37181: WEEK 1: LOGIC

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PLAN

- introduction, subject outline
- truth tables
- logical equivalence
- tautology
- quantified statements
- negation of quantified statements
- SAT and P=?NP

Definition

A *statement* is a sentence that can (theoretically) be assigned a value of *true* or *false*.

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Eg:

1. Um, like, whatever
2. All positive integers are prime
3. All lectures are recorded at UTS
4. In the year 4000BC, at this exact location, it was raining on the 5th of March at 10am
5. When will this lecture end?

We can build up more complicated statements out of simpler ones using *logical connectives* like *and* and *or*.

Eg:

1. Murray is a statistician and Murray has brown hair.
2. Murray is a statistician or Murray has brown hair.

English (or any natural human language) can be imprecise, so instead of using our “*intuitition*” we **define** what “*and*” and “*or*” and “*not*” mean using *truth tables*.

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p	q	$p \wedge q$
1	1	
1	0	
0	1	
0	0	

p	q	$p \vee q$
1	1	
1	0	
0	1	
0	0	

p	$\neg p$
1	
0	

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p	q	$p \vee q$
1	1	
1	0	
0	1	
0	0	

p	$\neg p$
1	
0	

Teenager speech is more precise: Eg: "Maths is awesome — NOT"

We can use truth tables to decide the truth values of more complicated statements, like $\neg p \vee q$:

p	q	$\neg p$	$\vee$	q
1	1			
1	0			
0	1			
0	0			

.

TRUTH TABLES FOR COMPOUND STATEMENTS

We can use truth tables to decide the truth values of more complicated statements, like $\neg p \vee q$:

p	q	$\neg p$	$\vee$	q
1	1			
1	0			
0	1			
0	0			

.

p	q	$\neg(p$	$\vee$	$q)$
1	1			
1	0			
0	1			
0	0			

.

Note that this is different to saying $\neg(p \vee q)$, since the truth values are not the same

Complete the truth tables for these statements:

p	q	$\neg$	$(p \wedge q)$
1	1		
1	0		
0	1		
0	0		

p	q	$\neg p$	$\vee$	$\neg q$
1	1			
1	0			
0	1			
0	0			

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Eg:

p	q	$p \vee \neg q$	$\neg(q \wedge \neg p)$
1	1		
1	0		
0	1		
0	0		

IMPLIES

In mathematics and logic we have a very specific meaning for "*p implies q*", or "*if p then q*", notation $p \rightarrow q$.

We define it using the following table:

p	q	$p \rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

IMPLIES

In mathematics and logic we have a very specific meaning for “*p implies q*”, or “*if p then q*”, notation $p \to q$.

We define it using the following table:

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

You may think that in English, “*if it is raining then I get wet*” means that the rain *caused* me to get wet. But in mathematics *if-then* has the meaning defined above: if “*I am wet*” is true and “*it is raining*” is false, the implication is still true. (I could be at a swimming pool).

Show that $p \to q$ is logically equivalent to $\neg p \vee q$.

A statement that is true for all truth value assignments is called a *tautology*.

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Eg:

p	q	$((p \to q) \wedge p) \to q$
1	1	
1	0	
0	1	
0	0	

TAUTOLOGY

Eg:

p	q	r	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
1	1	1	
1	1	0	
1	0	1	
1	0	0	
0	1	1	
0	1	0	
0	0	1	
0	0	0	

YOUR TURN

Decide which of these are tautologies:

1. $((p \to q) \wedge \neg q) \to \neg p$
2. $(p \to q) \leftrightarrow (q \to p)$
3. $(p \to q) \leftrightarrow (\neg q \to \neg p)$

ANOTHER WAY TO WRITE TAUTOLOGIES

In Humanities/Law you might see tautological statements written in this form. Some rules have names.

$$\begin{array}{l} p \to q \\ p \\ \hline q \end{array}$$

(Modus ponens)

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$$\begin{array}{l} p \to q \\ \underline{p \qquad\quad} \\ q \end{array}$$

(Modus ponens)

$$\begin{array}{l} p \to q \\ \underline{\neg q \qquad\quad} \\ \neg p \end{array}$$

(Modus tollens)

> If I am an axe murderer, then I can use an axe.
> I cannot use an axe.
> Therefore, I am not an axe murderer.

Which style of argument is this? (Write it in symbols).

PAUSE

Let F be a statement that is always false (has truth table 0, for example, $F = q \wedge \neg q$).

Then the statement

$$(\neg p \to F) \to p$$

is a tautology. Check it:

p	F	
1		
0		

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p	F	
1		
0		

It says, if not p implies something that is false, then it must be p (is true). This argument form is known as *proof by contradiction*. We will study this more when we start *proofs*

VARIABLES

Statements can contain *variables*.

Eg:

- $P(x)$: "the number x is greater than or equal to 3"
- $Q(x)$: "x lives in Queensland"

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The *universe of discourse* is the set of objects over which the statement could be defined.

- for $P(x)$ the universe of discourse could be $\mathbb{R}$ or $\mathbb{Z}$ or $\mathbb{N}$ (we would need to be told)
- for $Q(x)$ the universe might be all people, or all students at QUT.

We have the symbols $\forall$ =“for all” and $\exists$ =“there exists”.

Eg: Let the universe of discourse be $\mathbb{Z}$.

- $\forall x, x^2 > x$ reads as “for all integers x, x^2 is greater than x”

  Is this true?

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- $\forall x, x^2 > x$ reads as “for all integers x, x^2 is greater than x”

  Is this true?

- $\exists x, x^2 \leqslant x$ reads as “there exists (there is) some integer x whose square is smaller than or equal to itself”

  Is this true?

QUANTIFIERS

Rather than say " Let the universe of discourse be" we often hide this (make it *implicit*), or write

- $\forall x \in \mathbb{Z}, x^2 > x$

PRACTICE

Let $B(x)$ be the statement "x lives in Bondi".

Let $C(x)$ be the statement "x lives in Cabramatta".

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Let $C(x)$ be the statement "x lives in Cabramatta".

Write these in symbols:

- "All UTS students live in Bondi"
- "All UTS students either live in Cabramatta or do not live in Bondi"

Can you work out, *intuitively* what the meaning of

- $\neg(\forall x, B(x))$

is?

Formally, to negate a quantified statement you switch $\forall$ and $\exists$ at the front, then negate the proposition.

$$\neg\left(\forall x P(x)\right) = \exists x \neg P(x)$$

$$\neg\left(\exists x P(x)\right) = \forall x \neg P(x)$$

PRACTICE

Check the course notes, and Week 1 homework sheet, to practice turning English sentences into symbolic statements, and backwards, and negating them.

3-SAT is the following problem: on input an expression of the form

$$(x_1 \vee y_1 \vee z_1) \wedge (x_2 \vee y_2 \vee z_2) \wedge \ldots (x_n \vee y_n \vee z_n)$$

where x_i, y_i, z_i are propositions p or $\neg p$, answer yes or no: there is some assignment of truth values to the variables which makes the whole statement true.

For example

$$(p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (p \vee \neg q \vee r)$$

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$$(p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (p \vee \neg q \vee r)$$

If I tell you a particular truth assignment, like $p = 0, q = 1, r = 0$ etc, you can easily compute (in a number of steps polynomial in n) the truth value of the statement.

If an instance of a solution can be *verified* in polynomial time (number of steps), we say a problem is in NP.

If a solution can be found in polynomial time (number of steps), we say the problem is in *P*.

[1] hashtag NP-complete

If an instance of a solution can be *verified* in polynomial time (number of steps), we say a problem is in NP.

If a solution can be found in polynomial time (number of steps), we say the problem is in *P*.

No-one knows if you can always find a truth assignment, or show there is none, making a general 3-SAT expression true, in polynomially many steps. If you can, you will get $1M

3-SAT is an important problem, even though it may seem abstract and useless, because Cook and Levin showed that every other candidate to solve the P=NP problem is related to this one. [1]

[1] hashtag NP-complete

COMING UP

In your workshop tomorrow/Friday/Monday, lots of practice to fully understand the content presented today.

After the workshop and before the next lecture (eg: this weekend, or make some time Mon-Tue) do the homework sheet to consolidate your learning, and be ready for the quiz.

Next lecture:

- proof methods: direct, contrapositive, contradiction