Contents

Vectors and the Geometry of Space

The Dot Product

The dot product, or scalar product, of vectors u and v:

$$u \cdot v = |u|\,|v| \cos\theta \qquad \theta = \arccos \frac{u \cdot v}{|u|\,|v|}$$

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$$

For two non-zero vectors u and v;

$$\begin{aligned} u \cdot v > 0 &\quad\iff\quad \theta < 90^\circ \\ u \cdot v = 0 &\quad\iff\quad \theta = 90^\circ \\ u \cdot v < 0 &\quad\iff\quad \theta > 90^\circ \end{aligned}$$

The vector projection of u onto v is equal to the scalar component of u in the direction of v times the direction of v. Using the definition of the dot product, this can be rearranged into a form more easily evaluated by hand.

$$\text{proj}_v u = (|u| \cos\theta)\,\frac{v}{|v|} = \frac{u \cdot v}{|v|^2}\,v$$

The Cross Product

The cross product $u \times v$ where n is a unit vector normal to the plane formed by u and v. Because n is a unit vector, the magnitude of $u \times v$ is readily apparent. $|u \times v|$ is the area of the parallelogram formed by the vectors.

$$u \times v = (|u|\,|v| \sin\theta)\,n$$

$$|u \times v| = |u|\,|v| \sin\theta$$

Two non-zero vectors u and v are parallel if and only if $u \times v = 0$

$$u \,||\, v \quad\iff\quad u \times v = 0$$

Calculating the cross product as a determinant:

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{matrix} i\,(u_2 v_3 - u_3 v_2) - \\ j\,(u_1 v_3 - u_3 v_1) + \\ k\,(u_1 v_2 - u_2 v_1) \end{matrix}$$

The triple scalar product, or box product, is the volume of the parallelepiped formed by vectors u, v and w. It can be calculated as a determinant:

$$(u \times v) \cdot w = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Equations of Lines and Planes in Space

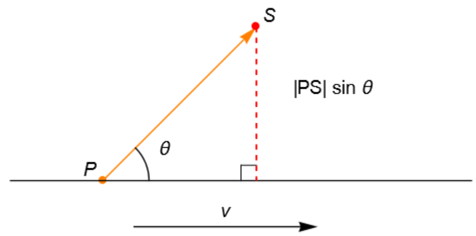
The standard parameterization of the line through $P\,(x_0, y_0, z_0)$ parallel to a nonzero vector $v = \langle v_1, v_2, v_3 \rangle$

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3$$

The distance from a point S to a line that passes through point P parallel to vector v is

$$d = |\overrightarrow{PS}| \sin\theta$$

$$d = \frac{|\overrightarrow{PS} \times v|}{|v|}$$



A plane can be defined by its normal vector n. The plane containing point P_0 consists of all points where n and $\overrightarrow{P_0P}$ are perpendicular, where P is any point. The equation of a plane through $P_0\,(x_0, y_0, z_0)$ normal to a nonzero vector $n = \langle a, b, c \rangle$ can be expressed as

$$\begin{aligned}
&n \cdot \overrightarrow{P_0P} = 0 \\
&a\,(x - x_0) + b\,(y - y_0) + c\,(z - z_0) = 0 \\
&ax + by + cz = d, \\
&\quad \text{where } d = ax_0 + by_0 + cz_0
\end{aligned}$$

If P is a point on a plane with normal vector n, then the distance from any point S to the plane is the length of the vector projection of $\overrightarrow{PS}$ onto n:

$$d = \left| \overrightarrow{PS} \cdot \frac{n}{|n|} \right|$$

The angle between two intersecting planes is defined to be the acute angle between their normal vectors.

$$\theta = \arccos \frac{n_1 \cdot n_2}{|n_1||n_2|}$$

Vector-Valued Functions and Space Curves

Curves in Space and Their Tangents

The vector function $r\left(r\right)=f\left(t\right)i+g\left(t\right)j+h\left(t\right)k$ has a derivative at t if f, g and h have derivatives at t.

$$r'\left(t\right)=\frac{dr}{dt}=\frac{df}{dt}i+\frac{dg}{dt}j+\frac{dh}{dt}k$$

Let u and v be differentiable vector functions of t, and f be any differentiable scalar function.

Dot Product:	$\frac{d}{dt}\left[u(t)\cdot v(t)\right]=u'(t)\cdot v(t)+u(t)\cdot v'(t)$
Cross Product:	$\frac{d}{dt}\left[u(t)\times v(t)\right]=u'(t)\times v(t)+u(t)\times v'(t)$
Chain Rule:	$\frac{d}{dt}\left[u\left(f(t)\right)\right]=f'(t)\;u'\left(f(t)\right)$

Arc Length and Curvature

The length of a smooth curve $r(t)=\langle\,x(t),y(t),z(t)\,\rangle$, $a\le t\le b$ from $t=a$ to $t=b$ is

$$L=\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt=\int_a^b|v|\;dt$$

Arc length parameterization with starting point P_{t_0}

$$s(t)=\int_{t_0}^t\left|v\left(\tau\right)\right|\;d\tau$$

The velocity vector $v=dr/dt$ is tangent to the curve $r(t)$, therefore

$$T=\frac{v}{|v|}$$

is a unit vector tangent to the (smooth) curve, called the *unit tangent vector*.

If T is the unit vector of a smooth curve, then the curvature function of the curve is

$$\kappa=\left|\frac{dT}{ds}\right|=\left|\frac{dT}{dt}\,\frac{dt}{ds}\right|=\frac{1}{|ds/dt|}\left|\frac{dT}{ds}\right|=\frac{1}{|v|}\left|\frac{dT}{dt}\right|$$

At a point where $\kappa\neq0$, the principal unit normal vector for a smooth curve in the plane is

$$N=\frac{1}{\kappa}\frac{dT}{ds}=\frac{dT/dt}{|dT/dt|}$$

The principal normal vector will point to the concave side of the curve.

Partial Derivatives

Functions of Several Variables

Level curves for a given z are found my setting $z = f(x, y)$ to be a constant, then finding the resulting curve in the x, y plane.

Limits of Functions of Several Variables

$$\lim_{(x_1,\ldots,x_n)\to(a_1,\ldots,a_n)} f(x_1,\ldots,x_n) = L$$

means that, given an $\epsilon > 0$ we can find a $\delta > 0$ such that

$$|f(x_1,\ldots,x_n) - L| < \epsilon \quad \text{whenever} \quad 0 < \sqrt{(x_1-a_1)^2 + \cdots + (x_n-a_n)^2} < \delta$$

Interestingly, any polynomial is continuous on $\mathbb{R}^2$, and any rational function is continuous on its domain. We can verify this by looking at the properties of limits; the limit of a sum is the sum of the limits, the limit of a product is the product of the limits, etc.

Partial Derivatives

A partial derivative is the derivative of a function with respect to one variable while the other variables are fixed.

$$\frac{\partial}{\partial x_i} f(x_1,\ldots,x_i,\ldots x_n) = \lim_{h\to 0} \frac{f(x_1,\ldots,x_i+h,\ldots x_n) - f(x_1,\ldots,x_i,\ldots x_n)}{h}$$

The partial derivative can be notated in different ways:

$$\frac{\partial}{\partial x} f(x,y) = \frac{\partial f}{\partial x} = f_x(x,y) = f_x = D_x f$$
$$\frac{\partial^2}{\partial^2 x} f(x,y) = \frac{\partial f}{\partial^2 x} = f_{xx}(x,y) = f_{xx} = D_{xx} f$$
$$\frac{\partial^2}{\partial x\,\partial y} f(x,y) = \frac{\partial f}{\partial x\,\partial y} = f_{xy}(x,y) = f_{xy} = D_{xy} f$$

It is important to note that $\partial f/\partial x$ *cannot* be interpreted as a ratio of differentials as it would with the total derivative df/dx

Clairaut's Theorem, or, symmetry of second derivatives: If the second partial derivatives of a function are continuous, the order of differentiation does not matter. This can also be extended to higher order partial derivatives if their functions are continuous.