Inner-products, Orthogonal Exansions and L2

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We can project one the basis by taking the dot product with e_i

Minimizing the L^2 Error

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Suppose we wish to approximate

$$F(x) \approx F_N(x) = \sum_{i=1}^{N} c_i f_i(x)$$

what are best choices for c_i

We can [illegible] the [illegible]

$$(F(x), f) \approx \sum_{i=1} c_i (f_j, f_i$$

<u>Theorem</u>: Suppose we define $F_N(x)$ as

above

$$F_N(x) = \sum_{i=1}^{N} c_i f_i(x) \quad \text{where } c_i = \frac{(F(x), f_i(x))}{\|f_i\|^2}$$

$$E = \| F(x) - F_N(x) \| = \sqrt{\int_a^b (F(x) - F_N(x))^2 \, dx}$$

then of all the approximations (choices of c_i's)

F_N minimizes the error, E.

Note:

E is the L_2 or RMS error.

L_1 error does a good! job of compressing images,

but hard to find coefficients.

Pf:

Let

$$\tilde{F}_N = \sum_{i=1}^{N} a_i f_i(x)$$

Lemma:

$$\|\tilde{F}_N\|^2 = \sum_{i=1}^{N} a_i^2 \|f_i\|^2$$

Pf:

$$\|\tilde{F}_N\|^2 = \left\langle \sum_{i=1}^{n} a_i f_i , \sum_{j=1}^{N} a_j f_j \right\rangle =$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j (f_i, f_j)$$

$$= \sum_{i=1}^{N} a_i^2 \|f_i\|$$

Look at

$$[E(a_n)]^2 = \|F(x) - \tilde{F}_N(x)\|^2$$

$$= \left(F(x) - \tilde{F}_N(x), F(x) - \tilde{F}_N(x)\right)$$

$$= \left(F(x), F(x)\right) - 2\left(F(x), \tilde{F}_N(x)\right) + \left(\tilde{F}_N(x), \tilde{F}_N(x)\right)$$

$$= \|F(x)\|^2 + \|\tilde{F}_N(x)\|^2 - 2\left(F(x), \tilde{F}_N(x)\right)$$

but $\left(F(x), \hat{F}_N(x)\right) = \left(F(x), \sum_{j=1}^{N} a_j f_j(x)\right)$

$$= \sum_{i=1}^{N} a_i \left(F(x), f_i(x)\right)$$

So

$$\left[E(a_n)^2\right] = \|F(x)\|^2 + \sum_{i=1}^{N}\left[a_i^2 \|f_i\|^2 - 2a_i\left(F(x), f_i(x)\right)\right]$$

We wish to minimize this as a function of a_i's
can minimize terms separately! (Diagonalized)

Lemma #2:

$P(a) = Aa^2 + Ba$ and $A > 0$. Then P is minimized for

$$a = \frac{-B}{2A}$$

Pf:

$$P(a) = A\left(a - \frac{B}{2A}\right)^2 - \frac{B^2}{4A}$$

From the lemma we know

$[E(a_n)]^2$ is minimized for $a_i = -\frac{(-2(f(x), f_i(x))}{2\|f_i\|^2}$

or $a_i = \frac{\langle f(x), f_i(x)\rangle}{\|f_i\|^2} = c_i$ from Fourier!

So $F_N(x)$ is the L^2 minimizer

Corollaries

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Corollary #1

Bessel's Inequality

$$\|F(x)\|^2 \geq \sum_{i=1}^{N} c_i^2 \|f_i(x)\|^2 = \|F_N(x)\|^2$$

Proof: When $a_i = c_i$,

$$[E(a_n)]^2 = E^2 = \|F(x)\|^2 + \sum_{i=1}^{N} c_i^2 \|f_i\|^2 - 2c_i(F(x), f_i)$$

but $(F(x), f_i) = c_i \|f_i\|^2$

So

$$E^2 = \|F(x)\|^2 - \sum_{i=1}^{N} c_i^2 \|f_i\|^2 \geq 0$$

which proves Bessel's Inequality

Corollary #2:

Suppose as $N \to \infty$, $E \to 0$ (approx is good, converges)

then we have

Parseval's Theorem

$$\boxed{\|F(x)\|^2 = \sum_{i=1}^{\infty} c_i^2 \|f_i(x)\|^2}$$

What Does Get Us ?

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