Homotopy, Modular Forms, and the Spectrum $Q(\ell)$
Final Report

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1 Introduction

Elliptic Curves and Modular Forms feel like they go together from the moment you hear both descriptions. Modular Forms remind one so much of functions of elliptic curves that one might even be inclined to tell you modular forms are such functions. The truth, however, is much more convoluted and entangles elliptic curves and modular forms in such a way that you certainly cannot talk about one without the other.

The goal of this project was, among other things, to produce $\ker(\psi_d + 1)$; that is to find all situations where f is a Modular Form (with level structure), such that $f + \psi_d(f)$ is a Modular Form (without level structure). To do this, it was important we understand, first and foremost, what Modular Forms are and what this map ψ_d is. From there, using special properties of modular forms, Isogenies (of elliptic curves), and operators called *Atkin-Lehner* involutions, we produced a computation of $\ker(\psi_d + 1)$, for any level of the graded ring of modular forms. This work originates from the computation of the homotopy groups of the spectrum $Q(2)$ via the Bockstein spectral sequence associated to it.

2 Background

2.1 Elliptic Curves

Elliptic curves have been the main characters in a number of mathematical stories over the years, each of which could be nearly freestanding. In this report, we take two perspectives of Elliptic Curves: smooth projective curves over a perfect (algebraically closed) field K or objects called 'complex tori.' These two descriptions of elliptic curves are entirely equivalent and are equally useful interpretations in different situations. Complex tori, as their name implies, are defined solely over the complex numbers. For the purposes of this paper, Elliptic Curves just give rise to our important map ψ_d, and the complex tori will do the trick.

Before we define elliptic curves, we first need a notion of a *lattice*,

Definition 1. (Silverman 2) A lattice Λ, defined over $\mathbb{C}$, is an abelian group $\omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ where $\omega_1 \neq \omega_1$ are both nonzero and in the upper half plane $\mathcal{H}$.

Note that for any nonzero $m \in \mathbb{Z}$, and any lattice Λ, $\Lambda \supset m\Lambda$ implies that $\Lambda = m\Lambda$. Note also that any lattice is *homothetic* to any other lattice. That is, by scaling and translating, any lattice becomes any other lattice. It is then that we arrive at the fact that,

Proposition 2. *Given any lattice $\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$, there exists a $\Lambda_\tau = \mathbb{Z} \oplus \tau\mathbb{Z}$ isomorphic to Λ. This Λ_τ is defined precisely so that $\tau = \frac{\omega_1}{\omega_2}$, and τ is certainly in $\mathcal{H}$.*

Now that we know lattices are actually given by any $\tau \in \mathcal{H}$, we will treat all lattices as Λ_τ lattices. Notice also that lattices are sort of like spans of two linearaly independent complex numbers. This makes the next idea of a complex torus a little more natural.

Definition 3 (Complex Torus)**.** Let $\Lambda_\tau = \mathbb{Z} \oplus \tau\mathbb{Z}$ be a lattice. Then a <u>complex torus</u> $E_\tau$is said to be $\mathbb{C}/\Lambda_\tau$.

We also establish, without proof, that every elliptic curve is isomorphic to a complex torus. This allows us to define a group action of area preserving transformations (that is an $\mathrm{SL}_2(\mathbb{Z})$-action) on elliptic curves, through complex tori. However, we are not interested in all area preserving transformations. Specifically, we are interested in a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ that is upper triangular mod N where $N \in \mathbb{N}$. This subgroup is called $\Gamma_0(N)$, and its action preserves the N-torsion data of a given elliptic curve. In this paper, we are most concerned with 2-torsion data, so our congruence subgroup of choice is $\Gamma_0(2)$.

The $\Gamma_0(2)$-action on complex tori (elliptic curves) $E = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$ descends from the $SL_2(\mathbb{Z})$-action and is thus given by,

$$\begin{aligned}
&\rho_\gamma : \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau) \to \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\gamma(\tau)) \\
&\rho_\gamma :: z + \Lambda_\tau \mapsto \frac{z}{c\tau + d} + \Lambda_{\gamma(\tau)}, \ \gamma \in \Gamma_0(2).
\end{aligned}$$

THe important structure that $\Gamma_0(2)$ provides is that this action will preserve the 'enhanced' structure of elliptic curves that we consider, namely that the maps will preserve points of order two. Structure preserving maps of elliptic curves and enhanced elliptic curves are called <u>isogenies</u>, and one such isogeny is our map ψ_d. This map comes from the natural projection $\pi : E \to E/_{< P >}$ where P is a point of order 2. This map ψ_d is defined initially on elliptic curves with $\Gamma_0(2)$ level structure, but pulls-back to a map of modular forms. This map is called the *Dual Isogeny*. The dual isogeny is the 'dual' of the map π above, which exists thanks to nice properties that surjective morphisms in the category of algebraic varieties enjoy. But, in preserving the data of enhanced elliptic curves (with 2-torsion data), the dual isogeny produces another subgroup of order 2, of the new curve. That is, $\psi_d : \mathcal{M}_0(2) \to \mathcal{M}_0(2) :: \left(E/_P, \hat{P}\right) \mapsto (E, P)$.

2.2 Modular Forms

We now turn our focus to modular forms. In a first course in abstract algebra, we might be most interested in symmetry groups and their actions on regular polygons. Here, we study an important class of functions $f : \mathcal{H} \to \mathbb{C}$ that are invariant under area preserving transformations. As was mentioned in the introduction, modular forms are not **just** functions. They manifest much more structure than is captured in their description as functions, and such structure becomes important at higher levels. For the purposes of this paper, modular forms are holomorphic (complex differentiable with other special properties) functions on the upper half plane.

Definition 4 (Modular Forms)**.** (Diamond and Shurman 1) Let $f : \mathcal{H} \to \mathbb{C}$ be a complex differentiable function, and let $\gamma \in SL_2(\mathbb{Z})$. Then, f is called a *modular form (of weight k)* if

$$f(\gamma\tau) = (c\tau + d)^k f(\tau) \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

, and f is complex differentiable at infinity.

We can define several congruence subgroups of $SL_2(\mathbb{Z})$ by the matrix entries of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, e.g. $\Gamma_0(N)$ is a congruence subgroup with $c \equiv 0$ mod N. They are defined by $\Gamma(N) \subset \Gamma$ for some integer N, which is referred to as the subgroup's level. We can then similarly define modular forms using these subgroups to get modular forms with level structure.

One of the important reasons we care about $SL_2(\mathbb{Z})$-'invariance' of these functions is because of how this structure relates to Elliptic Curves. Roughly speaking, this rough invariance that modular forms (without level structure) obey allows them to be regarded as 'functions' on isomorphism classes of elliptic curves.

Now, sometimes we find it useful to attach other data to Elliptic Curves. For example, one particularly natural piece of data to associate to any elliptic curve E are

1. A point of order N $P \in E$, written as (E, P)
2. A subgroup $H \leq E$ of order N, written similarly

While $SL_2(\mathbb{Z})$-modular forms are 'functions' on normal elliptic curves, we can find *congruence subgroups* $\Gamma \leq SL_2(\mathbb{Z})$ such that, with proper definitions, $\Gamma-$invariant modular forms become 'functions' on elliptic curves with the associated data above (with proper choice of Γ for the given situation).

In our case, we consider elliptic curves with a point of order 2. In this case the congruence subgroup we care about is $\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | c \cong 0 \mod 2 \right\}$.

2.3 Atkin-Lehner

Given $f \in M_k(\Gamma_0(N))$ and $Q|N$ s.t. $gcd(N, \frac{N}{Q}) = 1$ the Q'th Atkin-Lehner operator is the matrix $W_Q = \begin{bmatrix} aQ & b \\ cN & dQ \end{bmatrix}$ with (a, b, c, d) chosen s.t. $det(W_Q) = Q$.

It then turns out that $\psi_d(f)(\tau) = 2^{\frac{w(f)}{2}} f|_{W_2}$ where the $|_\gamma$ denotes a matrix acting on a modular form in the usual sense of $f|_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}(\tau) = f\left(\dfrac{a\tau + b}{c\tau + d}\right)$. This relation is extremely important in our work.

3 Results

The major theorem that most of our results employed was the following:

Theorem 5. *For $f \in M_k(\Gamma_0(2))$, $f \in M_k(SL_2(\mathbb{Z})) \iff \psi_d(f)(\tau) = f(2\tau)$*

Proof. ($\Longrightarrow$)

As $\psi(f)(\tau)$ is just $f(\frac{-1}{2\tau})$ without the automorphy factor, and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$, $\psi(f)(\tau) = \tau^{-k} f(\frac{-1}{2\tau}) = \tau^{-k}(2\tau)^k f(2\tau) = 2^k f(2\tau)$

($\Longleftarrow$)

Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(2)$, we already know that $f(\tau+1) = f(\tau)$. Thus, it is sufficient to prove that $f(\frac{-1}{\tau}) = \tau^{-k} f(\tau)$. (Deo and Medvedovsky 250) demonstrate that $f|_k W_2 = (\det W_2)^{\frac{k}{2}} j(W_2, \tau)^{-k} f(\frac{-1}{2\tau}) = 2^{\frac{k}{2}} f(2\tau)$. Simplifying this expression, we arrive at

$$2^{\frac{k}{2}} f(2\tau) = 2^{\frac{k}{2}} (2\tau)^{-k} f(\frac{-1}{2\tau})$$
$$f(\frac{-1}{2\tau}) = (2\tau)^k f(2\tau)$$

Then, replacing 2τ with τ, we have demonstrated that $f(\frac{-1}{\tau}) = \tau^k f(\tau)$, i.e. f is weight-k invariant under $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Since it is also weight-k invariant under $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and the two matrices generate $SL_2(\mathbb{Z})$, f is weight-k invariant under the full modular group $SL_2(\mathbb{Z})$ and is therefore a fully modular form. Thus $f \in M_k(SL_2(\mathbb{Z})) \iff \psi(f)(\tau) = 2^k f(2\tau)$. $\square$

From this, some important restrictions follow in a straightforward manner

Theorem 6. *The following restrictions exist on* $ker(\psi_d + 1 - \phi_f)$.

1. *For* $f \in M_k(\Gamma_0(2))$, $\psi_d(f) + f \in M_k(SL_2(\mathbb{Z})) \iff 2^k(f(\tau) - f(2\tau) - \psi_d(f)(2\tau)) + \psi_d(f)(\tau) = 0$

2. *If* $f \in M_k(SL_2(\mathbb{Z}))$ *and* $\psi_d(f) + f \in M_k(SL_2(\mathbb{Z}))$, *then* $f = 0$.

Proof. (Proof of (1.))

($\Longrightarrow$)

If $\psi(f) + f$ is fully modular, then $\psi(\psi(f) + f)(\tau) = 2^k(\psi(f) + f)(2\tau)$ by the above result. Additionally, since $\psi^2(f) = 2^k f$, $\psi(\psi(f) + f) = \psi^2(f) + \psi(f) = 2^k f + \psi(f)$. Thus since the two expressions must be equal, subtracting them gives zero. Thus $2^k f(\tau) + \psi(f)(\tau) - 2^k \psi(f)(2\tau) - 2^k f(2\tau) = 2^k(f(\tau) - f(2\tau) - \psi(f)(2\tau)) + \psi(f)(\tau) = 0$.

($\Longleftarrow$)

As was derived, the equation is equivalent to $\psi(\psi(f) + f)(\tau) = 2^k(\psi(f) + f)(2\tau)$. Theorem 5 then implies that $\psi(f) + f$ is fully modular.

(Proof of (2.))

By the above result once again, $2^k(f(\tau) - f(2\tau) - \psi(f)(2\tau)) + \psi(f)(\tau) = 0$. Then, since f is fully modular, we can simplify the $\psi(f)$'s to get $2^k f(\tau) - 2^k f(2\tau) - 2^{2k} f(4\tau) + 2^k f(\tau) = 2^k(f(\tau) - 2^k f(4\tau)) = 0$. Thus $f(\tau) - 2^k f(4\tau) = 0$.

Now, in terms of q-expansions (assuming $f = \sum_{i=0}^{\infty} a_i q^i$) we would have that $f(q) - 2^k f(q^4) = 0$. If we expand out the L.H.S. in q we arrive at $(a_0 - 2^k a_0) + a_1 q + a_2 q^2 + a_3 q^3 + (a_4 - 2^k a_1) q^4 + ... = 0$.

Thus we have the following conditions:

$$\begin{cases} a_0 - 2^k a_0 = 0 \\ a_1 = 0 \\ a_2 = 0 \\ a_3 = 0 \end{cases}$$

And since the first condition obviously implies that $a_0 = 0$ since we work in characteristic 0, we have that $a_i = 0, 0 \leq i \leq 3$. I claim that this forces every other coefficient to be zero. For, if all $a_i = 0, i < n$ and n is not a multiple of 4 then the n'th q-coefficient of $g = f(q) - 2^k f(q^4)$ would just be a_n (since $n \neq 4i$). Since $g = 0$, a_n must be zero as well. If 4 divides n, say $n = 4i$, then we have the n'th q-coefficient of g as $a_n - 2^k a_i$. However, $i < n$, so $a_i = 0$. Thus since $a_n - 2^k a_i = 0$ and $a_i = 0$, $a_n = 0$. Thus, by induction, $a_i = 0, \forall i \geq 0$. All together, this implies that $f = 0$. □

In particular, the first restriction allows for an easy method to calculate a basis for $K := \{f \in M_k(\Gamma_0(2)) | \psi_d(f) + f \in M_k(SL_2(\mathbb{Z}))\}$ which is a $\mathbb{C}$ vector subspace if we're working with coefficients in $\mathbb{C}$ or is a $\mathbb{Z}_3$ submodule if we're working with 3-adic coefficients of $M_k(\Gamma_0(2))$.

We also have results similar to those of last year, namely that $ker(\psi_d + 1)$ is intimiately related to $im(\psi_d - 1)$.

Theorem 7 (Main Theorem)**.** *For $f \in M_k(\Gamma_0(2))$, $\psi(f) + f \in M_k(SL_2(\mathbb{Z})) \iff \exists g \in M_k(SL_2(\mathbb{Z})) : f = \psi(g) - g$.*

Proof. ($\Longleftarrow$) If $f = \psi(g) - g$ for g fully modular, then $\psi(f) = 2^k g - \psi(g)$ so $\psi(f) + f = 2^k g - \psi(g) + \psi(g) - g = (2^k - 1)g$. Since $M^\bullet(SL_2(\mathbb{Z}))$ is a $\mathbb{C}-$vector space, $(2^k - 1)g \in M_k(SL_2(\mathbb{Z}))$. ($\Longrightarrow$) If $\psi(f) + f \in M_k(SL_2(\mathbb{Z}))$ then let $g = \psi(f) + f$. Then $\psi(g) = 2^k f + \psi(f)$. However, since $g = \psi(f) + f$, $\psi(f) = g - f$. Thus $\psi(g) = 2^k f + g - f$ so $(2^k - 1)f = \psi(g) - g$. Dividing both sides by $2^k - 1$ finishes the proof. □

However, since the final step requires division by $2^k - 1$ (which, for all even k is divisible by 3), this does not nicely extend to $\mathbb{Z}_3$ where 3 is not invertible. However, if g already has a factor of $3^{val_3(2^k - 1)}$ then we can safely cancel from both sides! The following result describes the $3 - adic$ divisibility of g for $g \in M_k(SL_2(\mathbb{Z}))$.

Theorem 8 (3-Adic Obstructions)**.** *If $g \in M_k(SL_2(\mathbb{Z}))$ and $2^k - 1 | \psi(g) - g$, then, letting $g(q) = \sum_{i=0}^{\infty} a_i q^i$, $2^k - 1 | a_i, \forall i \geq 1$. In particular, the only obstructions that arise when passing to $\mathbb{Z}_3$ occur at a_0.*

Proof. First, say $f = \psi(g) - g$ so that $2^k - 1 | f$. Then by Theorem 5, $f(q) = 2^k g(q^2) - g(q)$. Expanding out f, then gives

$$f = (2^k - 1)a_0 - a_1 q + (2^k a_1 - a_2)q^2 + ...$$

Then, since $f \equiv 0 \mod (2^k - 1)$, $a_1 \equiv 0 \mod (2^k - 1)$. Similarly, $\forall i$ odd $a_i \equiv 0 \mod (2^k - 1)$. Assume $a_i \equiv 0 \mod (2^k - 1), \forall 0 < i < n$. Then, for i even (say $i = 2j$), the ith q-coefficient of f is $2^k a_j - a_i$. Mod 3, however, we have that $2^k a_j - a_i \equiv 0 \mod (2^k - 1)$. Since $j < i$, inductively we have that $a_j \equiv 0 \mod (2^k - 1)$. Thus $2^k a_j - a_i \equiv -a_i \equiv 0 \mod (2^k - 1)$. Thus, by induction $\forall i > 0, a_i \equiv 0 \mod (2^k - 1)$. □

Thus, to determine if a given f is in the image of $\psi_d - 1$, we would need to study the relationship between f and g. The next two results describe this connection, culminating in a description of where exactly the obstructions preventing $im(\psi_d - 1)$ to be fully equal to $ker(\psi_d + 1 - \phi_f)$.

Lemma 9. *If $f \in M_k(\Gamma_0(2))$ and $g = \psi(f) + f \in M_k(SL_2(\mathbb{Z}))$ with $f = \sum_{i=0}^{\infty} a_i q^i$, $\psi(f) = \sum_{i=0}^{\infty} a_i' q^i$ and $g = \sum_{i=0}^{\infty} b_i q^i$ then $a_0' = 0$ and, consequently, $a_0 = b_0$.*

Proof. Applying Theorem 6.1 to our situation, we have that

$$2^k(f(q) - f(q^2) - \psi(f)(q^2)) + \psi(f) = 0$$

And, upon examining the constant term of the resulting expression we get

$$\begin{aligned} 2^k(a_0 - a_0 - a_0') + a_0' &= 0 \\ -(2^k - 1)a_0' &= 0 \\ a_0' &= 0 \end{aligned}$$

Where the last simplification follows from the characteristic being 0.
Thus, since $g = \psi(f) + f$, $b_0 = a_0 + a_0' = a_0$. □

Lemma 10. *$v_3(2^k - 1) = 1 + v_3(k)$ for k even where v_3 denotes the 3-adic valuation, i.e. the number of times 3 divides a number.*

Proof. According to(Sloane and Inc.) this seems to be a standard fact. □

Observation 11 (Obstructions)**.** *Combining Theorem 8, Lemma 9, and Lemma 10 we notice that $f \in M_k(\Gamma_0(2))$ satisfying $\psi(f) + f \in M_k(SL_2(\mathbb{Z}))$ is in the image of $\psi - 1$ if and only if, for $f = \sum_{i=0}^{\infty} a_i q^i$, $val_3(a_0) = 1 + val_3(k)$.*

3.1 Computations (via MAGMA)

The following procedure computes the basis described above at a given weight k.

Procedure 12 (Computation of a Basis)**.** *Inputs: k, the weight of modulars forms to examine; R, the ring to work over.*

1. *Generate a basis $\mathcal{B} = \{f_1, ..., f_n\}$ of weight k $\Gamma_0(2)$-modular forms as products of special forms q_2 and q_4.*
2. *Write a general weight k modular form f as $\sum\limits_{i=1}^{n} x_i f_i$.*
3. *Compute the q-expansion of $2^k(f(q) - f(q^2) - \psi_d(f)(q^2)) + \psi_d(f)(q) = \sum\limits_{i=1}^{n}(\sum\limits_{j=1}^{n} c_{i,j} x_i) q^i$.*
4. *Denoting $M := \begin{pmatrix} c_{1,1} & c_{2,1} & \cdots & c_{n,1} \\ c_{1,2} & c_{2,2} & .. & c_{n,2} \\ .. & .. & .. & .. \\ c_{1,n} & c_{2,n} & .. & c_{n,n} \end{pmatrix}$, find the Smith Normal Form $M = SDQ$, and take the submatrix of Q with columns starting where the first 0 appears on the diagonal of D. The columns of Q now represents the basis of the nullspace of M and thus gives a basis $\bar{\mathcal{B}} = \{\bar{f_1}, \bar{f_2}, ..., \bar{f_m}\}$ of K.*

4 Future Work

We have a couple of different directions we'd like to go in the future for this project:

4.1 Homological Computations

Since these computations are part of determining varying pages of a spectral sequence, these maps and kernels are all part of some more complication computation of a homology group. Specifically, these computations arise from a complex

$$M^\bullet(SL_2(\mathbb{Z})) \xrightarrow[g]{\alpha \mapsto (\alpha, 2^{\frac{deg(\alpha)}{2}} - 1)^{-1}(\psi_d - 1)(\alpha))} M^\bullet(SL_2(\mathbb{Z})) \oplus M^\bullet(\Gamma_0(2)) \xrightarrow[f]{(\alpha,\beta) \mapsto \psi_d(\beta) + \beta - \alpha} M^\bullet(\Gamma_0(2))$$

The groups we are interested in are ${}^{ker(f)}\!/_{im(g)}$. Now, $ker(f)$ is just $\{(\psi_d(\beta) + \beta, \beta) | \beta \in K\}$ by definition. Since we already are able to compute a basis for $ker(f)$, it would be nice to get something similar for $im(g)$, and to hopefully use the fact that $im(g) \subseteq ker(f)$ as $\mathbb{Z}_3-$modules as well as the Structure Theorem for Modules over PIDs to get a nice representation for the quotient.

4.2 Hecke Operators

Atkin Lehner involutions and the structure that arise from it have been shown to be quite useful. In the future, we will try to connect this work to those of Hecke Operators, which are related but different from the Atkin Lehner operators. Since they are both double coset operators on modular forms, we may be able to find some similar interesting properties. We will see if the more general properties of Hecke Operators can help us make more connections with our problem. We are currently trying to find restrictions on Fourier coefficients if any f is in the aforementioned kernels or cokernels. We can also look at Hecke Operators through elliptic curves instead of modular forms to potentially gain more insight.

4.3 Atkin-Lehner Involutions

Atkin-Lehner involutions, while they are here approached as operators on modular forms, are quite natural objects that also arise at the level of the universal abelian scheme, and lift from the regular case of modular forms to the topological modular forms (TMF). Quotients of Shimura Curves by Atkin-Lehner involutions are once again Smooth Moduli Objects. Furthermore, when an Atkin-Lehner involution does lift, we have the relation $W^2 = [N]$ where $N = \det W$. Atkin-Lehner involutions also normalize the congruence subgroup $\Gamma_0(2)$ in $GL_2^+(\mathbb{Z})$. In effect, Atkin-Lehner involutions produce a rich theory not just on newforms, but on Abelian Schemes, and are exciting objects for further study.

5 References

Works Cited

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