Problem 1

(5+5+3+5 points) Let $f(x) = e^{i\omega x}$

a) Compute the Taylor series for f around $c = \frac{\pi}{2}$.

Answer:

$$
\begin{aligned}
f(x) &= e^{i\omega x} & f\left(\frac{\pi}{2}\right) &= e^{i\omega\frac{\pi}{2}} \\
f'(x) &= (i\omega)e^{i\omega x} & f'\left(\frac{\pi}{2}\right) &= (i\omega)e^{i\omega\frac{\pi}{2}} \\
f''(x) &= (i\omega)^2 e^{i\omega x} & f''\left(\frac{\pi}{2}\right) &= (i\omega)^2 e^{i\omega\frac{\pi}{2}} \\
f'''(x) &= (i\omega)^3 e^{i\omega x} & f'''\left(\frac{\pi}{2}\right) &= (i\omega)^3 e^{i\omega\frac{\pi}{2}} \\
f''''(x) &= (i\omega)^4 e^{i\omega x} & f''''\left(\frac{\pi}{2}\right) &= (i\omega)^4 e^{i\omega\frac{\pi}{2}} \\
f'''''(x) &= (i\omega)^5 e^{i\omega x} & f'''''\left(\frac{\pi}{2}\right) &= (i\omega)^5 e^{i\omega\frac{\pi}{2}} \\
f''''''(x) &= (i\omega)^6 e^{i\omega x} & f''''''\left(\frac{\pi}{2}\right) &= (i\omega)^6 e^{i\omega\frac{\pi}{2}}
\end{aligned}
$$

So,

$$
\begin{aligned}
f(x) &= f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \dots \\
&= e^{i\omega\frac{\pi}{2}} + (i\omega)e^{i\omega\frac{\pi}{2}}\left(x - \frac{\pi}{2}\right) + \frac{(i\omega)^2 e^{i\omega\frac{\pi}{2}}}{2!}\left(x - \frac{\pi}{2}\right)^2 + \dots \\
&= e^{i\omega\frac{\pi}{2}} \sum_{n=0}^{\infty} \frac{(i\omega)^n (x - \frac{\pi}{2})^n}{n!}
\end{aligned}
$$

b) Use the Taylor series truncated after the n-th term to compute $f(\pi)$ for $n = 1, \ldots, 5$ and a general ω .

Answer:

$$f_1(x) = e^{i\omega\frac{\pi}{2}}\left[1 + (i\omega)\left(x - \frac{\pi}{2}\right)\right]$$
$$f_2(x) = e^{i\omega\frac{\pi}{2}}\left[1 + (i\omega)\left(x - \frac{\pi}{2}\right) + \frac{(i\omega)^2(x - \frac{\pi}{2})^2}{2!}\right]$$
$$f_3(x) = e^{i\omega\frac{\pi}{2}}\left[1 + (i\omega)\left(x - \frac{\pi}{2}\right) + \frac{(i\omega)^2(x - \frac{\pi}{2})^2}{2!} + \frac{(i\omega)^3(x - \frac{\pi}{2})^3}{3!}\right]$$
$$f_4(x) = e^{i\omega\frac{\pi}{2}}\left[1 + (i\omega)\left(x - \frac{\pi}{2}\right) + \frac{(i\omega)^2(x - \frac{\pi}{2})^2}{2!} + \frac{(i\omega)^3(x - \frac{\pi}{2})^3}{3!} + \frac{(i\omega)^4(x - \frac{\pi}{2})^4}{4!}\right]$$
$$f_5(x) = e^{i\omega\frac{\pi}{2}}\left[1 + (i\omega)\left(x - \frac{\pi}{2}\right) + \frac{(i\omega)^2(x - \frac{\pi}{2})^2}{2!} + \frac{(i\omega)^3(x - \frac{\pi}{2})^3}{3!} + \frac{(i\omega)^4(x - \frac{\pi}{2})^4}{4!} + \frac{(i\omega)^5(x - \frac{\pi}{2})^5}{5!}\right]$$

c) Compare values calculated in b) with the actual value of $f(\pi)$ for $\omega = 1$ and create a plot for the errors of the real part and imaginary part as a function of n. (Hint: Use Euler's formula).

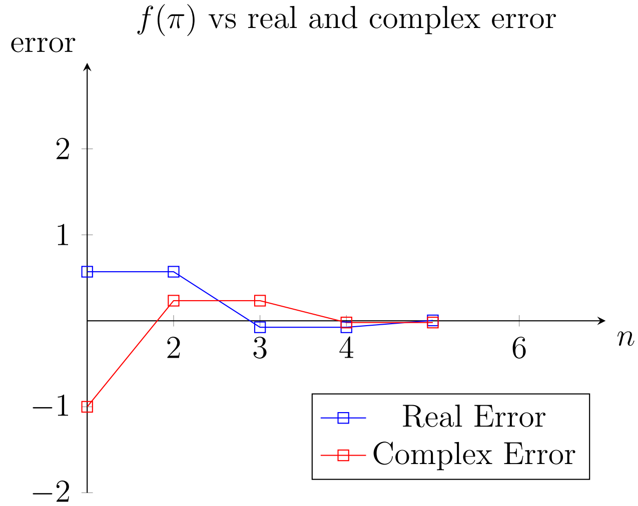
Answer: For $\omega = 1$ and $x = \frac{\pi}{2}$, $e^{i\omega x} = e^{i\frac{\pi}{2}} = i$

$$
\begin{aligned}
f(\pi) &= e^{i\pi} = -1 \\
f_1(\pi) &= i\left[1 + i\frac{\pi}{2}\right] \\
&= -\frac{\pi}{2} + i \\
f_2(\pi) &= i\left[1 + i\frac{\pi}{2} + \frac{i^2(\frac{\pi}{2})^2}{2!}\right] \\
&= i - \frac{\pi}{2} - i\frac{\pi^2}{8} \\
f_3(\pi) &= i\left[1 + i\frac{\pi}{2} + \frac{i^2(\frac{\pi}{2})^2}{2!} + \frac{i^3(\frac{\pi}{2})^3}{3!}\right] \\
&= i - \frac{\pi}{2} - i\frac{\pi^2}{8} + \frac{\pi^3}{48} \\
f_4(\pi) &= i\left[1 + i\frac{\pi}{2} + \frac{i^2(\frac{\pi}{2})^2}{2!} + \frac{i^3(\frac{\pi}{2})^3}{3!} + \frac{i^4(\frac{\pi}{2})^4}{4!}\right] \\
&= i - \frac{\pi}{2} - i\frac{\pi^2}{8} + \frac{\pi^3}{48} + i\frac{\pi^4}{384} \\
f_5(\pi) &= i\left[1 + i\frac{\pi}{2} + \frac{i^2(\frac{\pi}{2})^2}{2!} + \frac{i^3(\frac{\pi}{2})^3}{3!} + \frac{i^4(\frac{\pi}{2})^4}{4!} + \frac{i^5(\frac{\pi}{2})^5}{5!}\right] \\
&= i - \frac{\pi}{2} - i\frac{\pi^2}{8} + \frac{\pi^3}{48} + i\frac{\pi^4}{384} - \frac{\pi^5}{3840}
\end{aligned}
$$

Now,

$$
\begin{aligned}
f(\pi) - f_1(\pi) &\approx 0.5707 - i \\
f(\pi) - f_2(\pi) &\approx 0.5707 + 0.234i \\
f(\pi) - f_3(\pi) &\approx -0.0752 + 0.234i \\
f(\pi) - f_4(\pi) &\approx -0.0752 - 0.01997i \\
f(\pi) - f_5(\pi) &\approx 0.00452 - 0.01997i
\end{aligned}
$$

Plotting the errors:,



d) Show that the Taylor series for $f(x) = e^{i\omega x}$ around $c = \frac{\pi}{2}$ converges to f for $x \in [\frac{\pi}{2}, \pi]$

Answer: Error term $(E_n(x)) = \frac{f^{n+1}(\zeta_x)}{(n+1)!}(x-c)^{n+1}$

To show convergence for $x \in [\frac{\pi}{2}, \pi]$ i.e.:

$$\lim_{n\to\infty} E_n(x) \to 0$$

we can show convergence for $E(\frac{\pi}{2})$ and $E(\pi)$, and because of the Mean Value Theorem it will hold for everything in between. Namely, from Mean Value Theorem, we can deduce the following:

$$c \le \zeta_x \le x \to \frac{\pi}{2} \le \zeta_x \le \pi \to E\left(\frac{\pi}{2}\right) \le E\left(x\right) \le E\left(\pi\right)$$

So,

$$\lim_{n\to\infty} E_n(x) = \lim_{n\to\infty} e^{i\omega\zeta_x} \frac{(i\omega)^{n+1}(x-\frac{\pi}{2})^{n+1}}{(n+1)!}$$

For $x = \frac{\pi}{2}$ it is clearly 0. For $x = \pi$:

$$\lim_{n\to\infty} E_n(x) = \lim_{n\to\infty} e^{i\omega\zeta_x} \frac{(i\omega)^{n+1}(\frac{\pi}{2})^{n+1}}{(n+1)!}$$

As $n \to \infty$ the expression approaches 0 as $(n+1)!$ grows faster than the numerator. So, both $E(\frac{\pi}{2})$ and $E(\pi)$ approaches 0.

Alternative Method: An alternative solution is to break down $e^{i\omega\zeta_x}$ into $cos(\omega\zeta_x) + i \cdot sin(\omega\zeta_x)$ and show that both terms converge.

$$\begin{aligned}
\lim_{n\to\infty} E_n(x) &= \lim_{n\to\infty} e^{i\omega\zeta_x} \frac{(i\omega)^{n+1}(\frac{\pi}{2})^{n+1}}{(n+1)!} \\
&= \lim_{n\to\infty} \left[(cos(\omega\zeta_x) + i \cdot sin(\omega\zeta_x)) \frac{(i\omega)^{n+1}(\frac{\pi}{2})^{n+1}}{(n+1)!} \right] \\
&= \lim_{n\to\infty} \left[cos(\omega\zeta_x) \frac{(i\omega)^{n+1}(\frac{\pi}{2})^{n+1}}{(n+1)!} \right] + i \cdot \lim_{n\to\infty} \left[sin(\omega\zeta_x) \frac{(i\omega)^{n+1}(\frac{\pi}{2})^{n+1}}{(n+1)!} \right]
\end{aligned}$$

$cos(\omega\zeta_x)$ and $sin(\omega\zeta_x)$ are bounded.

$$-1 \le cos(\omega\zeta_x) \le 1$$
$$-1 \le sin(\omega\zeta_x) \le 1$$

Furthermore, for both parts, the numerators for the rest of the resulting equation are exponential, which grow much slower compared to the denominator, which is a factorial. Consequently, the limit for either part is 0.

$$\lim_{n\to\infty} E_n(x) = 0$$

This proves that the Taylor Series converges.

Problem 2

(5+5+2 points)

a) Compute the Taylor series for $f(x) = \sin(3x^2)$ around $c = 0$. (Hint: compute for $\sin(x)$ then substitute).

Answer: First, finding the taylor series for $\sin(x)$.

$$
\begin{aligned}
f(x) &= \sin(x) & f(0) &= \sin(0) = 0 \\
f'(x) &= \cos(x) & f'(0) &= \cos(0) = 1 \\
f''(x) &= -\sin(x) & f''(0) &= -\sin(0) = 0 \\
f'''(x) &= -\cos(x) & f'''(0) &= -\cos(0) = -1 \\
f''''(x) &= \sin(x) & f'''(0) &= \sin(0) = 0 \\
f'''''(x) &= \cos(x) & f''''(0) &= \cos(0) = 1
\end{aligned}
$$

We know,

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\
&= 0 + 1x + 0 + \frac{-1}{3!}x^3 + 0 + \frac{1}{5!}x^5 + 0 + ... \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}
\end{aligned}
$$

For $\sin(3x^2)$:

$$
\begin{aligned}
\sin(3x^2) &= \sum_{k=0}^{\infty} (-1)^k \frac{(3x^2)^{2k+1}}{(2k+1)!} \\
\sin(3x^2) &= \sum_{k=0}^{\infty} (-1)^k \frac{(x\sqrt{3})^{4k+2}}{(2k+1)!}
\end{aligned}
$$

b) The Taylor series for $f(x) = \frac{\sqrt{x+1}}{2}$ around $c = 0$ represents the function for $|x| \leq 1$. Show the Taylor expansion for $n = 1$ and the remainder term. Calculate the number of correct digits for $x = 0.0001$ and $x = -0.0001$.

Answer: $n = 1$ for the taylor series so we only need to get the first 2 terms of the series and the third term for remainder part:

$$f(x) = \frac{\sqrt{x+1}}{2} \qquad f(0) = \frac{1}{2}$$
$$f'(x) = \frac{1}{4\sqrt{x+1}} \qquad f'(0) = \frac{1}{4}$$
$$f''(x) = -\frac{1}{8\sqrt{(x+1)^3}} \qquad f''(0) = -\frac{1}{8}$$

Giving us,

$$\begin{aligned}
&\frac{f(c)}{0!}(x-c)^0 + \frac{f'(c)}{1!}(x-c)^1 \\
&= \frac{f(0)}{0!}(x)^0 + \frac{f'(0)}{1!}(x)^1 \\
&= \frac{1}{2} + \frac{1}{4}x
\end{aligned}$$

Remainder part:

$$\begin{aligned}
E_1(x) &= \frac{f^{(1+1)}(z)}{(1+1)!}x^{(1+1)} \\
&= \frac{f^2(z)}{2!}x^2 \\
&= -\frac{1}{2\cdot 8\sqrt{(z+1)^3}}x^2 \\
&= -\frac{1}{16\sqrt{(z+1)^3}}x^2
\end{aligned}$$

where z is in between c and x or x and c.
Now,

x	$T_1(x)$	$f(x)$	Correct digits
0.0001	0.500025	≈ 0.5000249994	6 (5 significant digits)
-0.0001	0.499975	≈ 0.49997499937	6 (5 significant digits)

c) Convert the following from one base to another and write down you calculations as an expansion.

Answer: We know Homer's scheme is:

```
while x > 0 do:
    a_i := x mod b
    x := x div b
    i := i + 1
```

i) $(530)_{10}$ to $(\ldots)_2$
Using Horner's scheme with $x = 530$, $i = 0$

$i = 0$:
$a_0 = 530 \mod 2 = 0$
$x = 530 \text{ div } 2 = 265$
$i = 0 + 1 = 1$

$i = 1$:
$a_1 = 265 \mod 2 = 1$
$x = 265 \text{ div } 2 = 132$
$i = 1 + 1 = 2$

$i = 2$:
$a_2 = 132 \mod 2 = 0$
$x = 132 \text{ div } 2 = 66$
$i = 2 + 1 = 3$

$i = 3$:
$a_3 = 66 \mod 2 = 0$
$x = 66 \text{ div } 2 = 33$
$i = 3 + 1 = 4$

This continues until $x \leq 0$.
Combining the digits we get: $(530)_{10} = (1000010010)_2$

ii) $(1.1011)_2$ to $(\ldots)_8$
First, split it into groups of 3: $(001\ .\ 101\ 100)_2$. Converting each of the groups into decimal base system:

$$(001)_2 = 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = (1)_{10}$$
$$(101)_2 = 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = (5)_{10}$$
$$(100)_2 = 1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0 = (4)_{10}$$

Converting each digit to octal using Horner's scheme (first $(1)_{10}$):

$i = 0$:
$a_0 = 1 \mod 8 = 1$
$x = 1 \text{ div } 8 = 0$
$i = 0 + 1 = 1$

Doing a similar process for the other digits gives us:

$$(1)_{10} = (1)_8$$
$$(5)_{10} = (5)_8$$
$$(4)_{10} = (4)_8$$

Finally, $(1.1011)_2 = (1.54)_8$