Problem 1

(3+3+3+3 points) Let $f(x) = x^3 - 7.5x^2 + 13.5x - 5$, and consider the starting points $x_0 = 0$, and $x_1 = 1$.

a) Check whether bisection, secant, and Newton's method can be applied.
Answer:
The conditions for each method to be applied are:
Bisection:

$\rightarrow$ $f \in C^0[x_0, x_1]$, because it is a polynomial

$\rightarrow$ $f(x_0) \cdot f(x_1) < 0$

$\rightarrow$ $f(0) \cdot f(1) = -5 \cdot (1 - 7.5 + 13.5 - 5) = -5 \cdot 2 = -10 < 0$

Newton:

$\rightarrow$ $f''(x)$ exists and is continuous (true as it is polynomial of degree 3)

$\rightarrow$ $f'(x) \neq 0$ at $[0, 1]$, i.e., roots of $f'(x)$ are outside the given range.
The roots of $f'(x)$ are ≈ 3.82288 and ≈ 1.17712, which are both outside the given range.

Secant:

$\rightarrow$ $f \in C^2[x_0, x_1]$

$\rightarrow$ $f(x_1) - f(x_0) \neq 0$, which is clear from above.

So, all the above methods can be applied.

b) Apply three steps of the bisection, Newton's and secant method. (For Newton's method start from $x_0 = 0$)
Answer:
Bisection Method:
Checking,

$$f(0) \cdot f(1) < 0$$

There does lie a root between 0 and 1.

Iteration 1:

$$c = \frac{0+1}{2} = 0.5$$

Bisect $[0, 1]$ into $[0, 0.5] \cup [0.5, 1]$ But first checking to see if $f(c) = 0$.

$$\begin{aligned}
&f(c)\\
&= f(0.5)\\
&= (0.5)^3 - 7.5(0.5)^2 + 13.5(0.5) - 5\\
&= 0.125 - 1.875 + 6.75 - 5\\
&= 0
\end{aligned}$$

So, the $f(c) = 0$, and we have found our root $c = 0.5$.

Newton's Method:

$$f'(x) = 3x^2 - 15x + 13.5$$

Starting from $x_0 = 0$

$$f(x_0) = -5 \qquad f'(x_0) = 13.5$$

$$x_2 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 + \frac{5}{13.5} \approx 0.37037$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \approx 0.48741$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = \approx 0.49986$$

Secant Method:

$$x_0 = 0 \qquad x_1 = 1$$

$$\begin{aligned}
x_2 &= x_1 - f(x_1) \cdot \frac{(x_0 - x_1)}{f(x_0) - f(x_1)} \\
&= 1 - 2 \cdot \frac{0 - 1}{-5 - 2} \approx 0.714286 \\
x_3 &= x_2 - f(x_2) \cdot \frac{(x_1 - x_2)}{f(x_1) - f(x_2)} \\
&= 0.714286 - 1.180758 \cdot \frac{(1 - 0.714286)}{2 - 1.180758} \approx 0.302491 \\
x_4 &= x_3 - f(x_3) \cdot \frac{(x_2 - x_3)}{f(x_2) - f(x_3)} \\
&= 0.302491 - (-1.574948) \cdot \frac{0.302491 - 0.714286}{-1.574948 - 1.180758} \approx 0.537841
\end{aligned}$$

c) Find the roots analytically and compare the errors of the results you computed in b).
Answer:
Now we know the coefficient of x^0 is -5 and the coefficient of x^3 is 1. So,$-5 \cdot 1 = -5$. Taking the factors of -5 gives us $\pm 1, \pm 5$. Let $x = 5$.

$$\begin{aligned} f(5) &= (5)^3 - 7.5(5)^2 + 13.5(5) - 5 \\ &= 125 - 187.5 + 67.5 - 5 \\ &= 0 \end{aligned}$$

So, $(x - 5)$ is a root.

$$\begin{array}{c|cccc} 5 & 1 & -7.5 & 13.5 & -5 \\ & \downarrow & 5 & -12.5 & 5 \\ \hline & 1 & -2.5 & 1 & 0 \end{array}$$

Giving us:

$$\begin{aligned} &(x-5)(x^2 - 2.5x + 1) \\ &= (x-5)(x^2 - 0.5x - 2x + 1) \\ &= (x-5)(x(x-0.5) - 2(x-0.5)) \\ &= (x-5)(x-2)(x-0.5) \end{aligned}$$

So, the roots are $x = 0.5, 2, 5$. The root closest to answer found in b is 0.5. Finally, finding the errors:

$$\begin{aligned} |\text{Bisection}| &= |0.5 - 0.5| = 0 \\ |\text{Newton}| &= |0.5 - 0.49986| = 0.00014 \\ |\text{Secant}| &= |0.5 - 0.537841| = 0.037841 \end{aligned}$$

d) Which of the above method's are expected to converge and why?
Answer:
All of the method's are expected to converge, because:
Bisection:

$\rightarrow$ if $f \in C^0[x_0, x_1]$ and $f(x_0) \cdot f(x_1) < 0$, it converges. And it also did converge.

Newton:

$\rightarrow$ $f(x_0) \cdot f(x_1) < 0$

$\rightarrow$ $f'(x) \neq 0$ at $[0, 1]$, i.e., roots of $f'(x)$ are outside the given range.
The roots of $f'(x)$ are ≈ 3.82288 and ≈ 1.17712, which are both outside the given range.

$\rightarrow$ $f''(x)$ exists and is continuous (true as it is polynomial) and $f''(x) > 0$ or $f''(x) < 0$ for $[0, 1]$

$\rightarrow$ x_0 also needs to be close enough to the root, which it is.

Secant:

$\rightarrow$ $f \in C^2[x_0, x_1]$

$\rightarrow$ x_0 and x_1 are close to the root

$\rightarrow$ Multiplicity of 1 is clear from c)

Problem 2

(8 points) Starting with $(0, 0)$ apply two iterations of Newton's method to solve the system of non-linear equations

$$-x^2 + x + 4y = 12$$
$$(x - 2)^2 + (2y - 3)^2 = 25$$

Answer:

$$f(x, y) = -x^2 + x + 4y - 12$$
$$g(x, y) = (x - 2)^2 + (2y - 3)^2 - 25$$

$$\vec{F}(x, y) = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}$$

Now, we calculate the Jacobian:

$$\mathbf{DF}(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} -2x+1 & 4 \\ 2(x-2) & 4(2y-3) \end{bmatrix}$$

We now have all the components required to start Newton's method:
Step 1:
Given $(x_0, y_0) = (0, 0)$.

$$\mathbf{DF}(0,0) = \begin{bmatrix} 1 & 4 \\ -4 & -12 \end{bmatrix}$$

$$\vec{F}(0,0) = \begin{bmatrix} -12 \\ -12 \end{bmatrix}$$

We have the following:

$$\mathbf{DF}(x_0,y_0) \cdot \begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \end{bmatrix} = -\vec{F}(x_0,y_0)$$

$$\begin{bmatrix} 1 & 4 \\ -4 & -12 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = 12 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \left(\frac{12}{-12+16} \right) \cdot \begin{bmatrix} -12 & -4 \\ 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = 3 \cdot \begin{bmatrix} -16 \\ 5 \end{bmatrix} = \begin{bmatrix} -48 \\ 15 \end{bmatrix}$$

Step 2:
$(x_1, y_1) = (-48, 15)$

$$\mathbf{DF}(-48,15) = \begin{bmatrix} 97 & 4 \\ -100 & 108 \end{bmatrix}$$

$$\vec{F}(-48,15) = \begin{bmatrix} -2304 \\ 3204 \end{bmatrix}$$

$$\mathbf{DF}(x_1,y_1) \cdot \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} = -\vec{F}(x_1,y_1)$$

$$\begin{bmatrix} 97 & 4 \\ -100 & 108 \end{bmatrix} \cdot \begin{bmatrix} x_2 + 48 \\ y_2 - 15 \end{bmatrix} = \begin{bmatrix} 2304 \\ -3204 \end{bmatrix}$$

$$\begin{bmatrix} x_2 + 48 \\ y_2 - 15 \end{bmatrix} = \left(\frac{1}{10876} \right) \cdot \begin{bmatrix} 108 & -4 \\ 100 & 97 \end{bmatrix} \cdot \begin{bmatrix} 2304 \\ -3204 \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{261648}{10876} \\ -\frac{80388}{10876} \end{bmatrix} + \begin{bmatrix} -48 \\ 15 \end{bmatrix} = \begin{bmatrix} -23.9426 \\ 7.6087 \end{bmatrix}$$

With this, two iterations are complete.

Problem 3

(5+5 points) Consider the values

x	-7	-6	0	5
y	-23	-54	-954	1

a) Derive the polynomial $p_\ell(x)$ in Lagrange form that interpolates the values at the given nodes.
Answer: First, we derive a table with index:

i	0	1	2	3
x	-7	-6	0	5
y	-23	-54	-954	1

Using this table, we can start deriving the nodes:

$$\begin{aligned}
L_0^3(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \\
&= \frac{(x-(-6))(x-0)(x-5)}{(-7-6)(-7-0)(-7-5)} \\
&= \frac{x^3+x^2-30X}{-84} \\
L_1^3(x) &= \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \\
&= \frac{(x+7)(x)(x-5)}{(-6+7)(-6)(-6-5)} \\
&= \frac{x(x^2+2x-35)}{66} \\
L_2^3(x) &= \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \\
&= \frac{(x+7)(x+6)(x-5)}{(7)(6)(-5)} \\
&= \frac{x^3+8x^2-23x-210}{-210} \\
L_3^3(x) &= \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \\
&= \frac{x(x^2+13x+42)}{660}
\end{aligned}$$

For Lagrange Interpolation, we know the following about the Collocation Matrix:

$$\Phi = I$$
$$\Phi\vec{\alpha} = \vec{P} \to \vec{\alpha} = \vec{P}$$

Therefore, we can calculate the polynomial as follows:

$$\begin{aligned}
P(x) &= y_0 \cdot L_0^3(x) + y_1 \cdot L_1^3(x) + y_2 \cdot L_2^3(x) + y_3 \cdot L_3^3(x) \\
&= 4x^3 + 35x^2 - 84x - 954
\end{aligned}$$

b) Use the polynomial and compute the interpolated value at -1. Apply Aitken's algorithm and recompute the interpolated value.
Answer:

$$P(-1) = -4 + 35 + 84 - 954 = -839$$

Again, we refer back to the table:

i	0	1	2	3
x	-7	-6	0	5
y	-23	-54	-954	1

Let, k is the step number, and i is the node number.
For $k = 0$,
For $i = 0$,

$$\frac{x - x_3}{x_0 - x_3} = \frac{-1-5}{-7-5} = \frac{1}{2}$$

For $i = 1$,

$$\frac{x - x_3}{x_1 - x_3} = \frac{-1-5}{-6-5} = \frac{6}{11}$$

For $i = 3$,

$$\frac{x - x_3}{x_2 - x_3} = \frac{-1-5}{0-5} = \frac{6}{5}$$

Now, we can apply one step of Aitken's:

$$\boxed{P_0^0 = -23} \xrightarrow{\frac{1}{2}} \boxed{P_0^1 = \tfrac{1}{2}(-23) + \tfrac{1}{2}(1) = -11}$$

$$\boxed{P_1^0 = -54} \xrightarrow{\frac{6}{11}} \boxed{P_1^1 = \tfrac{6}{11}(-54) + \tfrac{5}{11}(1) = -29}$$

$$\boxed{P_2^0 = -954} \xrightarrow{\frac{6}{5}} \boxed{P_2^1 = \tfrac{6}{5}(-954) - \tfrac{1}{5}(1) = -1145}$$

$$\boxed{P_3^0 = 1}$$

$1 - \frac{1}{2}$ $\quad 1 - \frac{6}{11}$ $\quad 1 - \frac{6}{5}$

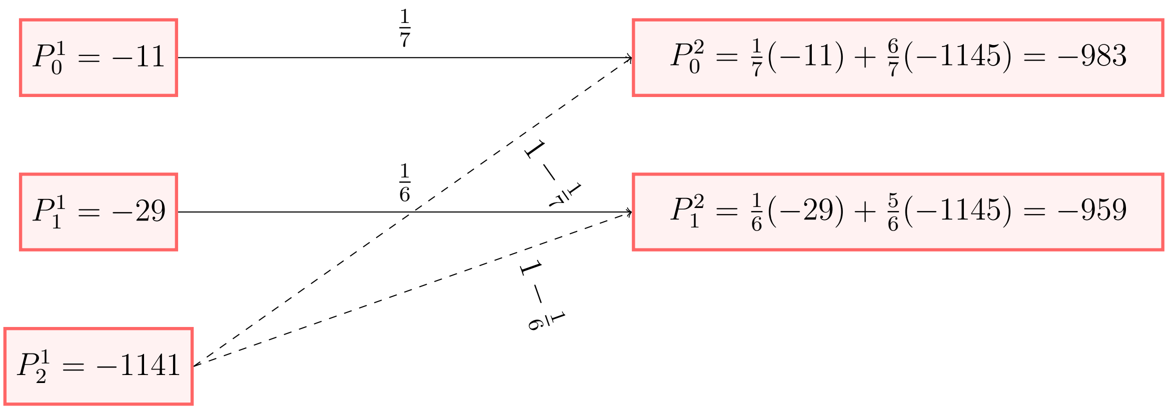
For $k = 1$,
For $i = 0$,

$$\frac{x - x_2}{x_0 - x_2} = \frac{-1-0}{-7-0} = \frac{1}{7}$$

For $i = 1$,

$$\frac{x - x_2}{x_1 - x_2} = \frac{-1-0}{-6-0} = \frac{1}{6}$$

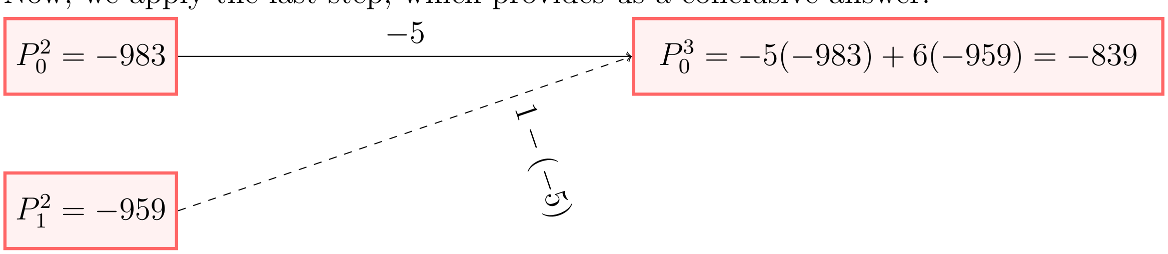
With this, we can apply another step of Aitken's:



For $k = 2$,
For $i = 0$,

$$\frac{x - x_1}{x_0 - x_1} = \frac{-1 + 6}{-7 + 6} = -5$$

Now, we apply the last step, which provides us a conclusive answer.



As we can see, the result is the same as what we found before.