KAIST

Fall 2023

November 26, 2023

Final Exam Part 2:

Take-home Team Project

Due: 12:00 PM December 19th (Tuesday)

(To be submitted to `chpark1111@kaist.ac.kr`)

N.B. The course's pedogogy: As more evident in the final exam questions to follow, the course's primary teaching method is what one might often refer to as Moore's *Texas method*. The latter method, in a nutshell, amounts to the small group discovery method, suggesting students are required to discover necessary principles and theorems all on their own by challenging and discussing unsolved problems without assuming any prerequisites related to them. All lecture notes posted as of Nov. 26th will guide and help you to solve the questions by exploiting Moore's Texas method contextualized in our course. Indeed, John Nash's notorious MIT exam is viewed as the epitome of Moore's method of Texas. It must be noted, moreover, that the exam's individual couterpart, Final Exam Part 3, will be distributed on the date of Dec. 11th in the final exam period, indicating that students are supposed to solve the rest of exam questions on an individual basis but using their own small group's already-discovered principles and theorems.

Please, answer all questions. This team project is worth a total of 1200 points.

PROBLEM (1200 points)

Consider an optimal growth model with uncertainty. For simplicity's sake, we leave out population growth and technology growth as well.

There is a representative infinitely lived household with preferences given by

$$\mathbb{E}\Big(\sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_t) \mid \mathcal{I}_0\Big), \tag{1}$$

where β is the subjective discount factor of the representative household, with $0 < \beta < 1$, c_t is consumption in period t, and $0 \leq n_t \leq 1$ is labour hours worked in period t. $\mathbb{E}(\cdot \mid \mathcal{I}_0)$ is the conditional expectation in period 0, i.e. "rational expecations" conditional on information accumulated up to period 0, $\mathcal{I}_0$. The period uitility function u is specified by

$$u(c_t, 1-n_t) = \log c_t + \chi \log(1-n_t) \tag{2}$$

where χ is a "disutility" parameter. Each period, the household is endowed with one unit of time and consumes "leisure good" $1 - n_t$.

The production function is given by

$$y_t = z_t(k_t)^{\alpha}(n_t)^{1-\alpha} \tag{3}$$

where y_t is output, k_t is the capital stock, and z_t is total factor productivity. Total factor productivity z_t follows a random process over time. It should be noted that the introduction of the random process z_t creates *uncertainty* in this economy. α represents capital income share, while $1 - \alpha$ represents labour income share.

The capital stock obeys the law of motion

$$k_{t+1} = (1 - \delta)k_t + i_t, \tag{4}$$

where i_t is investment and δ is the depreciation rate, with $0 < \delta < 1$, and k_0 is the initial capital stock, which is given. The resource constraint for this economy is

$$c_t + i_t = y_t. \tag{5}$$

1. [50 points] Set up the representative agent's optimum problem, i.e., the sequence problem (SP) for this economy.

2. [50 points] Suppose that $z_{\min} \leq z_t \leq z_{\max}$ and $0 < k_{\min} \leq k_t \leq k_{\max}$ for all $t \geq 0$. Let $K \equiv [k_{\min}, k_{\max}]$ and $Z \equiv [z_{\min}, z_{\max}]$. Denote $S_{K\times Z}$ by the set of all continuous and bounded functions defined on the set $K \times Z$. In other words,

$$S_{K\times Z} \equiv \{g(k,z) \mid g : K \times Z \longrightarrow \mathbb{R}, g(k,z) \text{ is continuous and bounded}\}.$$

Then, state correctly the *Feller property* of the conditional expectation with respect to the information set $\mathcal{I}_t$ generated by a $\mathcal{I}_t$-measurable random variable z_t.

3. [100 points] Suppose that the conditional expectation satisfies the above *Feller* property. Then define a mapping T on $S_{K\times Z}$ as follows: for any $g(k_{t+1}, z_{t+1}) \in S_{K\times Z}$ and any $t \geq 0$,

$$\begin{aligned} &T(g(k_{t+1}, z_{t+1})) \\ &= \max_{\{k_{t+1}, n_t\}} u(z_t(k_t)^{\alpha}(n_t)^{1-\alpha} + (1-\delta)k_t - k_{t+1}, 1 - n_t) + \beta\mathbb{E}(g(k_{t+1}, z_{t+1}) \mid \mathcal{I}_t). \end{aligned}$$

Then prove that $T(g(k_{t+1}, z_{t+1})) \in S_{K\times Z}$, i.e., $T(g(k_{t+1}, z_{t+1}))$ is a continuous and bounded function on $K \times Z$.

4. [100 points] Suppose you have successfully proved that the above T (it is called the Bellman operator) is a mapping of $S_{K\times Z}$ into $S_{K\times Z}$. Then prove that T is a contraction mapping with modulus β of $S_{K\times Z}$ into $S_{K\times Z}$.

5. [100 points] Write down the Bellman equation for this problem in any period t and then prove that the Bellman equation has a unique solution $V \in S_{K \times Z}$ –it is called the value function.

6. [100 points] Using the envelope theorem, derive the first-order conditions for an optimal household plan for consumption, labour supply, and capital accumulation.

7. [100 points] Now suppose that total factor productivity z_t follows AR(1) process, i.e.,

$$\log z_t = \rho \log z_{t-1} + \epsilon_t \tag{6}$$

where ρ is "autocorrelation coefficient," with $0 < \rho < 1$, and ϵ_t is "white noise," i.e. independent and normally distributed with mean zero and finite variance σ^2. We assume capital stock depreciates by 100 percent, i.e. $\delta = 1$. All remaining specifications for this economy are the same as in the previous problems, i.e. (3) and (5) are the same. Using the guess-and-verify method, derive the optimal policy functions for k_{t+1}, c_t, and n_t. Also determine the value function. Detail your derivations.

8. [300 points] Now suppose that total factor productivity z_t follows AR(1) process, i.e.,

$$\log z_t = \rho \log z_{t-1} + \epsilon_t \tag{7}$$

where ρ is "autocorrelation coefficient," with $0 < \rho < 1$, and ϵ_t is "white noise," i.e., independent and normally distributed with mean zero and finite variance σ^2. Unlike its conventional specification, we assume that the capital stock obeys the law of motion

$$k_{t+1} = (k_t)^{1-\delta} (i_t)^{\delta}, \tag{8}$$

where i_t is investment and δ is the depreciation rate, with $0 < \delta < 1$, and k_0 is the initial capital stock, which is given. All remaining specifications for this economy are the same as in the previous problems, i.e., (3) and (5) are the same. Using the guess-and-verify method, derive the optimal policy functions for k_{t+1}, c_t, and n_t. Also determine the value function. Detail your derivations (**Hint**: conjecture that

$$i_t^* = \lambda y_t, c_t^* = (1-\lambda) y_t \text{ for some unknown constant } \lambda$$

where the superscript '$*$' means an optimal solution of each variable). What could be the business-cycle implications of the present model? Compare them with those of the Brock-Mirman model discussed in class.

9. [300 points] Now suppose that total factor productivity z_t follows AR(1) process, i.e.,

$$\log z_t = \rho \log z_{t-1} + \epsilon_t \tag{9}$$

where ρ is "autocorrelation coefficient", with $0 < \rho < 1$, and ϵ_t is "white noise," i.e., independent and normally distributed with mean zero and finite variance σ^2. As in the previous problem, we assume that the capital stock obeys the law of motion

$$k_{t+1} = (k_t)^{1-\delta} (i_t)^{\delta}, \tag{10}$$

where i_t is investment and δ is the depreciation rate, with $0 < \delta < 1$, and k_0 is the initial capital stock, which is given. In addition, the period uitility function u is specified by

$$u(c_t, 1 - n_t) = \log\left(c_t - \chi (n_t)^{1+\omega}\right) \tag{11}$$

where χ is a "disutility" parameter and ω is also an unspecified parameter. All remaining specifications for this economy are the same as in the previous problems, i.e. (3) and (5) are the same. Using the guess-and-verify method, derive the optimal policy functions for k_{t+1}, c_t, and n_t. Also determine the value function. Detail your derivations.