Lecture #4: Higher-Order Functions

Functions are a method of abstraction that describes compound operations independent of the particular values of their arguments.

Lacking function definition would put us at the disadvantage of forcing us to work always at the level of the particular operations that happen to be primitives in the language, rather than in terms of higher-level operations.

Functions that manipulate functions (accept other functions as arguments or return functions as values) are called higher order functions.

Announcements

- Pair-programming demo (Pamela Fox & Patricia Ouyang).
- Homework 1 due Thursday.
- Project 1 (Hog) release today.
- Nine new tutorials added:
  - 2 on Wed. @4PM
  - 1 on Thu. @7AM
  - 1 on Thu. @8AM
  - 3 on Thu. @11AM
  - 2 on Thu. @12PM
- "Lost" sections starting Friday at 12–2PM and 4–6PM. See Piazza @239.
- Ask questions on the Piazza thread for today's lecture (@245).

Comments on Functions in General: Terminology

- The set of possible argument values of a function is known as its *domain*.
- The set of values that the function can return (all values that result from inputting some value from its domain) is called its *range*.
- The *codomain* of a function is a set of values that includes the range, and possibly other values.
- Thus, we might say that the square function has the real numbers as its domain, and the non-negative numbers as its range. We can choose to describe its codomain as the real numbers or as just the non-negative real numbers.

Documenting Functions

- Ideally, a *documentation comment* for a function provides enough information so that a programmer can use the function properly and understand what it does *without* having to read its body.
- It should make clear what inputs are valid or under what conditions the function may be called. This is the *precondition*.
- Likewise, it should make clear what the resulting output or effect of the function will be for correct inputs. This is the *postcondition*.
- Together, these are the *behavior* or *semantics* (meaning) of the function.

Two Design Principles

- Functions should do one well-defined thing (a complicated documentation comment might suggest your function does too much).
- *DRY* (Don't Repeat Yourself).
  - Multiple segments of code that look really similar to each other cry out for *refactoring*...
  - That is, for replacing the segments with simple calls to a single general function that states their shared structure just once, with parameters used to specialize to the various cases.

Functions As Templates

- If we think of a function body as a template for a computation, parameters are "blanks" in that template.
- For example:

```
def sum_squares(N):
    """Returns the sum of x**2 for all integers x with 1 <= x <= N."""
    k = 1
    sum = 0
    while k <= N:
        sum += k**2
        k += 1
    return sum
```

  is a template for an infinite set of computations that add squares of numbers up to 0, 1, 2, 3, . . . , in place of the N.
- But the `sum_squares` function is specialized to the summing k^2.
- A function for summing k^3, sin k, or $1/k$ would have the same structure, differing only in what comes after `sum +=`.
- How do we practice DRY here?

Functions on Functions

- Function parameters allow us to have templates with slots for *computations*:

```
def summation(N, term):
    k = 1
    sum = 0
    while k <= N:
        sum += term(k)
        k += 1
    return sum
```

- Generalizes sum_squares. We can write `sum_squares(5)` as:

```
def square(x):
    return x*x
summation(5, square)
```

- or (if we don't really need a "square" function elsewhere), we can create the function argument anonymously on the fly:

```
summation(5, lambda x: x*x)
```

To express certain general patterns as named concepts, we will need to construct functions that can accept other functions as arguments or return functions as values.

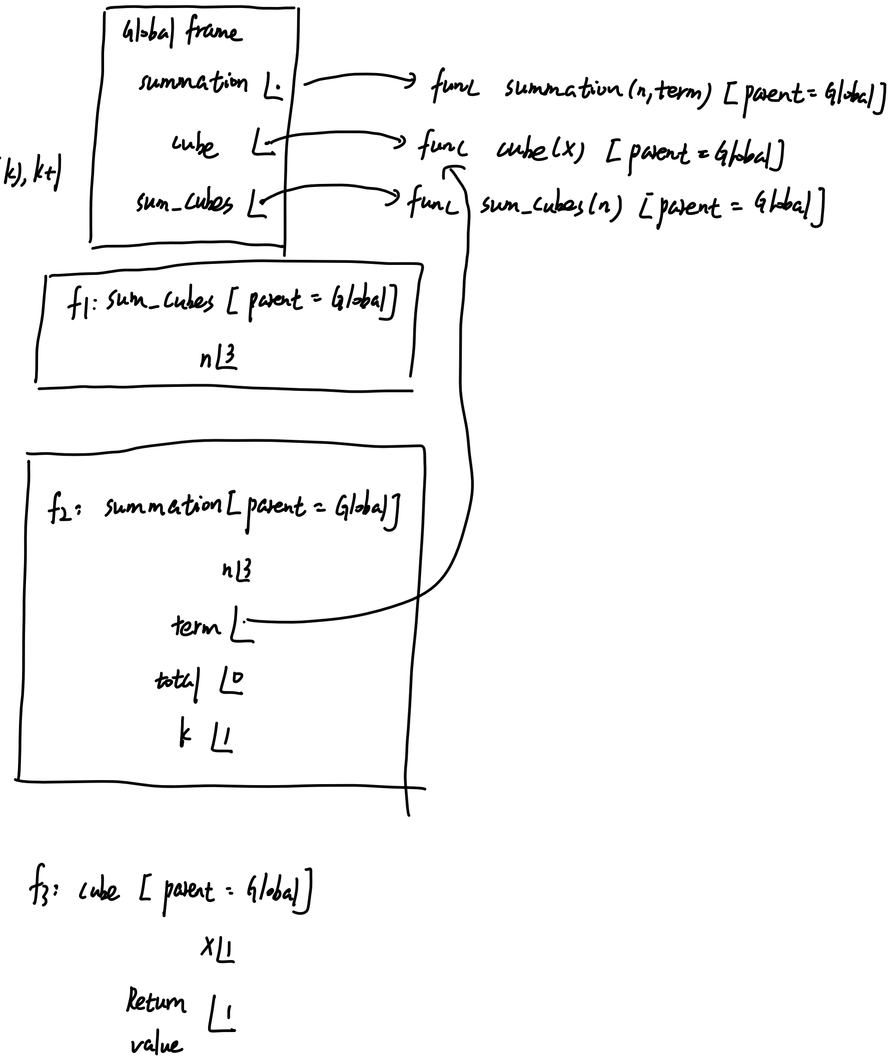
Higher-order functions

```
def summation(n, term):
    total, k = 0, 1
    while k <= n:
        total, k = total + term(k), k+1
    return total

def cube(x):
    return x*x*x

def sum_cubes(n):
    return summation(n, cube)

result = sum_cubes(3)
```



Quick Review of Lambda

- In Python, **lambda** is just an abbreviation.
- Writing lambda *PARAMS*: *EXPRESSION* is the same as writing NEWNAME, where NEWNAME is a name that appears nowhere else in the program and is defined by

```
def NEWNAME(PARAMS):
    return EXPRESSION
```

  evaluated in the same environment in which the original **lambda** was.
- There is no return: the body must be a single expression.
- Now we can write any number of summations succinctly:

```
summation(10, lambda x: x**3)       # Sum of cubes
summation(10, lambda x: 1 / x)      # Harmonic series
summation(10, lambda k: x**(k-1) / factorial(k-1))
                                    # Approximate e**x
```

```
def summation(n, term):
    total, k = 0, 1
    while k <= n:
        total, k = total + term(k), k+1
    return total

def cube(x):
    return x*x*x

def sum_cubes(n):
    return summation(n, cube)

result = sum_cubes(3)
```

每次循环都会创建一个新的frame.

Global

summation → func summation(n, term) [parent=Global]

cube → func cube(x) [parent=Global]

sum_cubes → func sum_cubes(n) [parent=Global]

f1: sum_cubes [parent=Global]

n 3

f2: summation [parent=Global]

n 3

cube →

Functions that Produce Functions

- Functions are *first-class values,* meaning that we can assign them to variables, pass them to functions, and return them from functions.
- Example: let's generalize the class of functions that—like

```
def h(x):  return sin(x) + cos(x)
```

—add the results of applying two functions to the same argument:

```
>>> def add_func(f, g):
...     """Return function that returns F(x)+G(x) for argument x."""
...     def adder(x):              #
...         return f(x) + g(x)     # or return lambda x: f(x) + g(x)
...     return adder               #

>>> from math import sin, cos, pi
>>> h = add_func(sin, cos)
>>> sin(pi/4) + cos(pi/4)
1.414213562373095
>>> h(pi / 4)
1.414213562373095
```

Generalize!

- Let's make a general function-combining function (that goes beyond addition):

```
>>> def combine_funcs(op):
...     """combine_funcs(OP)(f, g)(x) = OP(f(x), g(x))."""
...     def combined(f, g):
...         def val(x):
...             return op(f(x), g(x))
...         return val
...     return combined
```

- Now add_func itself can be constructed by a call to combine_funcs:

```
>>> from operator import add
>>> add_func = ??
>>> from math import sin, cos, pi
>>> h = add_func(sin, cos)
>>> h(pi / 4)
1.414213562373095
```

- What do the environments look like here? Think about it and try it out.

Generalize!

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...     def combined(f, g):
...         def val(x):
...             return op(f(x), g(x))
...         return val
...     return combined
```

- Now `add_func` itself can be constructed by a call to `combine_funcs`:

```
>>> from operator import add
>>> add_func = combine_funcs(add)
>>> from math import sin, cos, pi
>>> h = add_func(sin, cos)
>>> h(pi / 4)
1.414213562373095
```

- What do the environments look like here? Think about it and try it out.

Generalize!

- Let's make a general function-combining function (that goes beyond addition):

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...     def combined(f, g):
...         def val(x):
...             return op(f(x), g(x))
...         return val
...     return combined
```
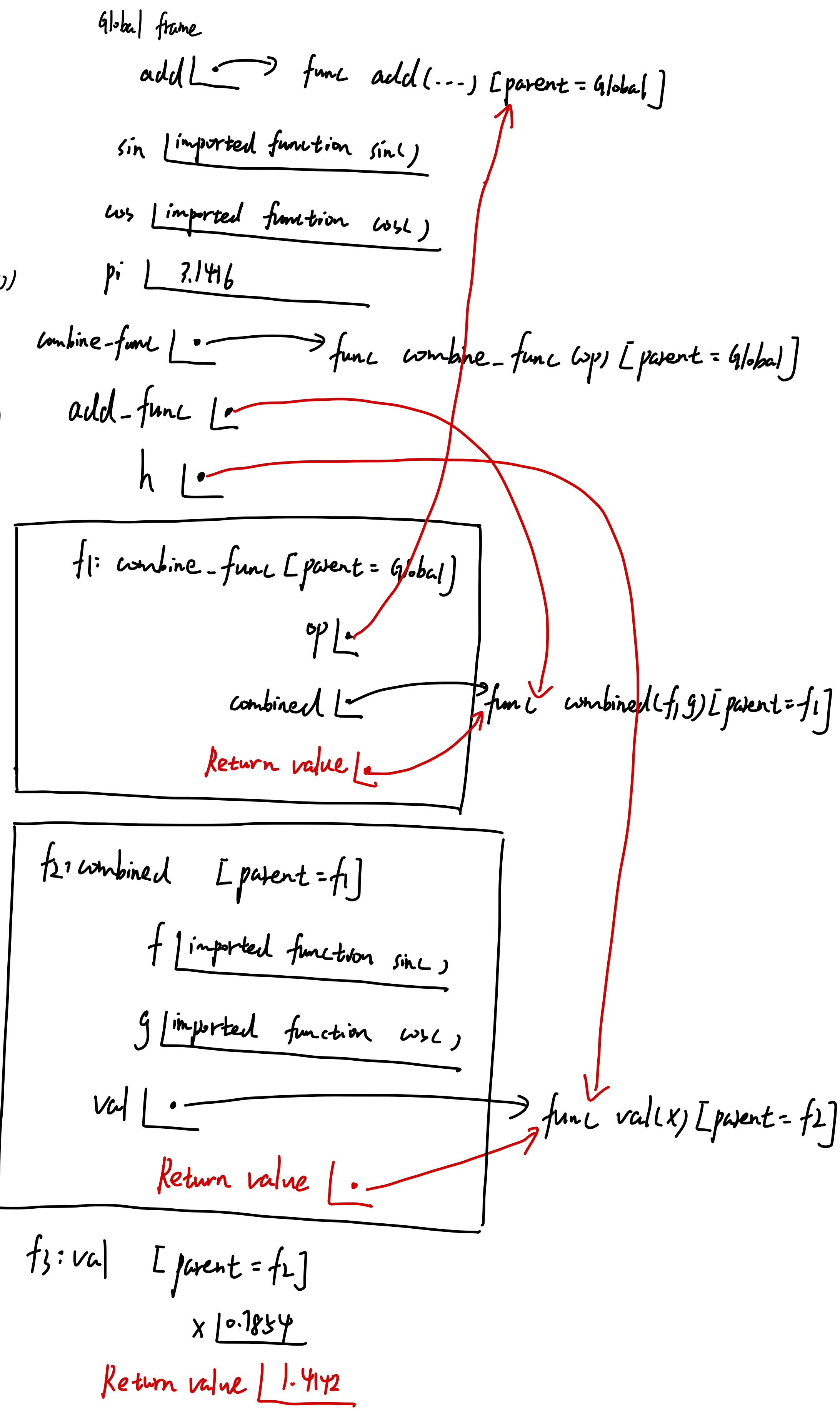

- Now add_func itself can be constructed by a call to combine_funcs:

```
>>> from operator import add
>>> add_func = combine_funcs(lambda x, y: x + y)
>>> from math import sin, cos, pi
>>> h = add_func(sin, cos)
>>> h(pi / 4)
1.414213562373095
```

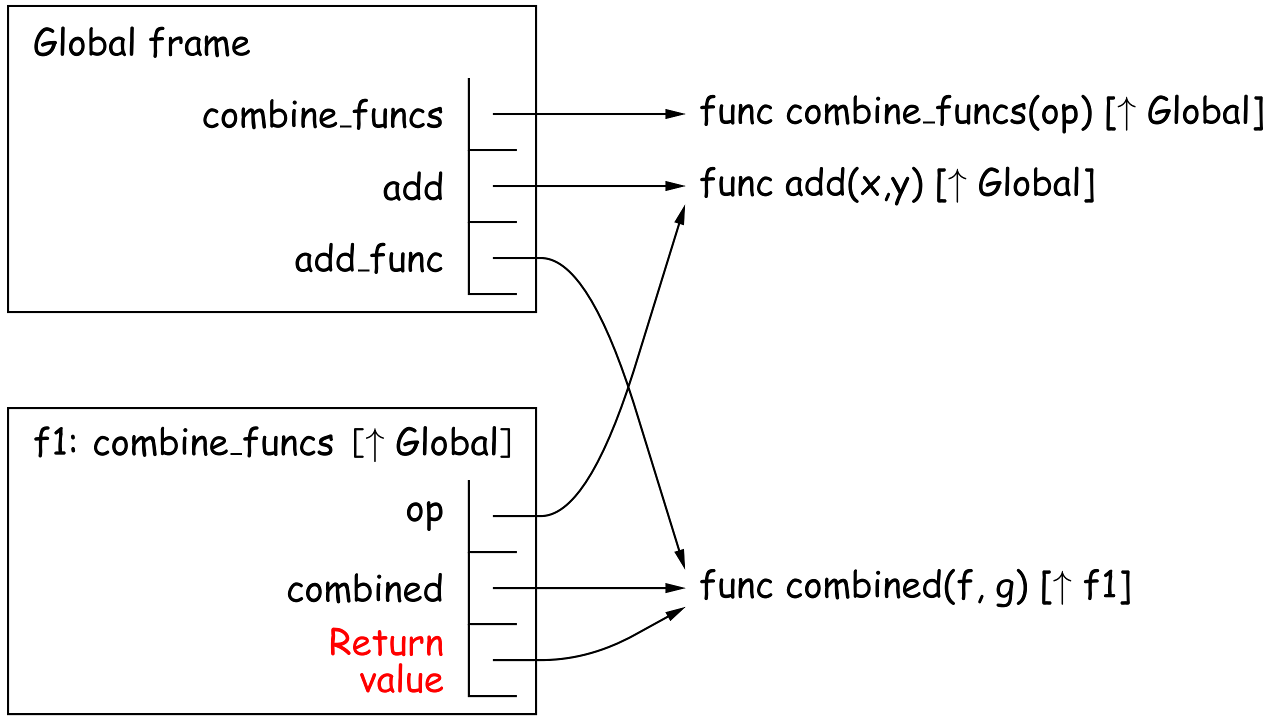
- What do the environments look like here? Think about it and try it out.

```python
from operator import add
from math import sin, cos, pi

def combine_func(op):
    def combined(f, g):
        def val(x):
            return op(f(x), g(x))
        return val
    return combined

add_func = combine_func(add)
h = add_func(sin, cos)
h(pi/4)
```



The Environment Picture (I)

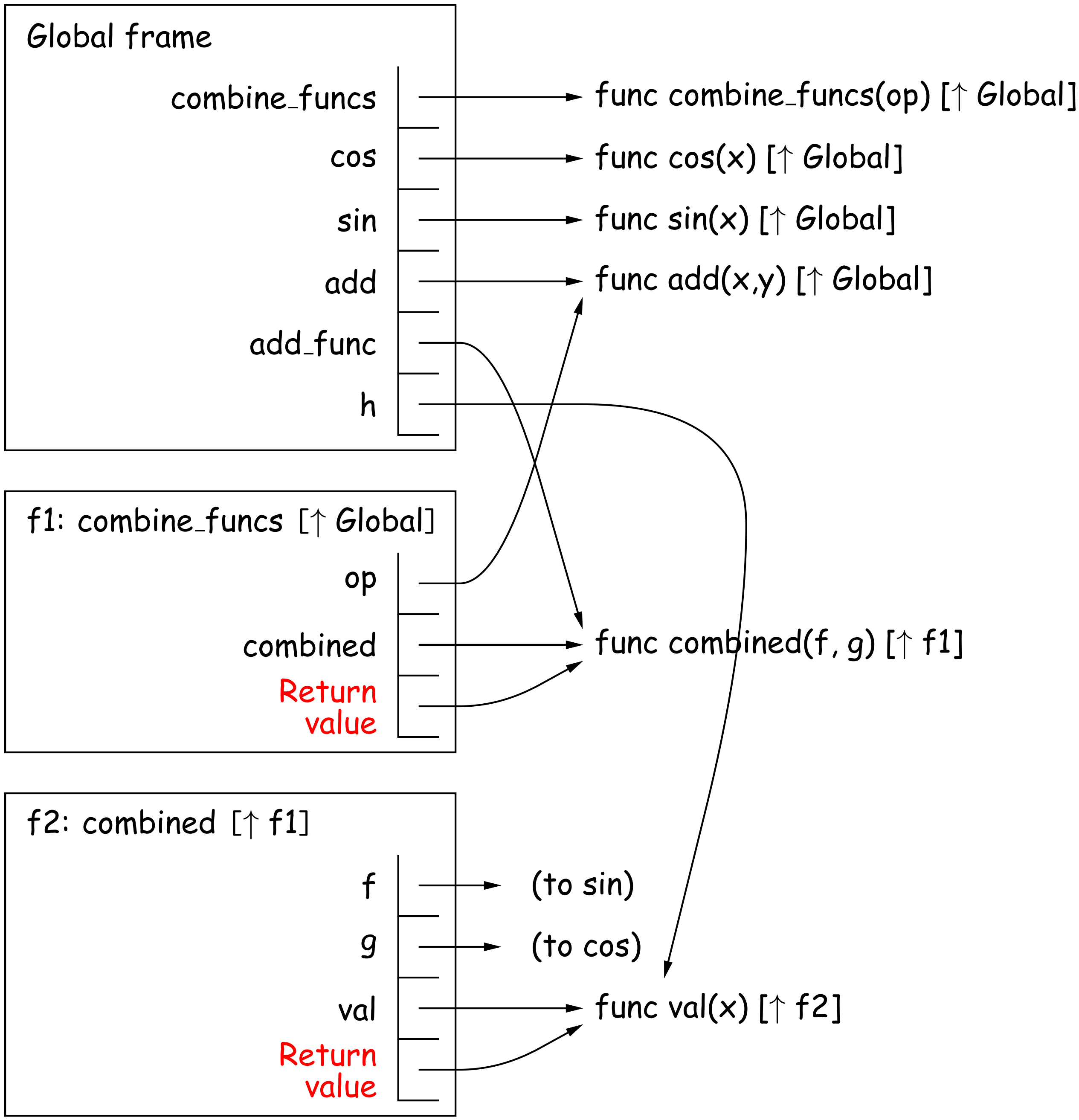
```
def combine_funcs(op):
  def combined(f, g):
    def val(x):
      return op(f(x), g(x))
    return val
  return combined
add_func = combine_funcs(add)
```



Legend: ↑ is short for "parent=".

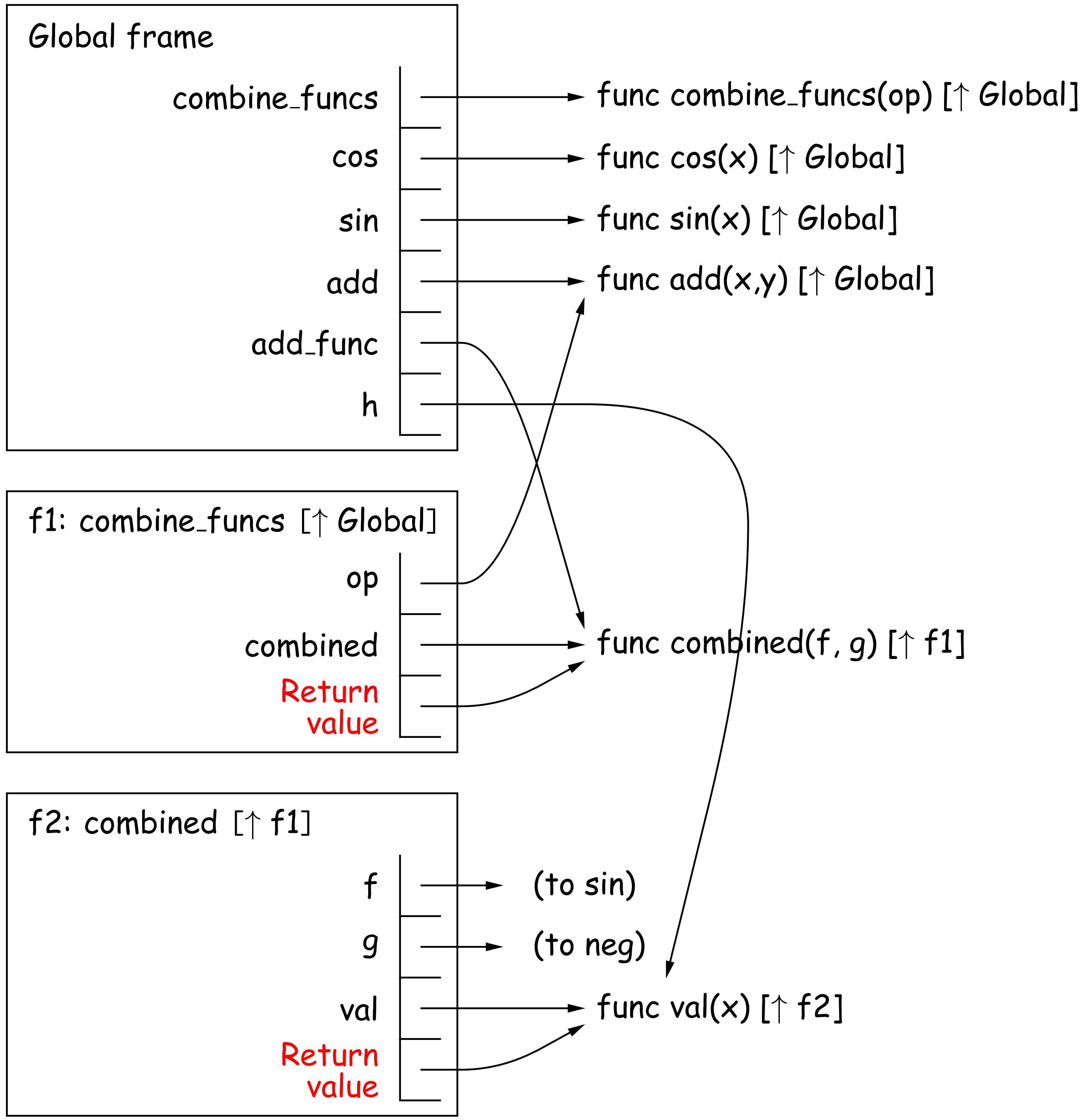
The Environment Picture (II)

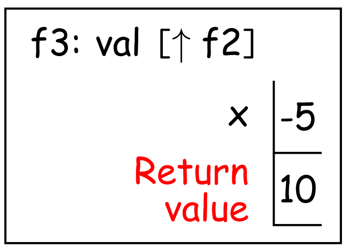
```
def combine_funcs(op):
  def combined(f, g):
    def val(x):
      return op(f(x), g(x))
    return val
  return combined
add_func = combine_funcs(add)
h = add_func(sin, cos)
```



The Environment Picture (III)

```
def combine_funcs(op):
  def combined(f, g):
    def val(x):
      return op(f(x), g(x))
    return val
  return combined
add_func = combine_funcs(add)
h = add_func(sin, cos)
h(-5)
```





+ local frames for calls to

- add (value of op),
- sin (value of f), and
- cos (value of g)

创建 golden ratio

```
def improve(update, close, guess=1):
    while not close(guess):
        guess = update(guess)
    return guess
```

迭代求精的通用表达式.
repetitive refinement

```
def golden_update(guess):
    return 1/guess + 1
```

```
def square_close_to_successor(guess):
    return approx_eq(guess * guess, guess + 1)
```

黄金比例的著名特性：通过反复叠加任何正数的倒数加上1来计算，而且比它的平方少1

```
def approx_eq(x, y, tolerance=1e-15):
    return abs(x - y) < tolerance
```

```
>>> improve(golden_update, square_close_to_successor)
1.61803398874
```

Global

improve → func improve(update, close, guess) default arguments guess 1 [parent = Global]

golden_update → func golden_update(guess) [parent = Global]

square_close_to_successor → func square_close_to_successor(guess) [parent = Global]

default arguments tolerance 0.001

f1: improve [parent = Global]

update

close

guess 1 → guess更新为2.0.

f2: square_close_to_successor [parent = Global]

注：编号代表返回顺序

guess 1

② Return value False

f3: approx_eq [parent = Global]

x 1

y 2

tolerance 0.001

① Return value False

f4: golden_update [parent = Global]

guess 1

Return value 2.0

Nested definition

```
def improve(update, close, guess=1):
        while not close(guess):
              guess = update(guess)
      return guess
```

```
def average(x, y):
      return (x + y) / 2

def sqrt_update(x, a):
      return average(x, a/x)
```

This two-argument update function is incompatible with improve (it takes two arguments, not one), and it provides only a single update, while we really care about taking square roots by repeated updates.

repeated application of the following update converges to the square root of a.

solution:

```
def sqrt(a):
      def sqrt_update(x):
            return average(x, a/x)
      def sqrt_close(x):
            return approx_eq(x*x, a)
      return improve(sqrt_update, sqrt_close)
```

Locally defined functions also have access to the name bindings in which they are defined.

sqrt_update $\xrightarrow{\text{access}}$ a   lexical scope

These functions are only in scope while sqrt is being evaluated

通过将 place function definitions inside the body of other definitions,
少了一个参数

result = sqrt(256)

carries with it some data: the value for a referenced in the environment in which it was defined

$\boxed{\text{locally defined functions}}$ → $\boxed{\text{closure 闭包}}$

An environment can consist of arbitrary long chain of frames, which always concludes with the global frame.

To enable lexical scoping
① Each user-defined function has a parent environment: the environment in which it was defined
② when a user-defined function is called, its local frame extends its parent environment

Two key advantages in lexical scoping:
① The name of a local function doesn't interfere with names external to the function in which it is defined, because the local function name will be bound in the current local environment in which it was defined, rather than the global environment
②局部函数可以访问外层函数环境，因为局部函数的函数体求值环境继承定义它的求值环境

Currying

use higher-order functions to convert a function that takes multiple arguments into a chain of functions that each take a single argument.

$f(x,y) \rightarrow g(x)(y)$

g: higher order function that takes in a single argument x and returns another function that takes in a single argument y.

Challenge: Conditional Function?

- Write a Python function, `if_func`, such that, for example

```
if_func(1/x, x > 0, 0)
```

always returns the same value as the conditional expression

```
1/x if x > 0 else 0
```

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  ```

  Answer: IMPOSSIBLE! Function calls *always* evaluate all their operands.

- But all is not lost, because we can define instead

  ```
  def if_func(then_expr, condition, else_expr):
      return then_expr() if condition else else_expr()
  ```

  and call

  ```
  if_func(lambda: 1/x, x > 0, lambda: 0)
  ```

- (The jargon term for those parameterless lambdas is *thunks*.)
- Why don't we need a thunk for the condition?

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  ```

- (The jargon term for those parameterless lambdas is *thunks.*)
- Why don't we need a thunk for the condition?
  Answer: Because the condition parameter must always be evaluated first anyway.