Ma 1A - Problem Set 1

1. Prove that $x + \frac{1}{x} \geq 2$ for all $x > 0$ and use this result to show that $\frac{a+b}{2} \geq \sqrt{ab}$ for all $a, b > 0$.

   Proof: First show that $x+\frac{1}{x} \geq 2$ for all $x > 0$. Start by multiplying both sides of the inequality by x, which is ok because we only care about $x \geq 0$ and don't have to worry about sign switching or multiplying by a negative number:

   $$x^2 + 1 \geq 2x$$

   Subtract $2x$ from both sides:

   $$x^2 - 2x + 1 \geq 0$$

   Now let's find the minimum value of the left side of the inequality. Take the derivative and set it equal to zero:

   $$\frac{d}{dx}(x^2 - 2x + 1) = 2x - 2 = 0$$

   $$2x = 2$$

   $$x = 1$$

   Thus we know that the minimum value of $x^2 - 2x + 1$ occurs at $x = 1$. Plug in $x = 1$ to the inequality to find the minimum value:

   $$(1)^2 - 2(1) + 1 \geq 0$$

   $$1 - 2 + 1 \geq 0$$

   $$0 \geq 0$$

   Which is true. Thus, because we know that the left side of the inequality is greater than or equal to 0 at its minumum, then we can infer that the left side of the inequality is greater than or equal to 0 at every value of x and conclude that the inequality is true.

   Now move on to the statement $\frac{a+b}{2} \geq \sqrt{ab}$ for all $a, b > 0$. Square both sides of the inequality:

   $$\frac{(a+b)^2}{4} \geq ab$$

2. Prove that $\sqrt{10}$ is irrational and deduce that $\sqrt{2} + \sqrt{5}$ is also irrational.

3. Prove, by induction or otherwise, that $\sum_{k=1}^{n} k^3 = (\sum_{k=1}^{n} k)^2$.

4. Each card in a pack has a number on one side and a letter on the other. Four cards are placed on the table:

$$\boxed{2}\,\boxed{3}\,\boxed{A}\,\boxed{B}$$

You are permitted to turn just two cards in order to test the following hypothesis: a card that has an even number on one side has a vowel on the other. Which two cards should you turn and why?

5. Which of the following relations on $\mathbb{N}$ are reflexive, which are symmetric and which are transitive:

(i) $a \mid b$ ('a divides b'),

(ii) $a \nmid b$ ('a does not divide b'),

(iii) a and b have the same remainder after division by 2025,

(iv) $\gcd(a, b) > 2025$?