One-Sided Jacobi method for computing Singular Value Decomposition

- The singular value decomposition, or SVD, is a very powerful technique for dealing with matrix problems in general.

$$\boldsymbol{A} = \boldsymbol{U\Sigma V}^T$$

$$\begin{array}{lll} \mathbf{U} & : m \times m & : \text{columns are left singular vectors} \\ \mathbf{\Sigma} & : m \times n & : \text{diagonal : singular values} \\ \mathbf{V} & : n \times n & : \text{columns are right singular vectors} \end{array}$$

e.g. for $m > n$

$$\mathbf{A} = \begin{bmatrix} \vdots & & \vdots & \vdots & & \vdots \\ \mathbf{u}_1 & \dots & \mathbf{u}_r & \mathbf{u}_{r+1} & \dots & \mathbf{u}_m \\ \vdots & & \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \\ 0 & \dots & \dots & \dots & \dots & 0 \\ \vdots & & & & & \vdots \\ 0 & \dots & \dots & \dots & \dots & 0 \end{bmatrix} \begin{bmatrix} \dots & \mathbf{v}_1^\top & \dots \\ & \vdots & \\ \dots & \mathbf{v}_r^\top & \dots \\ \dots & \mathbf{v}_{r+1}^\top & \dots \\ & \vdots & \\ \dots & \mathbf{v}_n^\top & \dots \end{bmatrix}$$

$\sigma_1 \geq \sigma_2 \geq \dots \sigma_r > 0$, $r = \text{rank}(\mathbf{A})$

- Applications are as diverse as least squares data fitting, Kalman filter, image compression, facial recognition, signal Processing, …
- The Kalman filter implementations, like square-root algorithms based on the Cholesky decomposition, are effective with well-conditioned problems but may fail if equations are ill-conditioned (due to roundoff errors or …), as those algorithms rely on matrix inversion in several places.
- SVD-based Kalman filter variant works much better in ill-conditioned cases. In the IMU example, a case of ill-conditioned problem is when multiple sets of the same measurements are available.
- By comparing various algorithms for calculating SVD, some researchers have identified **one-sided Jacobi rotation** algorithm outperforms the others for **embedded systems**. For example the article:

M. Alessandrini, … "Singular Value Decomposition in Embedded Systems Based on ARM Cortex-M Architecture"

- One sided Jacobi algorithm applies a sequence of plane rotations to the original matrix $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{S} = \boldsymbol{S}_0$, in order to reach the diagonal matrix.
- In other word, to rotate columns p and q of $\boldsymbol{A}$ through the angle θ so that they become orthogonal to each other.
- This algorithm produces a sequence $\boldsymbol{S}_1$, $\boldsymbol{S}_2$, $\boldsymbol{S}_3$, ... which eventually converge to a diagonal matrix with the eigenvalues on the diagonal

$$\boldsymbol{S}_{k+1} = \boldsymbol{J}_k^T(p,q,\theta)\,\boldsymbol{S}_k\,\boldsymbol{J}_k(p,q,\theta)$$

$$
\begin{array}{cc}
\begin{bmatrix}
1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & & \vdots & & \vdots \\
0 & \cdots & c & \cdots & s & \cdots & 0 \\
\vdots & & \vdots & \ddots & \vdots & & \vdots \\
0 & \cdots & -s & \cdots & c & \cdots & 0 \\
\vdots & & \vdots & & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 1
\end{bmatrix}
&
\begin{array}{c}
\\ \\ p \\ \\ q \\ \\ \\
\end{array}
\\
\begin{array}{ccccccc}
 & & p & & q & &
\end{array}
&
\end{array}
$$

$$c = \cos\theta \text{ and } s = \sin\theta$$

❖ In addition to calculating the SVD and the rank of $\boldsymbol{A}$ matrix, this code permute the columns of the U and V matrices so that the singular values on the diagonal of the $\boldsymbol{\Sigma}$ are in descending order.