37181: WEEK 5: RELATIONS, FUNCTIONS

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PLAN

- relations
- functions
- one-to-one
- onto
- Ackermann's function
- bijection

If A, B are sets we can define a new symbol (a, b) where $a \in A$ and $b \in B$.

This symbol is not the same as $\{a, b\}$, it is a new symbol. Also it is not the same as (b, a), the symbol has an *order*.

We call it an *ordered pair*.

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Define $A \times B = \{(a, b) \mid a \in A, b \in B\}$.

Cartesian Product

Eg: If $A = \underline{\{1, 2, 3\}}$ and $B = \underline{\{d, e\}}$ then $A \times B = \left\{\begin{matrix} (1, d) & (1, e) \\ (2, d) & (2, e) \\ (3, d) & (3, e) \end{matrix}\right\}$

AXIOM: If A, B are sets then so is $A \times B$

$A = \mathbb{R}, \; B = \mathbb{R}$ $\qquad \mathbb{R} \times \mathbb{R} = \{ (\; , \;), \ldots$

A subset of $A \times B$ is called a *relation* from A to B.

ϕ

We often use the notation $\mathscr{R}$ to denote a relation.

Eg: Let $A = \{1, 2, 3, 4\}$ and define $\mathscr{R} \subseteq A \times A$ by
$\mathscr{R} = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$.

Secretly: $(i, j) \in \mathcal{R}$

if and only if $i < j$

ORDERED PAIRS, RELATIONS

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We write $a\mathscr{R}b$ if $(a, b) \in \mathscr{R}$, and say "a is related to b". So for example $1\mathscr{R}3$.

What is another notation you could use for this relation?

$< = \{(\;\;),(\;\;),(\;\;)$

Let $A = \{1, 2, 3, 4\}$ define a relation $\mathscr{R} \subseteq A \times A$ which means "$\geqslant$"

$\geqslant = \{ (4, 1), (4, 2), (\quad),$
$(\quad)$

$a \equiv b \mod d$

$d \mid (b-a)$

Recall Homework Sheet 2 you learned the definition $\equiv \mod d$.

Let $\mathbb{Z}$ be our set and define $\mathscr{R}_d \subseteq \mathbb{Z} \times \mathbb{Z}$ by $a\mathscr{R}_d b$ if $a \equiv b \mod d$.

$d = 5$

$(5,5)$ $(6\,6)$ $(7\,7)$

$R_d = \{ (5\ 0), (0, 5), (1, 6), (6, 1), (2, 12)$

WE HAVE SEEN THIS BEFORE

Recall Homework Sheet 2 you learned the definition $\equiv \mod d$.

Let $\mathbb{Z}$ be our set and define $\mathscr{R}_d \subseteq \mathbb{Z} \times \mathbb{Z}$ by $a\mathscr{R}_d b$ if $a \equiv b \mod d$.

Ex: Write down some elements $a \in \mathbb{Z}$ such that $a\mathscr{R}_5 1$:

Definition

Let A be a set. Then $\mathscr{R} \subseteq A \times A$ is

- *reflexive* if for all $a \in A$, $a\mathscr{R}a$
- *symmetric* if for all $a, b \in A$, $a\mathscr{R}b$ implies $b\mathscr{R}a$
- *antisymmetric* if for all $a, b \in A$, $(a\mathscr{R}b$ and $b\mathscr{R}a$ implies $a = b)$
- *transitive* if for all $a, b, c \in A$, $a\mathscr{R}b$ and $b\mathscr{R}c$ implies $a\mathscr{R}c$

$A = \{1, 2, 3, 4\}$

Eg Qn is $\mathscr{R} = \emptyset$ symmetric yes

reflexive? No

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~~(3,1)~~

Ex: Let $A = \{1, 2, 3\}$ and

$$\mathscr{R} = \{(1,1), (2,2), (3,1), (1,3), (2,3), (3,2)\}.$$

Yes

Decide which of the four properties (reflexive, symmetric, antisymmetric, transitive) $\mathscr{R}$ satisfies.

No, (3,3) missing.

No

(2,3) & (3,1) but

(2,1) not

Definition

Let A be a set. Then $\mathscr{R} \subseteq A \times A$ is

- *symmetric* if for all $a, b \in A$, $a\mathscr{R}b$ implies $b\mathscr{R}a$
- *antisymmetric* if for all $a, b \in A$, $a\mathscr{R}b$ and $b\mathscr{R}a$ implies $a = b$

$A = \{1\ 2\ 3\}$

Ex: Construct an example (that means tell me a set A and some subset of $A \times A$) of a relation which is

- *both* symmetric and antisymmetric → $\{(1\ 1)\ (2\ 2)\ (\cancel{3\ 3})\}$
- *neither* symmetric nor antisymmetric

$\{(2,1)\ (3,1)\ (1,3)\}$

DEFINITIONS

These notions are extremely useful throughout mathematics.

For now, you should feel good if you can read the very abstract definitions (written in logic and set theory notation) and write down examples, prove/disprove some relation has them.

This will show you are “getting it” in this course.

Definition

Let A be a set. Then $\mathscr{R} \subseteq A \times A$ is

- an *equivalence relation* if it is reflexive, symmetric and transitive
- a *partial order* if it is reflexive, antisymmetric and transitive

$d \mid (b - a)$

Ex: Show that "$\equiv_d$ ~~mod d~~" is an equivalence relation on $\mathbb{Z}$.

1 $\forall a \in \mathbb{Z}$ $\quad (a - a) = 0 \quad d \mid 0$

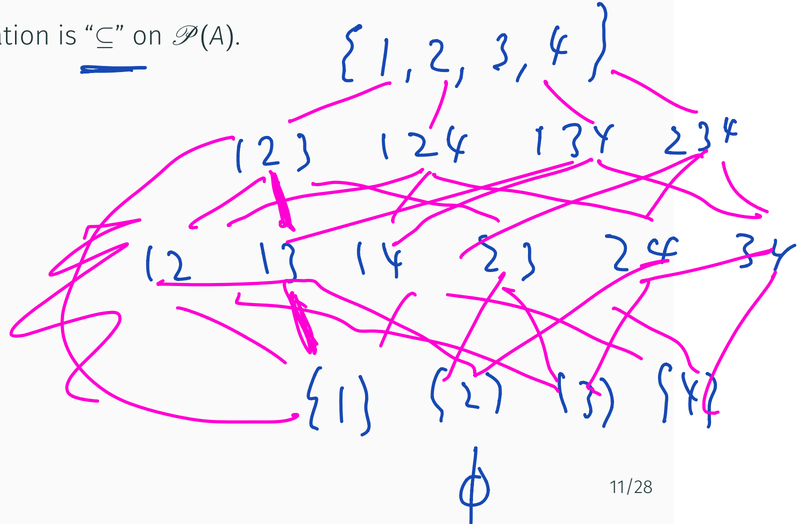
2 if $a \equiv_d b \to d \mid (b - a)$ $\quad b - a = -(a-b)$

Ex: Show that "$\leqslant$" is a partial order on $\mathbb{Z}$.

$\leq$ $\quad <$

Given a partial order on a set we can draw a nice picture called a *Hasse diagram*. Here is an example:

$A = \{1, 2, 3, 4\}$, relation is "$\subseteq$" on $\mathscr{P}(A)$.



FUNCTIONS

A *function* from A to B is a relation $f \subseteq A \times B$ in which every element of A appears exactly once as the first component of an ordered pair in the relation.

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Eg: Let S = the set of all students at UTS and $f \subseteq S \times \mathbb{N}$ where (s, n) means n is a student ID number for student s.

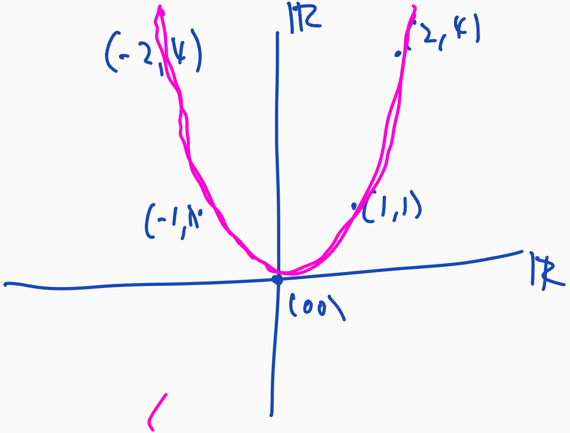
S

What if f was not a function?

What if $(s, 13645)$ and $(t, 13645)$ were both in f?

$\mathbb{R}_{\geq 0}$

Eg: sets are $A = \mathbb{R}$, $B = \mathbb{R}_{+} \cup \{0\}$, relation is $\{(x, x^2) \mid x \in \mathbb{R}\}$.



Eg: Define $f : \mathbb{R} \to \mathbb{Z}$ by floor

$$f(x) = \lfloor x \rfloor = \text{ the biggest integer less than or equal to } x.$$

2.731

2

Similarly we have $g : \mathbb{R} \to \mathbb{N}$ by

$$g(x) = \lceil x \rceil = \text{ the least integer greater than or equal to } x.$$

3

– ceiling

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Eg: Let $h : \mathbb{N} \to \mathbb{N}$ defined by

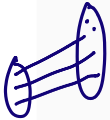
$$h(n) = \left\lceil \frac{n}{2} \right\rceil + 7.$$

If $n =$ your age, compute $h(n)$.

$$\cancel{\left\lceil \frac{41}{2} \right\rceil + 7 \quad 31}$$

https://en.wikipedia.org/wiki/Age_disparity_in_sexual_relationships. Should it be $\lceil\cdot\rceil$ or $\lfloor\cdot\rfloor$?

ONE-TO-ONE FUNCTIONS

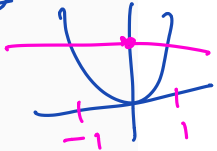


Definition

Let $f : A \to B$ be a function from a set A to a set B. We say f is *one-to-one* (or 1-1) if

$$\forall x \in A \forall y \in A[f(x) = f(y) \to x = y].$$

Eg $f(x) = x^2$



We want the student number function to be one-to-one.

Show that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5x + 3$ is one-to-one.

Proof: Given arbitrary $x, y \in \mathbb{R}$

if $f(x) = f(y)$

$\to 5x + 3 = 5\cancel{x}y + 3$

$\to 5x = 5y$

Direct

$\rightarrow x = y$

Definition

Let $f : A \to B$ be a function from a set A to a set B. We say f is *onto* if

$$\forall b \in B \exists a \in A[f(a) = b].$$

Ex: Show that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5x + 3$ is onto

Given arbitrary $b \in \mathbb{R}$

$\exists \left(a = \frac{b-3}{5} \right)$

so that $f(a) = 5\left(\frac{b-3}{5}\right) + 3$

b

Definition

Let $f : A \to B$ be a function from a set A to a set B. We say f is *onto* if

$$\forall b \in B \exists a \in A[f(a) = b].$$

Ex: Show that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5x + 3$ is onto

Ex: Show that $\lceil \cdot \rceil$ is onto and not 1-1.

$\lceil 1.5 \rceil = \lceil 1.6 \rceil$

$= 2$

$\forall b \in \mathbb{Z}$

$\exists b \in \mathbb{Z}$ $\quad \lceil b \rceil = b$

LOGIC AGAIN

Setting up our definitions using logical statements like this, it is easy to prove examples satisfy them or not.

$$\neg\forall x \forall y \in A[f(x) = f(y) \to x = y]$$
$$\leftrightarrow \exists x \exists y \in A[f(x) = f(y) \wedge x \neq y]$$

$$\neg\forall b \in B \exists a \in A[f(a) = b]$$
$$\leftrightarrow \exists b \in B \forall a \in A[f(a) \neq b]$$

Eg: $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$

EXERCISE

Let $A = \{a, b, c, d, e\}$, $B = \{b, d, e\}$, $C = \{f, g, a\}$. Give examples of functions

1. $f : A \to B$ which is onto and not 1-1
2. $g : A \to B$ which is 1-1 and not onto
3. $h : A \to B$ which is both 1-1 and onto
4. $i : B \to C$ which is onto and not 1-1
5. $j : B \to C$ which is 1-1 and not onto
6. $k : B \to C$ which is both 1-1 and onto

$\{(a, b)\ (b, d),\ (c, e)\ (d, \ )\ (e, \ )\}$

$\{(a, -\ (c, -\ (e, -\ (b, -\ (d, -$

$i := \{(b, -)\ (e, \ )\ (d, \ )$

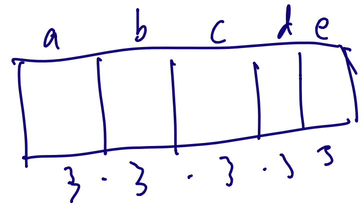
$-\ \{(b, f)\ (e, g)\ (d, a)\}$

bijection

$A = \{a\ b\ c\ d\ e\}$

$B = \{b\ c\ d\}$

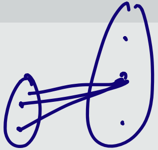
$f : A \to B$



Lemma

Let A, B be finite sets. If $f : A \to B$ is

- *1-1 then $|A| \leqslant |B|$.*
- *onto then $|B| \leqslant |A|$.*



Proof.

① If $f: A \to B$ is 1-1

$\square$

 $\mathbb{N} \times \mathbb{N}$ $A = \{ ((\cdot,\cdot)\cdot)$

Define a function $A : \mathbb{N}^2 \to \mathbb{N}$ using the following *recursive* definition.

$$\begin{aligned} A(0,n) &= n+1 & n \geqslant 0, \\ A(m,0) &= A(m-1,1) & m > 0, \\ A(m,n) &= A(m-1, A(m,n-1)) & m,n > 0. \end{aligned}$$

(a) Compute $A(1,3)$. $= A(0, A(1,2))$ $= A(1,2)+1$

(b) Compute $A(2,3)$. $= A(0, A(1,1)) + 1$

(c) Prove that $A(1,n) = n+2$ for all $n \in \mathbb{N}$. $= A(1,1)+2$

(d) Prove that $A(2,n) = 3 + 2n$ for all $n \in \mathbb{N}$. $= A(0, A(1,0))+2$

(e) Prove that $A(3,n) = 2^{n+3} - 3$ for all $n \in \mathbb{N}$.

(f) Find a formula for $A(4,n)$ $= A(1,0) + 3$

$= A(0,1) + 3$

$= 5.$

Definition

A function $f : A \to B$ is a *bijection* if it is both 1-1 and onto.

Eg: $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5x + 3$ is a bijection.

1. 1-1 ✓
2. onto ✓

We think if two sets are in bijection, they are the same size (you can match them up by pairs of elements).

True if both finite.

Claim: $\mathbb{N}$ and $\mathbb{Z}$ are in bijection.

?

$$f: \begin{array}{l} 2n \to n \\ 2n-1 \to -n \end{array}$$

~~$\begin{array}{ccc} & & -2 \\ & & -1 \\ 0 & — & 0 \\ 1 & — & 1 \\ 2 & — & 2 \\ 3 & — & 3 \\ 4 & — & 4 \\ 5 & & \\ 6 & & \\ \vdots & & \end{array}$~~

$n \to n$

$$\begin{array}{ccc} 0 & — & 0 \\ 1 & — & -1 \\ 2 & — & 1 \\ 3 & — & -2 \\ 4 & — & 2 \end{array}$$

 $\mathbb{Z} \to \mathbb{N} \to \mathbb{Q}$

We think if two sets are in bijection, they are the same size (you can match them up by pairs of elements). $\mathbb{N} \to \mathbb{Q}$

Claim: $\mathbb{N}_+$ and $\mathbb{Q}_+$ are in bijection.

$f(n) =$

$\mathbb{N}_+$ $\mathbb{Q}_+$

$1/1$	$1/2$	$1/3$	$1/4$
$2/1$	$2/2$	$2/3$	$2/4$
$3/1$	$3/2$	$3/3$	$3/4$

PUT THESE ON YOUR FORMULA SHEET

- ordered pair
- relation
- reflexive
- symmetric
- antisymmetric
- transitive
- equivalence relation
- partial order
- Hasse diagram
- function
- one to one
- onto
- bijection
- Ackermann's function

Finally, so far in this course, we have asked you to *accept* two "facts" or axioms:

WOP: Every nonempty subset of $\mathbb{N}$ has first element.

PMI: $P(n)$ statement.
if • $P(0)$ true, • $P(k) \to P(k+1)$ then $P(n)$ true for all $n \in \mathbb{N}$.

Axiom: true without following from any other fact.

Theorem

WOP implies PMI

Proof.

Assume $P(0)$ and $(P(k) \to P(k+1))$ are both true. Define

$$S = \{i \in \mathbb{N} \mid P(i) \text{ is false}\}.$$

So $\boxed{0 \notin S}$. Suppose S is nonempty

By WOP there is a first element, called $a \in S$.

□

This means $a-1$ is not in S.

so $P(a-1)$ must be true

 $\rightarrow$ $p(a_{-1+1})$ is true

Theorem

PMI implies WOP

Proof.

Team assignment .

□

NEXT

Next lecture:

- Countable
- Big O
- comparing speed of algorithms
- pigeonhole principle