DISCRETE MATH 37181 WORKSHEET 2

MURRAY ELDER

INSTRUCTIONS. Complete these problems in groups of 3-4 at the whiteboards. Quiz in the last 30 minutes (x:15 – x:45)

Recall:

Definition 1. Let $a, b \in \mathbb{Z}$. We say a *divides* b if $\exists s \in \mathbb{Z}$ such that $b = as$.

For example, 3 divides -18 since there exists -6 such that $-18 = 3 \cdot (-6)$, and 3 does not divide 14 since for all $s \in \mathbb{Z}$ $14 \neq 3s$.

Definition 2. $p \in \mathbb{N}, p > 1$ is called *prime* if $s \in \mathbb{N}$ divides p implies $s = 1$ or $s = p$.

1. Let the universe of discourse be all people, $C(x)$ the proposition that "x likes cheese" and $D(x)$ the proposition that "x cannot eat dairy foods".
   (a) The logical statement
   $$\forall x[D(x) \to \neg C(x)]$$
   has the following meaning in English:

   A. All people either cannot eat dairy or cannot eat cheese.

   B. If a person cannot eat dairy then they cannot eat cheese.

   C. For all people, if the person eats cheese then they like cheese.

   D. If a person cannot eat dairy then they don't like cheese.

   E. None of the above.

   (b) What is the negation of the statement $\forall x[D(x) \to \neg C(x)]$? Express in symbols, simplify then write in English.

2. Using a *direct* proof, prove that if $k, l \in \mathbb{Z}$ have the same parity[1] then $k + l$ is even.

Date: Week 2 workshop (Thursday 1, Friday 2, Monday 5 August).

[1]parity means being even or odd. So your proof can be Case 1: both even. Case 2: both odd.

3. For each of the following statements, either prove using either a *direct, contrapositive* or *contradiction* proof, or show that it is false by giving a counterexample.

   (a) For all $x \in \mathbb{Z}$, if 3 divides x then 3 divides x^2.

   (b) For all $x \in \mathbb{Z}$, if 3 divides x^2 then 3 divides x.

   (c) $\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}\, [(xy > 0) \vee (x = 0) \vee (y = 0)]$

   (d) For all $n \in \mathbb{Z}$, n is divisible by some prime number p.

   (e) For all $n \in \mathbb{N}$, $n^2 + 5n + 5$ is prime.

4. Write the output of the following (somewhat useless) code:

```
int j = 9;
for(int i=0; i<10; i++)
   j--;
print j
```

5. Determine whether or not the following statements is true or false. If false, give a counterexample. Assume the universe of discourse to be $\mathbb{Z}$.

   (a) $\forall x \exists y \exists z\ [x = 7y + 5z]$

   (b) $\forall x \exists y \exists z\ [x = 4y + 6z]$

6. Use the table on your formula sheet to *simplify* the expression $\neg\,(q \to (p \vee (q \wedge r)))$. Check your answer by drawing a truth table.

7. Is the formula $(p \vee q \vee r) \wedge (p \vee q \vee \neg r) \wedge (p \vee \neg q \vee r) \wedge (p \vee \neg q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (\neg p \vee q \vee \neg r) \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ satisfiable?

8. (Bonus question) Prove that $\sqrt{2}$ is not rational by following these steps.

   (a) Suppose (for contradiction) that $\sqrt{2}$ is rational. Then $\exists a, b \in \mathbb{Z}$ such that ...

   (b) If a, b have a common factor, we can divide top and bottom to get a smaller pair of numbers. So assume (here is the thing we will contradict later) that ...

   (c) Now multiply both sides of your answer to (a) by b then square both sides to get

   (d) It follows that a^2 is even, so by Lemma 4.3 of the lecture notes this implies a is even. Then $2b^2 = a^2 = (2s)^2 = 4s^2$ so divide both sides by 2 to get ...

   (e) What is the contradiction?

Sample formula sheet:

Some tautologies with names:

logic rule (tautology)	name
$\neg(\neg p) \quad \leftrightarrow \quad p$	double negative
$\neg(p \vee q) \quad \leftrightarrow \quad \neg p \wedge \neg q$	DeMorgan
$\neg(p \wedge q) \quad \leftrightarrow \quad \neg p \vee \neg q$	
$p \vee q \quad \leftrightarrow \quad q \vee p$	commutative
$p \wedge q \quad \leftrightarrow \quad q \wedge p$	
$p \vee (q \vee r) \quad \leftrightarrow \quad (p \vee q) \vee r$	associative
$p \wedge (q \wedge r) \quad \leftrightarrow \quad (p \wedge q) \wedge r$	
$p \vee (q \wedge r) \quad \leftrightarrow \quad (p \vee q) \wedge (p \vee r)$	distributive
$p \wedge (q \vee r) \quad \leftrightarrow \quad (p \wedge q) \vee (p \wedge r)$	
$p \vee p \quad \leftrightarrow \quad p$	idempotent
$p \wedge p \quad \leftrightarrow \quad p$	
$p \vee F \quad \leftrightarrow \quad p$	identity
$p \wedge T \quad \leftrightarrow \quad p$	
$p \vee (p \wedge q) \quad \leftrightarrow \quad p$	adsorption
$p \wedge (p \vee q) \quad \leftrightarrow \quad p$	
$p \to q \quad \leftrightarrow \quad \neg p \vee q$	useful ones
$p \vee \neg p \quad \leftrightarrow \quad T$	
$p \wedge \neg p \quad \leftrightarrow \quad F$	
$p \to q \quad \leftrightarrow \quad \neg q \to \neg p$	contrapositive
$(\neg p \to F) \to p$	contradiction

Definition. Let $a, b \in \mathbb{Z}$. Then a *divides* b if $\exists s \in \mathbb{Z}$ such that $b = as$.

Definition. Let $p \in \mathbb{N}, p > 1$. Then p is called *prime* if $s \in \mathbb{N}$ divides p implies $s = 1$ or $s = p$.

Brief solutions:

1. D. “If a person” in English means “if **any** person”, so the $\forall$ is hidden here. Literally, for all people, if a person cannot eat dairy then that person does not like cheese.

2. *Proof.* We will prove two cases: k, l both even, and k, l both odd.

   If k, l both even then $\exists b, c \in \mathbb{Z}$ with $k = 2b, l = 2c$. Then $k + l = 2b + 2c = 2(b + c)$ is even since $b + c \in \mathbb{Z}$.

   If k, l both odd then $\exists b, c \in \mathbb{Z}$ with $k = 2b+1, l = 2c+1$. Then $k+l = 2b+1+2c+1 = 2(b + c) + 2 = 2(b + c + 1)$ is even since $b + c + 1 \in \mathbb{Z}$. □

3. (a) *Proof.* Direct. By definition, if $3|x$ then $x = 3c$ for some $c \in \mathbb{Z}$ so $x^2 = (3c)^2 = 9c^2 = 3(3c^2)$ is divisible by 3. □

   (b) *Proof.* Contrapositive. Suppose 3 does not divide x. Then $x = 3c+i$ for $c \in Z$ and $i \in \{1, 2\}$ (that is, $i = 1$ or 2). Then $x^2 = (3c+i) = 9c^2+6ci+i^2 = 3(3c^2+2c)+i^2$.

   If $i = 1$ then $i^2 = 1$ and $x^2 = 3(3c^2 + 2c) + 1$ is not divisible by 3. If $i = 2$ then $i^2 = 4$ and $x^2 = 3(3c^2 + 2c + 1) + 1$is not divisible by 3. □

   (c) False, there exist $x = -1, y = 1$ such that $xy \leqslant 0$ and $x \neq 0$ and $y \neq 0$ ($i.e.$ the negation of the statement is true).

   (d) False, $n = 1$ is only divisible by ± 1.

   (e) False, if $n = 5$ then $n^2 + 5n + 5 = 55 = 5.10$ is not prime (has two divisors $\neq 1$).

4. It just prints once after it finishes the loop. Here is a table showing the value of i, j during the computation.

i	j
	9
0	8
1	7
2	6
3	5
4	4
5	3
6	2
7	1
8	0
9	-1

So the output is -1.

This question is here for the non-computer scientists in the class. Don’t be afraid of the syntax, just ask someone or look it up. This is C or C++ code that I found somewhere. `i++` means decrease the value of `i` by 1 and `j--`, well, guess. Obviously this is not a programming course (you should do one, or several!) but later in this course we will apply mathematics to understand computer code, so that’s why we want to get familiar with it here. For the computer programmers, we want you to be able to trace what the code does (draw a table like the one here) not just program and run without thinking what steps are happening.

5. (a) $\forall x \exists y \exists z\ [x = 7y + 5z]$ If $x = 1$ we need $7y + 5z = 1$ so $y = \frac{1-5z}{7}$ top should be a multiple of 7, yes $z = 3$ gives $1 - 15 = -14$ so $y = -2, z = 3$. For any other value of x, we just multiply everything by x: $y = -2x, z = 3x$. So its true.

   (b) $\forall x \exists y \exists z\ [x = 4y + 6z]$ This time the right hand side is always even, since we have 4 and 6, so it is false for $x = 1$.

6. Simplify means try to get rid of nested brackets, and negations out the front of brackets, etc.

$$\begin{array}{lll}
 & \neg(q \to (p \vee (q \wedge r))) & \\
\leftrightarrow & \neg(\neg q \vee (p \vee (q \wedge r))) & \text{using useful fact } a \to b \leftrightarrow \neg a \vee b \\
\leftrightarrow & \neg\neg q \wedge \neg(p \vee (q \wedge r)) & \text{using De Morgan} \\
\leftrightarrow & q \wedge (\neg p \wedge \neg(q \wedge r)) & \text{using double negative and De Morgan} \\
\leftrightarrow & q \wedge (\neg p \wedge (\neg q \vee \neg r)) & \text{using De Morgan} \\
\leftrightarrow & \neg p \wedge (q \wedge (\neg q \vee \neg r)) & \text{using commutative, associative, commutative} \\
\leftrightarrow & \neg p \wedge ((q \wedge \neg q) \vee (q \wedge \neg r)) & \text{distributive} \\
\leftrightarrow & \neg p \wedge (F \vee (q \wedge \neg r)) & \text{useful fact } a \wedge \neg a \text{ is always false} \\
\leftrightarrow & \neg p \wedge q \wedge \neg r & \text{identity}
\end{array}$$

Check we didn't make a mistake by drawing the truth table for both statements:

p	q	r	$\neg(q \to (p \vee (q \wedge r)))$	$\neg p \wedge q \wedge \neg r$
1	1	1		
1	1	0		
1	0	1		
1	0	0		
0	1	1		
0	1	0		
0	0	1		
0	0	0		

or by saying "$\neg p \wedge q \wedge \neg r$ is true only when $p = 0, q = 1, r = 0$, and $\neg(q \to (p \vee (q \wedge r)))$ is true only when $q \to (p \vee (q \wedge r))$ is false, which happens only when $q = 1$ and $p \vee (q \wedge r) = 0$ (implication only false when $1 \to 0$) which happens only when $p = 0$ and $q \vee r = 0$ which, since $q = 1$ already, means $r = 0$". Hmm I think just drawing the truth table might be easier than that!

7. No. To prove it, must check every possible value assignment (2^3 of them).

8. (a) Suppose (for contradiction) that $\sqrt{2}$ is rational. Then $\exists a, b \in \mathbb{Z}$ such that $\sqrt{2} = \frac{a}{b}$.

   (b) If a, b have a common factor, we can divide top and bottom to get a smaller pair of numbers. So assume (here is the thing we will contradict later) that the *greatest common divisor* [2] of a and b is 1.

   (c) Now multiply both sides of your answer to (a) by b then square both sides to get $b\sqrt{2} = a$, $2b^2 = a^2$.

   (d) It follows that a^2 is even, so by Lemma 4.3 of the lecture notes this implies a is even. Then $2b^2 = a^2 = (2s)^2 = 4s^2$ so divide both sides by 2 to get $b^2 = 2s^2$, which means b^2 is even so by Lemma 4.3 b is even as well.

[2] $\gcd(a, b) = 1$

(e) What is the contradiction? We assumed that we had reduced any common factors in a, b at the start, that is, $\gcd(a, b) = 1$ but now 2 divides both a and b, which is a contradicition. Therefore $\sqrt{2}$ is NOT rational.