KAIST

Fall 2023

Final Part III: Take-home Exam

Due: 12:00 PM December 19th (Tuesday)

(To be submitted to chpark1111@kaist.ac.kr)

Please, answer all questions *on an individual basis*. This exam is worth a total of 900 points.

PROBLEM (900 points)

Consider a representative consumer with initial assets A_0 and preferences given by:

$$\mathbb{E}(\sum_{t=0}^{\infty} \beta^t u(c_t) \mid \mathcal{I}_0),$$

where β is the subjective discount factor with $0 < \beta < 1$, c_t is consumption in period t, and $\mathbb{E}(\cdot \mid \mathcal{I}_0)$ is the conditional expectation in period 0, i.e. "rational expecations" conditional on information accumulated up to period 0, $\mathcal{I}_0$. The period uitility function u is supposed to be strictly increasing, strictly concave, and differentiable on $[0, \infty)$. The consumer's budget constraint is given by

$$A_{t+1} = (1 + r)(A_t - c_t + w_t)$$

where r is the one-period interest rate (assumed constant over time) and w_t is income in period t. The consumer's income process, $\{w_t\}$, is a random process where each w_t becomes known at the beginning of period t before making any decisions.

1. **[100 points]** Set up the representative consumer's optimum problem for this economy and according to the principle of optimality, write down the Bellman equation for this problem in any period t.

2. **[100 points]** Suppose that $w_{\min} \leq w_t \leq w_{\max}$ and $0 < A_{\min} \leq A_t \leq A_{\max}$ for all $t \geq 0$ and that the above conditional expectation satisfies the Feller property. Based upon the Bellman equation constructed in question 1, determine the Euler equation for this economy. You will have to explicitly invoke the Envelope theoreme (or the Benveniste-Schneikman formula) to determine the Euler equation.

3. **[100 points]** Show that the opitmal consumption in this model economy is a function of "permanent income" or "human wealth."

4. **[100 points]** Now suppose that the period utility function u is specified by

$$u(c_t) = -\frac{1}{2}(\bar{c} - c_t)^2$$

where $\bar{c}$ is constant. Then rewrite the Euler equation derived in Question 2 and the optimal consumption in Question 3. Interpret them using the ideas of conditional expectations.

5. **[200 points]** Now relax the assumption that the one-period gross interest rate $1+\ r$ is constant over time. In other words, we assume that the consumer's budget constraint is given by

$$A_{t+1} = R_{t+1}(A_t - c_t) \tag{1}$$

where the one period gross interest rate R_{t+1} is a random variable and is unknown when the period t consumption is made. Furthermore we assume that $\{R_{t+1}\}$ is an *i.i.d.* process, i.e. each R_{t+1} is stochastically independent of current information set $\mathcal{I}_t$ and has an identical probability distribution for all $t \geq 0$. Note that consumer's income process abstracts from his (flow) budget constraint (1). Suppose that the period utility function u is specified by

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}. \tag{2}$$

Guess that the value function form

$$V(A_t) = B\frac{A_t^{1-\gamma}}{1-\gamma}$$

where $B > 0$ is constant, and determine the optimal decision for consumption.

6. **[150 points]** In Question 5, introduce an income process $\{w_t\}$ $(t = 0, 1, \cdots, \infty)$ but the income process is an *i.i.d.* process, i.e. each w_t is stochastically independent of current information set $\mathcal{I}_t$ and has an identical probability distribution for all $t \geq 0$. Suppose also that the income process $\{w_t\}$ Is stochastically independent of $\{R_{t+1}\}$. Accordingly, the flow budget constraint (1) must be rewritten as follows:

$$A_{t+1} = R_{t+1}(A_t - c_t + w_t)$$

The period uility function remains to be the same as (2). Determine the corresponding Bellman and Euler equations for this problem. Is it possible to find a closed-form for the value function as in Question 5? Provide your arguments.

7. **[150 points]** Determine the optimal consumption function in Question 6. Is this revised model consistent with the *Permanent income Hypothesis* or the *Random Walk Hypothesis*? Provide your arguments.