# Introduction to Robotics

*Author: Daria Shutina*



## 23-02-03

### Sections of mechanics

**Statics** is concerned with the analysis of loads (force and torque, or "moment") acting on physical systems that do not experience an acceleration (a=0), but rather, are in static equilibrium with their environment.

**Kinematics** describes the motion of points, bodies (objects), and systems of bodies (groups of objects) without considering properties of objects (mass, density) or the forces that caused the motion.

**Kinetics** is concerned with the relationship between motion and its causes, specifically, forces and torques.

# 23-02-07

## Matrix operations

1. Addition
2. Scaling: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot 3 = \begin{pmatrix} 3a & 3b \\ 3c & 3d \end{pmatrix}$
3. Dot product (inner product): $(x_1 \dots x_n) \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (a_1)$
4. Multiplication -- consists of several dot product operations

   It is a right-associative.
5. Transposition: $A^T \ : \ a_{ij}^T = a_{ji}, \ \forall i, j$

   Also, $(ABC)^T = C^T B^T A^T$
6. Inverse
7. Determinant
8. Power (only for square matrices)
9. Trace -- sum of elements on the diagonal

   $\mathrm{tr} AB = \mathrm{tr} BA$

   $\mathrm{tr}(A + B) = \mathrm{tr} A + \mathrm{tr} B$
10. etc (maybe)

## Vector norm

A norm is a function $f \ : \ R^n \to R$ that satisfies 4 properties:

1. Non-negativity: $\forall x \in R^n \ : \ f(x) \geqslant 0$
2. Definiteness: $f(x) = 0 \Rightarrow x = 0$
3. Homogeneity: $\forall x \in R^n, t \in R \ : \ f(tx) = |t| f(x)$
4. Triangle inequality: $\forall x, y \in R^n \ : \ f(x + y) \leqslant f(x) + f(y)$

Euclidean vector norm (2-norm): $||x||_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$

$||x||_2 = \sqrt{x^T x}$

General $p$-norms, $p \geqslant 1$: $||x||_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$

$$||x||_\infty = \max_i |x_i|$$

## Matrix multiplication

- associative: $(AB)C = A(BC)$
- distributive: $A(B + C) = AB + AC$
- non-commutative: $AB \neq BA$

## Determinant

### Properties

- $\det AB = \det BA$
- $\det A^{-1} = \frac{1}{\det A}$
- $\det A^T = \det A$

## Inverse

Given a matrix $A$, its inverse is a matrix $A^{-1}$ such that $AA^{-1} = A^{-1}A = 1$.

$\exists A^{-1} \Rightarrow A$ is invertible and non-singular. Otherwise, it is singular.

Inverse matrices do not exist for non-square matrices.

### Properties

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$

# 23-02-10

# 2d rotation matrix

Counter-clockwise by the angle $\theta$:

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Clockwise by the angle $\theta$:

$$R^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Properties of $R$:

- $R \cdot R^T = E$
- $\det R = 1$

# Homogeneous coordinates

## Adding a constant

In order to be able to add a constant, we add `1` to the end of a matrix:

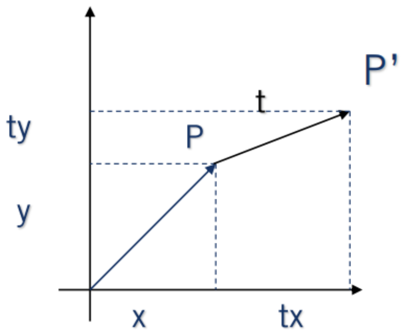
$$\begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x & y & 1 \end{pmatrix} = \begin{pmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{pmatrix}.$$

Such systems are called *homogeneous*. A homogeneous transformation matrix wil have a row `[0 0 1]` at the bottom, so there will be `1` at the bottom of the result matrix.

## 2d translation

The matrix have the form of $\begin{pmatrix} 1 & 0 & \dots & t_1 \\ 0 & 1 & \dots & t_2 \\ 0 & 0 & \dots & t_3 \\ & \dots & & \\ 0 & 0 & \dots & 1 \end{pmatrix}$.

$\begin{pmatrix} t_1 \\ \dots \\ t_n \end{pmatrix}$ is an addition of a constants to the initial coordinates.




$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{t} = (t_x, t_y) \rightarrow (t_x, t_y, 1)$$

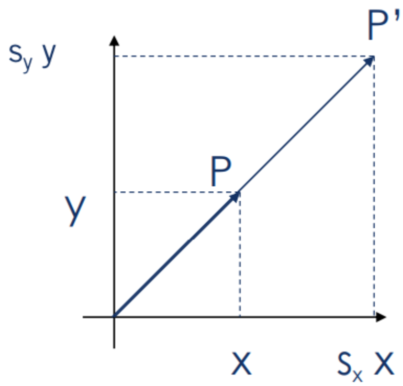
$$\mathbf{P'} \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \overset{t}{t_x} \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \overset{P}{x} \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ 0 & 1 \end{bmatrix} \cdot \mathbf{P} = \mathbf{T} \cdot \mathbf{P}$$

## Scaling

The matrix has a form of $\begin{pmatrix} s_1 & 0 & \dots & 0 & 0 \\ 0 & s_2 & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & s_n & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$.

$s_1, \dots, s_n$ are scalars for the initial coordinates.




$$\mathbf{P} = (\mathrm{x}, \mathrm{y}) \rightarrow \mathbf{P'} = (\mathrm{s}_\mathrm{x}\mathrm{x}, \mathrm{s}_\mathrm{y}\mathrm{y})$$

$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{P'} = (s_x x, s_y y) \rightarrow (s_x x, s_y y, 1)$$

$$\mathbf{P'} \rightarrow \begin{bmatrix} s_x x \\ s_y y \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{S}} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{S'} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \cdot \mathbf{P} = \mathbf{S} \cdot \mathbf{P}$$

## Linear independence

$I$ is a set of indexes.

The system $\{v_i\}_{i \in I}$ is *linearly dependent*, if one of the statements below are true:

1. $\nexists i \in I \;:\; v_i = \sum\limits_{j \neq i} v_j$
2. $\forall \{a_i\} \in R \;:\; \Sigma a_i v_i = 0 \;\Rightarrow\; a_i = 0, \forall i \in I.$

*Proof that* $1 \Leftrightarrow$ *2*:

## Matrix rank

Suppose we have a $m \times m$ matrix.

If its rank is $m$ then it is *full rank* and it has an inverse matrix.

If its rank is $< m$ then it is *singular* and does not have an inverse matrix. We also cannot restore the input.

## Eigenvectors and eigenvalues

Given a matrix $\mathcal{A}$ of size $n \times n$.

$v \in V/\{0\}$ is an *eigenvector* if $\exists\lambda \ : \ \mathcal{A}(v) = \lambda v$.

$\lambda$ is an *eigenvalue*.

$p_{\mathcal{A}}(\lambda) = \det(\mathcal{A} - \lambda E)$ is a *characteristic polynomial* of matrix $\mathcal{A}$.

### Th. (why do we need to substract $\lambda$ on the diagonal of the matrix)

$\lambda$ is an eigenvalue $\Leftrightarrow (\mathcal{A} - \lambda E)(v) = 0 \Leftrightarrow \det(A - \lambda E) = 0$

*Proof:*

$\mathcal{A} - \lambda v = 0 \ \Leftrightarrow \ (\mathcal{A} - \lambda E)v = 0 \ \Leftrightarrow Ker(\mathcal{A} - \lambda E) \neq \varnothing \underset{(*)}{\Leftrightarrow} \det(\mathcal{A} - \lambda E) = 0.$

For $(*)$, $\det(\mathcal{A} - \lambda E) = 0 \Leftrightarrow \mathcal{A} - \lambda E$ is not inversible $\Leftrightarrow Im(\mathcal{A} - \lambda E) \neq V \Leftrightarrow Ker(\mathcal{A} - \lambda E) \neq \varnothing$.

### Th. (relation between eigenvalue and characteristic polynomial)

$\lambda$ is an eigenvalue $\ \Leftrightarrow\ $ $\lambda$ is the root of the characteristic polynomial

*Proof:*

$\lambda$ is an eigenvalue $\ \Leftrightarrow\ \det(\mathcal{A} - \lambda E) = p_{\mathcal{A}}(\lambda) = 0$

## Example

$$\mathcal{A} = \begin{pmatrix} 3 & -1 \\ -3 & 5 \end{pmatrix}.$$

$$p_{\mathcal{A}}(\lambda) = \det \begin{pmatrix} 3-\lambda & -1 \\ -3 & 5-\lambda \end{pmatrix} = \lambda^2 - 8\lambda + 12 = 0.$$

$$\lambda_1 = 6,\ \lambda_2 = 2.$$

$$\mathcal{A}v_1 = \lambda_1 v_1 \ \Rightarrow \ v_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$\mathcal{A}v_2 = \lambda_2 v_2 \ \Rightarrow \ v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

## Diagonolize

Typically an $n \times n$ matrix has $n$ different eigenvalues and $n$ associated eigenvectors.

If there are $n$ independent eigenvectors, they can be used as a basis for $V$. Thus, we get a matrix $B = \begin{pmatrix} v_1 & \dots & v_n \end{pmatrix}$ of the size $n \times n$.

The diagonal matrix for $\mathcal{A}$ is $D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_2 & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & \lambda_n \end{pmatrix}$.

Finally, we come to $\mathcal{A} = BDB^{-1}$. You should be able to confirm this statement :^)

## Trace and determinant

$$p_{\mathcal{A}}(\lambda) = \det(\mathcal{A} - \lambda E) = \lambda^n - \color{blue}{\mathrm{tr}\mathcal{A}} \cdot \lambda^{n-1} + \dots + (-1)^n \cdot \color{red}{\det \mathcal{A}}.$$

$$\mathrm{tr}\mathcal{A} = \mathrm{tr}D = \lambda_1 + \dots + \lambda_n.$$

$$\det \mathcal{A} = \det D = \lambda_1 \cdot \dots \cdot \lambda_n.$$