

Simulation of soft tissues using innovative non-conforming Finite Element Methods (PhiFEM)

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Institution: University of Strasbourg

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Project description

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Environment and context

What is the environment?

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What is the environment?

- 1 Inria: where **digital health** is a main research topic
 - 2 MIMESIS:
 - real-time simulations for per-operative guidance
 - data-driven simulation dedicated to patient-specific modeling

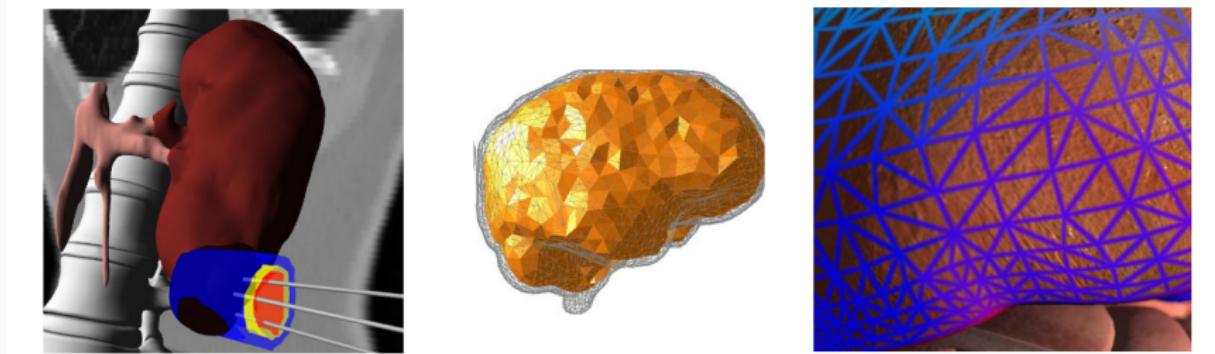


Figure : A few projects at MIMESI.

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Project description

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 - 3 Develop a ϕ -FEM technique for the linear elasticity equation.
 - 4 Use ϕ -FEM on body organ geometries.

Presentation of ϕ -FEM

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The classic FEM framework

What is FEM?

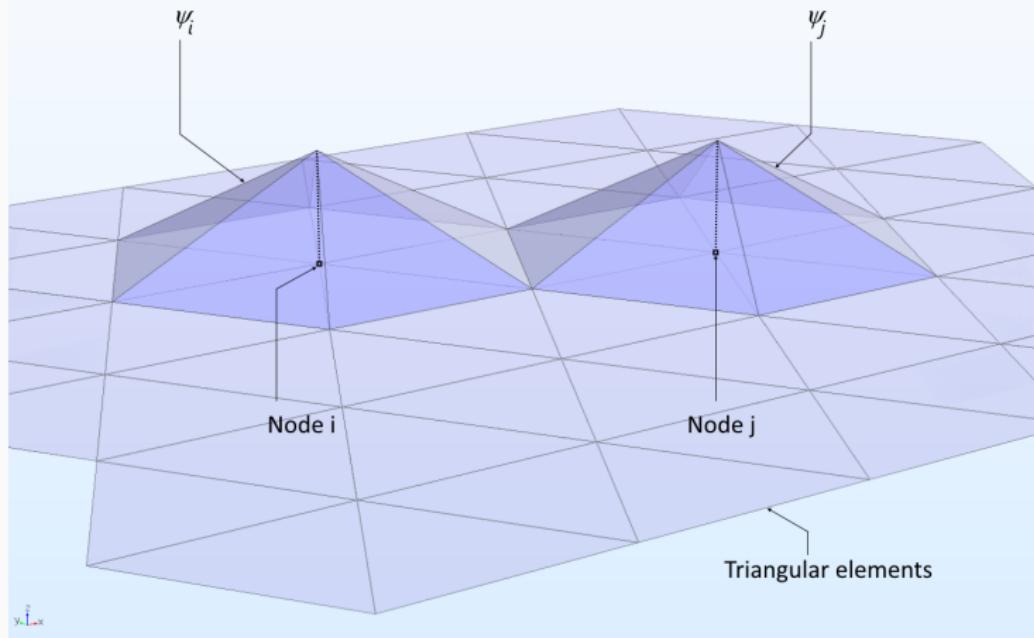


Figure : Nodal functions in the Finite Element Method (FEM) (Cyclopedia, 2017).

Presentation of ϕ -FEM

Immersed boundary methods

What are immersed boundary methods?

In the case of quasi-incompressibility for example, it is necessary to use hexahedral meshes in order to avoid locking phenomena. However, there is no 3D mesh generator capable of meshing any geometry in a hexahedral manner.

One answer is to use **immersed boundary methods**. They can handle:

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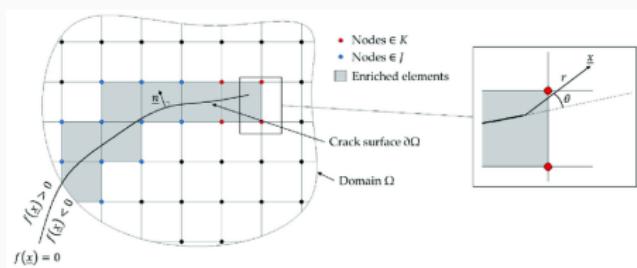


Figure : XFEM

- ## ■ XFEM (De Cicco and Taheri, 2018)

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Common examples are:

- XFEM (De Cicco and Taheri, 2018)
 - CutFEM (Burman and Hansbo, 2010)

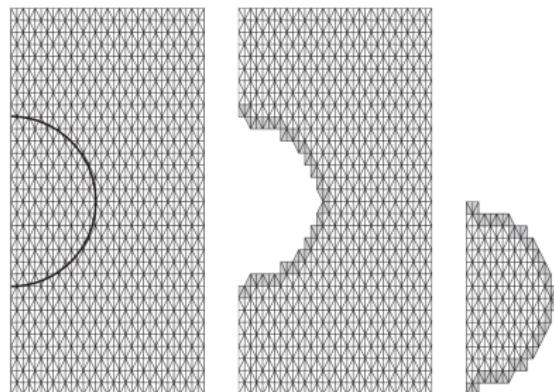


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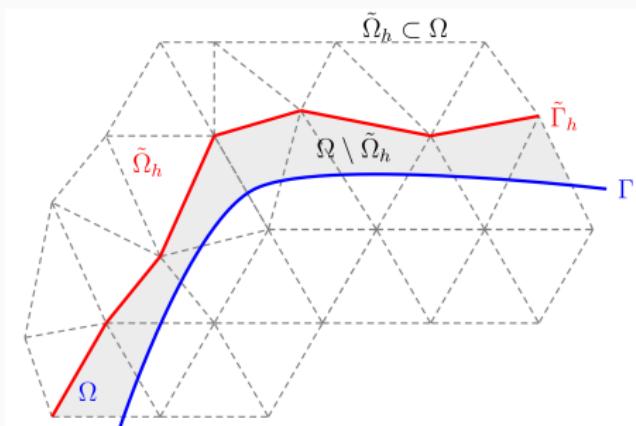


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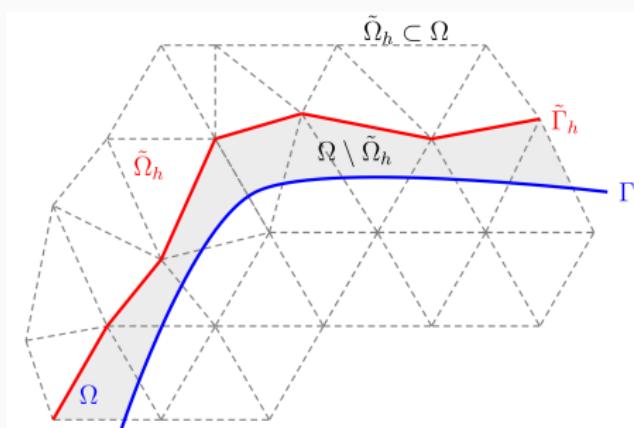


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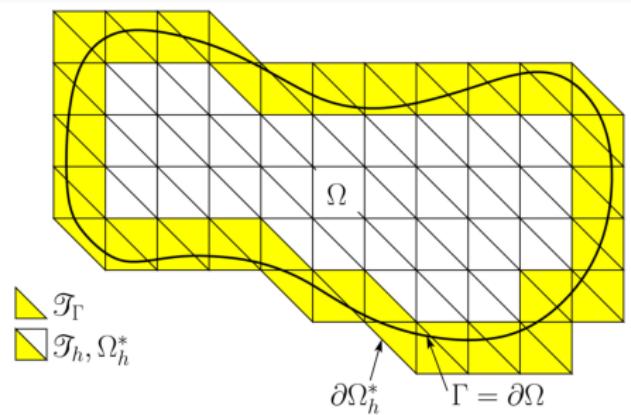


Figure : ϕ -FEM

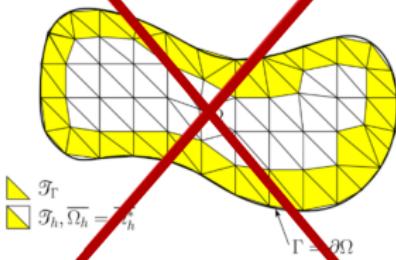
Note: These 3 techniques are efficient, but their implementation generates other difficulties, among other things, the quadrature (on the boundary). This lead to ϕ -FEM (Duprez and Lozinski, 2020).

The ideas behind ϕ -FEM

The main ideas are (for Dirichlet boundary conditions):

- 1 Define the domain using a level-set function ϕ .
 - 2 Then make that function carry the solution e.g. $u = \phi w + q$.

$$\exists ? u \text{ such that } \begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$



$\exists ? w$ such that $\begin{cases} \mathcal{L}(\phi w) = f & \text{in } \Omega \\ \text{where } \Omega = \{\phi < 0\}, \quad u = \phi w \end{cases}$

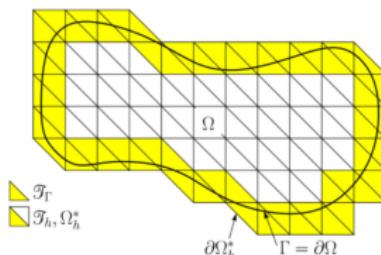


Figure : From Classic FEM (on the left) to ϕ -FEM (on the right)

Results

Results

The Poisson problem

Why the Poisson problem?

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

The Poisson problem (Poisson's equation) is used in various areas of Mathematics and Physics (Newtonian gravity, electrostatics, fluid dynamics, etc.).



Figure : Siméon Denis Poisson
(Wikipedia, 2020).

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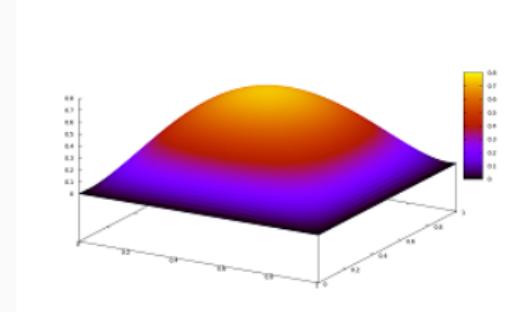


Figure : Displacement of a clamped membrane.



Figure : Siméon Denis Poisson
(Wikipedia, 2020).

It can model the displacement u of a clamped membrane to which a constant force f is applied.

Simulating this equation is critical if we are to understand the dynamics of human organs.

Theoretical framework for the Poisson problem

The Poisson problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1)$$

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Classic FEM

Find $u_h \in V_h^{(k)}$ (subspace of splines in $H_0^1(\Omega_h)$) such that

$$a_h(u_h, v_h) = l_h(v_h), \quad \forall v_h \in V_h^{(k)},$$

where

$$a_h(u, v) = \int_{\Omega_h} \nabla u \cdot \nabla v,$$

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ϕ -FEM (Duprez and Lozinski, 2020)

First, write $u_h = \phi_h w_h$. Then find $w_h \in V_h^{(k)}$ such that

$$a_h(w_h, v_h) = l_h(v_h) \text{ for all } v_h \in V_h^{(k)},$$

where

$$a_h(w, v) = \int_{\Omega_h} \nabla(\phi_h w) \cdot \nabla(\phi_h v) - \int_{\partial\Omega_h} \frac{\partial}{\partial n}(\phi_h w) \phi_h v + G_h(w, v),$$

$$l_h(v) = \int_{\Omega_h} f \phi_h v + G_h^{rhs}(v).$$

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$G_h(w, v)$ and $G_h^{rhs}(w, v)$ are stabilization terms, needed for the well-posedness of the discrete ϕ -FEM system.

$$G_h(w, v) := \sigma h \sum_{E \in \mathcal{F}_h^f} \int_E \left[\frac{\partial}{\partial n} (\phi_h w) \right] \left[\frac{\partial}{\partial n} (\phi_h v) \right] + \sigma h^2 \sum_{T \in \mathcal{T}_h^f} \int_T \Delta(\phi_h w) \Delta(\phi_h v),$$

$$G_h^{rhs}(v) := -\sigma h^2 \sum_{T \in \mathcal{T}_h^f} \int_T f \Delta(\phi_h v).$$

Numerical solution for the Poisson problem

Classic FEM

$$\begin{cases} \Omega = \left\{ (x, y) \in \mathbb{R}^2 : (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 < \frac{1}{8} \right\} \\ u(x, y) = - \left(\frac{1}{8} - (x - \frac{1}{2})^2 - (y - \frac{1}{2})^2 \right) \exp(x) \sin(2\pi y) \\ f(x, y) = - \frac{\partial^2 u}{\partial x^2}(x, y) - \frac{\partial^2 u}{\partial y^2}(x, y) \end{cases}$$

ϕ -FEM

$$\begin{cases} \mathcal{O} = [0, 1] \times [0, 1] \\ \phi(x, y) = -\frac{1}{8} + (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 \\ u(x, y) = \phi(x, y) \times \exp(x) \times \sin(2\pi y) \\ f(x, y) = - \frac{\partial^2 u}{\partial x^2}(x, y) - \frac{\partial^2 u}{\partial y^2}(x, y) \\ \sigma = 20 \end{cases}$$

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Poisson solution using Classic FEM

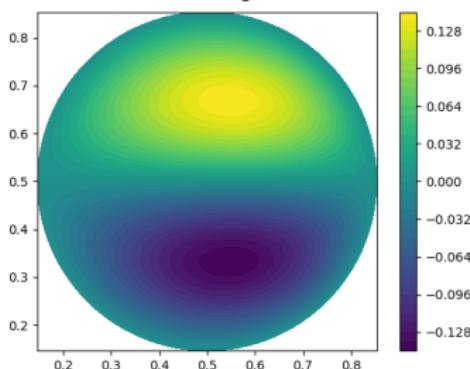


Figure : Classic FEM (39642 cells)

Poisson solution using Phi-FEM WITH stabilisation

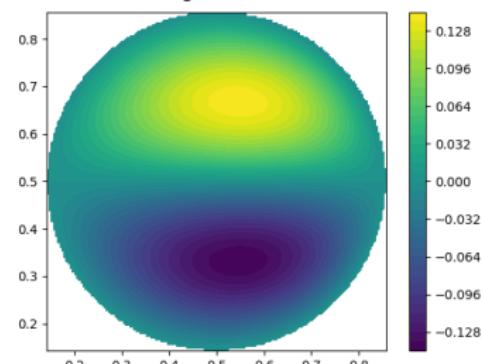


Figure : ϕ -FEM (39936 cells)

Convergence study for the Poisson Problem

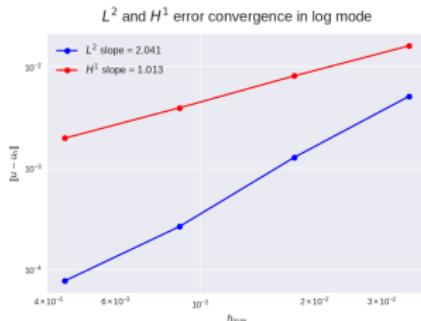


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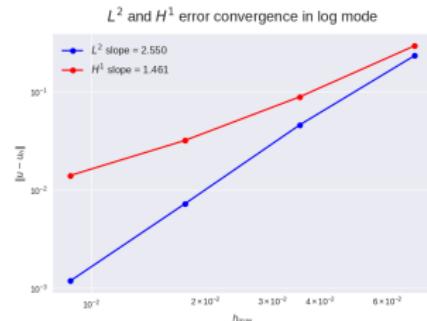


Figure : ϕ -FEM

Problem	Technique	L^2 slope	H^1 slope
Poisson	Classic FEM	2.041	1.013
Poisson	ϕ -FEM	2.550	1.461

Table : Convergence rates.

Convergence study for the Poisson Problem

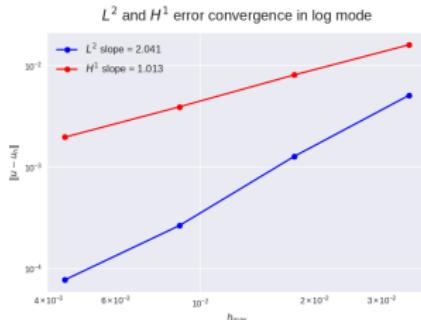


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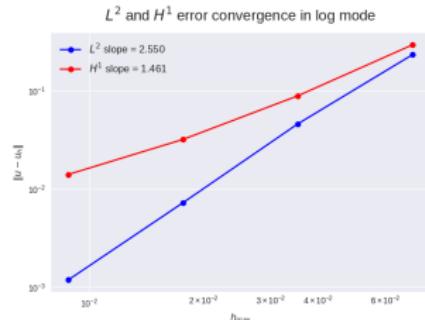


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Table : Convergence rates.

Note: The convergence rates are clearly better with ϕ -FEM. However, we can see that for relatively coarse meshes ($h_{\max} = 10^{-2}$ for example), the errors are smaller in Classic FEM. That said, if ϕ is taken in \mathbb{P}^2 as a second degree polynomial (as opposed to \mathbb{P}^1 as we did), the errors at $h_{\max} = 10^{-2}$ are the same as with Classic FEM (Duprez, 2020).

Results

The elasticity equation

Why use the equations of linear elasticity?

$$\begin{cases} \nabla \cdot \sigma(u) + f = 0 & \text{in } \Omega \\ \sigma(u) = \lambda(\nabla \cdot u)\mathcal{I} + \mu(\nabla u + \nabla u^T) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Analysis of structures is one of the major activities of modern engineering, which likely makes the PDE modeling the deformation of elastic bodies **the most popular PDE in the world** (FEniCS, 2020).

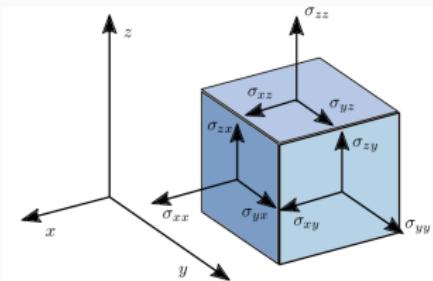


Figure : The nine quantities in the stress tensor $\sigma(\cdot)$ on a representative volume (Semanticscholar, 2020).

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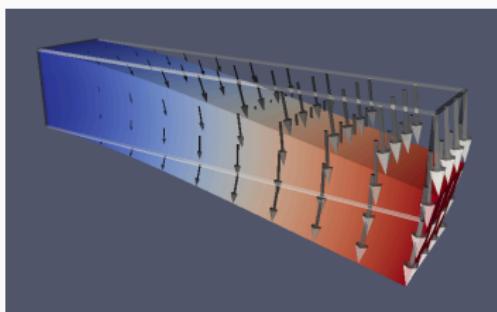


Figure : Gravity-induced deflection in a clamped beam (FEniCS, 2020).

It takes into account variables not considered in the Poisson problem; namely the direction of the displacement vector u . Also, if we added the Neumann condition $\sigma(u) \cdot n = g_N$, it would represent a force applied on the boundary.

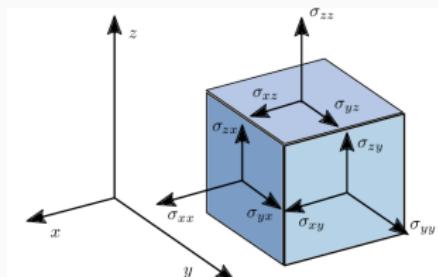


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Theoretical framework for the elasticity equation

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Classic FEM

First $V = [H_0^1(\Omega)]^d$, and $\gamma G = g$. Then find $u \in G + V$ such that

$$a(u, v) = l(v), \quad \forall v \in V,$$

where a and l are defined as

$$a(u, v) = \int_{\Omega} \sigma(u) : \varepsilon(v),$$

and

$$l(v) = \int_{\Omega} f \cdot v.$$

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ϕ -FEM (Duprez and Nzoyem, 2020)

First, $u_h = \phi w + g$. Then find $w \in V$ such that:

$$a(w, v) = l(v), \quad \forall v \in V,$$

where a and l are defined as

$$a(w, v) = \int_{\Omega} \sigma(\phi w) : \varepsilon(\phi v) - \int_{\partial\Omega} (\sigma(\phi w) \cdot n) \cdot (\phi v) + G(w, v),$$

and

$$l(v) = \int_{\Omega} f \cdot (\phi v) + \int_{\partial\Omega} (\sigma(g) \cdot n) \cdot (\phi v) - \int_{\Omega} \sigma(g) : \varepsilon(\phi v) + G_{rhs}(v).$$

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$$G(w, v) = \sigma_{pen} h^2 \sum_{E \in \mathcal{T}_h^r} \int_E (\nabla \cdot \sigma(\phi w)) \cdot (\nabla \cdot \sigma(\phi v)) + \sigma_{pen} h \sum_{F \in \mathcal{F}_h^r} \int_F [\sigma(\phi w) \cdot n] \cdot [\sigma(\phi v) \cdot n]$$

$$G_{rhs}(v) = -\sigma_{pen} h^2 \sum_{E \in \mathcal{T}_h^r} \int_E (f + \nabla \cdot \sigma(g)) \cdot (\nabla \cdot \sigma(\phi v)) - \sigma_{pen} h \sum_{F \in \mathcal{F}_h^r} \int_F [\sigma(g) \cdot n] \cdot [\sigma(\phi v) \cdot n]$$

Numerical solution for the elasticity equation

Classic FEM

$$\begin{cases} \Omega = \left\{ (x, y) \in \mathbb{R}^2 : (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 < \frac{1}{8} \right\} \\ u(x, y) = \begin{pmatrix} 2x + \sin(x) \exp(y) \\ \frac{x}{2} + \cos(x) - 1 \end{pmatrix} \\ f(x, y) = -\nabla \cdot \sigma(u(x, y)) \\ g(x, y) = u(x, y) \end{cases}$$

ϕ -FEM

$$\begin{cases} \mathcal{O} = [0, 1] \times [0, 1] \\ \phi(x, y) = -\frac{1}{8} + (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 \\ u(x, y) = \begin{pmatrix} 2x + \sin(x) \exp(y) \\ \frac{x}{2} + \cos(x) - 1 \end{pmatrix} \\ f(x, y) = -\nabla \cdot \sigma(u(x, y)) \\ g(x, y) = u(x, y) + \phi(x, y) \begin{pmatrix} \sin(x) \\ \exp(y) \end{pmatrix} \\ \sigma_{pen} = 20 \end{cases}$$

Numerical solution for the elasticity equation

Classic FEM

$$\left\{ \begin{array}{l} \Omega = \left\{ (x, y) \in \mathbb{R}^2 : (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 < \frac{1}{8} \right\} \\ u(x, y) = \begin{pmatrix} 2x + \sin(x) \exp(y) \\ \frac{x}{2} + \cos(x) - 1 \end{pmatrix} \\ f(x, y) = -\nabla \cdot \sigma(u(x, y)) \\ g(x, y) = u(x, y) \end{array} \right.$$

ϕ -FEM

$$\left\{ \begin{array}{l} \mathcal{O} = [0, 1] \times [0, 1] \\ \phi(x, y) = -\frac{1}{8} + (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 \\ u(x, y) = \begin{pmatrix} 2x + \sin(x) \exp(y) \\ \frac{x}{2} + \cos(x) - 1 \end{pmatrix} \\ f(x, y) = -\nabla \cdot \sigma(u(x, y)) \\ g(x, y) = u(x, y) + \phi(x, y) \begin{pmatrix} \sin(x) \\ \exp(y) \end{pmatrix} \\ \sigma_{pen} = 20 \end{array} \right.$$

Elasticity solution using Classic FEM

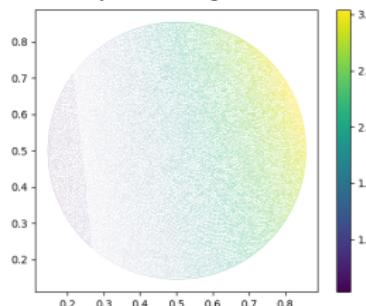


Figure : Classic FEM

Elasticity solution using Phi FEM

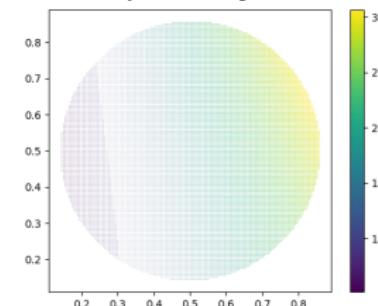


Figure : ϕ -FEM

Numerical solution for the elasticity equation (cont.)

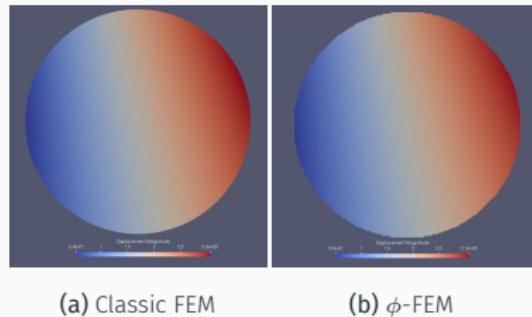


Figure : Solution (deformation) magnitude in Paraview.

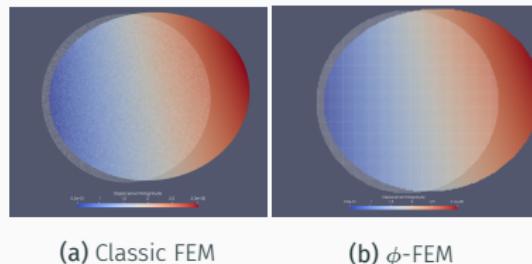


Figure : Solution warped by vector in Paraview.

Convergence study for the elasticity equation

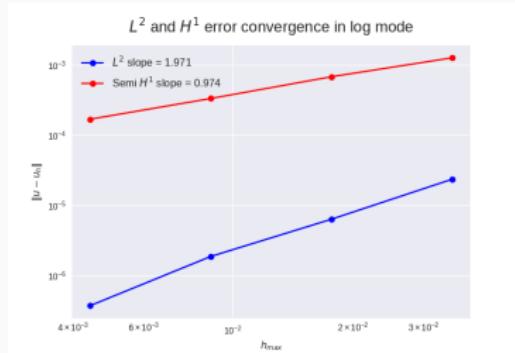


Figure : Classic FEM

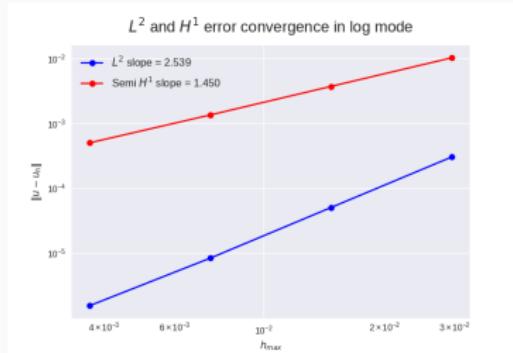


Figure : ϕ -FEM

Problem	Technique	L^2 slope	H^1 slope
Elasticity	Classic FEM ϕ -FEM	1.971 2.539	0.974 1.450

Table : Convergence rates.

Convergence study for the elasticity equation

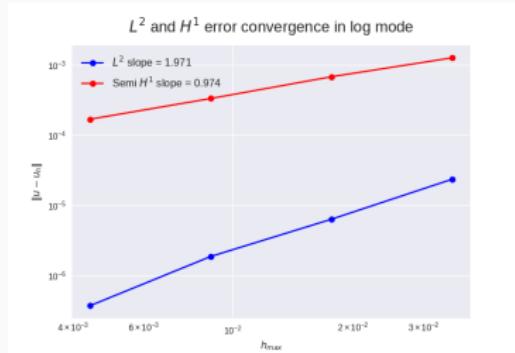


Figure : Classic FEM

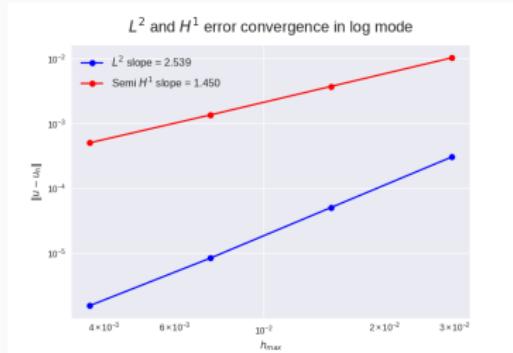


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Note: The convergence rates are clearly better with ϕ -FEM. However, our tests were conducted on relatively coarse meshes ($h_{\max} = 10^{-2}$), in which the errors are smaller in Classic FEM.

Project summary

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Work done

What has been done?

Based on the objectives, the deadlines, and the time estimates we set early in the project:

- ✓ Understanding ϕ -FEM : November 3rd, 2020 : **10 hours**

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- Test the technique on complex geometries.
 - Benchmark the technique for speed efficiency.
 - Deploy the numerical implementation into the SOFA software.
 - Implement the Neumann case (Duprez et al., 2020).

Project summary

Delivered documents

What did I deliver?

As promised:

- ## 1 A typewritten report

What did I deliver?

As promised:

- 1 A typewritten report
 - 2 A Python code base

All are available on this GitHub repository: <https://github.com/master-csmi/2020-m2-mimesis>

References

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Thank you for your kind attention ☺ !
Questions ?