Textbook: any thing called

Discrete Mathematics

37181: WEEK 1: LOGIC

A/Prof Murray Elder, UTS Wednesday 24 July 2019

- introduction, subject outline
- truth tables
- · logical equivalence
- tautology
- quantified statements
- $\boldsymbol{\cdot}$ negation of quantified statements
- SAT and P=?NP

Assessment Quizzes -30% Team assignment

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LOGIC

Definition

A <u>statement</u> is a sentence that can (th<u>eoretically</u>) be assigned a value of *true* or *false*.

LOGIC

Definition

A *statement* is a sentence that can (theoretically) be assigned a value of *true* or *false*.

Eg:

- 1. Um, like, whatever
- 2. All positive integers are prime
- 3. All lectures are recorded at UTS
- 4. In the year 4000BC, at this exact location, it was raining on the 5th of March at 10am
- 5. When will this lecture end?

LOGICAL CONNECTIVES

We can build up more complicated statements out of simpler ones using *logical connectives* like *and* and *or*.

Eg:

- 1. Murray is a statistician and Murray has brown hair.
- 2. Murray is a statistician or Murray has brown hair.

PRECISE MEANING: TRUTH TABLE

English (or any natural human language) can be imprecise, so instead of using our "intuitition" we define what "and" and "or" and "not" mean using truth tables.

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р	q	$p \wedge q$	р	q	$p \vee q$	_		
1	1	1	1	1	I		р	$\neg p$
1	0	0	1	0	1		1	0
0	1	Ò	0	1	1		0	0
0	0	0 0	0	0	1 0		,	

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1	1		1	1		р	$\neg p$
1	0		1	0		1	
0	1		0	1		0	
0	0		0	0			

Teenager speech is more precise: Eg: "Maths is awesome — NOT"

TRUTH TABLES FOR COMPOUND STATEMENTS

We can use truth tables to decide the truth values of more complicated statements, like $\neg p \lor q$:

р	q	$\neg p$	\vee	q
1	1	0	١	
1	0	0	0	
0	1	١	(
0	0	1		
			(2)	

TRUTH TABLES FOR COMPOUND STATEMENTS

We can use truth tables to decide the truth values of more complicated statements, like $\neg p \lor q$:

		$\neg p$		q	_	р	q	$\neg(p$	\vee	q)
1	1	00	- (1	1	0 0 0 1	ı	
						1	0	0	i	
0	1	t	1			0	1	0	i	
0	0	1	١			0	0)	D	
		(0)	2					(2)		

Note that this is different to saying $\neg(p \lor q)$, since the truth values are not the same

Ex: 7(pva) (

Complete the truth tables for these statements:

р	q	_	$(p \land q)$	р	q	$\neg p$	\vee	$\neg q$
1	1	0	l	1	1	0	0	Ø
1	0	(b	1	0	0	(- 1
0	1	\	Ø	0	1	1	(0
0	0	\	b 0	0	0	0 0 1 1	(1
						0		

le Morgan's Law.

LOGICALLY EQUIVALENT

When two (compound) statements have the same truth values we say they are logically equivalent.

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When two (compound) statements have the same truth values we say they are *logically equivalent*.

Eg:

					V
р	q	$p \vee \neg q$	$\neg(q \land \neg p)$		72 V(77p)
1	1			_	
1	0				
0	1				
0	0				
			l .		L

IMPLIES

In mathematics and logic we have a very specific meaning for "p implies q", or "if p then q", notation $p \rightarrow q$.

We define it using the following table:

р	q	$p \rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

IMPLIES

In mathematics and logic we have a very specific meaning for "p implies q", or "if p then q", notation $p \rightarrow q$.

We define it using the following table:

р	q	$p \rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

You may think that in English, "if it is raining then I get wet" means that the rain caused me to get wet. But in mathematics if-then has the meaning defined above: if "I am wet" is true and "it is raining" is false, the implication is still true. (I could be at a swimming pool).

YOUR TURN









Show that $p \to q$ is logically equivalent to $\neg p \lor q$.

This is only ever O when...

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TAUTOLOGY

A statement that is true for all truth value assignments is called a *tautology*.

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Eg:

р	q	((p -	\rightarrow (q)∧p	$) \rightarrow q$
1	1		l	ı	5
1	0	•	9	0	ø
0	1	١		0	i
0	0	(0	1
		((2)	(3)

Modus Ponens

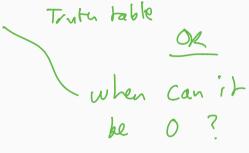
Eg:

р	q	r	$ [(p \rightarrow c)]$	q) ^ ($q \rightarrow r$	$(p) \rightarrow (p)$	$0 \rightarrow r$
1	1	1	(- 1	l	1	1
1	1	0	ι	ò	0	a 1	0
1	0	1	٥	0	l	4 1	1
1	0	0	٥	0	1	61	0
0	1	1	1	1	1	1	ī
0	1	0	1		Ö	i i	1
0	0	1	1	7	1	· ;	+
0	0	0			1	(1
		ı	'	1	•	1	1
			(1)	2	1	3	0

YOUR TURN

Decide which of these are tautologies:

- 1. $((p \rightarrow q) \land \neg q) \overleftrightarrow{\rightarrow} \neg p$
- 2. $(p \rightarrow q) \leftrightarrow (q \rightarrow p)$
- 3. $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$



$$7\rho = 1$$

$$() =$$

ANOTHER WAY TO WRITE TAUTOLOGIES

In Humanities/Law you might see tautological statements written in this form. Some rules have names.



(Modus ponens)

ANOTHER WAY TO WRITE TAUTOLOGIES

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 $\neg p$

$$\frac{p \to c}{q}$$

(Modus ponens)

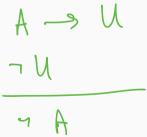
$$\begin{array}{ccc}
p \to q & \left(\left(p \to q \right) \wedge 72 \right) \\
 & \xrightarrow{\neg q} & \longrightarrow 79
\end{array}$$

(Modus tollens)

FROM WIKIPEDIA:

If I am an axe murderer, then I can use an axe.
I cannot use an axe.
Therefore, I am not an axe murderer.

Which style of argument is this? (Write it in symbols).



PAUSE

CONTRADICTION: PREVIEW

Let F be a statement that is always false (has truth table 0, for example, $F = q \land \neg q$).

Then the statement

$$(\neg p \to F) \to p$$

is a tautology. Check it:

$$\begin{array}{c|cccc}
p & F & (\neg p \rightarrow F) \\
\hline
1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{array}$$

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Let F be a statement that is always false (has truth table 0, for example, $F = q \land \neg q$).

Then the statement

$$(\neg p \to F) \to p$$

is a tautology. Check it:

It says, if not p implies something that is false, then it must be p (is true). This argument form is known as proof by contradiction. We will study this more when we start proofs

VARIABLES

Statements can contain variables.

Eg:

- P(x): "the number x is greater than or equal to 3"
 Q(x): "x lives in Queensland"

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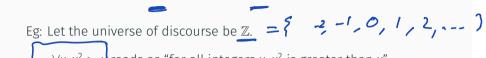
The *universe* of discourse is the set of objects over which the statement could be defined.

- for P(x) the universe of discourse could be $\mathbb R$ or $\mathbb Z$ or $\mathbb N$ (we would need to be told)
- for Q(x) the universe might be all people, or all students at QUT.

there is We have the symbols \forall ="for all" and \exists ="there exists".

QUANTIFIERS

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 $\cdot \forall x, x^2 > x$ reads as "for all integers x, x^2 is greater than x"

Eg: Let the universe of discourse be
$$\mathbb{Z}$$
.

• $\forall x, x^2 > x$ reads as "for all integers x, x^2 is greater than x "

Is this true?

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$$. \quad \exists \times \left(\times^2 > \times \right)$$

QUANTIFIERS

We have the symbols \forall ="for all" and \exists ="there exists".

Eg: Let the universe of discourse be \mathbb{Z} .

- $\forall x, x^2 > x$ reads as "for all integers x, x^2 is greater than x"
 - Is this true? Yes
- $\exists x, x^2 \le x$ reads as "there exists (there is) some integer x whose square is smaller than or equal to itself"

Is this true?



QUANTIFIERS

Rather than say "Let the universe of discourse be" we often hide this (make it *implicit*), or write

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for all x which are whole numbers, x^2 is shickly greater than x.
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Universe = all students at UTT.

Let $\underline{B(x)}$ be the statement "x lives in Bondi".

Let C(x) be the statement "x lives in Cabramatta".

PRACTICE Universe aft UTS students

Let B(x) be the statement "x lives in Bondi".

Let C(x) be the statement "x lives in Cabramatta".

Write these in symbols:

• "All UTS students live in Bondi,"
$$\forall x (\beta(x))$$

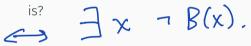
• "All UTS students either live in Cabramatta or do not live in Bondi" $\bigvee \times \left(C(x) \vee \neg \beta(x) \right)$

NEGATION OF QUANTIFIED STATEMENTS

Can you work out, intuitively what the meaning of

$$\cdot \neg (\forall x, B(x))$$





NEGATION OF QUANTIFIED STATEMENTS

Formally, to negate a quantified statement you switch \forall and \exists at the front, then negate the proposition.

$$\neg (\forall x P(x)) = \exists x \neg P(x)$$

$$\neg (\exists x P(x)) = \forall x \neg P(x)$$

PRACTICE

Check the course notes, and Week 1 homework sheet, to practice turning English sentences into symbolic statements, and backwards, and negating them.

3-SAT is the following problem: on input an expression of the form

$$(x_1 \vee y_1 \vee z_1) \wedge (x_2 \vee y_2 \vee z_2) \wedge \dots (x_n \vee y_n \vee z_n)$$

where x_i, y_i, z_i are propositions p or $\neg p$, answer yes or no: there is some assignment of truth values to the variables which makes the whole statement true.

For example

$$\begin{array}{c}
(p \lor q \lor \neg r) \land (\neg p \lor q \lor r) \land (p \lor \neg q \lor r) \\
p = 1 \\
q = 1 \\
r = xny thing$$

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For example

$$(p \lor q \lor \neg r) \land (\neg p \lor q \lor r) \land (p \lor \neg q \lor r)$$

If I tell you a particular truth assignment, like p=0, q=1, r=0 etc, you can easily compute (in a number of steps polynomial in n) the truth value of the statement.

If an <u>instance</u> of a solution can be <u>verified</u> in polynomial time (number of steps), we say a problem is in NP.

If a solution can be found in polynomial time (number of steps), we say the problem is in *P*.

¹hashtag NP-complete

If an instance of a solution can be *verified* in polynomial time (number of steps), we say a problem is in NP.

If a solution can be found in polynomial time (number of steps), we say the problem is in *P*.

No-one knows if you can always find a truth assignment, or show there is none, making a general 3-SAT expression true, in polynomially many steps. If you can, you will get \$1M

3-SAT is an important problem, even though it may seem abstract and useless, because Cook and Levin showed that every other candidate to solve the P=NP problem is related to this one. ¹

¹hashtag NP-complete

COMING UP

In your workshop tomorrow/Friday/Monday, lots of practice to fully understand the content presented today.

After the workshop and before the next lecture (eg: this weekend, or make some time Mon-Tue) do the homework sheet to consolidate your learning, and be ready for the quiz.

Next lecture:

 $\boldsymbol{\cdot}$ proof methods: direct, contrapositive, contradiction