IGL Results 2021

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1 Introduction

This is a collections of results I've discovered over the course of this semester, nicely typeset for clarity (especially considering my handwriting). The only conventions of note that I use are in saying that a form f is *fully modular* if it is properly modular on $SL_2(\mathbb{Z})$.

2 Definitions

Definition 1 (Atkin-Lehner Operator)**.** An *Atkin-Lehner Operator* on modular forms of level N is a matrix W_Q with $Q|N$ and $gcd(Q, \frac{N}{Q}) = 1$ of the form $\begin{bmatrix} aQ & b \\ cN & dQ \end{bmatrix} \in \Gamma_0(2)$ where a, b, c, d are such that $det(W_Q) = Q$

3 Results

Theorem 2. *If $f \in M_k(SL_2(\mathbb{Z})$ then $\psi_d(f)(\tau) = 2^k f(2\tau)$*

Proof. As $\psi_d(f)(\tau)$ is just $f(\dfrac{-1}{2\tau})$ without the automorphy factor, and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$, $\psi_d(f)(\tau) = \tau^{-k} f(\dfrac{-1}{2\tau}) = \tau^{-k}(2\tau)^k f(2\tau) = 2^k f(2\tau)$ $\square$

Lemma 3. *The claim above can be reversed, i.e. if $\psi_d(f)(\tau) = 2^k f(2\tau)$ for $f \in M_k(\Gamma_0(2))$ then $f \in M_k(SL_2(\mathbb{Z}))$ so f is, in fact, fully modular.*

Proof. Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(2)$, we already know that $f(\tau+1) = f(\tau)$. Thus, it is sufficient to prove that $f(\dfrac{-1}{\tau}) = \tau^{-k} f(\tau)$. (Deo and Medvedovsky 250) demonstrate that

$$f|_k W_2 = (\det W_2)^{\frac{k}{2}} j(W_2, \tau)^{-k} f(\frac{-1}{2\tau}) = 2^{\frac{k}{2}} f(2\tau)$$

. Simplifying this expression, we arrive at

$$2^{\frac{k}{2}}f(2\tau) = 2^{\frac{k}{2}}(2\tau)^{-k}f(\frac{-1}{2\tau})$$
$$f(\frac{-1}{2\tau}) = (2\tau)^k f(2\tau)$$

Then, replacing 2τ with τ, we have demonstrated that $f(\frac{-1}{\tau}) = \tau^k f(\tau)$, i.e. f is weight-k invariant under $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Since it is also weight-k invariant under $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and the two matrices generate $SL_2(\mathbb{Z})$, f is weight-k invariant under the full modular group $SL_2(\mathbb{Z})$ and is therefore a fully modular form. Thus $f \in M_k(SL_2(\mathbb{Z})) \iff \psi_d(f)(\tau) = 2^k f(2\tau)$. □

Theorem 4 (Constraining Equation)**.** *If $f \in M_k(\Gamma_0(2))$ and $\psi_d(f) + f \in M_k(SL_2(\mathbb{Z}))$ then $2^k(f(\tau) - f(2\tau) - \psi_d(f)(2\tau)) + \psi_d(f)(\tau) = 0$*

Proof. If $\psi_d(f)+f$ is fully modular, then $\psi_d(\psi_d(f)+f)(\tau) = 2^k(\psi_d(f)+f)(2\tau)$ by the above result. Additionally, since $\psi^2(f) = 2^k f$, $\psi_d(\psi_d(f)+f) = \psi^2(f)+\psi_d(f) = 2^k f + \psi_d(f)$. Thus since the two expressions must be equal, subtracting them gives zero. Thus $2^k f(\tau) + \psi_d(f)(\tau) - 2^k\psi_d(f)(2\tau) - 2^k f(2\tau) = 2^k(f(\tau) - f(2\tau) - \psi_d(f)(2\tau)) + \psi_d(f)(\tau) = 0$. □

Corollary 5. *The above claim can again be reversed. I.e. $\psi_d(f) + f$ is fully modular $\iff$ $2^k(f(\tau) - f(2\tau) - \psi_d(f)(2\tau)) + \psi_d(f)(\tau) = 0$.*

Proof. As was derived, the equation is equivalent to $\psi_d(\psi_d(f)+f)(\tau) = 2^k(\psi_d(f)+f)(2\tau)$. Lemma 3 then implies that $\psi_d(f) + f$ is fully modular. □

Theorem 6. *If $f \in M_k(SL_2(\mathbb{Z}))$ and $\psi_d(f) + f \in M_k(SL_2(\mathbb{Z}))$ then $f = 0$.*

Proof. By the above result once again, $2^k(f(\tau) - f(2\tau) - \psi_d(f)(2\tau)) + \psi_d(f)(\tau) = 0$. Then, since f is fully modular, we can simplify the $\psi_d(f)$'s to get

$$2^k f(\tau) - 2^k f(2\tau) - 2^{2k} f(4\tau) + 2^k f(\tau) = 2^k(f(\tau) - 2^k f(4\tau)) = 0.$$

Thus $f(\tau) - 2^k f(4\tau) = 0$.
Now, in terms of q-expansions (assuming $f = \sum_{i=0}^{\infty} a_i q^i$) we would have that $f(q) - 2^k f(q^4) = 0$. If we expand out the L.H.S. in q we arrive at $(a_0 - 2^k a_0) + a_1 q + a_2 q^2 + a_3 q^3 + (a_4 - 2^k a_1)q^4 + ... = 0$. Thus we have the following conditions:

$$\begin{cases} a_0 - 2^k a_0 = 0 \\ a_1 = 0 \\ a_2 = 0 \\ a_3 = 0 \end{cases}$$

And since the first condition obviously implies that $a_0 = 0$ since we work in characteristic 0, we have that $a_i = 0, 0 \leq i \leq 3$. I claim that this forces every other coefficient to be zero. For, if all $a_i = 0, i < n$ and n is not a multiple of 4 then the n'th q-coefficient of $g = f(q) - 2^k f(q^4)$ would just be a_n (since $n \neq 4i$). Since $g = 0$, a_n must be zero as well. If 4 divides n, say $n = 4i$, then we have the n'th q-coefficient of g as $a_n - 2^k a_i$. However, $i < n$, so $a_i = 0$. Thus since $a_n - 2^k a_i = 0$ and $a_i = 0$, $a_n = 0$. Thus, by induction, $a_i = 0, \forall i \geq 0$. All together, this implies that $f = 0$. □

Theorem 7 (Main Theorem)**.** *For $f \in M_k(\Gamma_0(2))$, $\psi_d(f) + f \in M_k(SL_2(\mathbb{Z})) \iff \exists g \in M_k(SL_2(\mathbb{Z})) : f = \psi_d(g) - g$.*

Proof. ($\Longleftarrow$) If $f = \psi_d(g) - g$ for g fully modular, then $\psi_d(f) = 2^k g - \psi_d(g)$ so $\psi_d(f) + f = 2^k g - \psi_d(g) + \psi_d(g) - g = (2^k - 1)g$. Since $M^\bullet(SL_2(\mathbb{Z}))$ is a $\mathbb{C}-$vector space, $(2^k-1)g \in M_k(SL_2(\mathbb{Z}))$. ($\Longrightarrow$) If $\psi_d(f) + f \in M_k(SL_2(\mathbb{Z}))$ then let $g = \psi_d(f) + f$. Then $\psi_d(g) = 2^k f + \psi_d(f)$. However, since $g = \psi_d(f) + f$, $\psi_d(f) = g - f$. Thus $\psi_d(g) = 2^k f + g - f$ so $(2^k - 1)f = \psi_d(g) - g$. Dividing both sides by $2^k - 1$ finishes the proof. $\square$

Lemma 8. *$v_3(2^k - 1) = 1 + v_3(k)$ for k even where v_3 denotes the 3-adic valuation, i.e. the number of times 3 divides a number.*

Proof. According to (Sloane and Inc.) this seems to be a standard fact. $\square$

Theorem 9. *For $g \in M_k(SL_2(\mathbb{Z}))$, if $2^k - 1 | \psi_d(g) - g$, then, letting $g(q) = \sum_{i=0}^{\infty} a_i q^i$, $2^k - 1 | a_i, \forall i \geq 1$. The converse of the statement also holds.*

Proof. First, say $f = \psi_d(g) - g$ so that $2^k - 1 | f$. Then by Theorem 2, $f(q) = 2^k g(q^2) - g(q)$. Expanding out f, then gives

$$f = (2^k - 1)a_0 - a_1 q + (2^k a_1 - a_2)q^2 + ...$$

Then, since $f \equiv 0 \mod (2^k - 1)$, $a_1 \equiv 0 \mod (2^k - 1)$. Similarly, $\forall i$ odd $a_i \equiv 0 \mod (2^k - 1)$. Assume $a_i \equiv 0 \mod (2^k - 1), \forall 0 < i < n$. Then, for i even (say $i = 2j$), the ith q-coefficient of f is $2^k a_j - a_i$. Mod 3, however, we have that $2^k a_j - a_i \equiv 0 \mod (2^k - 1)$. Since $j < i$, inductively we have that $a_j \equiv 0 \mod (2^k - 1)$. Thus $2^k a_j - a_i \equiv -a_i \equiv 0 \mod (2^k - 1)$. Thus, by induction $\forall i > 0, a_i \equiv 0 \mod (2^k - 1)$.

Now we prove the converse. For $g(q) = \sum_{i=0}^{\infty} a_i q^i \in M_k(SL_2(\mathbb{Z}))$ if $2^k - 1 | a_i \forall i > 0$ then since $(\psi_d - 1)(g)(q) = \psi_d(g)(q) - g(q) = 2^k g(q^2) - g(q)$. Thus

$$(\psi_d - 1)(g)(q) = \sum_{i=0}^{\infty} -a_{2i+1} q^{2i+1} + \sum_{i=0}^{\infty} (2^k a_i - a_{2i}) q^{2i} \tag{1}$$

$$= (2^k a_0 - a_0) + \sum_{i=0}^{\infty} -a_{2i+1} q^{2i+1} + \sum_{i=1}^{\infty} (2^k a_i - a_{2i}) q^{2i} \tag{2}$$

$$= (2^k - 1)a_0 + \sum_{i=0}^{\infty} -a_{2i+1} q^{2i+1} + \sum_{i=1}^{\infty} (2^k a_i - a_{2i}) q^{2i} \tag{3}$$

Since, for $i \geq 0, 2i + 1 > 0$ and for $i \geq 1, i, 2i > 0$ we have $2^k - 1 | a_{2i+1}, a_i, a_{2i}$ appearing in the sum. $2^k - 1 | (2^k - 1)a_0$ obviously. Therefore $2^k - 1 | (\psi_d - 1)(g)$.

Note that $2^k - 1$ can instead be replaced by $3^{val_3(k)+1}$. $\square$

Lemma 10. *If $f \in M_k(\Gamma_0(2))$ and $g = \psi_d(f) + f \in M_k(SL_2(\mathbb{Z}))$ with $f = \sum_{i=0}^{\infty} a_i q^i$, $\psi_d(f) = \sum_{i=0}^{\infty} a'_i q^i$ and $g = \sum_{i=0}^{\infty} b_i q^i$ then $a'_0 = 0$ and, consequently, $a_0 = b_0$.*

Proof. Applying Theorem 4 to our situation, we have that

$$2^k(f(q) - f(q^2) - \psi_d(f)(q^2)) + \psi_d(f) = 0$$

And, upon examining the constant term of the resulting expression we get

$$\begin{aligned} 2^k(a_0 - a_0 - a_0') + a_0' &= 0 \\ -(2^k - 1)a_0' &= 0 \\ a_0' &= 0 \end{aligned}$$

Where the last simplification follows from the characteristic being 0.
Thus, since $g = \psi_d(f) + f$, $b_0 = a_0 + a_0' = a_0$. □

Lemma 11 (Obstructions)**.** *For $f \in M_k(\Gamma_0(2))$ satisfying $\psi_d(f)+f \in M_k(SL_2(\mathbb{Z}))$ is in the image of $\psi_d - 1$ if and only if, for $f = \sum_{i=0}^{\infty} a_i q^i$, $val_3(a_0) \geq 1 + val_3(k)$.*

Proof. Let $g = \psi_d(f) + f \in M_k(SL_2(\mathbb{Z}))$. Then $\psi_d(g) - g = (\psi_d - 1) \circ (\psi_d + 1)(f) = (2^k - 1)f$ so $2^k - 1 | \psi_d(g) - g$. Therefore, if $g = \sum_{i=0}^{\infty} b_i q^i$, by Theorem 9 we know that $2^k - 1 | b_i, \forall i \geq 1$. Since $val_3(a_0) \geq 1 + val_3(k) = val_3(2^k - 1)$, by Lemma 10 we know then that $b_0 = a_0$ so $val_3(b_i) \geq val_3(2^k - 1) \forall i \geq 0$. (Implicitly we have used that $val_3(k) + 1 = val_3(2^k - 1)$ via Lemma 8.) Thus $\bar{g} = \frac{1}{2^k - 1}(\psi_d + 1)(f)$ exists. Finally $(\psi_d - 1)(\bar{g}) = \frac{1}{2^k - 1}(\psi_d - 1)(\psi_d + 1)(f) = f$ so $f \in im(\psi_d - 1)$!

For the converse, if $f \in im(\psi_d - 1)$ then $\exists g \in M_k(SL_2(\mathbb{Z}))$ such that $f = (\psi_d - 1)(g)$. Then, since $(\psi_d + 1)(f) = (2^k - 1)(f)$, Lemma 10 proves that $val_3(a_0) \geq val_3(k) + 1$. □

Theorem 12 (Image of $\psi_d + 1$)**.** *$g \in M_k(SL_2(\mathbb{Z})) \cap im(\psi_d + 1)$ $\mathbb{Z}_3$-adically if and only if, when writing $g(q) = \sum_{i=0}^{\infty} a_i q^i$, $val_3(a_i) \geq val_3(k) + 1, \forall i \geq 1$.*

Proof. By Theorem 9 we note that $3^{val_3(k)+1} | \psi_d(g) - g \iff val_3(a_i) \geq val_3(k) + 1, \forall i \geq 1$. To demonstrate that $g \in im(\psi_d + 1)$ we examine the 'inverse function' of ψ_d.

Lemma 13. $(\psi_d + 1)(\frac{1}{2^k - 1}(\psi_d - 1)(g)) = g$

Proof.

$$(\psi_d + 1)(\frac{1}{2^k - 1}(\psi_d - 1)(g)) = \frac{1}{2^k - 1}(\psi_d + 1)(\psi_d - 1)(g) = \frac{1}{2^k - 1}(\psi_d^2 - 1)(g) =$$
$$\frac{1}{2^k - 1}(2^k - 1)(g) = g$$

□

Thus, if $\frac{1}{2^k - 1}(\psi_d - 1)(g)$ is a $\mathbb{Z}_3$-adic modular form, then we know that g is in the image of $\psi_d + 1$. Moreover, since the map described above acts as an inverse for $\psi_d + 1$, we know that if such a form does not exist than *no* $\Gamma_0(2)$ modular form can map to g. $\frac{1}{2^k - 1}(\psi_d - 1)(g)$ is only defined if we do not have division by three occurring, i.e. the power of three in the denominator $2^k - 1$ is exactly matched by the power of three dividing $(\psi_d - 1)(g)$. However, the condition for these to cancel out is exactly Theorem 9! Thus $(\psi_d + 1)^{-1}(g)$ exists if and only if $3^{val_3(k)+1} | (\psi_d - 1)(g)$ which is true if and only if $val_3(a_i) \geq val_3(k) + 1, \forall i \geq 1$! □

We also have an algorithm to 'solve' for a basis of the kernel that proceeds as follows:

Theorem 14 (Algorithm)**.** *The following procedure will compute a 'basis' for the kernel.*

1. *Generate a basis $\mathcal{B} = \{f_1, ..., f_n\}$ of weight k $\Gamma_0(2)$-modular forms as products of special forms q_2 and q_4.*
2. *Write a general weight k modular form f as $\sum\limits_{i=1}^{n} x_i f_i$.*
3. *Compute the q-expansion of $2^k(f(q) - f(q^2) - \psi_d(f)(q^2)) + \psi_d(f)(q) = \sum\limits_{i=1}^{n}(\sum\limits_{j=1}^{n} c_{i,j} x_i) q^i$.*
4. *Denoting $M := \begin{pmatrix} c_{1,1} & c_{2,1} & \cdots & c_{n,1} \\ c_{1,2} & c_{2,2} & .. & c_{n,2} \\ .. & .. & .. & .. \\ c_{1,n} & c_{2,n} & .. & c_{n,n} \end{pmatrix}$, find the Smith Normal Form $M = SDQ$, and take the submatrix of Q with columns starting where the first 0 appears on the diagonal of D. The columns of Q now represents the basis of the nullspace of M and thus gives a basis $\bar{\mathcal{B}} = \{\bar{f_1}, \bar{f_2}, ..., \bar{f_m}\}$ of K.*

Proof. Application of Theorem 4. □

Finally, we compute the homology:

3.1 Computation of the Homology:

Theorem 15 (Homology)**.** *For a given weight k, $H_k(E^2) = {ker(\psi_d + 1 - \phi_f)}/{im(\psi_d - 1)} \cong \mathbb{Z}_3/3^{\nu_3(k)+1}\mathbb{Z}_3$ for $k > 2$ and even and 0 otherwise.*

Proof. For now, assume that $\exists f \in M_k(SL_2(\mathbb{Z})) : (\psi_d + 1)^{-1}(f) = \frac{1}{2^k - 1}(\psi_d - 1)(f) = \tilde{f}$ exists as well as that $f = 1 + O(q)$ (so, consequently, $\tilde{f} = 1 + O(q)$) and that it is a basis form for $M_k(SL_2(\mathbb{Z}))$. Then, given $g \in ker(\psi_d + 1 - \phi_f)$, we can decompose $g = n\tilde{f} + (g - n\tilde{f})$ (we will specify n later). Our first goal will be to demonstrate that $g - n\tilde{f} \in im(\psi_d - 1)$. We start by laying out a few definitions:

$$f = \sum_{i=0}^{\infty} a_i q^i \tag{4}$$

$$\tilde{f} = \sum_{i=0}^{\infty} \tilde{a}_i q^i \tag{5}$$

$$g = \sum_{i=0}^{\infty} b_i q^i \tag{6}$$

$$\bar{g} = (\psi_d + 1)(g) - nf = \sum_{i=0}^{\infty} \bar{b}_i q^i \tag{7}$$

Since $(\psi_d+1)(g) \in M_k(SL_2(\mathbb{Z}))$ and f is a modular forms basis element, $(\psi_d+1)(g) = nf + \bar{g}$, $\bar{g} \in M_k(SL_2(\mathbb{Z}))$. Immediately, we see that $\bar{g} = (\psi_d+1)(g) - nf = (\psi_d+1)(g) - (\psi_d+1)(n\tilde{f}) = (\psi_d+1)(g - n\tilde{f})$. Therefore, $g - n\tilde{f} \in ker(\psi_d + 1 - \phi_f)$. To apply Lemma 11, we must demonstrate that the constant term of $g - n\tilde{f} = b_0 - n\tilde{a_0}$ has $val_3(b_0 - n\tilde{a_0}) \geq 1 + val_3(k)$.

Then choose $n = b_0$ since, by Lemma 10 $\tilde{a_0} = a_0 = 1$ so $b_0 - n\tilde{a_0} = 0$. Then that guarantees that $g - n\tilde{f} \in im(\psi_d - 1)$ so our first claim is proven i.e.

$\forall g \in ker(\psi_d + 1 - \phi_f), \exists n \in \mathbb{Z}_3 / 3^{\nu_3(k)+1}\mathbb{Z}_3, \tilde{g} \in im(\psi_d - 1)$ so that $g = n\tilde{f} + \tilde{g}$. (We can choose $n \in \mathbb{Z}_3/3^{\nu_3(k)+1}\mathbb{Z}_3$ since $3^{\nu_3(k)+1}\tilde{f} \in im(\psi_d - 1)$.)

In turn, this demonstrates that $\forall [g] \in H_k(E^2), [g] = n[\tilde{f}]$ so $H_k(E^2) \cong \mathbb{Z}_3 / 3^{\nu_3(k)+1}\mathbb{Z}_3$ with $[\tilde{f}]$ as the generator.

Finally, such an $\tilde{f}$ exists. Let $f = \begin{cases} c_6^{\frac{k}{6}} & k \cong 0 \mod 6 \\ c_6^{\lfloor \frac{k}{6} \rfloor - 1} c_4^2 & k \cong 2 \mod 6 \\ c_6^{\lfloor \frac{k}{6} \rfloor} c_4 & k \cong 4 \mod 6 \end{cases}$ and set $\tilde{f} = (\psi_d + 1)^{-1}(f)$.

It has the right divisibility properties, i.e. $n\tilde{f} \in im(\psi_d - 1) \implies n \cong 0 \mod 3^{\nu_3(k)+1}$ since if there was a different multiple of $\tilde{f}$ in $im(\psi_d - 1)$ it would first have a smaller 3-adic divisibility i.e. $\nu_3(n) < \nu_3(k)+1$. Then $(\psi_d - 1)^{-1}(n\tilde{f}) = \frac{1}{2^k - 1}(\psi_d + 1)(n\tilde{f}) = \frac{n}{2^k - 1}(\psi_d + 1)(\tilde{f})$. However, since $\nu_3(n) < \nu_3(k)+1 = \nu_3(2^k - 1)$, $(\psi_d - 1)^{-1}(n\tilde{f})$ cannot exist $\mathbb{Z}_3$-adically. Therefore $\nu_3(n) \geq \nu_3(k)+1$ so $n \cong 0 \mod 3^{\nu_3(k)+1}$. $\square$

3.2 Cokernel

Lemma 16. $\nu_3((\psi_d - 1)(q_2^i q_4^j)) = \begin{cases} \nu_3(2^{2i} - 1), & j = 0 \\ 0, & j \neq 0 \end{cases}$

Proof. If $j = 0$ then $(\psi_d - 1)(q_2^i) = \psi_d(q_2)^i - q_2^i = (-2q_2)^i - q_2^i = ((-2)^i - 1)q_2^i$. Since the constant term of $q_2^i = (-\frac{1}{2})^i$, $\nu_3(((-2)^i - 1)q_2^i) = \nu_3((-2)^i - 1) = \begin{cases} \nu_3(2^i - 1), & i = 2k \\ \nu_3(2^i + 1), & i = 2k+1 \end{cases} = \nu_3(i) + 1 = \nu_3(2i) + 1 = \nu_3(2^{2i} - 1)$

If $j \neq 0$ then $(\psi_d - 1)(q_2^i q_4^j) = \psi_d(q_2)^i \psi_d(q_4)^j - q_2^i q_4^j$. However, since $\psi_d(q_4) = \mathcal{O}(q)$ the constant term of $(\psi_d - 1)(q_2^i q_4^j) = -(-\frac{1}{2})^i (\frac{1}{16})^j$ which is not divisible by 3. Therefore, $\nu_3((\psi_d - 1)(q_2^i q_4^j)) = 0$. $\square$

Lemma 17 (Characterization of Torsion). *Let* $[f] \in M_k(\Gamma_0(2)) / im(\psi_d + 1 - \phi_f)$. *Then* $3^{n_f}[f] = [0]$ *if and only if* $\exists g \in M_k(SL_2(\mathbb{Z}))$ *such that* $3^{n_f}(\psi_d - 1)(f)(q) \equiv g(q) - g(q^2) \mod 3^{\nu_3 k+1}$ *on the level of* q *expansions.*

Proof.

$$3^{n_f}[f] = [0] \iff 3^{n_f} f \in im(\psi_d + 1 - \phi_f) \iff$$
$$\exists f' \in M_k(\Gamma_0(2)), g \in M_k(SL_2(\mathbb{Z})) : 3^{n_f} f = (\psi_d + 1)(f') - g \iff f + g = (\psi_d + 1)(f') \iff$$
$$(\psi_d - 1)(3^{n_f} f + g) = (2^k - 1)f' \iff (\psi_d - 1)(3^{n_f} f + g) \equiv 0 \mod 3^{\nu_3(k)+1} \iff$$
$$3^{n_f}(\psi_d - 1)(f) \equiv g(q) - 2^k g(q^2) \mod 3^{\nu_3(k)+1} \iff$$
$$3^{n_f}(\psi_d - 1)(f) \equiv g(q) - g(q^2) \mod 3^{\nu_3(k)+1}$$

□

Corollary 18. *This conditions needs only to be checked for a finite set of q-expansion coefficients, i.e. the first $dim(MF_0(2)_k)$ q-expansion coefficients.*

Proof. $MF_0(2)_k$ is a free and finitely generated module. □

Corollary 19. *For $f \in M_k(\Gamma_0(2))$ and $(\psi_d - 1)(f) = a_0 + \mathcal{O}(q)$, $\nu_3(k) + 1 \geq n_f \geq \nu_3(k) + 1 - \nu_3(a_0)$.*

Proof. Since the constant term of $g(q) - g(q^2)$ is zero, any such n'_f must satisfy $3^{n'_f} a_0 \equiv 0 \mod 3^{\nu_3(k)+1}$, the least of which being $n'_f = \nu_3(k) + 1 - \nu_3(a_0)$. Then, since we may need more divisibility for subsequent coefficients, $n_f \geq n'_f = \nu_3(k) + 1 - \nu_3(a_0)$. That every class is $3^{\nu_3(k)+1}$ torsion is obvious. Therefore $\nu_3(k) + 1 \geq n_f \geq \nu_3(k) + 1 - \nu_3(a_0)$. □

Corollary 20. *A class $[f] \in M_k(im(\psi_d + 1 - \phi_f)) / im(\psi_d + 1 - \phi_f)_k$ has $n_f = 3^{\nu_3(k)+1}$ if and only if $\nu_3(a_0) = 0$ with a_0 as above.*

Proof. $\nu_3(k) + 1 \geq n_f \geq \nu_3(k) + 1 - \nu_3(a_0) \implies n_f = \nu_3(k) + 1$ and similarly for the reverse. □

Corollary 21. *For $f \in M_k(\Gamma_0(2))$, $3^{n_f}[f] = [0] \iff \exists g' \in M_k(SL_2(\mathbb{Z})) : 3^{n_f}(\psi_d - 1)(f) \equiv (\psi_d - 1)(g') \mod 3^{\nu_3(k)+1}$*

3.3 Twisted Multiplication

NOTE: THIS SECTION REFERS TO WORK THAT DOES NOT WORK $\mathbb{Z}_3$-adically!

Definition 22 (Twisted Multiplication)**.** With $f \in M_n(\Gamma_0(2))$ and $g \in M_m(\Gamma_0(2))$, define $f \star g \in M_{n+m}(\Gamma_0(2))$ to be $f \star g = fg + \frac{1}{2^{w(f)+w(g)} - 1}((2^{w(f)} - 1)f\psi_d(g) + (2^{w(g)} - 1)g\psi_d(f))$ if $w(f) + w(g) > 0$, and $f \star g = fg$ if $w(f) = w(g) = 0$.

The reason this twist is important is because of the following property:

Theorem 23 (Twisting with ψ_d)**.** $(\psi_d + 1)(f \star g) = (\psi_d + 1)(f) \cdot (\psi_d + 1)(g)$

Proof.

$$
\begin{gathered}
(\psi_d+1)(f\star g) = \\
(\psi_d+1)(fg + \frac{1}{2^{w(f)+w(g)}-1}((2^{w(f)}-1)f\psi_d(g) + (2^{w(g)}-1)g\psi_d(f))) = \\
\psi_d(f)\psi_d(g) + \frac{1}{2^{w(f)+w(g)}}((2^{w(f)}-1)2^{w(g)}\psi_d(f)g + (2^{w(g)}-1)2^{w(f)}\psi_d(g)f) + \\
fg + \frac{1}{2^{w(f)+w(g)}-1}((2^{w(f)}-1)f\psi_d(g) + (2^{w(g)}-1)g\psi_d(f)) = \\
\psi_d(f)\psi_d(g) + fg + \frac{1}{2^{w(f)+w(g)}}((2^{w(f)+w(g)}-1)(\psi_d(f)g + \psi_d(g)f)) = \\
\psi_d(f)\psi_d(g) + \psi_d(f)g + \psi_d(g)f + fg = \\
(\psi_d+1)(f)\cdot(\psi_d+1)(g)
\end{gathered}
$$

□

Theorem 24. *$(M^\bullet, +, \star)$ is a commutative ring w/o identity, and is an algebra over whatever the base ring R is.*

Proof. The flavor of this proof will exploit the above property rather than brute forcing through the symbols.

(Associativity). Note that

$$
\begin{gathered}
(\psi_d+1)(f\star(g\star h)) = (\psi_d+1)(f)\cdot(\psi_d+1)(g\star h) = \\
(\psi_d+1)(f)\cdot(\psi_d+1)(g)\cdot(\psi_d+1)(h) = \\
(\psi_d+1)(f\star g)\cdot(\psi_d+1)(h) = (\psi_d+1)((f\star g)\star h)
\end{gathered}
$$

Now if $w(f)+w(g)+w(h)>0$ then, since ψ_d+1 is a bijection on forms of weight >0, we have that $(\psi_d+1)(f\star(g\star h)) = (\psi_d+1)((f\star g)\star h)$ thus $f\star(g\star h) = (f\star g)\star h$. If $w(f)+w(g)+w(h)=0$ then $w(f)=w(g)=w(h)=0$ so $\star$ just descends to normal multiplication which is associative.

(Commutativity).

$$
(\psi_d+1)(f\star g) = (\psi_d+1)(f)\cdot(\psi_d+1)(g) = (\psi_d+1)(g)\cdot(\psi_d+1)(f) = (\psi_d+1)(g\star f)
$$

(Distributivity).

$$
\begin{gathered}
(\psi_d+1)((f+g)\star h) = (\psi_d+1)(f+g)\cdot(\psi_d+1)(h) = ((\psi_d+1)(f)+(\psi_d+1)(g))\cdot(\psi_d+1)(h) = \\
(\psi_d+1)(f)\cdot(\psi_d+1)(h) + (\psi_d+1)(g)\cdot(\psi_d+1)(h) = \\
(\psi_d+1)(f\star h) + (\psi_d+1)(g\star h) = (\psi_d+1)(f\star h + g\star h)
\end{gathered}
$$

The other side follows from commutativity.

(Scalar compatibility). $\forall c \in R$

$$
\begin{gathered}
(\psi_d+1)((cf)\star g) = (\psi_d+1)(cf)\cdot(\psi_d+1)(g) = \\
c(\psi_d+1)(f)\cdot(\psi_d+1)(g) = c(\psi_d+1)(f\star g) = (\psi_d+1)(c(f\star g))
\end{gathered}
$$

□

Now, since $\star$ satisfies all of the important properties of a ring multiplication[1], the upcoming important result follows:

Theorem 25 (Generation as an Algebra)**.** *Let $\tilde{c_4}$ and $\tilde{c_6}$ be 'inverses' of c_4 and c_6 respectively under $(\psi_d + 1)$.*[2] *Then, for $f \in ker(\psi_d + 1 - \phi_f)$ that is weight k, $\exists c_i$ s.t.*

$$f = \sum_{\substack{4i+6j=k \\ i,j\in\mathbb{Z}_{\geq 0}}} c_i \tilde{c_4}^{\star i} \star \tilde{c_6}^{\star j}$$

[3] *In other words, $ker(\psi_d + 1 - \phi_f)$ is generated, as an algebra, by $\tilde{c_4}$ and $\tilde{c_6}$ where the multiplication by $\star$.*

Proof. Since $f \in ker(\psi_d + 1 - \phi_f)$, $(\psi_d + 1)(f) = \bar{f}$ is a fully modular form. Then, since c_4 and c_6 generate the ring of modular forms (Diamond and Shurman 3.5.2, p. 101), we know that $\bar{f} = \sum_{\substack{4i+6j=k \\ i,j\in\mathbb{Z}_{\geq 0}}} c_i c_4^i c_6^j$ for some c_i. To reiterate, we also know that $\bar{f} = (\psi_d + 1)(f)$. Now, let's examine the following sum:

$$(\psi_d + 1)\left(\sum_{\substack{4i+6j=k \\ i,j\in\mathbb{Z}_{\geq 0}}} c_i \tilde{c_4}^{\star i} \star \tilde{c_6}^{\star j}\right) =$$

$$\sum_{\substack{4i+6j=k \\ i,j\in\mathbb{Z}_{\geq 0}}} (\psi_d + 1)(c_i \tilde{c_4}^{\star i} \star \tilde{c_6}^{\star j}) =$$

$$\sum_{\substack{4i+6j=k \\ i,j\in\mathbb{Z}_{\geq 0}}} c_i(\psi_d + 1)(\tilde{c_4}^{\star i} \star \tilde{c_6}^{\star j}) =$$

$$\sum_{\substack{4i+6j=k \\ i,j\in\mathbb{Z}_{\geq 0}}} c_i(\psi_d + 1)(\tilde{c_4}^{\star i})(\psi_d + 1) \cdot (\tilde{c_6}^{\star j}) =$$

$$\sum_{\substack{4i+6j=k \\ i,j\in\mathbb{Z}_{\geq 0}}} c_i c_4^i c_6^j = \bar{f}$$

But $\bar{f} = (\psi_d + 1)(f)$ and, since $(\psi_d + 1)$ is bijective on forms of weight > 0[4], it must be the case that

$$f = \sum_{\substack{4i+6j=k \\ i,j\in\mathbb{Z}_{\geq 0}}} c_i \tilde{c_4}^{\star i} \star \tilde{c_6}^{\star j}$$

□

[1] Though there is still the problematic possibility that division by 3 occurs in $\star$. At least in specific cases, it does not seem that $\tilde{c_4}^{\star i} \star \tilde{c_6}^{\star j}$ has division by three present.

[2] While the definition of the inverse of this map does include possible division by three, the forms $\tilde{c_4}$ and $\tilde{c_6}$ do not suffer from division by three.

[3] Here $f^{\star i}$ is shorthand for $f \star f \star f ...$ i times.

[4] Since we have an inverse function defined on these forms.

Importantly, note that the choice of c_4, c_6 were not important. As long as forms f, g generate modular forms as an algebra, then $\tilde{f}, \tilde{g}$ will generate the kernel under multiplication via $\star$.

Unfortunately, $\tilde{c_4}^{\star 3} \star \tilde{c_6}^{\star 7}$ has division by 3. Moreover, there are $\Gamma_0(2)$ modular forms with coefficients properly in $\mathbb{Z}_3$ (i.e. no division by 3) that are in $ker(\psi_d + 1 - \phi_f)$ but their expansions in the $\tilde{c_4}$, $\tilde{c_6}$ basis involve division by 3.

Thus, either we need a new definition of $\star$ such that $(\psi_d+1)(f \star g)$ is a polynomial in $(\psi_d+1)(f)$ and $(\psi_d+1)(g)$ (or at least some guarantee that $f, g \in ker(\psi_d+1-\phi_f) \implies f \star g \in ker(\psi_d+1-\phi_f)$ that never leads to division by 3 for $\tilde{c_4}^{\star i} \star \tilde{c_6}^{\star j}$ OR a different $\mathbb{Z}_3$ basis for $M^\bullet(SL_2(\mathbb{Z}))$ $\{f, g\}$ such that $\tilde{f}^{\star i} \star \tilde{g}^{\star j}$ never has division by 3.

To achieve the second part, the condition is equivalent to (I believe) asking that all of the non-constant terms in the q-expansion of $f^i \cdot g^j$ are divisible by $3^{val_3(i \cdot w(f) + j \cdot w(g)) + 1}$. This is because of the property that $(\psi_d + 1)(\tilde{f}^{\star i} \star \tilde{g}^{\star g}) = (\psi_d + 1)(\tilde{f})^i (\psi_d + 1)(\tilde{g})^j = f^i g^j$, so $\tilde{f}^{\star i} \star \tilde{g}^{\star j} = (\psi_d + 1)^{-1}(f^i g^j) = \dfrac{1}{2^{i \cdot w(f) + j \cdot w(g)} - 1}(\psi_d - 1)(f^i g^j)$. Applying Theorem 12, we see that this form contains no division by 3 if and only if we have the proper divisibility by 3 on all of the coefficients of $f^i g^j$. Recast in this light, the reason $\tilde{c_4}, \tilde{c_6}$ do not work is because the q-expansion of $c_4^3 c_6^7$ is $1 + 2808q + ...$ and $3^{val_3(54)+1} = 81 \nmid 2808$

4 Image of $\psi_d + 1$ and Divisibility

4.1 c_4^i and c_6^j

Let's start by demonstrating that $\forall i, j \geq 0$ $c_4^i, c_6^j \in im(\psi_d + 1)$. As was proven in Theorem 12, these forms are in the image of $\psi_d + 1$ if and only if, when we write $c_4^i, c_6^j = 1 + \sum_{i=0}^{\infty} a_i q^i$ then $val_3(a_i) \geq val_3(4i) + 1, val_3(6i) + 1$ respectively. Lets work with the former.

Since $c_4 = 1 + 240 \sum_{i=1}^{\infty} \sigma_3(i) q^i$, we can write $c_4^i = ((c_4 - 1) + 1)^i = \sum_{j=0}^{i} \binom{i}{j}(c_4 - 1)^i = 1 + \sum_{j=1}^{i} \binom{i}{j} 240^j (\sum_{n=1}^{\infty} \sigma_3(n) q^n)^j$. Then, we can see that $\binom{i}{j} 240^j | a_n$ so if we can demonstrate that $val_3(\binom{i}{j} 240^j) \geq val_3(4i) + 1$ then Theorem 12 will apply.

Lemma 26. $val_3(\binom{n}{k}) + k \geq val_3(n) + 1$.

Proof. Since $\dfrac{n}{\gcd\{n,k\}} | \binom{n}{k}$ we see that $val_3(\binom{n}{k}) \geq val_3(\dfrac{n}{\gcd\{n,k\}}) = val_3(n) - val_3(\gcd\{n,k\}) = val_3(n) - \min\{val_3(n), val_3(k)\}$.

We now break into the following two cases:

If $1 \le k \le val_3(n)$ then $val_3(k) \le val_3(n)$ so $\min\{val_3(n), val_3(k)\} = val_3(k)$. Then

$$val_3(\binom{n}{k}) \ge val_3(n) - val_3(k) \tag{8}$$

$$val_3(\binom{n}{k}) + val_3(k) \ge val_3(n) \tag{9}$$

$$val_3(\binom{n}{k}) + k - 1 \ge val_3(n) \tag{10}$$

$$val_3(\binom{n}{k}) + k \ge val_3(n) + 1 \tag{11}$$

Where the reasoning that allows us to deduce (6) from (5) is because $k - 1 \ge val_3(k)$.

Now, if $k \ge val_3(n) + 1$ then $val_3(\binom{n}{k}) + k \ge 0 + k \ge 0 + val_3(n) + 1 = val_3(n) + 1$. □

Therefore, since $val_3(n) + 1 = val_3(4n) + 1$ Theorem 12 applies to c_4^i and so $c_4^i \in im(\psi_d + 1)$ for all $i \ge 1$.

Now, we consider c_6^j for $j \ge 1$.

Similarly as to the above deduction, since $c_6 = 1 - 504 \sum_{i=1}^{\infty} \sigma_5(i) q^i$ we can decompose c_6^j as $((c_6 - 1) + 1)^j = 1 + \sum_{k=1}^{j} \binom{j}{k} 504^k (-\sum_{n=1}^{\infty} \sigma_5(n) q^n)^k$. Then, by Lemma 26, we note that $val_3(\binom{j}{k} 504^k) \ge val_3(\binom{j}{k}) + k + k \ge val_3(j) + 1 + k \ge val_3(j) + 1 + 1 = val_3(6j) + 1$. Thus $c_6^j \in im(\psi_d + 1)$ for all $j \ge 1$.

4.2 Generalizations

Generalizing the above, denote $c_{i,j,k} = \min\{c : c \ge 1,\ cc_4^i c_6^j \Delta^k \in im(\psi_d + 1)\}$. By the above section, $c_{i,0,0} = c_{0,j,0} = 1$. Theorem 12 allows us to rephrase the definition as the following.

Definition 27. Let $i, j \ge 0$ be two natural numbers. If we write $c_4^i c_6^j = 1 + \sum_{i=1}^{\infty} a_i q^i$ then $c_{i,j} = \min\{c : val_3(ca_i) \ge val_3(4i + 6j) + 1\}$.

For Δ^k, we note that $\Delta^k = q^k + O(q^{k+1})$ so, since $val_3(12k) + 1 = val_3(k) + 2$, we see that $c_{0,0,k} = 3^{val_3(k)+2}$. Similarly, since $c_4^i c_6^j = 1 + O(q)$, $c_4^i c_6^j \Delta^k = q^k + O(q^{k+1})$ thus, for $k > 1$, $c_{i,j,k} = val_3(4i + 6j + 12k) + 1$.

We can analyze one small nontrivial case:

Lemma 28. *If $val_3(4i + 6j) < 3$ then $c_{i,j} = 1$.*

I will transcribe casey's proof soon. □

5 References

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