UNIVERSITY OF STRASBOURG
UFR DE MATHÉMATIQUES ET INFORMATIQUE

Problem Set #1: RB for Linear Affine Elliptic

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1. Part 1 - Finite Element Approximation

Question a)

Let's show that $u^e(\mu) \in X^e \equiv H^1(\Omega)$ satifiies the weak form.

$$a(u^e(\mu), v; \mu) = l(v), \quad \forall v \in X^e \tag{7}$$

with

$$a(w, v; \mu) = \sum_{i=0}^{4} k^i \int_{\Omega_i} \nabla w \cdot \nabla v \, dA + \mathrm{Bi} \int_{\Gamma_{ext} \setminus \Gamma_{root}} wv \, dS$$

$$l(v) = \int_{\Gamma_{root}} v \, dS$$

Answer

Let's start by summarizing the domain names and boundaries. This is done in the picture below.

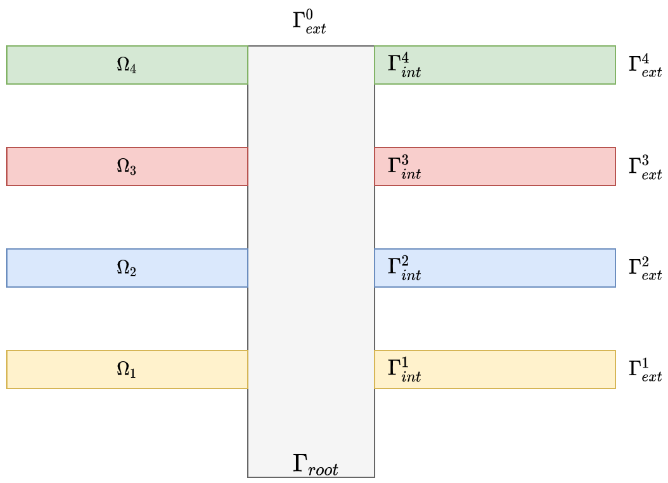

Figure 1: Domain and boundary denominations for the thermal fin

The heat transfer problem is governed by the following set of equations:

$$\begin{cases} -k^i \Delta u^i = 0 & \text{in} \quad \Omega^i \,, \, i = 0, ..., 4 \qquad (1) \\ u^0 = u^i & \text{on} \quad \Gamma^i_{int} \,, \, i = 1, ..., 4 \qquad (2) \\ -(\nabla u^0 \cdot n^i) = -k^i (\nabla u^i \cdot n^i) & \text{on} \quad \Gamma^i_{int} \,, \, i = 1, ..., 4 \qquad (3) \\ -(\nabla u^0 \cdot n^0) = -1 & \text{on} \quad \Gamma_{root} \qquad (4) \\ -k^i (\nabla u^i \cdot n^i) = \text{Bi} u^i & \text{on} \quad \Gamma^i_{ext} \,, \, i = 0, ..., 4 \qquad (5) \end{cases}$$

In the above set of equations, the parametrized exact solution $u^e(\mu)$ is called u for simplicity. u^i is the restriction of u to Ω^i with $i = 1, \ldots, 4$.
Using eq. (1), we get:

$$\begin{aligned} & -k^i \Delta u^i = 0 \quad \text{for } \; i = 0, \ldots, 4 \\ \implies & \int_{\Omega^i} -k^i \Delta u^i v = 0 \quad \forall v \in X^e \quad \text{for } \; i = 0, \ldots, 4 \end{aligned}$$

Using Green's formula, this yields:

$$\int_{\Omega^i} k^i \nabla u^i \cdot \nabla v - \int_{\partial \Omega^i} k^i \left(\nabla u^i \cdot n^i \right) v = 0 \qquad \text{for } \; i = 0, \ldots, 4 \qquad (*)$$

- When $i = 1, \ldots, 4$, we have $\partial \Omega^i = \Gamma^i_{int} \cup \Gamma^i_{ext}$. Using eq. (3) and (5), ($*$) becomes:

$$\int_{\Omega^i} k^i \nabla u^i \cdot \nabla v - \int_{\Gamma^i_{int}} \left(\nabla u^0 \cdot n^i \right) v + \text{Bi} \int_{\Gamma^i_{ext}} u^i v = 0$$

- When $i = 0$, we have $\partial \Omega^0 = \bigcup_{i=1}^4 \Gamma^i_{int} \cup \Gamma^0_{ext} \cup \Gamma_{root}$. Using eq. (3), (4), (5) and the fact that $k^0 = 1$, ($*$) becomes:

$$\int_{\Omega^0} k^0 \nabla u^0 \cdot \nabla v - \sum_{i=1}^{4} \int_{\Gamma^i_{int}} (\nabla u^0 \cdot n^0) \, v + \text{Bi} \int_{\Gamma^0_{ext}} u^0 v - \int_{\Gamma_{root}} v = 0$$

Now let's perform a summation of ($*$) over all the subdomains $\Omega^i, i = 0, \ldots, 4$.

$$\sum_{i=0}^{4} \int_{\Omega^i} k^i \nabla u^i \cdot \nabla v \underbrace{- \sum_{i=1}^{4} \int_{\Gamma^i_{int}} \left(\nabla u^0 \cdot n^i \right) v - \sum_{i=1}^{4} \int_{\Gamma^i_{int}} (\nabla u^0 \cdot n^0) \, v}_{= 0 \text{ because } n^i = -n^0 \text{ on } \Gamma^i_{int}} + \text{Bi} \sum_{i=0}^{4} \int_{\Gamma^i_{ext}} u^i v - \int_{\Gamma_{root}} v = 0$$

Noticing that $\bigcup_{i=0}^4 \Gamma^i_{ext} = \Gamma_{ext} - \Gamma_{root}$, we are left with:

$$\sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla u^i \cdot \nabla v + \text{Bi} \int_{\Gamma_{ext} \setminus \Gamma_{root}} uv - \int_{\Gamma_{root}} v = 0$$

By writing back $u^e(\mu) = u$, the above equation can be rewritten as:

$$\sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla u^e(\mu) \cdot \nabla v + \text{Bi} \int_{\Gamma_{ext} \setminus \Gamma_{root}} u^e(\mu) v = \int_{\Gamma_{root}} v \qquad \forall v \in X^e$$

This shows that $u^e(\mu)$ satisfies the week form (7) we set out to prove.

Question b)

Let's show that $u^e(\mu)$ is the argument that minimizes.

$$J(w) = \frac{1}{2}\sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla w \cdot \nabla w + \frac{\text{Bi}}{2}\int_{\Gamma_{ext}\backslash\Gamma_{root}} w^2 - \int_{\Gamma_{root}} w \tag{8}$$

over all functions w in X^e.

Answer

As we did in Question a), let's write $u^e(\mu) = u$ to simplify the notations.
We need to show that $J(w) > J(u)$ for all w in X^e such that $w \neq u$. To that effect, Let's write $w = w + u - u = u + v$ with $v = w - u \in X^e$.
For any $w \in X^e$ such that $w \neq u$ i.e $v \neq 0$, we have:

$$\begin{aligned}
J(w) &= J(u+v) \\
&= \frac{1}{2}\sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla(u+v)\cdot\nabla(u+v) + \frac{\text{Bi}}{2}\int_{\Gamma_{ext}\backslash\Gamma_{root}} (u+v)^2 - \int_{\Gamma_{root}} (u+v) \\
&= \frac{1}{2}\sum_{i=0}^{4} k^i \int_{\Omega^i} (\nabla u\cdot\nabla u + 2\nabla u\cdot\nabla v + \nabla v\cdot\nabla v) + \frac{\text{Bi}}{2}\int_{\Gamma_{ext}\backslash\Gamma_{root}} (u^2 + 2uv + v^2) - \int_{\Gamma_{root}} (u+v) \\
&= \underbrace{\frac{1}{2}a(u,u;\mu) - l(u)}_{J(u)} + \frac{1}{2}a(v,v;\mu) + \underbrace{a(u,v;\mu) - l(v)}_{0} \\
&= J(u) + \frac{1}{2}a(v,v;\mu) \qquad (*)
\end{aligned}$$

Let's show that the bilinear form a is coercive. For any $v \in X^e$ and any parameter μ, we have:

$$\begin{aligned}
a(v,v;\mu) &= \sum_{i=0}^{4} k^i \int_{\Omega^i} \nabla v\cdot\nabla v + \text{Bi}\int_{\Gamma_{ext}\backslash\Gamma_{root}} v^2 \\
&\geq \underbrace{\min_{0\leq i\leq 4} k^i}_{C_1(\mu)>0} \int_{\Omega} \nabla v\cdot\nabla v + \text{Bi}\int_{\Gamma_{ext}\backslash\Gamma_{root}} v^2 \\
&= C_1(\mu)\left(\|\nabla v\|^2_{L^2(\Omega)} + \frac{\text{Bi}}{C_1(\mu)}\|v\|^2_{L^2(\Gamma_{ext}\backslash\Gamma_{root})}\right) \qquad (**)
\end{aligned}$$

Now since Ω is a bounded, open and connected subset of $\mathbb{R}^d$, and $\text{Bi}/C_1(\mu) > 0$, a Poincaré-type inequality (Legoll, 2019, p.48) states there exists a constant $C_2 > 0$ such that

$$\forall v \in H^1(\Omega),\ \ \|\nabla v\|^2_{L^2(\Omega)} + \frac{\text{Bi}}{C_1(\mu)}\|v\|^2_{L^2(\Gamma_{ext}\backslash\Gamma_{root})} \geq C_2\|v\|^2_{H^1(\partial\Omega)}$$

The original inequality presented by F. Legoll has been adapted to suit our case by using the fact that the trace application on a part of the boundary $\gamma : H^1(\Omega) \to L^2(\Gamma_{ext}\backslash\Gamma_{root})$ remains continuous.
Using this result, the inequality $(**)$ is now:

$$a(v,v;\mu) \geq C_3(\mu)\|v\|^2_{H^1(\Omega)}$$

With $C_3(\mu) = C_1(\mu)C_2 > 0$, proving that a is strongly coercive.

We can now return to eq. ($*$). Thanks to the proven coercivity, $a(v, v; \mu) > 0$ since $v \neq 0$. And we have, for any $w \in H^1(\Omega) = X^e$ such that $w \neq u$

$$J(w) = J(u) + \frac{1}{2}a(v, v; \mu) > J(u) = J(u^e(\mu))$$

Therefore, $u^e(\mu)$ as indicated in (7) minimizes the functional J.

Part 2 - Reduced - Basis Approximation

a) Prove that, in the energy norm $|||\cdot||| = a(\cdot,\cdot;\mu)^{1/2}$

$$|||u(\mu) - u_N(\mu)||| \leq |||u(\mu) - w_N||| , \quad \forall w_N \in W_N$$

Let $w_N \in W_N \subset X^e$, we have

$$a(u - u_N, w_N; \mu) = a(u, w_N; \mu) - a(u_N, w_N; \mu)$$
$$= l(w_N) - l(w_N)$$
$$= 0$$

So, we have $a(u - u_N, w_N; \mu) = 0 \quad \forall w_N \subset W_N$.
W_N is a closed subspace of $X^e = H^1$ (Hilbert space thanks to the scalar product $a(\cdot,\cdot,\mu)$). Using the orthogonal projection theorem, we can conclude that

$$|||u(\mu) - u_N(\mu)||| \leq |||u(\mu) - w_N|||,$$

$\forall w_N \in W_N$

b) Proove that

$$T_{root}(\mu) - T_{root\,N}(\mu) = |||u(\mu) - u_N(\mu)|||^2$$

Exploiting the symmetric nature of $a(\cdot,\cdot,\mu)$, we write

$$a(u - u_N, u - u_N; \mu) = a(u,u) - a(u,u_N) - a(u_N,u) + a(u_N,u_N)$$

$$= a(u, \underbrace{u - 2u_N}_{\in X^e}) + a(u_N, \underbrace{u_N}_{\in W_N})$$

$$= l(u - 2u_N) + l(u_N)$$

$$= l(u) - l(u_N) \qquad (l \text{ is linear})$$

Since $l \equiv l^0$, we write $T_{root}(\mu) - T_{root\,N}(\mu) = l^0(u) - l^0(u_N)$
$$= |||u(\mu) - u_N(\mu)|||^2$$

c) Show that $u_N(\mu)$ as defined satisfies a set of $N\times N$ linear equations,

$$\underline{A}_N(\mu)\,\underline{u}_N(\mu) = \underline{F}_N$$

and that $T_{rad\,N}(\mu) = \underline{L}_N^T\,\underline{u}_N(\mu)$

Let's give expressions for $\underline{A}_N(\mu) \in \mathbb{R}^{N\times N}$ in terms of $\underline{A}^{\mathcal{N}}(\mu)$ and Z, $\underline{F}^N \in \mathbb{R}^N$ in terms of $\underline{F}^{\mathcal{N}}$ and Z, and $\underline{L}^N \in \mathbb{R}^N$ in terms of $\underline{L}^{\mathcal{N}}$ and Z.

We have $\quad a(u_N(\mu), v; \mu) = \ell(v) \quad \forall v \in W_N$

and $W_N = \text{span}\{\xi^1, \xi^2, \dots, \xi^N\}$.

Therefore, for any ξ^i

$$a(u_N(\mu), \xi^i; \mu) = \ell(\xi^i) \quad ; \quad i = 1, \dots, N$$

Now, $u_N(\mu) \in W_N \Rightarrow u_N(\mu) = \sum_{j=1}^{N} u_N^j\,\xi^j$

therefore, $a\left(\sum_{j=1}^{N} u_N^j\,\xi^j, \xi^i; \mu\right) = \ell(\xi^i) \quad i = 1, \dots, N$

$$\Rightarrow \sum_{j=1}^{N} a(\xi^j, \xi^i; \mu)\,u_N^j = \ell(\xi^i) \quad i = 1, \dots, N$$

$$\Rightarrow \sum_{j=1}^{N} \left(\underline{A}_N(\mu)\right)_{ij} \left(\underline{u}_N(\mu)\right)_j = \left(\underline{F}_N\right)_i$$

$$\Rightarrow \boxed{\underline{A}_N(\mu)\,\underline{u}_N(\mu) = \underline{F}_N}$$

with $\left(\underline{A}_N(\mu)\right)_{ij} = a(\xi^j, \xi^i) \quad , i,j = 1, \dots, N \qquad (*)$

$\left(\underline{u}_N(\mu)\right)_j = u_N^j \qquad j = 1, \dots, N \qquad (**)$

$\left(\underline{F}_N\right)_j = \ell(\xi^j) \qquad j = 1, \dots, N \qquad (***)$

We know that

$$Z = \left(u_N(\mu^0) \;\vdots\; u_N(\mu^1) \;\vdots\; \cdots \;\vdots\; u_N(\mu^N) \right)$$

$$= \left(\xi^0 \;\vdots\; \xi^1 \;\vdots\; \cdots \;\vdots\; \xi^N \right)$$

On peut aussi remarquer que Z c'est la matrice de passage de W_N à X^e.

This means $\xi^i = \sum_{k=1}^{\mathcal{N}} Z_{ki} \phi_k$ where ϕ_k, $k = 1, \cdots, \mathcal{N}$ are the nodal basis functions in X^e.

* Let's consider (*),

$$\left(\underline{A}_N(\mu)\right)_{ij} = a(\xi^i, \xi^i; \mu), \quad 1 \leq i,j \leq N$$

$$= a\left(\sum_{k=1}^{\mathcal{N}} Z_{ki} \phi_k , \sum_{l=1}^{\mathcal{N}} Z_{lj} \phi_l ; \mu \right)$$

$$= \sum_{k=0}^{\mathcal{N}} \sum_{l=0}^{\mathcal{N}} Z_{ki} Z_{lj} \; a(\phi_k, \phi_l)$$

$$= \sum_{k=0}^{\mathcal{N}} \sum_{l=0}^{\mathcal{N}} Z^T_{ik} \left(\underline{A}^{\mathcal{N}}(\mu)\right)_{kl} Z_{lj}$$

$$\Longrightarrow \boxed{\underline{A}_N(\mu) = Z^T \underline{A}^{\mathcal{N}}(\mu) Z}$$

* Let's consider (**)

$$(\underline{F}_N)_j = l(\xi^j) = l\left(\sum_k Z_{kj} \phi_k \right)$$

$$= \sum_k Z^T_{jk} (\underline{F}^{\mathcal{N}})_k$$

$$\Longrightarrow \boxed{\underline{F}_N = Z^T \underline{F}^{\mathcal{N}}}$$

* We have that

$$
\begin{aligned}
T_{root\,N}(\mu) &= \ell^O(u_N(\mu)) \\
&= \ell(u_N(\mu)) \\
&= \ell\Big(\sum_j u_N^j \xi^j\Big) \\
&= \sum_j (E_N)_j^T u_N^j \\
&= E_N^T u_N \\
&= (Z^T E^N)^T u_N
\end{aligned}
$$

$$\Rightarrow \boxed{L_N = Z^T E^N}$$

d) Let's show that the bilinear form $a(w,v;\mu)$ can be decomposed as

$$a(w,v,\mu) = \sum_{q=1}^{Q} \Theta^q(\mu)\, a^q(w,v), \quad \forall w,v \in X, \ \forall \mu \in D$$

for $Q = 6$ and give expressions for $\Theta^q(\mu)$ and the $a^q(w,v)$.

We have

$$
\begin{aligned}
a(w,v;\mu) &= \sum_{i=0}^{4} k^i \int_{\Omega_i} \nabla w \cdot \nabla v + Bi \int_{\Gamma_{ext}\backslash\Gamma_{root}} w v \\
&= \int_{\Omega^0} \nabla w \cdot \nabla v + \sum_{i=1}^{4} \mu_i \int_{\Omega^i} \nabla w \cdot \nabla v + \mu_5 \int_{\Gamma_{ext}\backslash\Gamma_{root}} w v \\
&= \sum_{q=1}^{Q} \Theta^q(\mu)\, a^q(w,v)
\end{aligned}
$$

with

$\theta^1(\mu) = 1$,	$a^1(w,v) = \int_{\Omega^0} \nabla w \cdot \nabla v$
$\theta^q(\mu) = \mu_{q-1}$,	$a^q(w,v) = \int_{\Omega^{q-1}} \nabla w \cdot \nabla v$, $q = 2, \dots, 5$
$\theta^6(\mu) = \mu_5$,	$a^6(w,v) = \int_{\Gamma_{ext} \setminus \Gamma_{rost}} w \cdot v$

Further, let's show that

$$\underline{A}^{\mathcal{N}}(\mu) = \sum_{q=1}^{Q} \theta^q(\mu)\, \underline{A}^{\mathcal{N}q}$$

$$\underline{A}^{N}(\mu) = \sum_{q=1}^{Q} \theta^q(\mu)\, \underline{A}_N^q$$

giving an expression for the $\underline{A}^{\mathcal{N}q}$ in terms of the nodal basis function, and ~~developping~~ a formula for the $\underline{A}_N^q$ in terms of the $\underline{A}^{\mathcal{N}q}$ and Z.

We know that

$$\begin{aligned}
\left(A^{\mathcal{N}}(\mu)\right)_{ij} &= a(\phi^i, \phi^j; \mu) \\
&= \sum_{q=1}^{Q} \theta^q(\mu)\, a^q(\phi^i, \phi^j) \\
&= \sum_{q=1}^{Q} \theta^q(\mu)\, \underline{A}^{\mathcal{N}q}
\end{aligned}$$

with $\boxed{\left(A^{\mathcal{N}q}\right)_{ij} = a^q(\phi^i, \phi^j)}$, $1 \le i,j \le \mathcal{N}$, q = 1, ...Q

$$\begin{aligned}
A^N(\mu) &= Z^T A^{\mathcal{N}}(\mu)\, Z \\
&= Z^T \left(\sum_{q=1}^{Q} \theta^q(\mu)\, A^{\mathcal{N}q} \right) Z \\
&= \sum_{q=1}^{Q} \theta^q(\mu)\, \underline{A}_N^q \quad \text{with } \boxed{\underline{A}_N^q = Z^T A^{\mathcal{N}q} Z}
\end{aligned}$$

$q = 1, \dots, Q$

e) $\boxed{\text{Let's show that the condition number of } \underline{A}_N(\mu) \text{ is bounded from above by } \gamma^e(\mu)/\alpha^e(\mu)}$

$\underline{A}_N(\mu)$ is symmetric and positive defined thanks to its coercivity. There fore $\boxed{\mathrm{Cond}\, \underline{A}_N(\mu) = \dfrac{\lambda_N(\underline{A}_N(\mu))}{\lambda_1(\underline{A}_N(\mu))}}$

where λ_1 and λ_N are the smallest and largest eigen values of $\underline{A}_N(\mu)$ respectively. Let's compare the above values to $\gamma^e(\mu)$ and $\alpha^e(\mu)$.

Let $v \in W_N \subset X^e$.

* a is coercive, there fore,

W_N is a subspace of X^e

$$\inf_{v \in W_N} \frac{a(v,v;\mu)}{\|v\|^2_{W_N}} \geq \inf_{v \in X^e} \frac{a(v,v;\mu)}{\|v\|^2_{X^e}} \geq \alpha^e_N(\mu) \qquad (*)$$

must be different from 0

* a is continues, there fore.

$$\sup_{v \in W_N} \sup_{v \in W_N} \frac{a(v,v;\mu)}{\|v\|_{W_N}\|v\|_{W_N}} \leq \sup_{v \in X^e} \sup_{v \in X^e} \frac{a(v,v;\mu)}{\|v\|_{X^e}\|v\|_{X^e}} \leq \gamma^e(\mu)$$

$$\Rightarrow \sup_{v \in W_N} \frac{a(v,v;\mu)}{\|v\|^2_{W_N}} \leq \gamma^e(\mu) \qquad (**)$$

Since the basis $(\xi^i)_{i=1,\dots,N}$ spanning W_N is orthonormalized, $\|v\|^2_{W_N} = v^T v$.

Hence the term

$$\frac{a(v,v;\mu)}{\|v\|^2_{W_N}} = \frac{v^T \underline{A}_N(\mu)\, v}{v^T v}$$

can be recognized as the Rayleigh Quotient $R(\underline{A}_N(\mu), v)$.

The Rayleigh Quotient

$$R(\underline{A}_N(\mu), v) = \frac{v^T A_N(\mu) v}{v^T v} = \frac{a(v, v, \mu)}{\|v\|^2_{W_N}}$$

is caraterized by

$$\inf_{v \in W_N} R(\underline{A}_N(\mu), v) = \lambda_1(\underline{A}_N(\mu))$$

$$\sup_{v \in W_N} R(\underline{A}_N(\mu), v) = \lambda_N(\underline{A}(\mu))$$

The inequalities (*) and (**) become

$$\lambda_1(\underline{A}_N(\mu)) \geq \alpha^e(\mu)$$

$$\lambda_N(\underline{A}(\mu)) \leq \gamma^e(\mu)$$

Therefore, the condition number

$$\text{Cond } \underline{A}_N(\mu) = \frac{\lambda_N(\underline{A}_N(\mu))}{\lambda_1(\underline{A}_N(\mu))} \leq \frac{\gamma^e(\mu)}{\alpha^e(\mu)}$$

References

- Legoll, F. (2019). *Partial Differential Equations and the Finite Element Method*. Retrieved from http://cermics.enpc.fr/~legoll/poly_EDP-EF_jan19.pdf.