DISCRETE MATH 37181 HOMEWORK 6

MURRAY ELDER

1. (a) Write down a precise definition for $f \in O(g)$ (g dominates f).

   (b) Write down a precise definition for a function $f : A \to B$ to be one-to-one.

   (c) Write down a precise definition for a function $f : A \to B$ to be onto.

   (d) Write down a precise statement of the pigeonhole principle.

   (e) Write down a precise statement of the generalised pigeonhole principle.

   (f) Write down a precise definition for an equivalence relation on a set A.

   (g) Write down a precise statement of a partition of a set A.

2. For each of the following functions, guess a Big-O form from one of those in Table 1 from the worksheet, then prove your guess.

   (a) $f(n) = 4n + 7$

   (b) $g(n) = 5n^2 + 3n \log_2 n$

   (c) $h(n) = 1 + 2 + \cdots + n$

3. Show that g dominates f in each of the following

   (a) $f(n) = 6n + 10, \quad g(n) = 0.05n^2$

   (b) $f(n) = 3n^2, \quad g(n) = 2^n + 2n$.

4. Prove that:

   If $S \subseteq \mathbb{N}_+$ and $|S| > 6$ then S contains three distinct elements x, y, z such that $x+y+z \equiv 0 \mod 3$.

Date: Week 6.

5. Let A, B be finite sets and $f : A \to B$ a function. Prove that if f is one-to-one then $|A| \leqslant |B|$.

6. Let A, B, C be sets and $f : A \to B, g : B \to C$. Recall that $g \circ f : A \to C$ is the function defined as $g \circ f(a) = g(f(a))$ for all $a \in A$.

   Prove that:

   If $f : A \to B, g : B \to C$ are one-to-one then $g \circ f$ is one-to-one.

Brief solutions:

1. (a) Let $f, g : \mathbb{N}_+ \to \mathbb{R}$. We say that g *dominates* f if there exist constants $m \in \mathbb{R}^+$ and $k \in \mathbb{Z}^+$ such that $|f(n)| \leqslant m|g(n)|$ for all $n \in \mathbb{N}, n \geqslant k$.

   (b) $\forall x, y \in A$ if $f(x) = f(y)$ then $x = y$.

   (c) $\forall y \in B\ \exists x \in A$ such that $f(x) = y$.

   (d) If m pigeons occupy n pigeonholes and $m > n$ then some pigeonhole has at least two pigeons in it.

   (e) If m pigeons occupy n pigeonholes and $m > kn$ then some pigeonhole has more than k pigeons in it.

   (f) A relation $\subseteq A \times A$ which is reflexive (includes (a, a) for every $a \in A$), symmetric (if $(a, b) \in \mathscr{R}$ then so is (b, a)) and transitive (if $(a, b), (b, c) \in \mathscr{R}$ then so is (a, c)).

   (g) See the worksheet from last week: a set of sets A_i so that A is the union of all the A_i and $A_i \cap A_j = \emptyset$ for all $i \neq j$. So (imprecisely) a partition is a way of dividing up the set into disjoint smaller sets which cover the whole set.

2. (a) Guess $O(n)$. There exists $m = 5, k = 7$ so that
$$f(n) = 4n + 7 \leqslant 4n + n \text{ (since } n \geqslant 7) \ = 5n$$
   for all $n \geqslant 7$.

   Guess $O(n^2)$ since the log is smaller than n. Proof: if $n \geqslant 1$ then $\log_2 n \leqslant n$ since $n \leqslant 2^n$ (prove this by induction to be very rigorous).
   Then
$$5n^2 + 3n \log_2 n \leqslant 5n^2 + 3n.n \text{ (since } n \geqslant \log_2 n) \ = 8n^2$$
   so $m = 8, k = 1$.

   (c) Induction we know this equals $\frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n$ so $O(n^2)$, quadratic:
$$\frac{1}{2}n^2 + \frac{1}{2}n \leqslant \frac{1}{2}n^2 + \frac{1}{2}n^2 = n^2$$
   for any $n \geqslant 1$ so $k = m = 1$.

3. (a) $f(n) = 6n + 10 \leqslant 6n + n$ (assuming $n \geqslant 10$)
$$= 7n = \frac{7}{0.05}(0.05n) = 140(0.05n) \leqslant 140(0.05n^2)$$
   if $n \geqslant 1$, so with $k = 10$ and $m = 140$ we have $f \in O(g)$.

Note there are many different ways to show this.

(b) I will show $3n^2 - 2n \leqslant 2^n$ for all $n \geqslant ?$ by induction. So I will have $m = 1$ and $k = ?$.
The statement $P(n) : 3n^2 - 2n \leqslant 2^n$ is true for $n = 1$, so we have $P(1)$. Assume $P(s)$. Then to show $P(s+1)$ we have:

$$3(s+1)^2 - 2(s+1) = 3(s^2 + 2s + 1) - 2s - 2 = 3s^2 + 6s + 3 - 2s - 2$$

$$= 3s^2 - 2s + 6s + 1 \leqslant 2^s + 6s + 1$$

by inductive assumption. To finish I need to show $6s + 1 \leqslant 2^s$, which I will prove separately below. However, this is only true for $s \geqslant ?$ so I will have to change the value of $k = ?$ to $k = 6$ and start over.

Lemma 1. *For all $n \geqslant 6$, $6n + 1 \leqslant 2^n$*

Proof. Induction: true for $n = 6$ since $37 < 2^6 = 64$. (also true for smaller, but 6 is fine for Big O proofs.)
Assume true for $t \geqslant 6$, then

$$6(t+1) + 1 = 6t + 1 + 6 < 6t + 1 + 6t + 1$$

(since $6 < 6t + 1$)

$$\leqslant 2^t + 2^t = 2^{t+1}.$$

Then by PMI $P(n)$ is true for all $n \geqslant 6$. □

Now I start again, using the lemma. Let $P(n)$ be the statement: $3n^2 - 2n \leqslant 2^n$
$P(6)$ since $LHS = 3.36 - 12 = 96$ and $RHS = 2^6 = 64$ Not true.
So I change my value of $k = ?$ again.

FINAL PROOF:
Let $P(n)$ be the statement: $3n^2 - 2n \leqslant 2^n$. $P(10)$ is true since $LHS = 3.100 - 20 = 280$ and $RHS = 2^{10} = 1024$. Assume $P(s)$. $P(s+1)$:

$$LHS = 3(s+1)^2 - 2(s+1) = 3(s^2 + 2s + 1) - 2s - 2 = 3s^2 + 6s + 3 - 2s - 2$$

$$= 3s^2 - 2s + 6s + 1 \leqslant 2^s + 6s + 1$$

by inductive assumption.

$$\leqslant 2^s + 2^s$$

by Lemma 1

$$= 2.2^s = 2^{s+1}$$

so by PMI $P(n)$ is true for all $n \geqslant 10$.

4. Pigeons are the elements in S, and pigeonholes are the remainders of each number mod 3, so a box labeled $0, 1$ and 2.
Since we have at least 7 pigeons going into 3 boxes, at least one box has 3 elements, call these x, y, z. If the three elements have the same remainder mod 3, then $x + y + z \equiv 0$ mod 3 (more detail: $x = 3p + i, y = 3q + i, z = 3r + i$ each with the same remainder i, then $x + y + z = 3(p + q + r + i)$.)

5. Suppose $|A| > |B|$. Let A be pigeons, B pigeonholes, and place each $a \in A$ into hole $f(a)$. Then since $|A| > |B|$ by PHP some hole has two elements of A in it, say $x, y \in A$, and $f(x) = f(y)$ since they are placed in the same hole. This shows f is not one-to-one. The result is the contrapositive.

6. Suppose $g \circ f(x) = g \circ f(y)$ for some $x, y \in A$. Then $g(f(x)) = g(f(y))$ and since g is one-to-one, this means $f(x) = f(y)$. But now since f is one-to-one this means $x = y$. This $g \circ f$ is one-to-one.