5ELEN018W - Robotic Principles
Lecture 9: More on Bode Plots and Linearisation - Robot Decision Making - Coursework
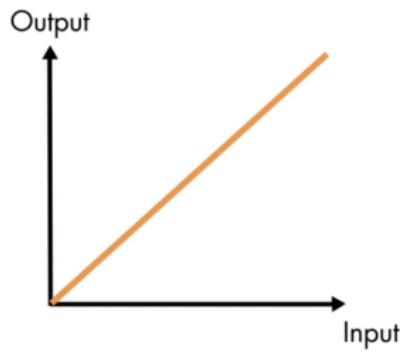
Dr Dimitris C. Dracopoulos
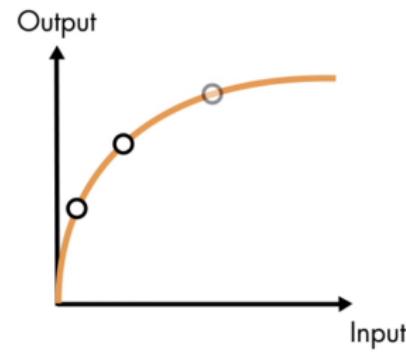
# Linear vs Nonlinear Systems

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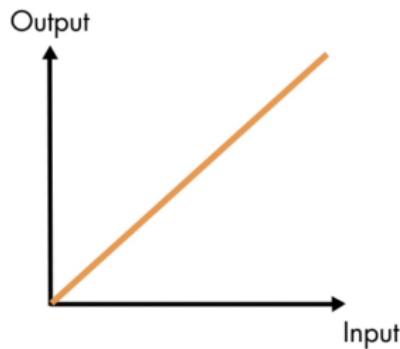
Linear System
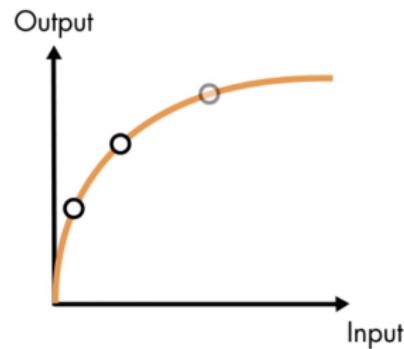
Nonlinear System

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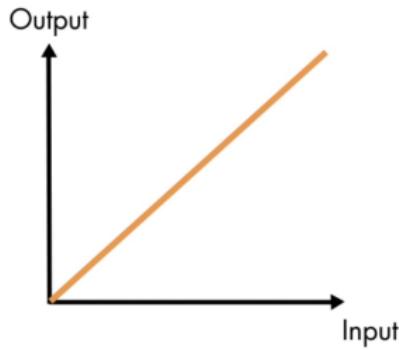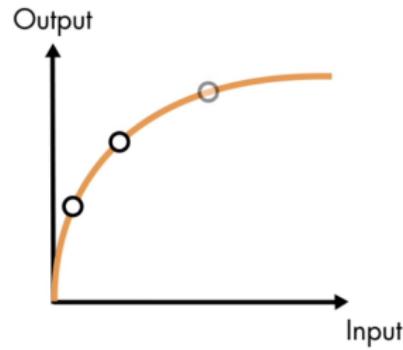
Linear System

Nonlinear System

- In real life all systems are nonlinear, however many of them can be linearised about their operation point.

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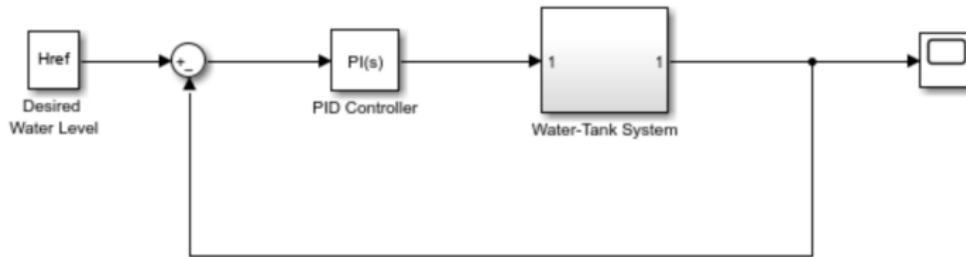

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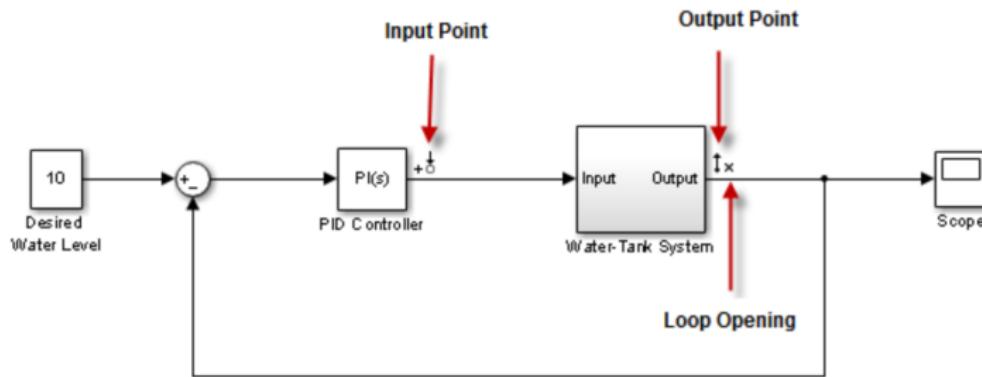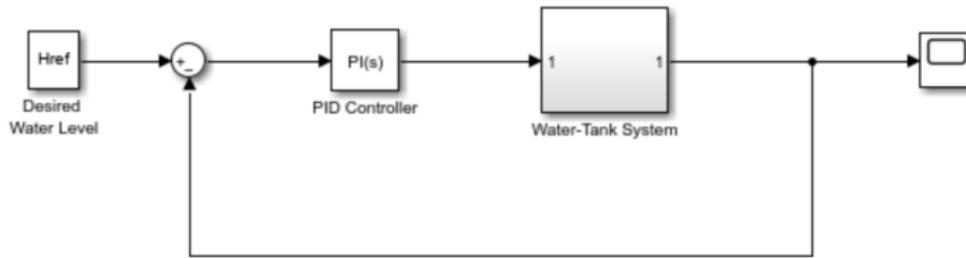
Linear systems are easier to analyse and prove mathematically their behaviour and properties.

How to Specify the Portion of a Model to Linearise
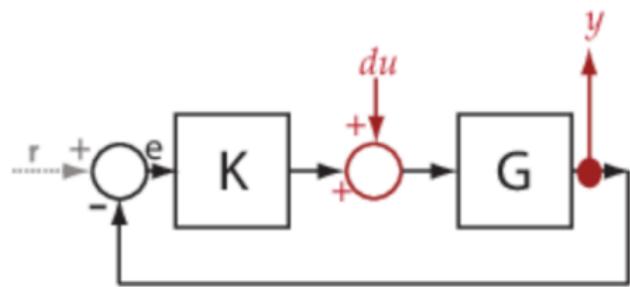
How to Specify the Portion of a Model to Linearise
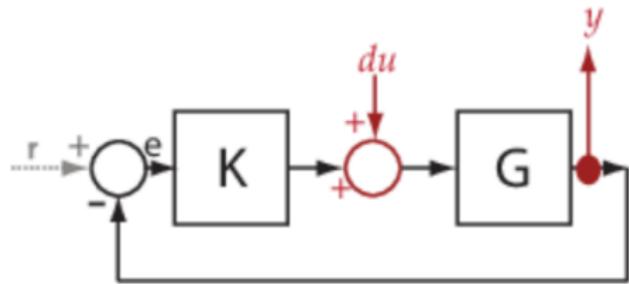
How to Specify the Portion of a Model to Linearise

Example 1
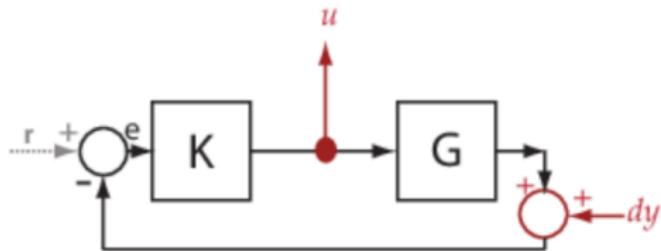
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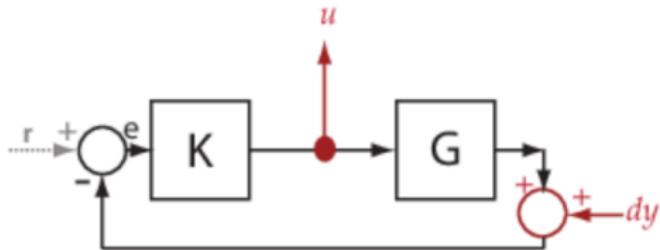
Show that the transfer function for $\frac{y}{du}$ is given by $\frac{G}{1+GK}$

Example 2

Example 2

Show that the transfer function for $\frac{u}{dy}$ is given by $\frac{-K}{1+KG}$

Bode Plots Revisited

Consider the watertank of the previous lecture and tutorial:

$$\frac{d}{dt}Vol = A\frac{dH}{dt} = bV - \alpha\sqrt{H} \qquad (1)$$

- $A = 20$
- $\alpha = 2$
- $b = 5$
- $H_{ref} = 10$ (reference signal - desired)
- Initial condition $H(0) = 1$
- PID parameters: $P = 1.599340, P_I = 0.079967, P_D = 0.$

Plot the Bode diagram.

How to model a Robot's Motion and Observations

Markov Models

Used to model dynamic systems.

- ▶ Speech/language processing
- ▶ Human behaviour (e.g. user modelling)
- ▶ Modelling psysical/biological processes
- ▶ Stock market
- ▶ ...

A class of probability models has to be defined, estimated from the observed behaviour of the dynamic system.

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- Transitions (dynamics): how the system moves from the current state s_t to the next state s_{t+1}.
- The transitions satisfy the first order Markov property:

$$P(s_{t+1}|s_t, s_{t-1}, \ldots, s_0) = P_1(s_{t+1}|s_t) \tag{2}$$

Markov Chains (cont'd)

Markov chains define a stochastic system which generates a sequence of states:

$$s_0 \longrightarrow s_1 \longrightarrow s_2 \longrightarrow \ldots$$

where s_0 is drawn from $P_0(s_0)$ and each s_{t+1} from one step transition probabilities $P_1(s_{t+1}|s_t)$.

- A Markov chain can be represented as a state transition diagram.

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- ▶ Since p_{ij} are probabilities:

$$0 \leq p_{ij} \leq 1 \qquad (4)$$

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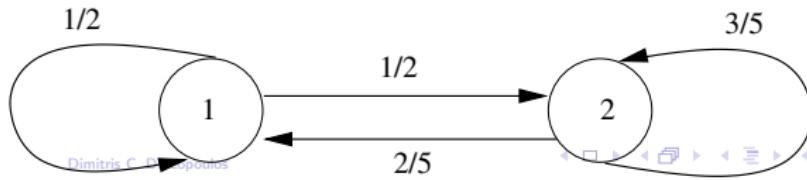
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$$P = [p_{ij}] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{2}{5} & \frac{3}{5} \end{bmatrix} \tag{5}$$

Markov Chain Problems

▶ *Prediction*: Probabilities that the system will be in state s_k after n transitions, given that at $n = 0$ is it in a known state.

▶ *Estimation*: Calculation of transition probabilities given some observed sequences of state transitions.

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$$\pi_j(n+1) = \sum_{i=1}^{N} \pi_i(n) p_{ij} \qquad n = 0, 1, 2, \ldots \qquad (7)$$

The Prediction Problem (cont'd)

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\begin{aligned}
\pi(1) &= \pi(0)P \\
\pi(2) &= \pi(1)P = \pi(0)P^2 \\
\pi(3) &= \pi(2)P = \pi(0)P^3
\end{aligned}
$$

$$(9)$$

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In general:

$$\pi(n) = \pi(0)P^n \qquad n = 0, 1, 2, \ldots \qquad (10)$$

Application to the Robot Maker Example

Assume that the robot maker starts with a successful robot, then $\pi_1(0) = 1$, $\pi_2(0) = 0$.

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After 1 week the robot maker is equally likely to be successful or unsuccessful.

After 2 weeks:

$$\pi(2) = \pi(1)P = [\frac{1}{2} \ \ \frac{1}{2}] \left[\begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ \frac{2}{5} & \frac{3}{5} \end{array} \right] = \left[\frac{9}{20} \ \ \frac{11}{20} \right]$$

so that the robot maker is slightly more likely to be unsuccessful.

Example: Successive State Probabilities Starting with a Successful Robot

n	0	1	2	3	4	5	...
$\pi_1(n)$	1	0.5	0.45	0.445	0.4445	0.44445	...
$\pi_2(n)$	0	0.5	0.55	0.555	0.5555	0.55555	...

As n becomes very large:

- $\pi_1(n)$ approaches $\frac{4}{9}$
- $\pi_2(n)$ approaches $\frac{5}{9}$

Ergodic Processes

In the previous example of the toymaker, as n becomes large, $\pi_1(n)$ approaches $\frac{4}{9}$ and $\pi_2(n)$ approaches $\frac{5}{9}$ independent of the starting state.

A Markov process is completely *ergodic* if:

▶ The limiting state probability distribution is independent of starting conditions.

For completely ergodic Markov processes π with components π_i is $\pi(n)$ as n approaches infinity. Also called absolute state probabilities.

Absolute State Probabilities

For ergodic Markov processes from (8):

$$\pi = \pi P \tag{11}$$

We also have:

$$\sum_{i=1}^{N} \pi_i = 1 \tag{12}$$

From (11), (12) the limiting state probabilities for any process can be found.

Ergodic Processes (cont'ed)

For the toymaker example, equations (11), (12) yield:

$$
\begin{aligned}
\pi_1 &= \frac{1}{2}\pi_1 + \frac{2}{5}\pi_2 \\
\pi_2 &= \frac{1}{2}\pi_1 + \frac{3}{5}\pi_2 \\
\pi_1 &+ \pi_2 = 1
\end{aligned}
\tag{13}
$$

The unique solution of the above is: $\pi_1 = \frac{4}{9}$, $\pi_2 = \frac{5}{9}$.

- ▶ Finding the limiting state probabilities involves the solution of a set of N linear equations.

Hidden Markov Models

▶ A hidden Markov model (HMM) is a model which we generate a sequence of outpouts in addition to the Markov state sequence:

$$s_0 \longrightarrow s_1 \longrightarrow s_2 \longrightarrow \ldots$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$x_0 \qquad x_1 \qquad x_2$$

▶ Only the outputs $\{x_0, x_1, \ldots, x_n\}$ are observed. The state sequence remains hidden.

Hidden Markov Problems to Solve

Two of the most interesting problems to solve, related to HMM are:

- ▶ Calculate the probability that our model generated the observation sequence $\{x_0, x_1, \ldots, x_n\}$?
  - $\rightarrow$ forward-backward algorithm
- ▶ How do we find the most likely hidden state sequence corresponding to these observations?
  - $\rightarrow$ dynamic programming