$$
\begin{aligned}
&\|\hat{e}(\mu)\|_X^2 \\
&= (\mathcal{C},\mathcal{C})_X + \sum_{q=1}^{Q_a}\sum_{n=1}^{N} \Theta_a^q(\mu) u_{N\,n}(\mu) \Bigg\{ \\
&\quad 2(\mathcal{C},\mathcal{A}_n^q)_X + \sum_{q'=1}^{Q_a}\sum_{n'=1}^{N} \Theta_a^{q'}(\mu) u_{N\,n'}(\mu) (\mathcal{A}_n^q,\mathcal{A}_{n'}^{q'})_X \Bigg\}
\end{aligned}
$$

- Compute $\underline{\boldsymbol{\mathcal{C}}} \in \mathbb{R}^{\boldsymbol{\mathcal{N}}}$ and $\underline{\boldsymbol{\mathcal{A}}}^q \in \mathbb{R}^{\boldsymbol{\mathcal{N}} \times \boldsymbol{N}}$, $\mathbf{1} \leq \boldsymbol{q} \leq \boldsymbol{Q_a}$, from

$$\begin{aligned} \underline{\underline{\mathbb{X}}}^{\boldsymbol{\mathcal{N}}} \underline{\boldsymbol{\mathcal{C}}} &= \underline{\boldsymbol{F}}^{\boldsymbol{\mathcal{N}}}; \\ \underline{\underline{\mathbb{X}}}^{\boldsymbol{\mathcal{N}}} \underline{\boldsymbol{\mathcal{A}}}^q &= -\underline{\underline{\mathbb{A}}}^{\boldsymbol{\mathcal{N}} q} \underline{\underline{\mathbb{Z}}}_{\boldsymbol{N}}, \quad \mathbf{1} \leq \boldsymbol{q} \leq \boldsymbol{Q_a}. \end{aligned}$$

- Form/Store

$$\begin{aligned} \underline{\boldsymbol{\mathcal{C}}}^T \underline{\underline{\mathbb{X}}}^{\boldsymbol{\mathcal{N}}} \underline{\boldsymbol{\mathcal{C}}} &\in \mathbb{R}; \\ (\underline{\boldsymbol{\mathcal{A}}}^q)^T \underline{\underline{\mathbb{X}}}^{\boldsymbol{\mathcal{N}}} \underline{\boldsymbol{\mathcal{C}}} &\in \mathbb{R}^{\boldsymbol{N}}, \qquad \mathbf{1} \leq \boldsymbol{q} \leq \boldsymbol{Q_a}; \\ (\underline{\boldsymbol{\mathcal{A}}}^{q'})^T \underline{\underline{\mathbb{X}}}^{\boldsymbol{\mathcal{N}}} \underline{\boldsymbol{\mathcal{A}}}^q &\in \mathbb{R}^{\boldsymbol{N} \times \boldsymbol{N}}, \quad \mathbf{1} \leq \boldsymbol{q}, \boldsymbol{q}' \leq \boldsymbol{Q_a}. \end{aligned}$$

$$
\begin{aligned}
&\|\hat{e}(\mu)\|_X^2 \\
&= \underline{\mathcal{C}}^T \underline{\mathbb{X}}^{\mathcal{N}} \underline{\mathcal{C}} + \sum_{q=1}^{Q_a} \Theta_a^q(\mu)(\underline{u}_N(\mu))^T \Bigg\{ \\
&\qquad 2(\underline{\mathcal{A}}^q)^T \underline{\mathbb{X}}^{\mathcal{N}} \underline{\mathcal{C}} + \sum_{q'=1}^{Q_a} \Theta_a^{q'}(\mu)\,(\underline{\mathcal{A}}^{q'})^T \underline{\mathbb{X}}^{\mathcal{N}} \underline{\mathcal{A}}^q\, \underline{u}_N(\mu) \Bigg\}
\end{aligned}
$$

- Rigorous, sharp, inexpensive error bound, $\mathbf{1 \le N \le N_{max}}$:

$$\begin{aligned} \|u(\mu) - u_N(\mu)\|_X &\le \Delta_N(\mu), \quad \forall \mu \in \mathcal{D}, \\ |||u(\mu) - u_N(\mu)|||_\mu &\le \Delta_N^{\mathbf{en}}(\mu), \quad \forall \mu \in \mathcal{D}. \end{aligned}$$

- Greedy Idea:
  - $\Delta_N^{(\mathbf{en})}(\mu)$ is measure of how “good” the RB approximation is.
  - Perform a successive search over $\Xi_{\mathbf{train}}$ to find “worst” approximation.
  - Update sample (and hence RB space) with parameter value resulting in “worst” approximation.
- Note: rigour, sharpness, and efficiency of error bound crucial.
- Lagrange spaces, i.e., $X_N = W_N$.

Given: – desired error tolerance $\varepsilon_{\mathbf{tol,min}}$,
– initial sample $S_1 = \mu_1^*$ (random or $\mu^{\mathbf{min,max}}$), and
– space $X_1 = \mathbf{span}\{u(\mu_1^*)\}$.

Greedy Algorithm

```
while
```
$\Delta_{N-1}^{\mathbf{max}} \geq \varepsilon_{\mathbf{tol,min}}$

$$N = N + 1;$$

$$\mu_N^* = \arg \max_{\mu \in \Xi_{\mathbf{train}}} \Delta_{N-1}(\mu);$$

$$\Delta_{N-1}^{\mathbf{max}} = \Delta_{N-1}(\mu_N^*);$$

$$S_N = S_{N-1} \cup \mu_N^*;$$

$$X_N = X_{N-1} + \mathbf{span}\{u^{\mathcal{N}}(\mu_N^*)\};$$

```
end
```

Comments

- $\boldsymbol{X_N} = \boldsymbol{X_N^{\mathbf{Greedy}}}$ are hierarchical.
- Sub-optimal solution to $\boldsymbol{L^\infty(\Xi_{\mathbf{train}})}$ optimization problem.
- Define the "true" error, $\mathbf{1} \leq \boldsymbol{N} \leq \boldsymbol{N_{\mathbf{max}}}$,

$$\bar{\varepsilon}_N^* = \arg \max_{\mu \in \Xi_{\mathrm{train}}} \|u(\mu) - u_N(\mu)\|_X,$$

then $\bar{\varepsilon}_N^*$ is bounded by

$$\bar{\varepsilon}_N^* \leq \Delta_N(\mu) \leq \varepsilon_{\mathrm{tol,min}}, \quad \forall \mu \in \Xi_{\mathrm{train}}.$$

- Condition on $\boldsymbol{N_{\mathbf{max}}}$ possible (hp-Reduced Basis).
- Perform Gram-Schmidt orthogonalization on $\boldsymbol{X_N}$.

Greedy, $L^\infty(\Xi_{\text{train}}, X)$, space "economization"

$$n_{\text{train}} \text{ contestants} \quad \Rightarrow \quad N_{\max} \ll n_{\text{train}} \text{ winners}$$

$$\in \Xi_{\text{train}} \qquad\qquad \mu_1^*, \ldots, \mu_{N_{\max}}^*$$

in which we *never form/calculate* most snapshots:

$$\|u(\mu) - u_N(\mu)\|_X \quad \text{replaced} \quad \Delta_N(\mu)$$

$$n_{\text{train}} \cdot O(\mathcal{N}^\bullet) \quad \text{by} \quad n_{\text{train}} \cdot O(Q^2 N^2)\ ^\dagger$$

note good *effectivity* of estimator is crucial.

Given: – desired error tolerance $\varepsilon_{\mathbf{tol,min}}$,

– initial sample $S_1^{\mathbf{out}} = \mu_1^{\mathbf{out},*}$ (random or $\mu^{\mathbf{min,max}}$), and

– space $X_1^{\mathbf{out}} = \mathrm{span}\{u(\mu_1^{\mathbf{out},*})\}$.

Greedy Algorithm

$$\mathtt{while}\ \Delta_{N-1}^{\mathbf{out,max}} \geq \varepsilon_{\mathbf{tol,min}}$$

$$N = N + 1;$$

$$\mu_N^{\mathbf{out},*} = \arg \max_{\mu \in \Xi_{\mathbf{train}}} ((\omega(\mu))^{-1} \Delta_{N-1}^{\mathbf{en}}(\mu);$$

$$\Delta_{N-1}^{\mathbf{out,max}} = ((\omega(\mu_N^{\mathbf{out},*}))^{-1} \Delta_{N-1}^{\mathbf{en}}(\mu_N^{\mathbf{out},*});$$

$$S_N^{\mathbf{out}} = S_{N-1}^{\mathbf{out}} \cup \mu_N^{\mathbf{out},*};$$

$$X_N^{\mathbf{out}} = X_{N-1}^{\mathbf{out}} + \mathrm{span}\{u^{\mathcal{N}}(\mu_N^{\mathbf{out},*})\};$$

$$\mathtt{end}$$

Comments

- $X_N^{\mathrm{out}} = X_N^{\mathrm{out,Greedy}}$ are hierarchical.
- Sub-optimal solution to $L^{\infty}(\Xi_{\mathrm{train}})$ optimization problem.
- Computational cost equivalent to W_N^*-Greedy
- Define the relative "true" error, $1 \leq N \leq N_{\max}$,

$$\bar{\varepsilon}_N^{\mathrm{out},*} = \arg \max_{\mu \in \Xi_{\mathrm{train}}} (\omega(\mu))^{-1} |||u(\mu) - u_N(\mu)|||_{\mu},$$

then $\bar{\varepsilon}_N^{\mathrm{out},*}$ is bounded by

$$\bar{\varepsilon}_N^{\mathrm{out},*} \leq (\omega(\mu))^{-1} \Delta_N^{\mathrm{en}}(\mu) \leq \varepsilon_{\mathrm{tol,min}}, \quad \forall \mu \in \Xi_{\mathrm{train}}.$$

Comments

- Direct control of (relative) RB error (Galerkin optimality)

$$\boldsymbol{\omega(\mu) \equiv |||u_N(\mu)|||_\mu, \quad \forall \mu \in \mathcal{D}.}$$

- Compliant case: direct control of (relative) error in the RB output prediction

$$\boldsymbol{\omega(\mu) \equiv |s_N(\mu)|, \quad \forall \mu \in \mathcal{D}.}$$

- Use of sharper bound since $\boldsymbol{\eta^{\mathrm{en}}_{\mathrm{max,UB}} \le \eta_{\mathrm{max,UB}}}$.
- Perform Gram-Schmidt orthogonalization on $\boldsymbol{X_N}$.