Simulation of soft tissues using innovative non-conforming Finite Elements Methods (PhiFEM)

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M2 CSMI Project Course 2020/2021

University of Strasbourg

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Project description

Project description

Environment and context

What is the environment?

1. **Inria:** where **digital health** is a main research topic
2. **MIMESIS:**
   - real-time simulations for per-operative guidance
   - data-driven simulation dedicated to patient-specific modeling

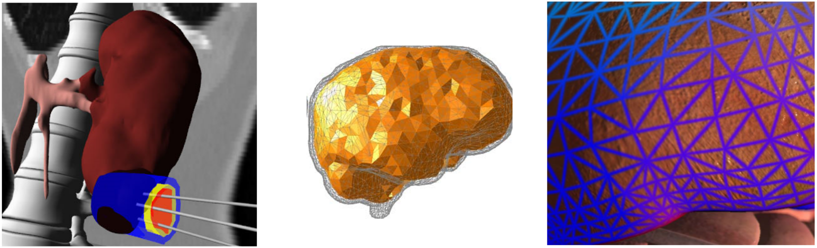


Figure : A few projects at MIMESI.

What are we trying to do?

The goal is to **develop a Finite Elements Method (FEM) adapted to computer-assisted surgery**.

The FEM should be:

- quick;
- precise;
- patient-specific.

Project description

Objectives

What is our goal?

- **Main objective**: develop a ϕ-FEM technique for the dynamics of soft tissues.
- **Intermediate objectives**:
  1. Understand the ϕ-FEM technique in question.
  2. Reproduce the results from a preliminary study (the Poisson problem).
  3. Develop a ϕ-FEM technique for the linear elasticity equation.
  4. Use ϕ-FEM on body organ geometries.

Presentation of ϕ-FEM

Presentation of ϕ-FEM

The classic FEM framework

What is FEM?

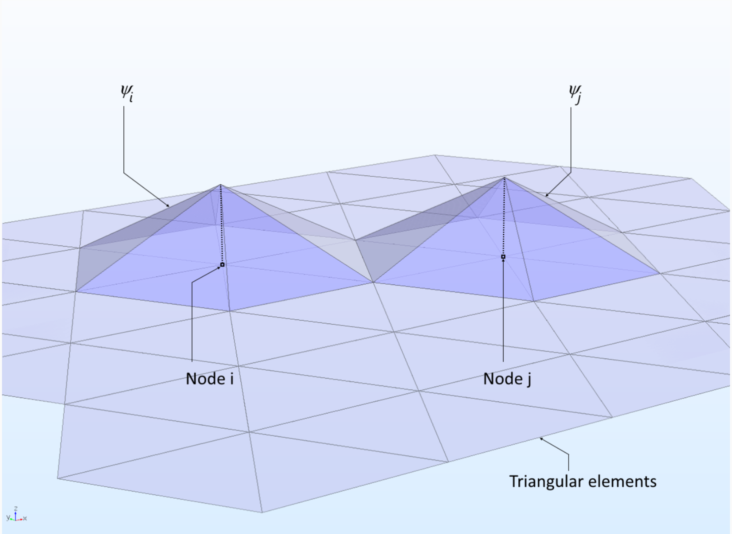


Figure : Finite Elements Method (FEM) principle (Cyclopedia, 2017).

Presentation of ϕ-FEM

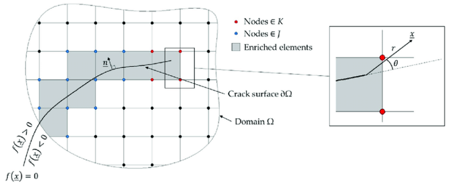
Immersed boundary methods

What are immersed boundary methods?

In the case of quasi-incompressibility for example, it is necessary to use hexahedral meshes in order to avoid locking phenomena. However, there is no 3D mesh generator capable of meshing any geometry in a hexahedral manner.

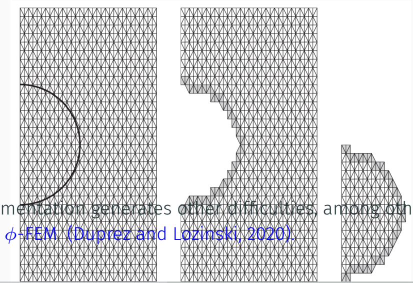
One answer is to use **immersed boundary methods**. They can handle:

- complex geometries;
- non-body conforming grids;
- moving/evolving boundaries.

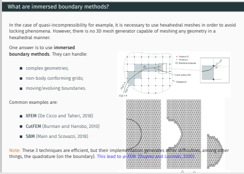


Common examples are:

- **XFEM** (De Cicco and Taheri, 2018)
- **CutFEM** (Burman and Hansbo, 2010)
- **SBM** (Main and Scovazzi, 2018)



Note: These 3 techniques are efficient, but their implementation generates other difficulties, among other things, the quadrature (on the boundary). This lead to ϕ-FEM (Duprez and Lozinski, 2020).



quasi-incompressibility: almost incompressible OR (for fluids) these are mixtures in which the density depends on the concentration of one of their components.locking: finite element methods produce results showing stiffness far greater then would be expected, rendering the results useless.hexahedral: geometric cell complex composed of 0-dimensional nodes, 1-dimensional edges, 2-dimensional quadrilaterals (residing in R3), and 3-dimensional hexahedra (cubes, parallelepipeds, etc.) Les 3 premieres methodes demandes des integrations sur la forntiere. PhiFEM ne demande pas de telle integrations; la quadrature se passe sur la frontiere fictive.

What is ϕ-FEM ?

The main ideas are (for Dirichlet boundary conditions):

1. Define the domain using a level-set function ϕ.
2. Then make that function carry the solution: $u = \phi w + g$.

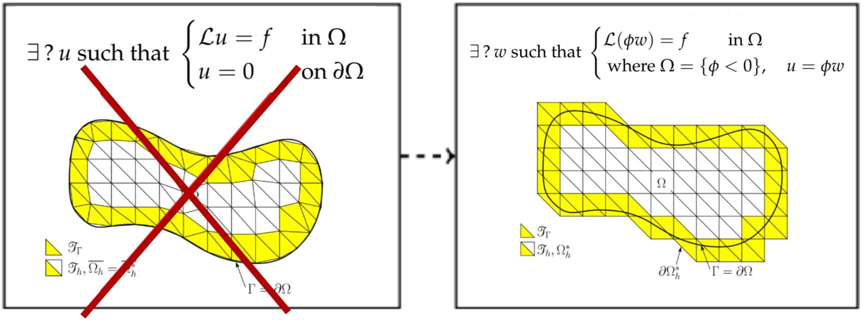


Figure : From Classic FEM (on the left) to ϕ-FEM (on the right)

Results

Results

The Poisson problem

Why the Poisson problem?

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

The Poisson problem (Poisson's equation) is used in various areas of Mathmetics and Physics (Newtonian gravity, electrostatics, fluid dynamics, etc.).

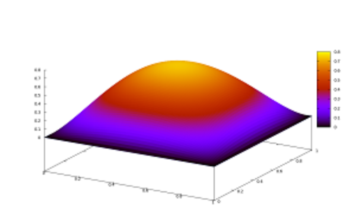


Figure : Displacement of a clampled membrane.

Figure : Siméon Denis Poisson (Wikipedia, 2020).

It can model the displacement u of a clamped membrane to which a constant force f is applied.

Simulating this equation is critical if we are to understand the dynamics of human organs.

Theoretical framework for the Poisson problem

The Poisson problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

Classic FEM

Find $u_h \in V_h^{(k)}$ (subspace of splines in $H_0^1(\Omega_h)$) such that

$$a_h(u_h, v_h) = l_h(v_h), \;\; \forall\, v_h \in V_h^{(k)},$$

where

$$a_h(u, v) = \int_{\Omega_h} \nabla u \cdot \nabla v \,,$$

$$l_h(v) = \int_{\Omega_h} fv \,.$$

ϕ-FEM (Duprez and Lozinski, 2020)

First, write $u_h = \phi_h w_h$. Then find $w_h \in V_h^{(k)}$ such that

$$a_h(w_h, v_h) = l_h(v_h) \text{ for all } v_h \in V_h^{(k)},$$

where

$$a_h(w, v) = \int_{\Omega_h} \nabla(\phi_h w)\cdot\nabla(\phi_h v) - \int_{\partial\Omega_h} \frac{\partial}{\partial n}(\phi_h w)\phi_h v + G_h(w, v) \,,$$

$$l_h(v) = \int_{\Omega_h} f\phi_h v + G_h^{rhs}(v).$$

$G_h(w, v)$ and $G_h^{rhs}(w, v)$ are stabilization terms, needed for the well-posedness of the discrete ϕ-FEM system.

$$G_h(w, v) := \sigma h \sum_{E\in\mathcal{F}_h^\Gamma} \int_E \left[\frac{\partial}{\partial n}(\phi_h w)\right]\left[\frac{\partial}{\partial n}(\phi_h v)\right] + \sigma h^2 \sum_{T\in\mathcal{T}_h^\Gamma} \int_T \Delta(\phi_h w)\Delta(\phi_h v) \,,$$

$$G_h^{rhs}(v) := -\sigma h^2 \sum_{T\in\mathcal{T}_h^\Gamma} \int_T f\Delta(\phi_h v) \,.$$

Numerical solution for the Poisson problem

Classic FEM

$$\begin{cases} \Omega = \left\{(x,y) \in \mathbb{R}^2 : \left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 < \frac{1}{8}\right\} \\ u(x,y) = -\left(\frac{1}{8} - \left(x-\frac{1}{2}\right)^2 - \left(y-\frac{1}{2}\right)^2\right) \exp(x)\sin(2\pi y) \\ f(x,y) = -\frac{\partial^2 u}{\partial x^2}(x,y) - \frac{\partial^2 u}{\partial y^2}(x,y) \end{cases}$$

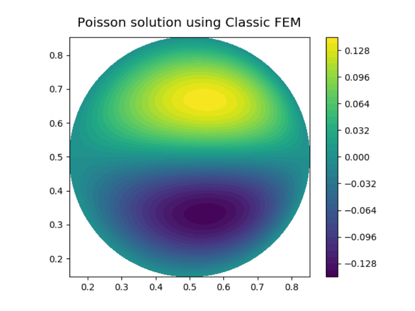


Figure : Classic FEM (39642 cells)

ϕ-FEM

$$\begin{cases} \mathcal{O} = [0,1] \times [0,1] \\ \phi(x,y) = -\frac{1}{8} + \left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 \\ u(x,y) = \phi(x,y) \times \exp(x) \times \sin(2\pi y) \\ f(x,y) = -\frac{\partial^2 u}{\partial x^2}(x,y) - \frac{\partial^2 u}{\partial y^2}(x,y) \\ \sigma = 20 \end{cases}$$

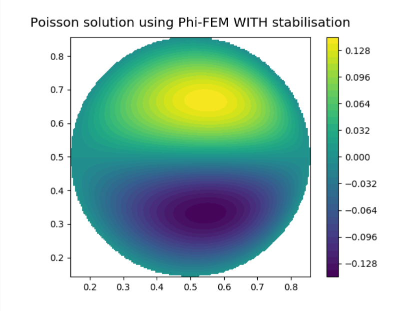
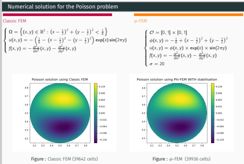


Figure : ϕ-FEM (39936 cells)



Remarque sur la non-smoothness de la solution sur les bords. C'est parcequ'un disque c'est facilement maillable!

Convergence study for the Poisson Problem

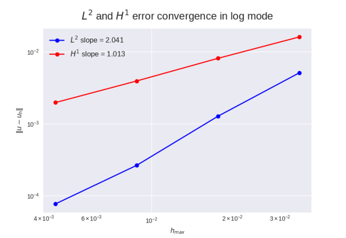


Figure : Classic FEM

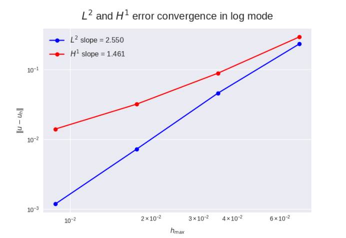


Figure : ϕ-FEM

Problem	Technique	L^2 slope	H^1 slope
Poisson	Classic FEM	2.041	1.013
	ϕ-FEM	2.550	1.461

Table : Convergence rates.

Note: The convergence rates are clearly better with ϕ-FEM. However, we can see that for relatively coarse mehses ($h_{max} = 10^{-2}$ for example), the errors are smaller in Classic FEM. That said, if ϕ is taken in $\mathbb{P}^2$ as a second degree polynomial (as opposed to $\mathbb{P}^1$ as we did), the errors at $h_{max} = 10^{-2}$ are the same as with Classic FEM (Duprez, 2020).

Results

The elasticity equation

Why use the equations of linear elasticity?

$$\begin{cases} \nabla \cdot \sigma(u) + f = 0 & \text{in } \Omega \\ \sigma(u) = \lambda(\nabla \cdot u)\mathcal{I} + \mu(\nabla u + \nabla u^T) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Analysis of structures is one of the major activities of modern engineering, which likely makes the PDE modeling the deformation of elastic bodies the most popular PDE in the world (FEniCS, 2020).

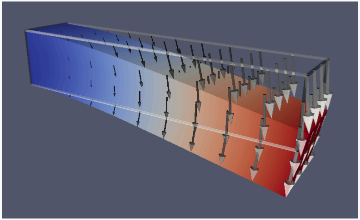


Figure : Gravity-induced deflection in a clamped beam (FEniCS, 2020).

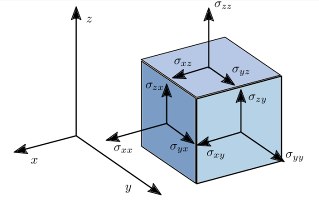


Figure : The nine quantities in the stress tensor $\sigma(\cdot)$ on a representative volume (Semanticscholar, 2020).

It takes into account variables not considered in the Poisson problem; namely the direction of the displacement vector u. Also, if we added the Neumann condition $\sigma(u) \cdot n = g_N$, it would represent a force applied on the boundary.

Theoretical framework for the elasticity equation

The elasticity equation:

$$\begin{cases} \nabla \cdot \sigma(u) + f = 0 & \text{in } \Omega \\ \sigma(u) = \lambda(\nabla \cdot u)\mathcal{I} + \mu(\nabla u + \nabla u^T) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{2}$$

Classic FEM

First $V = \left[H_0^1(\Omega)\right]^d$, and $\gamma G = g$. Then find $u \in G + V$ such that

$$a(u, v) = l(v), \quad \forall v \in V,$$

where a and l are defined as

$$a(u, v) = \int_\Omega \sigma(u) : \varepsilon(v),$$

and

$$l(v) = \int_\Omega f \cdot v.$$

ϕ-FEM (Duprez and Nzoyem, 2020)

First, $u_h = \phi w + g$. Then find $w \in V$ such that:

$$a(w, v) = l(v), \quad \forall v \in V,$$

where a and l are defined as

$$a(w, v) = \int_\Omega \sigma(\phi w) : \varepsilon(\phi v) - \int_{\partial\Omega} (\sigma(\phi w) \cdot n) \cdot (\phi v) + G(w, v),$$

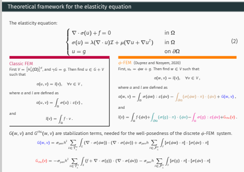
and

$$l(v) = \int_\Omega f \cdot (\phi v) + \int_{\partial\Omega} (\sigma(g) \cdot n) \cdot (\phi v) - \int_\Omega \sigma(g) : \varepsilon(\phi v) + G_{rhs}(v).$$

$G(w, v)$ and $G^{rhs}(w, v)$ are stabilization terms, needed for the well-posedness of the discrete ϕ-FEM system.

$$G(w, v) = \sigma_{pen} h^2 \sum_{E \in \mathcal{T}_h^\Gamma} \int_E (\nabla \cdot \sigma(\phi w)) \cdot (\nabla \cdot \sigma(\phi v)) + \sigma_{pen} h \sum_{F \in \mathcal{F}_h^\Gamma} \int_F [\sigma(\phi w) \cdot n] \cdot [\sigma(\phi v) \cdot n]$$

$$G_{rhs}(v) = -\sigma_{pen} h^2 \sum_{E \in \mathcal{T}_h^\Gamma} \int_E (f + \nabla \cdot \sigma(g)) \cdot (\nabla \cdot \sigma(\phi v)) - \sigma_{pen} h \sum_{F \in \mathcal{F}_h^\Gamma} \int_F [\sigma(g) \cdot n] \cdot [\sigma(\phi v) \cdot n]$$



Rappler qu'il s'agit d'une formulation variationnelle dans l'espace H1, et non l'approximation éléments finis. Ce qui fait qu'on peut enlever le subscript h.

Numerical solution for the elasticity equation

Classic FEM

$$\begin{cases} \Omega = \left\{(x,y) \in \mathbb{R}^2 : (x-\frac{1}{2})^2 + (y-\frac{1}{2})^2 < \frac{1}{8}\right\} \\ u(x,y) = \begin{pmatrix} 2x + \sin(x)\exp(y) \\ \frac{x}{2} + \cos(x) - 1 \end{pmatrix} \\ f(x,y) = -\nabla \cdot \sigma(u(x,y)) \\ g(x,y) = u(x,y) \end{cases}$$

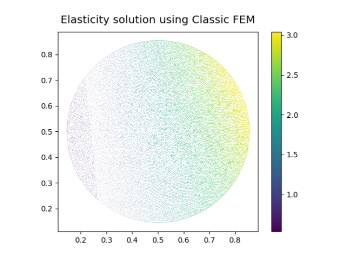


Figure : Classic FEM

ϕ-FEM

$$\begin{cases} \mathcal{O} = [0,1] \times [0,1] \\ \phi(x,y) = -\frac{1}{8} + (x-\frac{1}{2})^2 + (y-\frac{1}{2})^2 \\ u(x,y) = \begin{pmatrix} 2x + \sin(x)\exp(y) \\ \frac{x}{2} + \cos(x) - 1 \end{pmatrix} \\ f(x,y) = -\nabla \cdot \sigma(u(x,y)) \\ g(x,y) = u(x,y) + \phi(x,y) \begin{pmatrix} \sin(x) \\ \exp(y) \end{pmatrix} \\ \sigma_{pen} = 20 \end{cases}$$

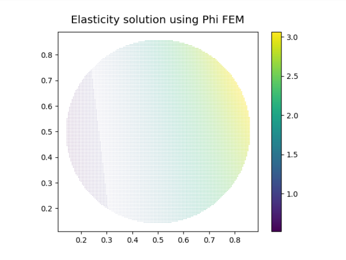
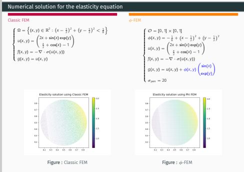
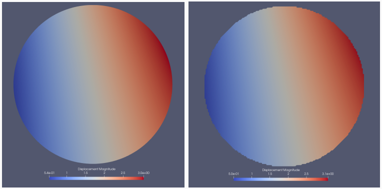


Figure : ϕ-FEM



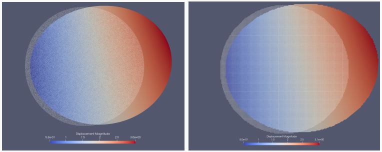
Signaler que g n’est pas egale à u sur les bords, ceci afin d’introduire une perturbation qui sera résolue par les termes de penalisation.

Numerical solution for the elasticity equation (cont.)



(a) Classic FEM (b) ϕ-FEM

Figure : Solution (deformation) magnitude in Paraview.



(a) Classic FEM (b) ϕ-FEM

Figure : Solution warped by vector in Paraview.

Convergence study for the elasticity equation

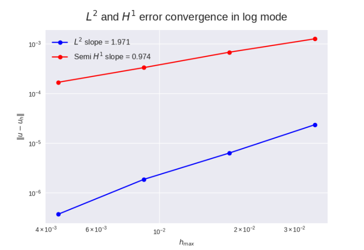


Figure : Classic FEM

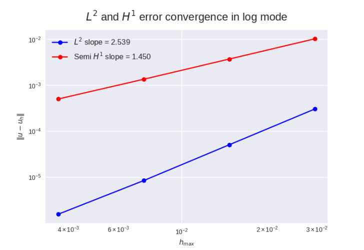


Figure : ϕ-FEM

Problem	Technique	L^2 slope	H^1 slope
Elasticity	Classic FEM	1.971	0.974
	ϕ-FEM	2.539	1.450

Table : Convergence rates.

Note: The convergence rates are clearly better with ϕ-FEM. However, our tests where conducted on relatively coarse mehses ($h_{max} = 10^{-2}$), in which the errors are smaller in Classic FEM.

2021-01-27

Convergence study for the elasticity equation

Figure : Classic FEM

Figure : ϕ-FEM

Problem	Technique	L^2 slope	H^1 slope
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Rappeller qu'il s'agit des semi-normes H1.

Project summary

Project summary

Work done

What has been done?

Based on the objectives, the deadlines, and the time estimates we set early in the project:

- ✓ **Understanding ϕ-FEM** : November 3rd, 2020 : 10 hours
- ✓ **Implementing the Poisson problem**: November 10, 2020 : 50 hours
- ✓ **Implementing the elasticity equation**: January 19, 2021 : 30 hours
- × **Running Simulations on organ geometries**: January 19, 2021 : 00 hours

Possible explorations:

- Test the technique on complex geometries.
- Benchmark the technique for speed efficiency.
- Deploy the numerical implementation into the SOFA software.
- Implement the Neumann case (Duprez et al., 2020).

Project summary

Delivered documents

What did I deliver?

As promised:

1. A typewritten report
2. A Python code base

All are available on this GitHub repository: `https://github.com/master-csmi/2020-m2-mimesis`

References

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Thank you for your kind attention ☺ !

Questions ?