RADIATIVE TRANFER IN 2D

We consider the P1 model for radiative transfer:

$$\begin{cases} \partial_t E + c\partial_x F = c\sigma_a(aT^4 - E) & (1.1) \\ \partial_t F + c\partial_x E = -c\sigma_c F & (1.2) \\ \rho C_v \partial_t T = c\sigma_a(E - aT^4) & (1.3) \end{cases} \qquad (1)$$

Where:

- $a = \frac{4\sigma}{c}$ with σ the Stefan–Boltzmann constant
- C_v is the thermal capacity of the medium
- c is the speed of light
- $T(t,x) > 0$ is the temperature of the medium
- $E(t,x) \in \mathbb{R}$ is the energy of the photons
- $F(t,x) \in \mathbb{R}$ is the flux of the photons
- $\rho(x) > 0$ is the density of the medium
- $\sigma_a(\rho,T) > 0$ is the absorption opacity
- $\sigma_c(\rho,T) > 0$ is the scattering opacity

Step 1: The coupling part

We consider the "equilibrium" part of (1). That is, the photons are not moving, and we only consider the equations that are affected by matter (i.e. equations (1.1) and (1.3) where terms with the medium's temperature are involved) (Franck, 2012, p. 160). This leads to all the terms with ∂_x in (1) to be equal to 0. That equation becomes:

$$\begin{cases} \partial_t E = c\sigma_a(aT^4 - E) \\ \partial_t F = 0 \\ \rho C_v \partial_t T = c\sigma_a(E - aT^4) \end{cases} \qquad (2)$$

Writing $\Theta = aT^4$, we solve (3) on each independent cell. The numerical scheme is given below.

$$\begin{cases} \dfrac{E_j^{q+1} - E_j^n}{\Delta t} = c\sigma_a\left(\Theta_j^{q+1} - E_j^{q+1}\right) \\ \dfrac{F_j^{q+1} - F_j^n}{\Delta t} = 0 \\ \rho_j C_v \mu_q \dfrac{\Theta_j^{q+1} - \Theta_j^n}{\Delta t} = c\sigma_a(E_j^{q+1} - \Theta_j^{q+1}) \end{cases}$$

Rewritten as:

$$\begin{cases} E_j^{q+1} = \alpha E_j^n + \beta \Theta_j^{q+1} \\ F_j^{q+1} = F_j^n \\ \Theta_j^{q+1} = \gamma \Theta_j^n + \delta E_j^{q+1} \end{cases}$$

Applying Cramer's rule, we get:

$$\begin{cases} E_j^{q+1} = \dfrac{\alpha E_j^n + \beta\gamma\Theta_j^n}{1-\beta\delta} \\ F_j^{q+1} = F_j^n \\ \Theta_j^{q+1} = \dfrac{\gamma\Theta_j^n + \alpha\delta E_j^n}{1-\beta\delta} \end{cases} \tag{3}$$

Where

- E_j^n, F_j^n and Θ_j^n are the value values of E_j, F_j and Θ_j on the cell at the beginning of the step.
- $\alpha = \frac{1}{\Delta t}\left(\frac{1}{\Delta t} + c\sigma_a\right)^{-1}$, $\beta = c\sigma_a\left(\frac{1}{\Delta t} + c\sigma_a\right)^{-1}$, $\gamma = \frac{\rho_j C_v \mu_q}{\Delta t}\left(\frac{\rho_j C_v \mu_q}{\Delta t} + c\sigma_a\right)^{-1}$, and $\delta = c\sigma_a\left(\frac{\rho_j C_v \mu_q}{\Delta t} + c\sigma_a\right)^{-1}$
- σ_a written above is a function of ρ_j and T_j^n. Thus, it is actually $\sigma_a(\rho_j, T_j^n)$.
- μ_q is such that $\mu_q = \dfrac{1}{T^{3,n} + T^n T^{2,q} + T^q T^{2,n} + T^{3,q}}$

Since this step is a fixed point method, we iterate on q until we get close enough to the fixed point $(E_j^*, F_j^*, \Theta_j^*)$, or more precisely (E_j^*, F_j^*, T_j^*). We then move to the next step.

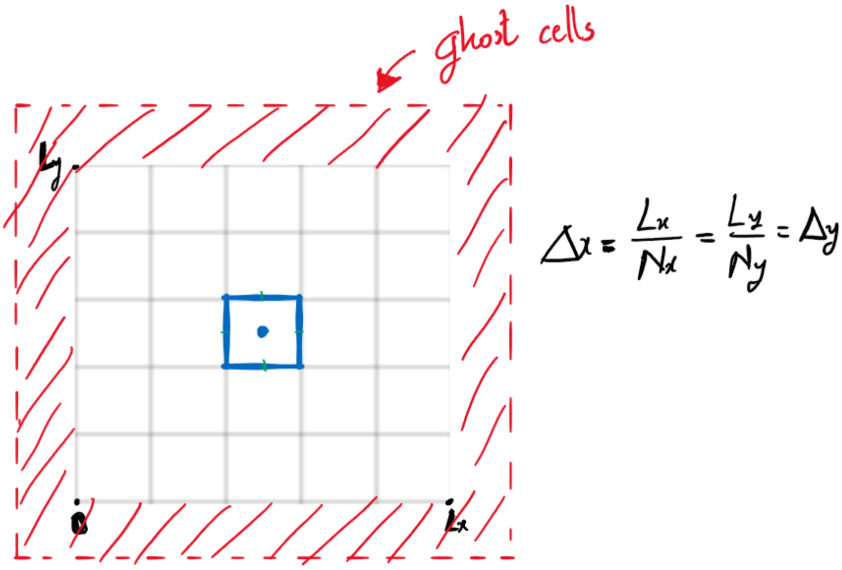
Step 2: The hyperbolic part in 2D

Once the first step converges, we solve (1.2) and (1.3) as if the radiation weren't coupled with the matter, hence considering only the two hyperbolic partial differential equations in (1). We do this with the values of E, F, and T on each cell updated from step 1. We write:

In 2D, $\mathbf{F} = (F_1, F_2)$ becomes a vector, and equations (1.1) and (1.2) become

$$\begin{cases} \partial_t E + c\nabla \cdot \mathbf{F} = 0 & (1.1) \\ \partial_t \mathbf{F} + c\nabla \cdot E = -c\sigma_c \mathbf{F} & (1.2) \end{cases}$$

Let's apply the finite volumes method where control volumes are identical with the grid cells. Let's use the following uniform grid ($\Delta x = \Delta y$) for discretization:



Let Ω_j be the volume of cell j and S_j its surface:

$$\begin{cases} \partial_t \int_{\Omega_j} E + c \int_{\Omega_j} \nabla \cdot \mathbf{F} = 0 \\ \partial_t \int_{\Omega_j} \mathbf{F} + c \int_{\Omega_j} \nabla \cdot E = -c\sigma_c \int_{\Omega_j} \mathbf{F} \end{cases}$$

Using the divergence theorem, we get:

$$\begin{cases} \partial_t \int_{\Omega_j} E + c \int_{S_j} (\mathbf{F}, \mathbf{n}) \, dS = 0 \\ \partial_t \int_{\Omega_j} \mathbf{F} + c \int_{S_j} E\mathbf{n} \, dS = -c\sigma_c \int_{\Omega_j} \mathbf{F} \end{cases}$$

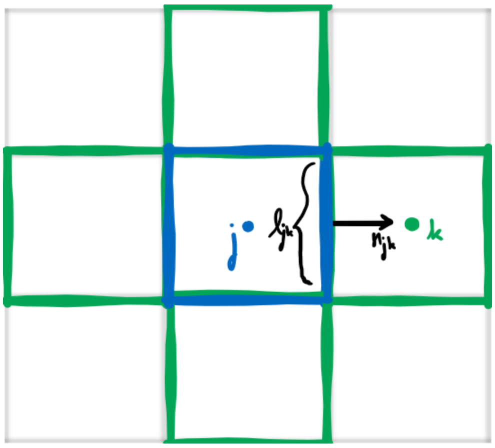
Averaging the values for $E(t,x)$ and $\mathbf{F}(t,x)$ over the cell, we have:

$$\begin{cases} \partial_t E_j + \frac{c}{|\Omega_j|} \int_{S_j} (\mathbf{F}, \mathbf{n}) \, dS = 0 \\ \partial_t \mathbf{F_j} + \frac{c}{|\Omega_j|} \int_{S_j} E\mathbf{n} \, dS = -\frac{c\sigma_c}{|\Omega_j|} \int_{\Omega_j} \mathbf{F} \end{cases}$$

Where:

$$E_j(t) = \frac{1}{|\Omega_j|} \int_{\Omega_j} E(t,x)$$
$$\mathbf{F}_j(t) = \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathbf{F}(t,x)$$

As for the discretization, let us consider a neighbouring cell k, we define:



$$(\mathbf{F}_{jk}, \mathbf{n}_{jk}) = l_{jk} M_{jk} \left(\frac{\mathbf{F}_j^n \cdot \mathbf{n}_{jk} + \mathbf{F}_k^n \cdot \mathbf{n}_{jk}}{2} - \frac{E_k^n - E_j^n}{2} \right)$$

$$(E_{jk} \mathbf{n}_{jk}) = l_{jk} M_{jk} \left(\frac{E_j^n + E_k^n}{2} - \frac{\mathbf{F}_k^n \cdot \mathbf{n}_{jk} - \mathbf{F}_j^n \cdot \mathbf{n}_{jk}}{2} \right) \mathbf{n}_{jk}$$

$$\mathbf{S}_j' = \frac{1}{|\Omega_j|} \left(\sum_k l_{jk} M_{jk}\, \mathbf{n}_{jk} \right) E_j^n$$

$$\mathbf{S}_j = -\left(\sum_k M_{jk} \sigma_{jk} \right) \mathbf{F}_j^{n+1}$$

$$M_{jk} = \frac{2}{2 + \Delta x \sigma_{jk}}$$

$$\sigma_{jk} = \frac{1}{2} \left(\sigma_c(\rho_j, T_j^n) + \sigma_c(\rho_k, T_k^n) \right)$$

Finally, we can write the scheme as:

$$\begin{cases} \dfrac{E_j^{n+1} - E_j^*}{\Delta t} + \dfrac{c}{|\Omega_j|} \displaystyle\sum_k (\mathbf{F}_{jk}, \mathbf{n}_{jk}) = 0 \\ \dfrac{\mathbf{F}_j^{n+1} - \mathbf{F}_j^*}{\Delta t} + \dfrac{c}{|\Omega_j|} \displaystyle\sum_k (E_{jk} \mathbf{n}_{jk}) - c\mathbf{S}_j' = c\mathbf{S}_j \end{cases}$$

Taking into account step 1, we can have:

$$\begin{cases} E_j^{n+1} = E_j^* + \alpha \displaystyle\sum_k (\mathbf{F}_{jk}, \mathbf{n}_{jk}) \\ \mathbf{F}_j^{n+1} = \beta \mathbf{F}_j^* + \boldsymbol{\gamma} E_j^n + \delta \displaystyle\sum_k (E_{jk} \mathbf{n}_{jk}) \\ T_j^{n+1} = T_j^* \end{cases} \tag{9}$$

With

$$\alpha = -\frac{c\Delta t}{|\Omega_j|}, \quad \beta = \frac{1}{\Delta t} \left(\frac{1}{\Delta t} + c \sum M_{jk} \sigma_{jk} \right)^{-1}, \quad \boldsymbol{\gamma} = \frac{c}{|\Omega_j|} \left(\sum l_{jk} M_{jk} \mathbf{n}_{jk} \right) \left(\frac{1}{\Delta t} + c \sum M_{jk} \sigma_{jk} \right)^{-1}$$
$$\text{and } \delta = -\frac{c}{|\Omega_j|} \left(\frac{1}{\Delta t} + c \sum M_{jk} \sigma_{jk} \right)^{-1}$$

We still need the CFL condition for stability.

$$\Delta t < \frac{\Delta x}{c}$$