Fields and Waves

Vatche Sahakian

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1 Classical Fields

1.1 A continuum limit

A field theory is described by a set of N fields:

$$\varphi^a(x^0, x^1, x^2, x^3) = \varphi^a(x) \quad a = 1 \dots N,$$

where

$$x^0 = ct, x^1 = x, x^2 = y, x^3 = z$$

and c is a characteristic speed of the theory (such and the speed of light or the speed of sound.) These fields are governed by a Langrangian functional, for example (and using $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ for notational convenience)

$$L = \frac{Y}{2} \int dx \left((\partial_0 \varphi)^2 - (\partial_1 \varphi)^2 \right).$$

Definition 1 (Action, Lagrangian Density)**.** *The action in a field theory is*

$$S = \int dt\ L = \int d^4 x \mathcal{L}[\varphi^a, \partial_\mu \varphi^a, x^\mu]$$

where $\mathcal{L}$ is the Lagrangian density, and is defined implicitly by the above.

The Lagrangian density of the earlier example is

$$\mathcal{L} = \frac{\sqrt{Y\mu}}{2} \left((\partial_0 \varphi)^2 - (\partial_1 \varphi)^2 \right).$$

Field theories can involve fewer or more dimensions than 4. It is the convention for this text that time-like dimensions are given the $x^0 = ct$ coordinate. A commonality among all field theories is that they carry energy that can be viewed as being stored at every spacetime point.

1.2 An Action Principle

The action principle is that the motion undergone by the system will be that which minimizes the action. Thus we wish to find φ^a such that the variation $\delta S = 0$ (with respect to φ^a. Following the usual derivation of Euler's equations we get the conditions

$$\frac{\partial \mathcal{L}}{\partial \varphi^a} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^a} \right).$$

This mirrors the case for discrete degrees of freedom:

$$\frac{\partial \mathcal{L}}{\partial q_k} = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right).$$

1.3 Complex Fields

A field theory consisting of two decoupled scalar fields can be re-written as being of one complex field. For example, for φ^1 and φ^2 we can write $\varphi = \varphi_1 + i\varphi_2$ and use the identities $\varphi_1 = \frac{1}{2}(\varphi + \varphi^*)$ and $\varphi_2 = \frac{1}{2i}(\varphi - \varphi^*)$ to re-write the Lagrangian in terms of φ and φ^*. If Euler's equations are used treating these as different fields, the resulting equations of motion will be equivalent.

2 Symmetries & Conservation Laws

2.1 Noether's Theorem

Noether's Theorem states that for every continuous symmetry of a physical system, there exists a conservation law. To understand this, we must first define what is meant by a continuous symmetry, and what is meant by a conservation law. First, a continuous system is some deformation of the system, both of the fields themselves and the coordinates, that preserves the action regardless of whether or not the field equations of motion are satisfied. Quantitatively we wrtie this as

$$\{\bar{\delta}\varphi^a, \delta x^\mu\} \implies \delta S = 0,$$

where $\{\bar{\delta}\varphi^a, \delta x^\mu\}$ is the symmetry (technically just a perturbation if the $\delta S = 0$ condition is not satisfied.) Propogating the effects of a given perturbation on S to calculate δS, and taking care to take into account the dependence of the fields on the coordinates we find that

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \varphi^a} \bar{\delta}\varphi^a + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^a} \partial_\mu \bar{\delta}\varphi^a + \partial_\mu(\delta x^\mu \mathcal{L}) \right).$$

Now we assume that this pertubation is a symmetry (so set $\delta S = 0$ and that the equations of motion are satisfied. Then we get that

$$\delta S = 0 = \int d^4x \partial_\mu j^\mu$$

where

$$j^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^a} \bar{\delta}\varphi^a + \delta x^\mu \mathcal{L}$$

is the conserved current associated with the symmetry. To see this, note that

$$\partial_\mu j^\mu = 0 \ \partial_0 j^0 + \partial_i j^i$$

so

$$\int d^3x \partial_0 j^0 + \int d^3x \vec{\nabla} \cdot \vec{j} = 0.$$

Using the divergence theorem and defining the Noether Change Q to be

$$Q = \int d^3x \frac{j^0}{c}$$

we can re-arrange this equation to see

$$\frac{dQ}{dt} = -\oint d\vec{A} \cdot \vec{j}.$$

Thus $\frac{j^0}{c}$ is the density of the Noether Charge which is conserved over regions in space with the j^i components describing the flux out of the regions.