1 The Fourier Transform

The Fourier Transform can be thought of as complex Fourier Series on an infinite interval. Consider an *absolutely integrable* function, that is $f(x)$, for $f \in \mathbb{C}$ such that

$$\int_{-\infty}^{\infty} |f(x)|\ dx \equiv \lim_{R\to\infty} \int_{-R}^{R} |f(x)\ dx = M < \infty.$$

Remember complex Fourier series of $f(x)$ on $x \in [-L, L]$

$$f(x) = \sum_{m=-\infty}^{\infty} c_m e^{im\frac{\pi x}{L}}$$

where

$$c_m = \int_{L}^{L} f(x) e^{-im\frac{\pi x}{L}}.$$

Define the Fourier Transform

$$\mathscr{F}\{f(x)\} = \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\ dx.$$

Let $\xi_m = \frac{m\pi}{L}$. Then

$$\hat{f}(\xi_m) = \lim_{L\to\infty} 2Lc_m$$

because

$$2Lc_m = \int_{-L}^{L} f(x) e^{-i\xi_m x}\ dx.^1$$

Note: c_m and $f(x)$ are descriptions of the same thing.
Question: If I know $f(x)$, I can find $\hat{f}(\xi)$. If I know $\hat{f}(\xi)$, can I recover $f(x)$?
Answer: Yes.
I know

$$\begin{aligned} f(x) &= \sum_{m=-\infty}^{\infty} c_m e^{i\xi_m x} \qquad \Delta\xi = \frac{\pi}{L} \\ &= \frac{1}{L} \sum_{m=-\infty}^{\infty} (Lc_m) e^{i\xi_m x} \qquad \xi = m\Delta\xi \\ &= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \Delta\xi (2Lc_m) e^{i\xi_m x}. \end{aligned}$$

Now, let $L \to \infty$, while holding ξ_m fixed. Then $\lim_{L\to\infty} 2Lc_m = \hat{f}(\xi_m)$. Then

$$f(x) = \lim_{\substack{L\to\infty \\ \text{or } \Delta\xi\to 0}} \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \Delta\xi \hat{f}(\xi_m) e^{i\xi x}.$$

This is a Riemann sum,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi_m x}\ d\xi_m.$$

This is the inverse Fourier Transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x}\ d\xi.$$

[1] This integral converges because f is absolutely integrable.

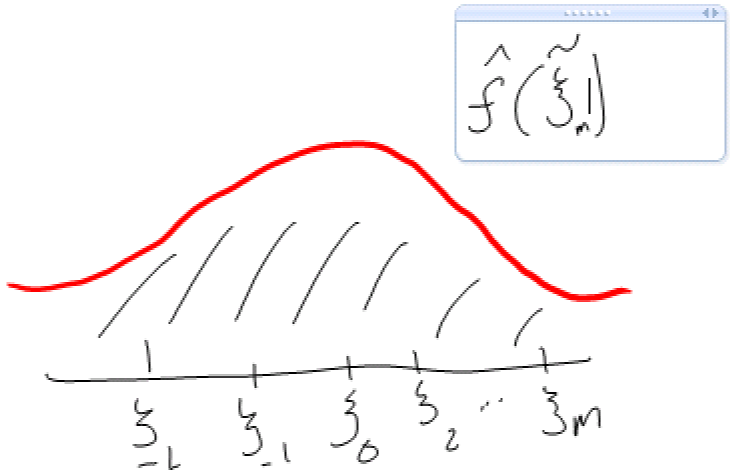

Figure 1: Riemann Sum.

1.1 Proof by example

Suppose

$$f(x) = \begin{cases} 1 & |x| \le d \\ 0 & |x| \ge d \end{cases}$$

Complex Fourier Series (assume $L > d$):

$$c_m = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\xi x}\,dx = \frac{1}{2L}\int_{-d}^{d} e^{-i\xi x}\,dx = \frac{2\sin(\frac{m\pi d}{L})}{m\pi/L}\frac{1}{2L}.$$

So

$$2Lc_m = \frac{2\sin(\xi_m d)}{\xi_m} \qquad \xi_m = \frac{m\pi}{L}.$$

Fourier Transform:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}\,dx = \int_{-d}^{d} e^{-i\xi i}\,dx = \frac{2\sin(\xi d)}{\xi}.$$

Also I claim

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin(\xi d)}{\xi}e^{i\xi x}\,dx.$$

The proof of this is left to the reader.

2 Is the Fourier Transform Useful?

Yes. The Fourier Transform can be used to solve ODEs and PDEs.

2.1 Example 1

Compute the Fourier Transform of

$$f(x) = H(x)e^{-ax}$$

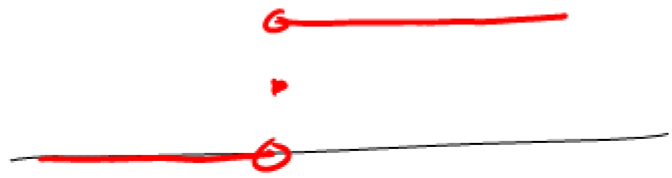

Figure 2: The Heaviside Function.

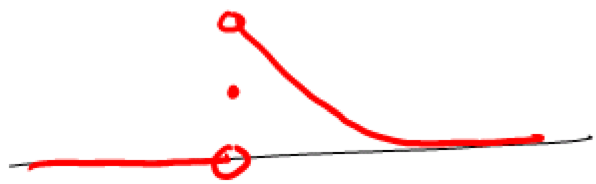

Figure 3: Graph of $H(x)e^{-ikx}$.

for $a > 0$, where

$$H(x) = \begin{cases} 1 & x > 0 \\ \frac{1}{2} & x = 0 \\ 0 & x < 0 \end{cases},$$

is Heaviside function, so

$$H(x)e^{-ax} = \begin{cases} e^{-ax} & x > 0 \\ \frac{1}{2} & x = 0 \\ 0 & x < 0 \end{cases}.$$

$$\begin{aligned}
\{f(x)\} &= \hat{f}(k) \\
&= \int_0^\infty e^{-ax}e^{-ikx}\ dx \\
&= \int_0^\infty e^{-(a+ik)x}\ dx \\
&= \frac{e^{-(a+ik)x}}{-(a+ik)}\Big[_{x=0}^{\infty} \\
&= \frac{1}{a+ik}.
\end{aligned}$$

2.2 Example 2

Suppose $\hat{y}(k) = \mathscr{F}\{y(x)\}$, wthat can we say about $\mathscr{F}\{y'(x)\}$?

$$\hat{y}(k) = \int_{-\infty}^{\infty} y(x)e^{-ikx}\ dx$$

and

$$\mathscr{F}\{y'(x)\} = \int_{-\infty}^{\infty} y'(x)e^{-ikx}\ dx.$$

We integrate by parts;

$$\begin{aligned} y'(x)dx &= du \\ e^{-ikx} &= v \\ y(x) &= u \\ -ike^{-ikx} &= dv. \end{aligned}$$

So

$$\mathscr{F}\{y'(x)\} = y(x)e^{-ikx}\Big[_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty}(-ikx)e^{-ikx}y(x)\ dx.$$

If $|y(x)| \to 0$ as $|x| \to \infty$ the first term vanishes.

$$\mathscr{F}\{y'(x)\} = ik\int_{-\infty}^{\infty} e^{-ikx}y(x)\ dx = ik\hat{y}(k),$$

the Fourier Transform turns differentiation into multiplication!

2.3 Example 3

Solve

$$\begin{aligned} &\text{DE: } y' + y = H(x)e^{-2x} \qquad -\infty < x < \infty \\ &\text{DC: } |y(x)| \to 0 \qquad \text{as } x \to \pm\infty. \end{aligned}$$

Solution: Fourier Transform both sides

$$\begin{aligned} \mathscr{F}\{y' + y\} &= \mathscr{F}\{H(x)e^{-2x}\} \\ \mathscr{F}\{y'\} + \mathscr{F}\{y\} &= \frac{1}{2+ik}. \end{aligned}$$

But

$$\begin{aligned} \mathscr{F}\{y'\} &= ik\hat{y} \\ \mathscr{F}\{y\} &= \hat{y} \end{aligned}$$

so

$$\begin{aligned} ik\hat{y} &= \frac{1}{2+ik} \\ (1+ik)\hat{y} &= \frac{1}{2+ik} \end{aligned}$$

So

$$\hat{y} = \frac{1}{(1+ik)}\frac{1}{(2+ik)}$$

But

$$\begin{aligned} y(x) &= \mathscr{F}^{-1}\{\hat{y}(k)\} \\ &= \mathscr{F}^{-1}\{\frac{1}{(1+ik)}\frac{1}{(2+ik)}\} \\ &= \mathscr{F}^{-1}\{\frac{1}{1+ik} - \frac{1}{2+ik}\} \\ &= H(x)e^{-x} - H(x)e^{-2x}. \end{aligned}$$

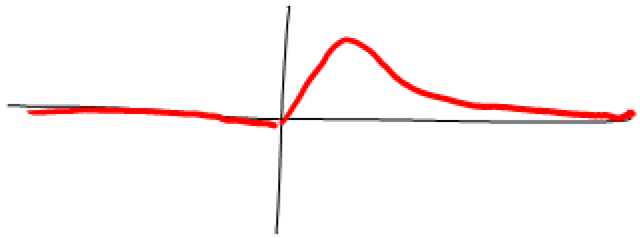

Figure 4: Sketch of the solution.

So

$$y(x) = H(x)[e^{-x} - e^{-2x}].$$

3 Transform of the Delta Function

$$\mathscr{F}\{\delta(x)\} = \int_{-\infty}^{\infty} \delta(x) e^{-ikx}\ dx = 1.$$