UNIVERSITY OF STRASBOURG
UFR DE MATHÉMATIQUES ET INFORMATIQUE

Problem Set #1: RB for Linear Affine Elliptic

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1. Part 1 - Finite Element Approximation

Question a)

Let's show that $u^e(\mu) \in X^e \equiv H^1(\Omega)$ satifies the weak form.

$$a(u^e(\mu), v; \mu) = l(v), \quad \forall v \in X^e \tag{7}$$

with

$$a(w, v; \mu) = \sum_{i=0}^{4} k^i \int_{\Omega_i} \nabla w \cdot \nabla v \, dA + \mathrm{Bi} \int_{\Gamma_{ext} \backslash \Gamma_{root}} wv \, dS$$

$$l(v) = \int_{\Gamma_{root}} v \, dS$$

Answer

Let's start by summarizing the domain names and boundaries. This is done in the picture below.

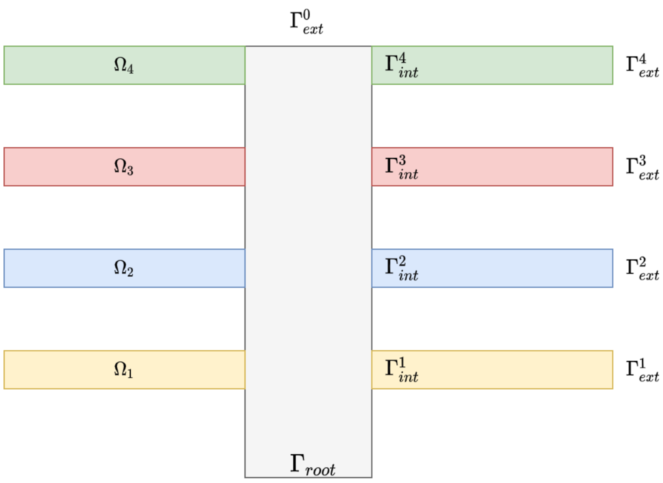

Figure 1: Domain and boundary denominations for the thermal fin

The heat transfer problem is governed by the following set of equations:

$$\begin{cases} -k^i\Delta u^i = 0 & \text{in} \quad \Omega^i \,,\, i = 0, ..., 4 & (1) \\ \ u^0 = u^i & \text{on} \quad \Gamma^i_{int} \,,\, i = 1, ..., 4 & (2) \\ -(\nabla u^0 \cdot n^i) = -k^i(\nabla u^i \cdot n^i) & \text{on} \quad \Gamma^i_{int} \,,\, i = 1, ..., 4 & (3) \\ -(\nabla u^0 \cdot n^0) = -1 & \text{on} \quad \Gamma_{root} & (4) \\ -k^i(\nabla u^i \cdot n^i) = \text{Bi}u^i & \text{on} \quad \Gamma^i_{ext} \,,\, i = 0, ..., 4 & (5) \end{cases}$$

In the above set of equations, the parametrized exact solution $u^e(\mu)$ is called u for simplicity. u^i is the restriction of u to Ω^i with $i = 1, \ldots, 4$.
Using eq. (1), we get:

$$\begin{aligned} & -k^i\Delta u^i = 0 \quad \text{for} \;\; i = 0, \ldots, 4 \\ \implies & \int_{\Omega^i} -k^i\Delta u^i v = 0 \quad \forall v \in X^e \quad \text{for} \;\; i = 0, \ldots, 4 \end{aligned}$$

Using Green's formula, this yields:

$$\int_{\Omega^i} k^i \nabla u^i \cdot \nabla v - \int_{\partial\Omega^i} k^i \left(\nabla u^i \cdot n^i\right) v = 0 \qquad \text{for} \;\; i = 0, \ldots, 4 \tag{$*$}$$

- When $i = 1, \ldots, 4$, we have $\partial\Omega^i = \Gamma^i_{int} \cup \Gamma^i_{ext}$. Using eq. (3) and (5), ($*$) becomes:

$$\int_{\Omega^i} k^i \nabla u^i \cdot \nabla v - \int_{\Gamma^i_{int}} \left(\nabla u^0 \cdot n^i\right) v + \text{Bi} \int_{\Gamma^i_{ext}} u^i v = 0$$

- When $i = 0$, we have $\partial\Omega^0 = \bigcup_{i=1}^4 \Gamma^i_{int} \cup \Gamma^0_{ext} \cup \Gamma_{root}$. Using eq. (3), (4), (5) and the fact that $k^0 = 1$, ($*$) becomes:

$$\int_{\Omega^0} k^0 \nabla u^0 \cdot \nabla v - \sum_{i=1}^{4} \int_{\Gamma^i_{int}} (\nabla u^0 \cdot n^0)\, v + \text{Bi} \int_{\Gamma^0_{ext}} u^0 v - \int_{\Gamma_{root}} v = 0$$

Now let's perform a summation of ($*$) over all the subdomains $\Omega^i, i = 0, \ldots, 4$.

$$\sum_{i=0}^{4} \int_{\Omega^i} k^i \nabla u^i \cdot \nabla v \underbrace{- \sum_{i=1}^{4} \int_{\Gamma^i_{int}} \left(\nabla u^0 \cdot n^i\right) v - \sum_{i=1}^{4} \int_{\Gamma^i_{int}} (\nabla u^0 \cdot n^0)\, v}_{=0 \text{ because } n^i = -n^0 \text{ on } \Gamma^i_{int}} + \text{Bi} \sum_{i=0}^{4} \int_{\Gamma^i_{ext}} u^i v - \int_{\Gamma_{root}} v = 0$$

Noticing that $\bigcup_{i=0}^4 \Gamma^i_{ext} = \Gamma_{ext} - \Gamma_{root}$, we are left with:

$$\sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla u^i \cdot \nabla v + \text{Bi} \int_{\Gamma_{ext} \setminus \Gamma_{root}} uv - \int_{\Gamma_{root}} v = 0$$

By writing back $u^e(\mu) = u$, the above equation can be rewritten as:

$$\sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla u^e(\mu) \cdot \nabla v + \text{Bi} \int_{\Gamma_{ext} \setminus \Gamma_{root}} u^e(\mu) v = \int_{\Gamma_{root}} v \qquad \forall v \in X^e$$

This shows that $u^e(\mu)$ satisfies the week form (7) we set out to prove.

Question b)

Let's show that $u^e(\mu)$ is the argument that minimizes.

$$J(w) = \frac{1}{2}\sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla w \cdot \nabla w + \frac{\text{Bi}}{2}\int_{\Gamma_{ext}\backslash\Gamma_{root}} w^2 - \int_{\Gamma_{root}} w \tag{8}$$

over all functions w in X^e.

Answer

As we did in Question a), let's write $u^e(\mu) = u$ to simplify the notations.
We need to show that $J(w) > J(u)$ for all w in X^e such that $w \neq u$. To that effect, Let's write $w = w + u - u = u + v$ with $v = w - u \in X^e$.
For any $w \in X^e$ such that $w \neq u$ i.e $v \neq 0$, we have:

$$\begin{aligned}
J(w) &= J(u+v) \\
&= \frac{1}{2}\sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla(u+v)\cdot\nabla(u+v) + \frac{\text{Bi}}{2}\int_{\Gamma_{ext}\backslash\Gamma_{root}} (u+v)^2 - \int_{\Gamma_{root}} (u+v) \\
&= \frac{1}{2}\sum_{i=0}^{4} k^i \int_{\Omega^i} (\nabla u\cdot\nabla u + 2\nabla u\cdot\nabla v + \nabla v\cdot\nabla v) + \frac{\text{Bi}}{2}\int_{\Gamma_{ext}\backslash\Gamma_{root}} \left(u^2 + 2uv + v^2\right) - \int_{\Gamma_{root}} (u+v) \\
&= \underbrace{\frac{1}{2}a(u,u;\mu) - l(u)}_{J(u)} + \frac{1}{2}a(v,v;\mu) + \underbrace{a(u,v;\mu) - l(v)}_{0} \\
&= J(u) + \frac{1}{2}a(v,v;\mu)
\end{aligned} \tag{$*$}$$

Let's show that the bilinear form a is coercive. For any $v \in X^e$ and any parameter μ, we have:

$$\begin{aligned}
a(v,v;\mu) &= \sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla v\cdot\nabla v + \text{Bi}\int_{\Gamma_{ext}\backslash\Gamma_{root}} v^2 \\
&\geq \underbrace{\min_{0\leq i\leq 4} k^i}_{C_1(\mu)>0} \int_{\Omega} \nabla v\cdot\nabla v + \text{Bi}\int_{\Gamma_{ext}\backslash\Gamma_{root}} v^2 \\
&= C_1(\mu)\left(\|\nabla v\|^2_{L^2(\Omega)} + \frac{\text{Bi}}{C_1(\mu)}\|v\|^2_{L^2(\Gamma_{ext}\backslash\Gamma_{root})}\right)
\end{aligned} \tag{$**$}$$

Now since Ω is a bounded, open and connected subset of $\mathbb{R}^d$, and $\text{Bi}/C_1(\mu) > 0$, a Poincaré-type inequality (Legoll, 2019, p.48) states there exists a constant $C_2 > 0$ such that

$$\forall v \in H^1(\Omega),\ \ \|\nabla v\|^2_{L^2(\Omega)} + \frac{\text{Bi}}{C_1(\mu)}\|v\|^2_{L^2(\Gamma_{ext}\backslash\Gamma_{root})} \geq C_2\|v\|^2_{H^1(\partial\Omega)}$$

The original inequality presented by F. Legoll has been adapted to suit our case by using the fact that the trace application on a part of the boundary $\gamma : H^1(\Omega) \to L^2(\Gamma_{ext}\backslash\Gamma_{root})$ remains continuous.
Using this result, the inequality ($**$) is now:

$$a(v,v;\mu) \geq C_3(\mu)\|v\|^2_{H^1(\Omega)}$$

With $C_3(\mu) = C_1(\mu)C_2 > 0$, proving that a is strongly coercive.

We can now return to eq. (∗). Thanks to the proven coercivity, $a(v, v; \mu) > 0$ since $v \neq 0$. And we have, for any $w \in H^1(\Omega) = X^e$ such that $w \neq u$

$$J(w) = J(u) + \frac{1}{2} a(v, v; \mu) > J(u) = J(u^e(\mu))$$

Therefore, $u^e(\mu)$ as indicated in (7) minimizes the functional J.