Introduction to Robotics

Author: Daria Shutina

23-02-03

Sections of mechanics

Statics is concerned with the analysis of loads (force and torque, or "moment") acting on physical systems that do not experience an acceleration (a=0), but rather, are in static equilibrium with their environment.

Kinematics describes the motion of points, bodies (objects), and systems of bodies (groups of objects) without considering properties of objects (mass, density) or the forces that caused the motion.

Kinetics is concerned with the relationship between motion and its causes, specifically, forces and torques.

Dynamics studies forces and their effect on motion.

23-02-07

Vector

Vector norm

A norm is a function $f : R^n \to R$ that satisfies 4 properties:

1. Non-negativity: $\forall x \in R^n \ : \ f(x) \geqslant 0$
2. Definiteness: $f(x) = 0 \Rightarrow x = 0$
3. Homogeneity: $\forall x \in R^n, t \in R \ : \ f(tx) = |t| f(x)$
4. Triangle inequality: $\forall x, y \in R^n \ : \ f(x+y) \leqslant f(x) + f(y)$

Euclidean vector norm (2-norm): $||x||_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$

$||x||_2 = \sqrt{x^T x}$

General p-norms, $p \geqslant 1$: $||x||_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$

$||x||_\infty = \max_i |x_i|$

Dot product

It is an operation between two vectors that results in a scalar (a single value).

$$[x_1 \dots x_n] \cdot \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \sum_1^n x_i y_i$$

Cross product

It is an operation between two vectors that results in a third vector perpendicular to the plane formed by the original vectors.

$$\mathbf{a} = \langle 1, 3, 4 \rangle \qquad \mathbf{b} = \langle 2, 7, -5 \rangle$$

$$\mathbf{a} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k} \qquad \mathbf{b} = 2\mathbf{i} + 7\mathbf{j} - 5\mathbf{k}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} = \overbrace{\begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \mathbf{k}}^{\text{раскрытие вдоль строки матрицы}}$$

$$\mathbf{a} \times \mathbf{b} = (-15 - 28)\mathbf{i} - (-5 - 8)\mathbf{j} + (7 - 6)\mathbf{k}$$

$$\mathbf{a} \times \mathbf{b} = -43\mathbf{i} + 13\mathbf{j} + \mathbf{k}$$

Properties:

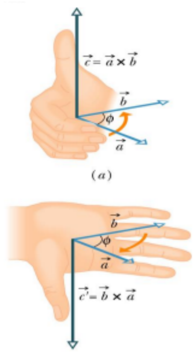
- $a \times a = 0$
- $||a \times b|| = ||a|| \cdot ||b|| \cdot \sin\theta$ where θ is the angle between a and b
- if a and b are parallel, then $a \times b = 0$ $(since \ \sin\theta = 0)$
- cross product is not commutative: $a \times b = -b \times a$
- cross product is not associative: $a \times (b \times c) \neq (a \times b) \times c$
- cross product is distributive: $a \times (b + c) = a \times b + a \times c$

Direction of the result vector

$C = A \times B$. The result vector C is perpendicular to both the original vectors A and B.

To find the direction, we use a right-hand rule:

- Place A and B tail to tail
- Determine the position of the right hand:
  - Four fingers are parallel to the plane formed by A and B
  - The thumb is perpendicular to four fingers
- Curl four fingers from A to B using the smaller angle. The thumb points the direction of the result vector C.



Matrix operations

1. Addition
2. Scaling: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot 3 = \begin{pmatrix} 3a & 3b \\ 3c & 3d \end{pmatrix}$
3. Dot product (inner product): $(x_1 \dots x_n) \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (a_1)$
4. Multiplication -- consists of several dot product operations

   It is a right-associative.
5. Transposition: $A^T \ : \ a_{ij}^T = a_{ji}, \ \forall i, j$

   Also, $(ABC)^T = C^T B^T A^T$
6. Inverse
7. Determinant
8. Power (only for square matrices)
9. Trace -- sum of elements on the diagonal

   $\mathrm{tr} AB = \mathrm{tr} BA$

   $\mathrm{tr}(A + B) = \mathrm{tr} A + \mathrm{tr} B$
10. etc (maybe)

Multiplication

- associative: $(AB)C = A(BC)$
- distributive: $A(B + C) = AB + AC$
- non-commutative: $AB \neq BA$

Determinant

Properties:

- $\det AB = \det BA$
- $\det A^{-1} = \frac{1}{\det A}$
- $\det A^T = \det A$

Inverse

Given a matrix A, its inverse is a matrix A^{-1} such that $AA^{-1} = A^{-1}A = 1$.

$\exists A^{-1} \quad \Rightarrow A$ is invertible and non-singular. Otherwise, it is singular.

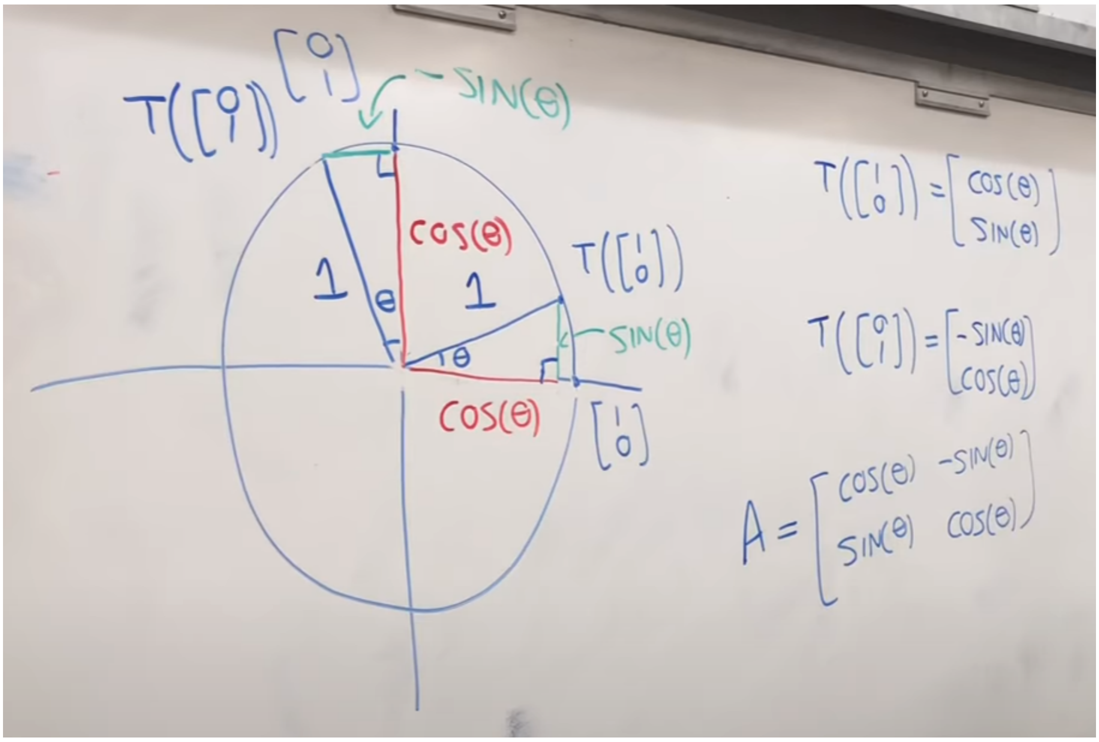
Inverse matrices do not exist for non-square matrices.

Properties:

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$

23-02-10

2d rotation matrix



A 2-dimensional vector consists of basic vectors $(1,0)$ and $(0,1)$ multiplied by some constants. Rotating the basic vectors, we rotate the original vector.

As a result, the rotation matrix $R = \begin{pmatrix} T[\begin{pmatrix} 1 \\ 0 \end{pmatrix}] & T[\begin{pmatrix} 0 \\ 1 \end{pmatrix}] \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Counter-clockwise by the angle θ:

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Clockwise by the angle θ:

$$R^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Properties of R:

- $R \cdot R^T = E$
- $\det R = 1$

Homogeneous coordinates

Adding a constant

In order to be able to add a constant, we add `1` to every vector:

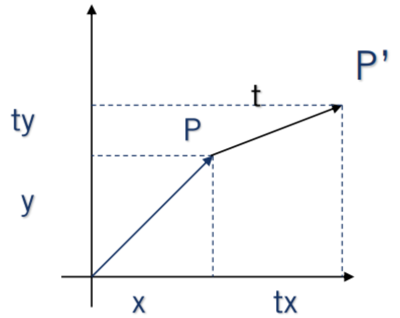
$$\begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} ax+by+c \\ dx+ey+f \\ 1 \end{pmatrix}.$$

Such systems are called *homogeneous*. A homogeneous transformation matrix will have a row `[0 0 1]` at the bottom, so there will be `1` at the bottom of the result matrix.

2d translation

The matrix have the form of $T = \begin{pmatrix} 1 & 0 & \dots & t_1 \\ 0 & 1 & \dots & t_2 \\ 0 & 0 & \dots & t_3 \\ & \dots & & \\ 0 & 0 & \dots & 1 \end{pmatrix}$.

$\begin{pmatrix} t_1 \\ \dots \\ t_n \end{pmatrix}$ is an addition of a constants to the initial coordinates.



$$\mathbf{P} = (x, y) \to (x, y, 1)$$

$$\mathbf{t} = (t_x, t_y) \to (t_x, t_y, 1)$$

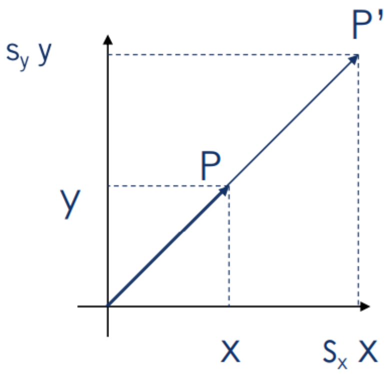
$$\mathbf{P'} \to \begin{bmatrix} x+t_x \\ y+t_y \\ 1 \end{bmatrix} = \overset{\qquad\quad \mathrm{t}}{\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}} \cdot \overset{\mathrm{P}}{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}$$

$$= \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ 0 & 1 \end{bmatrix} \cdot \mathbf{P} = \mathbf{T} \cdot \mathbf{P}$$

Scaling

The matrix has a form of $\begin{pmatrix} s_1 & 0 & \dots & 0 & 0 \\ 0 & s_2 & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & s_n & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$.

$s_1, \dots, s_n$ are scalars for the initial coordinates.



$$\mathbf{P} = (\mathrm{x}, \mathrm{y}) \to \mathbf{P'} = (\mathrm{s_x x}, \mathrm{s_y y})$$

$$\mathbf{P} = (x, y) \to (x, y, 1)$$

$$\mathbf{P'} = (s_x x, s_y y) \to (s_x x, s_y y, 1)$$

$$\mathbf{P'} \to \begin{bmatrix} s_x x \\ s_y y \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{S}} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{S'} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \cdot \mathbf{P} = \mathbf{S} \cdot \mathbf{P}$$

Linear independence

I is a set of indexes.

The system $\{v_i\}_{i \in I}$ is *linearly dependent*, if one of the statements below are true:

1. $\nexists i \in I \ : \ v_i = \sum\limits_{j \neq i} v_j$
2. $\forall \{a_i\} \in R \ : \ \Sigma a_i v_i = 0 \ \Rightarrow \ a_i = 0, \forall i \in I$.

Matrix rank

Suppose we have a $m \times m$ matrix.

If its rank is m then it is *full rank* and it has an inverse matrix.

If its rank is $< m$ then it is *singular* and does not have an inverse matrix. We also cannot restore the input.

Eigenvectors and eigenvalues

Given a matrix $\mathcal{A}$ of size $n \times n$.

$v \in V / \{0\}$ is an *eigenvector* if $\exists \lambda \ : \ \mathcal{A}(v) = \lambda v$.

λ is an *eigenvalue*.

$p_{\mathcal{A}}(\lambda) = \det(\mathcal{A} - \lambda E)$ is a *characteristic polynomial* of matrix $\mathcal{A}$.

Th. (why do we need to substract λ on the diagonal of the matrix)

λ is an eigenvalue $\Leftrightarrow (\mathcal{A} - \lambda E)(v) = 0 \Leftrightarrow \det(A - \lambda E) = 0$

Proof:

$\mathcal{A} - \lambda v = 0 \ \Leftrightarrow \ (\mathcal{A} - \lambda E)v = 0 \ \Leftrightarrow Ker(\mathcal{A} - \lambda E) \neq \varnothing \ \underset{(*)}{\Leftrightarrow} \ \det(\mathcal{A} - \lambda E) = 0.$

For $(*)$, $\det(\mathcal{A} - \lambda E) = 0 \Leftrightarrow \mathcal{A} - \lambda E$ is not inversible $\Leftrightarrow Im(\mathcal{A} - \lambda E) \neq V \Leftrightarrow Ker(\mathcal{A} - \lambda E) \neq \varnothing$.

Th. (relation between eigenvalue and characteristic polynomial)

λ is an eigenvalue $\Leftrightarrow$ λ is the root of the characteristic polynomial

Proof:

λ is an eigenvalue $\Leftrightarrow$ $\det(\mathcal{A} - \lambda E) = p_{\mathcal{A}}(\lambda) = 0$

Example

$\mathcal{A} = \begin{pmatrix} 3 & -1 \\ -3 & 5 \end{pmatrix}$.

$p_{\mathcal{A}}(\lambda) = \det \begin{pmatrix} 3-\lambda & -1 \\ -3 & 5-\lambda \end{pmatrix} = \lambda^2 - 8\lambda + 12 = 0.$

$\lambda_1 = 6,\ \lambda_2 = 2.$

$\mathcal{A}v_1 = \lambda_1 v_1 \ \Rightarrow\ v_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$. Normalized $\frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ $\quad\text{(length is equal to 1)}$

$\mathcal{A}v_2 = \lambda_2 v_2 \ \Rightarrow\ v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Normalized $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Diagonolization

Typically an $n \times n$ matrix has n different eigenvalues and n associated eigenvectors.

If there are n independent eigenvectors, they can be used as a basis for V. Thus, we get a matrix $B = (v_1 \quad \dots \quad v_n)$ of the size $n \times n$.

The diagonal matrix for $\mathcal{A}$ is $D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_2 & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & \lambda_n \end{pmatrix}$.

Finally, we come to $\mathcal{A} = BDB^{-1}$. You should be able to confirm this statement :^)

Trace and determinant

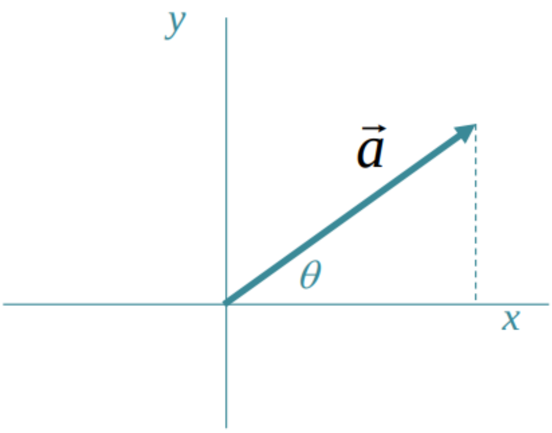
$$p_{\mathcal{A}}(\lambda) = \det(\mathcal{A} - \lambda E) = \lambda^n - \mathrm{tr}\mathcal{A} \cdot \lambda^{n-1} + \ldots + (-1)^n \cdot \det \mathcal{A}.$$

$$\mathrm{tr}\mathcal{A} = \mathrm{tr}D = \lambda_1 + \ldots + \lambda_n.$$

$$\det \mathcal{A} = \det D = \lambda_1 \cdot \ldots \cdot \lambda_n.$$

23-02-14

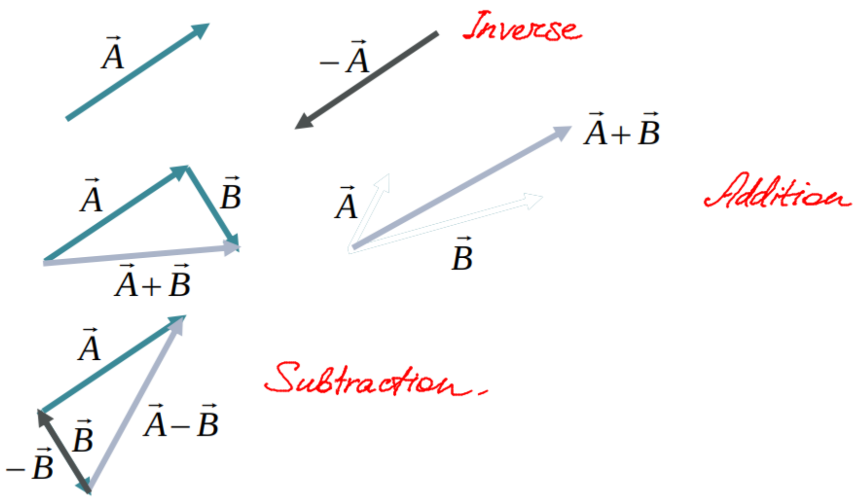
Vector math



$$a_x = a\cos\theta,\ a_y = a\sin\theta$$

$$a = \sqrt{a_x^2 + a_y^2}$$

$$\theta = \arctan\frac{a_y}{a_x}$$



23-02-17

3d rotation

Rotate over the angle α around the z axis:

$x' = x\cos\alpha - y\sin\alpha$

$y' = x\sin\alpha + y\cos\alpha$

$z' = z$

Counter-clockwise direction

- around z-axis:

$$R_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- around y-axis:

$$R_y(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- around x-axis:

$$R_z(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

A square matrix R is a **rotation matrix** if $R^T = R^{-1}$ and $\det R = 1$.

$Rv = v \ \Rightarrow v$ is a rotation matrix.

$Tr(R) = m - 2 + 2\cos\theta$ where m is the matrix's size and θ is the angle of rotation.

23-02-21

Quaternions

Usually, quaternions are expressed as a scalar and a 3-dimensional vector:

$q = (a, v) = (a, b, c, d)$ where a is a scalar, (b, c, d) is a vector.

$q = (a, b, c, d) = a + bi + cj + dk$

i, j, k are basis vectors of 3-dimensional matrix. They satisfy the rule:

$i^2 = j^2 = k^2 = ijk = -1$

$ij = k, jk = i, ki = j,$
$ji = -k, kj = -i, ik = -j$

Proof:

$$1 = (1,0,0,0) \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 1$$

$$i = (0,1,0,0) \longrightarrow \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = i$$

$$j = (0,0,1,0) \longrightarrow \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = j$$

$$k = (0,0,0,1) \longrightarrow \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = k$$

$$i^2 = \left(\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \right) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = -1$$

$ijk = -1 \underset{i\cdot\ldots}{\Rightarrow} -jk = -i \Rightarrow jk = i$

Operations

Addition / subtraction

$q_1 = (a_1, v_1)$ and $q_2 = (a_2, v_2)$

$q_1 \pm q_2 = (a_1 \pm a_2, v_1 \pm v_2)$

Addition / subtraction are commutative and associative

Multiplication

$q_1 = a_1 + b_1 i + c_1 j + d_1 k$

$q_2 = a_2 + b_2 i + c_2 j + d_2 k$

$q_1 \cdot q_2 = (a_1 + b_1 i + c_1 j + d_1 k) \cdot (a_2 + b_2 i + c_2 j + d_2 k) =$ just expand brackets

Multiplication is associative, but not commutative

The **norm** $||q|| = ||(a, v)|| = ||(a, b, c, d)|| = \sqrt{a^2 + b^2 + c^2 + d^2}$

Quaternion **conjugate** (сопряженная матрица) of $q = (a, v)$ is $q^* = (a, -v)$.

$q \cdot q^* = |q|^2 \Rightarrow$ Quaternion **inverse** $q^{-1} = \frac{q^*}{|q|^2}$

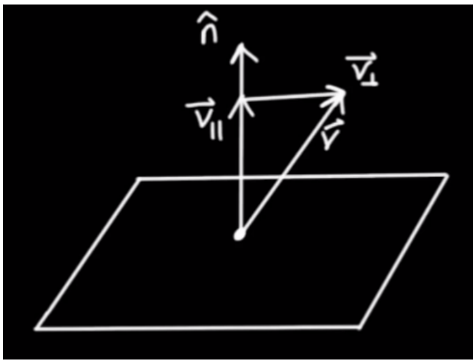
Get a 4-dimensional matrix

$$q = (a, b, c, d) = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}$$

23-02-24

3d-rotations

Rodriquez Rotation



We want to rotate vector v around axis n with the angle θ.

$v_{||}$ is a projection of v on axis n. $v_{\perp}$ is a projection on the plain.

$v = v_{||} + v_{\perp}$ -- the initial vector

$v' = v'_{||} + v'_{\perp}$ -- the result vector

$v'_{||} = v_{||} = (v \cdot n) \times n$ (n is normalized, value in brackets is a scalar)

$v' = \cos\theta \cdot v + (1 - \cos\theta) \cdot v_{ll} + \sin\theta \cdot (n \times v)$

Quaternions

Given a point $\bar{p}$. We want to rotate $\bar{p}$ by the angle θ around the axis $\bar{n}$.

$q = (\cos\frac{\theta}{2}, \frac{\bar{n}}{||n||}\sin\frac{\theta}{2})$.

The resulting point is equal to $p' = qpq^*$ $(\text{ if } q = (a, v), \text{ then } q^* = (a, -v)\ )$

Example

Given a point $P = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Rotate the point by 60° around the axis $n = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$p = (0, P) = (0 \quad 1 \quad 2 \quad 3)$.

$q = (\cos 30^\circ, \frac{n}{\sqrt{3}}\sin 30^\circ) = \begin{pmatrix} \cos 30^\circ & \frac{1}{\sqrt{3}}\sin 30^\circ & \frac{1}{\sqrt{3}}\sin 30^\circ & \frac{1}{\sqrt{3}}\sin 30^\circ \end{pmatrix}$

Intrinsic / extrinsic rotations

Intrinsic rotations, also known as body-fixed rotations or rotations in local coordinates, involve rotating an object around its own coordinate axes. This means that the axes of rotation are attached to the object itself and move with the object as it rotates.

Intrinsic rotations are often described using a sequence of rotations (such as yaw, pitch, and roll), where each rotation occurs around one of the object's local axes. The order of rotation matters.

Extrinsic rotations, also known as world-fixed rotations or rotations in global coordinates, involve rotating an object around fixed external coordinate axes. The order of rotation also matters.

23-02-28

For simplicity, we'll often use the scalar version of the equations.

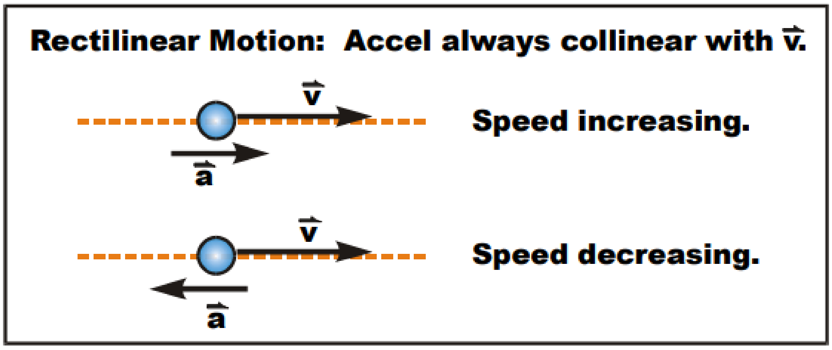
Mind:

All eqns refer to a known path, and vel / accel is along that path, only.

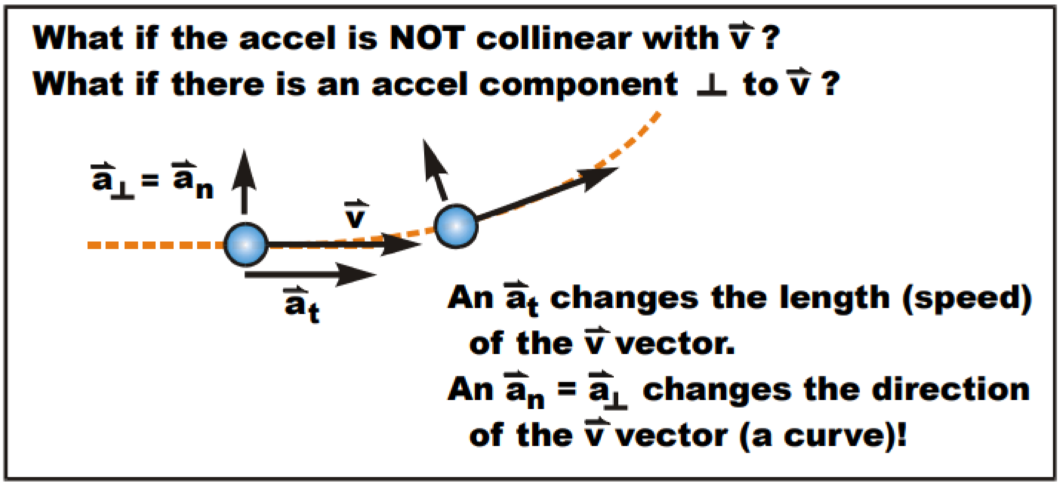
Elim dt from ① and ② gives ③ $a\,ds = v\,dv$

Defining Kinematic Eqns	
Scalar	**Vector**
① $v = \frac{ds}{dt}$	$\vec{v} = \frac{d\vec{r}}{dt}$
② $a = \frac{dv}{dt}$	$\vec{a} = \frac{d\vec{v}}{dt}$
③ $a\,ds = v\,dv$	

Key feature of ***straight line*** motion: Acceleration is always collinear with the velocity. Examples:



What if accel is NOT collinear with the velocity? You would have curvilinear motion. (Tangential / normal components of accel)



Straight line motion: $\bar{a}$ as a constant

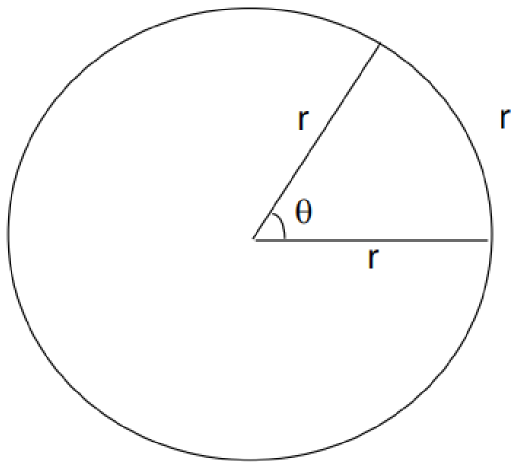
$$v = v_0 + at$$

$$s = s_0 + v_0 t + \frac{at^2}{2}$$

$$v^2 = v_0^2 + 2a(s - s_0)$$

Curvilinear motion

Measuring angles



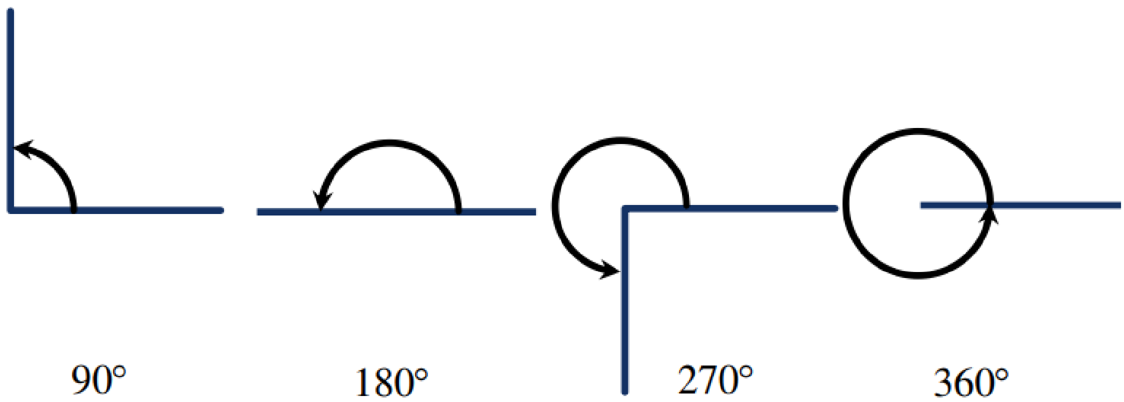
$\theta = 1\,rad = 57.3°$

$360° = 2\pi\,rad$

What is a radian?

- a unitless measure of angles
- the SI unit for angular measurement
- the arc / radius

1 radian is the angular distance covered when the arclength equals the radius



90°	180°	270°	360°
$\frac{\pi}{2}\,rad$	$\pi\,rad$	$\frac{3\pi}{2}\,rad$	$2\pi\,rad$
$\frac{1}{4}\,rev$	$\frac{1}{2}\,rev$	$\frac{3}{4}\,rev$	$1\,rev$

Absolute angle (segment angle) -- angle between a segment (a line or object) and the horizontal of the distal end (the outermost point) of the segment. In simple words, it is an angle between the line (it represents the object's direction) and the horizontal line.

Relative angle (joint angle) -- angle between the longitudinal axis of two adjacent segments. In simpler terms, it's the angle between the lines that represent the direction of two connected objects. The value $\leqslant 90$ is taken (it means if $\alpha > 90$, then $180 - \alpha$ is taken).

Angular and linear velocity

Angular velocity w is a measure of how quickly an object is rotating or spinning (around a particular axis). If you curl your fingers in the direction of rotation, your thumb points in the direction of the angular velocity vector.

$w = \frac{\Delta\theta}{\Delta t}\ (rad/s)$.

$\Delta\theta$ is in radians. It is the angle between two segments.

Lets have the initial and the final segments. Then:

$$w_f = w_i + at$$
$$\theta_f = \theta_i + w_i t + \frac{at^2}{2}$$
$$w_f^2 = w_i^2 + 2a(\theta_f - \theta_i)$$

Every spinning object has both angular and linear velocity. **Linear velocity** v is a measure of how quickly an object is moving along a straight path. The direction of linear velocity is along the path of motion. It is also called **tangential velocity**.

Linear velocity $v = wR\ (m/s)$.

The following formulas convert **angular parameters**, for a circle of radius r, to **linear parameters along the arc** (magnitudes):

$$s = \theta r$$
$$v = \omega r$$
$$\text{(tangential)}\quad a_t = \alpha r$$
$$\text{(centripetal)}\quad a_c = \omega^2 r \text{ or } v^2/r$$

Challenge (biking tour): Write $\vec{r} = r\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$
and take derivatives.
The radius r is fixed

Accelerations

Angular acceleration measures how quickly the angular velocity changes.

$$\alpha = \frac{\Delta w}{\Delta t}$$

Tangential acceleration is the measure of how quickly a tangential velocity changes. Its direction is perpendicular to the direction of a_c.

Tangential acceleration will work if an object is moving in a circular path.

$$a_t = \frac{\Delta v}{\Delta t}$$
$$a_t = \alpha r$$

Even if the velocity vector does not change the magnitude, its direction is constantly changing during angular motion. **Centripetal acceleration** is an acceleration towards the axis of rotation.

$$a_c = w^2 r = \frac{v^2}{r}$$

Since a_t and a_c are perpendicular, the **resultant linear acceleration** can be found using the Pythagorean Theorem:

$$a = \sqrt{a_t^2 + a_c^2}$$

23-03-07

Force and moment

Magnitude

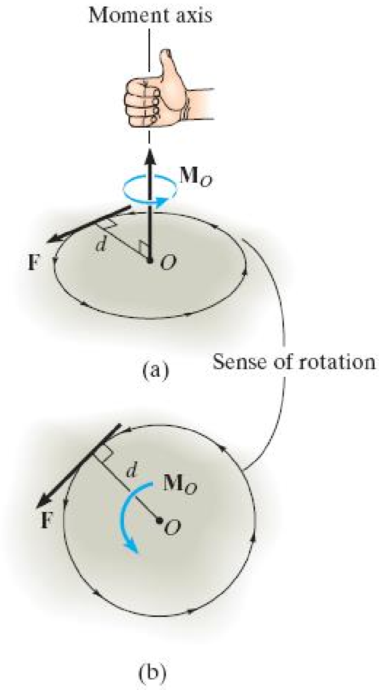
For magnitude of $\mathbf{M}_O$,

$$\mathbf{M}_O = F \cdot d \ (Nm)$$

where d = moment arm
= perpendicular distance
from O to the force's ***line of action***
(line through the point of application
parallel to the force)

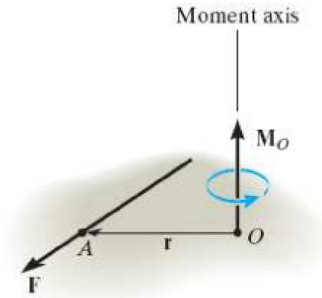
Direction

Direction using "right hand rule"



Moment of force F about point O can be expressed using cross product

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$$



Magnitude

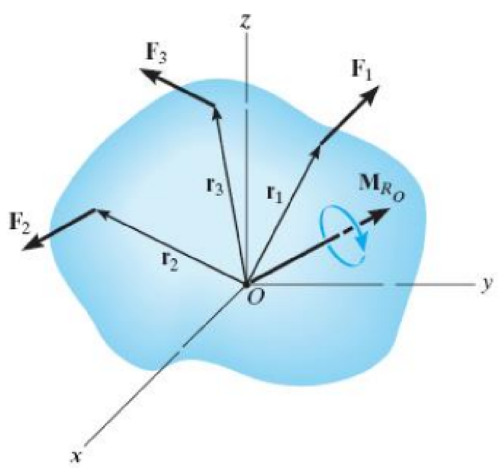
For magnitude of cross product,

$$M_O = r\,F\sin\theta$$

Resultant Moment of a System of Forces

Resultant moment of forces about point O can be determined by vector addition

$$\mathbf{M}_{R_O} = \sum(\mathbf{r} \times \mathbf{F})$$



Couple forces

Couple of forces are:

- two parallel forces at points p_1 and p_2
- same magnitude but opposite direction
- separated by perpendicular distance d
- resultant linear force is 0

Couple moment = $(r_1 - r_2) \times F$

Direction of the couple moment is determined by the right-hand rule. It is perpendicular to the plane containing the forces.

Polar representations

$x \to r\cos\theta$

$y \to r\sin\theta$

$z \to z$

23-03-14

LTI systems

A **signal** is an input or an output of the LTI system. It has a property called **amplitude** -- the strength of the signal.

The input/output signal is **bounded**, if its amplitude is finite. In other words, the input and the output do not grow uncontrollably. This ensures that the system's behavior is predictable and doesn't lead to any undesirable outcomes.

Damping

Damping refers to the mechanism that reduces the amplitude of oscillations in a system over time.

Positive damping occurs when the damping force makes the system return to its equilibrium position without overshooting. The amplitude of oscillations gradually decreases until the system comes to rest. For example, it is a door closer mechanism.

Negative damping (or overdamping) occurs when the damping force is strong enough to cause the system to return to its equilibrium position without oscillating. In other words, it is when the equilibrium position is overshot in a dynamic system.

Stability & unsuitability

LTI systems without input are

- *stable* if for any initial condition $x(0)$, the system's response $x(t)$ remains bounded as time t goes to infinity:

$$\forall x(0)\ \exists k > 0\ :\ ||x(t)|| < k, \forall t > 0$$

- *asymptotically stable* if for any initial condition $x(0)$, the system's response $x(t)$ converges to the origin (zero) as time t goes to infinity:

$$\lim_{t \to \infty} ||x(t)|| = 0$$

  The system's response gradually approaches and settles at the equilibrium point (i.e., zero) over time.

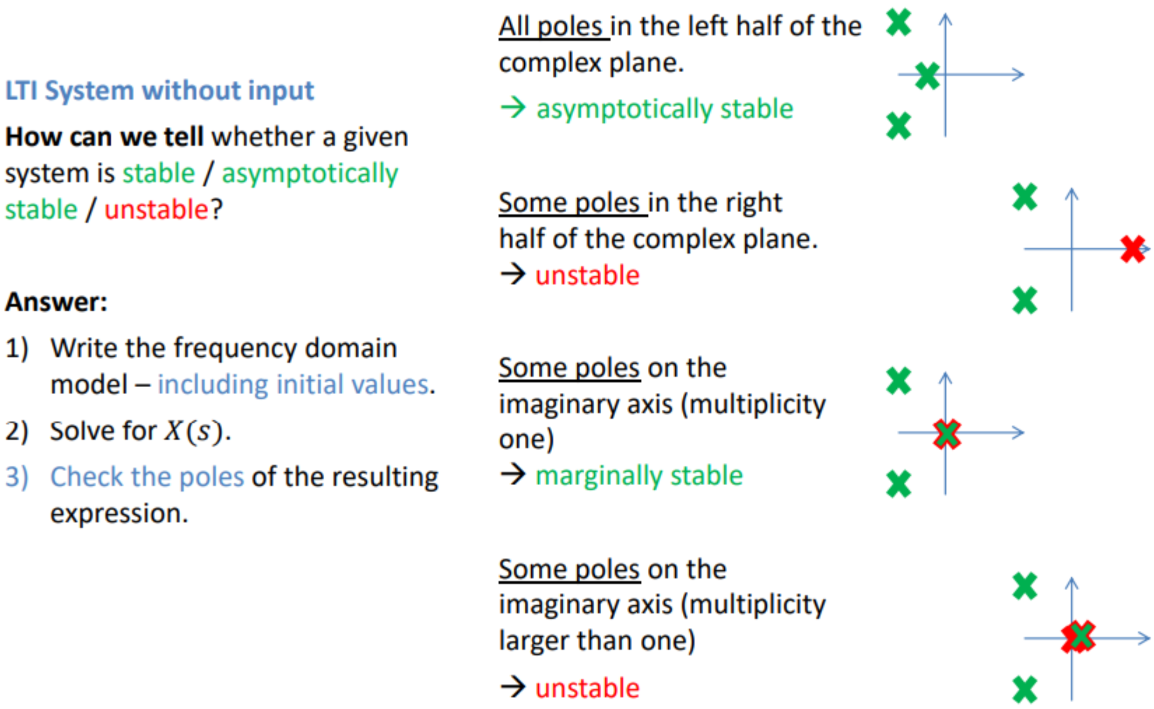
- *unstable* if its solutions do not remain bounded over time. The response of the system grows uncontrollably as time progresses.

LTI systems with input are considered **stable** if bounded input always lead to bounded output (BIBO criteria).

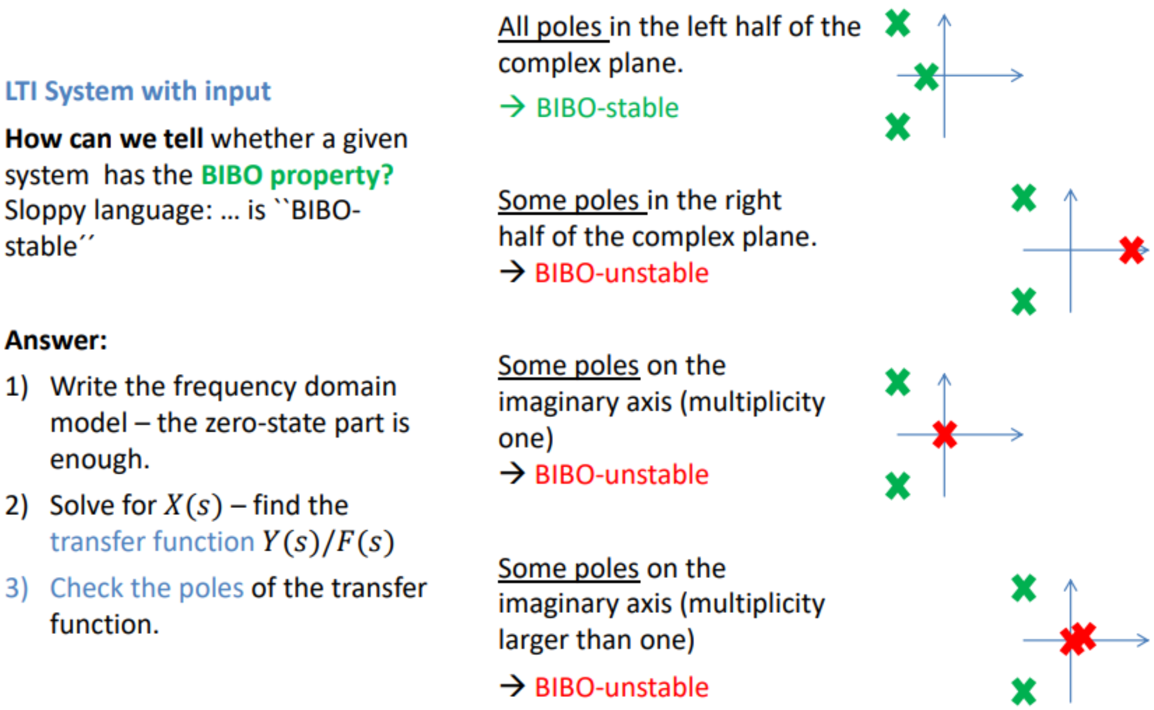
Check system's stability

The **poles of a system** refer to the values of the complex variable that make the denominator of the system's transfer function equal to zero.

For LTI systems without input:



For LTI systems with input:



Routh table

Finding poles can be problematic if a polynomial in the denominator is not factorized. You can find out the approximate position of the poles using the **Routh table**.

- if there is a negative or zero coefficient, the system is **not** BIBO-stable
- otherwise, use the Routh table:

This is how you start:

1) For a polynomial of order n, prepare a table with (n+1) rows , and lable them with s^n to s^0.
2) Enter the coefficients as indicated – always starting in the upper left corner.

This populates the first two rows. If you like, you can continue with additional zeros to the right – they do not hurt.

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

Find more rows!

Always the same pattern ...

But applied to different groups of numbers.

<u>Study carefully</u>:

The determinant is formed based on columns 1&2, then 1&3, then 1& 4, etc. until we run out of columns.

Then, move down by one row, and repeat!

Always devide by the entry in the lower left corner.

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$-\frac{\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$-\frac{\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$-\frac{\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$-\frac{\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$-\frac{\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Now, **the first column contains the information**: All entries positive → Transfer function is BIBO stable

It is allowed to reduce a complete row by a common positive factor