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CMPS 102
Homework 3

Problem 1:
1a) Show by induction that $(H_k)^2 = 2^k I_k$, where I_k is the identity matrix of dimension 2^k.

Base Case: $k = 0$. Then $H_0^2 = [1][1] = [1] = 2^0 \cdot I_0$.

Induction Step: Let $n \geq 0$ and $0 \leq n < k$. Suppose that $(H_n)^2 = 2^n I_n$, where I_n is the identity matrix of dimension 2^n.
Then $(H_k)^2 = \begin{bmatrix} H_{k-1} & H_{k-1} \\ H_{k-1} & -H_{k-1} \end{bmatrix} \begin{bmatrix} H_{k-1} & H_{k-1} \\ H_{k-1} & -H_{k-1} \end{bmatrix} = \begin{bmatrix} 2 \cdot H_{k-1}^2 & H_{k-1}^2 - H_{k-1}^2 \\ H_{k-1}^2 - H_{k-1}^2 & 2 \cdot H_{k-1}^2 \end{bmatrix}$

Let I_{k-1} denote the identity matrix of dimension 2^{k-1} and 0_{k-1} denote the 0 matrix of dimension 2^{k-1}. By the induction hypothesis, $2(H_{k-1})^2$ can be simplified as $2 \cdot 2^{k-1} I_{k-1}$. $H_{k-1}^2 - H_{k-1}^2$ can be simplified down to 0_{k-1} since the difference of any matrix with itself is the 0 matrix. Thus $(H_k)^2$ can be simplified as

$$\begin{bmatrix} 2 \cdot 2^{k-1} I_{k-1} & 0_{k-1} \\ 0_{k-1} & 2 \cdot 2^{k-1} I_{k-1} \end{bmatrix} = \begin{bmatrix} 2^k I_{k-1} & 0_{k-1} \\ 0_{k-1} & 2^k I_{k-1} \end{bmatrix} = 2^k \begin{bmatrix} I_{k-1} & 0_{k-1} \\ 0_{k-1} & I_{k-1} \end{bmatrix}$$

Notice that two identity matrices of dimensions $k-1 \times k-1$ lie within the main diagonal, and long the antidiagonal are two 0 matrices of dimensions $k-1 \times k-1$. This means that the matrix itself the matrix itself is the identity matrix with dimensions $2^k \times 2^k$.
Thus, $2^k \begin{bmatrix} I_{k-1} & 0_{k-1} \\ 0_{k-1} & I_{k-1} \end{bmatrix} = 2^k I_k$ as required.

1b) Note that Hadamard matrices are symmetric, i.e. $H_k = H_k^\top$. Thus by the above, $H_k H_k^\top = 2^k I_k$ as well. Use this fact for deriving a formula for the dot product between the i-th and j-th row of H_k, for $1 \leq i, j \leq 2^k$.

The dot product between the i-th and j-th is 0 whenever $i \neq j$ and the sum of the squares of all the matrix entries whenever $i = j$. In other words, the dot product between any two rows i and j is defined by the following summation:

$$\text{Dot product between the } i\text{th and } j\text{th row of } H = \begin{cases} i \neq j & 0 \\ i = j & \sum_{j=1}^{2^k} H_{ij}^2 \end{cases}$$

Problem 2: Consider the Coin Changing problem with the European coin set:

$$\{1, 2, 5, 10, 20, 50, 100, 200\}.$$

Prove that the Cashier's Algorithm is optimal given the above set of coins. Use the same proof method that was used for the American coin set in class.

Problem 3: Given a sorted array of distinct integers $A[1, ..., n]$, you want to find out whether there is an index i for which $A[i] = i$. Give a divide-and-conquer algorithm that runs in time $O(\log n)$.
Algorithm: