Carnegie Mellon University
MSCF Program
46-956 Introduction to Fixed Income

Fall 2021

Mini 1

Lecture Notes for Week 1

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August 31 & September 2, 2021

Page 1.1: Course Information

Please Read the Entire Syllabus Carefully! It is posted on Canvas.

Instructor: *Bill Hrusa*

Teaching Assistant: *Jose Luis Miranda Olvera*

Official Textbook: *Fixed Income Securities:Tools for Today's Markets* by Bruce Tuckman and Angel Serrat (3^{rd} ed). Exercises from Tuckman & Serrat are posted on Canvas under "Assignments".

Additional Recommended Textbook: *Fixed Income Securities: Valuation, Risk, and Risk Management* by Pietro Veronesi.

Page 1.2: Lectures and Discussion Sessions

Lectures: Tuesdays & Thursdays 3:35 - 5:05 PM

Discussion Sessions: Friday from 9:00 - 10:30 AM (1^{st} session on September 3). These sessions will be devoted to answering questions about lecture material, discussing homework problems, and talking about current market conditions and trading strategies (as time permits).

Trips to NY: I will be in NY on September 7, September 21, and October 5.

Page 1.3: Grading Components

Homework Assignments: 6 to hand in; Lowest Score will be Dropped, i.e. use the Best 5

Final Exam: Sunday October 17, 10:00 AM - 1:00 PM

Course Grade:

- 25% Homework Average
- 75% Final Exam Score

Make-Up Final: Students with a course grade of C+ or below can take a make-up final. The maximum course grade for students taking the make-up is B-. The make-up exam will not be used to lower your course grade.

Page 1.4: Homework Write-Ups

- Each student must turn in his or her own write-up. Homework will be due at 11:59 PM on Tuesdays.
- You may discuss the problems with one another, but must work independently on your write-up. If you consult with other students, please list their names on your write-up.
- You must explain clearly how you arrive at your answer, showing a sufficient number of intermediate steps, with the possible exception of a few short-answer questions.
- It is expressly forbidden to access course materials from previous renditions of 46-956 or 21-378.
- Late HWs cannot be accepted for credit. (Late means any time after 11:59 PM on the due date.)
- You should be sure to keep a copy of your solutions.

Page 1.5: Homework Submission

Homework solutions should be uploaded to the assignment on Canvas. The main body should be submitted as a single PDF file. Supporting spreadsheets (and other computer codes) can also be attached if relevant.

Page 1.6: Homework Grading

- Some of the homework problems might not be graded. (In fact, it will typically be the case that a few of the questions will not be graded.)
- Complete solutions to *all* problems will be posted on Canvas, so you can check your work on all problems. Solutions will be posted at 11:59 PM on the due date. Homework that is submitted after the solutions are posted cannot be accepted for credit.
- A subset of the problems will be selected for careful grading. It will not be announced in advance which problems will be graded.
- Each graded question is worth 10 points.
- For purposes of computing course grades, all homework assignments carry equal weight. (When I drop the lowest score, I will drop the score with the lowest percentage, rather than the lowest numerical score.)

Page 1.7: Securities

In this course, we shall use the term *security* to mean *any tradable financial instrument*. (In some contexts, the term "security" has a more precise legal meaning that might exclude some of the financial instruments we talk about from legally being considered securities.)

Some very important types of securities are:

- Equity Securities (e.g., common stocks)
- Debt Securities (e.g., bonds)
- Derivative Contracts

A financial derivative is a contract between two parties that specifies conditions on the values of underlying variables that determine how payments are to be exchanged between the parties. The underlying variables are often prices of assets. Some important examples of derivatives are put and call options on stocks, futures contracts, and interest rate swaps.

Page 1.8: A Bond is Really a Loan

An agent who purchases a bond is actually making a loan to the agency that issues the bond. The initial price paid for the bond is the *principal* of the loan. (Assuming that the issuing agency does not *default*) the bond holder receives *interest* payments as well as repayment of the principal. There are several common repayment schemes:

- The loan is repaid with a single lump-sum payment (principal plus interest) at maturity. (Zero Coupon Bond)
- There are periodic interest-only payments and the principal (together with a final interest payment) is paid at maturity. (Coupon Bond)
- There are periodic level payments with a portion of each payment reflecting interest on the outstanding principal and the rest of the payment is applied to reduce the outstanding principal balance. (Annuity or Self-Amortizing Loan)

Page 1.9: Negative Interest Rates

The "standard" situation with interest rates is that they are positive. In other words, the borrower in a loan is required to repay the lender *more* than the amount of money borrowed.

However, in certain parts of the world, there are currently important loan contracts with *negative* interest rates. For such contracts, the borrower repays less than the amount borrowed.

As of yesterday, 10-year government bonds had negative yields in France, Germany, and Switzerland. The yield on 5-year government bonds in Japan was also negative.

Page 1.10: Fixed-Income Securities

Literal Definition: A security whose future payments (dates and amounts) are known (with certainty) at the time that the security is issued.

It is standard practice to use the term *fixed income security* to also include derivatives written on debt securities. For such derivatives, the payment dates and amounts generally will not be known with certainty in advance. A better term might be *interest-rate product*.

We shall refer to securities whose future payments are known with certainty as *securities with deterministic cash flows* or *securities with fixed payments*. Many US treasuries fall into this category.

Page 1.11: Default Risk

Some securities (e.g., corporate bonds) that promise fixed future payments might not actually make the promised payments because the issuer of the security might default. *Default Risk*, *Credit Risk*, or *Counterparty Risk*, although extremely important, will not be treated with any degree of depth in this course. **Unless stated otherwise, we assume there is no risk of default**.

Page 1.12: Some Important Fixed-Income Securities

- Zero-Coupon Bonds
- Coupon Bonds
- Annuities
- Inflation Protected Bonds (TIPS)
- Floaters and Inverse Floaters
- Callable and Putable Bonds
- Interest Rate Swaps, Caps, Floors, Swaptions
- Interest Rate Futures (Eurodollar, SOFR, Fed Funds)
- Mortgage Backed Securities (MBS)
- Bond Options
- Bond Futures
- Options on Futures (Interest Rate, Bond)

Page 1.13: Interest Rates

Prices of fixed income securities are frequently described using *interest rates*. In practice, interest rates depend on many factors, including the initiation date of the loan, the length of the loan, and the schedule according to which payments are to be made.
Interest rates that will prevail at future dates are generally not known in advance.

Page 1.14: Models for Interest Rates

Much can be said about securities with deterministic cash flows without employing a model describing the manner in which interest rates will evolve in time. Roughly the first half of the course will be devoted to this topic. (Classical Material)

In most cases, in order to analyze fixed-income securities with uncertain payments, it is crucial to make a mathematical model that reflects the uncertainty. This is a very serious (and interesting) endeavor. The second half of the course will be devoted to this topic. We will consider only very simple models (discrete time and finite sample space), but the same ideas can be used with much more sophisticated models.

Page 1.15: Players in Fixed-Income Markets

- **Issuers** - include federal, state and local governments; agencies (GSE); corporations
- **Intermediaries** - facilitate the sale of securities - include dealers; investment banks; credit rating agencies
- **Investors** - include individuals; governments; insurance companies; pension funds; mutual funds; commercial banks

This course will focus on the mathematical principles used to analyze fixed-income securities rather than on "institutional practice". However, I will make a serious effort to respect actual market conventions as much as possible in examples and exercises.

Page 1.16: Risks for Fixed-Income Securities

In addition to default risk, there are many types of risks that need to be considered for fixed income securities. These include

- Inflation Risk
- Liquidity Risk
- Interest Rate Risk (Market Risk)
- Reinvestment Risk
- Currency Risk (FX Risk)
- Timing Risk (Call Risk)

Traders are most concerned with liquidity risk, interest rate risk, timing risk, and currency risk. Default risk can sometimes be hedged using *credit default swaps* (CDS). It is important note that all of the types of risk listed above, except for "Timing Risk" are relevant for securities with deterministic cash flows.

 Securities with Deterministic Cash Flows

Remark 1.1: Unless stated otherwise: Time will be measured in years and the present time is taken to be $t = 0$. We assume that there is no risk of default. We ignore transaction costs (including bid-ask spread) and we assume that all securities can be purchased or sold short in any amounts we please (including fractional shares). Moreover, we assume that the size of an order does not have any impact on the price per share.

Ideal Frictionless Liquid Market with no Default Risk

Page 1.18: Zero-Coupon Bonds

Characterized by a *face value* F and a *maturity* T. The holder receives a single payment of amount F at time T.

Building blocks for all securities with deterministic cash flows:
Every security with deterministic cash flows can be expressed as a portfolio of zero-coupon bonds.

They are also called *pure discount bonds*, *zeros*, or *ZCBs*.

Page 1.19: Coupon Bonds

Characterized by a *face value* F, a *maturity* T, an *annual coupon rate* q, and a number m of *coupon payments* per year.

The holder receives coupon payments of amount

$$F\frac{q}{m}$$

at each of the times $\frac{i}{m}$ for $i = 1, 2, \cdots, mT$ plus the face value at maturity. (Notice that that the holder receives $F(1 + \frac{q}{m})$ at time T.)

For the vast majority of bonds in the US $m = 2$, i.e. the coupon payments are made once every six months. Unless stated otherwise, we assume that $m = 2$.

Page 1.20: Par Value

Remark 1.2: The face value of a bond is often referred to as the *par value*. When a coupon bond is issued, the coupon rate is typically chosen so that the initial price of the bond is very close to the face value. A bond is said to trade *above par*, *at par*, or *below par*, depending on whether the current price of the bond is above, equal to, or below the face value. The terminology *premium to par* and *discount to par* is also commonly used.

Page 1.21: Price Quotations

Unless stated otherwise, bond prices will be given per \$100 of face value (or as a percentage of face value). I will use decimal amounts in lecture, homework exercises, and on the final exam. (Read about *ticks* on your own.)

Unless stated otherwise, when we encounter coupon bonds that were issued previously, we assume that a coupon payment has just been made and that the next coupon payment will be made in precisely $\frac{1}{m}$ years, where m is the number of coupon payments per year. The current price of the bond *does not* include the coupon payment that has just been made. When a bond trades between coupon payments there is *accrued interest* that is separate from the quoted bond price. (Accrued interest will be discussed next week.)

Page 1.22: Annuities

Characterized by a *maturity* T, a payment amount A, and a number m of payments per year. The holder receives payments of amount A at each of the times $\frac{i}{m}$ for $i = 1, 2, \cdots, mT$.

- A *perpetuity* is an annuity that has maturity $T = \infty$.
- There are also *lifetime annuities* (having unknown maturity) and annuities making variable payments.

Page 1.23: Discount Factors & the Time-Value of Money

Under typical economic conditions, it is better to receive \$1 now than in the future. Similarly, it is better to pay \$1 in the future than to pay \$1 now.

Question: How much would you pay today in order to receive \$1 t-years from today? (No risk of default.)

Although different investors may have different feelings about this, there is a single market price for this privilege, namely the *discount factor* for time t; it is denoted by

$$d(t).$$

Page 1.24: Discount Factors (Cont.)

We must have $d(t) > 0$ in order to avoid arbitrage.

In the US, it is almost always the case that

$$d(t) < 1 \text{ for } t > 0$$

and that $d(t)$ decreases as t increases.

Situations in which $d(t) > 1$ for some $t > 0$ or $d(t_2) > d(t_1)$ for some $t_2 > t_1$ correspond to some interest rate being negative. There are certain important economies (e.g., Japan and Germany – and even Denmark now!) that currently have some negative interest rates. There have been negative interest rates in the US (e.g. during the 2008 financial crisis), but these situations have not lasted for very long. Negative interest rates have been present in Japan and Germany for several years.

Page 1.25: Law of One Price

Remark 1.3: The price today for receiving an amount F t-years from today is $Fd(t)$. Consequently, the price today of a zero with face value F and maturity T should be $Fd(T)$.

Law of One Price: Two securities (or portfolios) with exactly the same future payments should have the same current price.

Remark 1.4: The price today for a security that will make payments of amounts F_i at each of the times t_i for $i = 1, 2, \cdots, N$ is given by

$$P = \sum_{i=1}^{N} F_i d(t_i) = F_1 d(t_1) + F_2 d(t_2) + \cdots + F_N d(t_N).$$

Page 1.26: Arbitrage

An extremely important concept in mathematical finance is the notion of **arbitrage**

By an *arbitrage strategy* we mean a trading strategy in which there is no input of capital, zero probability of a loss, and a strictly positive probability of a (strictly positive) profit.

By a *strong arbitrage* we mean an arbitrage strategy in which a strictly positive profit is certain (i.e. has probability 1). Clearly, every strong arbitrage strategy is an arbitrage strategy.

Although arbitrage opportunities do sometimes exist in the real world, they usually disappear shortly after they are discovered, because prices will adjust once there is heavy trading to attempt to make an arbitrage profit. Unless stated otherwise, we assume that arbitrage is not possible. The absence of arbitrage implies that the law of one price holds.

Page 1.27: Arbitrage (Cont.)

In general, the absence of arbitrage implies the absence of strong arbitrage, and the absence of strong arbitrage implies the law of one price. (It is possible to construct models in which the law of one price is satisfied, but there is strong arbitrage. It is also possible to construct models in which there is no strong arbitrage, but arbitrage is possible.) The concept of arbitrage will be treated in detail in MPAP (Multiperiod Asset Pricing).

Traders and other practitioners sometimes use the term arbitrage to indicate a strategy that requires no input of capital, the probability of a loss is small (but not necessarily zero), and the probability of a profit is high. If there is any doubt about what someone means by arbitrage, you should as whether the probability of a loss must really zero, or whether a loss is simply considered to be unlikely.

Page 1.28: Discount Factors and Replication

Example 1.1: Suppose that the following three bonds are trading:

- Bond #1: A zero with maturity 6 months and current price 97.53 (per $100 of face)
- Bond #2: A coupon bond with maturity 1 year, annual coupon rate 5% and current price 99.87
- Bond #3: A coupon bond with maturity 18 months and annual coupon rate 10% and current price 106.52

Bond #2 and Bond #3 pay coupons every 6 months.

Let us find the discount factors implicit in these bond prices.

Page 1.29: Example 1.1 (Cont.)

- For Bond #1: $97.53 = 100d(.5)$, which gives $d(.5) = .9753$.
- For Bond #2: coupon payments (per 100 face) are $100(.05)/2 = 2.5$, so that

$$99.87 = 2.5d(.5) + 102.5d(1) = 2.43825 + 102.5d(1),$$

  which gives $d(1) = .9506$.
- For Bond #3: coupon payments (per 100 face) are $100(.1)/2 = 5$, so that

$$106.52 = 5d(.5) + 5d(1) + 105d(1.5),$$

  which gives $d(1.5) = .9228$.

Page 1.30: Exapmle 1.1 (Cont.)

This illustrates the fact that discount factors should decrease as time increases.

Suppose that another bond is also trading:

- Bond #4: A zero coupon bond with maturity 1 year and current price 94.93.

We see that Bond #4 is trading below the price we calculate using the value for $d(1)$ computed above:

$$100d(1) = 95.06.$$

Page 1.31: Example 1.1 (Cont.)

We say that Bond #4 is *trading cheap* relative to the other bonds. (A bond that is trading at a price higher than the price calculated using other bonds is said to *trade rich* relative to the other bonds.)

There may be an opportunity to make a riskless profit by purchasing Bond #4 and selling a portfolio that *replicates* Bond #4 in terms of Bond #1 and Bond #2.

Page 1.32: Example 1.1 (Cont.)

If we purchase 100 face of Bond #2 and sell short 2.5 face of Bond #1, we create a zero with maturity 1 year and face value 102.5. Consequently, 100 face of Bond #4 can be replicated by purchasing 100/1.025 face of Bond #2 and shorting 2.5/1.025 face of Bond #1. The cost of the replicating portfolio (per 100 face) is

$$\frac{1}{1.025}99.87 - \frac{2.5}{102.5}97.53 = 95.06.$$

Ignoring transaction costs, one can make a riskless profit (immediately) of .13 per 100 face by purchasing Bond #4 and selling the replicating portfolio. This is an example of an *arbitrage strategy* (in fact, a strong arbitrage). Let us look at the associated cash flows:

Page 1.33: Example 1.1 (Cont.)

Time $t = 0$: Purchase 100 face of Bond #4, sell short 100/1.025 face of Bond #2, and purchase 2.5/1.025 face of Bond #1. Net cash flow is $-94.93 + (1/1.025)99.87 - (2.5/102.5)97.53 = .1254$.

Time $t = .5$: Pay coupon associated with the short sale of Bond #2 and receive payment from purchase of Bond #1. Net cash flow is $-(2.5/1.025) + (2.5/1.025) = 0$.

Time $t = 1$: Receive payment from Bond #4 and close out short position on Bond #2. Net cash flow is $100 - (1/1.025)102.5 = 0$.

Remark: On \$500 million face of Bond #4, the profit would be \$626,829.27. There may be practical reasons that a transaction cannot be executed at the posted prices. If this kind of trade could actually be executed in practice, prices of the bonds would adjust until the arbitrage opportunity disappeared.

Page 1.34: US Treasury Securities

T-Bills - Zeros with maturities of 28 days (4 weeks), 91 days (.25 years), 182 (.5 years) days, and 364 days (1 year) when issued.

T-Notes - Coupon bonds with maturities between 1 and 10 years when issued.

T-Bonds - Coupon bonds with maturities greater than 10 years when issued (currently 20 years and 30 years).

STRIPS - Bonds can be *stripped* into individual coupon payments and principal payments (C-strips and P-strips) thereby creating zeros of long maturities. (STRIPS stands for *Separate Trading of Registered Interest and Principal Securities*.)

Page 1.35: US Treasury Securities (Cont.)

All Treasury bills with maturities less than one year are issued weekly. One-year bills are issued every 4 weeks.

Treasury notes are currently being issued with maturities of 2, 3, 5, 7, and 10 years. Bonds are being issued with maturities of 20 and30 years. Notes and Bonds pay coupons every 6 months.

2, 3, 5, and 7-year notes are currently being issued monthly. 10-year notes and 20-year 30-year bonds are currently being issued 4 times per year.

The most recently issued treasury securities of each maturity are said to be *on the run*. Earlier issues are said to be *off the run*. On-the-run issues are typically much more liquid than off-the-run issues.

Page 1.36: US Treasury Securities (Cont.)

The treasury also issues floating rate notes (FRNs) with maturity 2 years and inflation-protected securities (TIPS) with maturities of 5, 10, and 30 years. FRNs pay coupons 4 times per year and TIPS pay coupons twice per year.

C-strips and P-strips can be used to re-assemble a bond. Coupon payments must come from a C-strip and the principal payment must come from a P-strip. Any C-strip paying the correct amount on the correct date can be used for a coupon payment. However, the P-strip must come from a bond having the exact same CUSIP number as the bond being assembled. (CUSIP stands for *Committee on Uniform Security Identification Procedures*.) For this reason, P-strips and C-strips paying the exact same amount on the exact same day might trade at different prices.

Page 1.37: Web Resources for US Treasury Info

www.treasurydirect.gov/indiv/indiv.htm

www.treasury.gov/resource-center/data-chart-center/Pages/index.aspx

www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=yield

www.bloomberg.com/markets/rates-bonds/government-bonds/us/

www.wsj.com/market-data/bonds

Page 1.38: Interest Rates

Discount factors are intrinsic (i.e., they do not depend on any kind of compounding convention). However, they are not very intuitive. Investors often find it useful to quantify the time-value of money using *interest rates*. There are numerous different conventions used to describe interest rates and it is essential to know what convention is being employed in each particular situation.

Page 1.39: Semiannual Compounding

Because most bonds in the US pay coupons every 6 months, bond investors in the US often focus on *semiannual compounding*: If an amount x is invested at an annual rate r compounded semiannually for T years, the value of the investment will be

$$x\left(1+\frac{r}{2}\right)^{2T}$$

after T years. If we know the initial investment, the final value of the investment, and the length of time the money was invested we can work backward and determine the interest rate. (When we use semiannual compounding, the maturity T should be a multiple of 6 months.)

Page 1.40: Semiannual Compounding (Cont.)

Indeed, if an initial investment of x grows to the amount w after T years, we have the equation

$$w = x\left(1 + \frac{r}{2}\right)^{2T},$$

which can be solved for r:

$$r = 2\left[\left(\frac{w}{x}\right)^{\frac{1}{2T}} - 1\right].$$

Page 1.41: Spot Rates

The *spot rate* for maturity t is the interest rate on a spot loan in which the lender gives money to the borrower at the time of the agreement ($t = 0$) and the loan is settled with a single lump-sum payment at time t. Following Tuckman & Serrat, the t-year spot rate (semiannual compounding) will be denoted by

$$\hat{r}(t).$$

Although spot rates can be described using any compounding convention, we shall use semiannual compounding unless stated otherwise. Notice that

$$\hat{r}(t) = 2\left[\left(\frac{1}{d(t)}\right)^{\frac{1}{2t}} - 1\right].$$

Page 1.42: Effective Spot Rates

It is also convenient to introduce the *effective spot rate* $\hat{R}(t)$ for time t defined by

$$\hat{R}(t) = \left(\frac{1}{d(t)}\right)^{\frac{1}{t}} - 1,$$

so that an amount x invested at time 0 is worth

$$x(1 + \hat{R}(t))^t$$

at time t. (We do not require t to be a multiple of 6 months.) This convention will prove especially useful in situations when cash flows arrive at dates that are not uniformly spaced.

Page 1.43: An Example

Example 1.2: Compute the spot rates and effective spot rates corresponding to the discount factors

$$d(.5) = .9753, \quad d(1) = .9506, \quad d(1.5) = .9228$$

from Example 1.1. We have

$$\hat{r}(.5) = 2\left[\left(\frac{1}{.9753}\right)^{1} - 1\right] = .05065,$$

$$\hat{r}(1) = 2\left[\left(\frac{1}{.9506}\right)^{\frac{1}{2}} - 1\right] = .05131,$$

$$\hat{r}(1.5) = 2\left[\left(\frac{1}{.9228}\right)^{\frac{1}{3}} - 1\right] = .05429,$$

Page 1.44: Example 1.2 (Cont.)

$$\hat{R}(.5) = \left(\frac{1}{.9753}\right)^2 - 1 = .05129,$$

$$\hat{R}(1) = \left(\frac{1}{.9506}\right) - 1 = .05197,$$

$$\hat{R}(1.5) = \left(\frac{1}{.9228}\right)^{\frac{2}{3}} - 1 = .055022.$$

Remark 1.5: Notice that in Example 1.2, the effective spot rates $\hat{R}(t)$ are greater than the corresponding spot rates $\hat{r}(t)$. This is always the case. Also in this example, the spot rates increase with t. This is not always the case.

Page 1.45: Continuously Compounded Spot Rates

It is sometimes convenient to express spot rates using *continuous compounding*. We denote the continuously compounded spot rate for maturity t by $\hat{r}_c(t)$. It satisfies the condition

$$e^{t\hat{r}_c(t)} = \frac{1}{d(t)},$$

which is equivalent to

$$\hat{r}_c(t) = -\frac{1}{t}\ln(d(t)).$$

Page 1.46: Discount Factors from Spot Rates

We can express the discount factor in terms of the spot rates via the formulas

$$d(t) = \frac{1}{\left(1+\frac{\hat{r}(t)}{2}\right)^{2t}} = \frac{1}{\left(1+\hat{R}(t)\right)^{t}} = e^{-t\hat{r}_c(t)}.$$

Except under extraordinary circumstances, discount factors in the real world decrease as maturity increases. (There are important models which allow discount factors to increase over some time intervals, but with very small probability.)

Page 1.47: Spot Rate Curves

The graph of $\hat{r}(t)$ versus t is called the *spot rate curve* or the *zero-coupon yield curve*. Spot rate curves can have different shapes depending on economic conditions. The graph of $\hat{R}(t)$ versus t is called the *effective spot rate curve*, and the graph of $\hat{r}_c(t)$ versus t is called the *continuously compounded spot rate curve*. We will look at spot rate curves in detail a bit later on.

You should verify for yourself as an exercise that for a given maturity $t > 0$ and a given value of $d(t)$ we have

$$\hat{r}_c(t) \leq \hat{r}(t) \leq \hat{R}(t),$$

with strict inequality except for $d(t) = 1$.

Page 1.48: Forward Interest Rates

A *forward loan* is an agreement to lend money at some future date. The interest rate (as well as the size of the loan) is set at the time of the agreement. (Nothing is paid by either party to enter into the agreement.) We really need 3 time variables and a compounding convention. We will use the notation

$$R^{for}_{\tau,\eta,T}$$

to indicate that the agreement is made at time τ, the loan is initiated at time η and settled at time T, and the compounding convention is "effective". We assume $\tau \leq \eta < T$. Notice that $T - \eta$ is the length of the loan. The amount to be repaid at time T, per \$1 borrowed at time η is

$$\left(1 + R^{for}_{\tau,\eta,T}\right)^{T-\eta}.$$

The forward rates $R^{for}_{\tau,\eta,T}$ are known at time τ, but not earlier.

Page 1.49: Forward Rates (Cont.)

Consider a forward loan in which it is agreed at time 0 to borrow \$1 at time η and repay the loan at time T, at the effective forward rate $R^{for}_{0,\eta,T}$. Let us put

$$A_T = \left(1 + R^{for}_{0,\eta,T}\right)^{T-\eta},$$

the amount to be repaid at time T. The forward loan can be replicated by purchasing a ZCB with maturity η and face value \$1 and shorting a ZCB with maturity T and face value A_T at time 0. By the no-arbitrage principle, the initial cost of the replicating portfolio must be zero, which implies that

$$d(\eta) - A_T d(T) = 0.$$

This tells us that

$$A_T = \frac{d(\eta)}{d(T)}.$$

Page 1.50: Forward Rates (Cont.)

Recalling the definition of A_T we find that

$$\left(1 + R^{for}_{0,\eta,T}\right)^{T-\eta} = \frac{d(\eta)}{d(T)} = \frac{(1 + \hat{R}(T))^T}{(1 + \hat{R}(\eta))^\eta}.$$

The above equation can easily be solved to obtain formulas for $R^{for}_{0,\eta,T}$ in terms of the discount factors or spot rates for maturities η and T.

It is much more important to understand how to replicate a forward loan using ZCBs than it is to "memorize" formulas for forward rates. Simply memorizing formulas could lead to an error because of a different convention being used to for compounding or for describing the time variables.

Page 1.51: Forward Rates (Cont.)

Remark 1.6: The third time variable in the specification of a forward interest rate is sometimes taken to be the length of the loan, rather than the maturity date of the loan. Be sure to ask what convention is being used if you have any doubt. Different practitioners use different conventions, and it is much better to ask a question than to make a pricing error.

Forward rates can also be quoted using other compounding conventions. For semiannual compounding, we use the notation

$$r^{for}_{\tau,\eta,T}$$

to denote the rate agreed upon at time τ for a loan to be initiated at time η and settled at time T. Here we assume that $T - \eta$ is a multiple of 6 months.

Page 1.52: Forward Rates (Continued)

Observe that

$$(1 + R^{for}_{\tau,\eta,T})^{T-\eta} = \left(1 + \frac{r^{for}_{\tau,\eta,T}}{2}\right)^{2(T-\eta)}.$$

Following Tuckman & Serrat we use the notation $f(t)$ to denote the (semiannually compounded) rate agreed upon at time 0 for a loan made at time $t - .5$ and settled with a single lump-sum payment at time t. Notice that

$$f(.5) = \hat{r}(.5).$$

Page 1.53: Forward Rates & Spot Rates

The forward rates $f(t)$ can be deduced from the spot rates $\hat{r}(t)$. For example we must have

$$\left(1+\frac{\hat{r}(1)}{2}\right)^2 = \left(1+\frac{f(.5)}{2}\right)\left(1+\frac{f(1)}{2}\right),$$

which can be solved for $f(1)$. More generally, we must have

$$\left(1+\frac{f(.5)}{2}\right)\times\left(1+\frac{f(1)}{2}\right)\times\cdots\times\left(1+\frac{f(t)}{2}\right) = \left(1+\frac{\hat{r}(t)}{2}\right)^{2t}.$$

Page 1.54: Forward Rates & Spot Rates (Cont.)

It is useful to observe that the forward rates can also be obtained from the equation

$$(1) \qquad \left(1+\frac{\hat{r}(t)}{2}\right)^{2t} = \left(1+\frac{\hat{r}(t-.5)}{2}\right)^{2t-1}\left(1+\frac{f(t)}{2}\right)$$

There are many other similar relationships, e.g.

$$\left(1+\frac{\hat{r}(t)}{2}\right)^{2t} = \left(1+\frac{\hat{r}(t-1)}{2}\right)^{2t-2}\left(1+\frac{f(t-.5)}{2}\right)\left(1+\frac{f(t)}{2}\right).$$

Page 1.55: An Example

Example 1.3: Find the forward rates $f(.5)$, $f(1)$, and $f(1.5)$ implied by the spot rates

$$\hat{r}(.5) = .05065, \quad \hat{r}(1) = .05131, \quad \hat{r}(1.5) = .05429$$

of Example 1.2. We observe first that $f(.5) = \hat{r}(.5) = .05065$. We next observe that

$$\left(1 + \frac{.05065}{2}\right)\left(1 + \frac{f(1)}{2}\right) = \left(1 + \frac{.05131}{2}\right)^2,$$

which gives $f(1) = .05197$. Finally, we observe that

$$\left(1 + \frac{.05065}{2}\right)\left(1 + \frac{.05197}{2}\right) \times \left(1 + \frac{f(1.5)}{2}\right) = \left(1 + \frac{.05429}{2}\right)^3,$$

which gives $f(1.5) = .06026$.

Page 1.56: An Important Observation

Remark 1.7: An upward sloping spot-rate curve corresponds to the forward rates $f(t)$ being above the spot rates $\hat{r}(t)$. A downward sloping spot-rate curve corresponds to the forward rates $f(t)$ being below spot rates $\hat{r}(t)$. To see why this is the case, let us fix $t \in \{.5, 1, 1.5, 2, 2.5, \cdots\}$ and put

$$\lambda = \frac{1 + \frac{\hat{r}(t)}{2}}{1 + \frac{\hat{r}(t-.5)}{2}}. \tag{2}$$

Observe that

$$\begin{aligned} \lambda > 1 \ \text{ if and only if } \ \hat{r}(t) > \hat{r}(t-.5), \\ \lambda < 1 \ \text{ if and only if } \ \hat{r}(t) < \hat{r}(t-.5). \end{aligned}$$

 Remark 1.7 (Cont.)

Using (1) and (2) we find that

$$\left(1 + \frac{f(t)}{2}\right) = \lambda^{2t-1}\left(1 + \frac{\hat{r}(t)}{2}\right),$$

which yields the desired conclusion. $\square$

Page 1.58: An Investment Scenario

Question: Consider two investors, say A and B, each with the same initial capital to invest. Investor A uses all of the initial capital to purchase a 6-month zero and holds this bond until maturity; at $t = .5$ she uses all of the money she receives from the 6-month zero to purchase another 6-month zero and hold it until maturity. Investor B uses uses all of the initial capital to purchase a one-year zero and holds the bond until maturity. Which investor will be better off in one year?

Page 1.59: Investment Secnario (Cont.)

Answer: It depends. Let's look at their capitals at time 1 (per $1 of initial investment).

$$\text{Investor B}: \quad \left(1+\frac{\hat{r}(1)}{2}\right)^2 = \left(1+\frac{\hat{r}(.5)}{2}\right)\left(1+\frac{f(1)}{2}\right),$$

$$\text{Investor A}: \quad \left(1+\frac{\hat{r}(.5)}{2}\right)\left(1+\frac{r_{.5,1}}{2}\right),$$

where $r_{.5,1}$ is the spot rate that will prevail at time $.5$ for 6-month loans (i.e., initiated at $t = .5$ and settled at $t = 1$.)

Remark 1.8: There is no way of knowing for sure at $t = 0$ whether $r_{.5,1}$ will be greater than, less than, or equal to $f(1)$.

Page 1.60: Interpretations of Interest Rates for ZCBs

Investing in a T-year zero can be interpreted as:

- **(i)**: a T-year investment in which the investor commits at $t = 0$ to leave the principal and all accumulated interested on deposit until $t = T$. The interest rate will be constant at $\hat{r}(T)$ over the entire period of investment and will be compounded semiannually; or
- **(ii)**: a T-year investment in which the investor commits at $t = 0$ to leave the principal and all accumulated interest on deposit until $t = T$. The interest rate will be $f(.5)$ during the first 6 months, $f(1)$ between $t = .5$ and $t = 1$, $f(1.5)$ between $t = 1$ and $t = 1.5$, $\cdots$, $f(T)$ between $t = T - .5$ and $t = T$.

Page 1.61: Interest Rates for ZCBs (Cont.)

However, the idea of *accumulated interest* on a zero is just a device to help us think about this kind of investment: If the investor wants to bail out at a time $t < T$, then he or she must accept the market price of the bond at time t (which might actually be below the initial purchase price).

Page 1.62: Dependence of Price on Maturity for Coupon Bonds

Suppose that we have two coupon bonds with the same coupon rate $q > 0$ and the same face value F. Bond #1 has maturity T, whereas Bond #2 has maturity $T + .5$.

Question: How do we tell quickly which bond should cost more at $t = 0$?

Page 1.63: Dependence of Price on Maturity (Cont.)

Answer:

- An investor who buys Bond #2 and shorts Bond #1 will have to pay F at time T and will receive $F(1 + \frac{q}{2})$ at time $T + .5$.
- An investor who agrees at time 0 to invest F between time T and time $T + .5$ (and pays nothing to enter the agreement) will receive $F(1 + \frac{f(T+.5)}{2})$ at time $T + .5$.
- Consequently if $q > f(T + .5)$ then Bond #2 will have a higher price than Bond #1 at $t = 0$. If $q < f(T + .5)$ then Bond #1 will have a higher price than Bond #2 at $t = 0$.

Page 1.64: A Reinvestment Scenario

Investors A and B have the same initial capital to invest at time 0.

- Investor A invests all of the initial capital in a 6-month zero and keeps rolling over the investment into new 6-month zeros every 6 months until maturity T.
- Investor B invests all of the initial capital in a coupon bond with maturity T and coupon rate $q > 0$, and all of the coupon payments will be invested in 6-month zeros and rolled over into new 6-month zeros every 6 months.

Page 1.65: Reinvestment Scenario (Cont.)

- If all forward rates $f(t)$ are higher than the corresponding 6-month spot rates $r_{t-.5,t}$ then investor B will be better off than investor A at time T.
- If all forward rates $f(t)$ are lower than the corresponding 6-month spot rates $r_{t-.5,t}$ then investor A will be better off than investor B at time T.

Here, $r_{t-.5,t}$ is the spot rate that will prevail at time $t - .5$ for loans initiated at time $t - .5$ and settled at time t.

You should try to write out a careful proof of this assertion. It is not difficult if you look at things the right way, but some thought may be required to find the right way.

Page 1.66: Additional Comment on Forward Rates

Question: "What kind of formula can be obtained for the forward rates $R^{for}_{\tau,\eta,T}$ when $\tau > 0$?"

Following the notation you will use in Stochastic Calculus: for $0 \leq t_1 \leq t_2$, let $B(t_1, t_2)$ denote the price at time t_1 for a ZCB that pays \$1 at time t_2. (Notice that $B(t_1, t_2)$ is not known until t_1.) If we use ZCBs to replicate a forward loan in which the rate is agreed upon at time τ, \$1 is received at time η and $(1 + R^{for}_{\tau,\eta,T})^{T-\eta}$ is repaid at time T, we find that

$$B(\tau, \eta) - (1 + R^{for}_{\tau,\eta,T})^{T-\eta} B(\tau, T) = 0.$$

This expression can be solved for the forward rate.