# Contents

# Motion

## Projectile Motion

The equations on the left are the standard kinematic equations; they express motion of constant acceleration on a straight line. The equations on the right are adapted to express motion on a parabola. Projectile motion can be defined as follows: An object is launched with initial velocity $\vec{v}_0$. Horizontal acceleration is zero, and vertical acceleration is $-g$, the downward force of gravity. $\theta_0$ is the initial launch angle with respect to the positive $x$-axis.

$$v = v_0 + at \qquad x = x_0 + (v_0 \cos\theta_0)\, t$$
$$x = x_0 + v_0 t + \tfrac{1}{2} a t^2 \qquad y = y_0 + (v_0 \sin\theta_0)\, t - \tfrac{1}{2} g t^2$$
$$v^2 = v_0^2 + 2a\,(x - x_0) \qquad v_y = v_0 \sin\theta_0 - gt$$
$$x = x_0 + \tfrac{1}{2}\,(v_0 + v)\, t \qquad v_y^2 = (v_0 \sin\theta_0)^2 - 2g\,(y - y_0)$$
$$x = x_0 + vt - \tfrac{1}{2} a t^2$$

The path trajectory of a projectile is parabolic and can be expressed as

$$y = (\tan\theta_0)\, x - \frac{g x^2}{2\,(v_0 \cos\theta_0)^2}$$

If $x_0 = y_0 = 0$, the object's horizontal range $R$ can be expressed as

$$R = \frac{v_0^2}{g} \sin 2\theta_0$$

## Relative Motion

A frame of reference is an arbitrary set of coordinate axes about which we make measurements. A passenger on a train is not moving relative to their seat, but they may be moving very quickly from the perspective of a tree next to the train tracks. Given two frames of reference, A and B, with constant velocity; the velocity of a particle P with respect to A is equal to the velocity P with respect to B, plus the velocity of B with respect to A . .

$$\vec{v}_{\mathrm{PA}} = \vec{v}_{\mathrm{PB}} + \vec{v}_{\mathrm{BA}}$$

The acceleration is equal from either frame of reference.

$$\vec{a}_{\mathrm{PA}} = \vec{a}_{\mathrm{PB}}$$

## Force and Motion

### Newton's Laws

| | | |
|---|---|---|
| First Law: | If there is no net force a body, it cannot change its velocity. | $\vec{F}_{\text{net}} = 0 \rightarrow \vec{a} = 0$ |
| Second Law: | The net force $\vec{F}_{\text{net}}$ on a body is the product of its mass $m$ and its acceleration $\vec{a}$. | $\vec{F}_{\text{net}} = m\vec{a}$ |
| Third Law: | Equal and opposite reactions. | $\vec{F}_{\text{BC}} = -\vec{F}_{\text{CB}}$ |

### Friction

The force of static friction $\vec{f}_s$ has a maximum value $f_{s,\ \text{max}}$ given by

$$f_{s,\ \text{max}} = \mu_s F_N$$

where $\mu_s$ is the coefficient of static friction and $F_N$ is the magnitude of the normal force. If the component of $\vec{F}$ parallel to the surface exceeds $f_{s,\ \text{max}}$, the body will begin to slide. Once the body begins to slide on the surface, the frictional force decreases to a constant value $f_k$ given by

$$f_k = \mu_k F_N$$

where $\mu_k$ is the coefficient of kinetic friction. Kinetic friction is less forceful than static friction.

### Drag Force

A body that is moving relative to a surrounding fluid experiences a drag force $\vec{D}$. The force points in the direction in which the fluid flows relative to the body, i.e., opposite of the body's motion relative to the fluid. The magnitude of $\vec{D}$ can be expressed as

$$D = \tfrac{1}{2} C \rho A v^2$$

where $C$ is the drag coefficient, $\rho$ is the fluid density, and $A$ is the effective cross-sectional area of the body, and $v$ is the relative speed.

### Terminal Speed

As an object in free fall builds speed, there comes a point where the magnitude of the drag force $D$ becomes equal to that of the force of gravity $F_g$. The object cannot fall any faster and is said to have reached terminal velocity $v_t$ expressed as

$$v_t = \sqrt{\frac{2F_g}{C\rho A}}$$

### Uniform Circular Motion

An object traveling along a circle of radius $R$ at constant speed $v$ is said to be in uniform circular motion. Speed is constant, therefore acceleration is constant. The constant magnitude of acceleration $\vec{a}$ can be expressed as

$$a = \frac{v^2}{R} \quad \text{and} \quad F_c = \frac{mv^2}{R}$$

Where $F_c$ is the magnitude of the centripetal force. Acceleration $\vec{a}$ and force $\vec{F_c}$ are always directed toward the center of the circle. This is what it means for a force to be centripetal; it is directed toward a center around which an object is moving.

The time for the object to complete the circle is

$$T = \frac{2\pi R}{v}$$

## Energy and Work

### Kinetic Energy and Work

The work done on an object over displacement $d$ can be expressed as

$$W = Fd\cos\phi = \vec{F} \cdot \vec{d}$$

where $\phi$ is the angle between $\vec{F}$ and $\vec{d}$, and both $\vec{F}$ and $\phi$ are constant.

Following the law of conservation of energy, the kinetic energy of an object is equal to the energy applied to that object in order to accelerate it to a given velocity. That concept is captured by the work-kinetic energy theorem: an increase in kinetic energy means that energy is being transferred to the object, and a decrease in kinetic energy means that energy is being transferred from the object; positive work and negative work, respectively.

$$K = \frac{1}{2}mv^2$$

$$\Delta K = K_f - K_i = W$$

$$K_f = K_i + W$$

## Gravity and Work

The work done on an object by the force of gravity can be expressed as

$$W_g = mgd\cos\phi$$

where $\phi$ is the angle between $\vec{F}_g$ and $\vec{d}$, and $\phi$ is constant.

If $K_f = K_i$, then by the work-kinetic energy theorem, the work applied in lifting an object can be expressed as

$$W_a = -W_g$$

## Variable Force and Work

Springs are a special case of variable force. In a system where a spring has one fixed end and one free end, the spring exerts a force proportional to the displacement of its free end. Hooke's Law encapsulates this behavior:

$$\vec{F}_s = -k\vec{d}$$

where $\vec{d}$ is the displacement of the spring's free end from its position at rest (neither compressed nor stretched), and $k$ is the constant of proportionality, the spring constant. Notice the $(-)$ in the expression; this is because the spring force always acts in the opposite direction of displacement.

The work done by a spring as it moves from one position to another along a single axis can be expressed as

$$W_s = \int_{x_i}^{x_f} -kx\,dx = -\frac{k}{2}\left(x_f^2 - x_i^2\right)$$

The work-kinetic energy theorem, as applied to spring force, states that the change in kinetic energy of an object in an ideal spring system is equal to the sum of the work applied and the work done by the spring.

$$\Delta K = K_f - K_i = W_a + W_s$$

The work done on an object by a general variable force is the sum of the work done along each axis.

$$W = \int_{x_i}^{x_f} F_x\,dx + \int_{y_i}^{y_f} F_y\,dy + \int_{z_i}^{z_f} F_z\,dz$$

where $a_i$ and $a_f$ are the initial and final positions with respect to a given axis, and $F_x$, $F_y$, and $F_z$ are functions that model force along the $x$, $y$, and $z$ axis.

### Power

Power is the rate of change of work with respect to time.

$$P_{\text{avg}} = \frac{W}{\Delta t} \qquad P = \frac{dW}{dt}$$

Given a constant force $\vec{F}$ acting on an object, the instantaneous power exerted on that object can be expressed as

$$\frac{d}{dt}\left[W = \vec{F} \cdot \vec{d}\right] = \left[P = \vec{F} \cdot \vec{v} = Fv\cos\phi\right]$$

where $\phi$ is the constant angle between force $\vec{F}$ and velocity $\vec{v}$.

## Potential Energy

### Conservation of Energy

TODO : define conservative force

The change in potential energy of an object is defined as the negative work done on that object. Using the general definition of work, we can define change in gravitational potential energy and elastic potential energy.

$$\Delta U = -W = -\left[\int_{x_i}^{x_f} F_x\,dx + \int_{y_i}^{y_f} F_y\,dy + \int_{z_i}^{z_f} F_z\,dz\right]$$

$$\Delta U = -\int_{y_i}^{y_f} mg\,dy = mg\left(y_f - y_i\right)$$

$$\Delta U = -\int_{x_i}^{x_f} -kx\,dx = \frac{k}{2}\left(x_f^2 - x_i^2\right)$$