Part 2 - Reduced - Basis Approximation

a) Prove that, in the energy norm $|||\cdot||| = a(\cdot,\cdot;\mu)^{1/2}$

$$|||u(\mu) - u_N(\mu)||| \leq |||u(\mu) - w_N||| , \quad \forall w_N \in W_N$$

Let $w_N \in W_N \subset X^e$, we have

$$a(u - u_N, w_N; \mu) = a(u, w_N; \mu) - a(u_N, w_N; \mu)$$
$$= l(w_N) - l(w_N)$$
$$= 0$$

So, we have $a(u - u_N, w_N; \mu) = 0 \quad \forall w_N \subset W_N$.
W_N is a closed subspace of $X^e = H^1$ (Hilbert space thanks to the scalar product $a(\cdot,\cdot,\mu)$). Using the orthogonal projection theorem, we can conclude that

$$|||u(\mu) - u_N(\mu)||| \leq |||u(\mu) - w_N|||,$$

$\forall w_N \in W_N$

b) Proove that

$$T_{root}(\mu) - T_{root\,N}(\mu) = |||u(\mu) - u_N(\mu)|||^2$$

Exploiting the symmetric nature of $a(\cdot,\cdot,\mu)$, we write

$$a(u - u_N, u - u_N; \mu) = a(u,u) - a(u,u_N) - a(u_N,u) + a(u_N,u_N)$$

$$= a(u, \underbrace{u - 2u_N}_{\in X^e}) + a(u_N, \underbrace{u_N}_{\in W_N})$$

$$= l(u - 2u_N) + l(u_N)$$

$$= l(u) - l(u_N) \quad (l \text{ is linear})$$

Since $l \equiv l^0$, we write $T_{root}(\mu) - T_{root\,N}(\mu) = l^0(u) - l^0(u_N)$
$$= |||u(\mu) - u_N(\mu)|||^2$$

c) Show that $\underline{u}_N(\mu)$ as defined satisfies a set of $N\times N$ linear equations,

$$\underline{A}_N(\mu)\,\underline{u}_N(\mu) = \underline{F}_N$$

and that $T_{root\,N}(\mu) = \underline{L}_N^T\,\underline{u}_N(\mu)$

Let's give expressions for $\underline{A}_N(\mu) \in \mathbb{R}^{N\times N}$ in terms of $\underline{A}^{\mathcal{N}}(\mu)$ and Z, $\underline{F}^N \in \mathbb{R}^N$ in terms of $\underline{F}^{\mathcal{N}}$ and Z, and $\underline{L}^N \in \mathbb{R}^N$ in terms of $\underline{L}^{\mathcal{N}}$ and Z.

We have $\quad a(u_N(\mu), v; \mu) = l(v) \quad \forall v \in W_N$

and $\quad W_N = \text{span}\{\xi^1, \xi^2, \dots, \xi^N\}$.

Therefore, for any ξ^i

$$a(u_N(\mu), \xi^i; \mu) = l(\xi^i) \; ; \quad i = 1, \dots, N$$

Now, $u_N(\mu) \in W_N \Rightarrow u_N(\mu) = \sum_{j=1}^{N} u_N^j\,\xi^j$

therefore, $a\left(\sum_{j=1}^{N} u_N^j\,\xi^j, \xi^i; \mu\right) = l(\xi^i) \quad i=1,\dots,N$

$$\Rightarrow \sum_{j=1}^{N} a(\xi^j, \xi^i; \mu)\,u_N^j = l(\xi^i) \qquad i=1,\dots,N$$

$$\Rightarrow \sum_{j=1}^{N} \left(\underline{A}_N(\mu)\right)_{ij}\left(\underline{u}_N(\mu)\right)_j = \left(\underline{F}_N\right)_i$$

$$\Rightarrow \boxed{\underline{A}_N(\mu)\,\underline{u}_N(\mu) = \underline{F}_N}$$

with $\left(\underline{A}_N(\mu)\right)_{ij} = a(\xi^j, \xi^i), \quad i,j = 1, \dots, N \qquad (*)$

$\left(\underline{u}_N(\mu)\right)_j = u_N^j \qquad j=1,\dots,N \qquad (**)$

$\left(\underline{F}_N\right)_j = l(\xi^j) \qquad j=1,\dots N \qquad (***)$

We know that

$$Z = \left(u_N(\mu^0) \;\vdots\; u_N(\mu^1) \;\vdots\; \cdots \;\vdots\; u_N(\mu^N) \right)$$

$$= \left(\xi^0 \;\vdots\; \xi^1 \;\vdots\; \cdots \;\vdots\; \xi^N \right)$$

this means $\xi^i = \sum_{k=1}^{\mathcal{N}} Z_{ki}\, \phi_k$ where ϕ_k, $k=1,\dots,\mathcal{N}$ are the nodal basis functions in X^e.

* Let's consider $(*)$,

$$\left(\underline{A}_N(\mu)\right)_{ij} = a(\xi^j, \xi^i; \mu), \quad 1 \le i,j \le N$$

$$= a\left(\sum_{k=1}^{\mathcal{N}} Z_{ki}\, \phi_k, \sum_{l=1}^{\mathcal{N}} Z_{lj}\, \phi_l ; \mu \right)$$

$$= \sum_{k=0}^{\mathcal{N}} \sum_{l=0}^{\mathcal{N}} Z_{ki}\, Z_{lj}\; a(\phi_k, \phi_l)$$

$$= \sum_{k=0}^{\mathcal{N}} \sum_{l=0}^{\mathcal{N}} Z^T_{ik} \left(\underline{A}^{\mathcal{N}}(\mu)\right)_{kl} Z_{lj}$$

$$\Rightarrow \boxed{\underline{A}_N(\mu) = Z^T \underline{A}^{\mathcal{N}}(\mu)\, Z}$$

* Let's consider $(***)$

$$\left(\underline{F}_N\right)_j = l(\xi^j) = l\left(\sum_k Z_{kj}\, \phi_k \right)$$

$$= \sum_k Z^T_{jk} \left(\underline{F}^{\mathcal{N}}\right)_k$$

$$\Rightarrow \boxed{\underline{F}_N = Z^T \underline{F}^{\mathcal{N}}}$$

We have that

$$
\begin{aligned}
T_{root\,N}(\mu) &= \ell^0(u_N(\mu)) \\
&= \ell(u_N(\mu)) \\
&= \ell\Big(\sum_j u_N^j \xi^j\Big) \\
&= \sum_j (E_N)_j^T u_N^j \\
&= E_N^T u_N \\
&= (Z_N^T E^{\mathcal{N}})^T u_N
\end{aligned}
$$

$$\Rightarrow \boxed{L_N = Z_N^T E^{\mathcal{N}}}$$

d) Let's show that the bilinear form $a(w,v;\mu)$ can be decomposed as

$$a(w,v,\mu) = \sum_{q=1}^{Q} \theta^q(\mu)\, a^q(w,v), \quad \forall w,v \in X, \ \forall \mu \in D$$

for $Q = 6$ and give expressions for $\theta^q(\mu)$ and the $a^q(w,v)$.

We have

$$a(w,v;\mu) = \sum_{i=0}^{4} k^i \int_{\Omega_i} \nabla w \nabla v + Bi \int_{\Gamma_{ext}\setminus\Gamma_{root}} w v$$

$$= \int_{\Omega^0} \nabla w \cdot \nabla v + \sum_{i=1}^{4} \mu_i \int_{\Omega^i} \nabla w \nabla v + \mu_5 \int_{\Gamma_{ext}\setminus\Gamma_{root}} w v$$

$$= \sum_{q=1}^{Q} \theta^q(\mu)\, a^q(w,v)$$

with

$\theta^1(\mu) = 1$,	$a^1(w,v) = \int_{\Omega^0} \nabla w \cdot \nabla v$
$\theta^q(\mu) = \mu_{q-1}$,	$a^q(w,v) = \int_{\Omega^{q-1}} \nabla w \cdot \nabla v$, $q = 2, \dots, 5$
$\theta^6(\mu) = \mu_5$,	$a^6(w,v) = \int_{\Gamma_{ext} \setminus \Gamma_{root}} w \cdot v$

Further, let's show that

$$\underline{A}^{\mathcal{N}}(\mu) = \sum_{q=1}^{Q} \theta^q(\mu)\, \underline{A}^{\mathcal{N}q}$$

$$\underline{A}^N(\mu) = \sum_{q=1}^{Q} \theta^q(\mu)\, \underline{A}_N^q$$

giving an expression for the $\underline{A}^{\mathcal{N}q}$ in terms of the nodal basis function, and ~~developping~~ developing a formula for the $\underline{A}_N^q$ in terms of the $\underline{A}^{\mathcal{N}q}$ and Z.

We know that

$$\begin{aligned}
\left(A^{\mathcal{N}}(\mu)\right)_{ij} &= a(\varphi^i, \varphi^j; \mu) \\
&= \sum_{q=1}^{Q} \theta^q(\mu)\, a^q(\varphi^i, \varphi^j) \\
&= \sum_{q=1}^{Q} \theta^q(\mu)\, \underline{A}^{\mathcal{N}q}
\end{aligned}$$

with $\boxed{\left(A^{\mathcal{N}q}\right)_{ij} = a^q(\varphi^i, \varphi^j)}$, $1 \le i,j \le \mathcal{N}$

$$\begin{aligned}
\left(A^N(\mu)\right)_{ij} &= Z^T A^{\mathcal{N}}(\mu)\, Z \\
&= Z^T \left(\sum_{q=1}^{Q} \theta^q(\mu)\, A^{\mathcal{N}q} \right) Z \\
&= \sum_{q=1}^{Q} \theta^q(\mu)\, \underline{A}_N^q \quad \text{with } \boxed{\underline{A}_N^q = Z^T A^{\mathcal{N}q} Z}
\end{aligned}$$

$q = 1, \dots, Q$

e) $\boxed{\text{Let's show that the condition number of } \underline{A}_N(\mu) \text{ is bounded from above by } \gamma^e(\mu)/\alpha^e(\mu)}$

$\underline{A}_N(\mu)$ is symmetric and positive defined thanks to its coercivity. Therefore $\boxed{\operatorname{Cond} \underline{A}_N(\mu) = \dfrac{\lambda_N(\underline{A}_N(\mu))}{\lambda_1(\underline{A}_N(\mu))}}$

where λ_1 and λ_N are the smallest and largest eigen values of $\underline{A}_N(\mu)$ respectively. Let's compare the above values to $\gamma^e(\mu)$ and $\alpha^e(\mu)$.

Let $v \in W_N \subset X^e$.

* a is coercive, therefore,

$$\inf_{v \in W_N} \frac{a(v, v; \mu)}{\|v\|^2_{W_N}} \geq \inf_{v \in X^e} \frac{a(v, v; \mu)}{\|v\|^2_{X^e}} \geq \alpha^e_N(\mu) \qquad (*)$$

* a is continues, therefore.

$$\sup_{v \in W_N} \sup_{v \in W_N} \frac{a(v; v; \mu)}{\|v\|_{W_N} \|v\|_{W_N}} \leq \sup_{v \in X^e} \sup_{v \in X^e} \frac{a(v, v; \mu)}{\|v\|_{X^e}\|v\|_{X^e}} \leq \gamma^e(\mu)$$

$$\Rightarrow \sup_{v \in W_N} \frac{a(v, v; \mu)}{\|v\|^2_{W_N}} \leq \gamma^e(\mu) \qquad (**)$$

Since the basis $(\xi^i)_{i=1,\dots,N}$ spanning W_N is orthonormalized, $\|v\|^2_{W_N} = v^T v$.

Hence the term

$$\frac{a(v, v; \mu)}{\|v\|^2_{W_N}} = \frac{v^T \underline{A}_N(\mu) v}{v^T v}$$

can be recognized as the Rayleigh Quotient $R(\underline{A}_N(\mu), v)$.

The Rayleigh Quotient

$$R(\underline{A}_N(\mu), v) = \frac{v^T A_N(\mu) v}{v^T v} = \frac{a(v, v, \mu)}{\|v\|^2_{W_N}}$$

is caraterized by

$$\inf_{v \in W_N} R(\underline{A}_N(\mu), v) = \lambda_1 (\underline{A}_N(\mu))$$

$$\sup_{v \in W_N} R(\underline{A}_N(\mu), v) = \lambda_N (\underline{A}(\mu))$$

The inequalities (*) and (**) become

$$\lambda_1 (\underline{A}_N(\mu)) \geq \alpha^e(\mu)$$

$$\lambda_N (\underline{A}(\mu)) \leq \gamma^e(\mu)$$

therefore, the condition number

$$\mathrm{Cond}\ \underline{A}_N(\mu) = \frac{\lambda_N(\underline{A}_N(\mu))}{\lambda_1(\underline{A}_N(\mu))}$$

$$\leq \frac{\gamma^e(\mu)}{\alpha^e(\mu)}$$