Homotopy, Modular Forms, and the Spectrum $Q(\ell)$
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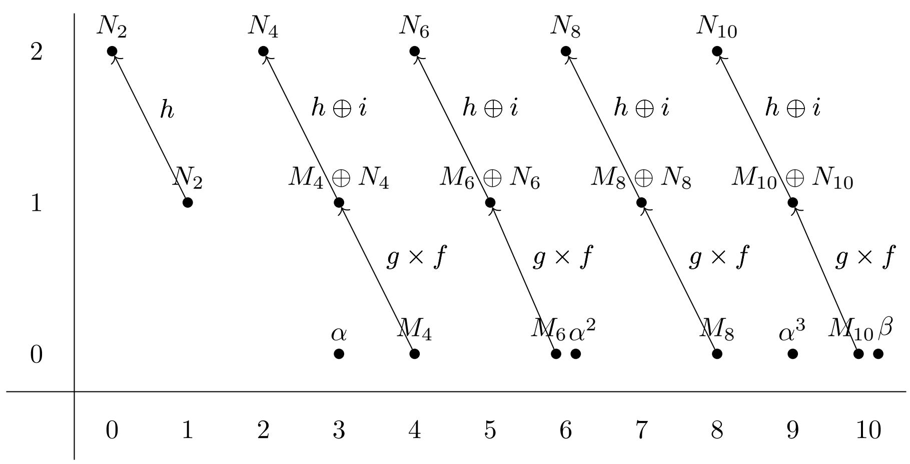
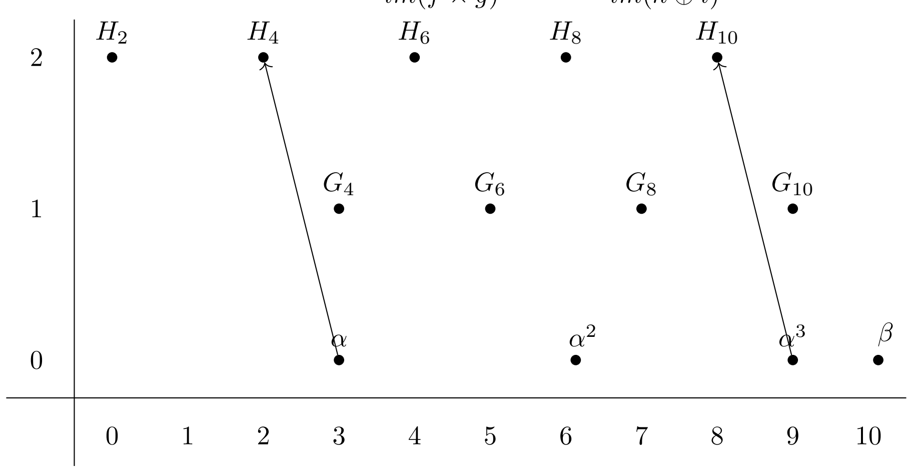
1 Introduction

The goal of this project was, among other things, to determine the various homology groups relating to the following specral sequence.

Let $M = M^{\bullet}(SL_2(\mathbb{Z}))$ and $N = M^{\bullet}(\Gamma_0(2))$, $f = (\psi_d - 1), g = (\psi_2 - 1), h = (\psi_d + 1), i = -\phi_f$



$G = {}^{ker(h \oplus i)}\!/_{im(f \times g)}$, $H = {}^{N}\!/_{im(h \oplus i)}$



Where the *Homology* at a given cell in a spectral sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is defined as ${}^{ker(g)}\!/_{im(f)}$.

To do this, it was important we understand, first and foremost, what Modular Forms are and what this map ψ_d is. From there, using special properties of modular forms, Isogenies (of elliptic curves), and operators called *Atkin-Lehner* involutions, we produced a computation of the middle homology groups, for any weight of the graded ring of modular forms. We also describe directions towards the computation of the upper homology groups. This work originates from the computation of the homotopy groups of the spectrum $Q(2)$ via the Bockstein spectral sequence associated to it.

2 Background

2.1 Elliptic Curves

Before we define elliptic curves, we first need a notion of a *lattice*,

Definition 1. (Silverman 2) A lattice Λ, defined over $\mathbb{C}$, is an abelian group $\omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ where $\omega_1 \neq \omega_1$ are both nonzero and in the upper half plane $\mathcal{H}$.

Note that for any nonzero $m \in \mathbb{Z}$, and any lattice Λ, $\Lambda \supset m\Lambda$ implies that $\Lambda = m\Lambda$. Note also that any lattice is *homothetic* to any other lattice. That is, by scaling and translating, any lattice becomes any other lattice. It is then that we arrive at the fact that,

Proposition 2. *Given any lattice $\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$, there exists a $\Lambda_\tau = \mathbb{Z} \oplus \tau\mathbb{Z}$ isomorphic to Λ. This Λ_τ is defined precisely so that $\tau = \frac{\omega_1}{\omega_2}$, and τ is certainly in $\mathcal{H}$.*

Now that we know lattices are actually given by any $\tau \in \mathcal{H}$, we will treat all lattices as Λ_τ lattices. Notice also that lattices are sort of like spans of two linearaly independent complex numbers. This makes the next idea of a complex torus a little more natural.

Definition 3 (Complex Torus)**.** Let $\Lambda_\tau = \mathbb{Z} \oplus \tau\mathbb{Z}$ be a lattice. Then a complex torus E_τ is said to be $\mathbb{C}/\Lambda_\tau$.

We also establish, without proof, that every elliptic curve is isomorphic to a complex torus. This allows us to define a group action of area preserving transformations (that is, an $\mathrm{SL}_2(\mathbb{Z})$-action) on elliptic curves, through complex tori. However, we are not interested in all area preserving transformations. Specifically, we are interested in a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ that is upper triangular $\mod N$ where $N \in \mathbb{N}$. This subgroup is called $\Gamma_0(N)$, and its action preserves the N-torsion data of a given elliptic curve. In this paper, we are most concerned with 2-torsion data, so our congruence subgroup of choice is $\Gamma_0(2)$.

The $\Gamma_0(2)$-action on complex tori (elliptic curves) $E = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$ descends from the $SL_2(\mathbb{Z})$-action and is thus given by,

$$\rho_\gamma : \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau) \to \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\gamma(\tau))$$
$$\rho_\gamma :: z + \Lambda_\tau \mapsto \frac{z}{c\tau + d} + \Lambda_{\gamma(\tau)},\ \gamma \in \Gamma_0(2).$$

The important structure that $\Gamma_0(2)$ provides is that this action will preserve the 'enhanced' structure of elliptic curves that we consider, namely that the maps will preserve points of order two. Structure preserving maps of elliptic curves and enhanced elliptic curves are called isogenies, and one such isogeny is our map ψ_d. This map comes from the natural projection $\pi : E \to {}^{E}\!/_{< P >}$ where P is a point of order 2. This map ψ_d is defined initially on elliptic curves with $\Gamma_0(2)$ level structure, but pulls-back to a map of modular forms. This map is called the *Dual Isogeny.* The dual isogeny is the 'dual' of the map π above, which exists thanks to nice properties that surjective morphisms in the category of algebraic varieties enjoy. But, in preserving the data of enhanced elliptic curves (with 2-torsion data), the dual isogeny produces another subgroup of order 2, of the new curve. That is, $\psi_d : \mathcal{M}_0(2) \to \mathcal{M}_0(2) :: \left({}^{E}\!/_{P}, \hat{P}\right) \mapsto (E, P)$.

2.2 Modular Forms

For the purposes of this paper, modular forms are holomorphic (complex differentiable with other special properties) functions on the upper half plane.

Definition 4 (Modular Forms)**.** (Diamond and Shurman 1) Let $f : \mathcal{H} \to \mathbb{C}$ be a complex differentiable function, and let $\gamma \in SL_2(\mathbb{Z})$. Then, f is called a *modular form (of weight k)* if

$$f(\gamma\tau) = (c\tau + d)^k f(\tau) \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

, and f is complex differentiable at infinity.

We can define several congruence subgroups of $SL_2(\mathbb{Z})$ by the matrix entries of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, e.g. $\Gamma_0(N)$ is a congruence subgroup with $c \equiv 0$ mod N. They are defined by $\Gamma(N) \subset \Gamma$ for some integer N, which is referred to as the subgroup's level. We can then similarly define modular forms using these subgroups to get modular forms with level structure.

One of the important reasons we care about $SL_2(\mathbb{Z})$-'invariance' of these functions is because of how this structure relates to Elliptic Curves. Roughly speaking, this rough invariance that modular forms (without level structure) obey allows them to be regarded as 'functions' on isomorphism classes of elliptic curves.

Now, sometimes we find it useful to attach other data to Elliptic Curves. For example, one particularly natural piece of data to associate to any elliptic curve E are

1. A point of order N $P \in E$, written as (E, P)
2. A subgroup $H \leq E$ of order N, written similarly

While $SL_2(\mathbb{Z})$-modular forms are 'functions' on normal elliptic curves, we can find *congruence subgroups* $\Gamma \leq SL_2(\mathbb{Z})$ such that, with proper definitions, $\Gamma-$invariant modular forms become 'functions' on elliptic curves with the associated data above (with proper choice of Γ for the given situation).

In our case, we consider elliptic curves with a point of order 2. In this case the congruence subgroup we care about is $\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, d \equiv 1 \mod 2, c \equiv 0 \mod 2 \right\}$.

For algebraic purposes, we can describe the space of modular forms and the map ψ_d via $\mathbb{Z}_3$ algebras. Firstly, we treat $M^\bullet(\Gamma_0(2))$ *the graded ring of* $\Gamma_0(2)$ *modular forms.* There are two special forms, $q_2 = -1/2 - 12q - 12q^2 - 48q^3 - 12q^4 - 72q^5 - 48q^6 - 96q^7 - 12q^8 - 156q^9 - 72q^{10} - 144q^{11} + O(q^{12})$ which is a weight 2 $\Gamma_0(2)$ modular form and $q_4 = 1/16 - q + 7q^2 - 28q^3 + 71q^4 - 126q^5 + 196q^6 - 344q^7 + 583q^8 - 757q^9 + 882q^{10} - 1332q^{11} + O(q^{12})$ which is a weight 4 form. Then $M^\bullet(\Gamma_0(2)) \cong \mathbb{Z}_3[q_2, q_4]$ with grading generated by setting q_2 to be degree 2, and q_4 degree 4. $\psi_d : M^\bullet(\Gamma_0(2)) \to M^\bullet(\Gamma_0(2))$ can then be described very nicely: $\psi_d(q_2) = -2q_2$ and $\psi_d(q_4) = q_2^2 - 4q_4$.

There is also a description of $M^\bullet(SL_2(\mathbb{Z}))$ in a similar vein. Let $c_4 = 1 + \sum_{n=1}^\infty \sigma_3(n)q^n$ and $c_6 = 1 - 504\sum_{n=1}^\infty \sigma_5(n)q^n$ (where $\sigma_k(n) = \sum_{d|n} d^k$) be two $SL_2(\mathbb{Z})$ modular forms of weight 4 and 6 respectively. However, one complication that arises is that this ring is not freely generated $\mathbb{Z}_3-$adically; we need another form $\Delta = \sum_{n=1}^\infty \tau(n)q^n$ (with τ being the Ramanujan Tau Function). Then $M^\bullet(SL_2(\mathbb{Z})) \cong \mathbb{Z}_3[c_4, c_6, \Delta]\big/(12^3\Delta - (c_4^3 - c_6^2))$. In terms of q_2 and q_4, $c_4 = 16(q_2^2 - 3q_4)$ and $c_6 = 32(2q_2^3 - 9q_2q_4)$, which then allows us to compute the action of ψ_d.

Since we can write c_4, c_6 (and consequently Δ) in terms of q_2, q_4 we have a natural inclusion ϕ_f : $M^\bullet(SL_2(\mathbb{Z})) \hookrightarrow M^\bullet(\Gamma_0(2))$.

Note: Throughout points in this paper, we also refer to M_k([Some group]) to denote the (additive) subgroup of degree k elements of $M^\bullet$([Some group]).

2.3 Atkin-Lehner

Given $f \in M_k(\Gamma_0(N))$ and $Q|N$ s.t. $gcd(N, \frac{N}{Q}) = 1$ the Q'th Atkin-Lehner operator is the matrix

$$W_Q = \begin{bmatrix} aQ & b \\ cN & dQ \end{bmatrix}$$

with (a, b, c, d) chosen s.t. $det(W_Q) = Q$.

It then turns out that $\psi_d(f)(\tau) = 2^{\frac{w(f)}{2}} f|_{W_2}$ where the $|_\gamma$ denotes a matrix acting on a modular form in the usual sense of $f|_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}(\tau) = f\left(\dfrac{a\tau + b}{c\tau + d}\right)$. This relation is extremely important in our work.

3 Results

3.1 ψ_d and (Fully) Modular Forms

Theorem 5. *If $f \in M_k(SL_2(\mathbb{Z})$ then $\psi_d(f)(\tau) = 2^k f(2\tau)$*

Proof. As $\psi_d(f)(\tau)$ is just $f(\dfrac{-1}{2\tau})$ without the automorphy factor, and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$, $\psi_d(f)(\tau) = \tau^{-k} f(\dfrac{-1}{2\tau}) = \tau^{-k}(2\tau)^k f(2\tau) = 2^k f(2\tau)$ □

Lemma 6. *The claim above can be reversed, i.e. if $\psi_d(f)(\tau) = 2^k f(2\tau)$ for $f \in M_k(\Gamma_0(2))$ then $f \in M_k(SL_2(\mathbb{Z}))$ so f is, in fact, fully modular.*

Proof. Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(2)$, we already know that $f(\tau + 1) = f(\tau)$. Thus, it is sufficient to prove that $f(\dfrac{-1}{\tau}) = \tau^{-k} f(\tau)$. (Deo and Medvedovsky 250) demonstrate that

$$f|_k W_2 = (\det W_2)^{\frac{k}{2}} j(W_2, \tau)^{-k} f(\frac{-1}{2\tau}) = 2^{\frac{k}{2}} f(2\tau)$$

. Simplifying this expression, we arrive at

$$2^{\frac{k}{2}} f(2\tau) = 2^{\frac{k}{2}} (2\tau)^{-k} f(\frac{-1}{2\tau})$$
$$f(\frac{-1}{2\tau}) = (2\tau)^k f(2\tau)$$

Then, replacing 2τ with τ, we have demonstrated that $f(\frac{-1}{\tau}) = \tau^k f(\tau)$, i.e. f is weight-k invariant under $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Since it is also weight-k invariant under $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and the two matrices generate $SL_2(\mathbb{Z})$, f is weight-k invariant under the full modular group $SL_2(\mathbb{Z})$ and is therefore a fully modular form. Thus $f \in M_k(SL_2(\mathbb{Z})) \iff \psi_d(f)(\tau) = 2^k f(2\tau)$. □

Theorem 7 (Constraining Equation)**.** *If $f \in M_k(\Gamma_0(2))$ and $\psi_d(f) + f \in M_k(SL_2(\mathbb{Z}))$ then $2^k(f(\tau) - f(2\tau) - \psi_d(f)(2\tau)) + \psi_d(f)(\tau) = 0$*

Proof. If $\psi_d(f) + f$ is fully modular, then $\psi_d(\psi_d(f) + f)(\tau) = 2^k(\psi_d(f) + f)(2\tau)$ by the above result. Additionally, since $\psi^2(f) = 2^k f$, $\psi_d(\psi_d(f) + f) = \psi^2(f) + \psi_d(f) = 2^k f + \psi_d(f)$. Thus since the two expressions must be equal, subtracting them gives zero. Thus $2^k f(\tau) + \psi_d(f)(\tau) - 2^k \psi_d(f)(2\tau) - 2^k f(2\tau) = 2^k(f(\tau) - f(2\tau) - \psi_d(f)(2\tau)) + \psi_d(f)(\tau) = 0$. □

Corollary 8. *The above claim can again be reversed. I.e. $\psi_d(f) + f$ is fully modular $\iff$ $2^k(f(\tau) - f(2\tau) - \psi_d(f)(2\tau)) + \psi_d(f)(\tau) = 0$.*

Proof. As was derived, the equation is equivalent to $\psi_d(\psi_d(f)+f)(\tau) = 2^k(\psi_d(f)+f)(2\tau)$. Lemma 3.1 then implies that $\psi_d(f) + f$ is fully modular. □

Theorem 9. *If $f \in M_k(SL_2(\mathbb{Z}))$ and $\psi_d(f) + f \in M_k(SL_2(\mathbb{Z}))$ then $f = 0$.*

Proof. By the above result once again, $2^k(f(\tau) - f(2\tau) - \psi_d(f)(2\tau)) + \psi_d(f)(\tau) = 0$. Then, since f is fully modular, we can simplify the $\psi_d(f)$'s to get

$$2^k f(\tau) - 2^k f(2\tau) - 2^{2k} f(4\tau) + 2^k f(\tau) = 2^k(f(\tau) - 2^k f(4\tau)) = 0.$$

Thus $f(\tau) - 2^k f(4\tau) = 0$.
Now, in terms of q-expansions (assuming $f = \sum_{i=0}^{\infty} a_i q^i$) we would have that $f(q) - 2^k f(q^4) = 0$. If we expand out the L.H.S. in q we arrive at $(a_0 - 2^k a_0) + a_1 q + a_2 q^2 + a_3 q^3 + (a_4 - 2^k a_1) q^4 + ... = 0$. Thus we have the following conditions:

$$\begin{cases} a_0 - 2^k a_0 = 0 \\ a_1 = 0 \\ a_2 = 0 \\ a_3 = 0 \end{cases}$$

And since the first condition obviously implies that $a_0 = 0$ since we work in characteristic 0, we have that $a_i = 0, 0 \leq i \leq 3$. I claim that this forces every other coefficient to be zero. For, if all $a_i = 0, i < n$ and n is not a multiple of 4 then the n'th q-coefficient of $g = f(q) - 2^k f(q^4)$ would just be a_n (since $n \neq 4i$).

Since $g = 0$, a_n must be zero as well. If 4 divides n, say $n = 4i$, then we have the n'th q-coefficient of g as $a_n - 2^k a_i$. However, $i < n$, so $a_i = 0$. Thus since $a_n - 2^k a_i = 0$ and $a_i = 0$, $a_n = 0$. Thus, by induction, $a_i = 0, \forall i \geq 0$. All together, this implies that $f = 0$. □

Theorem 10 (Main Theorem). *For $f \in M_k(\Gamma_0(2))$, $\psi_d(f) + f \in M_k(SL_2(\mathbb{Z})) \iff \exists g \in M_k(SL_2(\mathbb{Z})) : f = \psi_d(g) - g$.*

Proof. ($\Longleftarrow$) If $f = \psi_d(g) - g$ for g fully modular, then $\psi_d(f) = 2^k g - \psi_d(g)$ so $\psi_d(f) + f = 2^k g - \psi_d(g) + \psi_d(g) - g = (2^k - 1)g$. Since $M^\bullet(SL_2(\mathbb{Z}))$ is a $\mathbb{C}-$vector space, $(2^k - 1)g \in M_k(SL_2(\mathbb{Z}))$.
($\Longrightarrow$) If $\psi_d(f) + f \in M_k(SL_2(\mathbb{Z}))$ then let $g = \psi_d(f) + f$. Then $\psi_d(g) = 2^k f + \psi_d(f)$. However, since $g = \psi_d(f) + f$, $\psi_d(f) = g - f$. Thus $\psi_d(g) = 2^k f + g - f$ so $(2^k - 1)f = \psi_d(g) - g$. Dividing both sides by $2^k - 1$ finishes the proof. □

Lemma 11. *$v_3(2^k - 1) = 1 + v_3(k)$ for k even where v_3 denotes the 3-adic valuation, i.e. the number of times 3 divides a number.*

Proof. According to (Sloane and Inc.) this seems to be a standard fact. □

Theorem 12. *For $g \in M_k(SL_2(\mathbb{Z}))$, if $2^k - 1 | \psi_d(g) - g$, then, letting $g(q) = \sum_{i=0}^{\infty} a_i q^i$, $2^k - 1 | a_i, \forall i \geq 1$. The converse of the statement also holds.*

Proof. First, say $f = \psi_d(g) - g$ so that $2^k - 1 | f$. Then by Theorem 5, $f(q) = 2^k g(q^2) - g(q)$. Expanding out f, then gives

$$f = (2^k - 1)a_0 - a_1 q + (2^k a_1 - a_2)q^2 + ...$$

Then, since $f \equiv 0 \mod (2^k - 1)$, $a_1 \equiv 0 \mod (2^k - 1)$. Similarly, $\forall i$ odd $a_i \equiv 0 \mod (2^k - 1)$. Assume $a_i \equiv 0 \mod (2^k - 1), \forall 0 < i < n$. Then, for i even (say $i = 2j$), the ith q-coefficient of f is $2^k a_j - a_i$. Mod 3, however, we have that $2^k a_j - a_i \equiv 0 \mod (2^k - 1)$. Since $j < i$, inductively we have that $a_j \equiv 0 \mod (2^k - 1)$. Thus $2^k a_j - a_i \equiv -a_i \equiv 0 \mod (2^k - 1)$. Thus, by induction $\forall i > 0, a_i \equiv 0 \mod (2^k - 1)$.

Now we prove the converse. For $g(q) = \sum_{i=0}^{\infty} a_i q^i \in M_k(SL_2(\mathbb{Z}))$ if $2^k - 1 | a_i \forall i > 0$ then since $(\psi_d - 1)(g)(q) = \psi_d(g)(q) - g(q) = 2^k g(q^2) - g(q)$. Thus

$$(\psi_d - 1)(g)(q) = \sum_{i=0}^{\infty} -a_{2i+1} q^{2i+1} + \sum_{i=0}^{\infty} (2^k a_i - a_{2i}) q^{2i} \tag{1}$$

$$= (2^k a_0 - a_0) + \sum_{i=0}^{\infty} -a_{2i+1} q^{2i+1} + \sum_{i=1}^{\infty} (2^k a_i - a_{2i}) q^{2i} \tag{2}$$

$$= (2^k - 1)a_0 + \sum_{i=0}^{\infty} -a_{2i+1} q^{2i+1} + \sum_{i=1}^{\infty} (2^k a_i - a_{2i}) q^{2i} \tag{3}$$

Since, for $i \geq 0, 2i + 1 > 0$ and for $i \geq 1, i, 2i > 0$ we have $2^k - 1 | a_{2i+1}, a_i, a_{2i}$ appearing in the sum. $2^k - 1 | (2^k - 1)a_0$ obviously. Therefore $2^k - 1 | (\psi_d - 1)(g)$.

Note that $2^k - 1$ can instead be replaced by $3^{\nu_3(k)+1}$. □

Lemma 13. *If $f \in M_k(\Gamma_0(2))$ and $g = \psi_d(f) + f \in M_k(SL_2(\mathbb{Z}))$ with $f = \sum_{i=0}^{\infty} a_i q^i$, $\psi_d(f) = \sum_{i=0}^{\infty} a'_i q^i$ and $g = \sum_{i=0}^{\infty} b_i q^i$ then $a'_0 = 0$ and, consequently, $a_0 = b_0$.*

Proof. Applying Theorem 7 to our situation, we have that

$$2^k(f(q) - f(q^2) - \psi_d(f)(q^2)) + \psi_d(f) = 0$$

And, upon examining the constant term of the resulting expression we get

$$\begin{gathered} 2^k(a_0 - a_0 - a_0') + a_0' = 0 \\ -(2^k - 1)a_0' = 0 \\ a_0' = 0 \end{gathered}$$

Where the last simplification follows from the characteristic being 0.
Thus, since $g = \psi_d(f) + f$, $b_0 = a_0 + a_0' = a_0$. ☐

Lemma 14 (Obstructions)**.** *For $f \in M_k(\Gamma_0(2))$ satisfying $\psi_d(f)+f \in M_k(SL_2(\mathbb{Z}))$ is in the image of $\psi_d - 1$ if and only if, for $f = \sum_{i=0}^{\infty} a_i q^i$, $\nu_3(a_0) \geq 1 + \nu_3(k)$.*

Proof. Let $g = \psi_d(f) + f \in M_k(SL_2(\mathbb{Z}))$. Then $\psi_d(g) - g = (\psi_d - 1) \circ (\psi_d + 1)(f) = (2^k - 1)f$ so $2^k - 1 | \psi_d(g) - g$. Therefore, if $g = \sum_{i=0}^{\infty} b_i q^i$, by Theorem 12 we know that $2^k - 1 | b_i, \forall i \geq 1$. Since $\nu_3(a_0) \geq 1 + \nu_3(k) = \nu_3(2^k - 1)$, by Lemma 13 we know then that $b_0 = a_0$ so $\nu_3(b_i) \geq \nu_3(2^k - 1) \forall i \geq 0$. (Implicitly we have used that $\nu_3(k) + 1 = \nu_3(2^k - 1)$ via Lemma 11.) Thus $\bar{g} = \frac{1}{2^k - 1}(\psi_d + 1)(f)$ exists. Finally $(\psi_d - 1)(\bar{g}) = \frac{1}{2^k - 1}(\psi_d - 1)(\psi_d + 1)(f) = f$ so $f \in im(\psi_d - 1)$!

For the converse, if $f \in im(\psi_d - 1)$ then $\exists g \in M_k(SL_2(\mathbb{Z}))$ such that $f = (\psi_d - 1)(g)$. Then, since $(\psi_d + 1)(f) = (2^k - 1)(f)$, Lemma 13 proves that $\nu_3(a_0) \geq \nu_3(k) + 1$. ☐

Theorem 15 (Image of $\psi_d + 1$)**.** *$g \in M_k(SL_2(\mathbb{Z})) \cap im(\psi_d + 1)$ $\mathbb{Z}_3$-adically if and only if, when writing $g(q) = \sum\limits_{i=0}^{\infty} a_i q^i$, $\nu_3(a_i) \geq \nu_3(k) + 1, \forall i \geq 1$.*

Proof. By Theorem 12 we note that $3^{\nu_3(k)+1} | \psi_d(g) - g \iff \nu_3(a_i) \geq \nu_3(k) + 1, \forall i \geq 1$. To demonstrate that $g \in im(\psi_d + 1)$ we examine the 'inverse function' of ψ_d.

Lemma 16. $(\psi_d + 1)(\frac{1}{2^k - 1}(\psi_d - 1)(g)) = g$

Proof.

$$(\psi_d + 1)(\frac{1}{2^k - 1}(\psi_d - 1)(g)) = \frac{1}{2^k - 1}(\psi_d + 1)(\psi_d - 1)(g) = \frac{1}{2^k - 1}(\psi_d^2 - 1)(g) = \\ \frac{1}{2^k - 1}(2^k - 1)(g) = g$$

☐

Thus, if $\frac{1}{2^k - 1}(\psi_d - 1)(g)$ is a $\mathbb{Z}_3$-adic modular form, then we know that g is in the image of $\psi_d + 1$. Moreover, since the map described above acts as an inverse for $\psi_d + 1$, we know that if such a form does not exist than *no* $\Gamma_0(2)$ modular form can map to g. $\frac{1}{2^k - 1}(\psi_d - 1)(g)$ is only defined if we do not have division by three occurring, i.e. the power of three in the denominator $2^k - 1$ is exactly matched by the power of three dividing $(\psi_d - 1)(g)$. However, the condition for these to cancel out is exactly Theorem 12! Thus $(\psi_d + 1)^{-1}(g)$ exists if and only if $3^{\nu_3(k)+1} | (\psi_d - 1)(g)$ which is true if and only if $\nu_3(a_i) \geq \nu_3(k) + 1, \forall i \geq 1$! ☐

We also have an algorithm to 'solve' for a basis of the kernel that proceeds as follows:

Theorem 17 (Algorithm)**.** *The following procedure will compute a 'basis' for the kernel.*

1. *Generate a basis* $\mathcal{B} = \{f_1, ..., f_n\}$ *of weight* k $\Gamma_0(2)$*-modular forms as products of special forms* q_2 *and* q_4*.*

2. *Write a general weight* k *modular form* f *as* $\sum_{i=1}^{n} x_i f_i$*.*

3. *Compute the q-expansion of* $2^k(f(q) - f(q^2) - \psi_d(f)(q^2)) + \psi_d(f)(q) = \sum_{i=1}^{n}(\sum_{j=1}^{n} c_{i,j}x_i)q^i$*.*

4. *Denoting* $M := \begin{pmatrix} c_{1,1} & c_{2,1} & \cdots & c_{n,1} \\ c_{1,2} & c_{2,2} & .. & c_{n,2} \\ .. & .. & .. & .. \\ c_{1,n} & c_{2,n} & .. & c_{n,n} \end{pmatrix}$*, find the Smith Normal Form* $M = SDQ$*, and take the submatrix of* Q *with columns starting where the first 0 appears on the diagonal of* D*. The columns of* Q *now represents the basis of the nullspace of* M *and thus gives a basis* $\bar{\mathcal{B}} = \left\{\bar{f_1}, \bar{f_2}, ..., \bar{f_m}\right\}$ *of* K*.*

Proof. Application of Theorem 7. □

Finally, we can compute the homology.

3.2 Coefficient Divisibility

The following results will be used to describe a set of elements whose classes will generate homology groups at any given weight.

3.2.1 c_4^i and c_6^j

Let's start by demonstrating that $\forall i, j \geq 0$ $c_4^i, c_6^j \in im(\psi_d + 1)$. As was proven in Theorem 15, these forms are in the image of $\psi_d + 1$ if and only if, when we write $c_4^i, c_6^j = 1 + \sum_{i=0}^{\infty} a_i q^i$ then $\nu_3(a_i) \geq \nu_3(4i) + 1, \nu_3(6i) + 1$ respectively. Lets work with the former.

Since $c_4 = 1 + 240 \sum_{i=1}^{\infty} \sigma_3(i) q^i$, we can write $c_4^i = ((c_4 - 1) + 1)^i = \sum_{j=0}^{i} \binom{i}{j}(c_4 - 1)^i = 1 + \sum_{j=1}^{i} \binom{i}{j} 240^j (\sum_{n=1}^{\infty} \sigma_3(n) q^n)^j$. Then, we can see that $\binom{i}{j} 240^j | a_n$ so if we can demonstrate that $\nu_3(\binom{i}{j} 240^j) \geq \nu_3(4i) + 1$ then Theorem 15 will apply.

Lemma 18. $\nu_3(\binom{n}{k}) + k \geq \nu_3(n) + 1$.

Proof. Since $\frac{n}{\gcd\{n,k\}} | \binom{n}{k}$ we see that $\nu_3(\binom{n}{k}) \geq \nu_3(\frac{n}{\gcd\{n,k\}}) = \nu_3(n) - \nu_3(\gcd\{n,k\}) = \nu_3(n) - \min\{\nu_3(n), \nu_3(k)\}$.

We now break into the following two cases:

If $1 \le k \le \nu_3(n)$ then $\nu_3(k) \le \nu_3(n)$ so $\min\{\nu_3(n), \nu_3(k)\} = \nu_3(k)$. Then

$$\nu_3(\binom{n}{k}) \ge \nu_3(n) - \nu_3(k) \tag{4}$$
$$\nu_3(\binom{n}{k}) + \nu_3(k) \ge \nu_3(n) \tag{5}$$
$$\nu_3(\binom{n}{k}) + k - 1 \ge \nu_3(n) \tag{6}$$
$$\nu_3(\binom{n}{k}) + k \ge \nu_3(n) + 1 \tag{7}$$

Where the reasoning that allows us to deduce (6) from (5) is because $k - 1 \ge \nu_3(k)$.

Now, if $k \ge \nu_3(n) + 1$ then $\nu_3(\binom{n}{k}) + k \ge 0 + k \ge 0 + \nu_3(n) + 1 = \nu_3(n) + 1$. $\square$

Therefore, since $\nu_3(n) + 1 = \nu_3(4n) + 1$ Theorem 15 applies to c_4^i and so $c_4^i \in im(\psi_d + 1)$ for all $i \ge 1$.

Now, we consider c_6^j for $j \ge 1$.

Similarly as to the above deduction, since $c_6 = 1 - 504 \sum_{i=1}^{\infty} \sigma_5(i) q^i$ we can decompose c_6^j as $((c_6 - 1) + 1)^j = 1 + \sum_{k=1}^{j} \binom{j}{k} 504^k (-\sum_{n=1}^{\infty} \sigma_5(n) q^n)^k$. Then, by Lemma 18, we note that $\nu_3(\binom{j}{k} 504^k) \ge \nu_3(\binom{j}{k}) + k + k \ge \nu_3(j) + 1 + k \ge \nu_3(j) + 1 + 1 = \nu_3(6j) + 1$. Thus $c_6^j \in im(\psi_d + 1)$ for all $j \ge 1$.

3.2.2 Generalizations

Generalizing the above, denote $c_{i,j,k} = \min\{c : c \ge 1,\ c c_4^i c_6^j \Delta^k \in im(\psi_d + 1)\}$. By the above section, $c_{i,0,0} = c_{0,j,0} = 1$. Theorem 15 allows us to rephrase the definition as the following.

Definition 19. Let $i, j \ge 0$ be two natural numbers. If we write $c_4^i c_6^j = 1 + \sum_{i=1}^{\infty} a_i q^i$ then $c_{i,j} = \min\{c : \nu_3(c a_i) \ge \nu_3(4i + 6j) + 1\}$.

For Δ^k, we note that $\Delta^k = q^k + O(q^{k+1})$ so, since $\nu_3(12k) + 1 = \nu_3(k) + 2$, we see that $c_{0,0,k} = 3^{\nu_3(k)+2}$. Similarly, since $c_4^i c_6^j = 1 + O(q)$, $c_4^i c_6^j \Delta^k = q^k + O(q^{k+1})$ thus, for $k > 1$, $c_{i,j,k} = \nu_3(4i + 6j + 12k) + 1$.

Theorem 20. $\nu_3(c_4^i c_6^j - 1) = \begin{cases} \nu_3(4i + 6j) + 1, & \nu_3(4i + 6j) < 3 \\ \min\{\nu_3(i) + 1, \nu_3(4i + 6j)\} + 1, & \nu_3(4i + 6j) \ge 3 \end{cases}$

Proof. Since we have the ultrametric inequality $\nu_3(a + b) \ge \min\{\nu_3(a), \nu_3(b)\}$, we only need to consider this expression modulo $3^{\min\{\nu_3(2i+3j), \nu_3(i)+1\}+2} = 3^u$. We can also use the fact that $\nu_p\left(\binom{n}{k}\right)$ is the number of borrows performed when subtracting k from n in base p to prove that $\nu_p\left(\binom{n}{k}\right) \ge \nu_p(n) - \nu_p(k)$, which holds with equality if $k < p^{\nu_p(n)}$

We can now expand out $c_4^i c_6^j$ as follows: $c_4 = 1 + 240 S_3, c_6 = 1 - 504 S_5$, so $c_4^i c_6^j - 1 = (1 + 240 S_3)^i (1 - 504 S_5)^j - 1 \mod 3^u$. We can include the $-1 \mod 3^{\cdots}$ part since we don't care about the 3-valuation of the constant coefficient.

Let us first consider the expansion of c_4^i. We have that $(1 + 240 S_3)^i = \left(1 + \sum_{t=1}^{i} (3 \cdot 80)^t \binom{i}{t} S_3^t\right) \mod 3^u$. Our goal will be to strategically ignore certain terms of this expansion in order to make derivations simpler. Our first elimination comes from realizing that $u = \min\{\nu_3(4i + 6j), \nu_3(i) + 2\} + 1 \le \nu_3(i) + 3$. So if $t \ge \nu_3(i) + 3$ then $\nu_3((3 \cdot 80)^t \binom{i}{t} S_3^t) \ge \nu_3(3^t) = t \ge \nu_3(i) + 3 \ge u$. So we can ignore such terms as they will not affect our valuation. Thus, we will proceed assuming that $t < \nu_3(i) + 3$.

Since we work mod 3^u, we can again ignore these terms if $t+\nu_3\left(\binom{i}{t}\right) \geq u$ but since $t+\nu_3\left(\binom{i}{t}\right) \geq t+\nu_3(i)-\nu_3(t)$, we can deduce the former inequality if $t+\nu_3(i)-\nu_3(t) \geq u$. The simplified inequality holds if $t-\nu_3(t) \geq 3$, which one can check holds for all $t \geq 4$, so we have $(1+120S_3)^i \equiv 1+240iS_3+\binom{i}{2}(240S_3)^2 + 128000 \cdot 9i(i-1)(i-2)S_3^3 \mod 3^u$. For $(1-504S_5)^j$, we first note that $u \leq 3+\nu_3(\gcd(i,2i+3j)) \leq 3+\nu_3(\gcd(i,3j)) \leq \nu_3(j)+4$. Therefore, we can remove terms divisible by $\nu_3(j)+4$, which by a similar argument, shows that all but the $t=1$ term can be ignored, so we have $(1-504S_3)^j \equiv 1-504jS_3(\mod 3^u)$ After multiplying these expressions together and subtracting 1, we find that many of the products of terms can be ignored modulo 3^u, leaving us with
$c_4^i c_6^j - 1 = 3\cdot 80 \cdot i \cdot S_3 + 3^2 \cdot 80^2 \cdot \frac{i(i-1)}{2} \cdot S_3^2 + 3^2 \cdot 80^3 \cdot \frac{i(i-1)(i-2)}{2} S_3^3 - 3^2 \cdot 56 \cdot j \cdot S_5 \mod 3^u$, which after dividing the expression by 24 (which reduces the modulus to 3^{u-1}), and reducing the coefficients of the terms by the new modulus, we obtain
$(\frac{c_4^i c_6^j - 1}{24}) \equiv 10iS_3 - 3iS_3^2 + 3iS_3^3 - 21jS_5 \mod 3^{u-1}$

From here on, it suffices to show that this expression is congruent to $0(\mod 3^{u-2})$, and not congruent to $0 \mod 3^{u-1}$.

We can do the latter by checking that if the coefficient of q^2 is divisible by 3^{u-1}, then $i \equiv 2 \mod 3$, and that if $i \equiv 2 \mod 3$ then the coefficient of q must not be divisible by 3^{u-1} which proves that claim.
For the former, we can ignore the S_3^2 and S_3^3 terms, after which the expression reduces to showing that $10i \cdot S_3 - 21j \cdot S_5 \equiv 0 \mod 3^{u-2}$
As a reminder, $u-2 = \min(\nu_3(2i+3j), \nu_3(i)+1)$. Then since $-3j \equiv 2i \mod 3^{\nu_3(2i+3j)}$, we have that $10iS_3 - 21jS_5 \equiv 2i \cdot (5S_3 + 7S_5) \mod 3^{u-2}$. So after dividing both sides by $2i$, which amounts to dividing this expression by $2i$, and changing our modulus from 3^{u-2} to $(3^{u-2-\nu_3(i)}) \leq 3$, it remains to show that $5S_3 + 7S_5 \equiv S_5 - S_3 \equiv 0 \mod 3$. To prove this, note that the coefficient of q^n in S_{2k+1} is $\sigma_{2k+1}(n) = \sum_{d|n} d^{2k+1}$, so therefore, to prove that all q-coefficients of $S_5 - S_3$ are divisible by 3, it suffices to prove that the terms in the combined divisor sum, $d^5 - d^3$ are all divisible by 3. Since $d^5 - d^3 = 3 \cdot (2d^2 \cdot \binom{d+1}{3})$ for all positive integers d, this is true and so our original claim follows. ☐

Corollary 21. *In terms of $c_{i,j}$, $c_{i,j} = \begin{cases} 1, & \nu_3(4i+6j) < 3 \\ 3^{\nu_3(4i+6j)-\min\{\nu_3(i)+1,\nu_3(4i+6j)\}}, & \nu_3(4i+6j) \geq 3 \end{cases}$*

Proof. From the above Theorem 20 we can see that such a coefficient is necessary to ensure that the 3-adic valuation of all non-constant coefficients is greater than or equal to $\nu_3(4i+6j)+1$. ☐

3.3 Computation of the Middle Homology

Theorem 22 (Homology)**.** *For a given weight k, $H_k(E^2) = {ker(\psi_d + 1 - \phi_f)}/{im(\psi_d - 1)} \cong \mathbb{Z}_3/3^{\nu_3(k)+1}\mathbb{Z}_3$ for $k > 2$ and even and 0 otherwise.*

Proof. For now, assume that $\exists f \in M_k(SL_2(\mathbb{Z})) : (\psi_d+1)^{-1}(f) = \dfrac{1}{2^k-1}(\psi_d - 1)(f) = \tilde{f}$ exists as well as that $f = 1 + O(q)$ (so, consequently, $\tilde{f} = 1 + O(q)$) and that it is a basis form for $M_k(SL_2(\mathbb{Z}))$. Then, given $g \in ker(\psi_d + 1 - \phi_f)$, we can decompose $g = n\tilde{f} + (g - n\tilde{f})$ (we will specify n later). Our first goal will be to demonstrate that $g - n\tilde{f} \in im(\psi_d - 1)$. We start by laying out a few definitions:

$$f = \sum_{i=0}^{\infty} a_i q^i \tag{8}$$

$$\tilde{f} = \sum_{i=0}^{\infty} \tilde{a_i} q^i \tag{9}$$

$$g = \sum_{i=0}^{\infty} b_i q^i \tag{10}$$

$$\bar{g} = (\psi_d + 1)(g) - nf = \sum_{i=0}^{\infty} \bar{b_i} q^i \tag{11}$$

Since $(\psi_d + 1)(g) \in M_k(SL_2(\mathbb{Z}))$ and f is a modular forms basis element, $(\psi_d + 1)(g) = nf + \bar{g}$, $\bar{g} \in M_k(SL_2(\mathbb{Z}))$. Immediately, we see that $\bar{g} = (\psi_d+1)(g) - nf = (\psi_d+1)(g) - (\psi_d+1)(n\tilde{f}) = (\psi_d+1)(g - n\tilde{f})$. Therefore, $g - n\tilde{f} \in ker(\psi_d + 1 - \phi_f)$. To apply Lemma 14, we must demonstrate that the constant term of $g - n\tilde{f} = b_0 - n\tilde{a_0}$ has $\nu_3(b_0 - n\tilde{a_0}) \geq 1 + \nu_3(k)$.

Then choose $n = b_0$ since, by Lemma 13 $\tilde{a_0} = a_0 = 1$ so $b_0 - n\tilde{a_0} = 0$. Then that guarantees that $g - n\tilde{f} \in im(\psi_d - 1)$ so our first claim is proven i.e.

$\forall g \in ker(\psi_d + 1 - \phi_f), \exists n \in {}^{\mathbb{Z}_3}\!/_{3^{\nu_3(k)+1}\mathbb{Z}_3}, \tilde{g} \in im(\psi_d - 1)$ so that $g = n\tilde{f} + \tilde{g}$. (We can choose $n \in \mathbb{Z}_3/3^{\nu_3(k)+1}\mathbb{Z}_3$ since $3^{\nu_3(k)+1}\tilde{f} \in im(\psi_d - 1)$.)

In turn, this demonstrates that $\forall [g] \in H_k(E^2), [g] = n[\tilde{f}]$ so $H_k(E^2) \cong {}^{\mathbb{Z}_3}\!/_{3^{\nu_3(k)+1}\mathbb{Z}_3}$ with $[\tilde{f}]$ as the generator.

Finally, such an $\tilde{f}$ exists. Let $f = \begin{cases} c_6^{\frac{k}{6}} & k \equiv 0 \mod 6 \\ c_6^{\lfloor \frac{k}{6} \rfloor - 1} c_4^2 & k \equiv 2 \mod 6 \\ c_6^{\lfloor \frac{k}{6} \rfloor} c_4 & k \equiv 4 \mod 6 \end{cases}$ and set $\tilde{f} = (\psi_d + 1)^{-1}(f)$. From Theorem 20, we can see that $c_{i,j}$ of this form is exactly one so $\tilde{f} \in im(\psi_d - 1)$. (Since the power of c_4 is always < 3 either $\tilde{f}$ is a power of c_6 (in which case we immediately get the 3 divisibility needed, or $i \not\equiv 0 \mod 3$ so $\nu_3(4i + 6j) < 3$ and thus $c_{i,j} = 1$.

It has the right divisibility properties, i.e. $n\tilde{f} \in im(\psi_d - 1) \implies n \equiv 0 \mod 3^{\nu_3(k)+1}$ since if there was a different multiple of $\tilde{f}$ in $im(\psi_d - 1)$ it would first have a smaller 3-adic divisibility i.e. $\nu_3(n) < \nu_3(k)+1$. Then $(\psi_d - 1)^{-1}(n\tilde{f}) = \frac{1}{2^k - 1}(\psi_d + 1)(n\tilde{f}) = \frac{n}{2^k - 1}(\psi_d + 1)(\tilde{f})$. However, since $\nu_3(n) < \nu_3(k) + 1 = \nu_3(2^k - 1)$, $(\psi_d - 1)^{-1}(n\tilde{f})$ cannot exist $\mathbb{Z}_3$-adically. Therefore $\nu_3(n) \geq \nu_3(k) + 1$ so $n \equiv 0 \mod 3^{\nu_3(k)+1}$. $\square$

3.4 Computation of the Top Homology

Lemma 23. $\nu_3((\psi_d - 1)(q_2^i q_4^j)) = \begin{cases} \nu_3(2^{2i} - 1), & j = 0 \\ 0, & j \neq 0 \end{cases}$

Proof. If $j = 0$ then $(\psi_d - 1)(q_2^i) = \psi_d(q_2)^i - q_2^i = (-2q_2)^i - q_2^i = ((-2)^i - 1)q_2^i$. Since the constant term of $q_2^i = (-\frac{1}{2})^i$, $\nu_3(((-2)^i - 1)q_2^i) = \nu_3((-2)^i - 1) = \begin{cases} \nu_3(2^i - 1), & i = 2k \\ \nu_3(2^i + 1), & i = 2k + 1 \end{cases} = \nu_3(i) + 1 = \nu_3(2i) + 1 = \nu_3(2^{2i} - 1)$

If $j \neq 0$ then $(\psi_d - 1)(q_2^i q_4^j) = \psi_d(q_2)^i \psi_d(q_4)^j - q_2^i q_4^j$. However, since $\psi_d(q_4) = \mathcal{O}(q)$ the constant term of $(\psi_d - 1)(q_2^i q_4^j) = -(-\frac{1}{2})^i (\frac{1}{16})^j$ which is not divisible by 3. Therefore, $\nu_3((\psi_d - 1)(q_2^i q_4^j)) = 0$. $\square$

Lemma 24 (Characterization of Torsion). *Let* $[f] \in {}^{M_k(\Gamma_0(2))}\!/\!_{im(\psi_d + 1 - \phi_f)}$. *Then* $3^{n_f}[f] = [0]$ *if and only if* $\exists g \in M_k(SL_2(\mathbb{Z}))$ *such that* $3^{n_f}(\psi_d - 1)(f)(q) \equiv g(q) - g(q^2) \mod 3^{\nu_3 k+1}$ *on the level of* q *expansions.*

Proof.

$$\begin{gathered}
3^{n_f}[f] = [0] \iff 3^{n_f} f \in im(\psi_d + 1 - \phi_f) \iff \\
\exists f' \in M_k(\Gamma_0(2)), g \in M_k(SL_2(\mathbb{Z})) : 3^{n_f} f = (\psi_d + 1)(f') - g \iff f + g = (\psi_d + 1)(f') \iff \\
(\psi_d - 1)(3^{n_f} f + g) = (2^k - 1)f' \iff (\psi_d - 1)(3^{n_f} f + g) \equiv 0 \mod 3^{\nu_3(k)+1} \iff \\
3^{n_f}(\psi_d - 1)(f) \equiv g(q) - 2^k g(q^2) \mod 3^{\nu_3(k)+1} \iff \\
3^{n_f}(\psi_d - 1)(f) \equiv g(q) - g(q^2) \mod 3^{\nu_3(k)+1}
\end{gathered} \tag{12}$$

□

Corollary 25. *This conditions needs only to be checked for a finite set of q-expansion coefficients, i.e. the first* $dim(M_k(\Gamma_0(2)))$ *q-expansion coefficients.*

Proof. $M_k(\Gamma_0(2))$ is a free and finitely generated module. □

Corollary 26. *For* $f \in M_k(\Gamma_0(2))$ *and* $(\psi_d - 1)(f) = a_0 + \mathcal{O}(q)$, $\nu_3(k) + 1 \geq n_f \geq \nu_3(k) + 1 - \nu_3(a_0)$.

Proof. Since the constant term of $g(q) - g(q^2)$ is zero, any such n'_f must satisfy $3^{n'_f} a_0 \equiv 0 \mod 3^{\nu_3(k)+1}$, the least of which being $n'_f = \nu_3(k) + 1 - \nu_3(a_0)$. Then, since we may need more divisibility for subsequent coefficients, $n_f \geq n'_f = \nu_3(k) + 1 - \nu_3(a_0)$. That every class is $3^{\nu_3(k)+1}$ torsion is obvious. Therefore $\nu_3(k) + 1 \geq n_f \geq \nu_3(k) + 1 - \nu_3(a_0)$. □

Corollary 27. *A class* $[f] \in {}^{M_k(\Gamma_0(2))}\!/\!_{im(\psi_d + 1 - \phi_f)_k}$ *has* $n_f = 3^{\nu_3(k)+1}$ *if* $\nu_3(a_0) = 0$ *with* a_0 *as above.*

Proof. $\nu_3(k) + 1 \geq n_f \geq \nu_3(k) + 1 - \nu_3(a_0) \implies n_f = \nu_3(k) + 1$. □

Corollary 28. *For* $f \in M_k(\Gamma_0(2))$, $3^{n_f}[f] = [0] \iff \exists g' \in M_k(SL_2(\mathbb{Z})) : 3^{n_f}(\psi_d - 1)(f) \equiv (\psi_d - 1)(g')$ $\mod 3^{\nu_3(k)+1}$

Proof. An easy consequence of Equation 12. □

While we do not have a precise proof describing the cokernel we have noticed a conjectural pattern that we lay forth below:

Conjecture 29. ${}^{M_k(\Gamma_0(2))}\!/\!_{im(\psi_d + 1 - \phi_f)_k} \cong \left({}^{\mathbb{Z}}\!/\!_{3^{p_k}\mathbb{Z}}\right)^{n_k}$ *where* $p_k = 3^{\nu_3(k)+1}$ *and* $n_k = \left\lfloor \frac{k-2}{24} \right\rfloor + 1 + \begin{cases} 1, & k \equiv 14, 20, 22 \mod 24 \\ 0, & otherwise \end{cases}$

With the following computations (performed via MAGMA) providing evidence towards such a pattern:

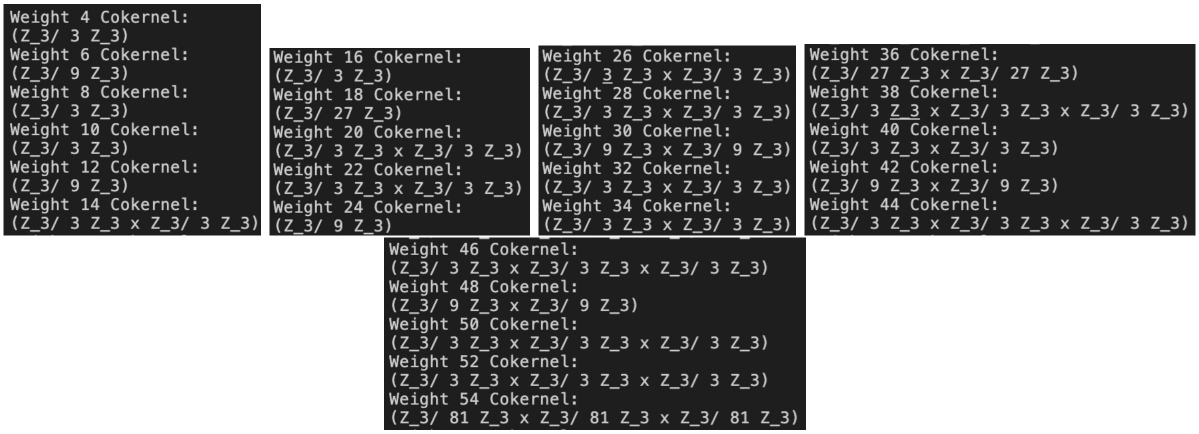


Figure 1: Example Computations of Cokernel Groups

This is a link to the code for our project.

4 Future Work

In the immediate future, we would like to work on concretely proving the pattern we found in the cokernels in Conjecture 28. Once we can do this, we can use this to generally compute the cokernel at any weight. Then we can derive the homology groups of the upper row $H_i = {}^{MF_0(2)}\!\big/_{im(\psi_d + 1 - \phi_f)_i}$. After that, we would be free to move on to the second page. The second page should contain similar work, computing kernels of other differentials. The second page should complete the spectral sequence, since once we reach the third page, the differentials become trivial. Either the arrows hit or come from the zero group there, so we do not need to compute homologies anymore. We would then use the final page to determine $\pi_k(Q(\ell))$.

5 References

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