DISCRETE MATH 37181 HOMEWORK 4

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Instructions. Try these sometime before the next lecture. Also finish off the worksheet if you did not get through all of it in the whiteboard workshop.

Confused about how to do an induction proof? There are loads of resources online, and in any textbook called Discrete Mathematics will have lots of worked examples. Your proofs should always look like this:

- start: Let $P(n)$ be the statement that ...
- show that $P(1)$ (or some small value) is true.
- assume $P(k)$ is true and prove that $P(k+1)$ is true. Usually you start by writing the LHS of $P(k+1)$, manipulate, use that $P(k)$ is true half way along, then get to =RHS.
- end: Thus by PMI $P(n)$ is true for all $n \geqslant 1$ (or whatever your small value was)

1. Prove (induction) that
$$1 + 3 + 5 + \cdots + (2n-1) = n^2$$
for all $n \in \mathbb{N}_+$.

Proof. Let $P(n)$ be the statement $\boxed{}$

Then $P(1)$ is true since $\boxed{}$

Assume $P(k)$ for $k \geqslant 1$. Then

$\boxed{}$

Thus by PMI $P(n)$ is true for all $n \in \mathbb{N}_+$. □

Date: Week 4.

2. Consider the statement about $n \in \mathbb{N}$:

$$\sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3}$$

   (a) Call this statement $P(n)$. Write out $P(1), P(2), P(3)$.

   (b) Prove (induction) that for all $n \in \mathbb{N}, n \geqslant 2$

$$\sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3}$$

3. (a) Prove (induction) that for all $n \geqslant 10$ $n! > 3^n$.

   (b) What is the smallest number you can replace 10 by so that the statement is still true?

4. Prove that for all $n \in \mathbb{N}_+$, $2^{2n} - 1$ is divisible by 3.

5. Prove that for some value of k large enough, $n^2 < 2^n$ for all $n > k$.

6. A very simple application of the idea of induction is to show that some algorithm (computer code) is correct. A *loop invariant* is some statement that, if it is true before one iteration of the loop, it must be true after one iteration.

   (a) Show that "$m + n$ is odd" is a loop invariant for this code.

```
    int uselesscode(int n, int m)
{
    while (m> 0 and m< 100)
       m:= m+4
       n:= n-2
    end while
    return n
 }
```

   (b) Give a different loop invariant involving m and n.

7. Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by the recursive definition

$$\begin{aligned} f(0) &= 1 \\ f(n) &= nf(n-1) \qquad n > 0. \end{aligned}$$

   The value of $f(7)$ is

A. 28

B. 5040

C. 2520

D. 420

E. none of the above.

Brief solutions:

1. *Proof.* Let $P(n)$ be the statement that $1 + 3 + 5 + \cdots + (2n-1) = n^2$.

   Then $P(1)$ is true since LHS$= 1$. RHS$= 1^2 = 1$ so $P(1)$ is true.

   Assume $P(k)$ for $k \geqslant 1$. Then

   for $P(k+1)$:

$$\begin{aligned}
\text{LHS} &= 1 + 3 + \cdots + (2k-1) + (2(k+1)-1) \\
&= 1 + 3 + \cdots + (2k-1) + (2k+1) \\
&= k^2 + (2k+1) && \text{using assumption } P(k) \text{ is true} \\
&= (k+1)^2 \\
&= \text{RHS}
\end{aligned}$$

   Thus by PMI $P(n)$ is true for all $n \geqslant 1$. □

2. (a) $P(1)$ LHS is not defined, sum starts at 1 and ends at 0. So $P(1)$ does not make sense.

   $P(2)$ LHS$= \sum_{i=1}^{1} i(i+1) = 1(2) = 2$. RHS$= \frac{2(1)(3)}{3} = 2$ so $P(2)$ is true.

   $P(3)$ (not needed for an induction proof, but question asks for it)
   LHS$= \sum_{i=1}^{2} i(i+1) = 1(2) + 2(3) = 2 + 6 = 8$. RHS$= \frac{3(2)(4)}{3} = 8$.

   (b) Let $P(n)$ be the statement about natural numbers n that

$$\sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3}.$$

   $P(2)$ is true as shown in part (a).

   Assume $P(k)$ is true for some $k \geqslant 2$.

   $P(k+1)$:

$$\begin{aligned}
\text{LHS} &= \textstyle\sum_{i=1}^{k+1-1} i(i+1) \\
&= \textstyle\sum_{i=1}^{k} i(i+1) \\
&= 1(2) + 2(3) + \cdots + (k-1)(k) + k(k+1) \\
&= \tfrac{k(k-1)(k+1)}{3} + k(k+1) \text{ using } P(k) \text{ is true} \\
&= k(k+1)\left(\tfrac{(k-1)}{3} + 1\right) \\
&= k(k+1)\left(\tfrac{k-1+3}{3}\right) \\
&= \tfrac{(k+1)k(k+2)}{3} \\
&= \text{RHS}
\end{aligned}$$

Thus by PMI $P(n)$ is true for all $n \geqslant 2$.

3. (a) Let $P(n)$ be the statement that $n! > 3^n$.

   $P(10)$: $10! = 3628800, 3^{10} = 59049$ so $P(10)$ is true.

   Assume $P(k)$ is true for $k \geqslant 10$. Then
$$\begin{array}{rcl} (k+1)! & = & (k+1)k! \\ & > & (k+1)3^k \text{ using } P(k) \text{ is true} \\ & \geqslant & 3 \cdot 3^k \text{since } k+1 \geqslant 11 > 3 \\ & = & 3^{k+1} \end{array}$$

   Thus by PMI $P(n)$ is true for all $n \geqslant 10$.

   (b) The inductive step of the proof requires $k+1 \geqslant 3$, so we need k at least 2. But using a calculator we see we need $k \geqslant 7$ for the base step to work.

4. Let $P(n)$ be the statement that $3 \mid (2^{2n} - 1)$.

   $P(1)$: $2^{2 \cdot 1} - 1 = 4 - 1 = 3$ so $P(1)$ is true.

   Assume $P(k)$ true. [I added the next bit after I started my proof] This means we can write $2^{2k} - 1 = 3p$ for some $p \in \mathbb{Z}$. This means $2^{2k} = 3p + 1$.

   Then $2^{2(k+1)} - 1 = 2^{2k+2} - 1 = 2^2 2^{2k} - 1 = 4 \cdot 2^{2k} - 1 = 4(3p+1) - 1 = 3(4p) + 4 - 1 = 3(4p) + 3 = 3(4p+1)$ so is divisible by 3.

   Thus by PMI $P(n)$ is true for all $n \geqslant 1$.

5. Seems to be true for 1 (and also 0) so we could try proving the statement for all $n \geqslant 1$. But its false for $n = 2$ and $n = 3$: $3^2 = 9, 2^3 = 8$. Try $4^2 = 16, 2^4 = 16$, false, $5^2 = 25, 2^5 = 32$. So we will try $n \geqslant 5$.
   Let $P(n)$ be the statement that $n^2 < 2^n$.
   $P(5)$ is true since $5^2 = 25 < 2^5 = 32$.
   Assume $P(k)$ is true for some $k \geqslant 5$.
   Then $P(k+1)$:
$$\begin{array}{rcl} \text{LHS} & = & (k+1)^2 \\ & = & k^2 + 2k + 1 \\ & \leqslant & k^2 + k^2 \text{ since } 2k+1 \leqslant k^2 \text{ when } k \geqslant 5. \text{ This needs its own proof (see below).} \\ & = & 2k^2 \\ & < & 2 \cdot 2^k \text{ using } P(k) \text{ is true} \\ & = & 2^{k+1} \\ & = & \text{RHS} \end{array}$$
   Thus by PMI $P(n)$ is true for all $n \geqslant 5$.

   Now I have to check the claim that $2k+1 \leqslant k^2$ when $k \geqslant 5$. I will prove this by induction too (could also use Calculus!)

   Let $Q(n)$ be the statement $2k+1 \leqslant k^2$.

$Q(5)$ is true since $11 \leqslant 25$.

Assume $Q(k)$ is true for $k \geqslant 5$.

Then show $Q(k+1)$:

$$\begin{aligned}
2(k+1)+1 &= 2k+3 \\
&= 2k+1+2 \\
&\leqslant k^2+2 \text{ using } Q(k) \text{ is true} \\
&= k^2+1+1 \\
&< k^2+2k+1 \text{ since } 2k>1 \\
&= (k+1)^2 \\
&= \text{RHS}
\end{aligned}$$

so true for all $n \geqslant 5$ by PMI.

6. (a) If $m+n = 2p+1$ before going into the loop, then afterwards the value is $m+4+n-2 = 2p+1+2$ is still odd. Thus "the sum being odd" is a loop invariant.

   (b) The sum begin even would also be a loop invariant, as would the sum being positive, etc.

7. $f(7) = 7f(6) = 7.6f(5) = 7.6.5f(4) = \cdots = 7!f(0) = 7! = 5040$.