So Long and Thanks for all the Fish

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0.1 Divergence Theorem

In $\mathbb{R}^n$, given a vector $\vec{u} = u_1\hat{i} + u_2\hat{j}$ its divergence is $\nabla\vec{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}$ and $\int\int_a \nabla \cdot u \; dV = \oiint \vec{u} \cdot \hat{n} \; ds$, where $\hat{n}$ is an outwardly pointed normal.

0.2 Heat Equation in $\mathbb{R}^n$

We will use $\mathbb{R}^2$, but the derivation generalizes. Let $u(\vec{x}, \vec{t})$ be the temperature in Ω. The heat energy is given by

$$Q = \iiint_\Omega c_p \rho u \; dV$$

where c_p is the specific heat per unit mass $= \frac{\text{Energy}}{\text{degree mass}}$ and $\rho =$ density $= \frac{\text{mass}}{\text{volume}}$.

Assume c_p and ρ are constant. Then

$$\frac{dQ}{dt} = c_p \rho \iiint_\Omega \frac{\partial u}{\partial t} \; dV$$

Fourier's Law of cooling says that the heat flux is proportional to the temperature gradient

$$\vec{q} = -k\nabla u$$

where $k =$ thermal conductivity $= \frac{\text{energy}}{\text{degree length}}$.

So

$$\begin{aligned} \frac{dQ}{dt} &= -\{\text{flux out of } \Omega \text{ of heat}\} \\ &= -\oiint_{\partial\Omega} \vec{q} \cdot \hat{n} \; ds \\ &\stackrel{\text{div}}{=} -\iiint_\Omega \nabla \cdot q \; dV \end{aligned}$$

But the volume Ω is arbitrary, s the two integrands must be equal.

$$c_p \rho = -\nabla \cdot \vec{q}$$

but

$$\vec{q} = -k\nabla u.$$

So

$$c_p \rho \frac{\partial u}{\partial t} = +k\nabla \cdot (\nabla u)$$

But

$$\nabla \cdot \nabla = \nabla^2 u$$

so

$$\frac{\partial u}{\partial \tau} = D\nabla^2 u$$
$$D = \frac{k}{c_p \rho}.$$

0.3 Steady States

Suppose we have the Dirichlet problem for the heat equation.

$$\begin{aligned} \text{DE: } & u_t = D\nabla^2 u \qquad \text{in } \Omega, t > 0 \\ \text{BC: } & u|_{\partial\Omega} = f(\vec{x}) \qquad \text{on } \partial\Omega, t > 0 \\ \text{IC: } & u(\vec{x}, 0) = g(\vec{x}) \qquad \text{in } \Omega \end{aligned}$$

The solution approaches a steady state, that is where $u_t = 0$. Call this state $\phi(\vec{x})$. ϕ satisfies Laplace's Equation

$$\begin{aligned} \nabla^2\phi &= 0 \qquad \text{in } \Omega \\ \phi &= f(\vec{x}) \qquad \text{on } \partial\Omega. \end{aligned}$$

This solution exists (hard!) and is unique (easier).

0.3.1 Proof of Uniqueness

Suppose we have two solutions, ϕ_1 and ϕ_2. Consider $\Psi = \phi_1 = \phi_2$. Ψ satisfies a homogereous Laplace Equation

$$\begin{aligned} \nabla^2\Psi &= 0 \qquad \text{in } \Omega \\ \Psi &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

Let's prove this with an energy method, but first, a vector identity.

$$\begin{aligned} \nabla \cdot (\Psi\nabla\Psi) &= \Psi\nabla^2\Psi + \nabla\Psi \cdot \nabla\Psi \\ &= \Psi\nabla^2\Psi + |\nabla\Psi|^2 \end{aligned}$$

Also, if Ψ is harmonic (i.e. satisfies $\nabla^2\Psi = 0$) then

$$\nabla \cdot (\Psi\nabla\Psi) = |\nabla\Psi|^2.$$

Now consider

$$\begin{aligned} \iiint_\Omega |\nabla\Psi|^2 \, dV &= \iiint_\Omega \nabla \cdot (\Psi\nabla\Psi) \, dV \\ &\overset{\text{div}}{=} \oiint_{\partial\Omega} \Psi\nabla\Psi \cdot \hat{n} \, dS \end{aligned}$$

But $\Psi = 0$ on $\partial\Omega$, so

$$\iiint_\Omega |\nabla\Psi|^2 \, dV = 0$$

which implies $|\nabla\Psi|^2 = 0 \implies \nabla\Psi = 0 \implies \Psi = \text{constant}$, but $\Psi = 0$ on $\partial\Omega \implies \Psi = 0$ identically. Therefore $\phi_1 = \phi_2$, so solutions to Laplace's Equation are unique.

0.4 Heat Equation in a Square

This was a problem on a previous final exam:

Solve

$$\begin{aligned}
\text{DE: } & u_t = \nabla^2 u \qquad 2 < x < \pi, 0 < y < \pi, t > 0 \\
\text{IC: } & u(x, y, 0) = f(x, y) \\
\text{BC: } & u(x, 0, t) = u(x, \pi, t) = 0 \qquad 2 < x < \pi, t > 0 \\
& u(0, y, t) = u(\pi, y, t) = 0 \qquad 0 < y < \pi, t > 0
\end{aligned}$$

The boundary equations asy that $u = 0$ on $\partial\square$.

0.4.1 Solution

Use separation of variables. Let

$$u(x, y, t) = T(t)\Psi(x, y)$$

$$\begin{aligned}
\text{DE: } u_t = \nabla^2 u \implies & T_t\Psi = T\nabla^2\Psi \\
\frac{T_t}{T} = & \frac{\nabla^2\Psi}{\Psi} = -\lambda
\end{aligned}$$

The T-equation

$$T_t + \lambda T = 0 \implies T(t) = e^{-\lambda t}.$$

The Ψ-equation we have actually seen before. From the BC on $\partial\square$, $u(x, y, t) = \Psi(x, y)T(t) = 0$, so Ψ vanishes on the boundary also.

$$\begin{aligned}
\text{DE: } & \nabla^2\Psi + \lambda\Psi = 0 \qquad \text{in } \Omega \\
\text{BC: } & \Psi(x, y) = 0 \quad \text{on } \partial\Omega
\end{aligned}$$

I will show that λ is real and positive - just assume it for the moment. Separate $\Psi(x, y) = X(x)Y(y)$.

$$\nabla^2\Psi = \Psi_{xx} + \Psi_{yy} = X_{xx}Y + XY_yy + \lambda XY = 0$$

Divide by X, Y

$$\underbrace{\frac{X_xx}{X}}_{=-\mu_1} + \underbrace{\frac{Y_yy}{Y}}_{=-\mu_2} = -\lambda$$

X-equation

$$X_xx + \mu_1 X = 0$$

BC'- $\implies$ $X(0) = X(\pi)$

Y-equation

$$Y_yy + \mu_2 Y = 0 \qquad Y(0) = Y(\pi) = 0$$

So

$$X(x) = X_n(x) = \sin(nx) \qquad n = 1, 2, 3, \ldots$$

$\mu_1 = n^2$ and

$$Y(y) = Y_m(y) = \sin(my) \qquad m = 1, 2, 3, \ldots$$

$\mu_2 = m^2$. So the solution for Ψ is

$$\Psi_{mn} = X_n(x)Y_m(x) = \sin(nx) \sim (mx)$$

$\lambda_{mn} = mu_1 + \mu_2 = n^2 + m^2$. So these are the eigenfuctions and eigenvalues of the Helmholtz equation.

Note

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{nm} \Psi_{nm}(x, y) e^{-\lambda_{nm} t}$$

We need to determine the c_{nm}s. What about the IC?

$$\begin{aligned} u(x, y, 0) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{nm} \Psi_{nm}(x, y) \\ &= f(x, y) \end{aligned}$$

I need an orthogonality condition.

0.4.2 Orthogonality

$$\begin{aligned} \langle\langle \Psi_{nm}, \Psi_{pq} \rangle\rangle &= \int_{y=0}^{\pi} \int_{x=0}^{\pi} \Psi_{nm} \Psi_{pq} \; dx \; dy \\ &= \begin{cases} 0 & n \neq p \text{ or } m \neq q \\ \frac{\pi^2}{4} & n = p \text{ and } m = q. \end{cases} \end{aligned}$$

Cheap proof:

$$\begin{aligned} \langle\langle \Psi_{nm}, \Psi_{pq} \rangle\rangle &= \int_{y=0}^{\pi} \int_{x=0}^{\pi} \sin(nx) \sin(my) \sin(px) \sin(qy) \; dx \; dy \\ &= \underbrace{\int_{y=0}^{\pi} \sin(my) \sin(qy) \; dy}_{=0m \quad m \neq q} \underbrace{\int_{x=0}^{\pi} \sin(nx) \sin(px) \; dx}_{=0 \quad n \neq p} \end{aligned}$$

So

$$f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{nm} \Psi_{nm}(x, y)$$

and

$$\langle\langle \Psi_{pq}, f(x, y) \rangle\rangle = c_{pq} \frac{\pi^2}{4}$$

So

$$c_{pq} = \frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} f(x, y) \sin(px) \sin(qy) \; dx \; dy$$

and

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} e^{-(n^2+m^2)t} c_{nm} \sin(nx) \sin(mx)$$

Theorem 1. *Suppose $\Psi(x, y)$ is a non-zero solution of $*$ for some eigenvalue λ. Then λ is real and positive.*

Proof. Consider

$$\begin{aligned}
\iint_\Omega \nabla\cdot(\Psi\nabla\Psi)\;dV &\overset{\text{div}}{=} \oint_{\partial\Omega} \Psi\nabla\Psi\cdot\hat{n}\;ds \\
&= 0 \qquad \text{because } \Psi|_{\partial\Omega}=0\text{; but} \\
&= \iint_{Omega} \Psi\nabla^2\Psi + |\nabla\Psi|^2\;dV \qquad \text{But } \nabla^2\Psi = -\lambda\Psi \\
&= -\lambda\iint \Psi^2\;dV + \iint |\nabla\Psi|^2\;dV \\
&= 0,
\end{aligned}$$

so

$$\lambda = \frac{\iint |\nabla\Psi|^2\;dX}{\iint |\Psi|^2\;dV}.$$

Thus λ is positive. To show it is real:

$$\begin{aligned}
\iint_\Omega \nabla\cdot(\Psi^*\nabla\Psi)\;dV &\overset{\text{div}}{=} \oint_{\partial\Omega} \Psi^*\nabla\Psi\cdot\hat{n}\;ds \\
&= 0 \qquad \text{because } \Psi|_{\partial\Omega}=0\text{; but} \\
&= \iint_{Omega} \Psi\nabla^2\Psi + |\nabla\Psi|^2\;dV \qquad \text{But } \nabla^2\Psi = -\lambda\Psi \\
&= -\lambda\iint \Psi^2\;dV + \iint |\nabla\Psi|^2\;dV \\
&= 0.
\end{aligned}$$

□

0.4.3 "Expensive" Proof of Orthogonality

Let Ψ, ϕ satisfy $\nabla^2\Psi+\lambda\Psi=0$, $\nabla^2\phi+\mu\phi=0$. We wish to show

$$\iint_\Omega \Psi\phi\;dV = 0 \qquad \text{if } \mu\neq\lambda$$

Proof. Consider

$$\iint \nabla\cdot[\phi\nabla\Psi-\Psi\nabla\phi]\;dV = \oint(\phi\nabla\Psi-\Psi\nabla\phi)\cdot\hat{n}\;ds = 0$$

since $\pi=\Psi=0$ on $\partial\Omega$. But

$$\begin{aligned}
0 &= \iint \nabla\cdot(\phi\nabla\Psi) - \nabla\cdot(\Psi\nabla\phi)\;dV \\
&= \iint \phi\nabla^2\Psi + \nabla\phi\cdot\nabla\Psi - \Psi\nabla^2\phi - \nabla\phi\cdot\nabla\Psi\;dV \\
&= \iint \phi\nabla^2\Psi - \Psi\nabla^2\phi\;dV \qquad \text{But } \nabla^2\Phi = -\lambda\Psi,\ \nabla^2\phi=-\mu\phi \\
&= -\iint \lambda\phi\Psi - \mu\Psi\phi\;dV \\
&= (\mu+\lambda)\iint \Psi\phi\;dV \\
&= 0
\end{aligned}$$

which implies either $\lambda = m$ or $\iint_{\Omega} \Psi\phi \, dV/0$. $\square$