Simulation of soft tissues using innovative non-conforming Finite Element Methods (PhiFEM)

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Course: M2 CSMI Project 2020/2021

Institution: University of Strasbourg

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Project description

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Environment and context

What is the environment?

1. **Inria:** where digital health is a main research topic

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2. **MIMESIS:**
   - real-time simulations for per-operative guidance
   - data-driven simulation dedicated to patient-specific modeling

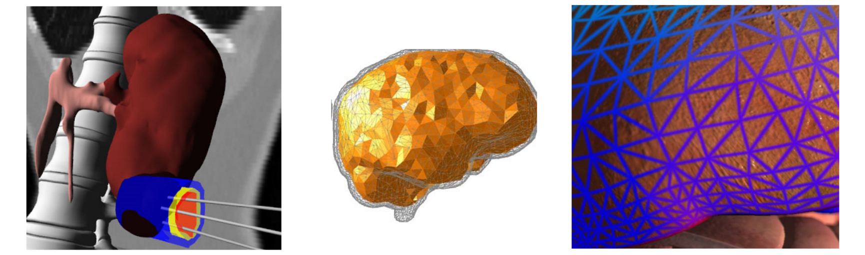

Figure : A few projects at MIMESI.

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  3. Develop a ϕ-FEM technique for the linear elasticity equation.

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  4. Use ϕ-FEM on body organ geometries.

Presentation of ϕ-FEM

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The classic FEM framework

What is FEM?

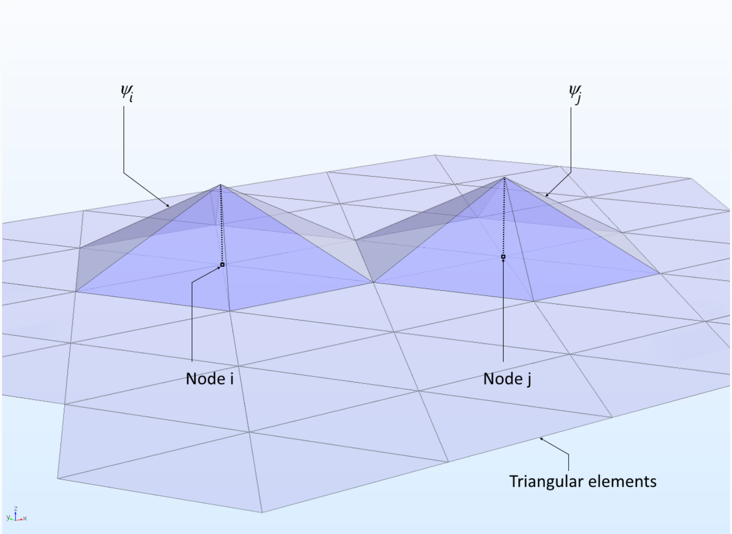

Figure : Nodal functions in the Finite Element Method (FEM) (Cyclopedia, 2017).

Presentation of ϕ-FEM

Immersed boundary methods

What are immersed boundary methods?

In the case of quasi-incompressibility for example, it is necessary to use hexahedral meshes in order to avoid locking phenomena. However, there is no 3D mesh generator capable of meshing any geometry in a hexahedral manner.

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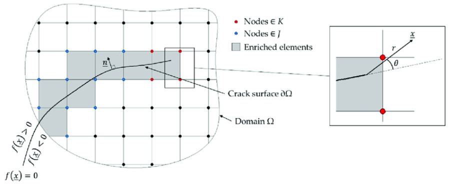

Figure : XFEM

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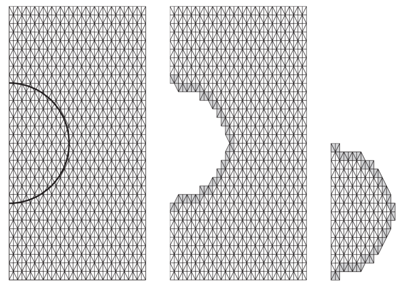

Figure : CutFEM

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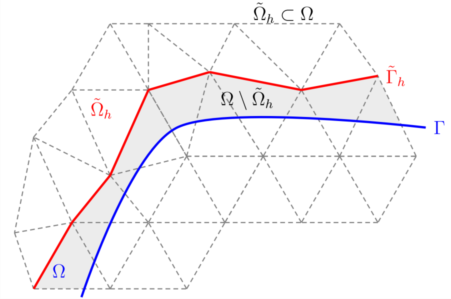

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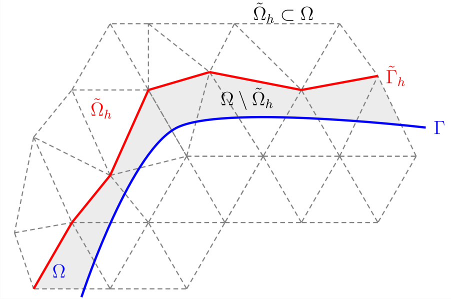

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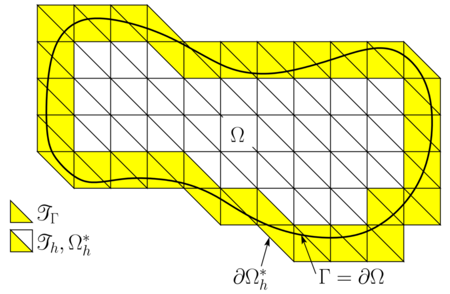

Figure : ϕ-FEM

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Note: These 3 techniques are efficient, but their implementation generates other difficulties, among other things, the quadrature (on the boundary). This lead to ϕ-FEM (Duprez and Lozinski, 2020).

The ideas behind ϕ-FEM

The main ideas are (for Dirichlet boundary conditions):

1. Define the domain using a level-set function ϕ.
2. Then make that function carry the solution e.g. $u = \phi w + g$.

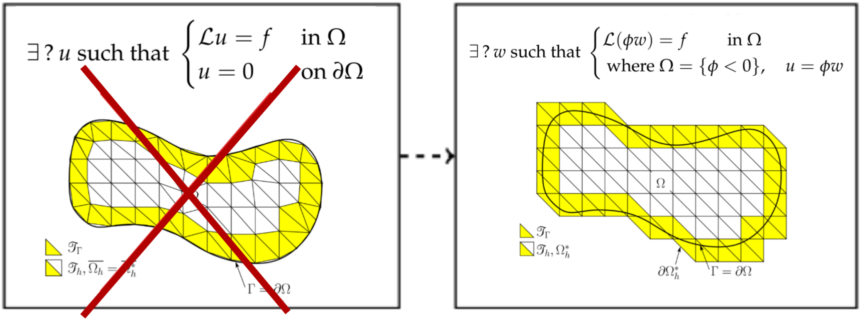

Figure : From Classic FEM (on the left) to ϕ-FEM (on the right)

Results

Results

The Poisson problem

Why the Poisson problem?

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

The Poisson problem (Poisson's equation) is used in various areas of Mathmetics and Physics (Newtonian gravity, electrostatics, fluid dynamics, etc.).

Figure : Siméon Denis Poisson (Wikipedia, 2020).

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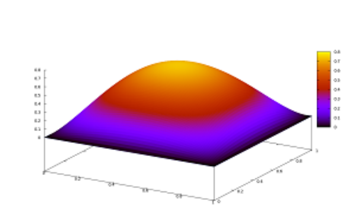

Figure : Displacement of a clampled membrane.

Figure : Siméon Denis Poisson (Wikipedia, 2020).

It can model the displacement u of a clamped membrane to which a constant force f is applied.

Simulating this equation is critical if we are to understand the dynamics of human organs.

Theoretical framework for the Poisson problem

The Poisson problem:

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Find $u_h \in V_h^{(k)}$ (subspace of splines in $H_0^1(\Omega_h)$) such that

$$a_h(u_h, v_h) = l_h(v_h), \;\; \forall\, v_h \in V_h^{(k)},$$

where

$$a_h(u, v) = \int_{\Omega_h} \nabla u \cdot \nabla v\,,$$

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ϕ-FEM (Duprez and Lozinski, 2020)

First, write $u_h = \phi_h w_h$. Then find $w_h \in V_h^{(k)}$ such that

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$$a_h(w, v) = \int_{\Omega_h} \nabla(\phi_h w)\cdot\nabla(\phi_h v) - \int_{\partial\Omega_h} \frac{\partial}{\partial n}(\phi_h w)\phi_h v + G_h(w, v) \,,$$

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$G_h(w, v)$ and $G_h^{rhs}(w, v)$ are stabilization terms, needed for the well-posedness of the discrete ϕ-FEM system.

$$G_h(w, v) := \sigma h \sum_{E\in\mathcal{F}_h^\Gamma} \int_E \left[\frac{\partial}{\partial n}(\phi_h w)\right]\left[\frac{\partial}{\partial n}(\phi_h v)\right] + \sigma h^2 \sum_{T\in\mathcal{T}_h^\Gamma} \int_T \Delta(\phi_h w)\Delta(\phi_h v)\,,$$

$$G_h^{rhs}(v) := -\sigma h^2 \sum_{T\in\mathcal{T}_h^\Gamma} \int_T f\Delta(\phi_h v)\,.$$

Numerical solution for the Poisson problem

Classic FEM

$$\begin{cases} \Omega = \left\{(x,y) \in \mathbb{R}^2 : \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 < \frac{1}{8}\right\} \\ u(x,y) = -\left(\frac{1}{8} - \left(x - \frac{1}{2}\right)^2 - \left(y - \frac{1}{2}\right)^2\right) \exp(x) \sin(2\pi y) \\ f(x,y) = -\frac{\partial^2 u}{\partial x^2}(x,y) - \frac{\partial^2 u}{\partial y^2}(x,y) \end{cases}$$

ϕ-FEM

$$\begin{cases} \mathcal{O} = [0,1] \times [0,1] \\ \phi(x,y) = -\frac{1}{8} + \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 \\ u(x,y) = \phi(x,y) \times \exp(x) \times \sin(2\pi y) \\ f(x,y) = -\frac{\partial^2 u}{\partial x^2}(x,y) - \frac{\partial^2 u}{\partial y^2}(x,y) \\ \sigma = 20 \end{cases}$$

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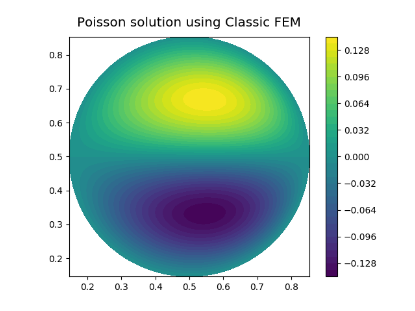


Figure : Classic FEM (39642 cells)

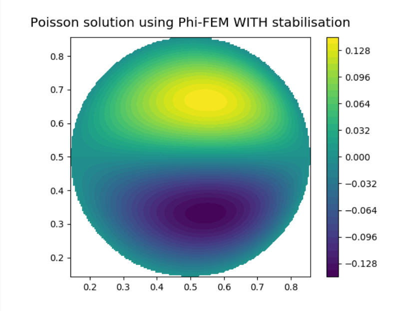
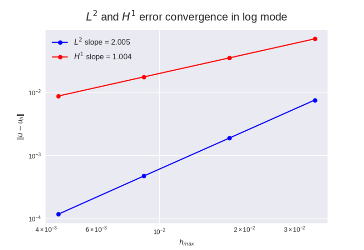


Figure : ϕ-FEM (39936 cells)

Convergence study for the Poisson Problem



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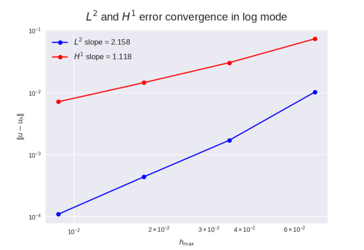

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Problem	Technique	L^2 slope	H^1 slope
Poisson	Classic FEM	2.005	1.004
	ϕ-FEM	2.158	1.118

Table : Convergence rates.

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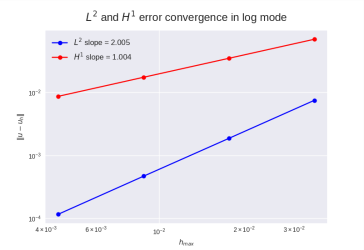

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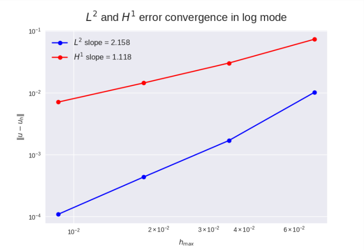

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Note: The convergence rates are clearly better with ϕ-FEM (Duprez, 2020).

Results

The elasticity equation

Why use the equations of linear elasticity?

$$\begin{cases} \nabla \cdot \sigma(u) + f = 0 & \text{in } \Omega \\ \sigma(u) = \lambda(\nabla \cdot u)\mathcal{I} + \mu(\nabla u + \nabla u^T) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Analysis of structures is one of the major activities of modern engineering, which likely makes the PDE modeling the deformation of elastic bodies the most popular PDE in the world (FEniCS, 2020).

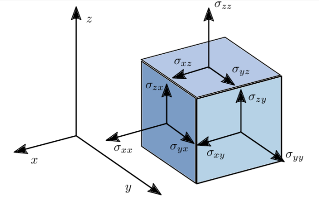

Figure : The nine quantities in the stress tensor $\sigma(\cdot)$ on a representative volume (Semanticscholar, 2020).

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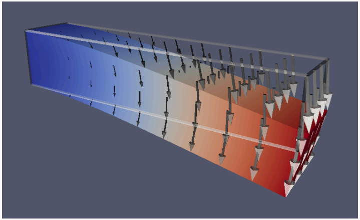

Figure : Gravity-induced deflection in a clamped beam (FEniCS, 2020).

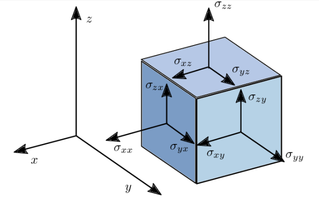

Figure : The nine quantities in the stress tensor $\sigma(\cdot)$ on a representative volume (Semanticscholar, 2020).

It takes into account variables not considered in the Poisson problem; namely the direction of the displacement vector u. Also, if we added the Neumann condition $\sigma(u) \cdot n = g_N$, it would represent a force applied on the boundary.

Theoretical framework for the elasticity equation

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Classic FEM

First $V = \left[H_0^1(\Omega)\right]^d$, and $\gamma G = g$. Then find $u \in G + V$ such that

$$a(u, v) = l(v), \quad \forall v \in V\,,$$

where a and l are defined as

$$a(u, v) = \int_\Omega \sigma(u) : \varepsilon(v)\,,$$

and

$$l(v) = \int_\Omega f \cdot v\,.$$

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ϕ-FEM (Duprez and Nzoyem, 2020)

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and

$$l(v) = \int_\Omega f \cdot (\phi v) + \int_{\partial\Omega} (\sigma(g) \cdot n) \cdot (\phi v) - \int_\Omega \sigma(g) : \varepsilon(\phi v) + G_{rhs}(v)\,.$$

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$G(w,v)$ and $G^{rhs}(w,v)$ are stabilization terms, needed for the well-posedness of the discrete ϕ-FEM system.

$$G(w,v)=\sigma_{pen}h^2\sum_{E\in\mathcal{T}_h^\Gamma}\int_E(\nabla\cdot\sigma(\phi w))\cdot(\nabla\cdot\sigma(\phi v))+\sigma_{pen}h\sum_{F\in\mathcal{F}_h^\Gamma}\int_F[\sigma(\phi w)\cdot n]\cdot[\sigma(\phi v)\cdot n]$$

$$G_{rhs}(v)=-\sigma_{pen}h^2\sum_{E\in\mathcal{T}_h^\Gamma}\int_E(f+\nabla\cdot\sigma(g))\cdot(\nabla\cdot\sigma(\phi v))-\sigma_{pen}h\sum_{F\in\mathcal{F}_h^\Gamma}\int_F[\sigma(g)\cdot n]\cdot[\sigma(\phi v)\cdot n]$$

Numerical solution for the elasticity equation

Classic FEM

$$\begin{cases} \Omega = \left\{(x,y) \in \mathbb{R}^2 : \left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 < \frac{1}{8}\right\} \\ u(x,y) = \begin{pmatrix} 2x + \sin(x)\exp(y) \\ \frac{x}{2} + \cos(x) - 1 \end{pmatrix} \\ f(x,y) = -\nabla \cdot \sigma(u(x,y)) \\ g(x,y) = u(x,y) \end{cases}$$

ϕ-FEM

$$\begin{cases} \mathcal{O} = [0,1] \times [0,1] \\ \phi(x,y) = -\frac{1}{8} + \left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 \\ u(x,y) = \begin{pmatrix} 2x + \sin(x)\exp(y) \\ \frac{x}{2} + \cos(x) - 1 \end{pmatrix} \\ f(x,y) = -\nabla \cdot \sigma(u(x,y)) \\ g(x,y) = u(x,y) + \phi(x,y) \begin{pmatrix} \sin(x) \\ \exp(y) \end{pmatrix} \\ \sigma_{pen} = 20 \end{cases}$$

Numerical solution for the elasticity equation

Classic FEM

$$
\begin{cases}
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\end{cases}
$$

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$$

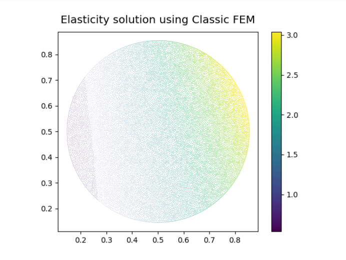


Figure : Classic FEM

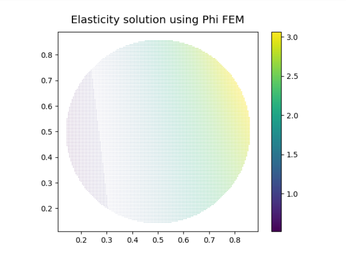
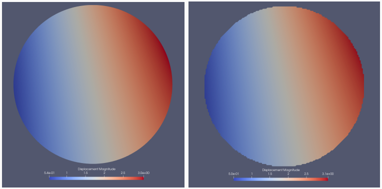


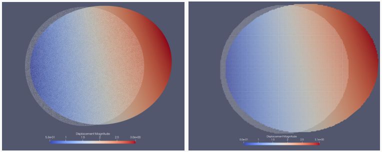
Figure : ϕ-FEM

Numerical solution for the elasticity equation (cont.)



(a) Classic FEM **(b)** ϕ-FEM

Figure : Solution (deformation) magnitude in Paraview.



(a) Classic FEM **(b)** ϕ-FEM

Figure : Solution warped by vector in Paraview.

Convergence study for the elasticity equation

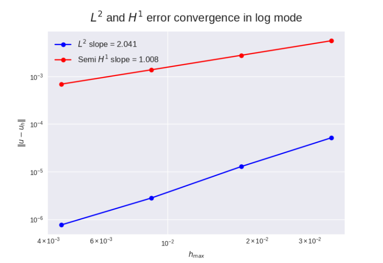


Figure : Classic FEM

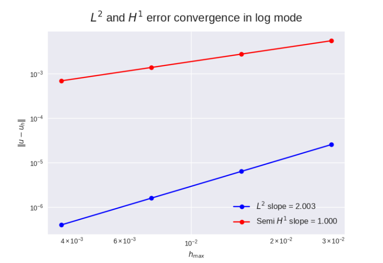


Figure : ϕ-FEM

Problem	Technique	L^2 slope	H^1 slope
Elasticity	Classic FEM	2.041	1.008
	ϕ-FEM	2.003	1.000

Table : Convergence rates.

Convergence study for the elasticity equation

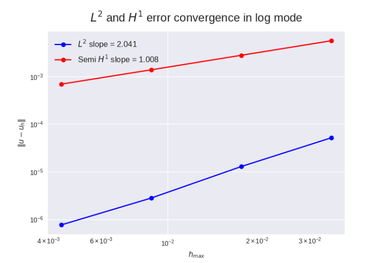
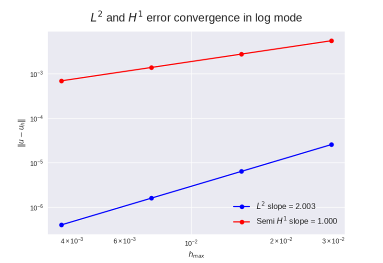


Figure : Classic FEM



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Elasticity	Classic FEM	2.041	1.008
	ϕ-FEM	2.003	1.000

Table : Convergence rates.

Note: The convergence rates are almost identical in both cases (Duprez, 2020).

Project summary

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Work done

What has been done?

Based on the objectives, the deadlines, and the time estimates we set early in the project:

- ✓ **Understanding ϕ-FEM** : November 3rd, 2020 : 10 hours

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Possible explorations:

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Possible explorations:

- Test the technique on complex geometries.
- Benchmark the technique for speed efficiency.
- Deploy the numerical implementation into the SOFA software.
- Implement the Neumann case (Duprez et al., 2020).

Project summary

Delivered documents

What did I deliver?

As promised:

1. A typewritten report

What did I deliver?

As promised:

1. A typewritten report
2. A Python code base

All are available on this GitHub repository: `https://github.com/master-csmi/2020-m2-mimesis`

References

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Thank you for your kind attention ☺ !
Questions ?