The following is a mock midterm exam. Try to solve it in 100min to practice for the final exam. This mock exam counts as a bonus homework. However, peer-to-peer assessment and grading will only be done on the full questions at the end (Mid 1-3, 30 points total).

1. Which of the following statements is/are giving the right order of convergence for the three nonlinear solvers: Newton, bisection and secant method (in that order)
   A. order of convergence 2, 1, and ≈1.6 B. order of convergence 3, 1, and ≈1.5 C. quadratic, superlinear, and linear D. order of convergence 2, 1, and 1.5 E. cubic, linear, and linear F. quadratic, linear, and superlinear

   Solution: A, F

2. What conditions need to hold for the bisection method to converge to a root of the function $f(x)$ within the interval $[a, b]$?

   A. $f'(a) \neq 0$ B. $f(a)f(b) < 0$ C. $f(a) < f(b)$ D. $f \in C^2([a, b])$ E. $f \in C^1([a, b])$ F. $f \in C([a, b])$

   Solution: B, F

3. What are the correct values for the coefficients a and b for the least squares approximation for $h(x) = a + bx$ and the data pairs $(0, 2)$ and $(1, 5)$? A. $a = 1$ B. $a = 2$ C. $a = -1$ D. $b = 0$ E. $b = -3$ F. $b = 3$

   Solution: B, F

4. What is the solution after the first step of the Secant Method for the function $f(x) = x^2 + 2x - 7$ with the initial iterations $x_1 = 1$ and $x_0 = 0$?

   A. $x_2 = 1/3$ B. $x_2 = 1/4$ C. $x_2 = 5/4$ D. $x_2 = 7/3$ E. $x_2 = 12/5$

   Solution: D

5. Given the following data:

x	0	2	3
y	1	3	2

   What is the collocation matrix for the Newton interpolation?

   A. $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix}$ B. $\begin{pmatrix} 2 & 3 & 3 \\ 2 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}$ C. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 3 & 0 & 0 \end{pmatrix}$ D. $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 6 \end{pmatrix}$ E. $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 3 \end{pmatrix}$

   Solution: E

6. Use central differencing to approximate the derivative of $f(x) = x^3$ at $x = 2$ and with a discretization step of $h = 1$.

   A. $f'(2) \approx 26$ B. $f'(2) \approx -11$ C. $f'(2) \approx 12$ D. $f'(2) \approx -9$ E. $f'(2) \approx 13$
   F. $f'(2) \approx 16$

   Solution: E

7. Use Taylor expansion for $f(x) = x^2$ at $x = 0$ to approximate $f(2)$. What is the error E if we only use the first three terms for the approximation of $f(2)$?

   A. $E = 1/2$ B. $E = 0.1$ C. $E = 0$ D. $E \approx 1$ E. $E > 0.2$ F. $E \approx -0.5$

   Solution: C

8. Using the trapezoidal rule, integrate $f(x) = x^3$ between $a = 0$ and $b = 2$ with a discretization step of $h = 1$. What is the approximated value of the integral?
A. 3 B. -4 C. 4 D. 5 E. 1.5 F. -6

Solution: D

Problem Mid 1

(6+4 points)
Consider the interpolation problem

i	0	1	2
x_i	0	π	$\frac{3\pi}{2}$
p_i	-1	2	-4

We try to fit the function $p(x) = a\sin(x) + b\cos^2(x)$.

a) Solve the resulting overdetermined system to find the free coefficients a and b by either using the 3×2 collocation matrix and its transpose or by writing down the system of normal equations. Both methods are based on the minimization of the sum of squared errors.

b) Use the same nodes values to do Newton interpolation. Calculate for both least square and the polynomial interpolation the value at $x = \frac{\pi}{2}$.

Solution:
a) We need to solve $\Phi\Phi^t \begin{pmatrix} a \\ b \end{pmatrix} = \Phi\vec{p}$ with the value vector $\vec{p}$ and coefficients a, b, and Φ the collocation matrix.
The collocation matrix is here
$$\Phi = \begin{pmatrix} \sin(x_0) & \cos^2(x_0) \\ \sin(x_1) & \cos^2(x_1) \\ \sin(x_2) & \cos^2(x_2) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and so we get, using Φ and Φ^t,

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

Which gives us $a = 4$ and $b = \frac{1}{2}$.

b) For Newton we get the Newton polynomials $P_0(x) = 1$, $P_1(x) = (x - x_0) = x$, and $P_2(x) = (x - x_0)(x - x_1) = x(x - \pi) = x^2 - \pi x$. We have the collocation matrix Φ_N for the Newton method

$$\begin{pmatrix} P_0(x_0) & P_1(x_0) & P_2(x_0) \\ P_0(x_1) & P_1(x_1) & P_2(x_1) \\ P_0(x_2) & P_1(x_2) & P_2(x_2) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & \pi & 0 \\ 1 & \frac{3\pi}{2} & \frac{3\pi^2}{4} \end{pmatrix}$$

which gives us (after solving the linear system of equations via forward substitution) the coefficients $\alpha_0 = p_0 = -1$, $\alpha_1 = \frac{3}{\pi}$, and $\alpha_2 = \frac{-10}{\pi^2}$.

For $x = \frac{\pi}{2}$ we get:
for least squares $y = 4$
for Newton $y = 3$

Problem Mid 2

(3+4+3 points)
We want to evaluate the integral $\int_0^1 2x^2 - 2x - 1 \, \mathrm{d}x$ numerically with an error less than 1/6.

a) How many intervals n do we need to satisfy this error bound when using the composite trapezoid rule?

b) Apply the composite trapezoid rule with the number of intervals that you found in (a) to obtain an approximation for the value of the integral.

c) With the help of the recursive trapezoidal rule, use the Romberg algorithm to find R_2^2. How large is the error compared to the analytical solution? Why do you observe this error?

Solution:

a) We need to get the error $|\int_0^1 2x^2 - 2x - 1 \, \mathrm{d}x - T_h(f,P)| = \frac{1}{12}|(b-a)h^2 f''(\xi)| < \frac{1}{6}$.
$f''(x) = 4$ which is the same everywhere (for any ξ), and $(b-a) = 1$. So we get $\frac{1}{12}4h^2 = \frac{1}{3}h^2 < \frac{1}{6}$, and with $h = \frac{b-a}{n} = \frac{1}{n}$ we get $\frac{1}{n^2} < \frac{1}{2}$, so $n > \sqrt{2}$, e.g. $n = 2$.

b) We need two intervals, so we split $[0,1]$ into two, i.e. $h = 0.5$ and $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$. We get for this specific partition P

$T_h(f,P) = h(\frac{f(x_0)+f(x_1)}{2}) + h(\frac{f(x_1)+f(x_2)}{2}) = \frac{1}{4}(-1-1.5) + \frac{1}{4}(-1.5-1) = -\frac{5}{4}$.

c) Using the trapezoidal rule for $n = 1$ (i.e. $h = 1$), $n = 2$ (as in b), i.e. $h = 0.5$) and $n = 4$ (i.e. $h = 0.25$) we get values $R_0^0 = -1$, $R_1^0 = -\frac{5}{4}$, and $R_2^0 = -\frac{21}{16}$. Using the Romberg algorithm, we get $R_1^1 = R_1^0 + \frac{1}{3}(R_1^0 - R_0^0) = -\frac{4}{3}$ and $R_2^1 = R_2^0 + \frac{1}{3}(R_2^0 - R_1^0) = -\frac{4}{3}$. Since they are both the same, and the Romberg algorithm uses the difference between two earlier stages (multiplied by a changing weight), we can conclude without any further computation that $R_2^2 = R_2^1 + \frac{1}{15}(R_2^1 - R_1^1) = -\frac{4}{3}$.
The function is a polynomial. We can easily integrate. The result is $[\frac{2}{3}x^3 - x^2 - x]_0^1 = -\frac{4}{3}$ and the error of R_2^2 is zero. Since the second Romberg column R_*^1 already has an error of order h^4 it depends on the forth derivative. This is zero in the case of a second order polynomial. So the algorithm is exact (up to potential rounding errors) already for the second column and therefore also for the third.

Problem Mid 3

(3+4+3 points)

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 8 & 8 \\ 3 & 8 & 11 \end{pmatrix}$

a) Show that the matrix A is positive definite. Why is this relevant for the rest of this problem?

b) Compute the LU decomposition of A.

c) Compute the Cholesky decomposition of A.

Solution:

a) There are several ways to show this. We could use the definition directly:
$\vec{v}^t A\vec{v} > 0$ for all $\vec{v} \in \mathbb{R}\backslash 0$ (without zero vector).
This is not straight forward. We could show that all the eigenvalues are larger than zero. This also takes some work. We can also show that all leading principal minors of A are all positive. That is not so difficult.
The first one is just 1 (top left value). The second is
$\left|\begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix}\right| = 4$ The final one, the determinant of the full matrix, is also 4. All are larger zero, so the matrix is positive definite. We need this to be able to find the Cholesky decomposition. Actually, if we can find the Cholesky decomposition the matrix is positive definite (therefore we could do c) to show a)). Since we can do Cholseky decomposition, we can also do LU decomposition.

b) The LU decomposition is given by

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0.5 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

b) The Cholesky decomposition is given by

$$\tilde{L} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 1 & 1 \end{pmatrix}$$

$$\tilde{L}^t = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$