Model predictive control: theory and practice

MPC for LTI systems

Conceptual issues in MPC

- Recursive feasibility and stability

Linear MPC

Linear MPC: basic formulation

For the following linear, discrete time, time invariant model

$$M:\begin{cases} x(k+1) = Ax(k) + B\,u(k) \\ y(k) = Cx(k) \end{cases}, x(k) \in \mathbb{R}^n, u(k) \in \mathbb{R}^m, y(k) \in \mathbb{R}^p$$

$A \in \mathbb{R}^{n,n}, B \in \mathbb{R}^{n,m}, C \in \mathbb{R}^{p,n}, (A,B) \rightarrow$ stabilizable

we want to design a predictive controller in order to obtain the system state regulation to the origin in the presence of:

input saturation constraints

linear state constraints

Linear MPC: basic formulation

For the considered control problem it will be shown that the MPC controller is obtained by solving a suitable **Quadratic Programme** (**QP**) problem at each sampling time

To this aim we will proceed in two steps:

- cost function → quadratic w.r.t. $U(k)$
- constraints → linear w.r.t. $U(k)$

The geometric structure of QP will be exploited in order to show that the predictive controller is a piecewise linear function of the system state

Basic formulation: the cost function

The considered control objectives can be taken into account through the following cost function:

$$J(x(k\mid k),U(k)) = \sum_{i=0}^{H_p-1} L(x(k+i\mid k),u(k+i\mid k)) + \Phi(x(k+H_p\mid k))$$

$$= \sum_{i=0}^{H_p-1} x(k+i\mid k)^T Q\, x(k+i\mid k) + u(k+i\mid k)^T R\, u(k+i\mid k) +$$

$$+ x(k+H_p\mid k)^T P\, x(k+H_p\mid k) =$$

$$= \sum_{i=0}^{H_p-1} \left\|x(k+i\mid k)\right\|_Q^2 + \left\|u(k+i\mid k)\right\|_R^2 + \left\|x(k+H_p\mid k)\right\|_P^2,$$

$$U(k) = \begin{bmatrix} u(k\mid k) & u(k+1\mid k) & \dots & u(k+H_c-1\mid k)\end{bmatrix}^T$$

$$H_p = H_c, Q = Q^T \geq 0, R = R^T > 0, P = P^T \geq 0$$

Basic formulation: the cost function

It can be shown that the cost function:

$$J(x(k\mid k),U(k)) = \sum_{i=0}^{H_p-1} \left\|x(k+i\mid k)\right\|_Q^2 + \left\|u(k+i\mid k)\right\|_R^2 + \left\|x(k+H_p\mid k)\right\|_P^2$$

is quadratic w.r.t. $U(k)$, i.e. is of the form:

$$J(x(k\mid k),U(k)) = \frac{1}{2}U(k)^T H U(k) + x(k\mid k)^T F U(k) + \bar{J}$$

where $H > 0$

Basic formulation: the cost function

In order to derive the quadratic form of the cost function recalling that

$$x(k+i\mid k) = A^i x(k\mid k) + \sum_{j=0}^{i-1} A^{i-j-1} B u(k+j\mid k)$$

then the predicted state sequence

$X(k) = [x(k\mid k), x(k+1\mid k), \dots, x(k+H_p\mid k)]^T$

can be expressed as:

$$X(k) = \mathcal{A}x(k\mid k) + \mathcal{B}U(k)$$

$$\mathcal{A} = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^{H_p} \end{bmatrix}, \mathcal{B} = \begin{bmatrix} B & 0 & 0 & \cdots & 0 \\ AB & B & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A^{H_p-2}B & A^{H_p-3}B & A^{H_p-4}B & \cdots & B \\ A^{H_p-1}B & A^{H_p-2}B & A^{H_p-3}B & \cdots & AB \end{bmatrix}, \dots$$

Basic formulation: the cost function

… defining the matrices:

$$\mathcal{Q} = \begin{bmatrix} Q & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & Q & 0 \\ 0 & \cdots & 0 & P \end{bmatrix} \in \mathbb{R}^{nH_p \times nH_p}, \mathcal{R} = \begin{bmatrix} R & 0 & \cdots & 0 \\ 0 & R & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & R \end{bmatrix} \in \mathbb{R}^{mH_p \times mH_p}$$

the cost function can be rewritten as:

$$J(x(k\mid k),U(k)) = X^T(k)\mathcal{Q}X(k) + U^T(k)\mathcal{R}U(k)$$

and substituting the expression of $X(k)$ we get

$$J(x(k\mid k),U(k)) =$$
$$x^T(k\mid k)\mathcal{A}^T\mathcal{Q}\mathcal{A}x(k\mid k) + 2x^T(k\mid k)\mathcal{A}^T\mathcal{Q}\mathcal{B}U(k) + U^T(k)(\mathcal{B}^T\mathcal{Q}\mathcal{B} + \mathcal{R})U(k)$$

…

Basic formulation: the cost function

$$J(x(k|k),U(k)) =$$
$$x^T(k|k)\mathcal{A}^T\mathcal{Q}\mathcal{A}x(k|k) + 2x^T(k|k)\mathcal{A}^T\mathcal{Q}\mathcal{B}U(k) + U^T(k)(\mathcal{B}^T\mathcal{Q}\mathcal{B}+\mathcal{R})U(k)$$

… posing

$$H = 2(\mathcal{B}^T\mathcal{Q}\mathcal{B}+\mathcal{R})$$
$$F = 2\mathcal{A}^T\mathcal{Q}\mathcal{B}$$
$$\bar{J} = x^T(k|k)\mathcal{A}^T\mathcal{Q}\mathcal{A}x(k|k)$$

the cost function can be rewritten as:

$$J(x(k|k),U(k)) = \frac{1}{2}U(k)^T H U(k) + x(k|k)^T F U(k) + \bar{J}$$

which is quadratic in $U(k)$ (note that $H > 0$)

Basic formulation: the cost function

In the absence of constraints the minimizer is given by:

$$U(k) = U^O(k) = -H^{-1}F\,x(k|k) \underset{K=H^{-1}F}{\underset{\uparrow}{=}} -Kx(k|k)$$

which corresponds to a (static) state feedback control law

Since the system is time invariant, such a solution is also the result of the application of the (unconstrained) RH strategy

A suitable choice of matrices Q, R and P guarantees (asymptotic) stability of the controlled system

Basic formulation: saturation constraints

The second point of the analysis of the basic formulation is to show that input saturation constraints and linear state constraints lead to linear constraints on $U(k)$

Saturation Constraints

For simplicity, consider the case $m = 1$ (single input system) and $H_p = H_c = 2$

In this case we have $U(k) = [u(k|k),\ u(k+1|k)]^T$ saturation constrainits are of the form:

$$u_{\min} \le u(k|k) \le u_{\max}$$
$$u_{\min} \le u(k+1|k) \le u_{\max}$$

Basic formulation: saturation constraints

After some manipulations we get:

$$\left.\begin{array}{l} u_{\min} \le u(k|k) \le u_{\max} \\ u_{\min} \le u(k+1|k) \le u_{\max} \end{array}\right\} \rightarrow \begin{array}{l} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u(k|k) \\ u(k+1|k) \end{bmatrix} \le \begin{bmatrix} u_{\max} \\ u_{\max} \end{bmatrix} \\ -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u(k|k) \\ u(k+1|k) \end{bmatrix} \le \begin{bmatrix} u_{\min} \\ u_{\min} \end{bmatrix} \end{array}$$

$$\rightarrow \underbrace{\begin{bmatrix} I_2 \\ -I_2 \end{bmatrix}}_{L_u}\begin{bmatrix} u(k|k) \\ u(k+1|k) \end{bmatrix} \le \underbrace{\begin{bmatrix} u_{\max} \\ u_{\max} \\ u_{\min} \\ u_{\min} \end{bmatrix}}_{W_u} \rightarrow L_u U(k) \le W_u$$

input saturation constraints lead to linear constraints on $U(k)$

Basic formulation: state constraints

Let us now consider the linear state constraints ($H_p = 2$)

$$L_1 x(k+1 \mid k) \le W_1$$
$$L_2 x(k+2 \mid k) \le W_2$$

$$L_1 x(k+1 \mid k) \le W_1,$$
$$\rightarrow x(k+1 \mid k) = Ax(k \mid k) + Bu(k \mid k)$$

$$L_2 x(k+2 \mid k) \le W_2,$$
$$\rightarrow x(k+2 \mid k) = A^2 x(k \mid k) + ABu(k \mid k) + Bu(k+1 \mid k)$$

$$\underbrace{\begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}\begin{bmatrix} B & 0 \\ AB & B \end{bmatrix}}_{L_x}\begin{bmatrix} u(k \mid k) \\ u(k+1 \mid k) \end{bmatrix} \le \underbrace{-\begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}\begin{bmatrix} A \\ A^2 \end{bmatrix} x(k \mid k) + \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}}_{W_x}$$

Basic formulation: linear constraints

Now rearranging both input and state constraints we obtain

$$\underbrace{\begin{bmatrix} L_u \\ L_x \end{bmatrix}}_{L}\begin{bmatrix} u(k \mid k) \\ u(k+1 \mid k) \end{bmatrix} \le \underbrace{\begin{bmatrix} W_u \\ W_x \end{bmatrix}}_{W}$$

$$LU(x) \le W$$

which represents a linear set of constraints on $U(k)$

Linear MPC

Similar results are obtained for general values of the prediction horizon (even in the case of different control horizon)
(for details see: *G. C. Goodwin,., M. M. Seron, J. A. De Doná, "Constrained Control and Estimation: an optimisation approach", Springer Verlag, 2005*)

To conclude, the optimization problem involved in MPC design is:

$$\min_{U(k)} J(x(k \mid k), U(k)) = \min_{U(k)} \frac{1}{2} U(k)^T HU(k) + x(k \mid k)^T FU(k)$$
$$s.t.$$
$$LU(k) \le W$$

which is a QP on $U(k)$

QP solution

$$\min_{U(k)} J(x(k \mid k), U(k)) = \min_{U(k)} \frac{1}{2} U(k)^T HU(k) + x(k \mid k)^T FU(k)$$
$$s.t.$$
$$LU(k) \le W$$

Matrix H is the **hessian** of QP

If $H > 0$, then the QP problem is convex

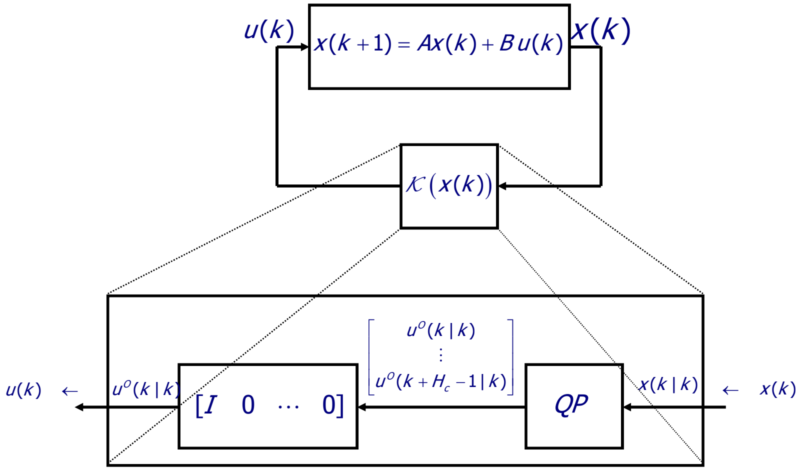
There are several effective numerical algorithm that can be employed in the solution of (convex) QP problems:

- "active set" algorithms
- "primal-dual" interior point algorithms

MatLab → `quadprog`

RH principle and QP

The RH controller $\mathcal{K}(\cdot)$ as the solution of QP



Stability analysis

Some conceptual issues …

Until now MPC seems to be a very powerful tool to solve in an efficient way constrained control problems

However, since the feedback control action is realized through the RH principle via the solution to a FHOCP, the two following issues arise:

- is the FHOCP always feasible at every point of the state space ?
- is the closed loop system stable (i.e. does the MPC controller $\mathcal{K}(x)$ stabilize the origin of the controlled system ?)

Some conceptual issues: loss of feasibility

Consider the linear system

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

subject to the constraints $|u(k)| \le 0.5, \|x(k)\|_\infty \le 5$

The MPC controller designed using the cost function

$$J = \sum_{i=0}^{H_p-1} x(k+i|k)^T Q\, x(k+i|k) + u(k+i|k)^T R\, u(k+i|k)$$

$$H_p = H_c = 3, Q = I_2, R = 10$$

leads to an unfeasible FHOCP at $k = 2$

Some conceptual issues: loss of stability

Consider the linear system

$$x(k+1) = \begin{bmatrix} 2 & 1 \\ 0 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$$

subject to the constraints $|u(k)| \le 1, \|x(k)\|_\infty \le 10$

Analyze the MPC controller designed using the cost function

$$J = \sum_{i=0}^{H_p-1} x(k+i|k)^T Q\, x(k+i|k) + u(k+i|k)^T R\, u(k+i|k)$$

$$Q = I_2$$

with different values of H_p, R

Some conceptual issues: loss of stability

① R=10, H_p=2

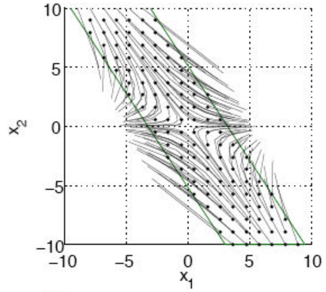
② R=2, H_p=3

③ R=1, H_p=4

* Initial points leading to trajectories that converge to the origin

• Intitial points that diverge

①



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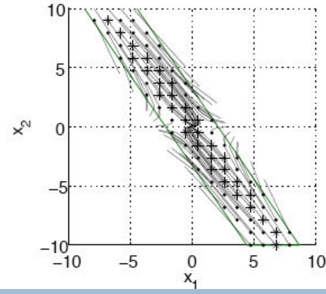

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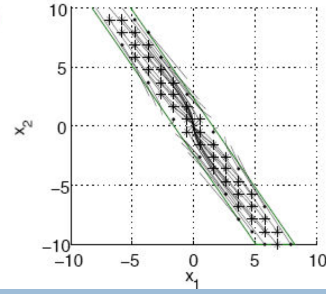
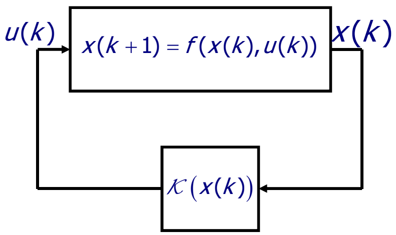


Remarks

- Guaranteeing stability of closed loop systems based on (finite horizon) optimization scheme (e.g. RH) is not a trivial task
- A noticeable aspect is that the obtained finite horizon sequence is optimal → optimality can then be turned into a notion of stability by utilizing the **optimal cost function** (i.e. $J^O(x(k))$) as a **Lyapunov function** for asymptotic stability of the controlled system:



$$\to x(k+1) = f(x(k), \mathcal{K}x(k))$$

Remarks

- However, the optimization is defined over a finite future horizon, yet stability properties must hold over an infinite horizon
- A commonly used "trick" is to add an appropriate weighting on the terminal state $\Phi(\cdot)$ in the cost function to account for impact of events that lie beyond the end of the fixed horizon
- Moreover, an (invariant) terminal constraint set $\mathbb{X}_F$ under a suitable known and stabilizing terminal control law $\mathcal{K}_F(x(k))$ is also introduced in the optimization problem in order to guarantee its feasibility

Reminder: Lyapunov direct method

Let (x_e) be an equilibrium of the dynamic system

$$x(k+1)=f(x(k),u(k))$$

and let $\mathcal{X}\subseteq\mathbb{R}^n$ a domain containing x_e. If there exists a class C^1 fuction $V(x(k))$ which is positive definite in $x_e\in\mathcal{X}$ and such that

$$\Delta V(x)=V(x(k+1))-V(x(k))=V(f(x(k),u_e))-V(x(k))$$

is negative definite in $x_e\in\mathcal{X}$ then
the equilibrium (x_e) is asymptotically stable

The function $\boldsymbol{V(x)}$ which satisfies the criterion assumptions is said **Lyapunov function** for asymptotic stability

Stability analysis for linear MPC

Definitions

Useful concepts

Feasible initial set

The set $\mathbb{S}_{Hp}$ of **feasible initial states** is the set of initial states $\boldsymbol{x_0}\in\mathbb{X}$ for which there exist feasible state and control sequences for the optimization problem $\mathcal{P}_{Hp}(x)$

Positively invariant set **positively invariant**

A set $\mathbb{S}\subset\mathbb{R}^n$ is said for the system

$$x(k+1)=f(x(k),u(k)),k\geq 0$$

under the control $u(k)=\mathcal{K}(x(k))$ (or positively invariant for the closed loop system $x(k+1)=f(x(k),\mathcal{K}(x(k)))$ if

$$f(x,\mathcal{K}(x))\in\mathbb{S},\ \forall x\in\mathbb{S}$$

General formulation

$$\mathcal{P}_{H_p}(x):\begin{cases}\min\limits_{U(k)} J(x(k\,|\,k),U(k))=\\ =\min\limits_{U(k)}\sum\limits_{i=0}^{H_p-1} x(k+i\,|\,k)^T Qx(k+i\,|\,k)+u(k\,|\,k)^T Ru(k\,|\,k)+\Phi(x(k+H_p\,|\,k))\\ s.t.\\ x(k+1)=Ax(k)+Bu(k)\\ U(k)\in\mathbb{U}\\ x(k+i\,|\,k)\in\mathbb{X},i=1,\dots,H_p-1\\ x(k+H_p\,|\,k)\in\mathbb{X}_F\subset\mathbb{X}\end{cases}$$

"End terminal constraint" (Kwon & Pearson, 1977)

$$\min_U J(x(k\,|\,k),U)=$$
$$=\min_U \sum_{i=0}^{H_p-1} x(k+i\,|\,k)^T Qx(k+i\,|\,k)+u(k+i\,|\,k)^T Ru(k+i\,|\,k)$$

s.t.

$$\begin{cases} x(k+i+1\,|\,k)=Ax(k+i\,|\,k)+Bu(k+i\,|\,k),\ i\ge 0\\ U(k)\in\mathbb{U}\\ x(k+i\,|\,k)\in\mathbb{X},\ i=1,\dots,H_p-1\\ u(k+i\,|\,k)=0,\ i=H_c,\dots,H_p-1\\ x(k+H_p\,|\,k)=0 \quad\leftarrow \end{cases}$$

"End terminal constraint" (Kwon & Pearson, 1977)

Sketch of the proof

Suppose feasibility of $\mathcal{P}_{Hp}(x)$ at time k let

$$U^O=[u^O(k\,|\,k),u^O(k+1\,|\,k),\dots,u^O(k+H_c-1\,|\,k)]$$

be the minimizer and let

$$X^O=[x^O(k\,|\,k),x^O(k+1\,|\,k),\dots,x^O(k+H_p-1\,|\,k),$$
$$V(x)=J(U^O,x(k\,|\,k))$$

be the corresponding state sequence and the cost fuction respectively

apply $u^O(k\,|\,k)$, go to time $k+1$ and consider the feasible (but not optmal) sequence

$$U_1=[u^O(k+1\,|\,k),\dots,u^O(k+H_c-1\,|\,k),0]$$

and its corresponding state sequence

$$X_1=[x^O(k+1\,|\,k),\dots,x^O(k+H_p-1\,|\,k),0]$$

"End terminal constraint" (Kwon & Pearson, 1977)

… at time $k+1$ the cost function obtained with sequence U_1 (not optimal) is

$$J(U_1,x(k+1\,|\,k+1))=J(U^O,x(k\,|\,k))-x^T(k\,|\,k)Qx(k\,|\,k)-u^T(k\,|\,k)Ru(k\,|\,k)$$

therefore:

$$V(x(k+1\,|\,k+1))=J(U_1^O,x(k+1\,|\,k+1))\le J(U_1,x(k+1\,|\,k+1))$$

where $U_1^O=[u^O(k+1\,|\,k+1),u^O(k+1\,|\,k+1),\dots,u^O(k+H_c\,|\,k+1)]$

Let us compute the increment

$$\Delta V(k)=V(x(k+1\,|\,k+1))-V(x(k\,|\,k))\le -x^T(k\,|\,k)Qx(k\,|\,k)-u^T(k\,|\,k)Ru(k\,|\,k)<0$$

recalling that $Q\ge 0$, $R>0$:

It is proven that $\Delta V(x)<0$ and since $V(x)>0$ asymptotic stability of the controlled system is proven

Moreover since $\{V(x)\}^{\infty}_{k=0}\to 0$ for $k\to\infty \Rightarrow x(k)\to 0$

"Terminal weighting matrix" *(Kwon et al.,1983)*

$$\min_U J(x(k\,|\,k),U)=$$
$$=\min_U \sum_{i=0}^{H_p-1} x(k+i\,|\,k)^T Qx(k+i\,|\,k)+u(k+i\,|\,k)^T Ru(k+i\,|\,k)$$
$$+x(k+H_p\,|\,k)^T Px(k+H_p\,|\,k)$$

s.t.

$$\begin{cases} x(k+i+1\,|\,k)=Ax(k+i\,|\,k)+Bu(k+i\,|\,k),\ i\ge 0\\ U(k)\in\mathbb{U}\\ x(k+i\,|\,k)\in\mathbb{X},\ i=1,\dots,H_p-1\\ u(k+i\,|\,k)=0,\ i=H_c,\dots,H_p-1 \end{cases}$$

Hp: stable system

"Terminal weighting matrix" (Kwon et al.,1983)

Matrix P is chosen as the positive definite solution of the discrete time Lyapunov equation:

$$A^T PA + Q = P$$

The proof is similar to the case of the "End terminal constraint"

Negative definiteness of the increment $\Delta V(x)$ can be shown as follows

$$V(x(k+1|k+1)) = J(U_1^O, x(k+1|k+1)) \le J(U_1, x(k+1|k+1))$$

$$\Delta V(k) = V(x(k+1|k+1)) - V(x(k|k)) \le$$

$$-x^T(k|k)Qx(k|k) - u^T(k|k)Ru(k|k) + x^T(k+1+H_p|k)Px(k+1+H_p|k) +$$

$$+x^T(k+H_p|k)(Q-P)x(k+H_p|k)$$

Terminal weighting matrix" (Kwon et al.,1983)

... $\Delta V(x)$ can be written as

$$\Delta V(k) = V(x(k+1|k+1)) - V(x(k|k)) \le$$

$$-x^T(k|k)Qx(k|k) - u^T(k|k)Ru(k|k) + M_k$$

where

$$M_k = x^T(k+1+H_p|k)Px(k+1+H_p|k) + x^T(k+H_c|k)(Q-P)x(k+H_p|k) =$$

$$= (Ax(k+H_p|k))P(Ax(k+H_p|k)) + x^T(k+H_p|k)(Q-P)x(k+H_p|k) =$$

$$= x^T(k+H_p|k)(A^T PA + Q - P)Px(k+H_p|k) = 0$$

Therefore $\Delta V(x) < 0$

A possible solution ...

Idea: consider the infinite horizon cost of the LQ controller

$$J = \sum_{i=0}^{\infty} x(k+i|k)^T Qx(k+i|k) + u(k+i|k)^T Ru(k+i|k)$$

and split it into two parts:

up to time $i = H_p$ where constraints are active

$$J = \sum_{i=0}^{H_p-1} x(k+i)^T Qx(k+i) + u(k+i)^T Ru(k+i) + \dots$$

for $k > H_p$ where constraints are not active

$$\dots + \sum_{i=H_p}^{\infty} x(k+i)^T Qx(k+i) + u(k+i)^T Ru(k+i)$$

"Invariant terminal set" (Skokaert & Rawlings, 1996)

$$\min_U J(x(k|k), U) =$$

$$= \min_U \sum_{i=0}^{H_p-1} x(k+i|k)^T Qx(k+i|k) + u(k+i|k)^T Ru(k+i|k) + x(k+H_p|k)^T Px(k+H_p|k)$$

s.t.

$$\begin{cases} x(k+i+1|k) = Ax(k+i|k) + Bu(k+i|k),\ i \ge 0 \\ U(k) \in \mathbb{U} \\ x(k+i|k) \in \mathbb{X},\ i = 1,\dots,H_p - 1 \\ u(k+i|k) = -K_{LQ}x(k+i|k),\ i = H_c,\dots,H_p - 1 \\ x(k+H_p|k) \in \Omega_{LQ} \supset 0 \end{cases}$$

Hp: Stabilizable system

"Invariant terminal set" (Skokaert & Rawlings, 1996)

The set Ω_{LQ} is chosen as a positive invariant set w.r.t. the (unconstrained) LQ controller designed according to the cost function

$$J=\sum_{i=0}^{\infty} x(k+i\mid k)^T Qx(k+i\mid k)+u(k+i\mid k)^T Ru(k+i\mid k)$$

The proof is similar to the one of the terminal weighting matrix case

P is chosen as the solution of the Riccati equation:

$$(A-BK_{LQ})^T P(A-BK_{LQ})+Q+K_{LQ}^T RK_{LQ}=P$$

"Invariant terminal set" (Skokaert & Rawlings, 1996)

How can be defined the set Ω_{LQ}

We can start using ellipsoidal shape

$$\mathcal{E}_W=\left\{x : x^T Wx\le 1\right\}, W\ge 0$$

In order to obtain positive invariance

$$x(k+1)^T Wx(k+1)\le x(k)^T Wx(k)$$
$$\rightarrow x(k)^T[(A-BK_{LQ})]^T W(A-BK_{LQ})x(k)\le x(k)^T Wx(k)$$
$$\rightarrow [(A-BK_{LQ})]^T W(A-BK_{LQ})-W\le 0$$

Moreover constraint satisfaction must be guaranteed

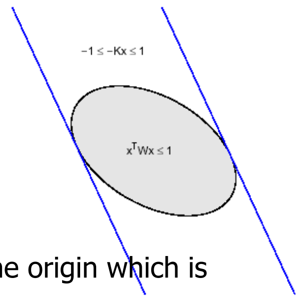
"Invariant terminal set" (Skokaert & Rawlings, 1996)

Therefore, the maximum volume ellipsoid which satisfies constraints is looked for

In the case of saturation constraints

$-1\le u(\cdot)\le 1 \rightarrow -1\le -K_{LQ}x(\cdot)\le 1$

The ellipsoid is the one reported in the picture



- The maximum volume ellipsoid centered at the origin which is contained in the strip must be determined
- This problem can be solved with standard convex optimization techniques

"Invariant terminal set" (Skokaert & Rawlings, 1996)

In this case the set Ω_{LQ} is represented by the ellipsoid $\mathcal{E}_W$ and the terminal state constraint is no more linear (solution with SOCP)

If polyhedral descriptions of Ω_{LQ} are used the constraint is linear

"Contraction constraint" (Zheng, 1995)

$$\min_U J(x(k\mid k),U) =$$

$$= \min_U \sum_{i=0}^{H_p} x(k+i\mid k)^T Qx(k+i\mid k) + u(k+i\mid k)^T Ru(k+i\mid k)$$

s.t.

$$\begin{cases} x(k+i+1\mid k) = Ax(k+i\mid k) + Bu(k+i\mid k),\ i \geq 0 \\ U(k) \in \mathbb{U} \\ x(k+i\mid k) \in \mathbb{X},\ i = 1,\dots,H_p-1 \\ u(k+i\mid k) = u(k+H_c-1\mid k), i = H_c,\dots,H_p-1 \\ x(k+1\mid k)^T Px(k+1\mid k) \leq \lambda^2 x(k\mid k)^T Px(k\mid k),\ \lambda<1,\ P: A^T PA + Q = P \end{cases}$$

$Hp:$ stable system

"Contraction constraint" (Zheng, 1995)

Sketch of the proof:

Given system stability it can be shown that constraints are always feasible (provided that suitable softening on the state constraints are introduced)

In particular, since in such a context the zero sequence is always feasible it can be shown that for a given λ in the interval $\lambda^* \leq \lambda < 1$ the contraction constraint is feasible for sure

$$x^T(k+1\mid k)Px(k+1+\mid k) = x^T(k\mid k)A^T P\,Ax(k\mid k) = x^T(k\mid k)(P-Q)x(k\mid k)$$

$$= x^T(k\mid k)Px(k\mid k) - x^T(k\mid k)Qx(k\mid k) = x^T(k\mid k)Px(k\mid k)\left(1 - \frac{x^T(k\mid k)Qx(k\mid k)}{x^T(k\mid k)Px(k\mid k)}\right) \leq$$

$$\leq x^T(k\mid k)Px(k\mid k)\left(\underbrace{1 - \frac{\underline{\sigma}(Q)}{\overline{\sigma}(P)}}_{<1}\right) = \lambda^2 x^T(k\mid k)Px(k\mid k)$$

"Contraction constraint" (Zheng, 1995)

Note that stability of the system to be controlled ensures feasibility of the zero input sequence:

$$J(x(k\mid k),U^O) \leq J(x(k\mid k),0) = \sum_{i=0}^{H_p} x(k+i\mid k)^T Qx(k+i\mid k) =$$

$$\sum_{i=1}^{H_p} x(k+i\mid k)^T (A^i)^T QAx(k+i\mid k) + x(k\mid k)^T Qx(k\mid k)$$

$$\leq x^T(k\mid k)Px(k\mid k)\left(1 - \frac{\underline{\sigma}((A^i)^T QA)}{\overline{\sigma}(P)}\right) \leq \lambda^2 x^T(k-1\mid k-1)Px(k-1\mid k-1)\left(1 - \frac{\underline{\sigma}((A^i)^T QA)}{\overline{\sigma}(P)}\right)$$

$$\lambda^{2k} x^T(0)Px(0)\left(1 - \frac{\underline{\sigma}((A^i)^T QA)}{\overline{\sigma}(P)}\right)$$

therefore $\lim_{k\to\infty} J(k) = 0$ and, since $Q > 0$, it follows $\lim_{k\to\infty} x(k) = 0$

"Contraction constraint" (Zheng, 1995)

Remarks

- Contraction constraints introduces quadratic constraints → solution with SOCP
- In order to reduce th online computational burden (avoiding SOCP):
  - solve QP without taking into account the contraction constraint
    - if the contraction constraint is satisfied → OK
    - If the constraint is not satisfied → add a penality term $\rho x^T(k)Px(k)$ in the cost function and tune (i.e. increase r) until the constraint is not satisfied

Stability analysis for nonlinear MPC

Problem formulation

Given the discrete time, time invariant model

$$M: x(k+1) = f(x(k), u(k))$$
$$x(k) \in \mathbb{R}^n, u(k) \in \mathbb{R}^m$$

which admits the origin of the state space as an equilibrium state for null input, i.e.:

$$f(0,0) = 0$$

consider the design of a predictive controller that aims at regulating at the origin the system state in the presence of input and state constraint

The optimization problem

… to this aim the feedback controller

$$u(k) = \mathcal{K}(x(k))$$

will be designed using the RH strategy based on the following optimization problem

$$\mathcal{P}_{H_p}(x): \begin{cases} \min\limits_{U(k)} J(x(k|k), U(k)) = \min\limits_{U(k)} \sum\limits_{i=0}^{H_p-1} L(x(k+i|k), u(k+i|k)) + \Phi(x(k+H_p|k)) \\ s.t. \\ x(k+1) = f(x(k), u(k)) \\ U(k) \in \mathbb{U} \\ x(k+i|k) \in \mathbb{X}, i = 1, \dots, H_p - 1 \\ x(k+H_p|k) \in \mathbb{X}_F \subset \mathbb{X} \end{cases}$$

Assumptions on $\mathcal{P}_{Hp}(x)$: part I

Let us recall the assumptions already made on the model, on the cost function terms and on the constraint sets

- $f(\cdot)$, $L(\cdot)$, $\Phi(\cdot) \in C^1$
- $\mathbb{U} \subset \mathbb{R}^m$ compact, $\mathbb{X} \subset \mathbb{R}^n$, $\mathbb{X}_F \subset \mathbb{R}^n$ closed, all containing the origin on their interiors

For simplicity, it will be assumed $H_p = H_c < \infty$

The minimizer of problem $\mathcal{P}_{Hp}(x)$ and the corresponding state sequence at time k will be indicated as:

$$U^O(k) = [u^O(k|k), u^O(k+1|k), \dots, u^O(k+H_p-1|k)]^T$$

$$X^O(k) = [x^O(k|k), x^O(k+1|k), \dots, x^O(k+H_p|k)]^T$$

Assumptions on $\mathcal{P}_{Hp}(x)$: part II

The following Assumptions on problem $\mathcal{P}_{Hp}(x)$ will be made:

1. $L(x,u)$ satisfies

   $L(0,0) = 0$ & $\|L(x,u)\| > \gamma(\|x\|)$ $\forall x \in \mathbb{S}_{Hp}$, $u \in \mathbb{U}$ where

   $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$, $\gamma(t) > 0$, $\forall t > 0$, $\lim_{t\to\infty} \gamma(t) = \infty$

2. There exists an "auxiliary" controller $u(k) = \mathcal{K}_F(x(k))$ such that $\mathcal{K}_F(x(k)) \in \mathbb{U}$, $\forall x \in \mathbb{X}_F$

3. $\mathbb{X}_F$ is a positively invariant set for $x(k+1) = f(x(k),\mathcal{K}_F(x(k)))$

4. $\Phi(x)$ satisfies

   a. $\Phi(0) = 0$, $\Phi(x) \geq 0$ $\forall x \in \mathbb{X}_F$

   b. $\Phi(f(x,\mathcal{K}_F(x)) - \Phi(x) \leq - L(x, \mathcal{K}_F(x))$, $\forall x \in \mathbb{X}_F$

Assumptions on $\mathcal{P}_{Hp}(x)$: part II

Remarks

An example of function $\boldsymbol{L(x,u)}$ satisfying hypothesis 1. is the quadratic form:

$$L(x,u) = x^T Q x + u^T R u,\ Q = Q^T \succeq 0,\ R = R^T \succ 0$$

where $\gamma(t) = \lambda_{\min}(Q)t^2$ and $\lambda_{\min}(Q) = \min_i \lambda_i(Q)$

An example of the "triple" $\Phi(x)$, $\mathcal{K}_F(x)$, $\mathbb{X}_F$ satisfying hypotheses 2. 3. and 4. is:

$$\Phi(x) = 0$$
$$\mathcal{K}_F(x(k)) = 0$$
$$\mathbb{X}_F = \{0\}$$

Asymptotic stability of MPC

Suppose that Assumptions 1. – 4. are met, then the **closed loop system**

$$x(k+1) = f(x(k),\mathcal{K}(x(k)))$$

obtained using the control law:

$$u(k) = \mathcal{K}(x(k))$$

as the result of the application of the RH strategy on problem $\mathcal{P}_{Hp}(x)$

is asymptotically stable

Sketch of the proof

Let $x(0) = x_0 \in \mathbb{S}_{Hp}$ be the initial state at time instant $k = 0$ and

$U_0^O = [u_0^O, u_1^O, \ldots, u_{Hp-1}^O]^T$ and $X_0^O = [x_0^O, x_1^O, x_2^O, \ldots, x_{Hp}^O]^T$

be the minimizer and the corresponding state sequence respectively obtained solving problem $\mathcal{P}_{Hp}(x)$

At the next time instant $k = 1$ consider the following sequence which is certainly feasible but, in general, not optimal:

$U_1 = [u_1^O, \ldots, u_{Hp-1}^O, \mathcal{K}_F(x_{Hp}^O)]^T$

the corresponding state sequence is

$X_1 = [x_1^O, x_2^O, \ldots, f(x_{Hp}^O, \mathcal{K}_F(x_{Hp}^O))]^T$

Note that all the components of X_1 belong to $\mathbb{S}_{Hp}$.
Iterating the same reasoning for the future time instants $k = 2, 3, \ldots$ feasibility of the sequences X_2, X_3,... is obtained thus **proving** the (positive) **invariance of** $\mathbb{S}_{Hp}$

Sketch of the proof

… the next step consists in proving that the optimal cost function $J^O(x)$ is a Lyapunov function for asymptotic stability of the closed loop system

$$x(k+1) = f(x(k), \mathcal{K}(x(k)))$$

What we need to show are the two properties:

1) $J^O(x) > 0$
2) $\Delta J^O(x) = J^O(x(k+1)) - J^O(x(k)) < 0$

Recalling that $J^O(x) = L(x(k|k), U^O)$, we have that satisfaction of hypotesis 1. leads to

$L(0,0) = 0$, $\Phi(0) = 0 \Rightarrow J^O(x) = 0$

$J^O(x) > L(x(k|k), U^O) \geq \gamma(\|x(k|k)\|)\ \forall x \in \mathbb{S}_{Hp}$

$\Rightarrow$ **$J^O(x) = 0$ positive definite**

Sketch of the proof

… now consider the change in value:

$$\Delta J^O(x) = J^O(x(k+1)) - J^O(x(k)) = J^O(X^O_{k+1}, U^O_{k+1}) - J^O(X^O_k, U^O_k) \leq$$
$$\leq J^O(X_{k+1}, U_{k+1}) - J^O(X^O_k, U^O_k) =$$
$$= -L(x(k|k), u(k|k)) + L(x(k+H_p|k), K_F(x(k+H_p|k))) +$$
$$+\Phi(f(x(k+H_p|k), K_F(x(k+H_p|k))) - \Phi(f(x(k+H_p|k))$$

On the basis of 4. the sum of the last three terms is nonpositive. Therefore using Hypothesis 1:

$$\Delta J^O(x) \leq -L(x(k|k), u(k|k)) < -\gamma(\|x\|) < 0, \forall x \in \mathbb{S}_{H_p}$$

i.e. the increment **$\Delta J^O(x)$ is negative definite**

It is then possible to conclude that the optimal cost function **$J^O(x)$** is a **Lyapunov function** for asymptotic stability of the origin of the closed loop system, moreover the **region of attraction** is $\mathbb{S}_{Hp}$