# DISCRETE MATH 37181 WORKSHEET 1

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Instructions. Complete these problems in groups of 3-4 at the whiteboards.

1. Each group member: write your name on the top of the whiteboard, and say hi to your teammates.

2. Draw truth tables for the following statements.

   (a) $((p \to q) \wedge p) \to \neg q$

   (b) $((p \to q) \wedge \neg q) \to \neg p$

   (c) $\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$

   (d) $((p \to q) \wedge (q \to r)) \to (p \to r)$

   (e) $(p \to q) \leftrightarrow (\neg q \to p)$

   (f) $s \leftrightarrow (p \to ((\neg p) \vee s))$

   (g) Which of (a)–(g) are *tautologies*?

3. For each of the following quantified statements, write a quantified statement which is logically equivalent to the negation and only uses the symbol $\neg$ after the quantifiers. [1] For (b),(e),(f), translate the statement into logical symbols, find the negation, then put your answer back in English.

   (a) $\forall x \exists y\, (x^2 > y \vee x < 2y)$

   (b) Every person has someone who loves them [2]

   (c) $\forall x \forall y\, (x < y \to \exists z (x < y < z))$

   (d) $\exists x \forall y \exists z\, (z > y \to z < x^2)$

*Date*: Week 1 workshop (Thursday 25, Friday 26, Monday 29 July).

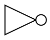
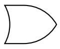
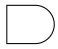
[1]For example, $\neg\, (\forall x \forall y (x > y))$ becomes $\exists x \exists y (x \leqslant y)$.

[2]define $L(x, y)$ to mean "$x$ loves $y$". Note $L(x, y)$ does not always have the same truth value as $L(y, x)$

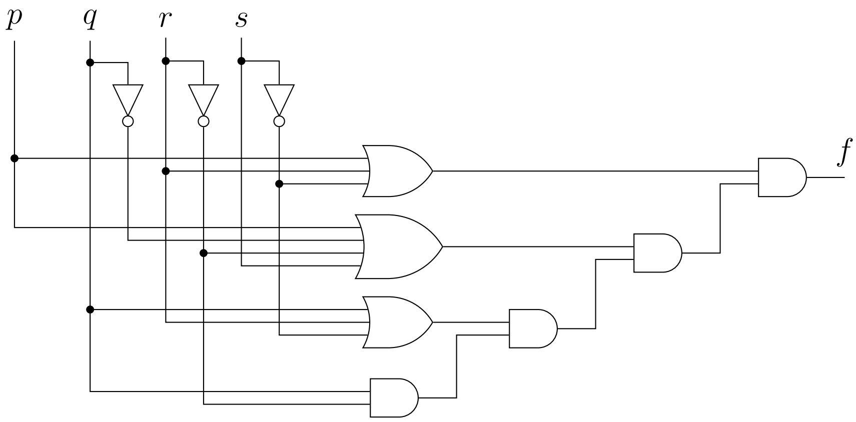
(e) Every person loves at least two people

(f) Every person loves at least two people who do not love each other.

4. A *logic circuit* has input wires labeled $p, q, r, s, \ldots$, *logic gates*:

not:  or:  and : 

and an output wire labeled $f$. An example is given here:




(a) On input $p = 0, q = 1, r = 0, s = 1$, what is the output at $f$?

(b) Write down the logical expression corresponding to the circuit diagram above.

(c) Draw a logic circuit representing this formula: [3]

$$(\neg(p \wedge q) \to \neg p) \vee q$$

5. Is the following formula *satisfiable*?

$$(p \vee q \vee \neg r) \wedge (r \vee w \vee \neg q) \wedge (\neg w \vee \neg p \vee \neg r)$$

6. Use propositional logic (*i.e.* convert to symbols) to determine which of the following are valid arguments:

(a) If I do not work hard, I will sleep. If I am worried, I will not sleep. Therefore, if I am worried, I will work hard.

(b) If it is raining, I am wet. I am dry. Therefore it is not raining. [4]

[3] hint: $p \to q$ is the same as $\neg p \vee q$

[4] "I am dry" is the same as "I am not wet"

Brief solutions:

2. (a)

| $p$ | $q$ | $((p \to q)$ | $\wedge$ | $p)$ | $\to$ | $\neg q$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | | 0 | 0 |
| 1 | 0 | 0 | 0 | | 1 | 1 |
| 0 | 1 | 1 | 0 | | 1 | 0 |
| 0 | 0 | 1 | 0 | | 1 | 1 |

(b)

| $p$ | $q$ | $((p \to q) \wedge \neg q)$ | $\to$ | $\neg p$ |
|---|---|---|---|---|
| 1 | 1 | | 1 | |
| 1 | 0 | | 1 | |
| 0 | 1 | | 1 | |
| 0 | 0 | | 1 | |

(c)

| $p$ | $q$ | $\neg(p \wedge q)$ | $\leftrightarrow$ | $(\neg p \vee \neg q)$ |
|---|---|---|---|---|
| 1 | 1 | | 1 | |
| 1 | 0 | | 1 | |
| 0 | 1 | | 1 | |
| 0 | 0 | | 1 | |

(d)

| $p$ | $q$ | $r$ | $((p \to q) \wedge (q \to r))$ | $\to$ | $(p \to r)$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | | 1 | |
| 1 | 1 | 0 | | 1 | |
| 1 | 0 | 1 | | 1 | |
| 1 | 0 | 0 | | 1 | |
| 0 | 1 | 1 | | 1 | |
| 0 | 1 | 0 | | 1 | |
| 0 | 0 | 1 | | 1 | |
| 0 | 0 | 0 | | 1 | |

(g) b,c,d only.

3. (a) $\forall x \exists y\, (x^2 > y \vee x < 2y)$

$$\neg \forall x \exists y \left(x^2 > y \vee x < 2y\right)$$
$$\exists x \forall y \neg \left(x^2 > y \vee x < 2y\right)$$
$$\exists x \forall y \left(\neg(x^2 > y) \wedge \neg(x < 2y)\right)$$
$$\exists x \forall y \left((x^2 \leqslant y) \wedge (x \geqslant 2y)\right)$$

(b) Every person has someone who loves them

Let $L(x, y)$ be the statement "$x$ loves $y$", and the universe of discourse all people.

$$\forall x \exists y L(y, x)$$

Negation:

$$\exists x \forall y \neg L(y, x)$$

There is a person that nobody loves. (Note: including themself).

(c) $\forall x \forall y\, (x < y \to \exists z (x < y < z))$ This is equivalent to

$$\forall x \forall y\, (\neg(x < y) \lor \exists z(x < y < z))$$

Negation:

$$\exists x \exists y \neg\, (\neg(x < y) \lor \exists z(x < y < z))$$
$$\exists x \exists y\, ((x < y) \land \neg \exists z(x < y < z))$$
$$\exists x \exists y\, ((x < y) \land \forall z \neg (x < y < z))$$
$$\exists x \exists y\, ((x < y) \land \forall z \neg ((x < y) \land (y < z)))$$
$$\exists x \exists y\, ((x < y) \land \forall z((x \geqslant y) \lor (y \geqslant z)))$$

There are two numbers $x, y$ so that $x < y$ and either $x \geqslant y$ (which cannot be true so we just have) or $y$ is bigger than or equal to all numbers $z$.
There are two numbers $x, y$ so that $x < y$ and $y$ is bigger than or equal to all numbers $z$. This is false ($\infty$ is not included in the universe of all real numbers).
The original statement was true: for all pairs of numbers $(x, y)$, if $x < y$ then you can find some $z = y + 1$ say so that $x < y < z$ is true.

(d) $\exists x \forall y \exists z\, (z > y \to z < x^2)$

$$\forall x \exists y \forall z\, \big((z > y) \land (z \geqslant x^2)\big)$$

(e) Every person loves at least two people

$$\forall x \exists y \exists z\, [L(x, y) \land L(x, z) \land (y \neq z)]$$

(This includes the possibility that $x = z$ or $x = y$, that is, one of the people $x$ loves is themself.)
Negation:

$$\neg \forall x \exists y \exists z\, [L(x, y) \land L(x, z) \land (y \neq z)]$$
$$\exists x \forall y \forall z \neg\, [L(x, y) \land L(x, z) \land (y \neq z)]$$
$$\exists x \forall y \forall z \neg L(x, y) \lor \neg L(x, z) \lor (y = z)$$

There is a person that either does not love anyone, or loves only one person. That is, $\forall y \forall z$, as you consider all pairs of people $(y, z)$, if $x$ loves both of them, then $y = z$.

(f) Every person loves at least two people who do not love each other.

$$\forall x \exists y \exists z\, [L(x, y) \land L(x, z) \land (y \neq z) \land \neg L(y, z) \land \neg L(z, y)]$$

Negation:

$$\exists x \forall y \forall z\, [\neg L(x, y) \lor \neg L(x, z) \lor (y = z) \lor L(y, z) \lor L(z, y)]$$

There is a person that either loves at most one person, or if they more than one person, then in each pair of people that $x$ loves, one of the people loves the other one.

4. (a) On input $p = 0, q = 1, r = 0, s = 1$, what is the output at $f$?

0. From the top *or* gate, all inputs are 0, and this feeds into the top *and* gate.

(b) Write down the logical expression corresponding to the circuit diagram above.

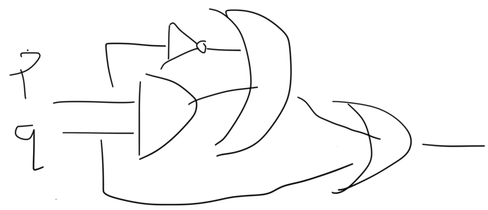
$$(p \lor r \lor \neg s) \land [(p \lor \neg q \lor \neg r \lor s) \land [(q \lor r \lor \neg s) \land (q \land \neg r))]$$

(c) Draw a logic circuit representing this formula:

$$(\neg(p \wedge q) \to \neg p) \vee q$$

is equivalent to

$$((p \wedge q) \vee \neg p) \vee q$$




5. Try some values: $p = 1, r = 1, w = 0$ makes the entire formula true (for any $q$).

6. (a) If I do not work hard, I will sleep. If I am worried, I will not sleep. Therefore, if I am worried, I will work hard.
$h$ = work hard, $s$ = sleep, $w$ = worried.

$$((\neg h \to s) \wedge (w \to \neg s)) \to (w \to h)$$

| $h$ | $s$ | $w$ | $((\neg h \to s)$ | $\wedge$ | $(w \to \neg s))$ | $\to$ | $(w \to h)$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | | | | 1 | 1 |
| 1 | 1 | 0 | | | | 1 | 1 |
| 1 | 0 | 1 | | | | 1 | 1 |
| 1 | 0 | 0 | | | | 1 | 1 |
| 0 | 1 | 1 | | | | 1 | 0 |
| 0 | 1 | 0 | | | | 1 | 1 |
| 0 | 0 | 1 | | | | 1 | 0 |
| 0 | 0 | 0 | | | | 1 | 1 |

A quick way to show this is to argue as follows: how could this statement be false? Only if $w \to h$ is false (and $(\neg h \to s) \wedge (w \to \neg s)$ is true).
So only need to check two rows of the truth table.

Alternatively, $\neg h \to s$ is logically equivalent to $\neg s \to h$ (contrapositive), so $((w \to \neg s) \wedge (\neg s \to h)) \to (w \to h)$ is syllogism.

(b) If it is raining, I am wet. I am dry. Therefore it is not raining.

$$((r \to w) \wedge \neg w) \to (\neg r)$$

Valid – this is modus tollens.