Probability and Random Processes

Author: Daria Shutina

Organization stuff

nstege@jacobs-university.de

Teams link: https://teams.microsoft.com/_?tenantId=f78e973e-5c0b-4ab8-bbd7-9887c95a8ebd#/school/conversations/General?groupId=9b2928da-a70d-41b9-9ba8-c828df8332ab&threadId=19:y5mgGU61uWlfOP95M0fwVjqsEZQoBiP6g7s3xbOW7nw1@thread.tacv2&ctx=channel

22-09-07

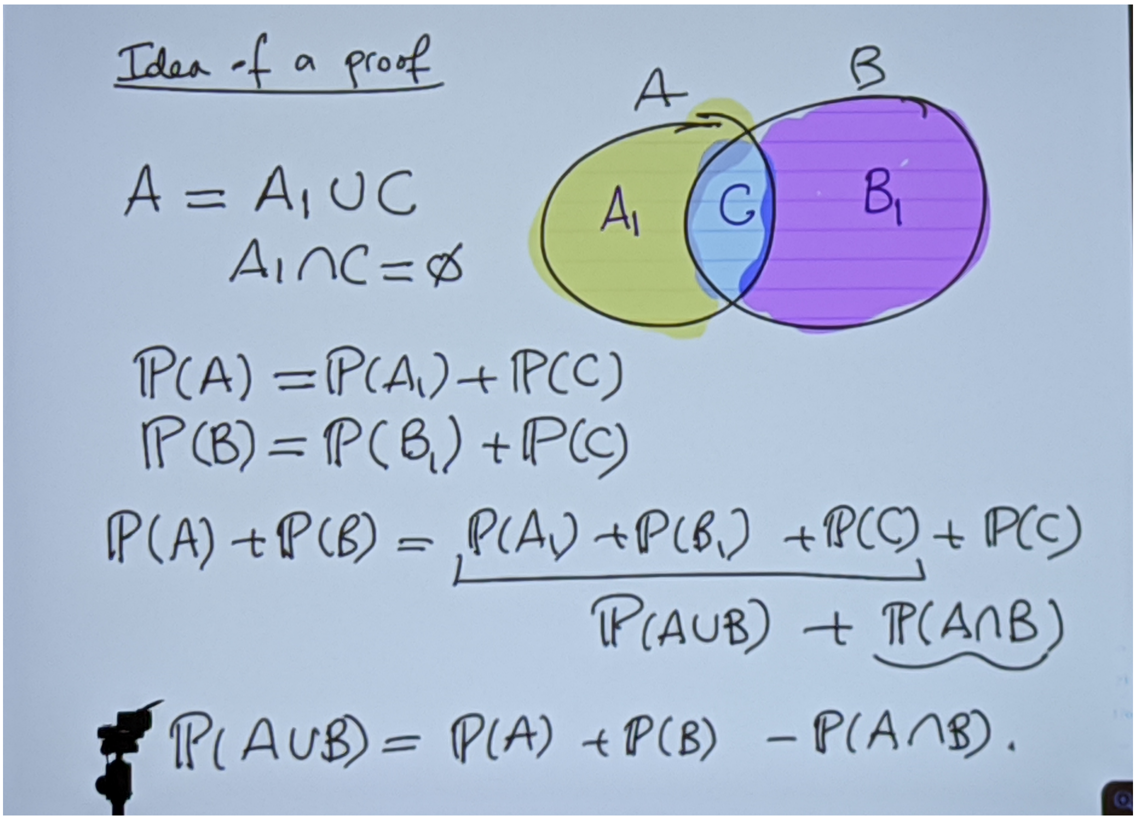
Ω is a sample space. $A \in \Omega$ is an event

$P[\Omega] = 1$

$A \cap B = \varnothing \;\Rightarrow\; P[A \cup B] = P[A] + P[B]$

The Union Law

$P[A \cup B] = P[A] + P[B] - P[A \cap B]$



22-09-15

Inclusion-exclusion principle

$$P(A_1 \cup \ldots \cup A_n) = \sum_{1 \leqslant i \leqslant n} A_i - \sum_{i<j} A_i \cap A_j + \sum_{i<j<k} A_i \cap A_j \cap A_k - \ \ldots \ + (-1)^{n-1} P(\bigcap_{1 \leqslant i \leqslant n} A_i)$$

Example of envelopes

There are n envelopes and n numbers. We want to put number i into an envelope i. What is the probability that at least one number will be in the right envelope?

$P(A_i) = \frac{1}{n}, \forall i$

$P(A_i \cap A_j) = \frac{(n-2)!}{n!}, \forall i, j,$ since 2 numbers are fixed and others can be in any order. There are n! options to order n elements.

$P(A_i \cap A_j \cap A_k) = \frac{(n-3)!}{n!}$

$\ldots$

$P(A_1 \cap \ldots \cap A_n) = \frac{1}{n!}$

$P(A_1 \cup \ldots \cup A_n) = \frac{1}{n} n \ - \ \frac{(n-2)!}{n!} \binom{n}{2} + \ \ldots \ + (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2!} + \frac{1}{3!} - \ldots + (-1)^{n-1} \frac{1}{n!}$

If $n \to \infty$, then P limits to $\frac{1}{e} \approx 0.37$ (the formula in a line higher is a Taylor series of e^x where $x = 1$).

Secretary problem (Задача о разборчивой невесте)

There is one position and n candidates, and we need to choose the best candidate. With what probability can at least one candidate be the best?

It is the same as $P(A_1 \cup \ldots \cup A_n)$ which limits to $\frac{1}{e} \approx 0.37$. So if you have a lot of candidates, it could be a good strategy to reject first 30%.

22-09-16

Geometric probability

Ω -- sample space

$A \subset \Omega$ -- set of favorable outcomes. $P(A) = \frac{\#A}{\#\Omega}$

Monte Carlo method -- randomly choose objects for a lot of times.

Ways to count probability:

- by length (intervals)
- by volume/area

Example: meet between noon and 1pm

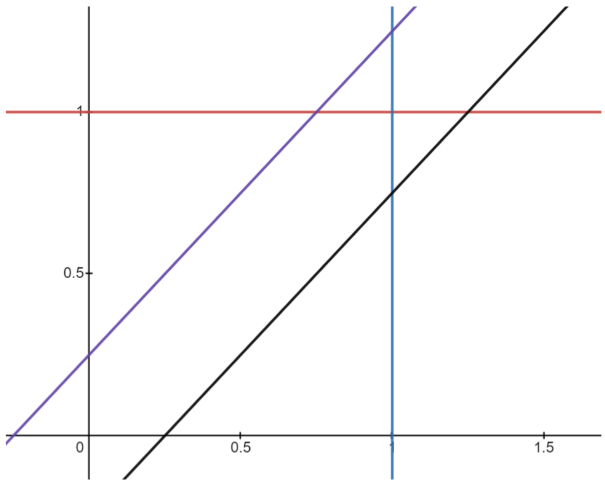
Alex and Anna want to meet between 12pm and 1pm. They choose independently at what time to show up. When sb comes, he/she waits for 15 minutes and then leaves. What is the probability that they meet?

T_1 -- time that Alex shows up. $0 \leqslant T_1 \leqslant 1$

T_2 -- time that Anna show up $0 \leqslant T_2 \leqslant 1$

$0 \leqslant |T_1 - T_2| \leqslant \frac{1}{4}$

$P = \frac{S_1}{S_0} = (1 - 0.75) \cdot 2 = \frac{1}{2}$



Example 2: Buffon's needle

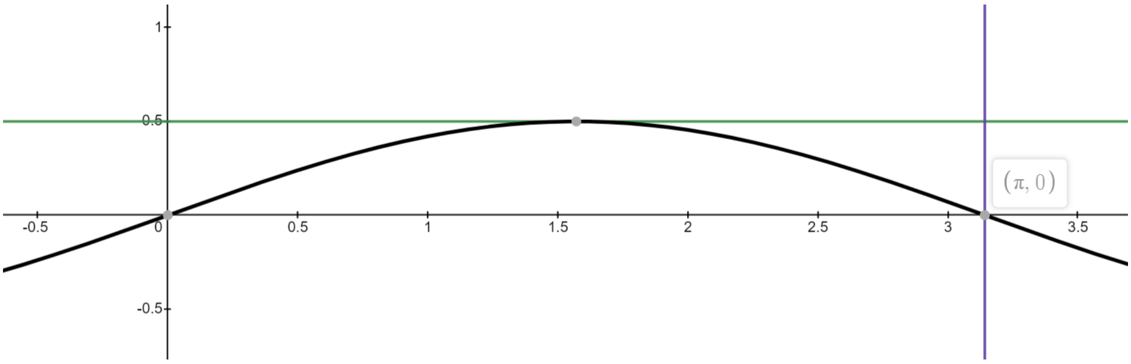
There are parallel lines in a distance of 1. There is a needle of length 1.

Need to keep track of:

- y -- coordinate of the center of the needle. $0 \leqslant y \leqslant 1$

- α -- angle that the needle forms with the line (in the plane which is parallel to the surface). $0 \leqslant \alpha \leqslant \pi$

$P = \frac{\int_0^\pi \frac{sinx}{2} dx}{\frac{1}{2}\pi} = \frac{-cosx|_0^\pi}{\pi} = \frac{2}{\pi}$



Example 3: Bertrand's paradox /todo

A chord of a circle of radius 1 is chosen. What is the probability that the length of the chord would be at least $\sqrt{3}$?

The points are on the circle. The coordinate of a point is between 0 and 2π. The distance between two points should be at least 120°.

We can peek the middle point p inside the circle, then draw a chord, where p is a center of the chord.

Now the length of the chord is $2\sqrt{1-x^2}$, where x is a distance between p and the center of the circle. Should be $\geqslant \sqrt{3} \Rightarrow x \leqslant \frac{1}{2} \Rightarrow P = \frac{1}{4}$

wtf

22-09-21

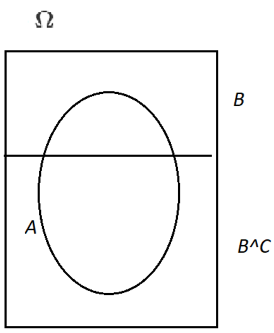
The conditional probability

$A, B \ : \ P(A) \neq 0, \ P(B) \neq 0.$

$P[A|B] = \frac{P[A \cap B]}{P[B]}$, where $P[A|B]$ -- "A given B" -- probability of A in case B has already occurred.

Conditioning: special case Ω

$P[A] = P[A \cap B] + p[A \cap B^c] = P[A|B] \cdot P[B] + P[A|B^c] \cdot P[B^c]$



In general, $P[A] = \sum\limits_{i=1}^{n} p[A \cap B_i] = \sum\limits_{i=1}^{n} P[A|B_i]P[B_i]$

Example: 5 coins

There are 5 coins: two double-headed, one double-tailed and two normal. One of the coins is randomly taken. What is the probability that we get a head?

B_1 -- "got double-headed", B_2 -- "got double-tailed", B_3 -- "got normal".

$P[H] = \sum\limits_{1}^{3} P[A|B_i]P[B_i] = \underbrace{1 \cdot \frac{2}{5}}_{B_1} + \underbrace{0 \cdot \frac{1}{5}}_{B_2} + \underbrace{\frac{1}{2} \cdot \frac{2}{5}}_{B_3} = 0.6$

Independent events

A and B are independent, when $\;P[A|B] = P[A]\;$ or $\;P[A \cap B] = P[A] \cdot P[B]$

In general, $A_1, \ldots, A_n$ are independent, when $\forall \{i_1, .., i_k \mid i_j \in [1..n], \forall j\} \;:\; P[\bigcap\limits_{j=1}^{k} A_{i_j}] = \prod\limits_{j=1}^{k} P[A_{i_j}]$

Example: a pair of dies

A pair of dies is rolled.

A: the first die's score $\geqslant 3$. $\;P[A] = \frac{1}{2}$

B: The second die's score $\geqslant 5$. $\;P[B] = \frac{1}{3}$

C: Aim of the scores $= 6$. $\;P[C] = \frac{5}{36}$

$P[A \cap B] = \frac{1}{6},\;\; P[A \cap C] = \frac{1}{12},\;\; P[B \cap C] = \frac{1}{36}$

A and B are independent: $\;P[A \cap B] = \frac{1}{6}$

A and C are dependent: $P[A \cap C] \neq \frac{5}{72}$

B and C are dependent: $\;P[B \cap C] \neq \frac{5}{108}$

Random walks

Random walk means you can move eigther to the right, either to the left with the probability of $\frac{1}{2}$.

Example: an X-axis \todo

There is an X-axis and numbers 0,1,2,3. We can move to the right or to the left with the prob. of $\frac{1}{2}$. If we get to 0 or 3, we stop. What is the probability that we stop in 0?

p_i = Prob. of stopping in 0 if we start in i.

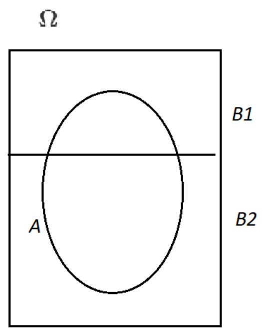
$p_0 = 1,\; p_3 = 0$

$p_1 = p_2 \cdot \frac{1}{2} + \frac{1}{2} = \frac{p_2+1}{2}$

$p_2 = p_3 \cdot \frac{1}{2} + p_1 \cdot \frac{1}{2}$

22-09-23

$\Omega = B_1 \cup B_2,\ \ B_1 \cap B_2 = \varnothing$



$P[B_1|A] = \frac{P[B_1 \cap A]}{P[A]} = \frac{P[A|B_1] \cdot P[B_1]}{P[A \cap B_1] + P[A \cup B_2]} = \frac{P[A|B_1] \cdot P[B_1]}{P[A|B_1] \cdot P[B_1] + P[A|B_2] \cdot P[B_2]}$

Example 1: transmission channel

Two types of messages can be sent: 0 and 1. We assume that 40% of the time 1 is got.

$\Omega = I_0 \cup I_1;$ I_1 = "input is 1", I_0 = "input is 0";

O_1 = "output is 1", O_0 = "output is 0";

$P[I_1] = 0.4,\ P[I_0] = 0.6$

$P[O_1|I_1] = 0.9,\ P[O_0|I_0] = 0.8$

$P[O_0|I_1] = 1 - P[O_1|I_1] = 0.1$

$P[O_0] = P[O_0] \cap P[I_0] + P[O_0] \cap P[I_1] = P[O_0|I_0] \cdot P[I_0] + P[O_0|I_1] \cdot P[I_1] = 0.52$

Random variables

Let (Ω, P) be a space. A function $X : \Omega \to R$

Example 1: three coin throwings

$X : \Omega \to \texttt{[\# of heads]}$

Example 2: three throwings of a fair die

$X_1 : \Omega \to \texttt{[sum of outcomes]}$

$X_2 : \Omega \to \{\ \texttt{0,1 : [two similar numbers in a row?]}\}$

The probability mass function of a randim variable X with X of values $x_1, \ldots$: $\quad p(x_i) = P[X = x_i]$

22-09-28

Random variables

Let (Ω, P) be a probability space.

$X\ :\ \Omega \to R$ is a *real-valued random variable*.

$X\ :\ \Omega \to R^n$ is a *vector-valued random variable*.

The probability *mass function* of a random variable X with values $x_1, .., x_n$ is defined by $p(x_i) = P[X = x_i]$

Mandatory conditions:

$0 \leqslant p(x_i) \leqslant 1$

$\sum\limits_1^n p(x_i) = 1$

Bernoulli random variables

A random variable X is called the Bernoulli random variable if it takes only two values: 0 and 1, and $P[X = 1] = p,\ P[X = 0] = 1 - p$

For example, $X = \{\ \text{'}\mathtt{got\ a\ head}\text{'},\ \text{'}\mathtt{got\ a\ tail}\text{'}\ \}$

Bernoulli distribution

The coins if thown n times. p is the prob. of getting a head. What is the probability that we get k heads?

X -- how many times there was a heads.

$$p[X = k] = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k},\ k \in [0, n] \\ 0,\ k \notin [0, n] \end{cases}$$

Geometric distribution

The coin is thrown untill a heads shows up. The prob. of getting a head s is p.

X -- first occurance of a heads.

$$p[X = k] = \begin{cases} (1-p)^{k-1} p,\ k \geqslant 1 \\ 0,\ k < 1 \end{cases}$$

Example: more than k heads

$P[X > k] = \sum\limits_{i=k+1}^{\infty} p(1-p)^{i-1} = p(1-p)^k \cdot (1 + (1-p) + (1-p)^2 + \ldots) = p(1-p)^k \cdot \frac{1}{1-(1-p)} = (1-p)^k$

Poisson distribution

Particles are created at a random time.

The probability that a practicle is created in a time segment $[t_0, t_1]$ depends on $\Delta t = t_0 - t_1$ and does not depend on the amount of already created particles. Also, the prob. is $\approx \lambda \cdot \Delta t$ for small Δt.

$$p[X=k] = \begin{cases} \frac{\lambda^k}{k!}e^{-\lambda},\ k \geqslant 0 \\ 0,\ k < 0 \end{cases}$$

Example: participants and phones

There are n people in a room. p is the prob. that a person's phone rings. What is the probability that k phones will ring?

$\lambda = \frac{p}{n}$ -- constant; $\lambda \to 0,\ n \to \infty$

$P[X=0] = p^0(1-p)^n = (1+\frac{-\lambda}{n}^{n}) \xrightarrow[n\to\infty]{} e^{-\lambda}$

$P[X=1] = \binom{n}{1}p(1-p)^{n-1} = np(1-p)^{n-1} = \frac{\lambda(1+\frac{-\lambda}{n})^n}{1+\frac{-\lambda}{n}} \xrightarrow[n\to\infty]{} \lambda e^{-\lambda}$

$P[X=k] = \binom{n}{k}p^k(1-p)^{n-k} = \frac{n(n-1)...(n-k+1)}{k!} \cdot \frac{\lambda^k}{n^k} \cdot \frac{(1+\frac{-\lambda}{n})^n}{(1+\frac{-\lambda}{n})^k} = \frac{\lambda^k}{k!}e^{-\lambda} \cdot \frac{n(n-1)...(n-k+1)}{(n-\lambda)^k} \xrightarrow[n\to\infty]{} \frac{\lambda^k}{k!}e^{-\lambda}$

Distribution function

Let $X\ :\ \Omega \to R$ be a random variable.

$F_X\ :\ R \to [0,1]$ -- the *probability distribution function* of X.

$F_X(t) = P[X \leqslant t]$

The probability distribution function properties:

1. F_X is increasing: $\forall t_1 < t_2\ :\ F_X(t_1) \leqslant F_X(t_2)$
2. F_X is right-continuous: $\lim\limits_{s\to t+} F_X(s) = F_X(t)$
3. $\lim\limits_{s\to +\infty} F_X(s) = 1$
4. $\lim\limits_{s\to -\infty} F_X(s) = 0$

Remark:

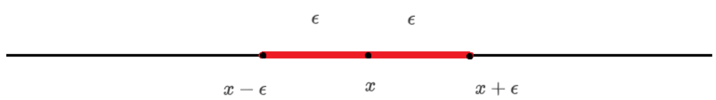
$P[x > t] = 1 - P[x \leqslant t] = 1 - F_X(t)$

$P[x < t] = F_X(t-) = \lim\limits_{s\to t-} F_X(s)$

Density function

Captures the density on the axis. There are no distinct points, so we use a delimiter to count the probability.

$$f(x) = \lim_{\epsilon\to 0} \frac{\texttt{mass} \in (x-\epsilon, x+\epsilon)}{2\epsilon}$$



$f(x) \geqslant 0,\ \forall x$

$\int\limits_{-\infty}^{+\infty} f(x) = 1$

Continuous random variables

$X : \Omega \to R$ is called continuous, if $\exists\, f_X : R \to R \geqslant 0, \forall t,$ such that $F_X(t) = \int\limits_{-\infty}^{t} f_X(x)dx$

$P[X > t] = 1 - F_X(t)$

$P[X \leqslant t] = \lim\limits_{s \to t+} F_X(s)$

$P[X < t] = \lim\limits_{s \to t-} F_X(s)$

$P[X = t] = F_X(t) - \lim\limits_{s \to t-} F_X(s)$

$F_X(t+\epsilon) - F_X(t) = \int\limits_{t}^{t+\epsilon} f_X(x)dx = \epsilon \cdot f_X(t)$

$\lim\limits_{\epsilon \to 0} \frac{F_X(t+\epsilon) - F_X(t)}{\epsilon} = f_X(t)$

Example

Find a probability density function (f_X) that satisfies the following properties:

$$\begin{cases} f_X(t) \geqslant 0, \forall t \\ \int\limits_{-\infty}^{+\infty} f_X(x)dx = 1 \end{cases}$$

$f_X = \frac{e^{-|t|}}{2}$

$\int\limits_{-\infty}^{+\infty} \frac{e^{-|x|}}{2} dx = (-1)e^{-x}|_0^{+\infty} = 1$

22-10-07

Uniform distribution

A random variable has a uniform distribution over the interval $[a, b]$ if its probability density function is given by

$$f_x(t) = \begin{cases} \frac{1}{b-a},\ t \in [a, b] \\ 0,\ \text{otherwise} \end{cases}$$

In other words, a random variable X is uniformly distributed in the interval $I = [a, b]$ if the probability that X belongs to a segment $I \in [a, b]$is proportional to the length of I.

The distribution function of a random variable with a uniform distribution is given by

$$F_x(t) = \begin{cases} 0,\ t \leqslant a \\ \frac{t-a}{b-a},\ a \leqslant t \leqslant b \\ 1,\ t \geqslant b \end{cases}$$

Proof:

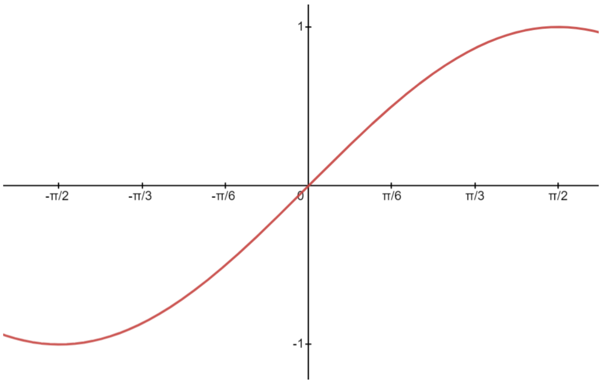
$t \leqslant a \;\Rightarrow\; \int\limits_{-\infty}^{t} f_x(x)dx = 0$

$a \leqslant t \leqslant b \;\Rightarrow\; \int\limits_{-\infty}^{t} f_X(x)dx = \frac{x}{b-a}|_a^t = \frac{t-a}{b-a}$

$t \geqslant b \;\Rightarrow\; \int\limits_{-\infty}^{t} f_x(x)dx = \frac{x}{b-a}|_a^b = 1$

Example 1: $Y = X^2$ **\todo**

Example 2: $Y = \sin X$ **\todo**



$P[-\frac{\pi}{4} \leqslant x \leqslant \frac{\pi}{4}] = \frac{\frac{\pi}{2}}{\pi} = 0.5$

$P[\frac{-\sqrt{2}}{2} \leqslant Y \leqslant \frac{\sqrt{2}}{2}] = \frac{1}{2}$

Let us find a distribution function for Y:

$-1 \leqslant t \leqslant 1$

$F_Y(t) = P[Y \leqslant t] = P[\sin X \leqslant t] = P[X \leqslant \arcsin t] = \frac{\frac{\pi}{2}+\arcsin t}{\pi} = \frac{1}{2} + \frac{\arcsin t}{\pi}$

$f_y(t) = \frac{d}{dt}F_y(t) = (\frac{\arcsin t}{\pi})' = \frac{1}{\pi}\frac{1}{\sqrt{1-t^2}}$

Exponential random variables

A continuous random variable X has an exponential distribution with a parameter λ if

$$f_X(t) = \begin{cases} \lambda e^{-\lambda t},\ t \geqslant 0 \\ 0,\ t < 0 \end{cases}$$

$\int\limits_{-\infty}^{+\infty} f_x(t)dt = \int\limits_{0}^{+\infty} \lambda e^{-\lambda t}dt = (-1)e^{-\lambda t}|_0^{+\infty} = 1$

The distribution function for a random variable with an exponential distribution is given by

$$F_X(t) = \begin{cases} 1 - e^{-\lambda t},\ t \geqslant 0 \\ 0,\ t < 0 \end{cases}$$

Proof:

$\int\limits_0^t \lambda e^{-\lambda x} dx = (-1)e^{-\lambda x}|_0^t = 1 - e^{-\lambda t}$

$P[X > t] = 1 - F_X(t) = e^{-\lambda t}$

$P[X > t_1 + t_2 | X > t_1] = \frac{P[X > t_1 + t_2 \ \&\& \ X > t_1]}{P[X > t_1]} = \frac{P[X > t_1 + t_2]}{P[X > t_1]} = \frac{e^{-\lambda(t_1+t_2)}}{e^{-\lambda t_1}} = e^{-\lambda t_2} = P[X > t_2]$

Gaussian (normal) random variables

A continuous random variable X is said to have Gaussian or normal distribution with parameters (μ, σ^2) if the probability density function is given by

$$f_X(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

22-10-19

Expectation

A discrete variable X with values $x_1, \ldots, x_n$ obtained with probabilities $p_1, \ldots, p_n$.

$$E(X) = \sum_1^n p_i x_i$$

For Bernoulli variables

$E(X) = p \cdot 1 + (1-p) \cdot 0 = p$

For binomial variables

$E(X) = \sum\limits_0^n k \cdot P(X = k) = \sum\limits_1^n k \cdot \binom{n}{k} \cdot p^k (1-p)^{n-k} = \sum\limits_1^n \frac{k \cdot n!}{k!(n-k)!} \cdot p^k (1-p)^{n-k} = \sum\limits_1^n \frac{(n-1)! \cdot n}{(k-1)!(n-k)!} \cdot p^k (1-p)^{n-k} =$

$= n \cdot \sum\limits_1^n \binom{n-1}{k-1} \cdot p^k (1-p)^{n-k} = n \cdot \sum\limits_0^{n-1} \binom{n-1}{k} p^{k+1} (1-p)^{(n-1)-k} = np \cdot (p + 1 - p)^{n-1} = np$

For geometric distribution

$E(X) = \sum\limits_1^\infty i \cdot p \cdot (1-p)^{i-1} = \sum\limits_1^\infty p(1-p)^{i-1} + \sum\limits_2^\infty p(1-p)^{i-1} + \sum\limits_3^\infty p(1-p)^{i-1} + \ldots =$

$= \sum\limits_1^\infty P[X \geqslant i] = \sum\limits_1^\infty (1-p)^{i-1} = \frac{1}{p}$

Theorem

X takes values $0, 1, 2, \ldots$. Then $E(X) = \sum\limits_1^\infty P[X \geqslant i]$

Proof:

$E(X) = \sum\limits_0^\infty i \cdot p_i = p_1 1 \ + \ p_2 2 \ + \ p_3 3 \ + \ p_4 4 \ + \ \ldots \ + \ p_2 2 \ + \ p_3 3 \ + \ p_4 4 \ + \ \ldots \ + \ p_3 3 \ + \ p_4 4 \ + \ \ldots = \sum\limits_1^\infty P[X \geqslant i]$

22-10-21

Properties of expectation

Let X, Y be random variables and c a constant.

1. *Linearity:* $E(cX + y) = cE(X) + E(Y)$
2. *Comparison:* $X \leqslant Y \;\Rightarrow\; E(X) \leqslant E(Y)$
3. $f \;:\; R \to R \implies E(X) = \sum\limits_{1}^{n} p_i \cdot f(x_i)$

Example 1: envelops and numbers

Suppose n letters are placed in n envelopes. X -- # of letters placed in the right envelope. Find $E(X)$.

Ex. n=3
envelops 1,2,3

$123 \xrightarrow{X} 3$
$132 \longrightarrow 1$
$231 \longrightarrow 0$
$213 \longrightarrow 1$
$312 \longrightarrow 0$
$321 \longrightarrow 1$

X	0	1	3
P	1/3	1/2	1/6

$E(X) = 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{2} + 3 \cdot \frac{1}{6} = 1$

A general case:

$X = X_1 + \ldots + X_n$, where $X_i = \begin{cases} 1, & \text{if i-th number is in the i-th letter} \\ 0, & \text{otherwise} \end{cases}$

$E(X_i) = \frac{(n-1)!}{n!} \cdot 1 + (1 - \frac{(n-1)!}{n!}) \cdot 0 = \frac{1}{n}$

$E(X) = \sum\limits_{1}^{n} E(X_i) = n \cdot \frac{1}{n} = 1$

Example 2: elevator stops

There are m people in an elevator. The elevator goes up a building with n floors and stops at each floor where at least one person wants to get off. X -- # of stops. Find $E(X)$.

$X_i = \begin{cases} 1, & \text{if elevator stops at i-th floor} \\ 0, & \text{otherwise} \end{cases}$

$X = X_1 + \ldots + X_n$

The prob. that the nobody goes off on the i-th floor is $\underbrace{(\frac{n-1}{n})^m}_{X_i = 0}$

$\Rightarrow$ the prob. that $\geqslant 1$ people go off is $\underbrace{1 - (\frac{n-1}{n})^m}_{X_i = 1}$.

$E(X_i) \;=\; (1 - (\frac{n-1}{n})^m) \cdot 1 \;+\; (\frac{n-1}{n})^m \cdot 0 \;=\; 1 - (\frac{n-1}{n})^m$

$E(X) = \sum\limits_{1}^{n} E(X_i) = n \cdot (1 - (\frac{n-1}{n})^m)$

22-10-28

Expectation as an integral

X is a continuous random variable.

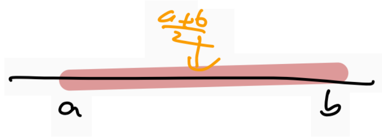
$f(X)$ is a probability density function

$$E(X) = \int\limits_{-\infty}^{+\infty} xf(x)dx$$

Example 1

$$f(x) = \begin{cases} \frac{1}{b-a}, \ a \leqslant x \leqslant b \\ 0, \ \text{otherwise} \end{cases}$$

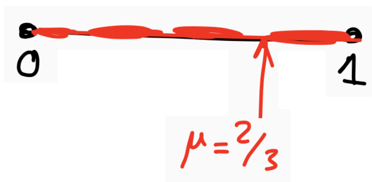
$$E(X) = \int\limits_{-\infty}^{+\infty} xf(x)dx = \int\limits_{a}^{b} \frac{x}{b-a}dx = \frac{x^2}{2(b-a)}|_a^b = \frac{b+a}{2}$$



Example 2

$$f(x) = \begin{cases} 2x, \ 0 \leqslant x \leqslant 1 \\ 0, \ \text{otherwise} \end{cases}$$

$$E(X) = \int\limits_{-\infty}^{+\infty} xf(x)dx = \int\limits_{0}^{1} 2xdx = \frac{2x^3}{3}|_0^1 = \frac{2}{3}$$



Expectation of a non-negative random variable

X is a non-negative continuous random variable

$F_X(t)$ is a distribution function. $F_X(t) = P[X \leqslant t]$

$$E(X) = \int\limits_{0}^{\infty} P[X \geqslant t]dt = \int\limits_{0}^{\infty} (1 - F_X(t))dt$$

Example

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, \ x > 0 \\ 0, \ x \leqslant 0 \end{cases}$$

$$F(t) = \begin{cases} 1 - e^{-\lambda t}, \ t > 0 \\ 0, \ t \leqslant 0 \end{cases}$$

$$E(X) = \int\limits_{0}^{\infty} (1 - (1 - e^{-\lambda t})) = \frac{e^{-\lambda t}}{-\lambda}|_0^{\infty} = \frac{1}{\lambda}$$

Example: $E[\sin x]$ over $[0, \pi]$

$$f(x) = \begin{cases} \frac{1}{\pi}, \ 0 \leqslant x \leqslant \pi \\ 0, \ \text{otherwise} \end{cases}$$

We choose one number from a segment $[0, \pi]$ with a prob. of $\frac{1}{\pi}$.

$$E(\sin x) = \int\limits_0^\pi \sin x \frac{1}{\pi} dx = \frac{-\cos x}{\pi}|_0^\pi = \frac{2}{\pi}$$

Variance

-- мера разброса значений случайной величины относительно её матожидания.

X is a random variable.

$Var(X) = E[(X - E(X))^2]$

$Var(X) = E(X^2) - E^2(X)$

Properties of variance

1. $Var(cX) = c^2 Var(X)$
   $Var(cX) = E(c^2X^2) - (E(cX))^2 = c^2E(X^2) - c^2E^2(X) = c^2Var(X)$
2. $Var(X) \geqslant 0$
   $E(X^2) \geqslant E^2(X) \ \Rightarrow \ Var(X \geqslant 0)$
3. $Var(X + a) = Var(X)$
4. $Var(X + a) = E[\ (\ (x + a) - \underbrace{E[x + a]}_{=E[X]+E[a]=E[X]+a}\ )^2\ ] = E[(x - E[X])^2] = Var(X)$
5. X, Y are independent $\Longrightarrow Var(X + Y) = Var(X) + Var(Y)$
   Proof is here (№6).

Example

$$f(x) = \begin{cases} 1, \ 0 \leqslant x \leqslant 1 \\ 0, \ \text{otherwise} \end{cases}$$

$$Var(X) = E(X^2) - E^2(X) = \int\limits_0^1 x^2 dx - (\int\limits_0^1 x dx)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

22-11-02

Joint probability mass function

n discrete random variables $x_1, \dots, x_n$

jpmf is defined by $p(x_1, \dots, x_n) = P[X_1 = x_1, \dots, X_n = x_n]$

Example: chicken eggs

A chicken lays N eggs, where N has a Poisson distribution with a parameter λ.

Each egg independently hatches with probability p and do not hatch with probability $1-p$.

X -- # of eggs that hatch.

Y -- # of eggs that do not hatch.

Find the joint probability mass function of X and Y.

$P[N=n] = e^{-\lambda}\frac{\lambda^n}{n!}, \;\; n = 0,\; 1,\; 2, \dots$

Imagine we do not have Y:

$$P[X=k] = \sum_{n=0}^{\infty} \underbrace{P[X=k|N=n]}_{\text{binomial distribution}} \cdot P[N=n] = \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} e^{-\lambda}\frac{\lambda^n}{n!} = \frac{e^{-\lambda}p^k\lambda^k}{k!} \underbrace{\sum_{n=k}^{\infty} \frac{(\lambda(1-p))^{n-k}}{(n-k)!}}_{=e^{\lambda(1-p)}} = \frac{p^k\lambda^k}{k!}e^{-\lambda p}$$

Now Y appears:

$$P[X=k, Y=j] = P[X=k, Y=j|N=k+j] \cdot P[N=k+j] =$$

$$= \binom{k+j}{k} p^k (1-p)^j \cdot e^{-\lambda}\frac{\lambda^{k+j}}{(k+j)!} = \frac{e^{-\lambda p}(\lambda p)^k}{k!} \cdot \frac{e^{-\lambda(1-p)}(\lambda(1-p))^j}{j!} = P[X=k] \cdot P[Y=j]$$

Theorem (expectation of joint prob)

$X_1, \dots, X_n$ are *independent* random variables.

$E(X_1 \dots X_n) = E(X_1) \cdot \dots \cdot E(X_n)$

Proof:

XY	x_1y_1	x_1y_2	x_2y_1	x_2y_2
	p_1q_1	p_1q_2	p_2q_1	p_2q_2

$Y \backslash X$	x_1	x_2	
y_1	p_1q_1	p_2q_1	q_1
y_2	p_1q_2	p_2q_2	q_2
	p_1	p_2	

$$\mathbb{E}[XY] = x_1y_1p_1q_1 + x_1y_2p_1q_2 + x_2y_1p_2q_1 + x_2y_2p_2q_2$$

$$= (x_1p_1 + x_2p_2)(y_1q_1 + y_2q_2)$$

$$= \mathbb{E}[X]\,\mathbb{E}[Y].$$

Example(matrix A and $E(det(A))$)

A is a 2×2 matrix whose entries are independent random variables with uniform distribution over $\{1,2,3\}$. Find $E(det(A))$.

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$E(A_{ij}) = (1+2+3)\cdot\frac{1}{3} = 2,\ \forall i,j$

$E(det(A)) = E(A_{11}A_{22} - A_{12}A_{21}) = E(A_{11})E(A_{22}) - E(A_{12})E(A_{22}) = 0$

Covariance

-- мера зависимости двух случайных величин.

X, Y are random variables.

$Cov\ (\ X,\ Y\ ) = E\ [\ (\ X - E[X]\ )\ (\ Y - E[Y]\ )\ ]$

Properties of covariance

1. $Cov(X,X) = Var(X)$

   $E((X-E_x)(X-E_x)) \underset{(2)}{=} E(X^2) - E^2(X) = Var(X)$
2. $Cov(X,Y) = E(XY) - E(X)\cdot E(Y)$

   $((X-E_X)(Y-E_Y)) = E(XY) - E_XE_Y - E_XE_Y + E_XE_Y = E(XY) - E_XE_Y$
3. X, Y are independent $\Longrightarrow Cov(X,Y) = 0$

   $Cov(X,Y) \underset{(2)}{=} E_XE_Y - E_XE_Y = 0$
4. X, Y are independent $\Longrightarrow Var(X+Y) = Var(X) + Var(Y)$

   $E((X+Y)^2) - (E(X)+E(Y))^2 = E(X^2) + E(Y^2) + 2E(XY) - E^2(X) - 2E(X)E(Y) - E^2(Y) = Var(X) + Var(Y) - 2Co$

X, Y are uncorrelated $\Longleftrightarrow\ E(XY) = E(X)E(Y)\ \Longleftrightarrow\ Cov(X,Y) = 0$

Example

$X_1, \ldots, X_n$ -- independent random variables with Bernoulli distribution

$S_n = X_1 + \ldots + X_n$

What is the probability mass function of S_n?

$P(X_i = 1) = p;\ P(X_i = 0) = 1-p$

$S_n \in \{\ 0,\ 1, \ldots, n\ \}$

$$P(S_n = k) = \sum \underbrace{P(X_1, = 1, \ldots, X_u = 1, X_{u+1} = 0, \ldots, X_k = 1, \ldots, X_n = 0)}_{(*)\,=\,Randomly\ choose\ k\ variables\ that\ are\ 1,\ others\ are\ 0}$$

$P(X_1, = 1, \ldots, X_u = 1, X_{u+1} = 0, \ldots, X_k = 1, \ldots, X_n = 0) \underset{X_i\ are\ ind.}{=} P(X_1 = 1)\cdot\ldots\cdot P(X_u = 1)\ldots P(X_{u+1} = 0)\ldots = p^k(1-p)^{n-k}$

$\Rightarrow\ S_n = \binom{n}{k}p^k(1-p)^{n-k}$

$E(S_n) = \sum_1^n E(X_i) = np$

$Var(S_n) \underset{X_i \, are \, ind.}{=} \sum_1^n Var(X_i) = \sum_1^n \underbrace{E(X_i^2)}_{=p} - \underbrace{E^2(X_i)}_{=p^2} = np(1-p)$

$E(X^2) = p \cdot 1^2 + (1-p) \cdot 0^2 = p$ (proof)

22-11-04

Joint probability density function (JPDF) of two random variables

A two-dimensional box is a subset of R^2 defined by $J = [a,b] \times [c,d]$

X, Y -- random continuous variables.

JPDF is a non-negative $f \ : \ R \to R$ such that:

(version 1) $\forall s, t \in R \ : \ P[X \leqslant s, Y \leqslant t] = \int\limits_{-\infty}^{s} \int\limits_{-\infty}^{t} f(x,y) dy dx$

(version 2) for a two-dimensional box $J = [s_1, s_2] \times [t_1, t_2]$ $P[s_1 \leqslant X \leqslant s_2, t_1 \leqslant Y \leqslant t_2] = \int\limits_{s_1}^{s_2} \int\limits_{t_1}^{t_2} f(x,y) dy dx$

(version 3) $\forall B, B \subset R \ : \ P[X, Y \in B] = \int \int_B f(x,y) dx dy$

$\int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} f(x,y) dx dy = 1$

Example 1

X, Y have a JPDF given by

$f(x,y) = \begin{cases} c(x^2+y^2), \ 0 \leqslant x, y \leqslant 1 \\ 0, \ otherwise \end{cases}$

1. Find c

   $\int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} f(x,y) dx dy = \int\limits_0^1 \int\limits_0^1 c(x^2+y^2) dx dy = \frac{2c}{3} = 1; \ c = \frac{3}{2}$

2. Find PDFs for X and Y

   $f_{X,Y}(x,y) = \begin{cases} \frac{3}{2}(x^2+y^2), \ 0 \leqslant x, y \leqslant 1 \\ 0, \ otherwise \end{cases}$

   $P_X(x_0) = \sum_y P_{X,Y}(x_0, y)$

   $f_X(x_0) = \int\limits_{-\infty}^{+\infty} f_{X,Y}(x_0, y) dy = \begin{cases} \int\limits_0^1 \frac{3}{2}(x^2+y^2) dy, \ 0 \leqslant y \leqslant 1 \\ 0, \ otherwise \end{cases} = \begin{cases} \frac{3}{2}(x^2+\frac{1}{3}), \ 0 \leqslant y \leqslant 1 \\ 0, \ otherwise \end{cases}$

   $P_Y(y_0) = \sum_x P_{X,Y}(x, y_0); \ f_Y(y_0) = \int\limits_{-\infty}^{+\infty} f_{X,Y}(x, y_0) dx = \begin{cases} \frac{3}{2}(y^2+\frac{1}{3}), \ 0 \leqslant y \leqslant 1 \\ 0, \ otherwise \end{cases}$

Example 2: point from a square $[0,1]^2$

A point $P=(x,y)$ is chosen informly and randomly from a square $[0,1]\times[0,1]$.

X, Y denote coordinates of P. Find PDFs for X and Y.

$$f_{X,Y}(x,y)=\begin{cases}\frac{1}{s_{[0,1]^2}}=1,\ 0\leqslant x,y\leqslant 1\\ 0,\ otherwise\end{cases}$$

$$f_X(x)=\begin{cases}1,\ 0\leqslant x\leqslant 1\\ 0,\ otherwise\end{cases}$$

$$f_Y(y)=\begin{cases}1,\ 0\leqslant y\leqslant 1\\ 0,\ otherwise\end{cases}$$

Example 2: point from a circle

A point $P=(x,y)$ is chosen randomly from a circle $x^2+y^2\leqslant 1$

$$f_{X,Y}(x,y)=\begin{cases}\frac{1}{S_{circle}}=\frac{1}{\pi},\ x^2+y^2\leqslant 1\\ 0,\ otherwise\end{cases}$$

$$-1\leqslant x\leqslant 1;\quad f_X(x)=\int\limits_{-\infty}^{+\infty}f_{X,Y}(x,y)dy=\int\limits_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\frac{1}{\pi}dy=\frac{y}{\pi}\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}=\frac{2\sqrt{1-x^2}}{\pi}$$

$$f_X(x)=\begin{cases}\frac{2\sqrt{1-x^2}}{\pi},\ -1\leqslant x\leqslant 1\\ 0,\ otherwise\end{cases}$$

$$f_Y(y)=\begin{cases}\frac{2\sqrt{1-y^2}}{\pi},\ -1\leqslant y\leqslant 1\\ 0,\ otherwise\end{cases}$$

Independence

Continuous random variables X, Y are independent if $f_{X,Y}(x,y)=f_X(x)f_Y(y)$

Example 1: exponential variables

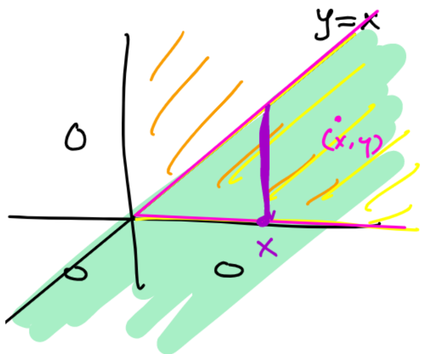
X and Y are independent exponential random variables with parameters λ and β. Find the prob. of the event that $Y\leqslant X$.

$$f_X(x)=\begin{cases}\lambda e^{-\lambda x},\ x\geqslant 0\\ 0,\ x<0\end{cases}$$

$$f_Y(y)=\begin{cases}\beta e^{-\beta y},\ y\geqslant 0\\ 0,\ y<0\end{cases}$$

$$f_{X,Y}(x,y)=f_X(x)f_Y(y)=\begin{cases}\lambda\beta e^{-(\lambda x+\beta y)},,\ x,y\geqslant 0\\ 0,\ otherwise\end{cases}$$

$$P[Y\leqslant x]=\int\limits_0^{+\infty}\int\limits_0^{x}\lambda\beta e^{-(\lambda x+\beta y)}dydx=\int\limits_0^{+\infty}\lambda e^{-\lambda x}-\lambda e^{-(\lambda+\beta)x}dx=\frac{\beta}{\lambda+\beta}$$



Example 2: choose from the interval (0, 1)

X and Y are chosen randomly and *independently* from the interval $(0,1)$. Define $Z = \frac{Y}{X}$. Compute the prob. distribution function of Z.

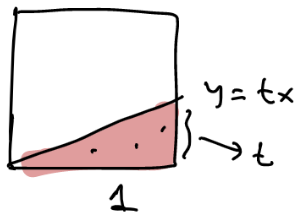
$f_X(x) = \begin{cases} 1, \ 0 < x < 1 \\ 0, \ otherwise \end{cases}$

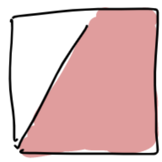
$f_Y(y) = \begin{cases} 1, \ 0 < y < 1 \\ 0, \ otherwise \end{cases}$

$f_{X,Y}(x,y) = \begin{cases} 1, \ 0 < x, y < 1 \\ 0, \ otherwise \end{cases}$

$Z = \frac{Y}{X} \in [0, +\infty)$

$F_Z(t) = P[Z \leqslant t] = P[Y \leqslant tX] = \begin{cases} S_{triangle} = \frac{t}{2}, t < 1 \\ S_{trapezoid} = 1 - \frac{1}{2t}, t \geqslant 1 \end{cases}$





22-11-09

JPDF of n random variables

$X_1, \ldots, X_n$ -- continuous random variables.

$f(X_1, \ldots, X_n)$ is a joint probability density function if

$$\forall B \subset R^n \ : \ P[(X_1, \ldots, X_n) \in B] = \int \int_B f(x_1, \ldots, x_n) dx_1 \ldots dx_n$$

Conditional probability mass function

X and Y are discrete random variables.

$p_Y(y) = P[Y = y] > 0$

The conditional probability mass function of X with condition $Y = y$ is defined by

$$p_{X|Y}(x|y) = P[X = x | Y = y] = \frac{P[X = x, Y = y]}{P[Y = y]} = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

Example 1

Find prob. of X give Y.

	$Y=-1$	$Y=0$	$Y=1$	
$X=-1$	$1/10$	0	$1/10$	2/10
$X=0$	$1/10$	$2/10$	$2/10$	5/10
$X=1$	$3/10$	0	0	3/10
	5/10	2/10	3/10	

$X\|Y$	$Y=-1$	$Y=0$	$Y=1$
$X=-1$	$\frac{1}{5}$	0	$\frac{1}{3}$
$X=0$	$\frac{1}{5}$	1	$\frac{2}{3}$
$X=1$	$\frac{3}{5}$	0	0

Example 2: a coin and streaks of Heads.

A coin is flipped 3 times.

N = # of Heads, S = length of the longest streak of Heads.

Determine the joint probability mass function of N given S and S given N.

N \ S	0	1	2	3	
0	1/8	0	0	0	1/8
1	0	3/8	0	0	3/8
2	0	1/8	2/8	0	3/8
3	0	0	0	1/8	1/8
	1/8	4/8	2/8	1/8	

HHH (3,3) HHT (2,2) HTH (2,1) HTT (1,1)

THH (2,2) THT (1,1) TTH (1,1) TTT (0,0)

N|S

N \ S	0	1	2	3
0	1	0	0	0
1	0	3/4	0	0
2	0	1/4	1	0
3	0	0	0	1

S|N

	0	1	2	3
0	1	0	0	0
1	0	1	0	0
2	0	1/3	2/3	0
3	0	0	0	1

Independence based on conditional prob

X and Y are *independent* random variables if $p_{X|Y}(x,y) = p_X(x), \forall y$

22-11-11

Conditional expectation

X, Y are random variables.

$x_1, \dots, x_n$ -- values for X.

$E(X|Y=y)$ -- cond. expectation of X given Y.

$E(X|Y=y) = \sum\limits_i x_i p_{X|Y}(x_i, y)$

Example

$p_{X,Y}(x,y)$ is defined by the table:

	$Y=-1$	$Y=0$	$Y=1$
$X=-1$	$1/10$	0	$1/10$
$X=0$	$1/10$	$2/10$	$2/10$
$X=1$	$3/10$	0	0

$X\|Y$	$Y=-1$	$Y=0$	$Y=1$
$X=-1$	$1/5$	0	$1/3$
$X=0$	$1/5$	1	$2/3$
$X=1$	$3/5$	0	0

$E[X|Y=-1] = (-1)\cdot\frac{1}{5} + 0\cdot\frac{1}{5} + 1\cdot\frac{3}{5} = \frac{2}{5}$

$E(X|Y=0) = \frac{1}{5}$

$E(X|Y=1) = (-1)\cdot\frac{1}{3} + 0\cdot\frac{2}{3} + 1\cdot 0 = -\frac{1}{3}$

22-11-16

\todo

Example

X is a standard Normal random variable. Estimate $N(t)$.

$f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$

$N(t) = P[|X| \geqslant t] \underset{f_X(x)\ is\ symmetric}{=} 2P[X \geqslant t] = 2\int\limits_t^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx \leqslant$

$\leqslant \frac{2}{\sqrt{2\pi}}\int\limits_t^\infty \frac{x}{t}e^{-\frac{x^2}{2}}dx = \frac{2}{\sqrt{2\pi}t}\cdot(-e^{-\frac{x^2}{2}}|_t^\infty) = \frac{2}{\sqrt{2\pi}t}e^{-\frac{t^2}{2}}$

Markov's inequality

Useful for large t (otherwise, the upper bound for P will be bigger than 1, senseless)

A positive random variable X has expectation μ. Then

$$N(t) = P[X \leqslant t] \ \leqslant \ \frac{\mu}{t}, \ \ \forall t > 0$$

Proof:

Compare X with another random variable Y, such that

$$Y = \begin{cases} 0, \ X \leqslant t \\ t, \ X \geqslant t \end{cases}$$

$$Y \leqslant X, \ \forall t > 0 \ \Rightarrow \ E(Y) \leqslant E(X), \forall t > 0$$

$$\left.\begin{aligned} E(Y) = 0 \cdot P[X \leqslant t] + t \cdot P[X \geqslant t] \\ E(X) = \mu \end{aligned}\right\} \ \Rightarrow \ P[X \geqslant t] = \frac{\mu}{t}, \ \forall t > 0$$

Example

The probability of a coin to land on heads is $\frac{1}{6}$. The coin is flipped 24 times.

Find a bound of the probability it lands on heads at least 20 times.

X -- the amount of heads.

$E(X) = 24 \cdot \frac{1}{6} = 4$

$P[\#head \geqslant 20] = \frac{E(X)}{20} = \frac{1}{5}$

Chebyshev's inequality

Random variable X has finite expectations and finite variance. Then

$$P(|X - E(X)| > t) \leqslant \frac{Var(X)}{t^2}$$

\todo insert a pic from slides

We can estimate the probability that a random variable is outside the segment