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CMPS 102
Homework 1

Problem 1: In a binary tree all nodes are either internal or they are leaves. In our definition, internal nodes always have two children and leaves have zero children. Prove that for such trees, the number of leaves is always one more than the number of internal nodes.

Proof:
If T is a tree containing n internal nodes, then T contains $n + 1$ leaves

I. Base Case
If T has just one internal node, then T has two leaves, thus the number of leaves in T is one greater than the number of internal nodes in T.

II. Induction Step
Let $n \geq 1$. Assume that all trees containing n internal nodes contain $n + 1$ leaves. Generate another internal node on T by taking an arbitrary leaf node in T and attaching two leaf nodes to it. By generating another internal node, T now contains $n + 1$ internal nodes. In the process of generating a new internal node, one leaf node is eliminated from T and two leaf nodes are attached to T, so T now contains $((n + 1) - 1) + 2$ leaf nodes. By the induction hypothesis, a tree containing $n + 1$ internal nodes contains $(n + 1) + 1$ leaves.
Thus, $((n + 1) - 1) + 2 = (n + 1) + 1$ as required.

Problem 2: For $n \geq 0$ consider $2^n \times 2^n$ matrices of 1s and 0s in which all elements are 1, except one which is 0 (The 0 is at an arbitrary position). Operation: At each step, we can replace three 1s forming an "L" with three 0s (The L's can have an arbitrary orientation).

Proof:
If M is a $2^n \times 2^n$ matrix consisting of all 1s and one 0, there exists a sequence of "L" operations such that replacing three 1s in an "L" pattern in M will give us the 0 matrix.

I. Base Case
$n = 0$. A $2^0 \times 2^0$ matrix is a 1×1 matrix. The only entry can be a 0, and thus applying the "L" operation 0 times is sufficient to obtain the 0 matrix.

II. Induction Step
Let $n \geq 0$. Let M be a $2^n \times 2^n$ matrix where M contains exactly one 0 and the rest 1s. Assume there exists a sequence of "L" operations to obtain a 0 matrix for M. Let M' be a $2^{n+1} \times 2^{n+1}$ matrix containing exactly one 0 entry with 1s for the rest of the entries.
(continued on the next page.)

$$M' = \begin{bmatrix} 0 & \dots & 1 & 1 & \dots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \dots & 1 & 1 & \dots & 1 \\ 1 & \dots & 1 & 1 & \dots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \dots & 1 & 1 & \dots & 1 \end{bmatrix}$$

Since M' is a $2^{n+1} \times 2^{n+1}$ matrix, M' can be evenly divided into four quadrants with each quadrant containing $2^n \times 2^n$ matrix entries. This means that the initial 0 entry in M' must be within one of the four quadrants.