INVERSE PROBLEM FOR RADIATIVE TRANSFER APPLIED TO MEDICAL TOMOGRAPHY

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CONTEXT

- Proposed by the MOCO Team at IRMA

- Inverse problems encountered in:
  - medical imaging
  - computer vision
  - radar, etc.

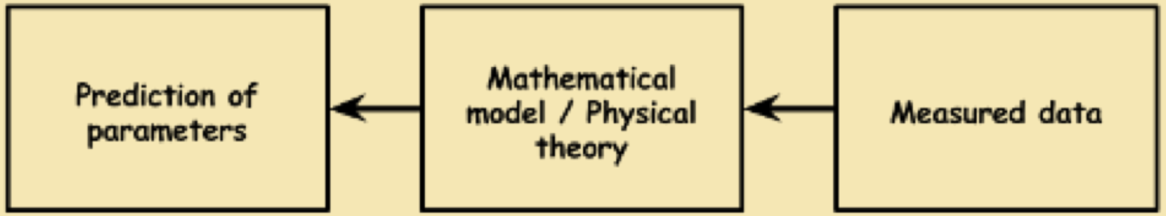


Figure 1: Inverse problem (Subramaniyam, 2018)

They are hard to solve, that's why we will be introducing a machine learning approach.

DESCRIPTION

We send an infrared light beam through an organ and we measure the signal on part of that organ. Knowing the initial conditions and the signal at all times, we can infer the influence of the organ's optical properties.

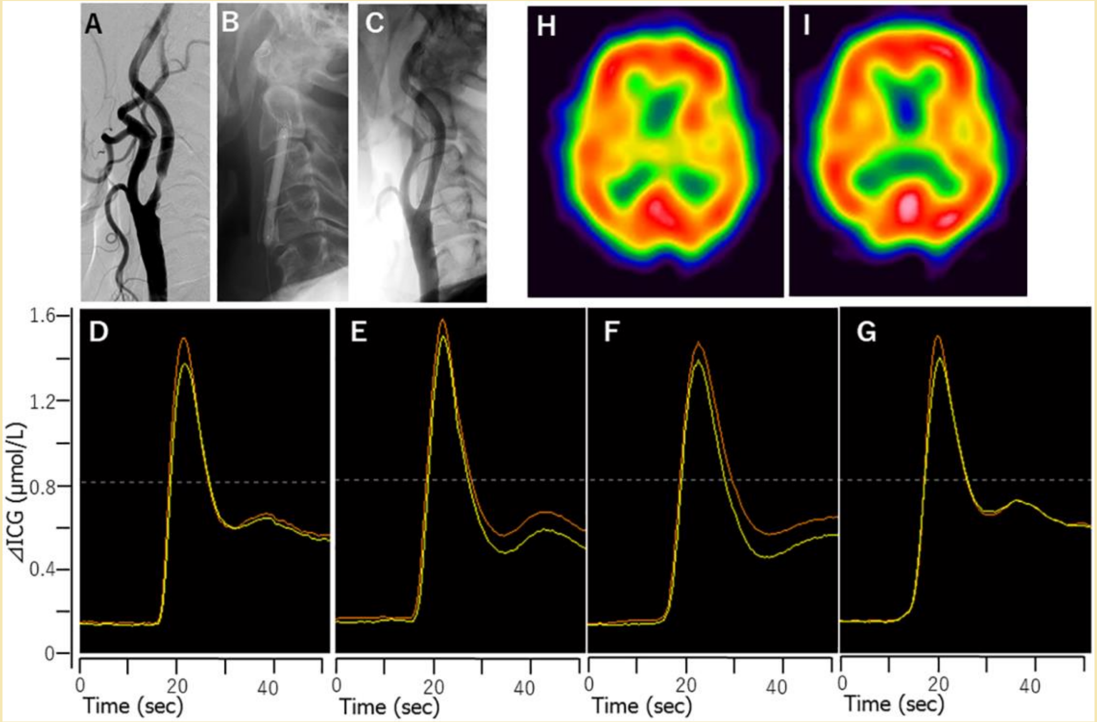


Figure 2: Near-infrared spectroscopy (Nakagawa et al., 2017)

Objectives

- Solve the forward problem for thermal radiation
- Get ready for AI prediction using neural networks

Tasks

- Read about the mathematical model to be used
- Implement the numerical scheme using the finite volumes method
- Run test cases
- Analyze the data
- Write the reports
- Present the work that was done

ROADMAP

Mathematical basis

Light interacts with matter through:

- Emission with opacity σ_e
- Absorption with opacity σ_a
- Scattering with opacity σ_c

With major simplifications:

- Local thermodynamic equilibrium
- Radiative equilibrium, etc.

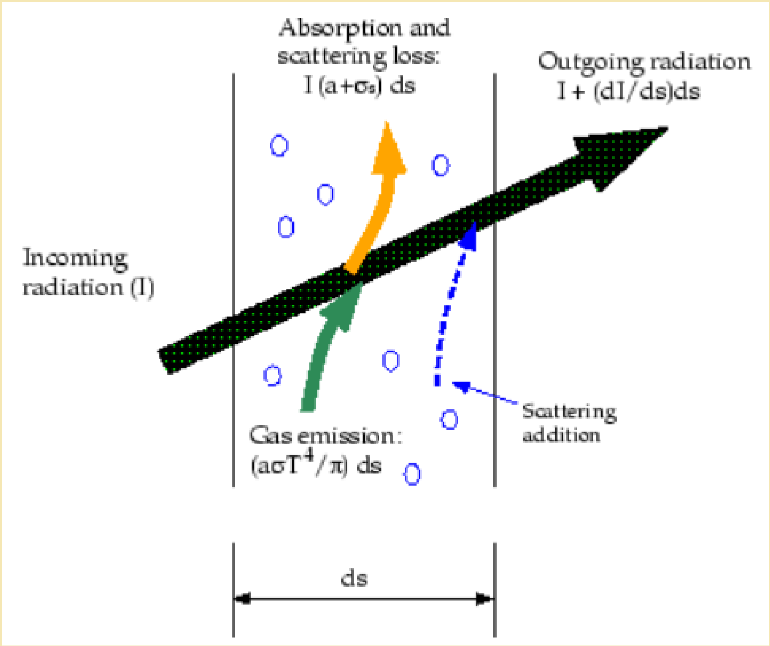


Figure 3: Radiative transfer phenomenon in a gas (Ansys Fluent, n.d)

The radiative transfer equation (RTE)

$$\frac{1}{c}\frac{\partial}{\partial t} I(t,x,\Omega,\nu) + \Omega\cdot\nabla_x I(t,x,\Omega,\nu)$$
$$= \sigma_a(\rho,\Omega,\nu)\big(B(\nu,T) - I(t,x,\Omega,\nu)\big)$$
$$+ \frac{1}{4\pi}\int_0^{\infty}\int_{S^2} \sigma_c(\rho,\Omega,\nu)\,p(\Omega' \to \Omega)\big(I(t,x,\Omega',\nu) - I(t,x,\Omega,\nu)\big)\,d\Omega'\,d\nu \qquad (RTE)$$

Emission/absorption in the direction Ω

Gain/loss due to scattering

ROADMAP

Mathematical basis

Other simplifications lead to this P_1 model:

$$\begin{cases} \partial_t E + c\partial_x F = c\sigma_a(aT^4 - E) \\ \partial_t F + \dfrac{c}{3}\partial_x E = -c\sigma_c F \\ \rho C_v \partial_t T = c\sigma_a(E - aT^4) \end{cases}$$

Where:

- $a = \frac{4\sigma}{c}$ with σ the Stefan–Boltzmann constant
- C_v is the thermal capacity of the medium
- c is the speed of light
- $T(t,x) > 0$ is the temperature of the medium
- $E(t,x) \in \mathbb{R}$ is the energy of the photons
- $F(t,x) \in \mathbb{R}$ is the flux of the photons
- $\rho(x) > 0$ is the density of the medium
- $\sigma_a(\rho, T) > 0$ is the absorption opacity
- $\sigma_c(\rho, T) > 0$ is the scattering opacity

ROADMAP

Solving the problem

- Using the finite volumes method
- Using a "splitting" scheme
- CFL condition for stability: $\Delta t < \frac{\Delta x}{c}$

Step 1 (the coupling part)

- With $\Theta = aT^4$
- On each independent cell j:
- At each time iteration step n (the *big* loop)
- We iterate on q (the *small* loop)

$$\begin{cases} \dfrac{E_j^{q+1} - E_j^n}{\Delta t} = c\sigma_a\left(\Theta_j^{q+1} - E_j^{q+1}\right) \\ \dfrac{F_j^{q+1} - F_j^n}{\Delta t} = 0 \\ \rho_j C_v \mu_q \dfrac{\Theta_j^{q+1} - \Theta_j^n}{\Delta t} = c\sigma_a\left(E_j^{q+1} - \Theta_j^{q+1}\right) \end{cases}$$

Where: $\mu_q = \dfrac{1}{T^{3,n} + T^n T^{2,q} + T^q T^{2,n} + T^{3,q}}$

- Until we get to the fixed point $(E_j^*, F_j^*, \Theta_j^*)$

Step 2 (the hyperbolic part)

- We iterate on n (the *big* loop)

$$\begin{cases} \dfrac{E_j^{n+1} - E_j^*}{\Delta t} + c\,\dfrac{F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}}{\Delta t} = 0 \\ \dfrac{F_j^{n+1} - F_j^*}{\Delta t} + c\,\dfrac{E_{j+\frac{1}{2}} - E_{j-\frac{1}{2}}}{\Delta t} = cS_j \\ \rho_j C_v \mu_q \dfrac{T_j^{n+1} - T_j^*}{\Delta t} = 0 \end{cases}$$

Where:

$$F_{j+\frac{1}{2}} = M_{j+\frac{1}{2}}\left(\frac{F_{j+1}^n + F_j^n}{2} - \frac{E_{j+1}^n - E_j^n}{2}\right), \quad F_{j-\frac{1}{2}} = M_{j-\frac{1}{2}}\left(\frac{F_j^n + F_{j-1}^n}{2} - \frac{E_j^n - E_{j-1}^n}{2}\right)$$

$$E_{j+\frac{1}{2}} = M_{j+\frac{1}{2}}\left(\frac{E_{j+1}^n + E_j^n}{2} - \frac{F_{j+1}^n - F_j^n}{2}\right), \quad E_{j-\frac{1}{2}} = M_{j-\frac{1}{2}}\left(\frac{E + E_{j-1}^n}{2} - \frac{F_j^n - F_{j-1}^n}{2}\right)$$

$$S_j = -\frac{1}{2}\left(M_{j+\frac{1}{2}}\sigma_{j+\frac{1}{2}} + M_{j-\frac{1}{2}}\sigma_{j-\frac{1}{2}}\right)F_j^{n+1}$$

$$M_{j+\frac{1}{2}} = \frac{2}{2 + \Delta x\sigma_{j+\frac{1}{2}}}, \quad M_{j-\frac{1}{2}} = \frac{2}{2 + \Delta x\sigma_{j-\frac{1}{2}}}$$

$$\sigma_{j+\frac{1}{2}} = \frac{1}{2}\left(\sigma_c\left(\rho_j, T_j^n\right) + \sigma_c\left(\rho_{j+1}, T_{j+1}^n\right)\right), \quad \sigma_{j-\frac{1}{2}} = \frac{1}{2}\left(\sigma_c\left(\rho_{j-1}, T_{j-1}^n\right) + \sigma_c\left(\rho_j, T_j^n\right)\right)$$

- Till the final time

RESULTS

Test case 1: The transport limit

We take $\sigma_a = \sigma_c = 0$ and we place a Gaussian function at the center of the domain.

$$\frac{\partial}{\partial t}(E+F) + c\frac{\partial}{\partial x}(E+F) = 0 \qquad \frac{\partial}{\partial t}(E-F) - c\frac{\partial}{\partial x}(E-F) = 0$$

$E + F$ is transported at the speed c while $E - F$ is transported at the speed $-c$.

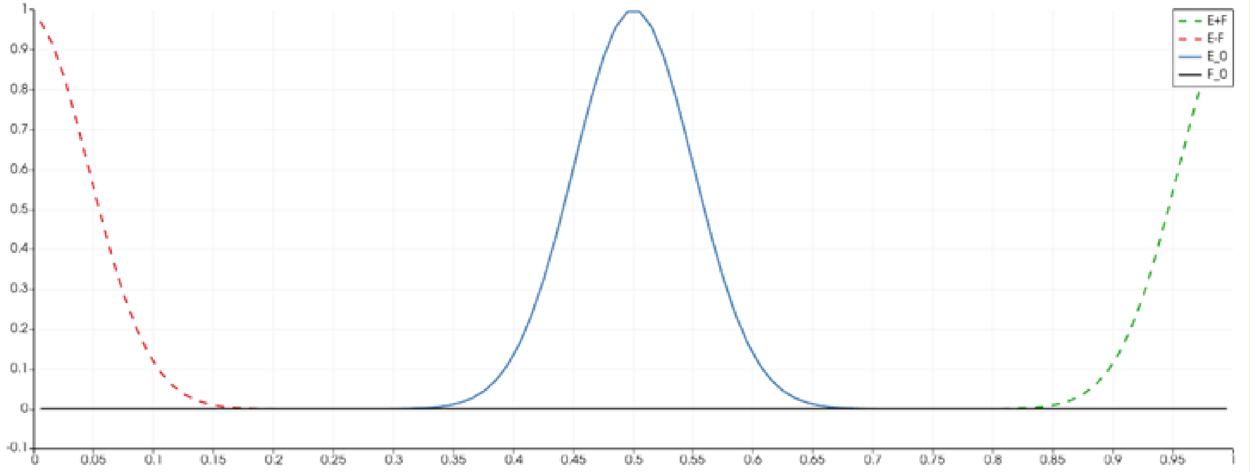


Figure 4: Transport limit with speed c=1

RESULTS

Test case 2: The diffusion limit

We take $\sigma_a = 0, \sigma_c = c = 1000$ and $t_0 = 0.002$.

$$E(t,x) = \frac{1}{\sqrt{4\pi(t+t_0)}} e^{-\frac{\left(x-\frac{1}{2}\right)^2}{4(t+t_0)}}$$

is solution to the diffusion equation

$$\begin{cases} \dfrac{\partial}{\partial t} E(t,x) - \dfrac{\partial^2}{\partial x^2} E(t,x) = 0, \qquad t > 0, \;\; x \in [0,1] \\ E_0(x) = E(0,x) = \dfrac{1}{\sqrt{4\pi t_0}} e^{-\frac{\left(x-\frac{1}{2}\right)^2}{4t_0}} \end{cases}$$

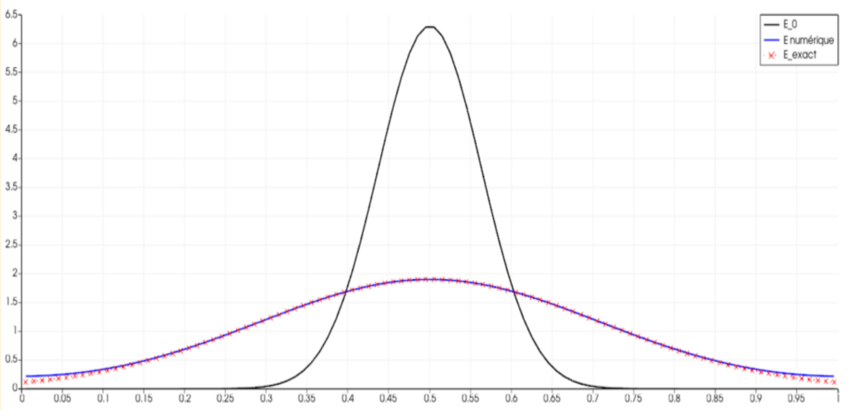


Figure 5: Spatial visualization for the diffusion limit

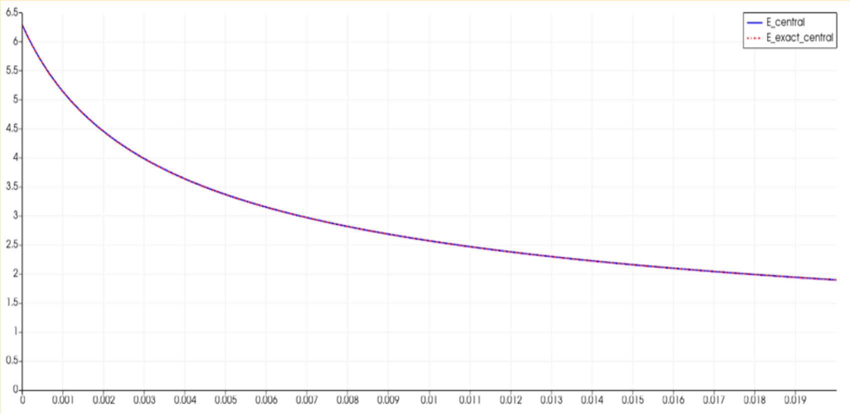


Figure 6: Temporal decrease in energy at the center of the domain for the diffusion limit

RESULTS

Test case 3: Olson-Auer-Hall

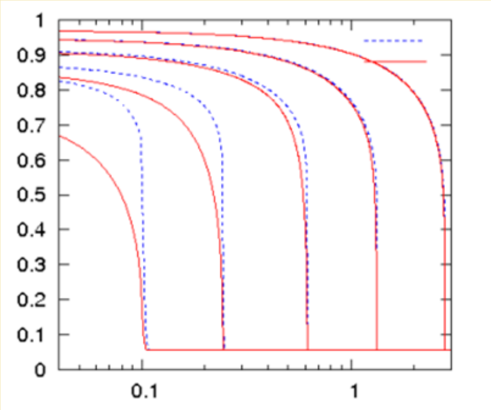


Figure 7: Expected results for t=1/c, 3/c, 10/c, 30/c, and 100/c (Franck, 2012, p.171)

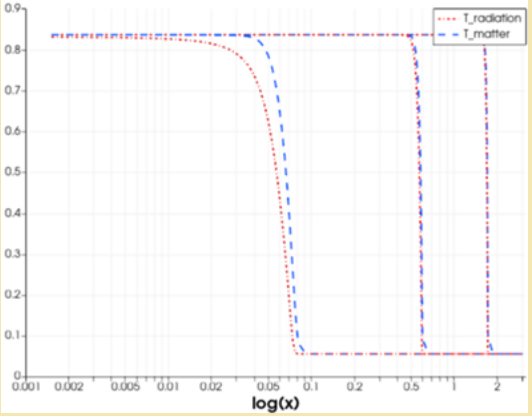


Figure 8: Obtained results for t=0.1/c, 1/c, and 3/c

The domain $[0,3]$ is heated by a blackbody to the left (with $T_r = 0.056234\ keV$). As the body gets hotter, its absorption opacity increases, causing thermal equilibrium between the radiation and the matter (Franck, 2012, p.171)

RESULTS

Influence of the density

It seems that as the density increases:

- The radiative energy E increases
- It doesn't change the flux F
- The temperature varies less. The equilibrium temperature seems to be higher.

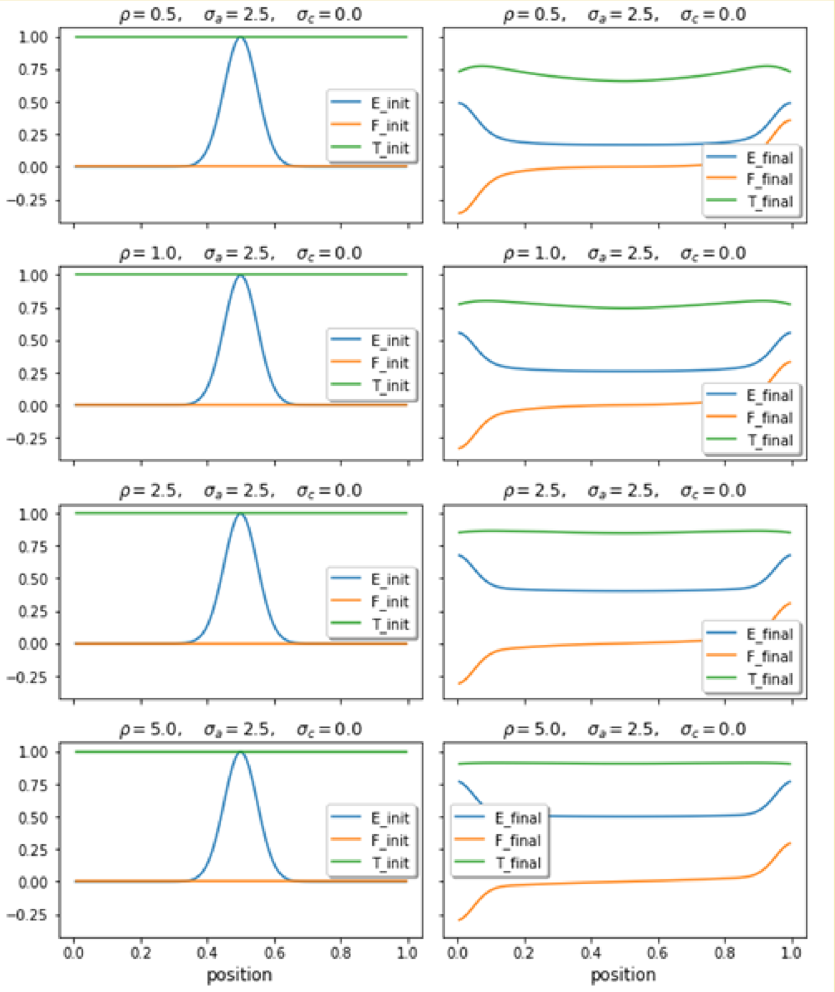


Figure 9: Influence of the density on E, F, and T for $\sigma_a = 2.5$ and $\sigma_c = 0$

RESULTS

Influence of the absorption opacity

It seems that as the opacity σ_a increases:

- The diffusion phenomenon is faster, but the energy E keeps the same pic value
- The flux F is attenuated during transport
- The temperature change is more important, and the thermal equilibrium is faster

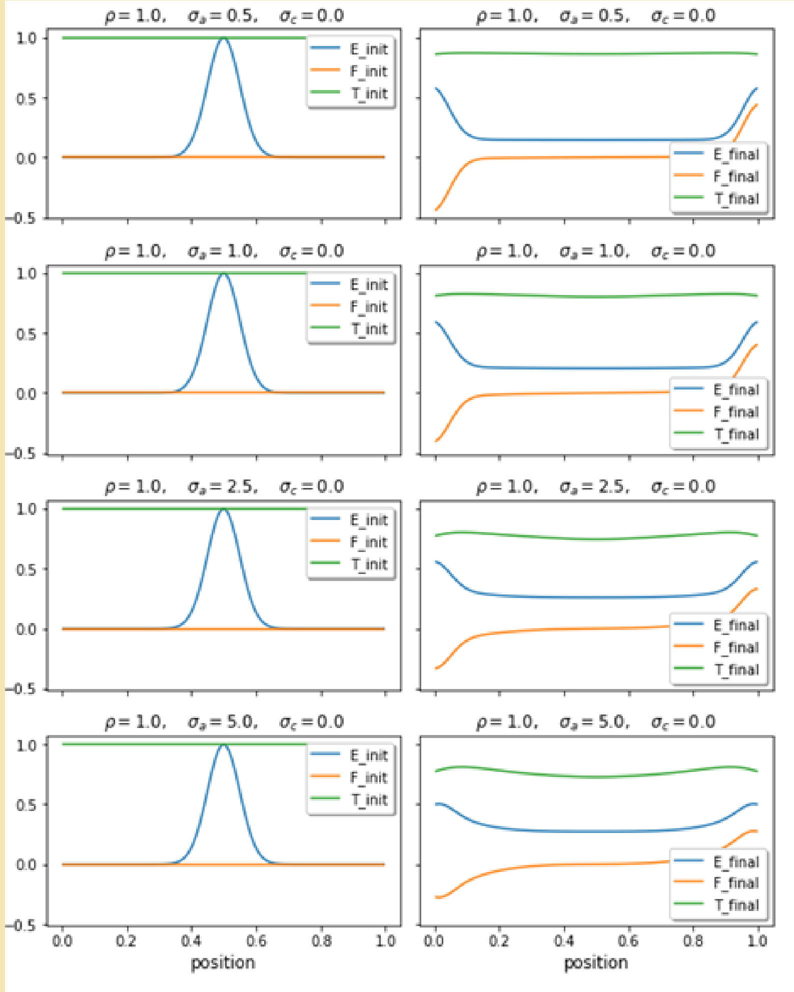


Figure 10: Influence of σ_a for $\rho = 1$ and $\sigma_c = 0$

RESULTS

Influence of the scattering opacity

It seems obvious that an increase in σ_c:

- Decreases the values of both the energy and the flux.
- However, to the naked eye, it is less obvious to notice any change in temperature.

These are just conjectures. With the help of a neural network, we could confirm or invalidates these trends.

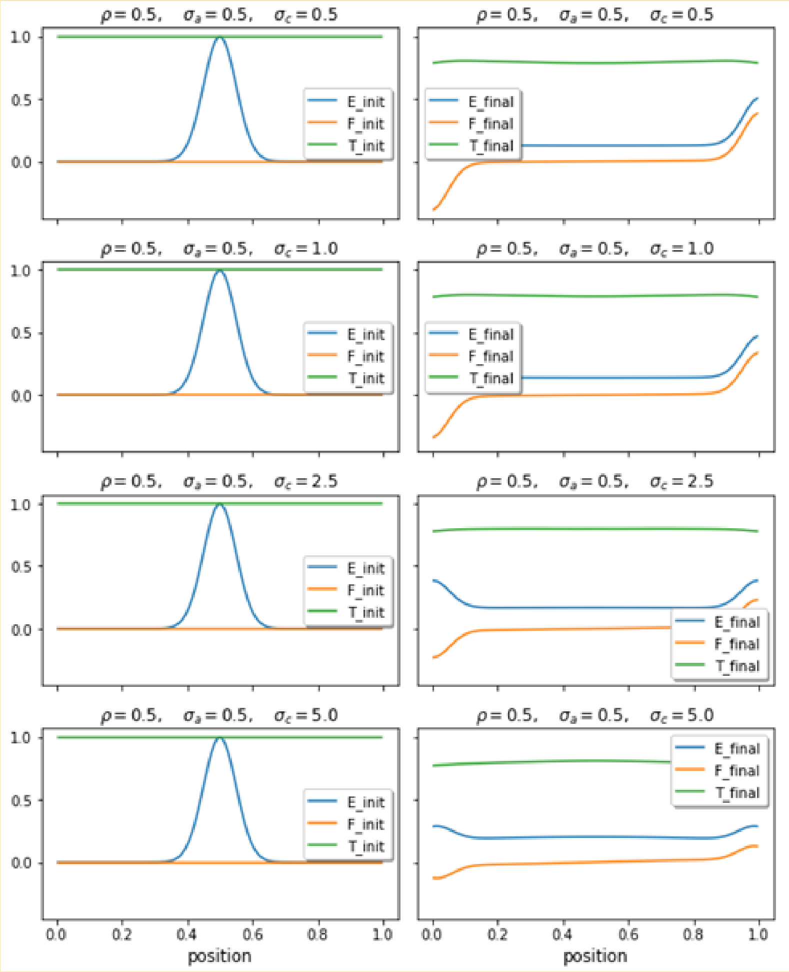
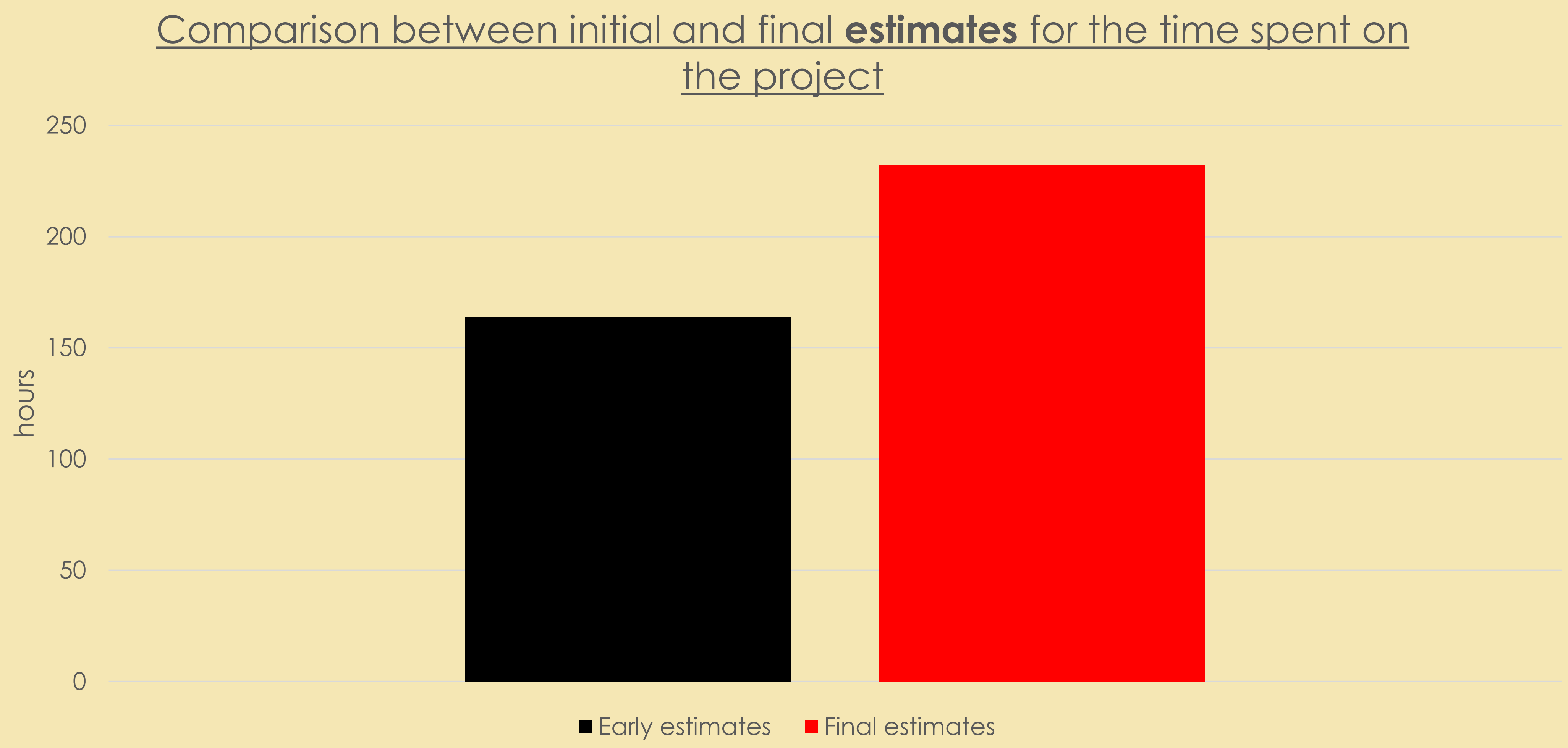


Figure 11: Influence of σ_c for $\rho = 0.5$ and $\sigma_a = 0.5$

MILESTONES


Comparison between initial and final estimates for the time spent on the project
250
200
150
100
50
0
hours
Early estimates
Final estimates

CHALLENGES

- Notations and nomenclature (stellar atmosphere in astrophysics, nuclear fusion by inertial confinement, etc…)
- Performance in the C++ implementation of the scheme

PERSPECTIVES

- Since the P_1 model only works well for $f = \frac{\|F\|}{cE} \leq 0.57$ (Turpault, 2003, p.22), we could use a more precise model for the radiative transfer equation (P_n, M_1, etc..).
- We could train a neural network that will then predict the optical properties of the medium.

THANK YOU !

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