Homotopy, Modular Forms, and the Spectrum $Q(\ell)$

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Introduction

Goal

To compute kernels and cokernels of maps on rings of modular forms that arise from spectral sequences that compute stable homotopy groups of the spectrum $Q(2)$.

0.1 Modular Forms

$f : \mathcal{H} \to \mathbb{C}$ is defined to be a modular form of weight k if $f(\gamma\tau) = (c\tau+d)^k f(\tau)$ for $\gamma \in SL_2(\mathbb{Z})$ and f is complex differentiable at infinity.
We can define several congruence subgroups of $SL_2(\mathbb{Z})$ by the matrix entries of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, e.g. $\Gamma_0(N)$ is a congruence subgroup with $c \equiv 0$ mod N. They are defined by $\Gamma(N) \subset \Gamma$ for some integer N, which is referred to as the subgroup's level. We can then similarly define modular forms using these subgroups to get modular forms with level structure.

0.2 Elliptic Curves

Some objects related to Modular Forms are *Elliptic Curves* which are quotients of the complex plane $\mathbb{C}$ by a 'lattice' or abelian group $\Lambda_\tau = \tau\mathbb{Z} \oplus \mathbb{Z}$ where τ is in the upper half-plane $\mathcal{H}$. Elliptic curves can also be separated into $SL_2(\mathbb{Z})$-orbits under the $SL_2(\mathbb{Z})$ action given by

$$\gamma : \Lambda_\tau \to \Lambda_{\frac{a\tau+b}{c\tau+d}},$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL_2(\mathbb{Z})$. Then just as there is level structure over modular forms above, there is a level structure over elliptic curves given by elliptic curves enhanced with cyclic subgroups of order N. Then for $N = 2$ we can take the elliptic curves with cyclic subgroups of order 2 called $\mathcal{M}_0(2)$. This object has the map $\psi_d : \mathcal{M}_0(2) \to \mathcal{M}_0(2)$, to be defined over modular forms.

1. Atkin-Lehner Connection

Now this map, ψ_d, has a distractingly simple description: $\psi_d(f)(\tau) = f\left(\frac{-1}{2\tau}\right)$ (up to some factors of τ that don't matter). Thus to understand ψ it suffices to examine the above transformation on $\Gamma_0(2)$ modular forms. Thankfully, this sort of transformation is already slightly studied:

Definition 1: (Atkin-Lehner Operator)

Given $f \in M_k(\Gamma_0(N))$ and $Q|N$ s.t. $gcd(N, \frac{N}{Q}) = 1$ the Q'th Atkin-Lehner operator is the matrix $W_Q = \begin{bmatrix} aQ & b \\ cN & dQ \end{bmatrix}$ with (a, b, c, d) chosen s.t. $det(W_Q) = Q$.

It then turns out that $\psi_d(f)(\tau) = 2^{\frac{w(f)}{2}} f|_{W_2}$ where the $|_\gamma$ denotes a matrix acting on a modular form in the usual sense of $f|_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}(\tau) = f\left(\frac{a\tau+b}{c\tau+d}\right)$.
Similarly, there are previously studied impressive results regarding the conditions under which the Dedekind-eta function and eta quotients are modular forms. We originally examined modular forms of these kinds, but Atkin-Lehner operators were more useful and more versatile for our work.

2. Methods and/or Results

2.1 Theoretical Results

The major theorem that most of our results employed was the following:

Theorem 1

For $f \in M_k(\Gamma_0(2))$, $f \in M_k(SL_2(\mathbb{Z})) \iff \psi_d(f)(\tau) = f(2\tau)$

Where the forward implications is valid since modular forms are invariant under $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$ and the reverse implication is because modular forms in $\Gamma_0(2)$ are automatically $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$-invariant and $\psi_d(f)(\tau) = f(2\tau)$ implies that it is also $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$-invariant. Since $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ generate $SL_2(\mathbb{Z})$, f is then fully invariant under the action of $SL_2(\mathbb{Z})$ so that $f \in M_k(SL_2(\mathbb{Z}))$.
From this, some important restrictions follow in a straightforward manner

Theorem 2

- For $f \in M_k(\Gamma_0(2))$, $\psi_d(f) + f \in M_k(SL_2(\mathbb{Z})) \iff 2^k(f(\tau) - f(2\tau) - \psi_d(f)(2\tau)) + \psi_d(f)(\tau) = 0$
- If $f \in M_k(SL_2(\mathbb{Z}))$ and $\psi_d(f) + f \in M_k(SL_2(\mathbb{Z}))$, then $f = 0$.

In particular, the first restriction allows for an easy method to calculate a basis for $K := \{f \in M_k(\Gamma_0(2)) | \psi_d(f) + f \in M_k(SL_2(\mathbb{Z}))\}$ which is a $\mathbb{C}$ vector subspace if we're working with coefficients in $\mathbb{C}$ or is a $\mathbb{Z}_3$ submodule if we're working with 3-adic coefficients in $M_k(\Gamma_0(2))$.

2.2 Computations (via MAGMA)

The following procedure computes the basis described above at a given weight k.

Procedure 1: (Computation of a Basis)

1. Generate a basis $\mathcal{B} = \{f_1, ..., f_n\}$ of weight k $\Gamma_0(2)$-modular forms as products of special forms q_2 and q_4.
2. Write a general weight k modular form f as $\sum_{i=1}^{n} x_i f_i$.
3. Compute the q-expansion of $2^k(f(q) - f(q^2) - \psi_d(f)(q^2)) + \psi_d(f)(q) = \sum_{i=1}^{n}(\sum_{j=1}^{n} c_{i,j} x_i) q^i$.
4. Denoting $M := \begin{pmatrix} c_{1,1} & c_{2,1} & \cdots & c_{n,1} \\ c_{1,2} & c_{2,2} & .. & c_{n,2} \\ .. & .. & .. & .. \\ c_{1,n} & c_{2,n} & .. & c_{n,n} \end{pmatrix}$, find the Smith Normal Form $M = SDQ$, and take the submatrix of Q with columns starting where the first 0 appears on the diagonal of D. The columns of Q now represents the basis of the nullspace of M and thus gives a basis $\bar{\mathcal{B}} = \{\bar{f}_1, \bar{f}_2, ..., \bar{f}_m\}$ of K.

Future Directions

We have a couple of different directions we'd like to go in the future for this project:

2.3 Homological Computations

Since these computations are part of determining varying pages of a spectral sequence, these maps and kernels are all part of some more complication computation of a homology group. Specifically, these computations arise from a complex

$$M^\bullet(SL_2(\mathbb{Z})) \xrightarrow[g]{\alpha \mapsto (\alpha, 2^{\lfloor \frac{deg(\alpha)}{2} \rfloor} - 1)^{-1}(\psi_d - 1)(\alpha))} M^\bullet(SL_2(\mathbb{Z})) \oplus M^\bullet(\Gamma_0(2)) \xrightarrow[f]{(\alpha,\beta) \mapsto \psi_d(\beta) + \beta - \alpha} M^\bullet(\Gamma_0(2))$$

The groups we are interested in are ${}^{ker(f)}/_{im(g)}$. Now, $ker(f)$ is just $\{(\psi_d(\beta) + \beta, \beta) | \beta \in K\}$ by definition. Since we already are able to compute a basis for $ker(f)$, it would be nice to get something similar for $im(g)$, and to hopefully use the fact that $im(g) \subseteq ker(f)$ as $\mathbb{Z}_3-$modules as well as the Structure Theorem for Modules over PIDs to get a nice representation for the quotient.

2.4 Oldforms

Atkin Lehner involutions and the structure that arise from it have been shown to be quite useful. In the future, we will try to connect this work to those of Hecke Operators and oldforms, which are related but different from the newforms that Atkin Lehner operators deal with. In particular, we will see how the more general properties of oldforms can help us make more connections with our problem. We are currently trying to find restrictions on fourier coefficients if any f is in the aforementioned kernels or cokernels. We can also look at Hecke Operators through elliptic curves instead of modular forms to potentially gain more insight. Actions of Atkin-Lehner involutions on moduli spaces produce further smooth moduli spaces. Studying the effects of Atkin-lehner involution on moduli of elliptic curves could be helpful in exploring the effects of these involutions on higher level objects like the ring of modular forms with level structure and TMF.

References

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