1 Homework Issues

Turn it in tomorrow. A head start:

$$
\begin{aligned}
g'' - 2g' + 2g &= \delta(x) \\
\mathscr{F}\{g'' - 2g' + 2g\} &= \mathscr{F}\{\delta(x)\} \\
(-k^2 - 2ik + 2)\hat{g} &= 1 \\
\hat{g} &= \frac{1}{-k^2 - 2ik + 2}
\end{aligned}
$$

Note that

$$
\begin{aligned}
(ik)^2 - 2ik + 2 &= (ik - i)^2 + 1 \\
&= [(ik - 1 + i)(ik - 1 - i)]
\end{aligned}
$$

so

$$
\begin{aligned}
\hat{g}(k) &= \frac{1}{(ik - 1 + i)(ik - 1 - i)} \\
&= \frac{A}{ik - 1 + i} + \frac{B}{ik - 1 - i} \\
&= \frac{-\frac{1}{2i}}{(ik - 1 + i} + \frac{\frac{1}{2i}}{(ik - 1 - i)} \\
&= \frac{1}{2i}\left[\frac{1}{(ik - 1 - i} - \frac{1}{ik - 1 + i}\right]
\end{aligned}
$$

We then need to figure out the inverse transform of this:

$$
\begin{aligned}
\mathscr{F}\{H(x)e^{-ax}\} &= \int_0^\infty e^(- a - ik)x \\
&= \frac{e^{(-a-ik)x}}{-a - ik}\Big|_{x=0}^{\infty} \\
&= \frac{1}{a + ik} \qquad \Re\{a\} > 0
\end{aligned}
$$

Suppose $a = \alpha + i\beta$. Then

$$
= \frac{e^{-\alpha - i\beta - ikx}}{-a - ik} = e^{-\alpha x}[\ldots
$$

2 Fun Fact

$$
I = \int_{-\infty}^{\infty} e^{-ax^2/2}\, dx = \sqrt{\frac{2\pi}{a}}.
$$

Proof:

$$
\begin{aligned}
I^2 &= \int_{-\infty}^{\infty} e^{-ax^2/2}\ dx \int_{-\infty}^{\infty} e^{-ay^2/2}\ dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{a}{2}(x^2+y^2)}\ dx\ dy \\
&= \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} e^{-a\frac{r^2}{2}}\ r\ dr\ d\theta \\
&= \frac{2\pi}{a}\int_0^{\infty} e^{\frac{-ar^2}{2}} a\ r\ dr \\
&= \frac{2\pi}{a}(-e^{-ar^2/2})\Big|_{r=0}^{\infty} \\
I^2 &= \frac{2\pi}{a} \\
I &= \sqrt{\frac{2\pi}{a}}
\end{aligned}
$$

3 Cauchy Problem for the Heat Equation

$$
\begin{aligned}
&\text{DE: } u_t = Du_x x \qquad -\infty < x < \infty, t > 0 \\
&\text{IC: } u(x,0) = f(x) \qquad -\infty < x < \infty \\
&\text{BC: } \max|u(x,t)| \text{ is bounded for all } t > 0
\end{aligned}
$$

Also, assume $\max|f(x)|$ is bounded. Do we always need this boundary condition? Yes, for a physical solution. How do we solve this thing? (Picture 1 in Notebook). Solution: Use the Fourier Transform.

$$
\begin{aligned}
\mathscr{F}\{u(x,t)\} &= \hat{u}(k,t) \\
\mathscr{F}\{u_t(x,t)\} &= \hat{u}_t(k,t) \\
\mathscr{F}\{u_x x(x,t)\} &= (ik)^2\hat{u}(k,t) \\
&= -k^2\hat{u}(k,t),
\end{aligned}
$$

So

$$
\begin{aligned}
u_t = Du_x x &\implies \hat{u}_t = -Dk^2\hat{u}, \\
\hat{u}(k,t) &= A(k)e^{-Dk^2 t} \qquad (*)
\end{aligned}
$$

From the IC

$$\mathscr{F}\{u(x,0)\} = \hat{u}(k,0) = \mathscr{F}\{f(x)\} = \hat{f}(k),$$

but from *,

$$\hat{u}(k,0) = A(k) = \hat{f}(k) \implies \hat{u}(k,t) = \hat{f}(k)e^{-Dk^2 t}.$$

We need to known

$$\mathscr{F}\{e^{(-(Dt)k^2}\}.$$

4 Fourier Transform of a Gaussian

$$\mathscr{F}\{e^{-ax^2/2}\} = \int_{-\infty}^{\infty} e^{-ax^2/2+ikx}\ dx$$

Let's complete the square

$$\begin{aligned}
\frac{ax^2}{2} + ikx &= \frac{a}{2}[x^2 + \frac{2ik}{a}x] \\
&= \frac{a}{2}[(x + \frac{ik}{a})^2 - (\frac{ik}{a})^2] \\
&= \frac{a}{2}(x + \frac{ik}{a})^2 + \frac{k^2}{2a})
\end{aligned}$$

So

$$\begin{aligned}
\mathscr{F}\{e^{-ax^2/2}\} &= \int_{-\infty}^{\infty} e^{-\frac{a}{2}(x+\frac{ik}{a})^2 + \frac{k^2}{2a})}\ dx \\
&= e^{-\frac{k^2}{2a}} \int_{x=-\infty}^{\infty} e^{-\frac{a}{2}(x+\frac{ik}{a})^2}\ dx
\end{aligned}$$

Let

$$\begin{aligned}
z &= x + \frac{ik}{a} \\
dz &= dx.
\end{aligned}$$

What is $\infty + \frac{ik}{a}$? Answer: ∞. See picture 2 in notebook. This function is analytic everywhere. So

$$\begin{aligned}
\mathscr{F}\{e^{-ax^2/2}\} &\overset{1}{=} e^{-\frac{k^2}{2a}} \int_{-\infty}^{\infty} e^{-\frac{a}{2}z^2}\ dz \\
&= e^{-\frac{k^2}{2a}} \sqrt{\frac{2\pi}{a}}
\end{aligned}$$

So

$$\mathscr{F}\{e^{ax^2/2}\} = \sqrt{\frac{2\pi}{a}} e^{-k^2/2a}. \tag{☺}$$

The Fourier Transform of a Gaussian is a Gaussian! Also

$$\mathscr{F}\{\frac{1}{2\sqrt{\pi b}} e^{-x^2/4b}\} = e^{-bk^2}.$$

In ☺, set

$$b = \frac{1}{2a} \implies a = \frac{1}{2b}$$

and multiply by

$$\sqrt{\frac{a}{2\pi}} = \frac{1}{2\sqrt{\pi b}}.$$

Finally, we have ($b = Dt$)

$$\begin{aligned}
\mathscr{F}^{-1}\{e^{-Dk^2t}\} &= G(x,t) \\
&= \frac{1}{2\sqrt{\pi Dt}} e^{-x^2/4Dt}.
\end{aligned}$$

This is the Green's function (or the "Kernel") of the heat equation.

[1] with a little help from Math 136.

4.1 Solutions of the Cauchy problems for...

i) $f(x) = \delta(x)$ delta

ii) $f(x) = f(x)$ arbitrary

iii) $f(x) = H(x)$ Heaviside

i) If $f(x) = \delta(x)$, $\mathscr{F}\{\delta(x)\} = 1$. Then

$$u(x,t) = G(x,t)\frac{e^{-x^2/4Dt}}{2\sqrt{\pi Dt}}.$$

See picture 3 in notebook. Width scales like $2\sqrt{Dt}$ spreading. Height scales like $\frac{1}{2\sqrt{\pi Dt}}$ decreasing. Area = 1. Self similar diffusion - it spreads out, but maintains its characteristic shape.

ii) $f(x) = f(x)$

$$\begin{aligned} \hat{u}(k,t) &= \hat{f}(k,t)e^{-Dtk^2} \\ &\Longrightarrow \\ u(x,t) &= f(x) \star G(k,t) = \frac{1}{2\sqrt{\pi Dt}}\int_{-\infty}^{\infty} f(x-y)e^{-y^2/4Dt}\ dy \end{aligned}$$

Poisson Integral Formula for the solution to the Cauchy problem. It is a continuous superposition of the solutions to the problem. It is a sum of many delta functions.

iii) $f(x) = H(x)$

$$u(x,t) = \frac{1}{2\sqrt{\pi Dt}}\int_{-\infty}^{\infty} H(x-y)e^{-y^2/4Dt}\ dy.$$

$$H(x-y) = \begin{cases} 1 & x > y \\ 0 & x < y \end{cases}.$$

So

$$u(x,t) = \frac{1}{2\sqrt{\pi Dt}}\int_{-\infty}^{x} e^{-y^2/4Dt}\ dy.$$

Let $z = \frac{y}{2\sqrt{Dt}}$ then $dz = \frac{dy}{2\sqrt{Dt}}$. So

$$u(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\frac{x}{2\sqrt{Dt}}} e^{-z^2}\ dz.$$

Remember the error function

$$\mathrm{erf}(w) = \frac{2}{\sqrt{\pi}}\int_0^w e^{-s^2}\ ds.$$

See picture 4 in notebook.

$$\begin{aligned} u(x,t) &= \frac{1}{2}[\frac{2}{\sqrt{\pi}}\int_{-\infty}^{0} e^{-w^2}\ dw + \int_0^{\frac{x}{2\sqrt{Dt}}} e^{-w^2}\ dw \\ &= \frac{1}{2}[-\mathrm{erf}(-\infty) + \mathrm{erf}(\frac{x}{2\sqrt{Dt}})] \\ &= \frac{1}{2}[1 + \mathrm{erf}(\frac{x}{2\sqrt{Dt}})] \end{aligned}$$

See picture 5 in notebook.