Numerical methods

Author: Daria Shutina

23-02-02

Approaches to solving a problem:

- iteration: $x_0 = c;\ x_n = f(x_{n-1})$
- interpolation -- choose a function which is the closest to the initial function
- integration -- use integrals

Taylor series

Given a function $f\ :\ R \to R$, which is infinitely differentiable at $c \in R$. The Taylor series of f at c is:

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(x_0)}{n!}(x-c)^n$$

If $c = 0$, then it is called ***the Maclaurin series***.

Note: A power series have an interval/radius of convergence. $f^{(n)} \in \text{radius of conv.}$

Given a function $f = \sum a_n x^n$. Then a radius of convergence of f is $R = \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}}$

Note: The smaller the difference between x and c, the faster the Taylor series converge.

Taylor theorem

version 1

$f \in C^{n+1}([a,b])$ (n+1 times continuously differentiable in [a, b])

Then for $\forall c \in [a,b]$ we have that $f = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + \underbrace{\frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x-c)^{n+1}}_{E_n(x)\,-\,remainder}$, where ξ_x is between x and c and depends on x.

version 2

$f \in C^{n+1}([a,b])$

For $x, x+h \in [a,b]$ $f(x+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}h^k + \frac{f^{(n+1)}(\xi_x)}{(n+1)!}h^{n+1}$, where $E_n(x) = O(h^{n+1})$

23-02-03

Mean value theorem \todo

For $n = 0$ $f(x) = f(c) + f'(\xi_x)(x-c)$

$x := b,\ c := a\ \Rightarrow\ f(b) = f(a) + f'(\xi_x)(b-a)\ \Rightarrow\ f'(\xi_x) = \frac{f(b)-f(a)}{b-a}$

\todo картинка график

Definition: The Taylor series *represents* f at $(.)x$ iff the Taylor series converge at $(.)x$.

$f = e^x$

$c = 0,\ e^x = \sum\limits_0^n \frac{x^k}{k!} + \frac{e^{\xi_x}}{(n+1)!} x^{n+1} (*)$

For $\forall x \in R\ \exists s \in R_0^+ \ :\ |x| \leqslant s \ \wedge \ |\xi_x| \leqslant s$

e^x is monotone increasing $\Rightarrow e^{\xi_x} \leqslant e^s \ \Rightarrow \ \lim\limits_{n\to\infty} |\frac{e^{\xi_x}}{(n+1)!} x^{n+1}| \leqslant e^s \cdot \lim\limits_{n\to\infty} |\frac{s^{n+1}}{(n+1)!}| = 0 \ \Rightarrow \ (*)$ represents e^x at x.

$f = \ln(1+x)$ **\todo**

$c = 0$

$f^{(k)}(x) = (-1)^{k-1}(k-1)! \frac{1}{(1+x)^k}$

$g(x) = \sum\limits_0^n \frac{(-1)^{k-1}}{k} x^k + \frac{(-1)^n}{n+1} \cdot \frac{1}{(1+\xi_x)^{n+1}} \cdot x^{n+1}$

$\lim\limits_{n\to\infty} E_n(x) = \underbrace{\lim\limits_{n\to\infty} \frac{(-1)^n}{n+1}}_{\to 0} \cdot \lim\limits_{n\to\infty} (\frac{x}{\xi_x + 1})^{n+1} = 0 \ \Rightarrow \ 0 < \frac{x}{\xi_x+1} < 1$

$\Rightarrow \ x \leqslant 1,\ if\ \xi_x \in [0, x]$ and $x > -1,\ if\ \xi_x \in [x, 0] \ \Rightarrow \ g$ represents f at $x \in (-1, 1)$.

Counting $\cos(0,1)$

$f = \cos x$

$g(x) = \sum\limits_0^n (-1)^k \frac{x^{2k}}{(2k)!} + (-1)^{n+1} \cos \xi_x \frac{x^{2(n+1)}}{(2(n+1))!},\ c = 0$

$|(-1)^{n+1} \underbrace{\cos \xi_x}_{\leqslant 1} \cdot \frac{x^{2(n+1)}}{(2(n+1))!}| \leqslant |\frac{x^{2(n+1)}}{(2(n+1))!}|$

$|\frac{0{,}1^{2(n+1)}}{(2(n+1))!}| \underset{n\to\infty}{\to} 0 \ \Rightarrow \ g$ represents f at $(.)0,1$.