# **Numerical methods**

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#### **Numerical methods**

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### 23-02-02

## Org stuff

#### **Grade:**

- 100% exam
- bonus 10% via homeworks

### Now the lecture starts

Approaches to solving a problem:

- iteration:  $x_0 = c$ ;  $x_n = f(x_{n-1})$
- interpolation -- choose a function which is the closest to the initial function
- integration -- use integrals

## **Taylor series**

Given a function f:R o R, which is infinitely differentiable at  $c\in R$ . The Taylor series of f at c is:

$$f(x) = \sum_{n=0}^{+\infty} rac{f^{(n)}(x_0)}{n!} (x-c)^n$$

If c=0, then it is called **the Maclaurin series**.

*Note:* A power series have an interval/radius of convergence.  $f^{(n)} \in \text{radius of conv}$ .

Given a function 
$$f=\sum a_nx^n$$
 . Then a radius of convergence of  $f$  is  $R=rac{1}{\displaystyle\lim_{n o\infty}\sup\sqrt[n]{|a_n|}}$ 

*Note:* The smaller the difference between x and c, the faster the Taylor series converge.

## **Taylor theorem**

#### version 1

 $f\in C^{n+1}([a,b])$  (n+1 times continuously differentiable in [a, b])

Then for 
$$\forall c \in [a,b]$$
 we have that  $f = \sum\limits_{k=0}^n \frac{f^{(k)}(c)}{k!} (x-c)^k + \underbrace{\frac{f^{(n+1)}(\xi_x)}{(n+1)!} (x-c)^{n+1}}_{E_n(x)-remainder}$ , where  $\xi_x$  is

between x and c and depends on x.

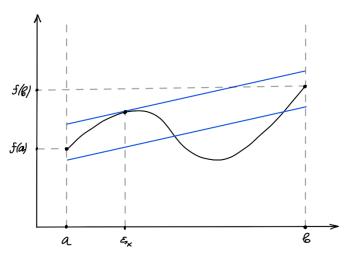
### version 2

$$f\in C^{n+1}([a,b])$$
 For  $x,x+h\in [a,b]$   $f(x+h)=\sum\limits_{k=0}^{n}rac{f^{(k)}(x)}{k!}h^k+rac{f^{(n+1)}(\xi_x)}{(n+1)!}h^{n+1}$  , where  $E_n(x)=O(h^{n+1})$ 

## 23-02-03

### Mean value theorem

For 
$$n=0$$
  $f(x)=f(c)+f'(\xi_x)(x-c)$   $x:=b,\ c:=a\ \Rightarrow\ f(b)=f(a)+f'(\xi_x)(b-a)\ \Rightarrow\ f'(\xi_x)=rac{f(b)-f(a)}{b-a}$ 



**Definition:** The Taylor series *represents* f at (.)x iff the Taylor series converge at (.)x.

$$f=e^x$$
 
$$c=0,\ e^x=\sum\limits_0^nrac{x^k}{k!}+rac{e^{\xi x}}{(n+1)!}x^{n+1}(*)$$
 For  $orall x\in R\ \exists s\in R_0^+\ :\ |x|\leqslant s\ \land\ |\xi_x|\leqslant s$ 

 $e^x \text{ is monotone increasing} \Rightarrow e^{\xi_x} \leqslant e^s \ \Rightarrow \ \lim_{n \to \infty} |\tfrac{e^{\varsigma_x}}{(n+1)!} x^{n+1}| \leqslant e^s \cdot \lim_{n \to \infty} |\tfrac{s^{n+1}}{(n+1)!}| = 0 \ \Rightarrow \ (*)$  represents  $e^x$  at x.

$$f = \ln(1+x) \text{ \tag{todo}}$$

$$c = 0$$

$$f^{(k)}(x) = (-1)^{k-1}(k-1)! \frac{1}{(1+x)^k}$$

$$g(x) = \sum_0^n \frac{(-1)^{k-1}}{k} x^k + \frac{(-1)^n}{n+1} \cdot \frac{1}{(1+\xi_x)^{n+1}} \cdot x^{n+1}$$

$$\lim_{n \to \infty} E_n(x) = \varprojlim_{n \to \infty} \frac{(-1)^n}{n+1} \cdot \lim_{n \to \infty} (\frac{x}{\xi_x+1})^{n+1} = 0 \ \Rightarrow \ 0 < \frac{x}{\xi_x+1} < 1$$

$$\Rightarrow \ x \leqslant 1, \ if \ \xi_x \in [0,x] \ \text{and} \ x > -1, \ if \ \xi_x \in [x,0] \ \Rightarrow \ g \ \text{represents} \ f \ \text{at} \ x \in (-1,1).$$

## Counting cos(0,1)

$$\begin{split} f &= \cos x \\ g(x) &= \sum_{0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} + (-1)^{n+1} \cos \xi_x \frac{x^{2(n+1)}}{(2(n+1))!}, \ c = 0 \\ &|(-1)^{n+1} \underbrace{\cos \xi_x}_{\leqslant 1} \cdot \frac{x^{2(n+1)}}{(2(n+1))!}| \leqslant |\frac{x^{2(n+1)}}{(2(n+1))!}| \\ &|\frac{0,1^{2(n+1)}}{(2(n+1))!}| \underset{n \to \infty}{\to} 0 \ \Rightarrow \ g \ \text{represents} \ f \ \text{at} \ (.)0,1. \end{split}$$

### 23-02-09

## **Base representation**

Every number  $x \in N$  can be written in the following form as a unique expansion with the resect to the base b, where  $b \in N/\{0\}$ , using digits  $a_i$ :

$$x = \sum_{i=0}^{n} a_i b^i$$
.

For a real number  $x \in R$  we can write:  $x = \sum\limits_{i=1}^{+\infty} a_{-i} b^{-i}$  .

#### **General remarks:**

• A number with simple representation in one base may be complicated to represent in another base:

$$0.1_{10} = (0.0001100110011...)_2$$

- ullet b=2 is binary, b=8 is octal, b=16 is hexadecimal
- To convert from base b to base 10, we perform the dolowwing computation:

$$y_b=\overline{a_n\ldots a_0}_b=\sum\limits_{i=0}^n a_nb^n=x_{10}$$

- ullet Conversion 2 o 8: three digits with base 2 represent one digit with base 8
- ullet Conversion 2 
  ightarrow 16: four digits with base 2 represent one digit with base 16

### **Example**

$$b=2;\ 1011_2=1\cdot 2^0+1\cdot 2^1+0\cdot 2^2+1\cdot 2^3=11_{10}$$

## **Euclid's algorithm**

Euclid's algorithm converts  $x_{10}$  to  $y_b$ .

- 1. Input  $x_{10}$
- 2. Determine  $\min n : x < b^{n+1}$
- 3. for i := n to 0 do:

$$a_i = x$$
 div  $b^i$ 

$$x=x \mod b^i$$

4. Output result  $\overline{a_n a_{n-1} \dots a_0} = y_b$ .

#### **Problems:**

- 1. Step 2 is inefficient
- 2. Division by large numbers can be problematic

### **Example**

```
egin{aligned} &1.\,13_{10}\longrightarrow y_2\ &2.\,\min n=3\,:\,13<2^4\ &3.\,i=3;\;\;a_3=1,\;x=5\ &i=2;\;\;a_2=1,\;x=1\ &i=1;\;\;a_1=0,\;x=1\ &i=0;\;\;a_0=1,\;x=0\ &4.\,\overline{a_3a_2a_1a_0}=1101_2 \end{aligned}
```

### Horner's scheme

- no division by large numbers
- no need in finding the amount of digits for  $y_b$  (aka n in Euclid's algorithm)
- applicable to real numbers, but needs a condition when we need to stop if representation with new base is infinite

```
1. Input x_{10}. i:=0.
```

```
2. 1 while (x > 0) {
    a[i++] = x % b;
    x /= b;
    4 }
```

3. 
$$\overline{a_n a_{n-1} \dots a_0} = y_b$$

## 23-02-10

## **Number normalization**

```
x=0.a_1a_2\dots a_k\cdot b^n with a_i\in\{0,\dots,b-1\}. a_1,\dots,a_k are digits. b is the base.
```

k is called *precision*. This is the actual amount of digits.

$$\overline{a_1 \dots a_k}$$
 is called *mantissa*.  $a_1 \neq 0$ .

n is called *exponent*. This is the distance from the current position of the floating point.  $n>0\Rightarrow$  move right, otherwise move left.

This form is called *normalization*. It makes representation of a number unique.

### **Examples**

 $0.099 \iff 0.99 \cdot 10^{-1}$ . We do not save leading zeros in the mantissa.

$$32.213 \Leftrightarrow 0.32213 \circ 10^2$$
.

$$1.101 \Leftrightarrow 0.1101 \circ 2^1$$
.

## **Single precision**

There are 4 bytes = 32 bits.

1 bit is for the sign of the number.

1 bit for the sign of the exponent.

7 bits for for the exponent. 7-bit largest number is  $\underbrace{1\dots1_2}_7=127$ .  $2^{127}\approx 10^{38}$ , so we have numbers from  $-10^{38}$  to  $10^{38}$ .

23 bits for mantissa. Actually, we are "able" to store 24 bits of a number, because, as we know,  $a_1 \neq 0 \Rightarrow$  it is always equal to 1 at base  $b=2 \Rightarrow$  it can be omitted.

We can have a better representation using *double precision* (8 bytes = 64 bits).

### **Problems with double numbers**

### **Example 1: sum of numbers**

Adding numbers is commutative, but not always associative:  $z + (y + w) \neq (z + y) + w$ .

For example, let us take  $x=y=0.00000033=0.33\cdot 10^{-6}$  ,  $z=0.00000034=0.34\cdot 10^{-6}$  ,  $w=1.000000=0.1\cdot 10^{1}$  .

$$((x+y)+z)+w=0.1\cdot 10^{-5}+0.1\cdot 10^{1}=1.000001$$

 $x+\left(y+\left(z+w
ight)
ight)=1.000000$ , since z+w can have only 7 significant digits in the mantissa.

Thus, it would be better to add numbers in increasing order of their size.

### **Example 2: subtraction of numbers**

Let us compute  $x - \sin x$  with x close to 0. For example,  $x = \frac{1}{15}$ .

$$x = 0.6666666667 \cdot 10^{-1}$$

$$\sin x = 0.6661729492 \cdot 10^{-1}$$

$$x - \sin x = 0.0004937175 \cdot 10^{-1} = 0.4937175000 \cdot 10^{-4}$$

Three zeros at the end of the mantissa mean a precision loss.

Thus, it would be better to avoid subtracting numbers of similar size.

The better idea is to use the Taylor series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

## Th. (about the lost of precision during subtraction)

Let x, y be normalized floating point numbers with x > y > 0 and base b = 2.

If  $\exists p,q\in N_0: 2^{-p}\leqslant 1-\frac{y}{x}\leqslant 2^{-q}$ , then at most p and at least q significant bits (digits at base 2) are lost during subtraction.