Optimal Execution of Portfolio Transactions

– an especially influential research paper by Almgren and Chriss

theory paper ≥ 500

Big Paper

X: the number of shares of a (single) stock to be sold

T: the time by which the shares are to be sold

– time 0 is now

卖 sold

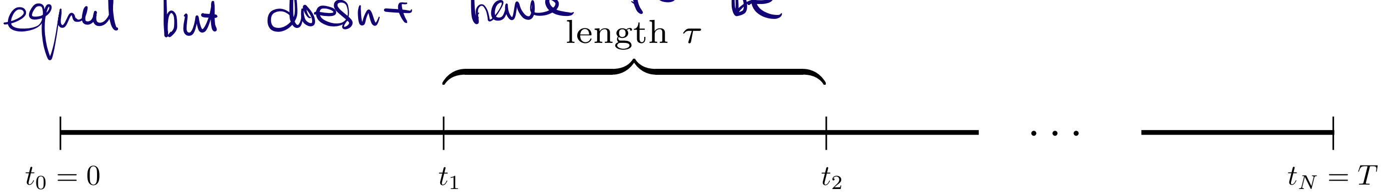
Assume the horizon is divided into periods of equal length

N: the number of periods

$\tau := T/N$: the length of each period

Having periods of different lengths τ_k can easily be included in the model.

equal but doesn't have to be



S_k: stock price at time t_k

Considerations of market impact aside,

the price is assumed to follow an arithmetic random walk:

$$S_k = S_{k-1} + \sigma \sqrt{\tau}\, \xi_k$$

length of time period 不变

where ξ_k is a random variable with mean 0 and variance 1

and $\xi_1, \ldots, \xi_T$ are independent

Notes:

- σ is the stock's volatility

  that is, the price's standard deviation *per unit time period*

- having nonzero means $E[\xi_k]$ can easily be included in the model

  as can having differing volatilities σ_k.

- the random walk is arithmetic rather than geometric

  – although this is an unavoidable shortcoming of the model,

  the time scales at which the model is applied

  makes this be not a significant issue

$X_0 = X$ units

$X_1 = X - y_1$

\# hodings

$$S_k = S_{k-1} + \sigma\sqrt{\tau}\,\xi_k$$

S_0 — S_1 — S_2 — $\cdots$ — S_N

$t_0 = 0$ — y_1 — t_1 — y_2 — t_2 — $\cdots$ — $t_N = T$

$\downarrow$ sell

y_k: amount to be sold between t_{k-1} and t_k

temporary market impact:

$\tau = \frac{T}{N}$

Rather than receiving S_{k-1} per unit sold, instead receive

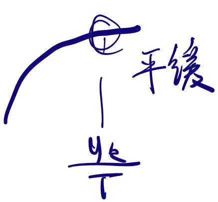
sell $\Rightarrow$ depress the price!

$$\tilde{S}_k := S_{k-1}^{\text{卖}} - h(y_k/\tau) \quad \text{rate!}$$

$h(v) = \alpha v^{\frac{1}{2}}$ typically

where h is a non-negative function chosen by the user.

(Appropriate examples include $h(v) = \alpha\, v^p$ for constants $\alpha, p \geq 0$.)



(Note: y_k/τ is the rate at which the stock is sold during the period.)

$$\underbrace{\text{total amount received}}_{\text{“captured value”}} = \sum_{k=1}^{N} \tilde{S}_k y_k \quad \text{at } T_k$$

Let x_k be number of shares still held at time t_k

– thus, $x_0 = X$, $x_N = 0$ and $x_k = x_{k-1} - y_k$

$$\text{captured value} = \sum_{k=1}^{N} \tilde{S}_k y_k$$

$$= \sum_{k=1}^{N} \left(S_{k-1} - h\left(\frac{y_k}{\tau}\right)\right) y_k$$

how much selling

$$= \underbrace{\sum_{k=1}^{N} S_{k-1} y_k} - \sum_{k=1}^{N} h\left(\frac{y_k}{\tau}\right) y_k$$

No randomness

by definition of x_k $\longrightarrow$ $=$

$$\underbrace{\sum_{k=1}^{N} S_{k-1}\left(x_{k-1} - x_k\right)}$$

by rearrangement, and using $x_0 = X$, $x_N = 0$ $\longrightarrow$ $=$

$$\underbrace{S_0 X + \sum_{k=1}^{N-1} \left(S_k - S_{k-1}\right) x_k}$$

by using $S_k = S_{k-1} + \sigma\sqrt{\tau}\,\xi_k$ $\longrightarrow$ $=$

$$S_0 X + \sigma\sqrt{\tau} \sum_{k=1}^{N-1} \xi_k\, x_k$$

Thus,

$$\text{captured value} \;=\; S_0X + \sigma\sqrt{\tau}\sum_{k=1}^{N-1}\xi_k\,x_k - \sum_{k=1}^{N} h\left(\frac{y_k}{\tau}\right)\,y_k$$

and so

E[ ] 0

$$\text{E[captured value]} \;=\; S_0X - \sum_{k=1}^{N} h\left(\frac{y_k}{\tau}\right)\,y_k$$

$$\text{Var[captured value]} \;=\; \sigma^2\tau\sum_{k=1}^{N-1}x_k^2$$

Observe that if $v \mapsto h(v)v$ is a convex function,
then E[captured value] is a concave function of $y_1, \ldots, y_n$

⊙caused.

just as one wants
for the problem $\max_y$ E[captured value]

In particular, if $h(v) = \alpha\, v^p$ for constants $\alpha, p \geq 0$,
then E[captured value] is concave.

“Efficient frontier of optimal execution”

$$\max \quad \mathrm{E}[\text{captured value}]$$

concave. (sold total value)

$$\text{s.t.} \quad \mathrm{Var}[\text{captured value}] \le s^2$$

user chosen constant

$$\Updownarrow$$

$$\max_{x,y} \quad S_0 X - \sum_{k=1}^{N} h\left(\frac{y_k}{\tau}\right) y_k$$

$$\begin{aligned} \text{s.t.} \quad & \|x\| \le \frac{s}{\sigma\sqrt{\tau}} \\ & x_1 = X - y_1 \\ & x_k = x_{k-1} - y_k \quad \text{for } k = 2, \ldots N-1 \\ & y_N = x_{N-1} \\ & y \ge 0 \end{aligned}$$

If $h(v) = \alpha v$ (where $\alpha > 0$),
then the objective is concave quadratic,
and the problem can be recast as an SOCP.

It is common, however, to rely on temporary impact functions
of the form $h(v) = \alpha\sqrt{v}$ (where $\alpha > 0$),
in which case the objective can still be handled by CVX,
although the optimization problem is no longer an SOCP.

Typically in practice, only temporary market impact is considered,
but now suppose that in addition to temporary market,
we want to include permanent impact:

$$S_k = S_{k-1} + \sigma\sqrt{\tau}\,\xi_k - \underbrace{\tau g(y_k/\tau)}_{\substack{\text{impact depends on} \\ \text{rate of selling } y_k/\tau \\ \text{and on length of selling } \tau}}$$

Notes:

- permanent impact generally is much less than temporary impact
- to achieve a model appropriate for CVX,
  there has to be relation between g and h,
  such as both being linear:
  $$h(v) = \alpha v, \qquad g(v) = \beta v \qquad (\text{where } \alpha, \beta > 0)$$

Going through the same gymnastics as before gives

captured value =

$$S_0 X + \sum_{k=1}^{N-1} \left(\underbrace{\sigma\sqrt{\tau}\xi_k}_{\text{random}} - \tau\, g\!\left(\frac{y_k}{\tau}\right)\right) x_k \; - \; \sum_{k=1}^{N} h\left(\frac{y_k}{\tau}\right) y_k$$

Thus,

$$\mathrm{E}[\text{captured value}] = S_0 X - \tau \sum_{k=1}^{N-1} g\left(\frac{y_k}{\tau}\right) x_k - \sum_{k=1}^{N} h\left(\frac{y_k}{\tau}\right) y_k$$

$$\mathrm{Var}[\text{captured value}] = \sigma^2 \tau \sum_{k=1}^{N-1} x_k^2$$

$h\left(\frac{y_k}{\tau}\right)$

If $h(v) = \alpha v$ and $g(v) = \beta v$, we have

$$\mathrm{E}[\text{captured value}] = S_0 X - \beta \sum_{k=1}^{N-1} y_k x_k - \frac{\alpha}{\tau} \sum_{k=1}^{N} y_k^2$$

But now the previous development hits a (slight) roadblock,
because functions $(x_k, y_k) \mapsto -x_k y_k$ are not concave,
unlike functions $y_k \mapsto -y_k^2$

Resurrection lies in permanent impact being less than temporary impact.

To see how, substitute $x_{k-1} - x_k$ for the first occurrence of y_k:

换 y_k !

$$\text{E[captured value]} = S_0 X - \beta \sum_{k=1}^{N-1} y_k x_k - \frac{\alpha}{\tau} \sum_{k=1}^{N} y_k^2$$

$x_0 = X \quad x_N = 0$

we can put N rather than $N-1$ because $x_N = 0$

$$= S_0 X - \beta \sum_{k=1}^{N} \underbrace{(x_{k-1} - x_k) x_k}_{= \underbrace{.5\left(x_{k-1}^2 - x_k^2 - (x_{k-1} - x_k)^2\right)}_{= .5\left(x_{k-1}^2 - x_k^2 - y_k^2\right)}} - \frac{\alpha}{\tau} \sum_{k=1}^{N} y_k^2$$

cancellation

$$= S_0 X - \frac{\beta}{2} X^2 - \left(\frac{\alpha}{\tau} - \frac{\beta}{2}\right) \sum_{k=1}^{N} y_k^2$$

消 x_k !

which is concave if $\alpha \geq \beta\tau/2$

(i.e., if temporary impact is appropriately larger than permanent impact)

Selling Multiple Securities Simultaneously

basket

Motivation: Selling one of the stocks
might depress the price not only of that stock
but also the prices of related stocks to be sold.

Ignoring such relationships would lead
to suboptimal execution of portfolio liquidation.

X is now a vector, as are S_k, y_k and x_k

(For example, the i^{th} coordinate of X
is the number of shares to be sold of the i^{th} stock.)

Assume for notational simplicity that $\tau = 1$

Considerations of market impact aside,

assume $S_k = S_{k-1} + \Theta_k$

where Θ is a random vector with mean $\vec{0}$ and covariance matrix V

(and where $\Theta_1, \ldots, \Theta_N$ are independent)

temporary market impact:

Realized prices of shares:

$$\tilde{S}_k(y_k) := S_{k-1} - H(y_k)$$

where $H : \mathbb{R}^n \to \mathbb{R}^n_+$ (i.e., $H(y_k)$ is an n-vector with all non-negative entries)

For simplicity, we will focus on the natural generalization to the linear impact function $h(v) = \alpha v$ in the single asset setting, this being the function $H(v) = Mv$ for any psd matrix M.

symmetric

$$
\begin{aligned}
\text{captured value} &= \sum_{k=1}^{N} \tilde{S}_k(y_k)^T y_k \\
&= S_0^T X + \sum_{k=1}^{N-1} \Theta_k^T x_k - \sum_{k=1}^{N} H(y_k)^T y_k \\
&= S_0^T X + \sum_{k=1}^{N-1} \Theta_k^T x_k - \sum_{k=1}^{N} y_k^T M y_k
\end{aligned}
$$

Thus,

$$
\mathrm{E}[\text{captured value}] = S_0 X - \sum_{k=1}^{N} y_k^T M y_k
$$

$$
\mathrm{Var}[\text{captured value}] = \sum_{k=1}^{N-1} x_k^T V x_k
$$

which give convex, quadratic optimization problems with regards to the efficient frontier.

Similarly, permanent impact can be incorporated with a function $G : \mathbb{R}^n \to \mathbb{R}^n_+$

$$S_k = S_{k-1} + \Theta_k - G(y_k)$$

Choosing $G(v) = \hat{M} v$ for some $n \times n$ symmetric matrix $\hat{M}$ results in

$$\text{captured value} = S_0^T X + \sum_{k=1}^{N-1} \left(\Theta_k^T - y_k^T \hat{M} \right) x_k - \sum_{k=1}^{N} y_k^T M y_k$$

$$\text{E[captured value]} = S_0^T X - \sum_{k=1}^{N-1} y_k^T \hat{M} x_k - \sum_{k=1}^{N} y_k^T M y_k$$

$$= S_0^T X - .5\, X^T \hat{M} X + .5 \sum_{k=1}^{n} y_k^T \hat{M} y_k - \sum_{k=1}^{N} y_k^T M y_k$$

$$= S_0^T X - .5\, X^T \hat{M} X + \sum_{k=1}^{n} y_k^T (.5\, \hat{M} - M) y_k$$

– thus, for concavity, we want to choose M and $\hat{M}$ so that $M - .5\, \hat{M}$ is a psd matrix

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Choosing $G(v) = \hat{M} v$ for some $n \times n$ symmetric matrix $\hat{M}$ results in

$$\text{captured value} \;=\; S_0^T X + \sum_{k=1}^{N-1} \left(\Theta_k^T - y_k^T \hat{M} \right) x_k \;-\; \sum_{k=1}^{N} y_k^T M y_k$$

$$\mathrm{E}[\text{captured value}] = S_0^T X - .5\, X^T \hat{M} X + \sum_{k=1}^{n} y_k^T (.5\, \hat{M} - M) y_k$$

$$\mathrm{Var}[\text{captured value}] = \sum_{k=1}^{N-1} x_k^T V x_k$$