Homotopy, Modular Forms, and the Spectrum $Q(\ell)$

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Outline Of Presentation

Modular Forms

In a first abstract algebra, symmetry groups of regular polygons are of primary importance. Here, symmetry groups of functions $f : \mathcal{H} \to \mathbb{C}$ are the objects of investigation.

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Definition 1 (Modular Form)

For complex differentiable functions $f : H \to \mathbb{C}$, and any $\gamma \in SL_2(\mathbb{Z})$, f is called a *modular form (of weight k)* if,

$$f(\gamma\tau) = (c\tau + d)^k f(\tau) \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and f is complex differentiable at infinity.

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Modular forms of weight k, with respect to a congruence group Γ, form rings written $M_k(\Gamma)$.

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Using this definition, the action of the congruence subgroup $\Gamma_0(2)$ on an elliptic curve $E = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$ is given by,

$$f_\gamma : \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau) \to \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\gamma(\tau))$$

$$f_\gamma :: z + \Lambda_\tau \mapsto \frac{z}{c\tau + d} + \Lambda_{\gamma(\tau)}, \ \gamma \in \Gamma_0(2).$$

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With this action, one arrives at $\mathcal{H}/\Gamma_0(2)$ which turns out to be precisely classes of elliptic curves (E, C) where C is the cyclic group of order two in E.

Problem Description

Our primary goal is to figure out under what conditions

$$\psi(f) + f \in M^{\bullet}(SL_2(\mathbb{Z}))$$

for

$$f \in M^{\bullet}(\Gamma_0(2))$$

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$(\psi_d + 1)^{-1}(f)$ *exists for* $f \in M_k(SL_2(\mathbb{Z}))$ *if and only if, when we write* $f = \sum_{i=0}^{\infty} a_i q^i$, $3^{\nu_3(k)+1} | a_i \ \forall i \geq 1$.

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(Trickier to prove, more useful though).

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These measure 'failure to be in image'.

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Theorem 8

$c_{i,j,0} = \max\{0, \nu_3(2i + 3j) - \nu_3(i) - 1\}$

How Did We Formulate Conjecture?

Used MAGMA to do many, many computations:

```
Weight: 54; Vals: 0,1,1,0,1
Weight: 108; Vals: 0,1,1,0,1,1,0,1,1,0
Weight: 162; Vals: 0,2,2,1,2,2,1,2,2,0,2,2,1,2
Weight: 216; Vals: 0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0
Weight: 270; Vals: 0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1
Weight: 324; Vals: 0,2,2,1,2,2,1,2,2,0,2,2,1,2,2,1,2,2,0,2,2,1,2,2,1,2,2,0
Weight: 378; Vals: 0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1
Weight: 432; Vals: 0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0
Weight: 486; Vals: 0,3,3,2,3,3,2,3,3,1,3,3,2,3,3,2,3,3,1,3,3,2,3,3,2,3,3,0,3,3,2,3,3,2,3,3,1,3,3,2,3
Weight: 540; Vals: 0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0
Weight: 594; Vals: 0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1
Weight: 648; Vals: 0,2,2,1,2,2,1,2,2,0,2,2,1,2,2,1,2,2,0,2,2,1,2,2,1,2,2,0,2,2,1,2,2,1,2,2,0,2,2,1,2,2,1,2,2,0,2,2,1,2,2,1,2,2,0
Weight: 702; Vals: 0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1
Weight: 756; Vals: 0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0
Weight: 810; Vals: 0,2,2,1,2,2,1,2,2,0,2,2,1,2,2,1,2,2,0,2,2,1,2,2,1,2,2,0,2,2,1,2,2,1,2,2,0,2,2,1,2,2,1,2,2,0,2,2,1,2,2,1,2,2,0,2,2,1,2,2,1,2,2,0,2,2,1,2
Weight: 864; Vals: 0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0
Weight: 918; Vals: 0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1
Weight: 972; Vals: 0,3,3,2,3,3,2,3,3,1,3,3,2,3,3,2,3,3,1,3,3,2,3,3,2,3,3,0,3,3,2,3,3,2,3,3,1,3,3,2,3,3,2,3,3,1,3,3,2,3,3,2,3,3,0,3,3,2,3,3,2,3,3,1,3,3,2,3,3,2,3,3,1,3,3
Weight: 1026; Vals: 0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,
Weight: 1080; Vals: 0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,
Weight: 1134; Vals: 0,2,2,1,2,2,1,2,2,0,2,2,1,2,2,1,2,2,0,2,2,1,2,2,1,2,2,0,2,2,1,2,2,1,2,2,0,2,2,1,2,2,1,2,2,0,2,2,1,2,2,1,2,2,0,2,2,1,2,2,1,2,2,0,2,2,1,2,2,1,2,2,0,2,
Weight: 1188; Vals: 0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,
Weight: 1242; Vals: 0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,
Weight: 1296; Vals: 0,2,2,1,2,2,1,2,2,0,2,2,1,2,2,1,2,2,0,2,2,1,2,2,1,2,2,0,2,2,1,2,2,1,2,2,0,2,2,1,2,2,1,2,2,0,2,2,1,2,2,1,2,2,0,2,2,1,2,2,1,2,2,0,2,2,1,2,2,1,2,2,0,2,
Weight: 1350; Vals: 0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,
Weight: 1404; Vals: 0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,
Weight: 1458; Vals:
0,4,4,3,4,4,3,4,4,2,4,4,3,4,4,3,4,4,2,4,4,3,4,4,3,4,4,1,4,4,3,4,4,3,4,4,2,4,4,3,4,4,3,4,4,2,4,4,3,4,4,3,4,4,1,4,4,3,4,4,3,4,4,2,4,4,3,4,4,3,4,4,2,4,4,3,4,4,3,4,4,0,4,4,
Weight: 1512; Vals:
0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,
Weight: 1566; Vals:
0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,
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Figure: Sequences we considered that demonstrated pattern.

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Future work to figure out what's going on! Most likely (and hopefully) there is a better basis.

Future Work

1. Fully characterize $ker(\psi_d + 1 - \phi_f)$.
2. Characterize $im(\phi_f \oplus ((2^{\lceil \frac{w}{2} \rceil} - 1)^{-1}(\psi_d - 1))$
3. Compute Homology.
4. Further research
   - Hecke Operators/elliptic curves over finite fields.
   - Dirichlet L Series

References I