Chain Rule (A Review)

Tuesday, September 07, 2010
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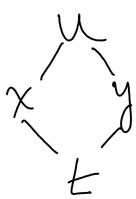
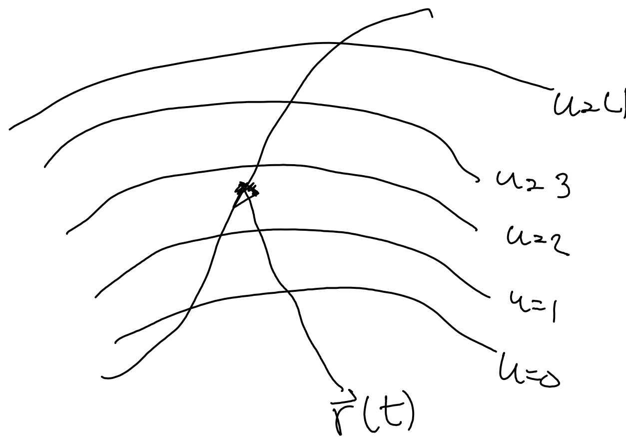
Suppose we have a function $u(x,y)$ which measures temperature in $\mathbb{R}^2$, and also a curve

$$\vec{r}(t) = \langle x(t), y(t) \rangle$$

which is a position vector along a curve parameterized by time t.

What is $\frac{du}{dt}$ along the curve $\vec{r}(t)$.





We Know

$$\boxed{\frac{du}{dt} = \underbrace{\frac{\partial u}{\partial x}\frac{dx}{dt}}_{\text{how fast it changes in } x,\text{ how fast we move in } x.} + \frac{\partial u}{\partial y}\frac{dy}{dt}}$$

Key to rest of lecture.

Suppose the path is along a <u>level curve</u> where

Suppose the path is along a level curve where

$u=$ constant.

Then

$$\frac{du}{dt}=0 \qquad \text{(temp. constant along path)}$$

$$=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}=0$$

What is $\frac{dy}{dx}$ on a level curve?

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=-\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}$$

$\Rightarrow$ $\boxed{\text{on a level curve} \quad \frac{dy}{dx}=\frac{-u_x}{u_y}}$

Can we use this to solve a PDE?

Yes! We can solve the first order linear PDE(*) for $u(x,y)$ using this identity

$$a(x,y)u_x+b(x,y)u_y=0 \qquad (*)$$

where a and b are given functions

Example: Solve

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$$a u_x + b u_y = 0 \quad \text{for } u(x,y)$$

Goal: Find the level curves! (Know where functions const.)
On these curves,

$$\frac{dy}{dx} = \frac{-u_x}{u_y} = \frac{b}{a} \qquad \left(u_x = -\frac{b}{a} u_y\right)$$

This is an ODE for $y(x)$.

$$y_x = \frac{b}{a} = \text{const}, \quad \text{so}$$

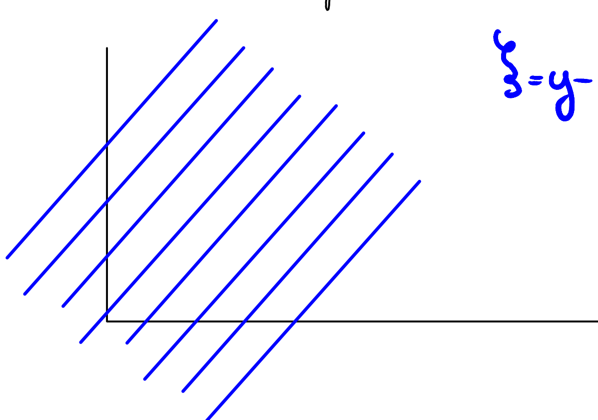
$$y = \int \frac{dy}{dx}\, dx = \int \frac{b}{a}\, dx = \frac{bx}{a} + c$$

$\boxed{y = \frac{bx}{a} + \xi \quad \text{are the level curves. They are parameterized by } \xi.}$

Claim the most general solution is

$$u(x,y) = F(\xi) = F\left(y - \frac{bx}{a}\right)$$

F is any function (should be differentiable, though there is a way around that Math 182).



$$\xi = y - \frac{bx}{a} = \text{constant}$$

("most general" can be proved using entropy)

Check:

$$u = F(\xi) \qquad \xi_x = \frac{-b}{a}, \quad \xi_y = 1$$

$$u_x = F'(\xi)\xi_x = -\frac{b}{a}F'(\xi)$$

$$u_y = F'(\xi)\xi_y = F'(\xi)\cdot 1$$

$$a u_x + b u_y = a\left[F'(\xi)\left(-\frac{b}{a}\right)\right] + bF'(\xi) = 0$$

This approach works in general unless you have more info

Example: Solve

$$y u_x - x u_y = 0$$

togethe w/ side condition:

a) $\quad a = u(x,0) = e^{-x^2}$ for $x \geq 0$

b) $\quad b = u(x,0) = x$ fo $-\infty < x < \infty$

(2 different problems)

Solution:

Find the level curves

$$\frac{dy}{dx} = \frac{-u_x}{u_y} = \frac{-x}{y}$$

Now solve this (seperable) equation:

$$\frac{dy}{dx} = \frac{-x}{y}$$

$$y\frac{dy}{dx} = -x$$

$$\int y\frac{dy}{dx}\,dx = \int -x\,dx$$

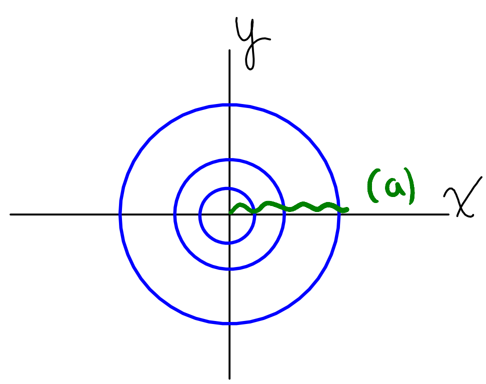
$$\frac{y^2}{2} = \frac{-x^2}{2} + C$$

$$y^2 = -x^2 + \xi$$

$$y = \sqrt{\xi - x^2}$$

so $\boxed{\xi = x^2 + y^2}$ so the level curves of u are circles!

$$\boxed{u(x,y) = F(\xi) = F(x^2+y^2)}$$



Now for the extra conditions:

$$u(x,0) = F(x^2) = e^{-x^2} \quad \text{for } x \geq 0$$

so $\boxed{F(z) = e^{-z}}$ for $z \geq 0$

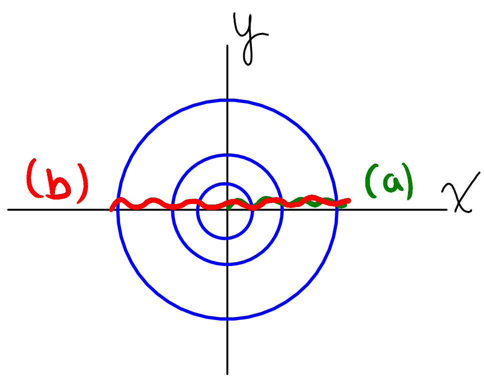
So $F(x^2+y^2) = e^{-(x^2+y^2)}$ and

$$\boxed{u(x,y) = e^{-(x^2+y^2)}}$$

This is a unique solution. Why? We specified the value of F on every level curve.

b)

$$u(x,y) = F(\xi) = F(x^2+y^2)$$



$u(x,0) = F(x^2) = x$ for $-\infty < x < \infty$

$$\left.\begin{array}{ll} \text{if } x=1 & F(1)=1 \\ x=-1 & F(1)=-1 \end{array}\right\} \text{this is a contradiction}$$

We have a problem – we specified F twice on every level curve (except $x^2+y^2=0$) and these specifications are inconsistent.

Strategy

Find the level curves

Find a function that is constant on every level curve

Specify the value consistently at every level curve.

Conservation Laws & The Transport Equation

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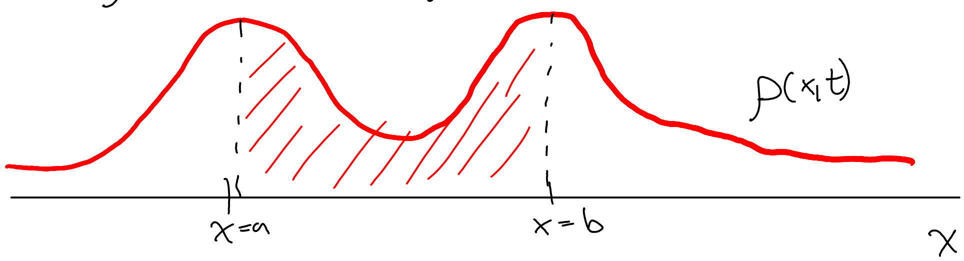
Consider an infinite freeway, and let

$$\rho(x,t) = \text{density of cars / length}$$



$$M(a,b) = \text{mass between } a \text{ \& } b$$
$$= \int_{x=a}^{x=b} \rho(x,t)\,dx.$$

What is

$$\frac{dM(a,b)}{dt} = \frac{d}{dt}\int_{x=a}^{x=b} \rho(x,t)\,dx$$
$$= \int_{x=a}^{x=b} \frac{d\rho}{dt}\,dx$$

(a, b not changing in time)

Suppose there are no on or off ramps
Then mass is conserved and

$$\frac{dM}{dt} = \begin{bmatrix}\text{Flux of cars} \\ \text{in at } a\end{bmatrix} - \begin{bmatrix}\text{Flux of cars} \\ \text{out at } b\end{bmatrix}$$

Let $Q(x)$ = flux of cars forward at x

[Mass/time]

So
$$\frac{dM}{dt} = Q(a) - Q(b)$$
$$= -\left[Q(b) - Q(a)\right]$$
$$= -\int_{x=a}^{x=b} \frac{\partial Q}{\partial x}\, dx$$

limits of an integral

Now
$$\frac{dM}{dt} = \int_{x=a}^{x=b} \frac{\partial \rho}{\partial t}\, dx = -\int_{x=a}^{x=b} \frac{\partial Q}{\partial x}$$

And
$$\int_{x=a}^{x=b} \left[\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial x}\right] dx = 0$$

this is true for any interval, (a,b). $\int_a^b F(\text{stuff})\,dx = 0$

if the integrand is anywhere approaching continuous (Lipshitz), then integrand is 0:

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial x} = 0}$$

This is the transport equation and it reflects conservation of mass.

Usually, we have a model for Q in terms

of ρ, x, t (and maybe some derivatives).

Example: Traffic in the urban planner's dreams.

Everybody goes at speed limit, C.

Then

$$Q(x) = C\rho = \frac{\text{length}}{\text{time}} \frac{\text{mass}}{\text{length}} = \frac{\text{mass}}{\text{time}} \text{ (flux)}$$

Thus

$$\rho_t + Q_x = 0$$

const.

$$\rho_t + \frac{\partial}{\partial x}(C\rho) = 0$$

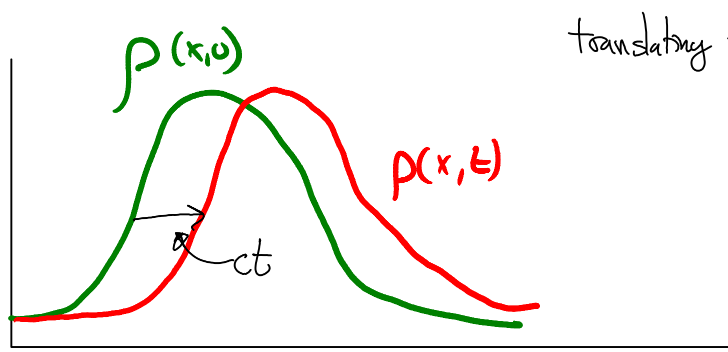
$$\boxed{\rho_t + C\rho_x = 0}$$

Advection Equation

We can solve this as in the first example of the lecture:

$$\rho = F(\xi) \qquad \xi = x - ct$$

$$\rho(x,t) = F(\xi) = F(x - ct)$$



translating to right at speed c.