University of Strasbourg

MIMESIS

Inria

PROJECT REPORT V0

Simulation of soft tissues using innovative non-conforming Finite Elements Methods

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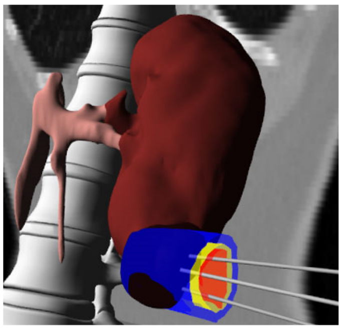

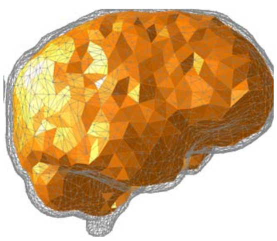

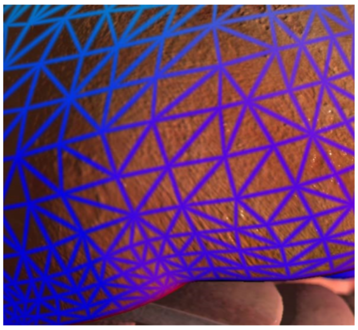

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Contents

Chapter 1

Introduction

1.1 Environment

Inria is the National Institute for Research in Digital Science and Technology. **Digital health** is one of their main research topics, and several teams are mobilized to face its challenges. The MIMESIS team focuses on **real-time simulations for per-operative guidance**. The team develops **numerical techniques for real-time computation** and **data-driven simulation dedicated to patient-specific modeling**. It's global objective is to create a synergy between clinicians and scientists in order to create new technologies capable of redefining healthcare, with a strong emphasis on clinical translation (MIMESIS, 2020).

1.2 Context

The context of this project is the **development of a Finite Elements Method (FEM) adapted to computer-assisted surgery**. In recent years, numerical models using FEM to simulate the soft-tissue mechanisms of the human body have attracted a great interest from the scientific community. Models by FEM are among others, tools that contribute to the development of medical devices such as prosthesis and orthosis. They have the potential to improve strategies in planning and surgical assistance. In the context of computer-assisted surgery, it is essential that the FEM technique used is (Duprez, 2020):

- **Quick**: given the context of real-time simulations
- **Precise**: in order to accurately guide the surgeon
- **Patient-specific**: involving complex body organ geometries

1.3 Objectives

In the case of quasi-incompressibility for example, it is necessary to use hexahedral meshes in order to avoid **locking** phenomena. However, there is no 3D mesh generator capable of meshing any geometry in an hexahedral fashion. In order to completely avoid the creation of meshes, it is possible to perform numerical simulations on a **non-conforming** mesh (one which does not coincide with the boundary of the organs), and to manage the boundary conditions by **penalization**, **stabilization** or using **Lagrange multipliers**. Such methods already exists (e.g. XFEM, CutFEM), but their implementation generates other difficulties, among other things, the quadrature (computation of the integrals appearing in the FEM formulation).

In a preliminary study (Duprez and Lozinski, 2020; Duprez et al., 2020)[1], a new approach overcoming the afore-mentioned difficulty was developed. This method, called ϕ-FEM, uses a **Level Set** function that cancels itself at the edges of the domain. The main objective of this project is to **develop a ϕ-FEM technique for the dynamic of soft tissues and potentially provide a mathematical proof of the method's convergence**. In order to achieve our main objective, we have divided the project into multiple milestones whose (intermediate) objectives are:

1. Understand the ϕ-FEM technique in question
2. Reproduce the results of (Duprez and Lozinski, 2020) in the case of the Poisson equation
3. Develop a ϕ-FEM technique for the linear elasticity equation
4. Perform simulations on body organ geometries

1.4 Tools Needed

A mastery of the following tools and technologies is required to accurately complete the tasks in this project

- Python: a basic knowledge is required in order to write scripts
- FEniCS: this python library offers a relatively easy interface to implement weak formulations for PDEs
- Docker: in order to setup the FEniCS environment
- Sympy: to perform symbolic derivatives, needed to test our programs

1.5 Mathematical tools involved

As for the mathematical aspect of the project, knowledge in several fields is needed. Those are:

- Partial differential equations (PDE): elasticity equations, biomechanics, etc.
- Scientific computing: building and implementation of numerical schemes based on FEM
- Numerical analysis: studying the convergence of a model to a mathematical solution

1.6 Deliverables

At the end of the project, the following will be submitted:

- A typewritten report
- A Python code base

[1]Henceforth, these two papers will simply be referred to as "the papers"

Chapter 2

Results

2.1 Presentation of the ϕ-FEM technique

2.1.1 The Classic FEM approach

As indicated by it's name, the ϕ-FEM tecnique is based on the classic FEM technique (Ern and Guermond, 2013, p.111). Let Ω be a bounded domain in $\mathbb{R}^d$ with a regular boundary $\partial\Omega$. Let $\mathcal{L}$ be a scalar elliptic operator (Evans, 1998). Given a function $f \in L^2(\Omega)$, and another function $g \in L^2(\partial\Omega)$, the problem is to solve

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

Except for the non-homogeneous Dirichlet problem [1], problems of the form (2.1) can be transformed into a weaker form called variational formulation:

$$\text{Seek } u \in V \text{ such that} \quad a(u,v) = f(v), \quad \forall v \in V \tag{2.2}$$

where V is a Hilbert space satisfying

$$H_0^1(\Omega) \subset V \subset H^1(\Omega)$$

Moreover, a is a bilinear form defined on $V \times V$, and f is a linear form defined on V.

The Poisson problem ($\mathcal{L} := -\Delta$) with homogeneous boundary condition is a classic exemple of a elliptical PDE formulated as in (2.5). It is defined as

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.3}$$

and its weak formulation

$$\text{Seek } u \in H_0^1(\Omega) \text{ such that} \quad a(u,v) = f(v), \quad \forall v \in H_0^1(\Omega). \tag{2.4}$$

Once the variational form (2.5) has been obtained, it has to be approximated [2]. Assuming the boundary $\partial\Omega = \Gamma$ is sufficiently smooth, let $\mathcal{T}_h$ be a quasi-uniform simplicial mesh on Ω of mesh size h. The domain occupied by the mesh elements is denoded as $\Omega_h = (\cup_{T\in\mathcal{T}_h} T)^o$. Let's consider,

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[1] The non-homogeneous Dirichlet case can be brought back to the 2.5 by writing $u = u_g + \Phi$, where u_g is a lifting function, and Φ satisfies 2.5 where $V = H_0^1(\Omega)$

[2] In order to have uniform notations everywhere in the document, let's consider the notations from (Duprez and Lozinski, 2020)

for an integer $k \geq 1$, the finite element space

$$V_h^{(k)} = \{v_h \in H^1(\Omega_h) : v_h|_T \in \mathbb{P}_k(T)\ \forall T \in \mathcal{T}_h\}$$

where $\mathbb{P}_k(T)$ stands for the space of polynomials in d variables of degree $\leq k$ viewed as functions on T. The FEM approximation can now be derived from (2.5) as

$$\text{Seek } u_h \in V_h^{(k)} \text{ such that } \quad a_h(u_h, v_h) = l_h(v_h), \quad \forall v_h \in V \tag{2.5}$$

where the bilinear form a_h and the linear form l_h are defined on $V_h^{(k)} \times V_h^{(k)}$ and $V_h^{(k)}$ respectively. In the case of the Poisson problem (2.3), a_h and l_h are defined as:

$$\begin{aligned} a_h(u, v) &= \int_{\Omega_h} \nabla u \cdot \nabla v \\ l_h(v) &= \int_{\Omega_h} f v \end{aligned} \tag{2.6}$$

2.1.2 XFEM and CutFEM

XFEM, CutFEM and ϕ-FEMare all part of a family of numerical techniques **immerses boundary methods**. These techniques use a non-body conforming Cartesian grid. In general a body does not align with the grid, so some of the computational cells will be cut. the advantages of using an immersed boundary method is that grid generation is much easier, because a body does not necessarily have to fit a Cartesian grid. Another benefit is that grid complexity and quality are not significantly affected by the complexity of the geometry when carrying out a simulation on a non-boundary conforming Cartesian grid. Also, an immersed boundary method can handle moving boundaries, due to the stationary non-deforming Cartesian grid (Bandringa, 2010).

The extended finite element method (XFEM) (fig. 2.1) is specifically designed to treat discontinuities . It is a numerical method that enables a local enrichment of approximation spaces. The enrichment is realized through the partition of unity concept (Fries, n.d.). The method is particularly useful for the approximation of solutions with pronounced non-smooth characteristics in small parts of the computational domain, for example near discontinuities and singularities (Moës et al., 1999).

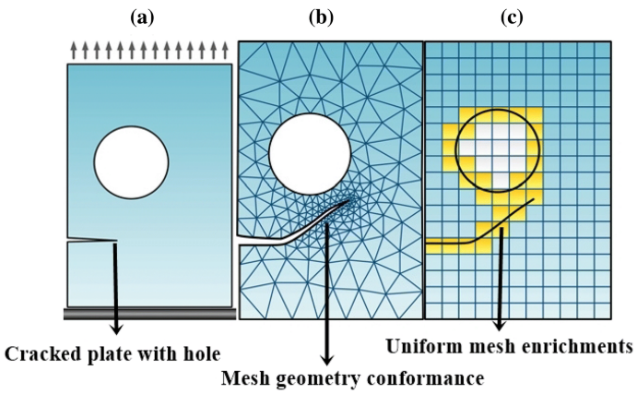


FIGURE 2.1: Extended finite element method (XFEM) applied to micro-crack propagation. (a) Crack propagation in a plate with hole, (b) the standard FEM mesh refinement and mesh geometry conformance and (c) the enriched-FEM with uniform mesh enrichments (Swati et al., 2019)

In the CutFEM approach (fig. 2.2), the boundary of a given domain is represented on a background grid. CutFEM partitions domains into an Ω_1 (exterior) and Ω_2 (interior); separate solution for Ω_1 and Ω_2; captures jumps; integrates over intersections of each element with subdomains; and finally, it weakly imposes boundary and jump conditions.

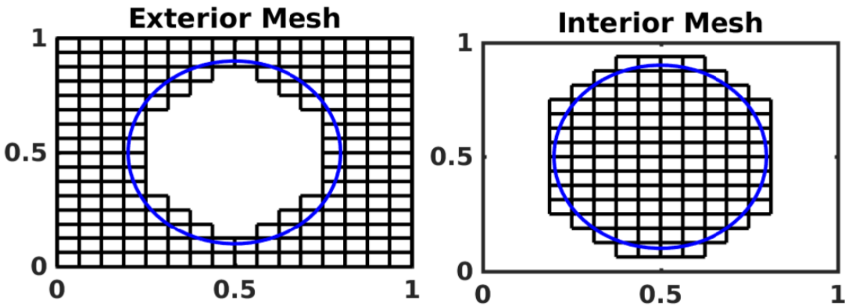


FIGURE 2.2: Illustration of Ω_1 (exterior) and Ω_2 (interior) subdomains in the CutFEM approach

While these methods (and others based on them) are effective, the integrals over Ω are kept in the discretization, which in practice, is cumbersome since one needs to implement integrations on the boundary Γ, and on parts of mesh elements cut by the boundary. Multiple attempts have been made to alleviate this practical difficulty with methods that do not require to perform the integration on the cut elements, but still need the integration on Γ. ϕ-FEM avoids any integration whatsoever on Γ.

2.1.3 Formal presentation of theϕ-FEM technique

As of November 17, 2020, ϕ-FEMhas only been developed and tested on the Poisson equation. In the coming paragraphs, we will present the general idea behind the technique, for any common elliptical equation (2.1).

The basis for ϕ-FEM have been laid down above. Here, the boundary condition is carried by the level-set function; and as hinted, all the integrations in ϕ-FEM is performed on the whole mesh elements, and there are no integrals on Γ (even if the mesh element in question is cut by the boundary Γ). As stated in the introduction, the domain's boundary is the region in space where the Level Set function ϕ vanishes. This Level Set function is chosen such that it remains strictly negative in the interior of the domain. The level-set function is supposed to be smooth, and to behave near Γ as the signed distance to Γ. As an indication, the following problem (eq. (2.1) with an homogeneous Dirichlet boundary condition)

$$\exists\,?\,u \text{ such that } \begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is reformulated in ϕ-FEM as (see fig. 2.3)

$$\exists\,?\,w \text{ such that } \begin{cases} \mathcal{L}(\phi w) = f & \text{in } \Omega \\ \text{where } \Omega = \{\phi < 0\}, & u = \phi w \end{cases}$$

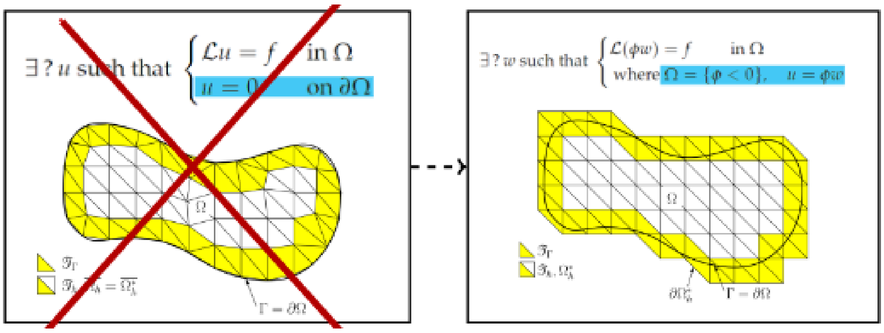


FIGURE 2.3: Conversion of a problem from classic FEM to *phi*-FEM in order to avoid complications due to meshing. Ω and $\partial\Omega$ are defined by the smooth level set function ϕ. $\partial\Omega^*$ is a fictitious boundary containing all the cells cut by ϕ, useful for the method's stabilization.

As indicated in fig. 2.3, the bounded domain's boundary $\partial\Omega = \Gamma$ is assumed to be smooth and given by a level-set function ϕ. As stated in the introduction, the domain's boundary will often be the region in space where the Level Set function ϕ vanishes. This Level Set function is chosen such that it remains strictly negative in the interior of the domain.

$$\Omega := \{\phi < 0\} \quad \text{and} \quad \Gamma := \{\phi = 0\}.$$

The level-set method allows for treatment of internal boundaries and interfaces without any explicit treatment of the interface geometry. Such a representations is typical in **immersed border techniques**, or **fictitious domain methods**. It is important for dealing with complex boundaries and problems with evolving surfaces or interfaces. This provides a convenient and an appealing mean for tracking moving interfaces, such as the ones of living organs.

Now let's provide a formal formulation for the Poisson equation (2.3) using the ϕ-FEMtechnique. As Ω is a bounded domain, we can write $\Omega \subset \mathcal{O} \subset \mathbb{R}^d$ $(d = 2, 3)$. Let $\mathcal{T}_h^{\mathcal{O}}$ be a quasi-uniform simplicial mesh on $\mathcal{O}$ of mesh size h. Let's consider, for an integer $l \geq 1$, the finite element space

$$V_{h,\mathcal{O}}^{(l)} = \{v_h \in H^1(\mathcal{O}) : v_h|_T \in \mathbb{P}_l(T)\ \forall T \in \mathcal{T}_h^{\mathcal{O}}\}$$

where $\mathbb{P}_l(T)$ stands for the space of polynomials in d variables of degree $\leq l$ viewed as functions on T. We define ϕ_h as

$$\phi_h := I_{h,\mathcal{O}}^{(l)}(\phi)$$

here $I_{h,\mathcal{O}}^{(l)}$ is the standard Lagrange interpolation operator on $V_{h,\mathcal{O}}^{(l)}$. Henceforth, this function will be used to approximate the physical domain and its boundary. Next we introduce the computational mesh $\mathcal{T}_h$ as the subset of $\mathcal{T}_h^{\mathcal{O}}$ composed of the triangles/tetrahedrons having a non-empty intersection with the approximate domain $\{\phi_h < 0\}$. We denote the domain occupied by $\mathcal{T}_h$ by Ω_h, *i.e.*

$$\mathcal{T}_h := \{T \in \mathcal{T}_h^{\mathcal{O}} : T \cap \{\phi_h < 0\} \neq \emptyset\} \quad \text{and} \quad \Omega_h = (\cup_{T \in \mathcal{T}_h} T)^o.$$

Now the function space will be defined as

$$V_h^{(k)} = \{v_h \in H^1(\Omega_h) : v_h|_T \in \mathbb{P}_k(T)\ \forall\ T \in \mathcal{T}_h\}.$$

The ϕ-FEM approximation is introduced as follows: find $w_h \in V_h^{(k)}$ such that:

$$a_h(w_h, v_h) = l_h(v_h) \text{ for all } v_h \in V_h^{(k)}, \tag{2.7}$$

where the bilinear form a_h and the linear form l_h are defined by

$$a_h(w,v) := \int_{\Omega_h} \nabla(\phi_h w)\cdot\nabla(\phi_h v) - \int_{\partial\Omega_h} \frac{\partial}{\partial n}(\phi_h w)\phi_h v + G_h(w,v) \tag{2.8}$$

and

$$l_h(v) := \int_{\Omega_h} f\phi_h v + G_h^{rhs}(v),$$

with G_h and G_h^{rhs} standing for

$$G_h(w,v) := \sigma h \sum_{E\in\mathcal{F}_h^\Gamma} \int_E \left[\frac{\partial}{\partial n}(\phi_h w)\right]\left[\frac{\partial}{\partial n}(\phi_h v)\right] + \sigma h^2 \sum_{T\in\mathcal{T}_h^\Gamma} \int_T \Delta(\phi_h w)\Delta(\phi_h v)\,,$$
$$G_h^{rhs}(v) := -\sigma h^2 \sum_{T\in\mathcal{T}_h^\Gamma} \int_T f\Delta(\phi_h v)\,,$$

where $\sigma > 0$ is an h-independent stabilization parameter, $\mathcal{T}_h^\Gamma \subset \mathcal{T}_h$ contains the mesh elements cut by the approximate boundary $\Gamma_h = \{\phi_h = 0\}$, *i.e.*

$$\mathcal{T}_h^\Gamma = \{T\in\mathcal{T}_h : T\cap\Gamma_h \neq \emptyset\}, \quad \Omega_h^\Gamma := \left(\cup_{T\in\mathcal{T}_h^\Gamma} T\right)^o.$$

and $\mathcal{F}_h^\Gamma$ collects the interior facets of the mesh $\mathcal{T}_h$ either cut by Γ_h or belonging to a cut mesh element

$$\mathcal{F}_h^\Gamma = \{E \text{ (an internal facet of } \mathcal{T}_h) \text{ such that } \exists T\in\mathcal{T}_h : T\cap\Gamma_h\neq\emptyset \text{ and } E\in\partial T\}.$$

The brackets inside the integral over $E \in \mathcal{F}_h^\Gamma$ in the formula for G_h stand for the jump over the facet E. The first part in G_h actually coincides with the ghost penalty as introduced in (Burman, 2010) for P_1 finite elements.

2.2 The Poisson problem

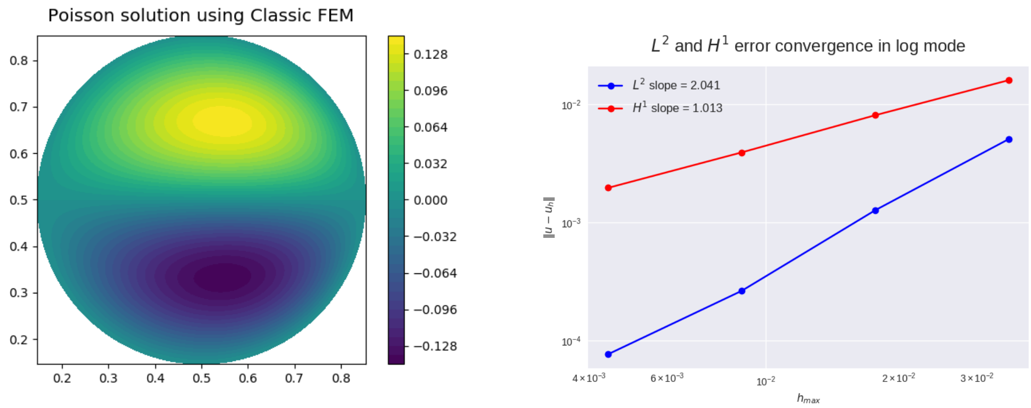
2.2.1 Using classic FEM

The basis for the weak formulation has been laid down in (2.6). The values for f, and the domain Ω need to be defined in order to run the test cases. Moreover, the exact solution u has to be known in order to perform a convergence study.

The classic FEM technique has been tested on a particular case. The case in question is the *test case 1* from (Duprez and Lozinski, 2020, p.15). The corresponding parameters are presented below [3], and the result in fig. 2.4.

$$\begin{cases} \Omega = \left\{(x,y)\in\mathbb{R}^2 : \left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 < \frac{1}{8}\right\} \\ u(x,y) = \left(\frac{1}{8} - \left(x-\frac{1}{2}\right)^2 - \left(y-\frac{1}{2}\right)^2\right)\exp(x)\sin(2\pi y) \\ f(x,y) = -\frac{\partial^1 u}{\partial x^2}(x,y) - \frac{\partial^1 u}{\partial y^2}(x,y) \end{cases} \tag{2.9}$$

[3]Notice that f is very cumbersome to compute. It will be computed using *Sympy*



(A) Solution (B) Convergence study

FIGURE 2.4: Results obtained when applying the classic FEM technique to the Poisson equation (2.10). These results confirm what we expect since and will serve as a reference for the following results. The red and blue plots in (B) indicate the relative error.

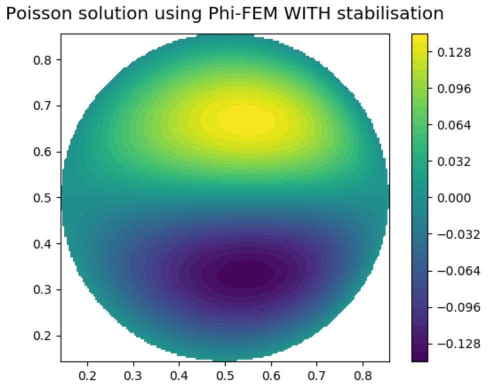
COMMENTS ON THE A PRIORI ERROR ESTIMATION -> NOT FOR NOW

2.2.2 Using ϕ-FEM

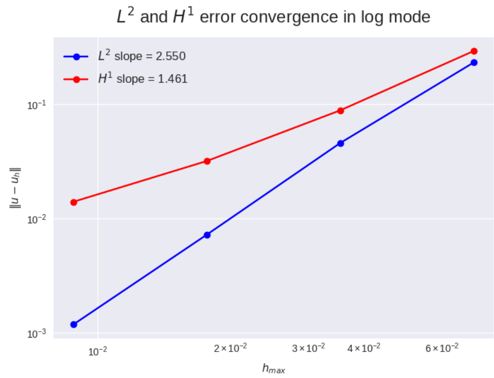
The basis for the Poisson problem in ϕ-FEM have been laid in (2.8). Multiple parameters must be defined in order to have a proper test case.

Let's repeat the same test case we did in classic FEM. The test case, defined in classic FEM as (2.9), is redefined below. The result is presented at fig. 2.5

$$\begin{cases} \mathcal{O} = [0,1] \times [0,1] \\ \phi(x,y) = -\frac{1}{8} + \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 \\ u(x,y) = \phi(x,y) \times \exp(x) \times \sin(2\pi y) \\ f(x,y) = -\frac{\partial^1 u}{\partial x^2}(x,y) - \frac{\partial^1 u}{\partial y^2}(x,y) \\ \sigma = 20 \end{cases} \tag{2.10}$$



(A) Solution



(B) Convergence study

FIGURE 2.5: Results obtained when applying the ϕ-FEM technique with stabilization to the Poisson equation. These results confirm what we expected, with better slopes compared to fig. 2.4.

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2.3 The elasticity equation

2.3.1 Problem description

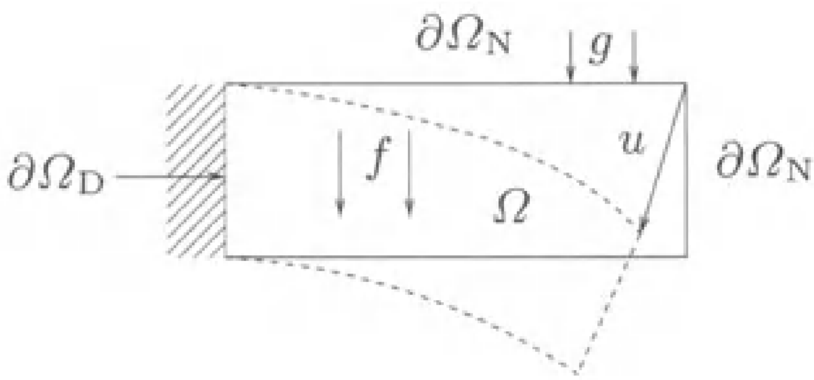

FIGURE 2.6: Illustration of the linear elasticity (small deformations) problem. The boundary $\partial\Omega_D$ is clamped, whereas a normal load $g : \partial\Omega_N \mapsto \mathbb{R}^3$ is imposed on $\partial\Omega_N$. f is the external force applied to the beam's particles i.e the gravity. f and g are assumed smooth enough(Ern and Guermond, 2013, p.153).

With $u : \Omega \mapsto \mathbb{R}^3$, σ and $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ respectively representing the displacement, the stress and the strain tensors, the model problem (illustrated at fig. 2.6) is as follows (Ern and Guermond, 2013, p.153):

$$\begin{cases} \nabla \cdot \sigma(u) + f = 0 & \text{in } \Omega \\ \sigma(u) = \lambda(\nabla \cdot u)\mathcal{I} + \mu(\nabla u + \nabla u^T) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega_D \\ \sigma(\mu) \cdot n = g & \text{on } \partial\Omega_N \end{cases} \tag{2.11}$$

Where λ and μ are the Lamé coefficients. With $d = 2, 3$, the test functions are taken in the functional space

$$V_{DN} = \{v \in \left[H^1(\Omega)\right]^d : v = 0 \text{ on } \partial\Omega_D\}$$

This leads to the following weak formulation in classic FEM

$$\text{Seek } u \in V_{DN} \text{ such that} \quad a(u, v) = l(v), \quad \forall v \in V \tag{2.12}$$

where a and l are defined as

$$a(u, v) = \int_\Omega \sigma(u) : \varepsilon(v)$$

and

$$l(v) = \int_\Omega f \cdot v + \int_{\partial\Omega_N} g \cdot v$$

Chapter 3

Milestones

Milestones	Steps	Tools involved	Deadline	Estimated number of hours	Effective number of hours
Understand the ϕ-FEM technique	**1. Read the documents related to ϕ-FEM** • Read the introductory paper • Read the Neumann boundary case		03/11/2020	20	
The Poisson equation	**1. Install FEniCS using Docker** • Install the most recent version • Test the installation with the demo case provided **2. Solve the Poisson equation using the classic FEM technique** • Use a simple domain (a unit disk) • Validate this step by differentiating a known solution and verifying the results **3. Perform the convergence study in norms L^2 and H^1** • According to the theory, the slopes must be respectively close to 2 and 1 **4. Solve the Poisson equation using the ϕ-FEM technique, without stabilising terms. Compare the results with the classic FEM technique** • Validate this step by comparison with the test cases in the paper **5. Repeat the preceding test, while applying stabilizing terms** • Validate this step by comparison with the paper • Repeat the exact test cases in the paper if necessary	Docker FEniCS	10/11/2020	25	
The elasticity equation	**1. Reformulate the elasticity equation using ϕ-FEM** • Take inspiration from the Poisson formulation **2. Solve the equation using FEniCS** • The method can be validated using academic cases as done in the papers • The method can also be validated on classical solid mechanics cases such as beams	Docker FEniCS	19/01/2021	25	
Simulations on organ geometries	**3. Find the geometries** **4. Integrate the results into SOFA**	Docker FEniCS SOFA	19/01/2021	25	

Chapter 4

Future works

Having tested the ϕ-FEM technique on the elasticity equation hence giving us a glimpse into its effectiveness on solid dynamics, it remains essential to completed the next steps in order to apply the technique to the real-time yet precise simulation of human organs. These steps could possibly be:

1. Test the technique on complex geometries
2. Benchmark the technique
3. Deploy the numerical implementation into the SOFA software

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