Let $\gcd(a,b)$ be the largest positive integer that divides both a and b. Assume $\gcd(0,0) = 0$. Negative a and b are allowed, but $\gcd(a,b)$ is always nonnegative.

When traveling from (a,b) to a next point, the gcd of the points is maintained. To show this, we first notice that any common divisor of a and b divides $a+b$ and $a-b$. So in particular, the greatest one, $\gcd(a,b)$, divides $a+b$ and $a-b$. Therefore, $\gcd(a,b)$ divides $\gcd(a+b,b)$ and $\gcd(a-b,b)$. Also, $a = (a+b)-b = (a-b)+b$, so by the same argument, $\gcd(a+b,b)$ and $\gcd(a-b,b)$ also divides $\gcd(a,b)$. Therefore they must all be equal.

So, if $\gcd(a,b) \neq \gcd(x,y)$, the answer is immediately "NO", because we can only reach points with the same gcd. The more surprising fact is that if $\gcd(a,b) = \gcd(x,y)$, then the answer is "YES"!

To show this, first note that the individual traversal operations are reversible. For example, from (a,b), if we go to $(a+b,b)$, we can go back to (a,b) because $(a,b) = ((a+b)-b,b)$. Specifically, the first and third kinds of traversals are reverse operations of each other, and the same is true for the second and fourth. So instead of saying "(a,b) can reach (x,y)", we say instead "(a,b) is connected to (x,y)" to emphasize the reversibility of the traversal.

Lemma 0.1. *A point (a,b) where $g = \gcd(a,b)$ is connected to one of the following points: $(g,0)$, $(0,g)$, $(-g,0)$, $(0,-g)$.*

Proof. For $g = 0$, the only point is $(0,0)$, so the result holds.

For a fixed $g > 0$, we'll prove the statement for all points (a,b) with $\gcd(a,b) = g$ by induction on increasing $|a|+|b|$. The number $|a|+|b|$ is always a positive multiple of g.

For the base case, if $|a|+|b| = g$, then (a,b) is already one of $(g,0)$, $(0,g)$, $(-g,0)$, $(0,-g)$, so the result holds.

Now suppose we want to show the result for (a,b) with $|a|+|b| > g$. Then:

- If $|a| \leq |b|$, then either $a-b$ or $a+b$ has smaller absolute value than b, so we go to either $(a,a+b)$ or $(a,a-b)$.

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- If $|a| > |b|$, then either $a - b$ or $a + b$ has smaller absolute value than a, so we go to either $(a + b, b)$ or $(a - b, b)$.

In any case, we end up with a smaller sum of absolute values, which by induction hypothesis is connected to one of $(g, 0)$, $(0, g)$, $(-g, 0)$, $(0, -g)$. Therefore the inductive case holds. $\square$

That $a - b$ or $a + b$ has smaller absolute value than $\max(|a|, |b|)$ follows from the fact that neither of a and b are zero (if one of them is zero and $|a| + |b| > g$ then $\gcd(a, b) > g$).

Now,

- $(g, 0)$ is connected to $(0, -g)$ because

$$(g, 0) \to (g, 0 - g) = (g, -g) \to (g + (-g), -g) = (0, -g)$$

- $(g, 0)$ is connected to $(0, g)$ because

$$(g, 0) \to (g, g + 0) = (g, g) \to (g - g, g) = (0, g)$$

- $(g, 0)$ is connected to $(-g, 0)$ because $(g, 0)$ is connected to $(0, -g)$ (using the first property), which is connected to $(-g, 0)$ (using the second property).

Therefore, $(g, 0)$, $(0, g)$, $(-g, 0)$ and $(0, -g)$ are all connected to each other.

So if $\gcd(a, b) = \gcd(x, y)$, (x, y) is connected to (a, b) because:

1. (a, b) is connected to one of $(g, 0)$, $(0, g)$, $(-g, 0)$, $(0, -g)$,
2. (x, y) is connected to one of $(g, 0)$, $(0, g)$, $(-g, 0)$, $(0, -g)$,
3. $(g, 0)$, $(0, g)$, $(-g, 0)$ and $(0, -g)$ are all connected to each other.

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