Simulation of soft tissues using innovative non-conforming Finite Elements Methods

Author : Desmond Roussel NZOYEM

Supervisors : Michel DUPREZ, Stéphane COTIN
Referent teacher : Christophe PRUD'HOMME

M2 CSMI Project Course 2020/2021

University of Strasbourg

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Project description

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Environment and context

What is the environment?

1. **Inria:** where digital health is a main research topic

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2. **MIMESIS:**
   - real-time simulations for per-operative guidance
   - data-driven simulation dedicated to patient-specific modeling

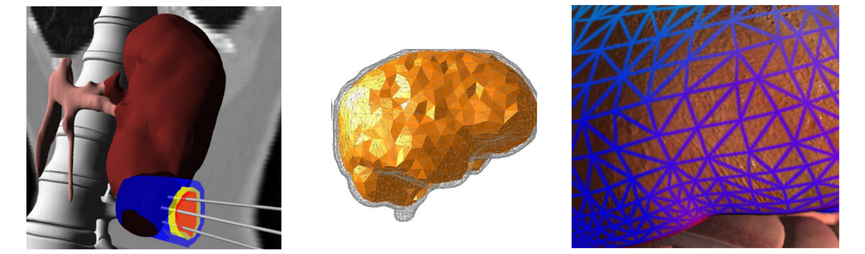

Figure 1: A few projects at MIMESI.

What are we trying to do?

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Project description

Objectives

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  3. Develop a ϕ-FEM technique for the linear elasticity equation.

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- **Intermediate objectives**:
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  2. Reproduce the results from a preliminary study (the Poisson equation).
  3. Develop a ϕ-FEM technique for the linear elasticity equation.
  4. Use ϕ-FEM on body organ geometries.

Presentation of ϕ-FEM

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The classic FEM framework

What is FEM?

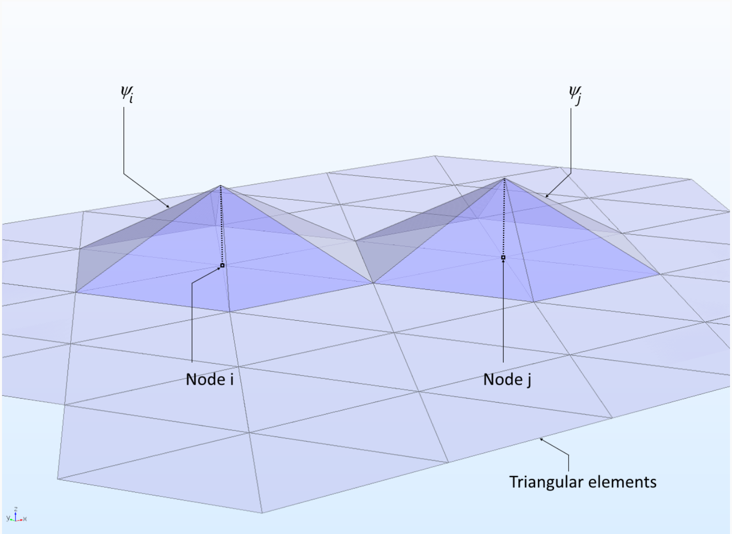

Figure 2: Finite Elements Method (FEM) principle (Cyclopedia, 2017).

Presentation of ϕ-FEM

Immersed boundary methods

What are immersed boundary methods?

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Common examples are:

- **XFEM** (De Cicco and Taheri, 2018)

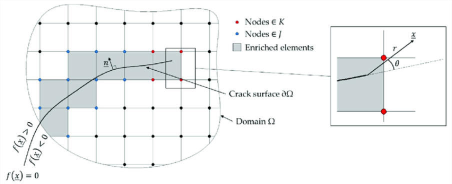

Figure 3: XFEM

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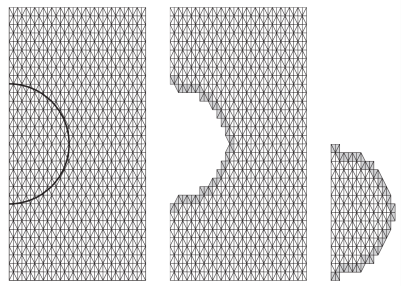

Figure 3: CutFEM

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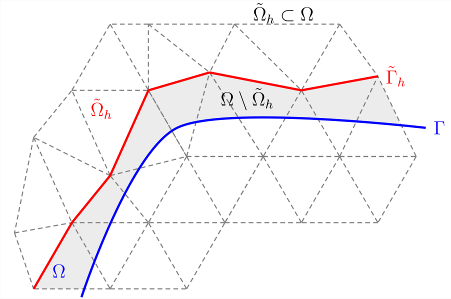

Figure 3: SBM

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- **CutFEM** (Burman et al., 2015)
- **SBM** (Atallah et al., 2020)
- **ϕ-FEM** (Duprez and Lozinski, 2020)

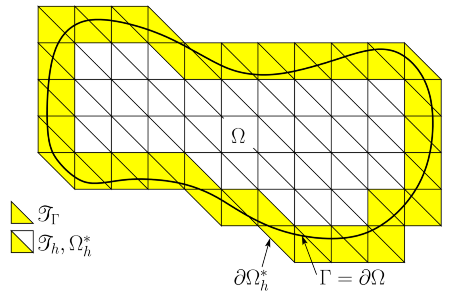

Figure 3: ϕ-FEM

What is ϕ-FEM ?

The main ideas are (for Dirichlet boundary conditions):

1. Define the domain using a level-set function ϕ.
2. Then make that function carry the solution: $u = \phi w + g$.

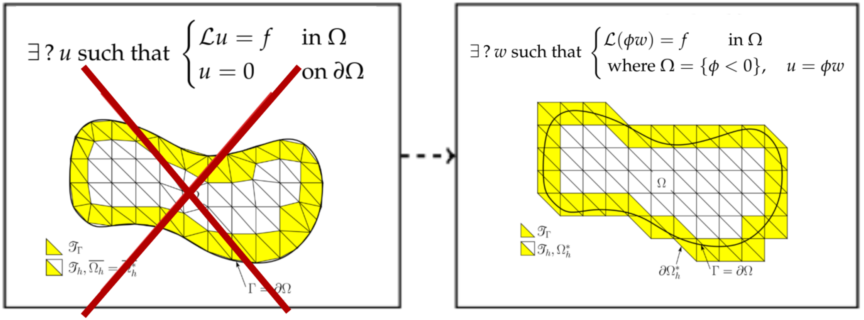

Figure 4: From Classic FEM (on the left) to ϕ-FEM (on the right)

Results

Results

The Poisson problem

Theoretical framework for the Poisson problem

The Poisson problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

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Classic FEM

Find $u_h \in V_h^{(k)}$ such that

$$a_h(u_h, v_h) = l_h(v_h), \ \ \forall\, v_h \in V_h^{(k)},$$

where

$$a_h(u, v) = \int_{\Omega_h} \nabla u \cdot \nabla v$$

$$l_h(v) = \int_{\Omega_h} fv\,.$$

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ϕ-FEM (Duprez and Lozinski, 2020)

First, write $u_h = \phi_h w_h$. Then find $w_h \in V_h^{(k)}$ such that

$$a_h(w_h, v_h) = l_h(v_h) \text{ for all } v_h \in V_h^{(k)},$$

where

$$a_h(w,v) = \int_{\Omega_h} \nabla(\phi_h w)\cdot\nabla(\phi_h v) - \int_{\partial\Omega_h} \frac{\partial}{\partial n}(\phi_h w)\phi_h v + G_h(w,v)$$

$$l_h(v) = \int_{\Omega_h} f\phi_h v + G_h^{rhs}(v).$$

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$$l_h(v) = \int_{\Omega_h} f\phi_h v + G_h^{rhs}(v).$$

$$G_h(w, v) := \sigma h \sum_{E\in\mathcal{F}_h^\Gamma} \int_E \left[\frac{\partial}{\partial n}(\phi_h w)\right]\left[\frac{\partial}{\partial n}(\phi_h v)\right] + \sigma h^2 \sum_{T\in\mathcal{T}_h^\Gamma} \int_T \Delta(\phi_h w)\Delta(\phi_h v)\,,$$

$$G_h^{rhs}(v) := -\sigma h^2 \sum_{T\in\mathcal{T}_h^\Gamma} \int_T f\Delta(\phi_h v)\,.$$

Numerical solution for the Poisson problem

Classic FEM

$$\begin{cases} \Omega = \left\{(x,y) \in \mathbb{R}^2 : \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 < \frac{1}{8}\right\} \\ u(x,y) = -\left(\frac{1}{8} - \left(x - \frac{1}{2}\right)^2 - \left(y - \frac{1}{2}\right)^2\right) \exp(x)\sin(2\pi y) \\ f(x,y) = -\frac{\partial^2 u}{\partial x^2}(x,y) - \frac{\partial^2 u}{\partial y^2}(x,y) \end{cases}$$

ϕ-FEM

$$\begin{cases} \mathcal{O} = [0,1] \times [0,1] \\ \phi(x,y) = -\frac{1}{8} + \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 \\ u(x,y) = \phi(x,y) \times \exp(x) \times \sin(2\pi y) \\ f(x,y) = -\frac{\partial^2 u}{\partial x^2}(x,y) - \frac{\partial^2 u}{\partial y^2}(x,y) \\ \sigma = 20 \end{cases}$$

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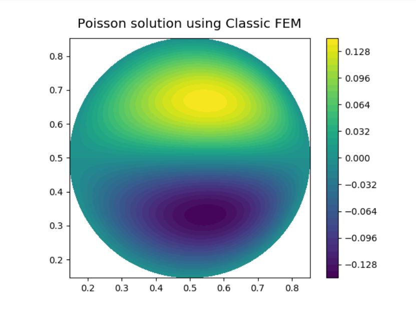

Figure 5: Classic FEM (39642 cells)

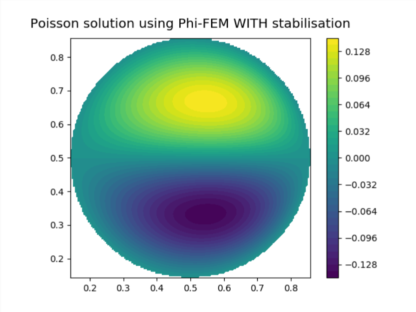

Figure 6: ϕ-FEM (39936 cells)

Convergence study for the Poisson Problem

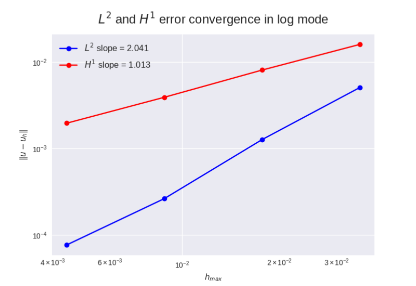

Figure 7: Classic FEM

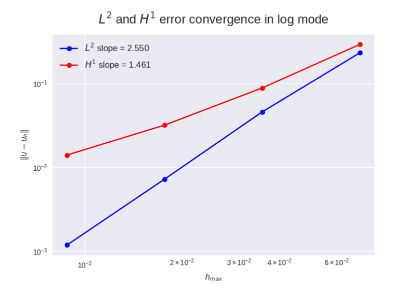

Figure 8: ϕ-FEM

Problem	Technique	L^2 slope	H^1 slope
Poisson	Classic FEM	2.041	1.013
	ϕ-FEM	2.550	1.461

Table 1: Convergence rates.

Results

The elasticity equation

Theoretical framework for the elasticity equation

The elasticity equation:

$$\begin{cases} \nabla \cdot \sigma(u) + f = 0 & \text{in } \Omega \\ \sigma(u) = \lambda(\nabla \cdot u)\mathcal{I} + \mu(\nabla u + \nabla u^T) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{2}$$

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Classic FEM

First $V = \left[H_0^1(\Omega)\right]^d$, and $\gamma G = g$. Then find $u \in G + V$ such that

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and

$$l(v) = \int_\Omega f \cdot (\phi v) + \int_{\partial\Omega} (\sigma(g) \cdot n) \cdot (\phi v) - \int_\Omega \sigma(g) : \varepsilon(\phi v) + G_{rhs}(v)$$

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$$G(w, v) = \sigma_{pen} h^2 \sum_{E \in \mathcal{T}_h^\Gamma} \int_E (\nabla \cdot \sigma(\phi w)) \cdot (\nabla \cdot \sigma(\phi v)) + \sigma_{pen} h \sum_{F \in \mathcal{F}_h^\Gamma} \int_F [\sigma(\phi w) \cdot n] \cdot [\sigma(\phi v) \cdot n]$$

$$G_{rhs}(v) = -\sigma_{pen} h^2 \sum_{E \in \mathcal{T}_h^\Gamma} \int_E (f + \nabla \cdot \sigma(g)) \cdot (\nabla \cdot \sigma(\phi v)) - \sigma_{pen} h \sum_{F \in \mathcal{F}_h^\Gamma} \int_F [\sigma(g) \cdot n] \cdot [\sigma(\phi v) \cdot n]$$

Numerical solution for the elasticity equation

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Numerical solution for the elasticity equation

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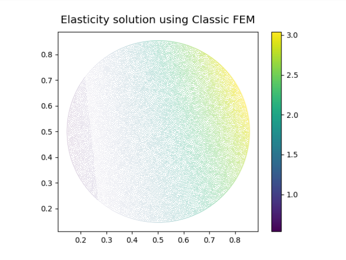


Figure 9: Classic FEM

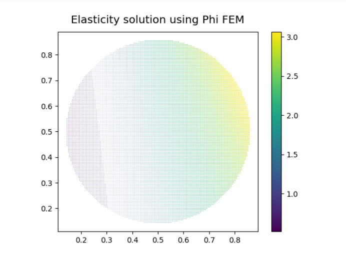
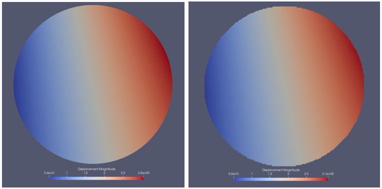


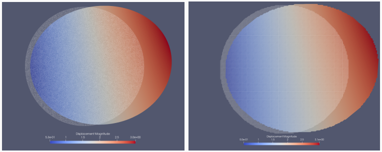
Figure 10: ϕ-FEM

Numerical solution for the elasticity equation (cont.)



(a) Classic FEM (b) ϕ-FEM

Figure 11: Solution (deformation) magnitude in Paraview.



(a) Classic FEM (b) ϕ-FEM

Figure 12: Solution warped by vector in Paraview.

Convergence study for the elasticity equation

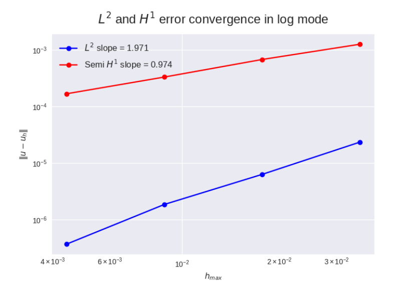


Figure 13: Classic FEM

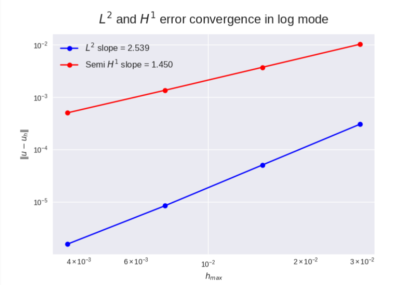


Figure 14: ϕ-FEM

Problem	Technique	L^2 slope	H^1 slope
Elasticity	Classic FEM	1.971	0.974
	ϕ-FEM	2.539	1.450

Table 2: Convergence rates.

Project summary

Project summary

Work done

What has been done?

Based on the objectives, the deadlines, and the time estimates we set early in the project:

- ✓ **Understanding ϕ-FEM** : November 3rd, 2020 : 10 hours

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Possible explorations:

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- Benchmark the technique for speed efficiency.

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- Test the technique on complex geometries.
- Benchmark the technique for speed efficiency.
- Deploy the numerical implementation into the SOFA software.

What has been done?

Based on the objectives, the deadlines, and the time estimates we set early in the project:

- ✓ **Understanding ϕ-FEM** : November 3rd, 2020 : 10 hours
- ✓ **Implementing the Poisson equation**: November 10, 2020 : 50 hours
- ✓ **Implementing the elasticity equation**: January 19, 2021 : 30 hours
- × **Simulations on organ geometries**: January 19, 2021 : 00 hours

Possible explorations:

- Test the technique on complex geometries.
- Benchmark the technique for speed efficiency.
- Deploy the numerical implementation into the SOFA software.
- Implement the Neuman case (Duprez et al., 2020).

Project summary

Delivered documents

What did I deliver?

As promised:

1. A typewritten report

What did I deliver?

As promised:

1. A typewritten report
2. A Python code base

All are available on this GitHub repository: *https://github.com/master-csmi/2020-m2-mimesis*

References

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De Cicco, Davide and Farid Taheri (2018). “Delamination buckling and crack propagation simulations in fiber-metal laminates using xFEM and cohesive elements”. In: *Applied Sciences* 8.12, p. 2440.

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Thank you for your kind attention ☺ !
Questions ?