

Introduction to Robotics

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Sections of mechanics

Statics is concerned with the analysis of loads (force and torque, or "moment") acting on physical systems that do not experience an acceleration ($a=0$), but rather, are in static equilibrium with their environment.

Kinematics describes the motion of points, bodies (objects), and systems of bodies (groups of objects) without considering properties of objects (mass, density) or the forces that caused the motion.

Kinetics is concerned with the relationship between motion and its causes, specifically, forces and torques.

23-02-07

Matrix operations

1. Addition

2. Scaling: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot 3 = \begin{pmatrix} 3a & 3b \\ 3c & 3d \end{pmatrix}$

3. Dot product (inner product): $(x_1 \dots x_n) \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (a_1)$

4. Multiplication -- consists of several dot product operations

It is a right-associative.

5. Transposition: $A^T : a_{ij}^T = a_{ji}, \forall i, j$

Also, $(ABC)^T = C^T B^T A^T$

6. Inverse

7. Determinant

8. Power (only for square matrices)

9. Trace -- sum of elements on the diagonal

$$\text{tr}AB = \text{tr}BA$$

$$\text{tr}(A + B) = \text{tr}A + \text{tr}B$$

10. etc (maybe)

Vector norm

A norm is a function $f : R^n \rightarrow R$ that satisfies 4 properties:

1. Non-negativity: $\forall x \in R^n : f(x) \geq 0$

2. Definiteness: $f(x) = 0 \Rightarrow x = 0$

3. Homogeneity: $\forall x \in R^n, t \in R : f(tx) = |t|f(x)$

4. Triangle inequality: $\forall x, y \in R^n : f(x + y) \leq f(x) + f(y)$

Euclidean vector norm (2-norm): $\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$

$$\|x\|_2 = \sqrt{x^T x}$$

General p -norms, $p \geq 1$: $\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}$

$$\|x\|_\infty = \max_i |x_i|$$

Matrix multiplication

- associative: $(AB)C = A(BC)$
- distributive: $A(B + C) = AB + AC$
- non-commutative: $AB \neq BA$

Determinant

Properties

- $\det AB = \det BA$
- $\det A^{-1} = \frac{1}{\det A}$
- $\det A^T = \det A$

Inverse

Given a matrix A , its inverse is a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$.

$\exists A^{-1} \Rightarrow A$ is invertible and non-singular. Otherwise, it is singular.

Inverse matrices do not exist for non-square matrices.

Properties

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$

23-02-10

2d rotation matrix

Counter-clockwise by the angle θ :

$$R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

Clockwise by the angle θ :

$$R^T = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

Properties of R :

- $R \cdot R^T = E$
- $\det R = 1$

Homogeneous coordinates

Adding a constant

In order to be able to add a constant, we add **1** to the end of a matrix:

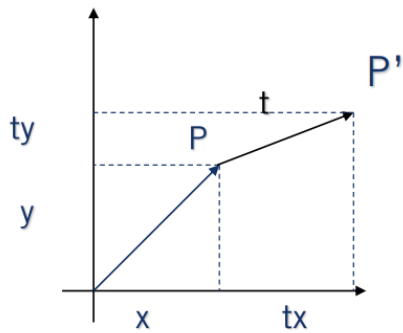
$$\begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x & y & 1 \end{pmatrix} = \begin{pmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{pmatrix}.$$

Such systems are called *homogeneous*. A homogeneous transformation matrix will have a row **[0 0 1]** at the bottom, so there will be **1** at the bottom of the result matrix.

2d translation

The matrix have the form of $\begin{pmatrix} 1 & 0 & \dots & t_1 \\ 0 & 1 & \dots & t_2 \\ 0 & 0 & \dots & t_3 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}$.

$\begin{pmatrix} t_1 \\ \dots \\ t_n \end{pmatrix}$ is an addition of a constants to the initial coordinates.



$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

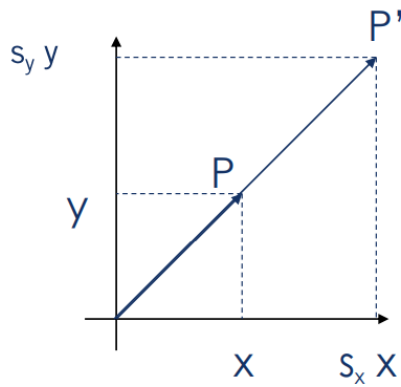
$$\mathbf{t} = (t_x, t_y) \rightarrow (t_x, t_y, 1)$$

$$\begin{aligned} \mathbf{P}' &\rightarrow \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ 0 & 1 \end{bmatrix} \cdot \mathbf{P} = \mathbf{T} \cdot \mathbf{P} \end{aligned}$$

Scaling

The matrix has a form of $\begin{pmatrix} s_1 & 0 & \dots & 0 & 0 \\ 0 & s_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & s_n & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$.

s_1, \dots, s_n are scalars for the initial coordinates.



$$\mathbf{P} = (x, y) \rightarrow \mathbf{P}' = (s_x x, s_y y)$$

$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{P}' = (s_x x, s_y y) \rightarrow (s_x x, s_y y, 1)$$

$$\mathbf{P}' \rightarrow \begin{bmatrix} s_x x \\ s_y y \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{S}} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{S}' & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \cdot \mathbf{P} = \mathbf{S} \cdot \mathbf{P}$$

Linear independence

I is a set of indexes.

The system $\{v_i\}_{i \in I}$ is *linearly dependent*, if one of the statements below are true:

1. $\nexists i \in I : v_i = \sum_{j \neq i} v_j$
2. $\forall \{a_i\} \in R : \sum a_i v_i = 0 \Rightarrow a_i = 0, \forall i \in I.$

Matrix rank

Suppose we have a $m \times m$ matrix.

If its rank is m then it is *full rank* and it has an inverse matrix.

If its rank is $< m$ then it is *singular* and does not have an inverse matrix. We also cannot restore the input.

Eigenvectors and eigenvalues

Given a matrix \mathcal{A} of size $n \times n$.

$v \in V/\{0\}$ is an *eigenvector* if $\exists \lambda : \mathcal{A}(v) = \lambda v$.

λ is an *eigenvalue*.

$p_{\mathcal{A}}(\lambda) = \det(\mathcal{A} - \lambda E)$ is a *characteristic polynomial* of matrix \mathcal{A} .

Th. (why do we need to subtract λ on the diagonal of the matrix)

λ is an eigenvalue $\Leftrightarrow (\mathcal{A} - \lambda E)(v) = 0 \Leftrightarrow \det(\mathcal{A} - \lambda E) = 0$

Proof:

$$\mathcal{A} - \lambda v = 0 \Leftrightarrow (\mathcal{A} - \lambda E)v = 0 \Leftrightarrow \text{Ker}(\mathcal{A} - \lambda E) \neq \emptyset \underset{(*)}{\Leftrightarrow} \det(\mathcal{A} - \lambda E) = 0.$$

For $(*)$, $\det(\mathcal{A} - \lambda E) = 0 \Leftrightarrow \mathcal{A} - \lambda E$ is not invertible $\Leftrightarrow \text{Im}(\mathcal{A} - \lambda E) \neq V \Leftrightarrow \text{Ker}(\mathcal{A} - \lambda E) \neq \emptyset$.

Th. (relation between eigenvalue and characteristic polynomial)

λ is an eigenvalue $\Leftrightarrow \lambda$ is the root of the characteristic polynomial

Proof:

$$\lambda \text{ is an eigenvalue} \Leftrightarrow \det(\mathcal{A} - \lambda E) = p_{\mathcal{A}}(\lambda) = 0$$

Example

$$\mathcal{A} = \begin{pmatrix} 3 & -1 \\ -3 & 5 \end{pmatrix}.$$

$$p_{\mathcal{A}}(\lambda) = \det \begin{pmatrix} 3 - \lambda & -1 \\ -3 & 5 - \lambda \end{pmatrix} = \lambda^2 - 8\lambda + 12 = 0.$$

$$\lambda_1 = 6, \lambda_2 = 2.$$

$$\mathcal{A}v_1 = \lambda_1 v_1 \Rightarrow v_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$\mathcal{A}v_2 = \lambda_2 v_2 \Rightarrow v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Diagonalization

Typically an $n \times n$ matrix has n different eigenvalues and n associated eigenvectors.

If there are n independent eigenvectors, they can be used as a basis for V . Thus, we get a matrix $B = (v_1 \ \dots \ v_n)$ of the size $n \times n$.

The diagonal matrix for \mathcal{A} is $D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_2 & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & \lambda_n \end{pmatrix}.$

Finally, we come to $\mathcal{A} = BDB^{-1}$. You should be able to confirm this statement :^)

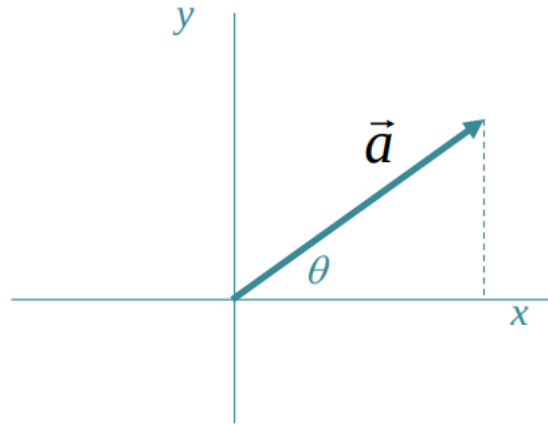
Trace and determinant

$$p_{\mathcal{A}}(\lambda) = \det(\mathcal{A} - \lambda E) = \lambda^n - \text{tr} \mathcal{A} \cdot \lambda^{n-1} + \dots + (-1)^n \cdot \det \mathcal{A}.$$

$$\text{tr} \mathcal{A} = \text{tr} D = \lambda_1 + \dots + \lambda_n.$$

$$\det \mathcal{A} = \det D = \lambda_1 \cdot \dots \cdot \lambda_n.$$

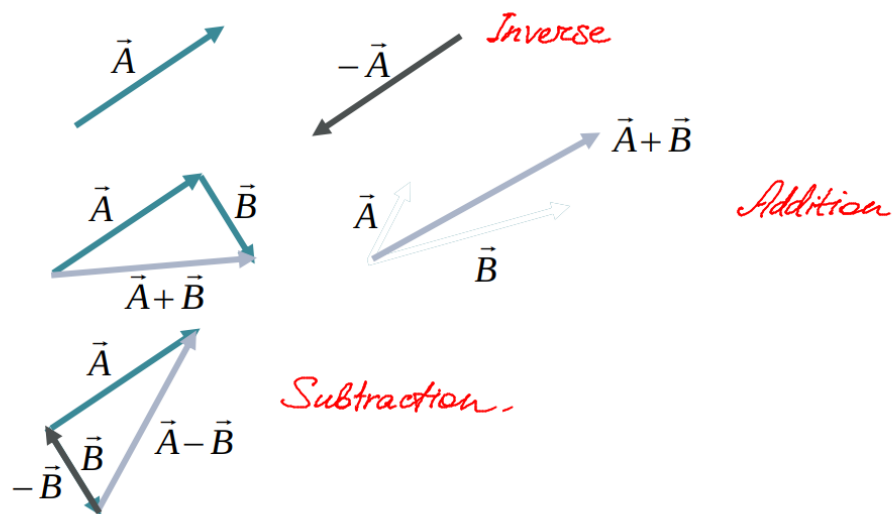
Vector math



$$a_x = a \cos \theta, \quad a_y = a \sin \theta$$

$$a = \sqrt{a_x^2 + a_y^2}$$

$$\theta = \arctan \frac{a_y}{a_x}$$



23-02-17

3d rotation

Rotate over the angle α around the z axis:

$$x' = x \cos \alpha - y \sin \alpha$$

$$y' = x \sin \alpha + y \cos \alpha$$

$$z' = z$$

Counter-clockwise direction

- around z-axis:

$$R_z(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 & 0 \\ \sin \alpha & \cos \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- around y-axis:

$$R_y(\alpha) = \begin{pmatrix} \cos \alpha & 0 & \sin \alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \alpha & 0 & \cos \alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- around x-axis:

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha & 0 \\ 0 & \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$