Numerical methods

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Org stuff

Grade:

- 100% exam
- bonus 10% via homeworks

Now the lecture starts

Approaches to solving a problem:

- iteration: $x_0 = c;\ x_n = f(x_{n-1})$
- interpolation -- choose a function which is the closest to the initial function
- integration -- use integrals

Taylor series

Given a function $f\ :\ R \to R$, which is infinitely differentiable at $c \in R$. The Taylor series of f at c is:

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(x_0)}{n!}(x-c)^n$$

If $c = 0$, then it is called ***the Maclaurin series***.

Note: A power series have an interval/radius of convergence. $f^{(n)} \in \textbf{radius of conv.}$

Given a function $f = \sum a_n x^n$. Then a radius of convergence of f is $R = \frac{1}{\lim\limits_{n\to\infty} \sup \sqrt[n]{|a_n|}}$

Note: The smaller the difference between x and c, the faster the Taylor series converge.

Taylor theorem

version 1

$f \in C^{n+1}([a,b])$ (n+1 times continuously differentiable in [a, b])

Then for $\forall c \in [a,b]$ we have that $f = \sum\limits_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + \underbrace{\frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x-c)^{n+1}}_{E_n(x) - remainder}$, where ξ_x is between x and c and depends on x.

version 2

$f \in C^{n+1}([a,b])$

For $x, x+h \in [a,b]$ $f(x+h) = \sum\limits_{k=0}^{n} \frac{f^{(k)}(x)}{k!}h^k + \frac{f^{(n+1)}(\xi_x)}{(n+1)!}h^{n+1}$, where $E_n(x) = O(h^{n+1})$

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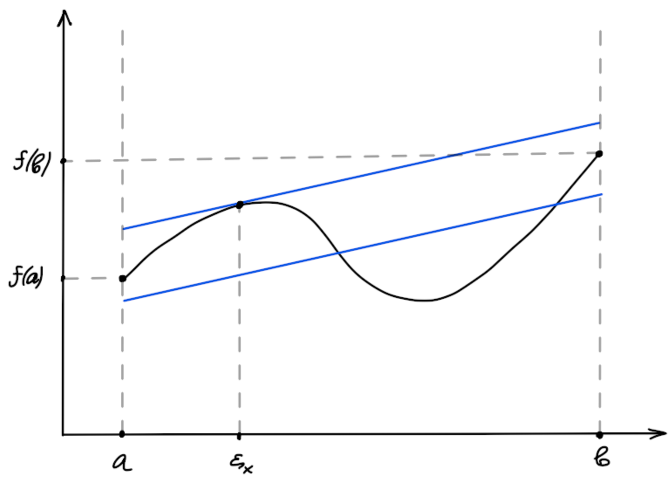
Mean value theorem

For $n = 0$ $f(x) = f(c) + f'(\xi_x)(x - c)$

$x := b,\ c := a\ \Rightarrow\ f(b) = f(a) + f'(\xi_x)(b - a)\ \Rightarrow\ f'(\xi_x) = \frac{f(b)-f(a)}{b-a}$



Definition: The Taylor series *represents* f at $(.)x$ iff the Taylor series converge at $(.)x$.

$f = e^x$

$c = 0,\ e^x = \sum_0^n \frac{x^k}{k!} + \frac{e^{\xi_x}}{(n+1)!}x^{n+1}(*)$

For $\forall x \in R\ \exists s \in R_0^+\ :\ |x| \leqslant s\ \wedge\ |\xi_x| \leqslant s$

e^x is monotone increasing $\Rightarrow e^{\xi_x} \leqslant e^s\ \Rightarrow\ \lim_{n\to\infty} |\frac{e^{\xi_x}}{(n+1)!}x^{n+1}| \leqslant e^s \cdot \lim_{n\to\infty} |\frac{s^{n+1}}{(n+1)!}| = 0\ \Rightarrow\ (*)$ represents e^x at x.

$f = \ln(1+x)$ **\todo**

$c = 0$

$f^{(k)}(x) = (-1)^{k-1}(k-1)!\frac{1}{(1+x)^k}$

$g(x) = \sum\limits_0^n \frac{(-1)^{k-1}}{k}x^k + \frac{(-1)^n}{n+1} \cdot \frac{1}{(1+\xi_x)^{n+1}} \cdot x^{n+1}$

$\lim\limits_{n\to\infty} E_n(x) = \underbrace{\lim\limits_{n\to\infty} \frac{(-1)^n}{n+1}}_{\to 0} \cdot \lim\limits_{n\to\infty} (\frac{x}{\xi_x+1})^{n+1} = 0 \Rightarrow 0 < \frac{x}{\xi_x+1} < 1$

$\Rightarrow x \leqslant 1,\ if\ \xi_x \in [0,x]$ and $x > -1,\ if\ \xi_x \in [x,0] \Rightarrow g$ represents f at $x \in (-1,1)$.

Counting $\cos(0,1)$

$f = \cos x$

$g(x) = \sum\limits_0^n (-1)^k \frac{x^{2k}}{(2k)!} + (-1)^{n+1}\cos\xi_x \frac{x^{2(n+1)}}{(2(n+1))!},\ c = 0$

$|(-1)^{n+1}\underbrace{\cos\xi_x}_{\leqslant 1} \cdot \frac{x^{2(n+1)}}{(2(n+1))!}| \leqslant |\frac{x^{2(n+1)}}{(2(n+1))!}|$

$|\frac{0{,}1^{2(n+1)}}{(2(n+1))!}| \underset{n\to\infty}{\to} 0 \Rightarrow g$ represents f at $(.)0,1$.

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Base representation

Every number $x \in N$ can be written in the folowing form as a unique expansion with the resect to the base b, where $b \in N/\{0\}$, using digits a_i:

$x = \sum\limits_{i=0}^n a_i b^i.$

For a real number $x \in R$ we can write: $x = \sum\limits_{i=1}^{+\infty} a_{-i}b^{-i}$.

General remarks:

- A number with simple representation in one base may be complicated to represent in another base:

  $0.1_{10} = (0.0001100110011\ldots)_2$
- $b = 2$ is binary, $b = 8$ is octal, $b = 16$ is hexadecimal
- To convert from base b to base 10, we perform the dolowwing computation:

  $y_b = \overline{a_n \ldots a_0}_b = \sum\limits_{i=0}^{n} a_n b^n = x_{10}$

Example

$b = 2;\ 1011_2 = 1 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdot 2^2 + 1 \cdot 2^3 = 11_{10}$

Euclid's algorithm

Euclid's algorithm converts x_{10} to y_b.

1. Input x_{10}
2. Determine $\min n\ :\ x < b^{n+1}$
3. for $i := n$ to 0 do:

   $a_i = x\ \texttt{div}\ b^i$

   $x = x\ \texttt{mod}\ b^i$
4. Output result $\overline{a_n a_{n-1} \ldots a_0} = y_b$.

Problems:

1. Step 2 is inefficient
2. Division by large numbers can be problematic

Example

1. $13_{10} \longrightarrow y_2$
2. $\min n = 3\ :\ 13 < 2^4$
3. $i = 3;\ \ a_3 = 1,\ x = 5$

   $i = 2;\ \ a_2 = 1,\ x = 1$

   $i = 1;\ \ a_1 = 0,\ x = 1$

$i = 0; \;\; a_0 = 1, \; x =$

4. $\overline{a_3 a_2 a_1 a_0} = 1101_2$

Horner's scheme

- no division by large number
- no need in finding the amount of digits for y_b (*aka* n in euclid's algorithm)

1. Input x_{10}. $i := 0$.
2.
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while (x > 0) {
    a[i++] = x % b;
    x /= b;
}
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3. $\overline{a_n a_{n-1} \dots a_0} = y_b$