down movements in the whole yield curve, and for changes in the slope and curvature of the yield curve as well. In practice, using multiple measures is useful (and can give a more realistic assessment of the risk in positions that are designed to be neutral to a single duration measure). Often, using multiple measures is not much different in its prescription than approximate matching of cash flows as we have discussed earlier.