Minimum Vertex Cover

in arbitrary conflict graph problems

1. Question-of-the-day

In an art museum, tight security is crucial to prevent the theft of valuable paintings. Cameras placed at hallway corners, where two hallways intersect, can effectively cover paintings in both hallways.
How can security personnel install these cameras most economically?

2. Overview

This project aims to learn and implement **minimum vertex cover** - a topic in graph theory that has applications in matching and optimization problems.

3. Definition & Terminologies

$G = (V, E)$

A graph **G** is a collection of **vertices V** and **edges E**

Constraint problems generalization

Arbitrary conflict graph
- **V (vertices)** — objects
- **E (edges)** — relationships (e.g. conflict, crossing, overlap)

Graph G

Vertex Cover (size = 3)

Minimum Vertex Cover (size = 2)

Vertex Cover

a subset S of its vertices that covers all edges of G such that for every edge (u, v) of the graph, either vertex u or v is in the vertex cover

$S \subseteq V, (u, v) \in E, u \in S \vee v \in S$

Minimum Vertex Cover

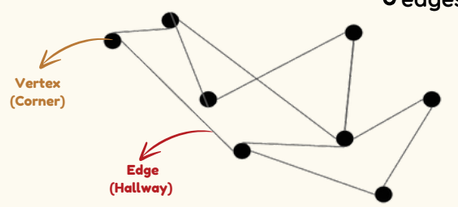
a vertex cover of G with the smallest size, denoted as $\tau(G)$

Characterization of Minimum Vertex Cover

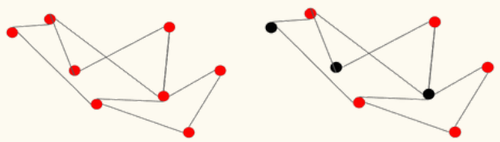
- **Covering Property: Every edge in the graph is incident to at least one vertex in the subset.**
- **Minimality: There is no smaller set of vertices that covers all the edges in the graph.**

4. Solution for our question

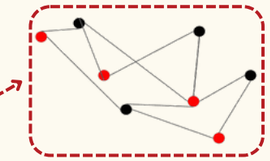
Model the museum map as a **graph**
- vertices → **hallways**
- edges → **corners**



Determine the vertex cover set to find out where to place the cameras



WANT TO MINIMIZE the cost of installing cameras while still covering the security of all properties → **Minimum Vertex Cover**



Just need to install 4 cameras in total!

References

CS 354: Unfulfilled Algorithmic Fantasies. (2019). Available at: https://web.stanford.edu/class/cs354/scribe/lecture09.pdf [Accessed 12 May 2024].
brilliant.org. (n.d.). Vertex Cover | Brilliant Math & Science Wiki. [online] Available at: https://brilliant.org/wiki/vertex-cover/.
Csorba, P., Hurkens, C.A.J. and Woeginger, G.J. (2012). The Alcuin Number of a Graph and Its Connections to the Vertex Cover Number. SIAM Review, [online] 54(1), pp.141–154. Available at: https://www.jstor.org/stable/41642576 [Accessed 12 May 2024].

Our team

Tran Nguyen Hanh Nguyen
Le Thuy Nguyen

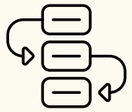
Special thanks to Professor Linh Tran & TA. Phuong Pham

5. Implementation

A program to find minimum vertex cover
- Language: **Python**
- Libraries: **matplotlib, networkx, tkinter**
- Time Complexity: **O((V+E) * log V)**

- **Graph Representation: using an adjacency matrix to check edge connectivity and manipulate graph structure.**
- **Covering Property Validation: utilizes dynamic programming and uses bit manipulation to efficiently generate and check subsets of vertices for covering all edges in the graph**
  - **iterates through all possible subsets of vertices of size k**
  - **checks if they cover all the edges in the graph**
- **Minimality Validation: employs a binary search approach to find the minimum vertex cover size by iteratively narrowing down the search space**
  - **starts with the smallest possible cover size left=1 and the largest possible cover size right=n (n: total number of vertices)**
  - **repeatedly calls the Covering Property Validation function to check if a cover of size m = midpoint of left and right exists**
    - **if it does, it updates the right boundary to m**
    - **otherwise, it updates the left boundary to m + 1**

6. Application

Field	Generalize into conflict graph	Goal	Vertex cover can...
Logistics and Scheduling 	In a project management scenario: • Vertices - Tasks that need to be completed. • Edges - drawn between two tasks if one task depends on the completion of the other.	Determine a schedule that satisfies all precedence constraints.	Find the minimum number of tasks that need to be scheduled such that all task dependencies are satisfied.
Network Design (Computer Networks) 	A network of computers where: • Vertex - Each computer • Edge - the corresponding computers can directly communicate with each other.	Monitor the network traffic efficiently.	Represent a set of computers that can monitor all communications in the network. --> Detecting anomalies or intrusions in the network.

7. Conclusion

- A vertex cover might be a good approach to a problem where all of the edges in a graph need to be included in the solution (e.g. **solving matching and optimization problems**).
- **Discussion & Limitation:**
  - **Vertex Cover Problem** is a well-known NP-hard optimization problem in graph theory**.** Our algorithm of utilizing dynamic programming and binary search is not the most efficient and researchers continue to explore efficient algorithms.
  - Our initial intention was to apply the minimum vertex cover concept to solve the river crossing problem.
    - Unfortunately, we've made progress only in determining the minimum boat capacity and identifying the objects for the first move. We welcome any thoughts or insights on this matter.