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Lecture Notes for Week 2

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Page 2.1: Discount Factors & Bond Prices

Discount Factor for time t:

$$d(t)$$

Price at time 0 in order to receive \$1 at time t (no risk of default).

Zero Coupon Bond: Single payment of amount F at maturity T. The price is given by

$$P = Fd(T).$$

Annuity: Makes equal payments of amount A, m times per year until maturity T. Assuming that the annuity has just been issued (or that a payment has just been made), the price is given by

$$P = A \sum_{i=1}^{mT} d\left(\frac{i}{m}\right).$$

Page 2.2 Discount Factors & Bond Prices (Cont.)

Coupon Bond: Coupon payments of amount $F\frac{q}{2}$ every 6 months prior to maturity plus a payment of $F(1+\frac{q}{2})$ at maturity T. q is the annual coupon rate. The price is given by

$$P = Fd(T) + F\frac{q}{2}\sum_{i=1}^{2T} d\left(\frac{i}{2}\right),$$

assuming that the bond has just been issued (or that a coupon has just been paid).

Page 2.3: Discount Factors & Interest Rates

Discount factors incorporate a rate per unit time for growth of capital as well as a length of time. (Intrinsic, but not so intuitive.)

Interest rates just describe a rate of growth per unit time, but we must worry about various conventions:

$\hat{r}(t)$ – t-year spot rate (semiannual compounding)

$\hat{R}(t)$ – t-year effective spot rate

$\hat{r}_c(t)$ – t-year continuously compounded spot rate

$$d(t) = \frac{1}{\left(1 + \frac{\hat{r}(t)}{2}\right)^{2t}} = \frac{1}{\left(1 + \hat{R}(t)\right)^{t}} = e^{-t\hat{r}_c(t)}.$$

The spot rates $\hat{r}(t)$, $\hat{R}(t)$, and $\hat{r}_c(t)$ apply to loans where the money is received at time 0 and repaid in a single lump sum at time t. These rates are known at time 0.

Page 2.4: Forward Rates

$f(t)$ – forward rate (agreed upon at time 0) for loans or investments between time $t - .5$ and time t

Remark 2.1: The forward rates $f(t)$ are known at time 0. The actual spot rates that will prevail at future dates are not known at time 0.

Remark 2.2: The forward rates $f(t)$ can be expressed in terms of discount factors through the formula

$$f(t) = 2\left[\frac{d(t-.5)}{d(t)} - 1\right].$$

We observed last time that

$$(1) \quad \left(1 + \frac{\hat{r}(t)}{2}\right)^{2t} = \left(1 + \frac{f(.5)}{2}\right)\left(1 + \frac{f(1)}{2}\right)\cdots\left(1 + \frac{f(t)}{2}\right).$$

Page 2.5: An Important Approximation Formula

Remark 2.3: The previous formula indicates that the spot rate $\hat{r}(t)$ is some kind of geometric average of the forward rates $f(.5), f(1), f(1.5), \cdots, f(t)$. In fact, the spot rate $\hat{r}(t)$ is usually very close numerically to the arithmetic average of the forward rates $f(.5), f(1), f(1.5), \cdots, f(t)$, i.e.

$$\hat{r}\left(\frac{N}{2}\right) \approx \frac{1}{N}\sum_{i=1}^{N} f\left(\frac{i}{2}\right).$$

Here N is a positive integer.

Example 2.1: In Example 1.3, we had

$$f(.5) = .05065,\ f(1) = .05197,\ f(1.5) = .06026, \hat{r}(1.5) = .05429.$$

Observe that

$$\frac{1}{3}[f(.5) + f(1) + f(1.5)] = .05429.$$

Page 2.6: Justification of Remark 2.3

Let N be a positive integer and put $t = \frac{N}{2}$. Let

$$x_i = f\left(\frac{i}{2}\right), \quad i = 1, 2, \cdots, N.$$

Solving (1) for $\hat{r}\left(\frac{N}{2}\right)$ we see that

$$\hat{r}\left(\frac{N}{2}\right) = g(x_1, x_2, \cdots, x_N),$$

where

$$g(\vec{x}) = 2\left[\left(1 + \frac{x_1}{2}\right)\left(1 + \frac{x_2}{2}\right)\cdots\left(1 + \frac{x_N}{2}\right)\right]^{\frac{1}{N}} - 2.$$

Observe that

$$\frac{\partial g}{\partial x_i} = \frac{1}{N}\left[\left(1 + \frac{x_1}{2}\right)\cdots\left(1 + \frac{x_N}{2}\right)\right]^{\frac{1}{N}-1} h_i(\vec{x}),$$

Page 2.7: Justification (Cont.)

where

$$h_i(\vec{x}) = \left(1 + \frac{x_1}{2}\right) \cdots \left(1 + \frac{x_{i-1}}{2}\right) \left(1 + \frac{x_{i+1}}{2}\right) \cdots \left(1 + \frac{x_N}{2}\right).$$

In particular, we have

$$\vec{\nabla} g(\vec{0}) = \frac{1}{N}(1, 1, \cdots, 1).$$

Since $g(\vec{0}) = 0$, the linear approximation for $g(\vec{x})$ about $\vec{0}$ is

$$g(\vec{x}) \approx \vec{\nabla} g(\vec{0}) \cdot \vec{x} = \frac{1}{N} \sum_{i=1}^{N} x_i.$$

[Analogs of Remark 2.1 that are exact, rather than approximate, hold if continuous compounding is used. You should check this out as an exercise.]

Page 2.8: Monotonicity of Forward and Spot Rates

Here is another really neat consequence of (1):

Remark 2.4: Assume that $f(t + .5) > f(t)$ for all $t = .5, 1, 1.5, \cdots, T$. Then $\hat{r}(t + .5) > \hat{r}(t)$ for all $t = .5, 1, \cdots, T$.

Remark 2.4 remains valid if $>$ is replaced throughout by $<$ (or by $\geq$ or by $\leq$). In other words a monotonicity condition on the forward rates implies the same monotonicity condition for the spot rates.

The converse implication is false. It can happen that the spot rates are monotonic, but the forward rates are not. (You should find an example yourself as an exercise.)

I recommend that you prove Remark 2.4 on your own as an exercise. I will post a *solution* on Canvas if you want.

Page 2.9: Forward Loans and Discount Factors

Remark 2.5: To compute payments for forward loans it is often simpler to use discount factors than interest rates. If we agree at $t = 0$ to borrow A_η at time η and repay the loan with a single payment of amount A_T at time T (with $T > \eta$), then we have

$$A_\eta d(\eta) - A_T d(T) = 0,$$

which gives

$$\frac{A_T}{A_\eta} = \frac{d(\eta)}{d(T)}.$$

Recall that nothing is paid initially to enter into a forward loan agreement.

Page 2.10: Bond Yields

Coupon Yield: q the annual coupon rate

Current Yield:

$$q\frac{F}{P} \text{ where } P \text{ is the current price of the bond}$$

The coupon yield and the current yield are, of course, identical if the bond is trading at par. In this case they provide a reasonable measurement of the investment yield over the period between the present time and the bond's maturity. For bonds that are not trading at par, these measures of return can be quite misleading, because the face value (which will be paid at maturity) does not match the initial investment.

Page 2.11: Yield to Maturity

It is very useful to consider a bond's *yield to maturity* which is defined to be the single interest rate that when used to discount all of the bond's future payments produces the bond's current market price.

If all of the relevant spot rates had the same value y, then the discount factors would be given by

$$d(t) = \frac{1}{\left(1 + \frac{y}{2}\right)^{2t}},$$

so that the price of the bond would be given by

$$P = F\frac{q}{2}\left[\frac{1}{\left(1+\frac{y}{2}\right)} + \frac{1}{\left(1+\frac{y}{2}\right)^2} + \cdots + \frac{1}{\left(1+\frac{y}{2}\right)^{2T}}\right] + \frac{F}{\left(1+\frac{y}{2}\right)^{2T}}.$$

Page 2.12: Yield to Maturity (Cont.)

We can work this process in reverse and ask which value of y makes the above equation correct. (Assuming that all payments are nonnegative, it can be shown that there is exactly one such y.) This value of y is called the *yield to maturity* of the bond. In general it must be computed numerically by solving the equation

$$P = F\frac{q}{2}\left[\frac{1}{\left(1+\frac{y}{2}\right)} + \frac{1}{\left(1+\frac{y}{2}\right)^2} + \cdots + \frac{1}{\left(1+\frac{y}{2}\right)^{2T}}\right] + \frac{F}{\left(1+\frac{y}{2}\right)^{2T}}$$

for y.

Page 2.13: Some Remarks on Yield to Maturity

Since the yield to maturity does not account for what an investor does with the coupon payments, several caveats are in order.

Remark 2.6: If two different bonds with the same maturity (but different coupon rates) have different yields to maturity, it is not necessarily the case that the security with the higher yield to maturity represents a “better investment”.

Remark 2.7: Even if a bond is held until maturity, the yield to maturity does not necessarily represent “the return” on the investment. In fact, when cash flows arrive at multiple dates, it is not clear how to define a return on the investment unless some assumptions are made concerning what happens to the payments received prior to maturity.

Page 2.14: Remarks on Yield to Maturity (Cont.)

Remark 2.8: However, it is true that if the yield to maturity remains unchanged between two successive coupon payments, then this common value represents the yield associated with holding the bond over that period of time.

Remark 2.9: It is theoretically possible to have a negative yield to maturity. This hardly ever occurs in practice in the US. Unless warned otherwise, you may assume that all bonds in this course have $y > 0$.

Page 2.15: Computing the Yield to Maturity

Let $\lambda = \frac{1}{1+\frac{y}{2}}$. Then we have

$$P = F\lambda^{2T} + F\frac{q}{2}\sum_{i=1}^{2T}\lambda^i. \tag{2}$$

Using the formula for summing a geometric series:

$$\sum_{i=1}^{N}\lambda^i = \frac{\lambda(1-\lambda^N)}{1-\lambda}, \quad \lambda \neq 1,$$

we find that

Page 2.16: Computing the YTM (Cont.)

$$P = F\left(\lambda^{2T} + \frac{q}{2}\frac{\lambda(1-\lambda^{2T})}{1-\lambda}\right). \tag{3}$$

We solve equation (3) for λ and recover y via the formula

$$y = 2\left(\frac{1}{\lambda} - 1\right).$$

For reasonably small maturities, it is simpler to use (2) directly rather than (3). For $T \geq 1.5$, we will typically need to solve numerically for λ.

Page 2.17: An Extremely Important Result

The following proposition concerning prices, yields to maturity and coupon rates is of crucial importance.

Proposition: For coupon bonds, we have

(i) $P > F$ if and only if $q > y$.

(ii) $P = F$ if and only if $q = y$.

(iii) $P < F$ if and only if $q < y$.

Remark 2.10: In practice, prices of coupon bonds are often quoted by giving the yield to maturity.

Page 2.18: Proof of Proposition

Proof: To prove the proposition, we observe that

$$\frac{\lambda}{1-\lambda} = \frac{2}{y},$$

so that (3) becomes

$$P = F\left[\lambda^{2T} + \frac{q}{y}(1-\lambda^{2T})\right]. \tag{4}$$

The desired conclusions follow from (4). Here's how: Keeping $y > 0$ fixed, we can think of the right-hand side of (2) as a linear function in the variable $x = \frac{q}{y}$ with positive slope. When $x = 1$ this function takes the value F; for $x > 1$ the value is greater than F and for $x < 1$ the value is less than F. $\square$

Page 2.19: Effective Yield to Maturity

It is sometimes useful to use the *effective yield to maturity* Y. If we have a security that will make payments $F_i > 0$ at each of the times $T_i > 0$ then the effective yield to maturity is the unique number Y satisfying

$$P = \sum_{i=1}^{N} \frac{F_i}{(1+Y)^{T_i}},$$

where P is the current price of the security.

Remark 2.11: If the payments F_i can be both positive and negative then it can happen that there is no yield to maturity or there can be more than one. (Exactly one is still possible.) See Assignment 2. The term *internal rate of return (IRR)* is sometimes used in place of yield to maturity. I will use "yield to maturity" in situations with nonnegative payments, and use "IRR" in the rare situations when we allow the payments to change sign.

Page 2.20: Continuously Compounded YTM

Of course, we can also use the continuous compounding convention for the YTM if we like. If we have a security that will make payments $F_i > 0$ at each of the times $T_i > 0$ then the continuously compounded yield to maturity is the unique number y_c satisfying

$$P = \sum_{i=1}^{N} F_i e^{-T_i y_c}$$

where P is the current price of the security.

Page 2.21: A Numerical Example

Example 2.2: Consider Bond #3 from Example 1.1. The maturity is 18 months, the coupon rate is $q = .1$ and the price (per 100 face) is 106.52. Since the bond is trading above par, we must have $y < .1$, by virtue of the proposition. To determine y precisely, we consider first the equation

$$106.52 = 105\lambda^3 + 5\lambda^2 + 5\lambda.$$

Solving numerically (by bisection, for example) we find $\lambda = .973635$, which gives

$$y = 2\left(\frac{1}{.973635} - 1\right) = .054158.$$

We could also determine y by solving

$$106.52 = 100\left[\lambda^3 + .05\left(\frac{\lambda - \lambda^4}{1 - \lambda}\right)\right].$$

Page 2.22: Some Important Facts

Yield-To-Maturity and Spot Rates: (i) The yield to maturity is alway between the lowest spot rate and the highest spot rate for times up to the maturity of the bond. (ii) If the term structure is flat (i.e. if the spot rates are constant), then the common spot rate is equal to the yield to maturity.

Pull to Par: As the time to maturity shortens, the prices of premium bonds fall to par and the prices of discount bonds rise to par. (To make this notion precise, we must use the *flat price*, which will be introduced shortly.)

Page 2.23: Annuities

Recall that an annuity is a security with fixed cash flows that pays the same amount, say A, at equally spaced times, until maturity T. Let us assume here that the payments will be made every 6 months, with the first payment to be made 6 months from today. (In practice, annuities frequently make monthly or quarterly payments.) If the current price of such an annuity is P, then the yield to maturity is defined to be the unique number y satisfying the equation

$$P = \sum_{i=1}^{2T} \frac{A}{\left(1 + \frac{y}{2}\right)^i} = A\lambda \frac{(1 - \lambda^{2T})}{1 - \lambda},$$

Here, as before, we have put

$$\lambda = \frac{1}{1 + \frac{y}{2}}.$$

Page 2.24: Perpetuities

A *perpetuity* or *perpetual annuity* is an annuity that makes payments indefinitely. Although they are not actually used in practice, they are useful theoretical devices. If the yield to maturity y of an annuity is constant and $T \to \infty$ we find

$$P \to A\frac{\lambda}{1-\lambda} = \frac{2A}{y},$$

so that

$$y = \frac{2A}{P}.$$

Page 2.25: Perpetuities (Cont.)

The formula for the price of a perpetuity in terms of the payment amount A and the yield to maturity y has a very natural financial interpretation. Since there is no repayment of principal, each payment A represents 6 months interest on the amount P at the annual rate y (compounded semiannually). Therefore

$$A = P\frac{y}{2},$$

which easily gives

$$P = \frac{2A}{y}.$$

For perpetuities making payments m times per year and having yield to maturity y (expressed according to compounding m times per year), the same logic gives

$$P = \frac{mA}{y}.$$

Page 2.26: Yield Curves

Zero-Coupon Yield Curve: Spot-rate curve

Par-Coupon Yield Curve: Yield to maturity of coupon bonds that are currently trading at par, plotted as a function of maturity

Annuity Yield Curve: Yield to maturity of annuities plotted as a function of maturity

Page 2.27: Yield to Maturity and Coupon Rate

Assuming that the bond has just been issued or that a coupon payment has just been made, the YTM of a coupon bond is the unique number $y > 0$ satisfying

$$P = F\lambda^{2T} + F\frac{q}{2}\sum_{i=1}^{2T}\lambda^{i},$$

where $\lambda = (1 + \frac{y}{2})^{-1}$. Using the formula for the price in terms of the discount factors, we have

$$Fd(T) + F\frac{q}{2}\sum_{i=1}^{2T} d\left(\frac{i}{2}\right) = F\lambda^{2T} + F\frac{q}{2}\sum_{i=1}^{2T}\lambda^{i}.$$

We can cancel out F in the equation above, but not q. The yield to maturity of a coupon bond generally depends on the coupon rate q and on the maturity T.

 # The Coupon Effect

In general, there is no simple relationship between the coupon rate q and the yield to maturity y for coupon bonds. However, if the spot rates are monotonic (i.e., always increasing or always decreasing) we can order the yields of coupon bonds having the same maturity if we know the ordering of the coupon rates. More precisely, we have the following proposition.

 Coupon Effect (Cont.)

Proposition: Consider two coupon bonds having the same maturity T, but different coupon rates $q^{(2)} \geq q^{(1)}$. (Both bonds are assumed to pay coupons twice per year, with the next coupon payment at time $t = .5$.) Let $y^{(1)}$ and $y^{(2)}$ denote the yields to maturity of these bonds.

(i) If $\hat{r}(t) \geq \hat{r}(t - .5)$ for $t = 1, 1.5, 2, \cdots, T$, then $y^{(1)} \geq y^{(2)}$.

(ii) If $\hat{r}(t) \leq \hat{r}(t - .5)$ for $t = 1, 1.5, 2, \cdots, T$, then $y^{(2)} \geq y^{(1)}$.

(The proposition remains valid if all inequalities are replaced with strict inequalities.)

This result follows from a more general one that you will prove as part of Assignment 2.

Page 2.30: Par Coupon Bonds

Remark 2.12: For a given maturity T, the yield to maturity of a coupon bond depends on the coupon rate. Yields are often quoted for par coupon bonds. For such bonds, the coupon rate is given by

$$q = \frac{2(1 - d(T))}{\sum_{i=1}^{2T} d(\frac{i}{2})};$$

moreover, by the proposition, the coupon rate is the same as the yield to maturity y. (Be sure that you know how to derive this formula yourself.)

Page 2.31: Yield to Maturity for Annuities

For an annuity making payments (of amount A) twice per year, the yield to maturity is the unique number $y > 0$ satisfying

$$P = A \sum_{i=1}^{2T} \lambda^i,$$

where $\lambda = (1 + \frac{y}{2})^{-1}$. Using the formula for the price in terms of the discount factors, we find that

$$A \sum_{i=1}^{2T} d\left(\frac{i}{2}\right) = A \sum_{i=1}^{2T} \lambda^i.$$

We can, of course, cancel out A and we see that the yield to maturity of an annuity is independent of the payment size. Of course, the yield maturity will generally depend on the maturity T.

Page 2.32: An Approximation for Annuity Yields

Remark 2.13: For annuities having very long (but finite) maturity T and making payments twice per year, the approximation

$$y \approx \frac{2}{\sum_{i=1}^{2T} d\left(\frac{i}{2}\right)}.$$

is reasonable.

This approximation comes from the formula

$$y = \frac{2A}{P}$$

for a perpetuity making payments of amount A every 6 months forever.

Recall that the price for the annuity with maturity T is given by

$$P = A \sum_{i=1}^{2T} d(\frac{i}{2}).$$

Page 2.33: Yield to Maturity of a Zero Coupon Bond

For a ZCB with maturity T, the yield to maturity is simply the spot rate $\hat{r}(T)$:

$$P = Fd(T) = \frac{F}{\left(1 + \frac{\hat{r}(T)}{2}\right)^{2T}} = \frac{F}{\left(1 + \frac{y}{2}\right)^{2T}}.$$

This is, of course, consistent with simply putting $q = 0$ in the formulas for coupon bonds.

Page 2.34: Yield Curves

For US Treasury securities, there are three *yield curves* that we will look at:

- Zero-Coupon Yield Curve: Same as the spot-rate curve
- Annuity Yield Curve: Plot of the yield to maturity for annuities making payments every 6 months, versus maturity of the annuity (also called the *self-amortizing yield curve*)
- Par-Coupon Yield Curve: Plot of the yield to maturity for par coupon bonds versus maturity of the bond

Each one of these yield curves completely determines the other two.

The par-coupon yield is always between the zero-coupon yield and the annuity yield, but whether the zero-coupon yield is above or below the annuity yield depends on the shape of the zero-coupon yield curve. (If the entire zero-coupon yield curve is upward sloping, then the zero-coupon yield is always higher than the annuity yield.)

Page 2.35: Some Comments on Constructing Yield Curves

In practice, it is useful to have a "smooth" discount function $d(t)$ available for all t between 0 and some maturity T. (This is equivalent to having a "nice" spot-rate curve.) To construct such a function in practice, one obtains as many "reliable" data points as possible (from prices of liquidly traded instruments) and uses some kind of interpolation argument. The details are quite involved. Indeed there will typically be many more unknowns than equations and choosing an appropriate interpolation method is essential. Then discount factors and forward rates can computed from the interpolated spot-rate curve. The discount factors are smoother than the spot rates, while the forward rates are more "jagged" than the spot rates. We will not carry out such an analysis in this course. It will be discussed in*Financial Computing III*.)

Yield curve data for US treasuries is available at

www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=yield

 Remarks on Yield Curves (Cont.)

Remark 2.14: Rather than interpolating to fit the yield curve (or discount function) exactly at all observed points, it is sometimes better to use a "least-squares" type of approximation based on a family of curves whose shape is adapted to the term structures produced by certain types of models. A very important example of such a family of curves is the Nelson-Siegel family of curves with four parameters a, b, c, λ:

$$a + b\frac{1 - e^{-\lambda t}}{\lambda t} + c\left(\frac{1 - e^{-\lambda t}}{\lambda t} - e^{-\lambda t}\right),$$

for fitting continuously compounded yield curves. This topic will be discussed in *Data Science* and *Financial Computing III*.

Page 2.37: Accrued Interest

Question: What happens when a bond trades in between coupon dates?

The "value" of a coupon bond must drop by the coupon payment amount $F\frac{q}{2}$ immediately after a coupon is paid. In order to quote prices that are "continuous" the price of a coupon bond is decomposed into

$$\text{“flat price”} + \text{“accrued interest”}.$$

It is the accrued interest that "jumps" at coupon payments.

Remark 2.15: The accrued interest should equal 0 just after a coupon is paid and should be equal to the full coupon amount $F\frac{q}{2}$ just before a coupon is paid.

Page 2.38: Accrued Interest (Cont.)

The flat price plus the accrued interest is called the *full price* or *invoice price* of the bond. It is the full price that is determined by the market. The choice of a formula for the accrued interest is simply a market convention.

$$P^{full} = P^{flat} + AI \; = \; PV(\text{future payments}).$$

The full price is determined by the market, the accrued interest is defined by a market convention, and the *flat price* is defined to be the full price minus the accrued interest. The convention that has been adopted for accrued interest is based on the notion of *simple interest*.

Remark 2.16: The terms *clean price* and *dirty price* are sometimes used for the flat price and full price, respectively. It is the flat price that is quoted.

Page 2.39: Simple Interest

The notion of compounding does not suit every situation. In the *money market* in which investors borrow or lend for short terms (usually less than one year), an investor commits to a fixed term and interest is paid at the end of the term. The simple interest convention

$$\mathit{Interest} \ = \ (\mathit{Principal}) \times (\mathit{Rate}) \times (\mathit{Time})$$

is typically used.

Page 2.40: Day Count Conventions

There is more than one convention for converting the time interval between two calendar dates into a fraction of a year. Such a convention is called a *day count convention*. For example, it is common practice with mortgages (or car loans) to treat each month as being $\frac{1}{12}$ of a year.

With simple interest, sometimes the convention *actual/360* is used and sometimes the convention *actual/actual* is used. Under actual/360, the interest earned in n days is $rn/360$ times the principal, whereas the interest for n days is $rn/365$ under the actual/actual convention (in a non-leap year). Here r is the annual simple interest rate.

Page 2.41: Day Count Example

Example 2.3: Money markets A and B are both quoting an annual simple interest rate of $r = .05$ for investments of term one year. Market A uses the actual/360 convention, whereas market B uses the actual/actual convention.

(i) If \$50,000 is deposited for one year the interest is

$$\frac{50,000(.05)365}{360} = 2,534.72 \text{ in } A, \quad 50,000(.05) = 2,500 \text{ in } B,$$

a difference of \$34.72.

(ii) If \$500 million is deposited for one year, the difference between the two accounts is \$347,222.22.

Page 2.42: Accrued Interest

For a coupon bond the accrued interest is computed using a simple interest convention by the formula

$$AI = \eta F \frac{q}{2},$$

where η is the fraction of a coupon period that elapses between the last coupon payment and settlement. For US treasury notes and bonds the actual/actual convention is used; more precisely accrued interest is computed according to the formula

$$AI = \frac{n}{N} F \frac{q}{2},$$

where N is the actual number of days between the last coupon payment and the next coupon payment and n is the number of days between the last coupon payment and settlement.

 Accrued Interest Example

Example 2.4: Suppose that there are 182 days between the last coupon date and the next coupon date. Assume that the last coupon date was 77 days ago and that settlement occurs today. On a bond with $F = \$10,000$ and $q = 5.5\%$, the accrued interest is

$$AI = \frac{10,000(77)(.055)}{182(2)} = 116.35.$$

Page 2.44: YTM Between Coupon Payments

Let τ denote the fraction of a coupon period between the present time and the next coupon payment (so that $\tau = 1 - \eta$) and let k denote the number of additional coupon payments remaining (after the next coupon). The yield to maturity y is determined by solving the equation

$$P^{full} = F\lambda^{k+\tau} + F\frac{q}{2}\sum_{i=0}^{k}\lambda^{\tau+i},$$

where $\lambda = (1 + \frac{y}{2})^{-1}$.

Page 2.45: YTM Between Coupon Payments (Cont.)

Remark 2.17: There are $k+1$ future coupon payments. Each future payment is discounted by

$$d(t) = \frac{1}{(1+\frac{y}{2})^{2t}},$$

where t is the time until the payment is received. (Here t need not be a multiple of 6 months.)

Page 2.46: Current Yield Between Coupon Payments

The current yield is defined to be $q \frac{F}{P^{flat}}$ for bonds in between coupon payments.

Remark 2.18: Coupon payments are not always made on the "official coupon date" because of holidays. This may lead to complications that can be significant in interpreting the yield to maturity when coupon payments do not arrive exactly 6 months apart. (See pages 60-61 in the second edition of Tuckman for an interesting example.)

Page 2.47: Conventions for T-Bills

Prices of T-Bills are often quoted in terms of the so-called *discount yield* y_d. The formula is

$$P = 100\left[1 - \frac{ny_d}{360}\right],$$

where P is the price per 100 face and n is the number of days between settlement and maturity. Notice that

$$y_d = \frac{(100 - P)360}{100n}.$$

Page 2.48: Conventions for T-Bills (Cont.)

Remark 2.19: The definition of discount yield has two peculiar features:

(i) It divides the dollar gain by 100 rather than P; and

(ii) it assumes a 360-day year.

The *bond-equivalent yield* y_{be} corrects these two defects. For $n \leq 182$, we have

$$y_{be} = \frac{(100 - P)365}{Pn}.$$

For $n > 182$, we have

$$P\left(1 + \frac{y_{be}}{2}\right) + \frac{y_{be}}{365}\left(n - \frac{365}{2}\right)\left(1 + \frac{y_{be}}{2}\right)P = 100.$$

Page 2.49: Conventions for Yields (Cont.)

The bond-equivalent yield is also called the *coupon-equivalent yield*.

Remark 2.20: There is a veritable "snake pit" of conventions concerning day counts, compounding, and yields. Software accounts properly for these in most cases, but you must be aware that subtle differences in conventions can lead to big dollar differences on large transactions. Be careful in practice!

Remark 2.21: Even those fixed income securities having no risk of default and whose payment dates and amounts are known with certainty at the time of issue face *interest rate risk*. For example, it can happen that the price of a zero coupon bond will drop below its original purchase price. Roughly speaking, as interest rates rise, the market prices of previously issued bonds will drop.

Page 2.51: Sensitivity Analysis (Cont.)

Financial news and changes in investors' views of the market leads to changes in bond prices and spot rates. The changes that can occur in the spot rate curve can be quite complex. In particular, it can happen that long-term rates and short-term rates move in opposite directions.

For certain applications, it may be possible to capture the essential features of changes in the spot rate curve by keeping track of one, or several, key *interest rate factors*. In other words, we think of changes in the spot rate curve as being driven by some (small) list of interest rate variables such as a long-term rate, a medium-term rate, and a short-term rate; or in very special situations by a parallel shift.

Page 2.52: Sensitivity Analysis (Cont.)

Clearly, we cannot retain all of the information about the spot-rate curve by monitoring just a few variables, but sometimes we can do extremely well.

ONE-FACTOR MODELS

We try to measure changes in the spot-rate curve by means of a single interest rate factor (variable) y.

Trade-Off: Very simple, but not always appropriate.

The challenge of measuring price sensitivity lies in quantifying what is meant by changes in interest rates.

 One Factor Models: Basic Approximations

Suppose that the price of a security is given by

$$P = f(y),$$

where y is some interest rate factor (or variable).

First-Order Approximation

$$\Delta P \approx f'(y)\Delta y$$

Second-Order Approximation

$$\Delta P \approx f'(y)\Delta y + \frac{1}{2}f''(y)(\Delta y)^2$$

Page 2.54: A Quick look at ZCBs

The price of a zero coupon bond with face value F and maturity T is given by

$$P = f(y) = F \cdot (1 + \frac{y}{2})^{-2T},$$

where y is the yield to maturity of the bond. (Recall that $y = \hat{r}(T)$.) Taking the derivative of f, we find that

$$\begin{aligned} f'(y) &= -2TF \cdot (1 + \tfrac{y}{2})^{-2T-1}(\tfrac{1}{2}) \\ &= -\left(\frac{T}{1+\frac{y}{2}}\right) F \cdot (1 + \tfrac{y}{2})^{-2T} \\ &= -\left(\frac{T}{1+\frac{y}{2}}\right) f(y). \end{aligned}$$

Page 2.55: ZCBs: First-Order Approximation

The first-order approximation gives

$$\Delta P = -\left(\frac{T}{1+\frac{y}{2}}\right) P \Delta y.$$

- The price decreases when "rates go up".
- Change in price is proportional to current price.
- Change in price is proportional to time to maturity.

Page 2.56: ZCBs (Cont.)

Dollar Change in Price

$$\Delta P = -\left[\left(\frac{T}{1+\frac{y}{2}}\right)P\right]\Delta y$$

Relative Change in Price

$$\frac{\Delta P}{P} = -\left(\frac{T}{1+\frac{y}{2}}\right)\Delta y$$

Back to the General Situation

- Bond is not necessarily a zero.
- y can be any single interest rate factor.

Page 2.57: Basis Points

Changes in interest rates are frequently measured in *basis points*. One basis point is equal to one one hundredth of a percentage points. In terms of decimals, one basis point equals .0001.

Example 2.5: Suppose that the two-year spot rate is 4.853%, i.e. $\hat{r}(2) = .04853$. If the rate increases by 2 basis points, then

$$\hat{r}(2) = 4.873\% = .04873.$$

Page 2.58: DV01

The abbreviation *DV01* stands for *dollar value of an '01*, i.e. the change in dollar value in price resulting from a change in the interest rate factor of one basis point. Sometimes other names are used for the same concept – however DV01 is by far the most common term.

The abbreviation *PVBP* stands for *price value of a basis point*. This means the same thing as DV01.

Since an increase in interest rates leads to a decrease in price for most of the basic fixed-income securities, a minus sign is included in the definition:

$$DV01 = -\frac{\Delta P}{10,000 \Delta y}.$$

Page 2.59: DV01 (Cont.)

Remark 2.22: DV01 is often quoted for some specified amount of face. Be careful to give units.

Warning: Occasionally the opposite sign convention is used. Be sure to check the sign convention if in doubt!

Page 2.60: First-Order Approximation and DV01

Recall that $P = f(y)$. The quantity $\Delta P/\Delta y$ represents the slope of a secant line to the graph of the price function f. If we have an analytical formula for $f(y)$ we can use techniques from calculus to compute the derivative $f'(y)$ which gives the slope of the tangent line to the graph of the price function. For small changes in y, the slope of the secant line is for all practical purposes the same as the slope of the tangent line. In such a case we can write

$$DV01 = -\frac{f'(y)}{10,000}.$$

 Yield-Based DV01

Remark 2.23: Most market participants use the term DV01 to mean yield-based DV01, i.e. DV01 computed when y is the yield to maturity of the security. Of course, other interest rate variables could be used. If you have any doubt about what "interest rate variable" is being used, be sure to insist on clarification.

Page 2.62: DV01 of a Call Option on a Bond

Example 2.6: Consider a European call option (with exercise date $T_E = 1$ year) struck at par on a coupon bond having maturity $T > 1$. Let P denote the current price of the option (per \$100 face of the bond) and let y denote the yield to maturity of the bond. Suppose that the yield of the bond is currently at $y = 4\%$ and we know that

$$P = 8.0866 \text{ when } y = 4.01\% \qquad P = 8.2148 \text{ when } y = 3.99\%.$$

We are interested in what happens as the yield of the bond moves away from 4%. We calculate

$$DV01 = -\frac{8.0866 - 8.2148}{10,000(.0401 - .0399)} = .0641.$$

This means that for every *increase* in rate of one basis point, we can expect the price of the option to *decrease* by approximately 6.41 cents for every \$100 face of the bond.

Page 2.63: Call Option (Cont.)

This observation is certainly **not** rocket science – we should expect the change in price to be approximately proportional to the change in rate (for small changes in the interest rate factor). Given two prices for nearby values of y we should be able to predict prices for other nearby values of y by a linear interpolation of the two given values.

However, we can use DV01 to draw some very important conclusions. In particular, we gain some insight into how to hedge a short position on the option from Example 2.6.

 Hedging a Call Option on a Bond

Example 2.7: Consider the call option of Example 2.6. Suppose that we know the DV01 of the underlying bond (per $100 face) as well:

Rate Level	Option DV01	Bond DV01
4.00%	.0641	.0857

Suppose also that an trader is short $100 million face of the call. How might she hedge the interest rate exposure by trading in the underlying bond?

The trader stands to lose money if rates fall, so bonds should be purchased. Let F be the face amount of bond to be purchased. She should choose F so that price change in the bond portfolio associated with a one basis point change in bond yield equals the price change in the option position. This leads to

Page 2.65: Hedging a Call (Cont.)

$$F \times \frac{.0857}{100} = 100,000,000 \times \frac{.0641}{100},$$

or

$$F = 100,000,000 \times \frac{.0641}{.0857} = 74,795,799.30.$$

In other words, the trader should purchase \$74,795,799.30 face of the underlying bond.

Remark 2.24: (i) If the DV01s of securities A and B have the same sign, then hedging a short position in security A requires a long position in security B.
(ii) If the DV01s of security A and B have opposite signs then hedging a short position in security A requires a short position in security B.

Page 2.66: Duration

Another popular measure of interest-rate risk is the so-called *duration* which is defined by

$$D = -\frac{1}{P}\frac{\Delta P}{\Delta y}.$$

Observe that duration is simply $10,000 \times DV01/P$. Duration measures the *relative change* in price resulting from a change in y. In fact, duration gives the percentage decrease in price due to an increase in the interest rate factor of 100 percentage points.

Page 2.67: Duration and DV01

When we apply sensitivity analysis in practice, the Δy values should be very small.

Once again, if we have an analytical expression for $P = f(y)$ we use calculus to compute $f'(y)$ and write

$$D = -\frac{f'(y)}{P} = -\frac{1}{P}\frac{dP}{dy}.$$

Notice that

$$DV01 = \frac{PD}{10,000}.$$

Remark 2.25: We don't really need both DV01 and duration. However, practitioners use both. Each is more convenient for certain purposes.

Page 2.68: Convexity

The *convexity* of a bond measures the how the interest rate sensitivity changes when the interest rate factor changes. The mathematical definition is

$$C = \frac{f''(y)}{P} = \frac{1}{P}\frac{d^2P}{dy^2}.$$

This assumes that we have an explicit expression for the price as a function of y. Without such a formula, convexity must be approximated numerically.