Vibrations of a Drum

Professor Bernoff

12/2/2010

The oscillations of a drum are governed by the wave equation;

$$\text{DE: } u_{tt} = c^2\nabla^2 \qquad \text{in } \Omega$$
$$\text{BC: } u = 0 \qquad \text{on } \partial\Omega, t > 0$$
$$\text{IC: } u_t(x, y, 0) = g(x, y).$$

Here $u(x, y, t)$ is the displacement of the membrane.

First, let's separate the t variable

$$u(x, y, t) = T(t)\Psi(x, y).$$

The DE tells

$$T_t t\Psi = c^2\nabla^2\Psi T.$$

So We claim that λ is real and positive. We now solve the T equation;

$$T_{tt} + c^2\lambda T = 0.$$

So

$$T(t) = A\cos(\omega t) + B\sin(\omega t)$$
$$\omega = c\sqrt{\lambda}$$

And now the Ψ-equation. Since

$$u(x, y, t) = T(t)\Psi(x, y) = 0 \qquad \text{on } \partial\Omega,$$

for a non-trivial solution

$$T(t) \neq 0 \implies \Psi(x, y) = 0 \qquad \text{on } \partial\omega.$$

This yields the **Helmholtz Problem**

$$\nabla^2\Psi + \lambda\Psi = 0 \qquad \text{in } \Omega$$
$$\Psi = 0 \qquad \text{on } \partial\Omega$$

This problem has a countable set of real positive eigenvalues for Ω being a simply connected compact domain.[1] Note $\lambda_n = \frac{\omega_n^2}{c^2}$ for $n = 1, 2, 3, \ldots$ yields the oscillation frequencies of the drum.

Question: Does knowing $\{\lambda_n\}$ tell you the shape of the drum? This lead to a very famous paper from Mark Kac; "Can you hear the shape of a drum?"

Time to play the Bongos. Consider Ω to be a disc of radius a centered at the origin, and parameterize it in polar form by r and θ. Let's find the eigenvalues of the Helmholtz Equation.

[1] The extent to which these restrictions can be relaxed is an open question in analysis.

Let $\Psi = \Psi(r, \theta)$.

$$\text{DE: } \nabla^2\Psi + \lambda\Psi = \Psi_{rr} + \frac{1}{r}\Psi_r + \frac{1}{r^2}\Psi_{\theta\theta} + \lambda\Psi = 0$$
$$\text{DC: } \Psi(a, \theta) = 0.$$

We proceed by separation of variables.

$$\Psi(r, \theta) = \mathbb{R}(n)\Theta(\theta)$$

so

$$R_{rr}\Theta + \frac{1}{r}R_r\Theta + \frac{1}{r^2}R\Theta_{\theta\theta} + \lambda R\Theta = 0.$$

Divide by $\frac{\mathbb{R}\Theta}{r^2}$.

$$\frac{R_{rr} + \frac{1}{r}R_r + \lambda R}{\frac{R}{r^2}} = -\frac{\Theta_{\theta\theta}}{\Theta} = \mu.$$

First we solve the Θ-equation.

$$\Theta_{\theta\theta} + \mu\Theta = 0 \qquad 0 \leq \Theta \leq 2\pi$$

I want Θ to be 2π-periodic. Claim

$$\Theta_0 = 1, \quad \mu_0 = 0$$
$$\Theta_n = D_n\cos(n\theta) + E_n\sin(n\theta), \quad \mu = n^2$$

are solutions. Now we solve the R-equation. For $\mu_n = n^2$, $n = 0, 1, 2, \ldots$, I see

$$R_{rr} + \frac{1}{r}R_r + (\lambda - \frac{n^2}{r^2})R = 0.$$

Recall that this is Bessel's Equation of order n.[2] Also, I want $R(0)$ bounded and

$$\Psi(a, \theta) = R(a)\Theta(\theta) = 0 \implies R(a) = 0.$$

We can scale out λ. Let $z = \sqrt{\lambda}r$, then $R = \tilde{R}$

$$\frac{d}{dz} = \frac{dr}{dz}\frac{d}{dr} = \frac{1}{\lambda}\frac{d}{dr} \iff \sqrt{\lambda}\frac{d}{dz} = \frac{d}{dr}.$$

So

$$\lambda\tilde{R}_{zz} + \frac{\lambda}{z}\tilde{R}_z + \left(\lambda - \frac{\lambda n^2}{z^2}\right)\tilde{R} = 0.$$

Divide by λ to obtain

$$\tilde{R}_{zz} + \frac{1}{z}\tilde{R}_z + \left(1 - \frac{n^2}{z^2}\right)\tilde{R} = 0.$$

This is a Bessel's Equation of order n.

$$\tilde{R}(z) = \beta J_n(z) + \gamma\mathbb{Y}_n(z).$$

Note that $\lim_{z\to 0}\mathbb{Y}_n(z) = -\infty \implies$ set $\gamma = 0$ (and $\beta = 1$). So

$$\tilde{R}(z) = J_n(z)$$

[2]Note that this is why we normally have integer orders. If we had a wedge, we would have fractional orders.

and

$$R(r) = J_n(\sqrt{\lambda} r).$$

Applying the BC at $r = a$

$$R(a) = J_n(\sqrt{\lambda} a) = 0 \implies \sqrt{\lambda} a = \alpha_{np}.$$

Where α_{np} is the pth positive zero of J_n. Example diagram on board. Let

$$\lambda_{np} = \left(\frac{\alpha_{np}}{a}\right)^2.$$

So the eigenfunctions and eigenvalues of Ψ are;

$$\begin{aligned}
n = 0: \;\; \Psi_{0p} &= \Theta_0 R = J_0(\alpha_{op}\frac{r}{a}) \\
\lambda_{0p} &= \left(\frac{\alpha_{0p}}{a}\right)^2 \\
n = 1, 2, 3, \ldots: \;\; \Psi^c_{np} &= \cos(n\theta) J_n(\alpha_{np}\frac{r}{a}) \\
\Psi^s_{np} &= \sin(n\theta) J_n(\alpha_{np}\frac{r}{a}) \\
\lambda_{np} &= \left(\frac{\alpha_{np}}{a}\right)^2.
\end{aligned}$$

So the oscillation modes are

$$\begin{aligned}
u(r,\theta,t) \;\; = \;\; & \cos(\omega_{np} t)\Psi^c_{np} \\
& \sin(\omega_{np} t)\Psi^c_{np} \\
& \cos(\omega_{np} t)\Psi^s_{np} \\
& \sin(\omega_{np} t)\Psi^c_{np} \\
& \sin(\omega_{0p} t)\Psi_{0p} \\
& \cos(\omega_{0p} t)\Psi_{0p}
\end{aligned}$$

where $\omega_{np} = c\sqrt{\lambda_{np}}$ and $n = 1, 2, 3, \ldots$.

Suppose I wish to love

$$\begin{aligned}
\text{DE: } & u_{tt} = c^2 \nabla^2 u \qquad r < a \\
\text{BC: } & u(a,\theta,t) = 0 \\
\text{IC: } & u(r,\theta,0) = f(r,\theta) \\
& u_t(r,\theta,0) = 0.
\end{aligned}$$

The solution must be expressed in terms of these modes

$$u(r,\theta,t) = \sum_{p=1}^{\infty} A_{0p} \underbrace{J_0(\alpha_{0p}\frac{r}{a})}_{\Psi_{0p}} \cos(\omega_{0p} t) + \sum_{n=1}^{\infty}\sum_{p=1}^{\infty} [A_{np} \underbrace{J_n(\alpha_{np}\frac{r}{a})\cos(n\theta)}_{\Psi^c_{np}} + B_{np} \underbrace{J_n(\alpha_{np}\frac{r}{a})\sin(n\theta)}_{\Psi^s_{np}}] \cos(\omega_{np} t).$$

Remember

$$\{1, \cos(n\theta), \sin(n\theta)\}$$

are orthogonal for

$$\langle h, g \rangle = \int_0^{2\pi} hg \; d\theta.$$

Also, the set

$$\{J_n(\alpha_{np}\frac{r}{a}\}$$

are orthogonal for the inner-product

$$[h, g] = \int_0^a hgr\ dr.$$

So for $\{\Psi_{0p}, \Psi_{np}^c, \Psi_{np}^s\}$ the functions are orthogonal

$$\begin{aligned}\langle\langle\Psi_1, \Psi_2\rangle\rangle &= \int_0^{2\pi}\int_0^a \Psi_1\Psi_2 r\ dr\ d\theta \\ &= \int_\Omega \Psi_1\Psi_2\ dA.\end{aligned}$$

So

$$\begin{aligned}A_{0p} &= \frac{\langle\langle\Psi_{0p}, f(r,\theta)\rangle\rangle}{\langle\langle\Psi_{0p}, \Psi_{0p}\rangle\rangle} \\ A_{np} &= \frac{\langle\langle\Psi_{np}^c, f(r,\theta)\rangle\rangle}{\langle\langle\Psi_{np}, \Psi_{np}\rangle\rangle} \\ B_{np} &= \ldots\end{aligned}$$