HSS310 Final

January 3, 2024

1 Part 1

1.1 Problem 1

Suppose there are n regions each of which consist of a firm and a household in the economy. The firms' production function and the households' utility and the law of motion for capital follows the same form as in problem 7. The total factor productivity of the firm in region i is z_i follows the AR(1) process with ρ_i. The total factor productivity is pairwise independent. Suppose there is a goverment in the economy that taxes each firm's production and redistributes it into each region in egalitarian manner. We can expect that the government's role will allow the value function to be the same regardless of the distribution of capital stock and the total factor productivity. In this problem, we prove that this is in fact the case.

The Bellman equation for this problem is written as:

$$\begin{aligned} V(\mathbf{k}_t, \mathbf{z}_t) = \max_{\mathbf{k}_{t+1}, \mathbf{n}_t, \mathbf{t}_t, \mathbf{g}_t} \{ \sum_i u_i \left(z_{i,t} k_{i,t}^{\alpha} n_{i,t}^{1-\alpha} - t_{i,t} + g_{i,t} - k_{i,t+1}, 1 - n_{i,t} \right) \\ + \beta E[V(\mathbf{k}_{t+1}, \mathbf{z}_{t+1}) | I_t] \} \end{aligned}$$

Where t_1 and t_2 are the tax rates for the firms in region 1 and 2 respectively, and g_1 and g_2 are the transfer payments for the households in region 1 and 2 respectively.

$$\text{subject to } \sum_i t_{i,t} = \sum_i g_{i,t} \tag{1}$$

This equation implies that $\sum_i s_{i,t} = 0$ where $-\mathbf{t}_t + \mathbf{g}_t = \mathbf{s}_t$. Which is a preservation of the total tax revenue and the total transfer payments.

This substitution simplfies the bellman equation into:

$$\begin{aligned} V(\mathbf{k}_t, \mathbf{z}_t) = \max_{\mathbf{k}_{t+1}, \mathbf{n}_t, \mathbf{t}_t, \mathbf{g}_t} \{ \sum_i u_i \left(z_{i,t} k_{i,t}^{\alpha} n_{i,t}^{1-\alpha} + s_{i,t} - k_{i,t+1}, 1 - n_{i,t} \right) \\ + \beta E[V(\mathbf{k}_{t+1}, \mathbf{z}_{t+1}) | I_t] \} \end{aligned}$$

The Euler equation for region i is:

$$\frac{1}{c_{i,t}} = \frac{\beta}{c_{i,t+1}} \alpha k_{i,t+1}^{\alpha-1} n_{i,t+1}^{1-\alpha} E\left[z_{i,t+1} | I_t \right] \tag{2}$$

The first order condition with respect to $n_{i,t}$ with envelope condition is:

$$\frac{\chi}{1-n_{i,t}} = \frac{1}{c_{i,t}} \cdot z_{i,t}(k_{i,t})^{\alpha}(1-\alpha)n_{i,t}^{-\alpha} \tag{3}$$

The gradient with respect to $\mathbf{s}_t$, we get:

$$0 = \sum_i u_{i,c}\left(z_{i,t}k_{i,t}^{\alpha}n_{i,t}^{1-\alpha} + s_{i,t} - k_{i,t+1}, 1-n_{i,t}\right) ds_{i,t} \tag{4}$$

In other words:

$$0 = \sum_i \frac{1}{c_{i,t}} ds_{i,t} \tag{5}$$

Since $\sum_i s_{i,t} = 0$, we have $\sum_i ds_{i,t} = 0$. These equations along with $0 = \sum_i \frac{1}{c_{i,t}} ds_{i,t}$ implies

$$\frac{1}{c_{i,t}} = \frac{1}{c_{j,t}} \quad \forall i, j \tag{6}$$

Apply the guess-and-verify method, where $V(\mathbf{k}_t, \mathbf{z}_t) = A + \sum_i B_i \log k_{i,t} + \sum_i C_i \log z_{i,t}$, then the Bellman equation is written as:

$$\begin{aligned} A + \sum_i B_i \log k_{i,t} + \sum_i C_i \log z_{i,t} = \max_{\mathbf{k}_{t+1}, \mathbf{n}_t, \mathbf{t}_t, \mathbf{g}_t} \{ & \sum_i u_i \left(z_{i,t}k_{i,t}^{\alpha}n_{i,t}^{1-\alpha} + s_{i,t} - k_{i,t+1}, 1-n_{i,t}\right) \\ & + \beta E[A + \sum_i B_i \log k_{i,t+1} + \sum_i C_i \log z_{i,t+1} | I_t]\} \end{aligned}$$

$$B_i = \frac{\alpha}{1-\alpha\beta} \tag{7}$$

$$C_i = \frac{1}{(1-\rho_i)} \tag{8}$$

The optimal policies are:

$$n_{i,t} = \frac{\alpha-1}{\alpha\beta\chi + \alpha - \chi - 1} \tag{9}$$

$$c_{i,t} = (1-\alpha\beta)(k_{i,t}^{\alpha}n_{i,t}^{1-\alpha}z_{i,t} + s_{i,t}) \tag{10}$$

$$k_{i,t+1} = \alpha\beta(k_{i,t}^{\alpha}n_{i,t}^{1-\alpha}z_{i,t} + s_{i,t}) \tag{11}$$

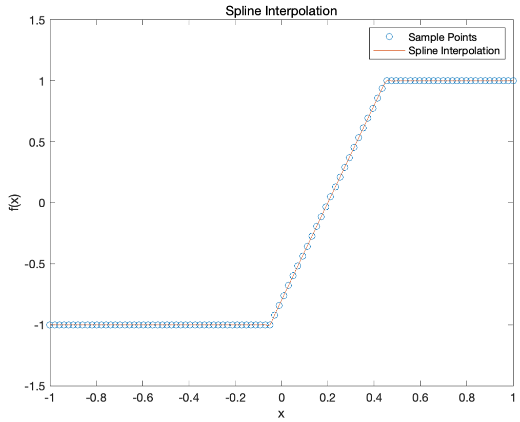
Since $c_{i,t}$ is the same for all i, we get:

$$s_{i,t} = \frac{1}{n}\sum_j z_{j,t}k_{j,t}^{\alpha}n_{j,t}^{1-\alpha} - z_{i,t}k_{i,t}^{\alpha}n_{i,t}^{1-\alpha} \tag{12}$$

In other words, the government distributes the tax revenue in consideration of different production outputs.

1.2 Problem 2

The spline interpolation result is as follows:

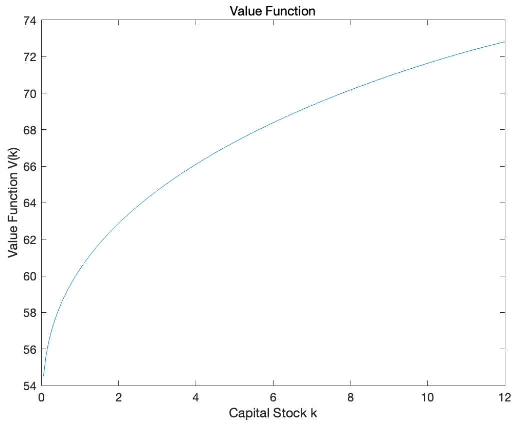


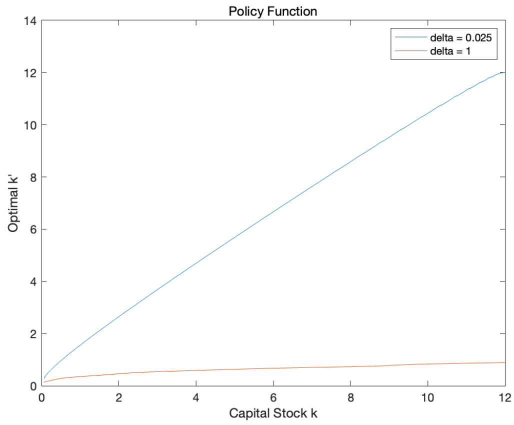
The matlab code is as follows:

```
f = @(x) min(max(-1, 4*(x - 0.2)), 1);

x = linspace(-1, 1, 100);
y = arrayfun(f, x);
xi = linspace(-1, 1, 1000);
yi = interp1(x, y, xi, 'spline');

figure;
plot(x, y, 'o', xi, yi, '-');
legend('Sample Points', 'Spline Interpolation');
title('Spline Interpolation');
xlabel('x');
ylabel('f(x)');
print('spline_interpolation', '-dpng');

A = 1;
beta = 0.99;
delta = 0.025;
alpha = 0.36;
kmin = 0.06;
kmax = 12;
tol = 0.01;
m = 300;

k = kmin:kmin:kmax;

[V, policy] = run_bellman(k, A, beta, delta, alpha, tol, m);
[V_trivial, policy_trivial] = run_bellman(k, A, beta, 1, alpha, tol, m);

figure;
plot(k, V);
title('Value Function');
xlabel('Capital Stock k');
ylabel('Value Function V(k)');
print('value_function', '-dpng');
```

```
figure;
plot(k, policy, k, policy_trivial);
title('Policy Function');
xlabel('Capital Stock k');
ylabel('Optimal k''');
legend('delta = 0.025', 'delta = 1');
print('policy_function', '-dpng');

function [V, policy] = run_bellman(k, A, beta, delta, alpha, tol, m)
n = length(k);
V = zeros(1, n);

for iter = 1:m
    Vnew = zeros(1, n);
    for i = 1:n
        obj = @(j) (log(A*k(i)^alpha + (1-delta)*k(i) - k(j)) + beta*V(j));

        jmax = find(k <= A*k(i)^alpha + (1-delta)*k(i), 1, 'last');
        [val, ip] = max(arrayfun(obj, 1:jmax));

        if A*k(i)^alpha + (1-delta)*k(i) - k(ip) <= 0
            val = -Inf;
        end

        Vnew(i) = val;
    end

    if max(abs(V - Vnew)) < tol
        break;
    end
    V = Vnew;
end

policy = zeros(1, length(k));
for i = 1:n
    obj = @(kp) -(log(A*k(i)^alpha + (1-delta)*k(i) - kp) + beta*interp1(k, V, kp, 'spline'));
    [kp_opt, ~] = fminbnd(obj, 0, A*k(i)^alpha + (1-delta)*k(i));
    policy(i) = kp_opt;
end
end
```

The results for the code is as follows:





2 Part 2

2.1

$$\max_{\{k_{t+1}, c_t, n_t, y_t, i_t\}} E^P\left(\sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_t) \,|\, I_0\right) \text{ subject to } y_t = i_t + c_t,$$

$$k_{t+1} = (1-\delta)k_t + i_t,$$

$$y_t = z_t k_t^{\alpha} n_t^{1-\alpha},$$

$$0 \le n_t \le 1,$$

$$k_0, z_0 : \text{ given},$$

$$\{z_t\} : \text{a random process},$$

$$P \equiv \{\text{subjective probability distribution at the individual level or}$$
$$\text{objective distribution in equilibrium (rational expectations)}\}.$$

2.2

The first-order condition and differentiation under conditional expectation sign dictates:

$$\partial k_{t+1} : -u_c\left(c_t, 1-n_t\right) + \beta E[V_k(k_{t+1}, z_{t+1})|I_t] = 0 \tag{13}$$

$$\partial n_t : -u_n\left(c_t, 1-n_t\right) + \beta E[V_k(k_{t+1}, z_{t+1}) z_t (k_t)^\alpha (1-\alpha) n_t^{-\alpha} |I_t] = 0 \tag{14}$$

where u_c is the marginal utility of consumption, and V_k is the partial derivative of V with respect to k.

The envelope condition dictates:

$$\partial k_t : V_k(k_t, z_t) = u_c\left(c_t, 1-n_t\right) \cdot \left(z_t \alpha k_t^{\alpha-1} n_t^{1-\alpha} + 1 - \delta\right) \tag{15}$$

Combining the above two equations, we have:

$$-u_c\left(c_t, 1-n_t\right) + \beta E[u_c\left(c_{t+1}, 1-n_{t+1}\right) \cdot \left(z_{t+1} \alpha k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha} + 1 - \delta\right) |I_t] = 0 \tag{16}$$

$$\frac{1}{c_t} = \beta E\left[\frac{1}{c_{t+1}} \cdot \left(z_{t+1} \alpha k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha} + 1 - \delta\right) |I_t\right] \tag{17}$$

$$\frac{1}{c_t} = \frac{\beta}{c_{t+1}} \left(\alpha k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha} E\left[z_{t+1}|I_t\right] + (1-\delta)\right) \tag{18}$$

From (19), we get:

$$\frac{\chi}{1-n_t} = \beta E\left[\frac{1}{c_{t+1}} \cdot \left(z_{t+1} \alpha k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha} + 1 - \delta\right) \cdot z_t (k_t)^\alpha (1-\alpha) n_t^{-\alpha} |I_t\right] \tag{19}$$

$$\frac{\chi}{1-n_t} = \frac{1}{c_t} \cdot z_t (k_t)^\alpha (1-\alpha) n_t^{-\alpha} \tag{20}$$

For capital accumulation, we have:

$$k_{t+1} = z_t k_t^\alpha n_t^{1-\alpha} + (1-\delta) k_t - c_t \tag{21}$$

2.3

Suppose $V(k_t, z_t) = A + B \log k_t + C \log z_t$, and for $\delta = 1.0$, we have:

$$\frac{1}{c_t} = \frac{\beta}{c_{t+1}} \alpha k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha} E\left[z_{t+1}|I_t\right]$$

$$k_{t+1} = z_t k_t^\alpha n_t^{1-\alpha} - c_t$$

Bellman equation is written as:

$$\begin{aligned} A + B \log k_t + C \log z_t = \max_{k_{t+1}, n_t} &\{\log\left(z_t k_t^\alpha n_t^{1-\alpha} - k_{t+1}\right)(1-n_t)^\chi \\ &+ \beta E[A + B \log k_{t+1} + C \log z_{t+1}|I_t]\} \end{aligned}$$

The first order condition with respect to k_{t+1} is:

$$0 = \frac{B \cdot \beta}{k_{t1}} - \frac{1}{k_t^\alpha \cdot n_t^{(1-\alpha)} \cdot z_t - k_{t1}} \tag{22}$$

The first order condition with respect to n_t is:

$$0 = -\frac{\chi}{1-n_t} + \frac{k_t^\alpha \cdot n_t^{(1-\alpha)} \cdot z_t \cdot (1-\alpha)}{n_t \cdot (k_t^\alpha \cdot n_t^{(1-\alpha)} \cdot z_t - k_{t1})} \tag{23}$$

Suppose the optimal choice is k_{t+1}^* and n_t^*, then we have:

$$k_{t+1}^* = \frac{\beta B}{1+\beta B} z_t k_t^\alpha n_t^{1-\alpha} \tag{24}$$

$$n_t^* = \frac{\alpha - 1}{\alpha\beta\chi + \alpha - \chi - 1} \tag{25}$$

$$\begin{aligned} A + B\log k_t + C\log z_t =& \log\left(z_t k_t^\alpha n_t^{*1-\alpha} - k_{t+1}^*\right) + \chi\log\left(1-n_t^*\right) \\ &+ \beta A + \beta B \log k_{t+1}^* + \beta C\rho \log z_t \end{aligned}$$

Using k_{t+1}^* we have:

$$\begin{aligned} A + B\log k_t + C\log z_t = \beta(A + B\log(&\frac{B\beta k_t^\alpha n_t^{*1-\alpha} z_t}{B\beta+1}) \\ &+ C\rho\log(z_t)) + \chi\log\left(1-n_t^*\right) + \log(k_t^\alpha n_t^{1-\alpha} z_t - \frac{B\beta k_t^\alpha n_t^{1-\alpha} z_t}{B\beta+1}) \end{aligned}$$

Then, the coefficients are:

$$A = \frac{(1-\alpha)\log(n_t^*) + (1-\alpha\beta)\chi\log(1-n_t^*) + \alpha\beta\log(\alpha\beta) + (1-\alpha\beta)\log(1-\alpha\beta)}{(1-\alpha\beta)(1-\beta)} \tag{26}$$

$$B = \frac{\alpha}{1-\alpha\beta} \tag{27}$$

$$C = \frac{1}{(1-\alpha\beta)(1-\rho)} \tag{28}$$

The optimal policy functions are:

$$n_t = \frac{\alpha - 1}{\alpha\beta\chi + \alpha - \chi - 1} \tag{29}$$

$$c_t = (1-\alpha\beta)k_t^\alpha n_t^{1-\alpha} z_t \tag{30}$$

$$k_{t+1} = \alpha\beta k_t^\alpha n_t^{1-\alpha} z_t \tag{31}$$

2.4

Bellman equation is written as:

$$V(k_t, z_t) = \max_{k_{t+1}, n_t} \left\{ u\left(z_t k_t^{\alpha} n_t^{1-\alpha} - i_t, 1 - n_t\right) + \beta E[V(k_{t+1}, z_{t+1})|I_t] \right\} \tag{32}$$

For simplification, we maximize w.r.t. i_t, since determining i_t is equivalent to determining k_{t+1}.

Suppose $V(k_t, z_t) = A + B\log k_t + C\log z_t$, then we have:

$$\begin{aligned} A + B\log k_t + C\log z_t = &\max_{i_t, n_t} \{\log\left(z_t k_t^{\alpha} n_t^{1-\alpha} - i_t\right) + \chi\log(1 - n_t) \\ &+ \beta A + \beta B\log(k_t)^{1-\delta}(i_t)^{\delta} + \beta\rho C\log z_t\} \end{aligned}$$

First order condition with respect to i_t:

$$0 = \frac{B \cdot \beta \cdot \delta}{i_t} - \frac{1}{-i_t + k_t^{\alpha} \cdot n_t^{(1-\alpha)} \cdot z_t} \tag{33}$$

First order condition with respect to n_t:

$$0 = -\frac{\chi}{1 - n_t} + \frac{k_t^{\alpha} \cdot n_t^{(1-\alpha)} \cdot z_t \cdot (1-\alpha)}{n_t \cdot (-i_t + k_t^{\alpha} \cdot n_t^{(1-\alpha)} \cdot z_t)} \tag{34}$$

After solving the above two equations, we have:

$$B = \frac{\alpha}{1 - \beta(1-\delta) - \alpha\beta\delta} \tag{35}$$

$$C = \frac{1}{(1 - \rho\beta)} \tag{36}$$

$$i_t^* = \frac{B\beta\delta k_t^{\alpha} n_t^{1-\alpha} z_t}{B\beta\delta + 1} \tag{37}$$

$$n_t^* = \frac{-B\alpha\beta\delta + B\beta\delta - \alpha + 1}{-B\alpha\beta\delta + B\beta\delta - \alpha + \chi + 1} \tag{38}$$

Therefore, for control variables, we have:

$$c_t = \frac{k_t^{\alpha} n_t^{1-\alpha} z_t(-\alpha\beta\delta + \beta\delta - \beta + 1)}{\beta\delta - \beta + 1} \tag{39}$$

$$n_t = \frac{\alpha\beta\delta - \alpha\beta + \alpha - \beta\delta + \beta - 1}{\alpha\beta\chi\delta + \alpha\beta\delta - \alpha\beta + \alpha - \beta\chi\delta + \beta\chi - \beta\delta + \beta - \chi - 1} \tag{40}$$

For state variables, we have:

$$k_{t+1} = k_t^{1-\delta} \left(\frac{\alpha\beta\delta k_t^{\alpha} n_t^{1-\alpha} z_t}{\beta\delta - \beta + 1} \right)^{\delta} \tag{41}$$

The law of motion for capital can be written as $k_{t+1} = k_t(\frac{i_t}{k_t})^{1-\delta}$ and a few observations can be made from this. First, the investment should be always nonzero, otherwise the capital stock will be zero in the next period. This is similar to the Brock-Mirman model with $\delta = 1$, where investment should be positive for capital accumulation. However, this is not true for the Brock-Mirman model with $0 < \delta < 1$, where the capital stock does not immediately become zero in the next period even if the investment is zero. Second, unlike the Brock-Mirman model, the investment is not additive to the capital stock, but rather multiplicative and scaled. The capital in the next time step exhibits constant returns to scale in capital and investment. This implies that the investment should be larger than the current capital stock k_t, otherwise the capital stock begins to shrink. The Brock-Mirman model differs in that the investment should larger than δk_t for the capital stock to grow. Third, the concavity of this investment-capital ratio implies diminishing marginal return of investment. This is also what differs from the Brock-Mirman model, where marginal return of investment is constant. This relation is referred to as *the capital adjustment cost* in our survey of the literature.

The business cycle implications are as follows:

$$\mathrm{Var}[\log i_t^*] = \mathrm{Var}[\log c_t^*] = \mathrm{Var}[\log y_t^*] \tag{42}$$

This result is the same as the Brock-Mirman model with $\delta = 1$.

2.5

The Bellman equation for this problem is written as:

$$V(k_t, z_t) = \max_{k_{t+1}, n_t} \left\{ \log\left(z_t k_t^{\alpha} n_t^{1-\alpha} - i_t - \chi(n_t)^{1+\omega}\right) + \beta E[V(k_{t+1}, z_{t+1})|I_t] \right\} \tag{43}$$

For simplification, we maximize w.r.t. i_t, since determining i_t is equivalent to determining k_{t+1}.

Suppose $V(k_t, z_t) = A + B\log k_t + C\log z_t$, then we have:

$$\begin{aligned} A + B\log k_t + C\log z_t = &\max_{i_t, n_t}\{\log\left(z_t k_t^{\alpha} n_t^{1-\alpha} - i_t - \chi(n_t)^{1+\omega}\right) \\ &+ \beta A + \beta B(1-\delta)\log(k_t) + \beta B\delta\log(i_t) + \beta\rho C\log z_t\} \end{aligned}$$

First order condition with respect to i_t:

$$i_t^* = \frac{B\cdot\beta\cdot\delta\cdot(-\chi\cdot n_t^{(\omega+1)} + k_t^{\alpha}\cdot n_t^{(1-\alpha)}\cdot z_t)}{B\cdot\beta\cdot\delta + 1} \tag{44}$$

First order condition with respect to n_t:

$$n_t^* = \left(\frac{1-\alpha}{\chi(1+\omega)}\right)^{\frac{1}{\omega+\alpha}} (k_t^{\alpha} z_t)^{\frac{1}{\omega+\alpha}} \tag{45}$$

$$y_t^* = \left(\frac{1-\alpha}{\chi(1+\omega)}\right)^{\frac{1-\alpha}{\alpha+\omega}} (k_t^{\alpha} z_t)^{\frac{\omega+1}{\omega+\alpha}} \tag{46}$$

Suppose $\overline{a} = \left(\frac{1-\alpha}{\chi(1+\omega)}\right)^{\frac{1}{\alpha+\omega}}$, then we have:

$$\log(y_t^* - \chi\cdot n_t^{(\omega+1)}) = \log\left(\overline{a}^{1-\alpha} - \chi\overline{a}^{1+\omega}\right) + \frac{\alpha(\omega+1)}{\omega+\alpha}\log(k_t) + \frac{\omega+1}{\omega+\alpha}\log(z_t) \tag{47}$$

$$
\begin{aligned}
A + B\log k_t + C\log z_t = &- \log(1 + B\beta\delta) + \log\left(\overline{a}^{1-\alpha} - \chi\overline{a}^{1+\omega}\right) \\
&+ \frac{\alpha(\omega+1)}{\omega+\alpha}\log(k_t) + \frac{\omega+1}{\omega+\alpha}\log(z_t) \\
&+ \beta A + \beta B(1-\delta)\log(k_t) + \beta B\delta\log(i_t) + \beta\rho C\log z_t
\end{aligned}
$$

$$
\begin{aligned}
A + B\log k_t + C\log z_t = &(B\beta\delta)\log(B\beta\delta) - (1 + B\beta\delta)\log(1 + B\beta\delta) + \beta A \\
&(1 + B\beta\delta)\log\left(\overline{a}^{1-\alpha} - \chi\overline{a}^{1+\omega}\right) + (1 + B\beta\delta)\frac{\alpha(\omega+1)}{\omega+\alpha}\log(k_t) + (1 + B\beta\delta)\frac{\omega+1}{\omega+\alpha}\log(z_t) \\
&\beta B(1-\delta)\log(k_t) + \beta\rho C\log z_t
\end{aligned}
$$

$$
B = \frac{\alpha(-\omega-1)}{\alpha\beta\delta\omega + \alpha\beta - \alpha - \beta\delta\omega + \beta\omega - \omega} \tag{48}
$$

$$
C = \frac{(\omega+1)(\alpha\beta\delta - \alpha\beta + \alpha + \beta\delta\omega - \beta\omega + \omega)}{(\alpha\beta\rho - \alpha + \beta\omega\rho - \omega)(\alpha\beta\delta\omega + \alpha\beta - \alpha - \beta\delta\omega + \beta\omega - \omega)} \tag{49}
$$

$$
i_t^* = \frac{\alpha\beta\delta(\omega+1)(k_t^{\alpha}n_t^{(1-\alpha)}z_t - \chi n_t^{(\omega+1)})}{\alpha\beta\delta - \alpha\beta + \alpha + \beta\delta\omega - \beta\omega + \omega} \tag{50}
$$

The optimal household polices are:

$$
n_t = \left(\frac{1-\alpha}{\chi(1+\omega)}\right)^{\frac{1}{\omega+\alpha}}(k_t^{\alpha}z_t)^{\frac{1}{\omega+\alpha}} \tag{51}
$$

$$
k_{t+1} = k_t^{\delta}\left(\frac{\alpha\beta\delta(\omega+1)(k_t^{\alpha}n_t^{(1-\alpha)}z_t - \chi n_t^{(\omega+1)})}{\alpha\beta\delta - \alpha\beta + \alpha + \beta\delta\omega - \beta\omega + \omega}\right)^{1-\delta} \tag{52}
$$

$$
c_t = y_t - i_t \tag{53}
$$

3 Part 3

3.1 Problem 1

The representative consumer's optimum problem is:

$$
\max_{\{A_{t+1}\}_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty}\beta^t u(A_t + w_t - \frac{A_{t+1}}{1+r})|I_0\right]
$$

subject to:

$$
\begin{gathered}
A_0, w_0 \text{ given} \\
0 \leq A_{t+1} \leq (1+r)(A_t + w_t) \quad \forall t \geq 0 \\
\{w_t\}_{t=0}^{\infty} \text{ is a random process}
\end{gathered}
$$

The Bellman equation is:

$$
V(A_t, w_t) = \max_{A_{t+1}}\left\{u(A_t + w_t - \frac{A_{t+1}}{1+r}) + \beta E[V(A_{t+1}, w_{t+1})|I_t]\right\}
$$

3.2 Problem 2

The first order condition with differentiation under conditional expectation sign dictates:

$$\partial_{A_{t+1}}: \frac{-1}{1+r}u'(A_t + w_t - \frac{A_{t+1}}{1+r}) + \beta E\left[\frac{\partial V(A_{t+1}, w_{t+1})}{\partial A_{t+1}}|I_t\right] = 0 \tag{54}$$

The envelope condition is:

$$\partial_{A_t} V(A_t, w_t) = u'(A_t + w_t - \frac{A_{t+1}}{1+r}) \tag{55}$$

The Euler equation is:

$$\frac{1}{1+r}u'(A_t + w_t - \frac{A_{t+1}}{1+r}) = \beta E\left[u'(A_{t+1} + w_{t+1} - \frac{A_{t+2}}{1+r})|I_t\right] \tag{56}$$

This equation becomes:

$$E\left[\frac{u'(c_{t+1})}{u'(c_t)}|I_t\right] = \frac{1}{\beta(1+r)} \tag{57}$$

3.3 Problem 3

Suppose $\beta = \frac{1}{1+r}$. Then the Euler equation becomes:

$$E\left[u'(c_{t+1})|I_t\right] = u'(c_t) \tag{58}$$

Taylor expansion of $u'(c_{t+1})$ around c_t gives:

$$E\left[u'(c_{t+1})|I_t\right] = u'(c_t) + u''(c_t)(E\left[c_{t+1}|I_t\right] - c_t) + E\left[O((c_{t+1} - c_t)^2)|I_t\right] \tag{59}$$

The Euler equation becomes:

$$E\left[c_{t+1}|I_t\right] = c_t - \frac{1}{u''(c_t)}E\left[O((c_{t+1} - c_t)^2)|I_t\right] \tag{60}$$

Suppose $c_t \approx c_{t+1}$, then $O((c_{t+1} - c_t)^2) \to 0$. Therefore, the Euler equation becomes:

$$E\left[c_{t+1}|I_t\right] = c_t \tag{61}$$

This implies that the optimal consumption is martingale and the following holds:

$$E\left[c_{t+k}|I_t\right] = c_t \tag{62}$$

Since

$$E\left[c_{t+k}|I_t\right] = E\left[E\left[c_{t+k}|I_{t+k-1}\right]|I_t\right] = E\left[c_{t+k-1}|I_t\right] = \cdots = E\left[c_{t+1}|I_t\right] = c_t \tag{63}$$

The intertemporal budget constraint with no-Ponzi scheme is:

$$\sum_{t=0}^{\infty} \frac{1}{(1+r)^t}c_t = A_0 + \sum_{t=0}^{\infty} \frac{1}{(1+r)^t}w_t \tag{64}$$

Take expectation on both sides, we have:

$$\sum_{t=0}^{\infty} \frac{1}{(1+r)^t}E\left[c_t|I_0\right] = A_0 + \sum_{t=0}^{\infty} \frac{1}{(1+r)^t}E\left[w_t|I_0\right] \tag{65}$$

Since $E[c_t|I_0] = E[c_0|I_0] = c_0$, we have:

$$c_0 = \frac{r}{1+r}\left(A_0 + \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} E\left[w_t|I_0\right]\right) \tag{66}$$

This implies that if the discount rate is equal to the interest rate, the consumption is equal to the expected present discounted sum of lifetime income. The expected return from the initial asset price also affects the consumption

3.4 Problem 4

$$u'(c) = -(\bar{c} - c) \tag{67}$$

Therefore, the Euler equation becomes:

$$E\left[c_{t+1}|I_t\right] = \frac{c_t}{\beta(1+r)} + \frac{\bar{c}(\beta(1+r)-1)}{\beta(1+r)} \tag{68}$$

If we assume that $\beta = \frac{1}{1+r}$, then the optimal consumption is martingale:

$$E\left[c_{t+1}|I_t\right] = c_t \tag{69}$$

This can be extended to time shift of k:

$$E\left[c_{t+k}|I_t\right] = c_t \tag{70}$$

Therefore, the expected life time consupmtion is:

$$\sum_{t=0}^{\infty} \frac{1}{(1+r)^t} E\left[c_t|I_0\right] = \frac{1+r}{r} c_0 \tag{71}$$

From the intertemporal budget constraint and no-Ponzi scheme, we have:

$$\sum_{t=0}^{\infty} \frac{1}{(1+r)^t} E\left[c_t|I_0\right] = A_0 + \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} E\left[w_t|I_0\right] \tag{72}$$

Therefore, the consumption is:

$$c_0 = \frac{r}{1+r}\left(A_0 + \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} E\left[w_t|I_0\right]\right) \tag{73}$$

In other words, if the discount rate is equal to the interest rate, the consumption is determined by the expected present value of permanent income and the inital asset.

3.5 Problem 5

The Bellman equation is:

$$V(A_t) = \max_{A_{t+1}} \left\{ u(A_t - \frac{A_{t+1}}{R_{t+1}}) + \beta E[V(A_{t+1})|I_t] \right\}$$

If $u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$, then the Euler equation becomes:

$$\frac{1}{R_{t+1}} c_t^{-\gamma} = \beta E\left[c_{t+1}^{-\gamma} | I_t\right] \tag{74}$$

Suppose $V(A_t) = B\frac{A_t^{1-\gamma}}{1-\gamma}$, then the Bellman equation becomes:

$$B\frac{A_t^{1-\gamma}}{1-\gamma} = \max_{A_{t+1}} \left\{ \frac{1}{1-\gamma}(A_t - \frac{A_{t+1}}{R_{t+1}})^{1-\gamma} + \beta E\left[B\frac{A_{t+1}^{1-\gamma}}{1-\gamma} | I_t\right]\right\} \tag{75}$$

This is equivalent to:

$$B\frac{A_t^{1-\gamma}}{1-\gamma} = \max_{c_t} \left\{ \frac{1}{1-\gamma} c_t^{1-\gamma} + \beta E\left[B\frac{(R_{t+1}(A_t - c_t))^{1-\gamma}}{1-\gamma} | I_t\right]\right\} \tag{76}$$

$$B\frac{A_t^{1-\gamma}}{1-\gamma} = \max_{c_t} \left\{ \frac{1}{1-\gamma} c_t^{1-\gamma} + \beta B\frac{(A_t - c_t)^{1-\gamma}}{1-\gamma} E\left[R_{t+1}^{1-\gamma} | I_t\right]\right\} \tag{77}$$

Since R_{t+1} is unknown at time t and independent of I_t, and the first order condition on c_t and we have:

$$c_t^{-\gamma} = \beta B \cdot (A_t - c_t)^{-\gamma} E\left[R_{t+1}^{1-\gamma}\right] \tag{78}$$

$$c_t = \frac{A_t}{1 + \left(\beta B \cdot E\left[R_{t+1}^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}} \tag{79}$$

$$B\frac{A_t^{1-\gamma}}{1-\gamma} = \frac{1}{1-\gamma} A_t^{1-\gamma} \left(1 + (B\beta \cdot E\left[R_{t+1}^{1-\gamma}\right])^{\frac{1}{\gamma}}\right)^{\gamma} \tag{80}$$

$$B = \frac{1}{\left(1 - \left(E\left[R_{t+1}^{1-\gamma}\right]\right)^{\frac{1}{\gamma}} \beta^{\frac{1}{\gamma}}\right)^{\gamma}} \tag{81}$$

Therefore, the value function becomes:

$$V(A_t) = \frac{1}{\left(1 - \left(E\left[R_{t+1}^{1-\gamma}\right]\right)^{\frac{1}{\gamma}} \beta^{\frac{1}{\gamma}}\right)^{\gamma}} \frac{A_t^{1-\gamma}}{1-\gamma} \tag{82}$$

The optimal consumption is:

$$c_t = \left(1 - \left(E\left[R_{t+1}^{1-\gamma}\right]\right)^{\frac{1}{\gamma}} \beta^{\frac{1}{\gamma}}\right) \cdot A_t \tag{83}$$

3.6 Problem 6

The Bellman equation is:

$$V(A_t) = \max_{c_t} \left\{ \frac{1}{1-\gamma} c_t^{1-\gamma} + \beta E[V(A_{t+1}) | I_t] \right\} \tag{84}$$

The first order condition on A_{t+1} is:

$$c_t^{-\gamma} = \beta E\left[R_{t+1} V'(A_{t+1}) | I_t\right] \tag{85}$$

The Benveniste-Scheinkman theorem with uncertainty dictates:

$$\partial_{A_t} V(A_t) = V'(A_t) = c_t^{-\gamma} \tag{86}$$

Time shift the above equation by 1 period, and with the first order condition, we have:

$$c_t^{-\gamma} = \beta E\left[R_{t+1} c_{t+1}^{-\gamma} | I_t\right] \tag{87}$$

The guess-and-verify method of linear policy $c_{t+1}^{-\gamma} = aA_t + bw_t + c$ failed, since it was impossible to find appropriate b. The difficulty may arise due to the inability to link w_t and w_{t+1} together.
Try guess-and-verify of $V(A_t) = B\frac{A_t^{1-\gamma}}{1-\gamma}$, then the Euler equation becomes:

$$c_t^{-\gamma} = \beta B \cdot E\left[R_{t+1}^{1-\gamma}(A_t + w_t - c_t)^{-\gamma} | I_t\right] \tag{88}$$

Since R_{t+1} is independent of I_t and w_t, we have:

$$c_t^{-\gamma} = \beta B \cdot E\left[R_{t+1}^{1-\gamma}\right] E\left[(A_t + w_t - c_t)^{-\gamma} | I_t\right] \tag{89}$$

Guess-and-verify of $V(A_t) = B\frac{A_t^{1-\gamma}}{1-\gamma}$ fails, since the right hand side is a mulinomial of c_t, and not a single term of the power of c_t. i.i.d and I_t independent random variables should not appear in the closed form of the value function, since the value function indicates the maximum utility achievable with the given state in the infinite horizon, The value function should not be affected by that random variable. In other words, the value function is invariant to w_t and R_{t+1}. This indicates the only possible functional form that the value function can take is $V(A_t) = B\frac{A_t^{1-\gamma}}{1-\gamma}$. Failure to find appropriate B indicates that the value function cannot be expressed in a closed form.

3.7 Problem 7

The Euler equation is:

$$c_t^{-\gamma} = \beta E\left[R_{t+1}\right] E\left[c_{t+1}^{-\gamma} | I_t\right], \tag{90}$$

In other words:

$$E\left[\frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}} | I_t\right] = \frac{1}{\beta E\left[R_{t+1}\right]} \tag{91}$$

Taking log and apply Jensen's inequality, we have:

$$\log\left(\frac{1}{\beta E\left[R_{t+1}\right]}\right) = \log\left(E\left[\frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}} | I_t\right]\right) \geq -\gamma E\left[\log(c_{t+1}) - \log(c_t)\right] \tag{92}$$

In other words:

$$\frac{1}{\gamma}\log\left(\beta E\left[R_{t+1}\right]\right) \leq E\left[\log(c_{t+1}) - \log(c_t) | I_t\right] \tag{93}$$

To put into words, the long run growth of consumption is lower bounded by log of the expected return.

Since the consumption is not martingale, this cannot be fit into the intertemporal budget constraint with no-Ponzi scheme as before. However, for the further analysis of short term change of consumption, we can conduct taylor expnasion on $u'(c_{t+1})$ around c_t, we have:

$$u'(c_t) = \beta R E_t \left[u'(c_t) + u''(c_t)(c_{t+1} - c_t) + \frac{1}{2} u'''(c_t)(c_{t+1} - c_t)^2 + ... \right] \tag{94}$$

Since $\frac{u''(c_t)}{u'(c_t)} = -\gamma$ and $\frac{u'''(c_t)}{u'(c_t)} = \gamma(\gamma+1)$, we have:

$$E_t \left[-\gamma(c_{t+1} - c_t) + \frac{1}{2}\gamma(\gamma+1)(c_{t+1} - c_t)^2 + ... \right] = \frac{1}{\beta E[R]} - 1 \tag{95}$$

Where if we assume that $c_t \approx c_{t+1}$, then the above equation becomes:

$$E_t \left[c_{t+1} \right] = \frac{\beta E[R] - 1}{\gamma \beta E[R]} + c_t \tag{96}$$

Therefore, if $\beta E[R] > 1$, the consumption is submartingale, and otherwise supermartingale. Also, γ affects the degree. Depending on the value of $\frac{\beta E[R]-1}{\gamma \beta E[R]}$, the consumption will either increase or decrease over time. However, when $\beta E[R] = 1$, the consumption is martingale, and the permanent income hypothesis holds.