FD-03: The Emergence of 3+1 Dimensions

From K_4 Spectral Structure to Spacetime

Johannes Wielsch

Independent Researcher

https://github.com/de-johannes/FirstDistinction

December 2025

Abstract

We prove that the complete graph K_4 has exactly one zero eigenvalue and three degenerate eigenvalues equal to 4. This spectral structure $(0, 4, 4, 4)$ is machine-verified and follows necessarily from K_4's topology. We then propose the physical hypothesis that these eigenvalues correspond to the $(3 + 1)$-dimensional structure of spacetime: three spatial dimensions from the degenerate eigenspace, and one temporal dimension from the unique zero eigenvalue (global symmetry). The mathematics is proven under `-safe -without-K` in Agda (7,938 lines). The physical interpretation remains a testable hypothesis.

1 Introduction

1.1 From Graph to Geometry

In FD-01, we proved that K_4 emerges necessarily from the concept of distinction. Now we ask: *What structure does K_4 impose?*

The answer lies in spectral graph theory: the eigenvalues of K_4's Laplacian encode its symmetries and constraints. We prove:

$$\text{spectrum}(L_{K_4}) = \{0, 4, 4, 4\} \tag{1}$$

One zero (connectivity), three fours (degeneracy). This is not a choice—it is forced by K_4's complete structure.

1.2 The Central Claim

1. **Mathematics (proven):** K_4 has eigenvalues $(0, 4, 4, 4)$
2. **Hypothesis (testable):** These eigenvalues correspond to spacetime's $(3 + 1)$ structure
3. **Prediction:** If this hypothesis is correct, dimensionality is not a free parameter

1.3 Methodology

All mathematical proofs formalized in Agda:

- `-safe`: Zero axioms, zero postulates
- `-without-K`: No uniqueness of identity proofs

Complete source: https://github.com/de-johannes/FirstDistinction

2 The Laplacian Matrix

2.1 Definition

Definition 2.1 (Laplacian Matrix)**.** For a graph $G = (V, E)$ with n vertices, the Laplacian is:

$$L = D - A \tag{2}$$

where:

- $D_{ij} = \deg(v_i) \cdot \delta_{ij}$ (degree matrix)
- $A_{ij} = 1$ if $(v_i, v_j) \in E$, else 0 (adjacency matrix)

2.2 K_4 Laplacian

For K_4, every vertex has degree deg $= 3$, and every pair is connected:

$$A_{K_4} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad D_{K_4} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \tag{3}$$

Therefore:

$$L_{K_4} = D_{K_4} - A_{K_4} = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix} \tag{4}$$

Remark 2.2. This matrix is:

- Symmetric: $L^\top = L$
- Positive semi-definite: $\mathbf{x}^\top L\mathbf{x} \geq 0$ for all $\mathbf{x}$
- Row sums zero: $\sum_j L_{ij} = 0$ (implies zero eigenvalue)

3 The Eigenvalue Problem

3.1 Characteristic Equation

To find eigenvalues, solve:

$$\det(L_{K_4} - \lambda I) = 0 \tag{5}$$

Expanding:

$$\det \begin{pmatrix} 3-\lambda & -1 & -1 & -1 \\ -1 & 3-\lambda & -1 & -1 \\ -1 & -1 & 3-\lambda & -1 \\ -1 & -1 & -1 & 3-\lambda \end{pmatrix} = 0 \tag{6}$$

3.2 Symmetry Exploitation

By symmetry, all rows and columns are equivalent. The matrix has the form:

$$L_{K_4} = 4I - J \tag{7}$$

where J is the 4×4 all-ones matrix.

Key insight: J has eigenvalues $\{4, 0, 0, 0\}$ (rank 1, with eigenvector $(1, 1, 1, 1)^\top$).

Therefore, $L_{K_4} = 4I - J$ has eigenvalues:

$$\lambda(L_{K_4}) = 4 - \lambda(J) = \{4-4, 4-0, 4-0, 4-0\} = \{0, 4, 4, 4\} \tag{8}$$

4 Machine-Verified Proof

Machine-Verified

Theorem 4.1 (K4 Spectrum). *The Laplacian matrix of K_4 has exactly four eigenvalues:*

$$\mathit{spectrum}(L_{K_4}) = \{0, 4, 4, 4\} \tag{9}$$

with multiplicities:

- $\lambda_0 = 0$ *(multiplicity 1)*
- $\lambda_1 = \lambda_2 = \lambda_3 = 4$ *(multiplicity 3)*

Proof sketch. **Step 1 (Zero eigenvalue):** The all-ones vector $\mathbf{v}_0 = (1, 1, 1, 1)^\top$ satisfies:

$$L_{K_4}\mathbf{v}_0 = \begin{pmatrix} 3-1-1-1 \\ -1+3-1-1 \\ -1-1+3-1 \\ -1-1-1+3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{10}$$

Hence $\lambda_0 = 0$.

Step 2 (Degeneracy): Consider vectors orthogonal to $(1, 1, 1, 1)^\top$, e.g.:

$$\mathbf{v}_1 = (1, -1, 0, 0)^\top \tag{11}$$
$$\mathbf{v}_2 = (1, 0, -1, 0)^\top \tag{12}$$
$$\mathbf{v}_3 = (1, 0, 0, -1)^\top \tag{13}$$

Direct computation shows $L_{K_4}\mathbf{v}_i = 4\mathbf{v}_i$ for $i = 1, 2, 3$.
Example:

$$L_{K_4}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3\cdot 1 - 1\cdot(-1) - 1\cdot 0 - 1\cdot 0 \\ -1\cdot 1 + 3\cdot(-1) - 1\cdot 0 - 1\cdot 0 \\ -1\cdot 1 - 1\cdot(-1) + 3\cdot 0 - 1\cdot 0 \\ -1\cdot 1 - 1\cdot(-1) - 1\cdot 0 + 3\cdot 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 0 \\ 0 \end{pmatrix} = 4\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \tag{14}$$

Step 3 (Completeness): The four eigenvalues account for the 4×4 matrix. The subspace of eigenvectors with $\lambda = 4$ has dimension 3.

Full proof: lines 2476–2540 of `FirstDistinction.agda`. □

4.1 Uniqueness of Zero

Machine-Verified

Corollary 4.2 (Unique Zero Eigenvalue). *The zero eigenvalue has multiplicity exactly 1, corresponding to the connected nature of K_4.*

Proof. The nullspace of L_{K_4} is spanned by the all-ones vector. This reflects global connectivity: all vertices are reachable from any other. If K_4 had multiple disconnected components, the zero eigenvalue would have higher multiplicity. Lines 2541–2570. □

4.2 Degeneracy of Four

Machine-Verified

Corollary 4.3 (Threefold Degeneracy)**.** *The eigenvalue $\lambda = 4$ has multiplicity exactly 3.*

Proof. The eigenspace for $\lambda = 4$ consists of vectors orthogonal to $(1,1,1,1)^\top$. This is a 3-dimensional subspace of $\mathbb{R}^4$. The three eigenvectors are linearly independent and span the orthogonal complement of the connectivity eigenvector. Lines 2571–2610. □

5 Graph-Theoretic Interpretation

5.1 The Zero Eigenvalue

Definition 5.1 (Algebraic Connectivity)**.** The second-smallest eigenvalue of a Laplacian is called the *algebraic connectivity* or *Fiedler value.*

For K_4, the Fiedler value is $\lambda_1 = 4$ (not zero). This indicates:

- K_4 is maximally connected (complete graph)
- Removing any edge still leaves the graph connected
- The gap between $\lambda_0 = 0$ and $\lambda_1 = 4$ is maximal for 4 vertices

5.2 The Degenerate Eigenspace

The 3-dimensional eigenspace for $\lambda = 4$ reflects:

- Three independent constraints (degrees of freedom)
- Orthogonal directions in the space of perturbations
- Equal resistance to perturbations in all three directions (isotropy)

6 The Physical Hypothesis

6.1 Mapping to Spacetime

Physical Hypothesis

Hypothesis 6.1 (Dimension Correspondence)**.** The eigenvalue structure $(0, 4, 4, 4)$ of K_4 corresponds to the $(3 + 1)$-dimensional structure of spacetime:

- **Zero eigenvalue (multiplicity 1):** Temporal dimension (global connectivity, breaking of symmetry)
- **Degenerate eigenvalue 4 (multiplicity 3):** Three spatial dimensions (isotropy, symmetry)

6.2 Rationale

Why zero → time?

- The zero eigenvalue corresponds to the eigenvector $(1, 1, 1, 1)^\top$: all vertices treated equally
- This represents *global symmetry*—a distinguished direction
- Time is the dimension in which irreversibility and asymmetry manifest (see FD-05: Time from Asymmetry)

Why $4 = 4 = 4 \to$ space?

- Threefold degeneracy implies three *equivalent* directions
- Space is isotropic: no preferred spatial direction
- The eigenvalue 4 reflects the constraint imposed by complete connectivity

6.3 Predictions

If Hypothesis 6.1 is correct:

1. Spatial dimensionality ($d = 3$) is not a free parameter—it is forced by K_4
2. Temporal dimensionality ($d = 1$) reflects the unique zero eigenvalue
3. Any theory requiring $d \neq 3$ spatial dimensions contradicts the K_4 structure
4. Higher-dimensional theories (e.g., $d = 10$ in string theory) would require justification beyond K_4

7 Alternative Graphs

7.1 Why Not K_3?

The spectrum of K_3:

$$\text{spectrum}(L_{K_3}) = \{0, 3, 3\} \tag{15}$$

This gives $(1 + 2)$ dimensions—not $(3 + 1)$. Moreover, K_3 fails to achieve closure (FD-01, Theorem 7.1).

7.2 Why Not K_5?

The spectrum of K_5:

$$\text{spectrum}(L_{K_5}) = \{0, 5, 5, 5, 5\} \tag{16}$$

This gives $(1 + 4)$ dimensions. However, K_5 is not forced by the genesis mechanism (FD-01, Theorem 7.2).

7.3 Uniqueness of K_4

Machine-Verified

Theorem 7.1 (K4 Dimensional Uniqueness)**.** *Among complete graphs K_n:*

- *Only K_4 yields exactly 3 degenerate non-zero eigenvalues*

- *Only K_4 satisfies both closure (FD-01) and $(3+1)$ spectrum*

Proof. The spectrum of K_n is $\{0, n, n, \ldots, n\}$ with n appearing $(n-1)$ times. For $(3+1)$ structure, we need $n-1=3$, hence $n=4$. Lines 2650–2700. □

8 Connection to Physics

8.1 Dimensionality Problem

In standard physics, the dimensionality of spacetime is an *input*:

- General relativity: $(3+1)$ is assumed
- String theory: 10 or 11 dimensions postulated
- Loop quantum gravity: $(3+1)$ dimensions assumed

The FD approach proposes: $(3+1)$ is *derived*, not assumed.

8.2 Kaluza-Klein and Compactification

Higher-dimensional theories often invoke *compactification*: extra dimensions are "rolled up" and unobservable. In the FD framework:

- K_4 provides exactly $(3+1)$ eigenvalues
- No extra dimensions exist to compactify
- The question shifts from "why don't we see extra dimensions?" to "why would extra dimensions exist?"

8.3 Isotropy of Space

The threefold degeneracy ($\lambda = 4 = 4 = 4$) implies:

- No preferred spatial direction (Copernican principle)
- Rotational symmetry (SO(3) naturally emerges)
- Equal expansion/contraction in cosmology (if K_4 structure is preserved)

9 Validation via Four-Part Structure

Machine-Verified

Theorem 9.1 (Dimension Four-Part Validation)**.** *The* $(3+1)$ *dimensional structure satisfies:*

1. ***Consistency:*** *Eigenvalue count matches graph size* $(4 = 1 + 3)$
2. ***Exclusivity:*** *Only* K_4 *gives* $(0, n, n, n)$ *with* $n - 1 = 3$
3. ***Robustness:*** *Perturbations preserve eigenvalue structure (symmetric matrix properties)*
4. ***Cross-Constraints:*** *Eigenvalue sum equals trace:* $0 + 4 + 4 + 4 = 12 = 4 \cdot \deg =$

$tr(L_{K_4})$

Proof.

- **Consistency:** $\dim(\ker L) + \dim(\text{eigenspace}_4) = 1 + 3 = 4$
- **Exclusivity:** Proven by checking K_3, K_4, K_5 spectra
- **Robustness:** Symmetric matrices have real eigenvalues; perturbations don't destroy degeneracy pattern
- **Cross-Constraints:** $\sum \lambda_i = \text{tr}(L) = \sum \deg(v_i) = 4 \cdot 3 = 12$

Lines 2700–2750. □

10 Experimental Tests

10.1 What Would Falsify This Hypothesis?

1. **Discovery of a fourth spatial dimension:** If experiments reveal $d = 4$ spatial dimensions, Hypothesis 6.1 fails
2. **Multiple time dimensions:** If $d_{\text{time}} > 1$, the unique zero eigenvalue cannot explain it
3. **Variable dimensionality:** If spacetime dimensionality varies with energy scale or location, K_4 structure is insufficient

10.2 Supportive Evidence

- All experiments confirm $d = 3$ spatial dimensions (no deviations at any scale)
- Time is observably unique and asymmetric (consistent with unique $\lambda = 0$)
- Isotropy of space matches degeneracy of $\lambda = 4$

10.3 Open Questions

- Can Lorentz signature $(-, +, +, +)$ be derived from eigenvalue signs?
- Does the eigenvalue 4 encode physical constants (e.g., coupling strengths)?
- Can quantum mechanics emerge from eigenspace structure?

11 Implications

11.1 What Is Proven

1. K_4 has eigenvalues $(0, 4, 4, 4)$ (machine-verified, zero axioms)
2. This spectral structure is unique among forced complete graphs
3. The degeneracy pattern is $1 + 3$

11.2 What Is Hypothesized

1. The eigenvalue structure corresponds to spacetime dimensions
2. Zero eigenvalue $\leftrightarrow$ time (global symmetry)
3. Three degenerate eigenvalues $\leftrightarrow$ three spatial dimensions (isotropy)

11.3 Philosophical Implications

If accepted, this result suggests:

- Dimensionality is not arbitrary—it follows from K_4
- The number 3 (spatial dimensions) is logically necessary
- Higher-dimensional theories require justification beyond minimal structure

12 Related Work

- **Spectral graph theory:** Chung (1997), Mohar (1991)
- **Emergent spacetime:** Hořava-Lifshitz gravity, loop quantum gravity, causal sets
- **Dimension from dynamics:** Carlip (2017)—dimensional reduction at small scales
- **String theory:** Postulates 10 or 11 dimensions, compactifies to 4

Our contribution: derivation of $(3+1)$ from K_4 spectral structure, with zero free parameters.

13 Verification

13.1 How to Verify

```
git clone https://github.com/de-johannes/FirstDistinction.git
cd FirstDistinction
agda --safe --without-K FirstDistinction.agda
```

Check lines 2476–2750 for eigenvalue proofs.

13.2 Proof Statistics

Metric	Value
Total lines	7,938
Laplacian construction	Lines 2420–2475
Eigenvalue proofs	Lines 2476–2540
Degeneracy analysis	Lines 2571–2610
Uniqueness	Lines 2650–2700
Axioms	0
Postulates	0

14 Conclusion

We have proven that K_4's Laplacian has spectrum $(0, 4, 4, 4)$: one zero, three fours. This is not a choice—it is forced by complete connectivity.

We hypothesize that this eigenvalue structure corresponds to spacetime's $(3+1)$ dimensions:

- One time dimension (unique zero eigenvalue, global symmetry)
- Three spatial dimensions (degenerate eigenvalue, isotropy)

If correct, dimensionality is not a free parameter. It is derived from the minimal structure forced by distinction itself.

The mathematics is proven. The physics remains to be tested.

Acknowledgments

This work benefited from AI assistance (Claude, ChatGPT, DeepSeek, Perplexity) for proof structuring and LaTeX formatting. All mathematical content is the author's responsibility.

References

[1] The Agda Team. *Agda Documentation*. https://agda.readthedocs.io/

[2] F. R. K. Chung. *Spectral Graph Theory*. American Mathematical Society, 1997.

[3] B. Mohar. *The Laplacian spectrum of graphs*. Graph Theory, Combinatorics, and Applications, 1991.

[4] S. Carlip. *Dimension and Dimensional Reduction in Quantum Gravity*. Classical and Quantum Gravity, 34(19), 2017.

[5] J. Polchinski. *String Theory*, Volume 1. Cambridge University Press, 1998.