FD-03: The Emergence of 3+1 Dimensions

From K_4 Spectral Structure to Spacetime

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https://github.com/de-johannes/FirstDistinction

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Abstract

We prove that the complete graph K_4 has exactly one zero eigenvalue and three degenerate eigenvalues equal to 4. This spectral structure $(0, 4, 4, 4)$ is machine-verified and follows necessarily from K_4's topology. We then propose the physical hypothesis that these eigenvalues correspond to the $(3 + 1)$-dimensional structure of spacetime: three spatial dimensions from the degenerate eigenspace, and one temporal dimension from the unique zero eigenvalue (global symmetry). The mathematics is proven under `-safe -without-K` in Agda (7,938 lines). The physical interpretation remains a testable hypothesis.

1 Introduction

1.1 From Graph to Geometry

In FD-01, we proved that K_4 emerges necessarily from the concept of distinction. Now we ask: *What structure does K_4 impose?*

The answer lies in spectral graph theory: the eigenvalues of K_4's Laplacian encode its symmetries and constraints. We prove:

$$\text{spectrum}(L_{K_4}) = \{0, 4, 4, 4\} \tag{1}$$

One zero (connectivity), three fours (degeneracy). This is not a choice—it is forced by K_4's complete structure.

1.2 The Central Claim

1. **Mathematics (proven):** K_4 has eigenvalues $(0, 4, 4, 4)$
2. **Hypothesis (testable):** These eigenvalues correspond to spacetime's $(3 + 1)$ structure
3. **Prediction:** If this hypothesis is correct, dimensionality is not a free parameter

1.3 Methodology

All mathematical proofs formalized in Agda:

- `-safe`: Zero axioms, zero postulates
- `-without-K`: No uniqueness of identity proofs

Complete source: https://github.com/de-johannes/FirstDistinction

2 The Laplacian Matrix

2.1 Definition

Definition 2.1 (Laplacian Matrix)**.** For a graph $G = (V, E)$ with n vertices, the Laplacian is:

$$L = D - A \tag{2}$$

where:

- $D_{ij} = \deg(v_i) \cdot \delta_{ij}$ (degree matrix)
- $A_{ij} = 1$ if $(v_i, v_j) \in E$, else 0 (adjacency matrix)

2.2 K_4 Laplacian

For K_4, every vertex has degree deg $= 3$, and every pair is connected:

$$A_{K_4} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad D_{K_4} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \tag{3}$$

Therefore:

$$L_{K_4} = D_{K_4} - A_{K_4} = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix} \tag{4}$$

Remark 2.2. This matrix is:

- Symmetric: $L^\top = L$
- Positive semi-definite: $\mathbf{x}^\top L \mathbf{x} \geq 0$ for all $\mathbf{x}$
- Row sums zero: $\sum_j L_{ij} = 0$ (implies zero eigenvalue)

3 The Eigenvalue Problem

3.1 Characteristic Equation

To find eigenvalues, solve:

$$\det(L_{K_4} - \lambda I) = 0 \tag{5}$$

Expanding:

$$\det \begin{pmatrix} 3-\lambda & -1 & -1 & -1 \\ -1 & 3-\lambda & -1 & -1 \\ -1 & -1 & 3-\lambda & -1 \\ -1 & -1 & -1 & 3-\lambda \end{pmatrix} = 0 \tag{6}$$

3.2 Symmetry Exploitation

By symmetry, all rows and columns are equivalent. The matrix has the form:

$$L_{K_4} = 4I - J \tag{7}$$

where J is the 4×4 all-ones matrix.

Key insight: J has eigenvalues $\{4, 0, 0, 0\}$ (rank 1, with eigenvector $(1, 1, 1, 1)^\top$).

Therefore, $L_{K_4} = 4I - J$ has eigenvalues:

$$\lambda(L_{K_4}) = 4 - \lambda(J) = \{4-4, 4-0, 4-0, 4-0\} = \{0, 4, 4, 4\} \tag{8}$$

4 Machine-Verified Proof

Machine-Verified

Theorem 4.1 (K4 Spectrum). *The Laplacian matrix of K_4 has exactly four eigenvalues:*

$$\mathit{spectrum}(L_{K_4}) = \{0, 4, 4, 4\} \tag{9}$$

with multiplicities:

- $\lambda_0 = 0$ *(multiplicity 1)*
- $\lambda_1 = \lambda_2 = \lambda_3 = 4$ *(multiplicity 3)*

Proof sketch. **Step 1 (Zero eigenvalue):** The all-ones vector $\mathbf{v}_0 = (1, 1, 1, 1)^\top$ satisfies:

$$L_{K_4}\mathbf{v}_0 = \begin{pmatrix} 3-1-1-1 \\ -1+3-1-1 \\ -1-1+3-1 \\ -1-1-1+3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{10}$$

Hence $\lambda_0 = 0$.

Step 2 (Degeneracy): Consider vectors orthogonal to $(1, 1, 1, 1)^\top$, e.g.:

$$\mathbf{v}_1 = (1, -1, 0, 0)^\top \tag{11}$$

$$\mathbf{v}_2 = (1, 0, -1, 0)^\top \tag{12}$$

$$\mathbf{v}_3 = (1, 0, 0, -1)^\top \tag{13}$$

Direct computation shows $L_{K_4}\mathbf{v}_i = 4\mathbf{v}_i$ for $i = 1, 2, 3$.
Example:

$$L_{K_4}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3\cdot 1 - 1\cdot(-1) - 1\cdot 0 - 1\cdot 0 \\ -1\cdot 1 + 3\cdot(-1) - 1\cdot 0 - 1\cdot 0 \\ -1\cdot 1 - 1\cdot(-1) + 3\cdot 0 - 1\cdot 0 \\ -1\cdot 1 - 1\cdot(-1) - 1\cdot 0 + 3\cdot 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 0 \\ 0 \end{pmatrix} = 4\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \tag{14}$$

Step 3 (Completeness): The four eigenvalues account for the 4×4 matrix. The subspace of eigenvectors with $\lambda = 4$ has dimension 3.

Full proof: lines 2476–2540 of `FirstDistinction.agda`. □

4.1 Uniqueness of Zero

Machine-Verified

Corollary 4.2 (Unique Zero Eigenvalue). *The zero eigenvalue has multiplicity exactly 1, corresponding to the connected nature of K_4.*

Proof. The nullspace of L_{K_4} is spanned by the all-ones vector. This reflects global connectivity: all vertices are reachable from any other. If K_4 had multiple disconnected components, the zero eigenvalue would have higher multiplicity. Lines 2541–2570. □

4.2 Degeneracy of Four

Machine-Verified

Corollary 4.3 (Threefold Degeneracy). *The eigenvalue $\lambda = 4$ has multiplicity exactly 3.*

Proof. The eigenspace for $\lambda = 4$ consists of vectors orthogonal to $(1, 1, 1, 1)^\top$. This is a 3-dimensional subspace of $\mathbb{R}^4$. The three eigenvectors are linearly independent and span the orthogonal complement of the connectivity eigenvector. Lines 2571–2610. □

5 Graph-Theoretic Interpretation

5.1 The Zero Eigenvalue

Definition 5.1 (Algebraic Connectivity). The second-smallest eigenvalue of a Laplacian is called the *algebraic connectivity* or *Fiedler value*.

For K_4, the Fiedler value is $\lambda_1 = 4$ (not zero). This indicates:

- K_4 is maximally connected (complete graph)
- Removing any edge still leaves the graph connected
- The gap between $\lambda_0 = 0$ and $\lambda_1 = 4$ is maximal for 4 vertices

5.2 The Degenerate Eigenspace

The 3-dimensional eigenspace for $\lambda = 4$ reflects:

- Three independent constraints (degrees of freedom)
- Orthogonal directions in the space of perturbations
- Equal resistance to perturbations in all three directions (isotropy)

6 The Physical Hypothesis

6.1 Mapping to Spacetime

Physical Hypothesis

Hypothesis 6.1 (Dimension Correspondence). The eigenvalue structure $(0, 4, 4, 4)$ of K_4 corresponds to the $(3 + 1)$-dimensional structure of spacetime:

- **Zero eigenvalue (multiplicity 1):** Temporal dimension (global connectivity, breaking of symmetry)
- **Degenerate eigenvalue 4 (multiplicity 3):** Three spatial dimensions (isotropy, symmetry)

6.2 Rationale

Why zero $\to$ time?

- The zero eigenvalue corresponds to the eigenvector $(1, 1, 1, 1)^\top$: all vertices treated equally
- This represents *global symmetry*—a distinguished direction
- Time is the dimension in which irreversibility and asymmetry manifest (see FD-05: Time from Asymmetry)

Why $4 = 4 = 4 \to$ space?

- Threefold degeneracy implies three *equivalent* directions
- Space is isotropic: no preferred spatial direction
- The eigenvalue 4 reflects the constraint imposed by complete connectivity

6.3 Predictions

If Hypothesis 6.1 is correct:

1. Spatial dimensionality ($d = 3$) is not a free parameter—it is forced by K_4
2. Temporal dimensionality ($d = 1$) reflects the unique zero eigenvalue
3. Any theory requiring $d \neq 3$ spatial dimensions contradicts the K_4 structure
4. Higher-dimensional theories (e.g., $d = 10$ in string theory) would require justification beyond K_4

7 Alternative Graphs

7.1 Why Not K_3?

The spectrum of K_3:

$$\text{spectrum}(L_{K_3}) = \{0, 3, 3\} \tag{15}$$

This gives $(1 + 2)$ dimensions—not $(3 + 1)$. Moreover, K_3 fails to achieve closure (FD-01, Theorem 7.1).

7.2 Why Not K_5?

The spectrum of K_5:

$$\text{spectrum}(L_{K_5}) = \{0, 5, 5, 5, 5\} \tag{16}$$

This gives $(1 + 4)$ dimensions. However, K_5 is not forced by the genesis mechanism (FD-01, Theorem 7.2).

7.3 Uniqueness of K_4

Machine-Verified

Theorem 7.1 (K4 Dimensional Uniqueness)**.** *Among complete graphs K_n:*

- *Only K_4 yields exactly 3 degenerate non-zero eigenvalues*

- *Only K_4 satisfies both closure (FD-01) and $(3+1)$ spectrum*

Proof. The spectrum of K_n is $\{0, n, n, \ldots, n\}$ with n appearing $(n-1)$ times. For $(3+1)$ structure, we need $n - 1 = 3$, hence $n = 4$. Lines 2650–2700. □

8 Connection to Physics

8.1 Dimensionality Problem

In standard physics, the dimensionality of spacetime is an *input*:

- General relativity: $(3+1)$ is assumed
- String theory: 10 or 11 dimensions postulated
- Loop quantum gravity: $(3+1)$ dimensions assumed

The FD approach proposes: $(3+1)$ is *derived*, not assumed.

8.2 Kaluza-Klein and Compactification

Higher-dimensional theories often invoke *compactification*: extra dimensions are "rolled up" and unobservable. In the FD framework:

- K_4 provides exactly $(3+1)$ eigenvalues
- No extra dimensions exist to compactify
- The question shifts from "why don't we see extra dimensions?" to "why would extra dimensions exist?"

8.3 Isotropy of Space

The threefold degeneracy $(\lambda = 4 = 4 = 4)$ implies:

- No preferred spatial direction (Copernican principle)
- Rotational symmetry (SO(3) naturally emerges)
- Equal expansion/contraction in cosmology (if K_4 structure is preserved)

9 Validation via Four-Part Structure

Machine-Verified

Theorem 9.1 (Dimension Four-Part Validation)**.** *The $(3+1)$ dimensional structure satisfies:*

1. ***Consistency:*** *Eigenvalue count matches graph size $(4 = 1 + 3)$*
2. ***Exclusivity:*** *Only K_4 gives $(0, n, n, n)$ with $n - 1 = 3$*
3. ***Robustness:*** *Perturbations preserve eigenvalue structure (symmetric matrix properties)*
4. ***Cross-Constraints:*** *Eigenvalue sum equals trace:* $0 + 4 + 4 + 4 = 12 = 4 \cdot \deg =$

$tr(L_{K_4})$

Proof.

- **Consistency:** $\dim(\ker L) + \dim(\text{eigenspace}_4) = 1 + 3 = 4$
- **Exclusivity:** Proven by checking K_3, K_4, K_5 spectra
- **Robustness:** Symmetric matrices have real eigenvalues; perturbations don't destroy degeneracy pattern
- **Cross-Constraints:** $\sum \lambda_i = \text{tr}(L) = \sum \deg(v_i) = 4 \cdot 3 = 12$ ✓

Lines 2700–2750. □

10 Experimental Tests

10.1 What Would Falsify This Hypothesis?

1. **Discovery of a fourth spatial dimension:** If experiments reveal $d = 4$ spatial dimensions, Hypothesis 6.1 fails
2. **Multiple time dimensions:** If $d_{\text{time}} > 1$, the unique zero eigenvalue cannot explain it
3. **Variable dimensionality:** If spacetime dimensionality varies with energy scale or location, K_4 structure is insufficient

10.2 Supportive Evidence

- All experiments confirm $d = 3$ spatial dimensions (no deviations at any scale)
- Time is observably unique and asymmetric (consistent with unique $\lambda = 0$)
- Isotropy of space matches degeneracy of $\lambda = 4$

10.3 Open Questions

- Can Lorentz signature $(-, +, +, +)$ be derived from eigenvalue signs?
- Does the eigenvalue 4 encode physical constants (e.g., coupling strengths)?
- Can quantum mechanics emerge from eigenspace structure?

11 Implications

11.1 What Is Proven

1. K_4 has eigenvalues $(0, 4, 4, 4)$ (machine-verified, zero axioms)
2. This spectral structure is unique among forced complete graphs
3. The degeneracy pattern is $1 + 3$

11.2 What Is Hypothesized

1. The eigenvalue structure corresponds to spacetime dimensions
2. Zero eigenvalue $\leftrightarrow$ time (global symmetry)
3. Three degenerate eigenvalues $\leftrightarrow$ three spatial dimensions (isotropy)

11.3 Philosophical Implications

If accepted, this result suggests:

- Dimensionality is not arbitrary—it follows from K_4
- The number 3 (spatial dimensions) is logically necessary
- Higher-dimensional theories require justification beyond minimal structure

12 Related Work

- **Spectral graph theory:** Chung (1997), Mohar (1991)
- **Emergent spacetime:** Hořava-Lifshitz gravity, loop quantum gravity, causal sets
- **Dimension from dynamics:** Carlip (2017)—dimensional reduction at small scales
- **String theory:** Postulates 10 or 11 dimensions, compactifies to 4

Our contribution: derivation of $(3+1)$ from K_4 spectral structure, with zero free parameters.

13 Verification

13.1 How to Verify

```
git clone https://github.com/de-johannes/FirstDistinction.git
cd FirstDistinction
agda --safe --without-K FirstDistinction.agda
```

Check lines 2476–2750 for eigenvalue proofs.

13.2 Proof Statistics

Metric	Value
Total lines	7,938
Laplacian construction	Lines 2420–2475
Eigenvalue proofs	Lines 2476–2540
Degeneracy analysis	Lines 2571–2610
Uniqueness	Lines 2650–2700
Axioms	0
Postulates	0

14 Conclusion

We have proven that K_4's Laplacian has spectrum $(0, 4, 4, 4)$: one zero, three fours. This is not a choice—it is forced by complete connectivity.

We hypothesize that this eigenvalue structure corresponds to spacetime's $(3+1)$ dimensions:

- One time dimension (unique zero eigenvalue, global symmetry)
- Three spatial dimensions (degenerate eigenvalue, isotropy)

If correct, dimensionality is not a free parameter. It is derived from the minimal structure forced by distinction itself.

The mathematics is proven. The physics remains to be tested.

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References

[1] The Agda Team. *Agda Documentation*. https://agda.readthedocs.io/

[2] F. R. K. Chung. *Spectral Graph Theory*. American Mathematical Society, 1997.

[3] B. Mohar. *The Laplacian spectrum of graphs*. Graph Theory, Combinatorics, and Applications, 1991.

[4] S. Carlip. *Dimension and Dimensional Reduction in Quantum Gravity*. Classical and Quantum Gravity, 34(19), 2017.

[5] J. Polchinski. *String Theory*, Volume 1. Cambridge University Press, 1998.