Percolation project

Yang Liu

Faculty of Natural and Mathematical Science
King's College London

Contents

Chapter 1

formula

1.1 basics

1.1.1 generating functions

$$G_0(x) = \sum_k p_k x^k \tag{1.1}$$

$$G_1(x) = \sum_k q_k x^k \tag{1.2}$$

Since $q_k = \frac{(k+1)p_{k+1}}{\langle k \rangle}$, the excess degree distribution defined by..., we can rewrite $G_1(x) = \sum_k \frac{kp_k}{\langle k \rangle} x^{k-1}$.

The generating function for the probability distribution of second, third, even d-th distance neighbours' total number of nodes is derived as following:

- $G^{(2)}(x) = G_0(G_1(x))$
- $G^{(d)}(x) = G^{(d-1)}(G_1(x))$ In other words,$G^{(d)}(x) = G_0(G_1(\cdots G_1(x)\cdots))$, with $d-1$ copies of G_1 nested inside a single G_0

$$H_0(x) = \sum_{s=1}^{\infty} \pi_s x^s \tag{1.3}$$

$$H_1(x) = \sum_{s=1}^{\infty} \rho_s x^s \tag{1.4}$$

$$H_0(x) = xG_0(H_1(x)) \tag{1.5}$$

$$H_1(x) = xG_1(H_1(x)) \tag{1.6}$$

$$S = 1 - \sum_s \pi_s = 1 - H_0(1) = 1 - G_0(H_1(1)) = 1 - G_0(G_1(H_1(1))) \tag{1.7}$$

1.1.2 percolation

u, the probability that a randomly chosen vertex does NOT belong to giant cluster. $\tilde{u}$, probability that a vertex chosen by following a random edge does NOT belong to giant cluster.

- $u = \sum_k p_k \tilde{u}^k$
- $\tilde{u} = \sum_k \frac{kp_k}{\langle k \rangle} \tilde{u}^{k-1}$
- $\tilde{u}^k = \sum_{k_1} \cdots \sum_{k_k} \prod_{\alpha=1}^{k} \left(\frac{k_\alpha p_{k_\alpha}}{\langle k \rangle} \tilde{u}^{k_\alpha - 1} \right)$

Notation design:

The initial node (center node) has degree m. $\mathbf{k}^{(d)}$ is the degree vector for nodes have distance d with the initial node. Thus, $\mathbf{k}^{(0)} = m$, $\mathbf{k}^{(1)} = (\alpha_1, \alpha_2, \cdots, \alpha_m)$.

Denote n_d as the total number of nodes in the d-th distance shell from the initial node, then $n_1 = m$, $n_2 = \sum_{i=1}^{m} \alpha_i - n_1 = ||\mathbf{k}^{(1)}||_1 - n_1$, here $|| \cdots ||_1$ is the so-called taxicab norm. Therefore, $\mathbf{k}^{(2)} = (\beta_1, \beta_2, \cdots, \beta_{n_2})$, $\mathbf{k}^{(3)} = (\gamma_1, \gamma_2, \cdots, \gamma_{n_3})$, $n_3 = \sum_{i=1}^{n_2} \beta_i - n_2$, $n_4 = \sum_{i=1}^{n_3} \gamma_i - n_3$, $\cdots$.

Then we can develop the joint probability for initial node does NOT belong to giant cluster with microscopic degree structure, say $\mathbf{k}^{(d)}$ as $d = 1, 2, \cdots$.

$$
\begin{aligned}
u &= \sum_{m=1}^{\infty} p_m \tilde{u}^m && (1.8) \\
&= \sum_{m=1}^{\infty} p_m \prod_{i=1}^{m} \left(\sum_{\alpha_i} \frac{\alpha_i p_{\alpha_i}}{\langle \alpha_i \rangle} \tilde{u}^{\alpha_i - 1} \right) && (1.9) \\
&= \sum_{m=1}^{\infty} p_m \sum_{\alpha_1, \cdots, \alpha_m} \left(\prod_{i=1}^{m} \frac{\alpha_i p_{\alpha_i}}{\langle \alpha_i \rangle} \right) \tilde{u}^{n_2} && (1.10) \\
&= \sum_{m=1}^{\infty} p_m \sum_{\mathbf{k}^{(1)}} \left(\prod_{i=1}^{m} \frac{\alpha_i p_{\alpha_i}}{\langle \alpha_i \rangle} \right) \tilde{u}^{n_2} && (1.11) \\
&= \sum_{m=1}^{\infty} p_m \sum_{\mathbf{k}^{(1)}} \left(\prod_{i=1}^{m} \frac{\alpha_i p_{\alpha_i}}{\langle \alpha_i \rangle} \right) \prod_{j=1}^{n_2} \left(\sum_{\beta_j} \frac{\beta_j p_{\beta_j}}{\langle \beta_j \rangle} \tilde{u}^{\beta_j - 1} \right) && (1.12) \\
&= \sum_{m=1}^{\infty} p_m \sum_{\mathbf{k}^{(1)}} \left(\prod_{i=1}^{m} \frac{\alpha_i p_{\alpha_i}}{\langle \alpha_i \rangle} \right) \sum_{\mathbf{k}^{(2)}} \left(\prod_{j=1}^{n_2} \frac{\beta_j p_{\beta_j}}{\langle \beta_j \rangle} \right) \tilde{u}^{n_3} && (1.13) \\
&= \sum_{m=1}^{\infty} p_m \sum_{\mathbf{k}^{(1)}} \left(\prod_{i=1}^{m} \frac{\alpha_i p_{\alpha_i}}{\langle \alpha_i \rangle} \right) \sum_{\mathbf{k}^{(2)}} \left(\prod_{j=1}^{n_2} \frac{\beta_j p_{\beta_j}}{\langle \beta_j \rangle} \right) \sum_{\mathbf{k}^{(3)}} \left(\prod_{l=1}^{n_3} \frac{\gamma_l p_{\gamma_l}}{\langle \gamma_l \rangle} \right) \tilde{u}^{n_4} && (1.14) \\
& && (1.15)
\end{aligned}
$$

Rearrange the above equation, then we will have the following general expression:

$$u = \sum_{\mathbf{k}^{(0)}} \cdots \sum_{\mathbf{k}^{(d)}} p_{\mathbf{k}^{(0)}} \prod_{i=1}^{n_1} \cdots \prod_{q=1}^{n_d} \left(\frac{\alpha_i p_{\alpha_i}}{\langle \alpha_i \rangle} \cdots \frac{\eta_q p_{\eta_q}}{\langle \eta_j \rangle} \right) \tilde{u}^{n_{d+1}} \tag{1.16}$$

$$= \sum_{\mathbf{k}^{(0)}} \cdots \sum_{\mathbf{k}^{(d)}} p_{\mathbf{k}^{(0)}} \prod_{i=1}^{n_1} \cdots \prod_{q=1}^{n_d} \left(c^{-d} \alpha_i p_{\alpha_i} \cdots \eta_q p_{\eta_q} \right) \tilde{u}^{n_{d+1}} \qquad \text{since } \langle \cdots \rangle = c \tag{1.17}$$

It is also intuitive that the conditional probability for initial node does NOT belong to giant cluster given $\mathbf{k}^{(d)}$ is simply

$$u|_{\mathbf{k}^{(d)}} = \tilde{u}^{n_{d+1}} \tag{1.18}$$

and $\tilde{u}$ is well defined by $\tilde{u} = \sum_k \frac{kp_k}{\langle k \rangle} \tilde{u}^{k-1}$, which can be graphically analysed.