Climate MS - Sea Ice/Ocean

Fracture of an ice floe: modelisation and simulation

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Context of the presentation

We already have a granular model for the Marginal Ice Zone. It deals with:

- real-time simulation
- 1 million floes
- floes scale from several meters to several kilometers
- variety of shapes
- simulation window of 100 kilometers

Context of the presentation: video

Outline of the presentation

1. A brief overview of brittle fracture models

2. Our brittle fracture model : theoretical results

3. Numerical methods and results

A brief overview of brittle fracture models

Griffith's criterion for brittle fracture

Assume that the crack path is known. Denote by $l(t)$ the crack at time t, and by k the toughness of the material.

Energy relase rate:

$$G(t, l(t)) = -\lim_{h \to 0} W(t, l(t) + h) - W(t, l(t)).$$

Then $G(t, l(t))$ follows the evolution law:

$$\begin{cases} G(t, l(t)) \leq k \\ l(t) \text{ increasing} \\ \dfrac{\mathrm{d}l}{\mathrm{d}t}(t) \neq 0 \Rightarrow G(t, l(t)) = k \end{cases}$$

Limitations:

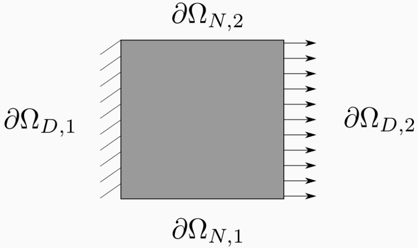
- it does not account for fracture initiation
- it does no predict fracture path

A variational model for brittle fracture

Denote by Ω a floe, apply a displacement U_D on $\partial\Omega_D$.
Competition between:

- elastic energy : $\frac{1}{2}\int_{\Omega\setminus\sigma} Ae(u) : e(u)\,\mathrm{d}x$,
- fracture energy : $k\mathcal{H}^1(\sigma)$,

with A the stiffness tensor, $e(u) = \frac{1}{2}\left(\nabla u + \nabla u^t\right)$ the symetric gradient and $\mathcal{H}^1(\sigma)$ the fracture's lengh.



A variational model for brittle fracture

Theorem (Dal Maso and Toader (02), Chambolle (03))

The functional :

$$E(u,\sigma) = \frac{1}{2}\int_{\Omega\setminus\sigma} Ae(u) : e(u)\,\mathrm{d}x + k\mathcal{H}^1(\sigma)$$

has a minimum over

$$\{(u,\sigma)\ |\sigma \subset \overline{\Omega}, \sigma \text{ closed },$$
$$u \in L^2(\Omega,\mathbb{R}^2), u = U_D \text{ on } \partial\Omega_D, e(u) \in L^2(\Omega, S^2(\mathbb{R}))\}$$

The solution is a minimum of the floe's energy.

Theorem from image segmentation.

Theorem (De Giorgi, Carriero, Leacci (89))

The Mumford-Shah functional:

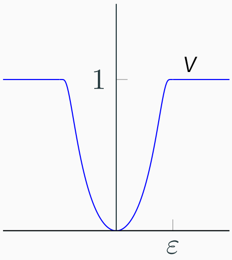
$$F(u,K) = \frac{1}{2}\int_{\Omega\setminus K} |\nabla u|^2 \,\mathrm{d}x + \int_\Omega |u-g|^2 + \alpha\mathcal{H}^1(K)$$

has a minimum over $\left\{(u,K) \,\middle|\, K \text{ closed }, u \in C^1(\Omega\setminus K)\right\}$.

The classical numerical approach : Ambrosio-Tortorelli / phase field approximation

We replace the unknown set σ by a function v :

$$v = 0 \text{ on the fracture}, \quad v = 1 \text{ on } \Omega \setminus \mathcal{V}_\varepsilon(\sigma).$$



The classical numerical approach : Ambrosio-Tortorelli / phase field approximation

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Theorem (Ambrosio and Tortorelli (90), Bourdin (98))

The sequence of functionals

$$F_\varepsilon(u,v) = \frac{1}{2}\int_\Omega v^2 Ae(u):e(u)\,\mathrm{d}x + \varepsilon\int_\Omega \|\nabla v\|^2\,\mathrm{d}x + \frac{1}{4\varepsilon}\int_\Omega (1-v)^2\,\mathrm{d}x$$

Γ*-converges to the functional* $F(u,\sigma)$

Major inconvenient in our case: it needs tight meshs.

Our brittle fracture model : theoretical results

Our take on the problem

We make restrictive hypothesis on the fracture: they are lines.

We study the minimum problem:

$$\inf_{\sigma \in \Sigma} \inf_{u \in V_\sigma} \int_{\Omega \setminus \sigma} Ae(u) : e(u) \, \mathrm{d}x + \mathcal{H}^1(\sigma)$$

with the variational spaces

$$\begin{aligned}
\Sigma &= \left\{ [a\,, b] \subset \mathbb{R}^2 \,\middle|\, a \in \partial\Omega, b \in \overline{\Omega},]a\,, b[\subset \Omega \right\} \\
V_\sigma &= \{ u \in H^1(\Omega \setminus \sigma, \mathbb{R}^2) \mid u = U_D \text{ on } \partial\Omega_D \setminus \sigma, \\
&\qquad\qquad\qquad\qquad (u^+ - u^-) \cdot \nu \geq 0 \text{ on } \sigma \}
\end{aligned}$$

Σ is endowed with the Hausdorff distance d_{H}.

Theorem (Labbé, Lattes, Weiss, Balasoiu)

If Ω is convex, the minimum problem:

$$\inf_{\sigma\in\Sigma}\inf_{u\in V_\sigma}\int_{\Omega\setminus\sigma} Ae(u):e(u)\,\mathrm{d}x+\mathcal{H}^1(\sigma)$$

has a solution over $\bigcup_{\sigma\in\Sigma}\{\sigma\}\times V_\sigma$.

Proof

Suppose $(\sigma_n)_{n\in\mathbb{N}}\subset\Sigma$ such that:

$$\inf_{u\in V_{\sigma_n}} E(u,\sigma_n)\to\inf_{\sigma\in\Sigma}\inf_{u\in V_\sigma} E(u,\sigma).$$

There exists $u_n\in V_{\sigma_n}$ such that: $E(u_n,\sigma_n)=\inf_{u\in V_{\sigma_n}} E(u,\sigma_n)$.
There exists $\sigma\in\Sigma$ such that $d_{\mathrm{H}}(\sigma_n,\sigma)\to 0$.

An existence result: proof

Proof.
Embeding: $\forall \sigma \in \Sigma, V_\sigma \subset V = L^2(\Omega, \mathbb{R}^2)^2 \times L^2(\Omega, \mathbb{R}^4)^2$.
Mosco convergence: $V_{\sigma_n} \to V_\sigma$

- $\forall \varphi \in V_\sigma, \exists \varphi_n \in V_{\sigma_n}, \varphi_n \to \varphi$ strongly
- $\forall \varphi_n \in V_{\sigma_n}$ bounded in V, $\exists \varphi \in V_\sigma, \varphi_n \to \varphi$ weakly

There exists $u \in V_\sigma$, such that $u_n \to u$ weakly.
Let $v \in V_\sigma$, there exists $v_n \in V_{\sigma_n}$ such that $v_n \to v$ strongly.
Use variational formulation for fixed $n \in \mathbb{N}$:

$$\int_{\Omega \setminus \sigma_n} Ae(u_n) : e(v_n - u_n)\,\mathrm{d}x \geq 0.$$

Go to the limit with strong convergence of v_n and weak convergence of u_n: u verifies the variational formulation. $\square$

A quasistatic framework: mitigation of the strong geometric hypothesis

Traction on $\partial\Omega_D$: $U_D(t) = tU_D$
Let $0 = t_0 < \cdots < t_i < \cdots < t_p = 1$ be a subdivision of $[0\,,1]$.
A discrete evolution satisfies:

1. $u_0 = 0\,\mathrm{Id}_\Omega$ and $\sigma_0 = \emptyset$,
2. $\forall i \in \{1, \ldots, p\}\,, \sigma_i \in \Sigma_{\sigma_{i-1}}$,
3. $\forall i \in \{1, \ldots, p\}\,, \forall \sigma \in \Sigma_{\sigma_{i-1}}, \forall u \in V_{t_i,\sigma}, \quad E(u_i, \sigma_i) \leq E(u, \sigma)$.

with the variational spaces:

$$\Sigma_\sigma = \left\{\sigma \cup [a\,,b] \subset \mathbb{R}^2 \,\middle|\, a = \mathrm{end}(\sigma), b \in \overline{\Omega},]a\,,b[\subset \Omega \setminus \sigma\right\}$$
$$V_{t,\sigma} = \{u \in H^1(\Omega \setminus \sigma, \mathbb{R}^2) \mid u = U_D(t) \text{ on } \partial\Omega_D \setminus \sigma,$$
$$(u^+ - u^-) \cdot \nu \geq 0 \text{ on } \sigma\}$$

A quasistatic framework: convergence result

Theorem (Dal Maso and Toader, Chambolle)

The discrete evolution converges to the continuous evolution:

1. $\sigma(0) = \emptyset$,
2. $\forall t \in [0\,,1]$, $\sigma(t)$ *is a rectifiable curve,*
3. $\forall t_1 < t_2$, $\sigma(t_1) \subset \sigma(t_2)$,
4. *for all rectifiable curves* $\tau \supset \bigcup_{s<t} \sigma(s)$*:*

$$\inf_{u \in V_{t,\sigma(t)}} E(u, \sigma(t)) \leq \inf_{u \in V_{t,\tau}} E(u, \tau).$$

Under investigation

There is room for improvement :

- Ω non convex -> Problem with fracture beeing tangent to the boundary
- Growing fracture -> Natural space of admissible fractures is open and allows for fracture autointersection.

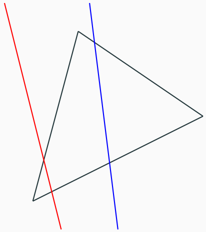
$$\Sigma_\sigma = \left\{ [a\,,b] \subset \mathbb{R}^2 \,\middle|\, a = \mathrm{End}(\sigma), b \in \overline{\Omega},]a\,,b[\subset \Omega \setminus \sigma \right\}$$

The listed problems emerge from the global minimization hypothesis.

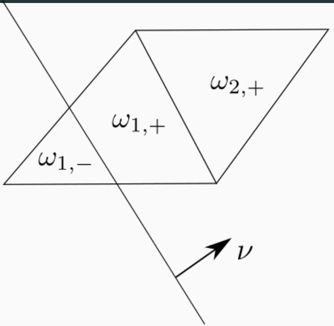
Numerical methods and results

Numerical scheme: requirements

The number of mesh in a floe does not deppend of the floe's size.
On large floes : large cells.
We need the admissible fracture set to be independent of the mesh's precision.



An ad-hod finite element method



Set:

$$T_{\tau,\sigma} = \{\omega \in \tau_2 \,|\, \omega \cap \sigma \neq \emptyset\}$$

$$\nu_{\tau,\sigma} = \{p \in \tau_0 \,|\, p \in \sigma\} \cup \{p \in \tau_0 \,|\, \exists \omega \in T_{\tau,\sigma}, p \in \overline{\omega}\}$$

For each finite element e_p based on a node p, define:

$$e^+_{\tau,p} = e_{\tau,p} \sum_{\omega \in T_{\tau,\sigma}} \mathbb{1}_{\omega^+}, \quad e^-_{\tau,p} = e_{\tau,p} \sum_{\omega \in T_{\tau,\sigma}} \mathbb{1}_{\omega^-},$$

Finite element space:

$$W_{\tau,\sigma} = \underset{p \in \tau_0 \setminus \nu_{\tau,\sigma}}{\mathrm{span}} (e_{\tau,p} u_x, e_{\tau,p} u_y) + \underset{p \in \nu_{\tau,\sigma}}{\mathrm{span}} (e^+_{\tau,p} u_x, e^-_{\tau,p} u_x, e^+_{\tau,p} u_y, e^-_{\tau,p} u_y).$$

A convergence result

Set:

$$\Sigma_n \subset \Sigma \text{ a discretization of } \Sigma$$

and

$$V = \bigcup_{\sigma \in \Sigma} V_\sigma \times \{\sigma\}, \quad V_n = \bigcup_{\sigma \in \Sigma_n} W_{\tau_n,\sigma} \times \{\sigma\}.$$

Define:

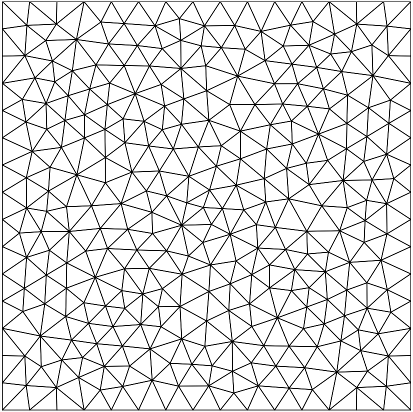
$$\begin{aligned} E_n : V &\to \overline{\mathbb{R}} \\ (u, \sigma) &\mapsto \begin{cases} E(u, \sigma) \text{ if } (u, \sigma) \in V_n \\ +\infty \text{ otherwise} \end{cases} \end{aligned}$$

Theorem (Labbé, Lattes, Weiss, Balasoiu)

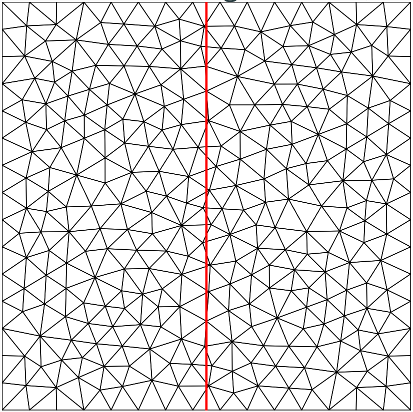
The sequence of functionals $(E_n)_{n \in \mathbb{N}}$ are equi-coercive, and Γ-converges to E (for the topology of V).

Numerical solution: initiation in finite time

Elastic behavior (small traction)

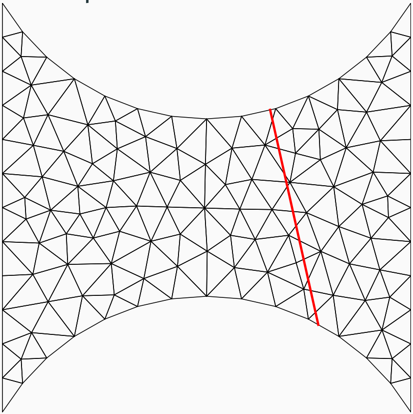


With fracture (huge traction)

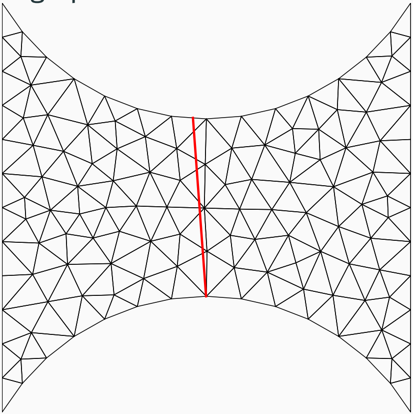


Numerical solution: influence of fracture boundary precision

Small precision

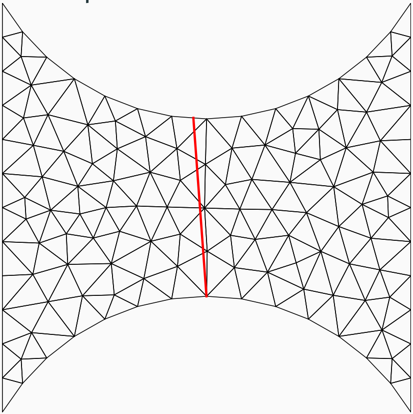


High precision

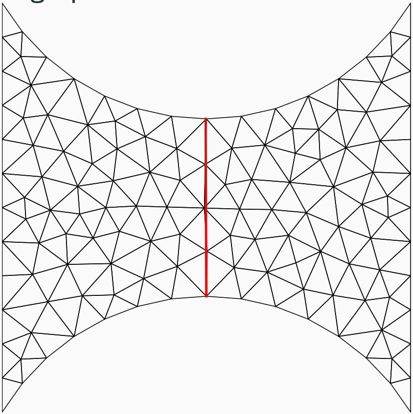


Numerical solution: influence of fracture angular precision

Small precision

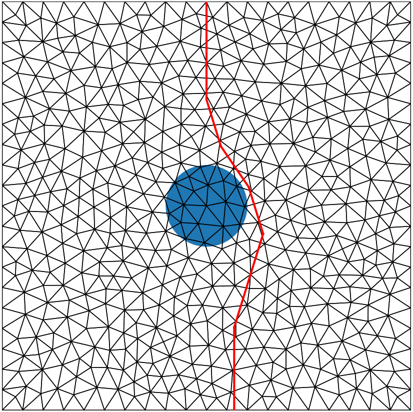


High precision



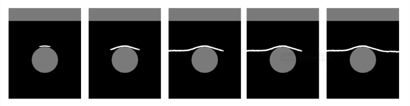
Numerical solution: path prediction 1/2

Rigid circular inclusion with $k_1 >> k$.

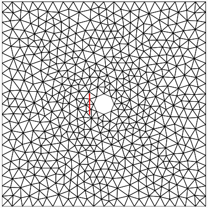


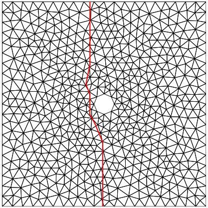
Circular hole inside floe.

Phase field solution



Our solution





References

Dimitri Balasoiu, Stéphane Labbé, Philippe Lattes, and Jérome Weiss. "An Efficient Variational Model for Brittle Fracture". In: (To appear).

Matthias Rabatel, Stéphane Labbé, and Jérôme Weiss. "Dynamics of an Assembly of Rigid Ice Floes". In: *Journal of Geophysical Research: Oceans* 120.9 (2015), pp. 5887–5909.