Homotopy, Modular Forms, and the Spectrum $Q(\ell)$

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Objects of Study

Modular Forms:
Given a group $\Gamma \leq SL_2(\mathbb{Z})$, a modular form of weight k with Γ level structure is a function $f : \mathbb{H} \to \mathbb{C}$ where $f(\tau)$ and $f(\gamma \cdot \tau)$ are nicely related, with $\gamma \in \Gamma$ and matrices acting on complex numbers by *fractional linear transformations*.
Modular forms have q-expansions, i.e. we have expansions $f = \sum_{i=0}^{\infty} a_i q^i$ for modular forms f. The a_i are *q-expansion coefficients*.
Homotopy Groups:
Given a space X, we can define $\pi_k(X)$ as *the different ways in which a k-sphere can be mapped into X "up to deformation"*

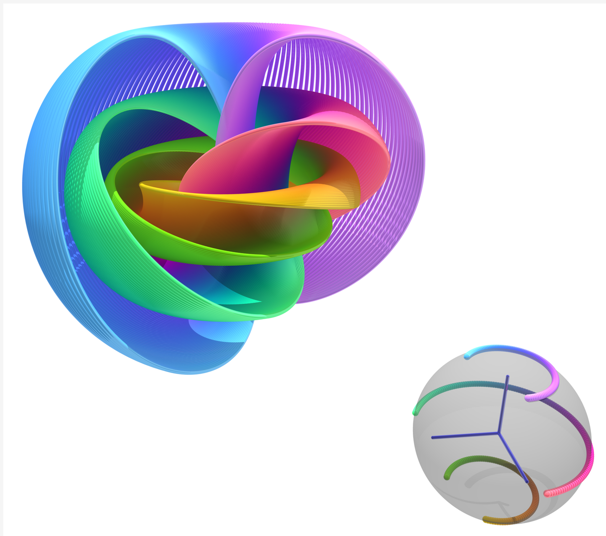


Figure 1: Example of a nontrivial map $S^3 \to S^2$, the *Hopf Fibration*

Problem Description

High Level: Compute $\pi_k(Q(\ell))$, *the homotopy groups of the spectrum* $Q(\ell)$, via spectral sequence:

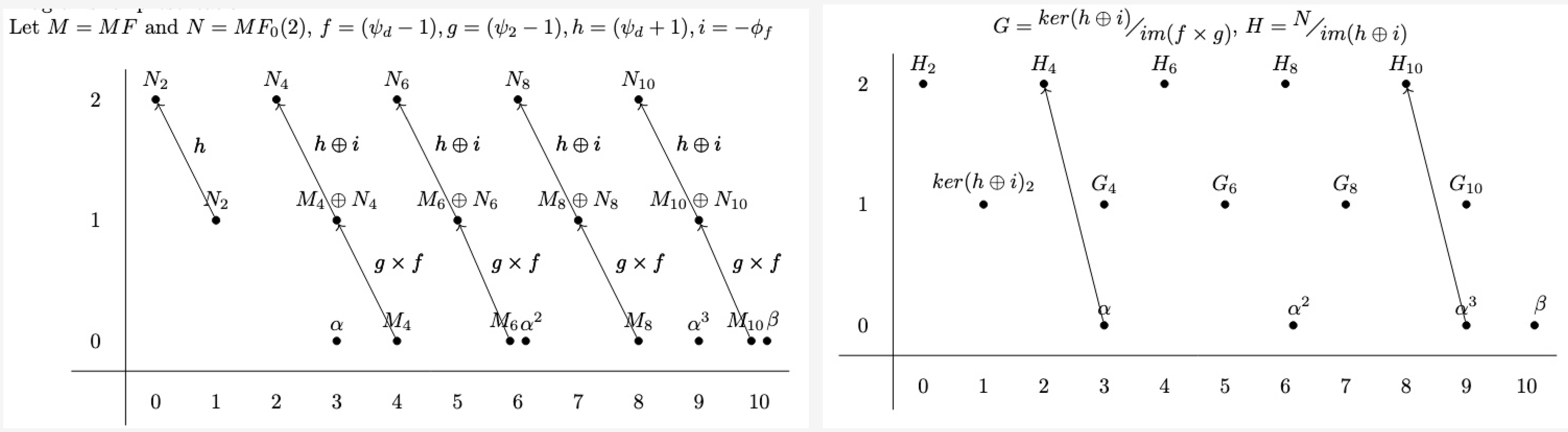


Figure 2: Spectral Sequence Pages

Computational Level: Compute Kernels and Cokernels of maps that are on rings of modular forms.
Idea: $Q(\ell)$ algebraically acts like 'modular forms' and we can piece together groups related to it, by breaking them up into pieces (groups in a spectral sequence), looking at how the pieces relate (compute differentials in spectral sequence), and see how well the relations capture all the relevant behavior (study failure to be injective/surjective via homology).

1. Coefficient Divisibility

- The largest power of 3 dividing all of the q coefficients becomes important in figuring out these quotients. Maps were 'almost isomorphisms' if the 3-divisibility was high enough.
- Luckily, modular forms often have number theoretic q coefficients, which makes computing that power of 3 much easier
- E.g. there are important forms c_4, c_6, where $c_4 = 1 + 240 \sum_{i=1}^{\infty} \sigma_3(i) q^i$, and $c_6 = 1 - 504 \sum_{i=1}^{\infty} \sigma_5(i) q^i$, where $\sigma_k(n) = \sum_{d|n} d^k$ and if we write $c_4^i c_6^j = 1 + \sum_{n=1}^{\infty} a_i q^i$, it turns out that $\nu_3(a_i) \geq \begin{cases} \nu_3(4i+6j)+1, & \nu_3(4i+6j) < 3 \\ \min\{\nu_3(i)+1, \nu_3(4i+6j)\}+1, & \nu_3(4i+6j) \geq 3 \end{cases}$
- Even better, for modular forms with $SL_2(\mathbb{Z})$-level structure, maps nicely preserved 3-divisibility.

2. First Page Homology

- To move spectral sequence forward, we needed to compute homology at all classes in the first page.
- For a class B, if we have a map $f : A \to B$ hitting it, and a map $g : B \to C$ leaving it, the homology is ${}^{ker(g)}/_{im(f)}$.
- This semester, we were able to fully calculate the homology at the middle cells, $G_i = {}^{ker(\psi_d + 1 - \phi_f)}/_{im((\psi_d - 1) \times (\psi_d + 1))} \cong {}^{\mathbb{Z}}/_{3^{\nu_3(i)+1}\mathbb{Z}}$ for even $i > 2$ and 0 otherwise.
- Proof for this exploited that all elements of the homology were classes of $\Gamma_0(2)$ forms f such that $(\psi_d + 1)(f)$ was a $SL_2(\mathbb{Z})$ form. This lead to many, many 3-divisibility simplifications due to nice symmetries of $SL_2(\mathbb{Z})$.
- Turns out class of element $f \in ker(\psi_d + 1 - \phi_f)$ in the quotient is entirely determined by constant coefficient.
- Similarly, it isn't difficult to tell that the bottom row homology is 0, since one of the maps 'looks like' just multiplication by a nonzero number. Then, ker([that map]) is 0.
- Top row homology is harder, since we lose the structure of $SL_2(\mathbb{Z})$ modular forms and have 'two independent variables' in $im(\psi_d + 1 - \phi_f)$.

3. Methods of Proof

- The groups we work with also carry a $\mathbb{Z}_3$-module structure, which makes computing kernels and co-kernels of map much easier.
- Working weight by weight, we can deal exclusively with free modules, so our maps can be represented by matrices M with entries in $\mathbb{Z}_3$. Since that ring is a PID, we can efficiently compute homologies via the Smith Normal Form of M.
- In the case of the middle homologies G_i, using a CAS called MAGMA, we noticed that that $G_i = {}^{\mathbb{Z}}/_{3^{\nu_3(i)+1}\mathbb{Z}}$. Since this is a single group, this suggested that such groups were generated by a single form f that was not in the image, but that $3^{\nu_3(i)+1} f$ was.
- Via our work with coefficient divisibility, we had already calculated forms at every weight that satisfied this property. It turned out that the math worked out, and it did generate the homology!
- In terms of the homologies of the upper classes, there are multiple factors of the homology groups at each weight, which makes the analysis less easy.

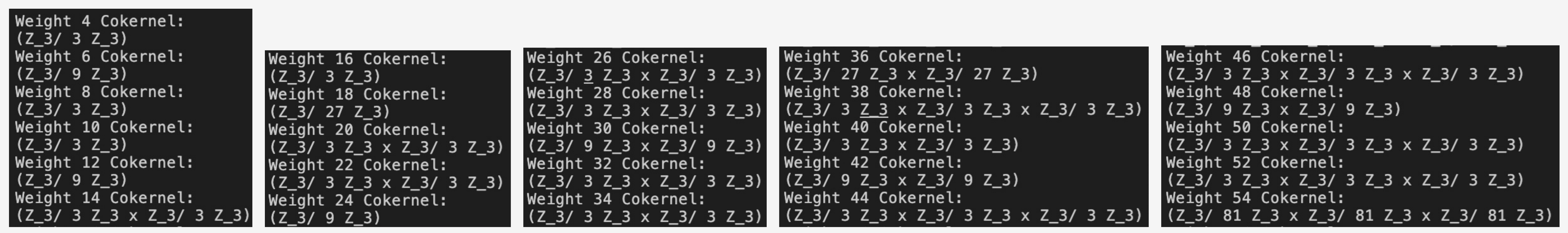
4. Further Work

Computation of the homology of the upper classes is important for two reasons:

- Cokernels give permanent classes in the homotopy groups $\pi_k(Q(\ell))$, i.e. they directly give topological information we desire
- As we can see on page two, we have differentials from *Greek letter element* classes into some of the cokernels. To determine if they survive, we need to understand the groups to determine if the classes map to zero in the homologies.

Thus we have two stages of further work:

1. Determine the upper homology groups $H_i = {}^{MF_0(2)}/_{im(\psi_d + 1 - \phi_f)_i}$.
   - We have computational results, however we have not determined a pattern yet nor a theoretical description of the behavior.



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Weight 4 Cokernel:
(Z_3/ 3 Z_3)
Weight 6 Cokernel:
(Z_3/ 9 Z_3)
Weight 8 Cokernel:
(Z_3/ 3 Z_3)
Weight 10 Cokernel:
(Z_3/ 3 Z_3)
Weight 12 Cokernel:
(Z_3/ 9 Z_3)
Weight 14 Cokernel:
(Z_3/ 3 Z_3 x Z_3/ 3 Z_3)
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Weight 16 Cokernel:
(Z_3/ 3 Z_3)
Weight 18 Cokernel:
(Z_3/ 27 Z_3)
Weight 20 Cokernel:
(Z_3/ 3 Z_3 x Z_3/ 3 Z_3)
Weight 22 Cokernel:
(Z_3/ 3 Z_3 x Z_3/ 3 Z_3)
Weight 24 Cokernel:
(Z_3/ 9 Z_3)
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Weight 26 Cokernel:
(Z_3/ 3 Z_3 x Z_3/ 3 Z_3)
Weight 28 Cokernel:
(Z_3/ 3 Z_3 x Z_3/ 3 Z_3)
Weight 30 Cokernel:
(Z_3/ 9 Z_3 x Z_3/ 9 Z_3)
Weight 32 Cokernel:
(Z_3/ 3 Z_3 x Z_3/ 3 Z_3)
Weight 34 Cokernel:
(Z_3/ 3 Z_3 x Z_3/ 3 Z_3)
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Weight 36 Cokernel:
(Z_3/ 27 Z_3 x Z_3/ 27 Z_3)
Weight 38 Cokernel:
(Z_3/ 3 Z_3 x Z_3/ 3 Z_3 x Z_3/ 3 Z_3)
Weight 40 Cokernel:
(Z_3/ 3 Z_3 x Z_3/ 3 Z_3)
Weight 42 Cokernel:
(Z_3/ 9 Z_3 x Z_3/ 9 Z_3)
Weight 44 Cokernel:
(Z_3/ 3 Z_3 x Z_3/ 3 Z_3 x Z_3/ 3 Z_3)
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Weight 46 Cokernel:
(Z_3/ 3 Z_3 x Z_3/ 3 Z_3 x Z_3/ 3 Z_3)
Weight 48 Cokernel:
(Z_3/ 9 Z_3 x Z_3/ 9 Z_3)
Weight 50 Cokernel:
(Z_3/ 3 Z_3 x Z_3/ 3 Z_3 x Z_3/ 3 Z_3)
Weight 52 Cokernel:
(Z_3/ 3 Z_3 x Z_3/ 3 Z_3 x Z_3/ 3 Z_3)
Weight 54 Cokernel:
(Z_3/ 81 Z_3 x Z_3/ 81 Z_3 x Z_3/ 81 Z_3)
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Figure 3: Example Computations of Cokernel Groups

2. Determine the kernels of the second page differentials, which should complete the spectral sequence!
3. (Use the final page to determine $\pi_k(Q(\ell))$, with possible group-extension problems getting in the way)