Last Time: Best L2 Approximation

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$$F_N(x) = \sum_{n=1}^{N} c_n f_n(x) \qquad c_n = \frac{(F(x), f_n(x))}{\|f_n(x)\|^2}$$

where $\{f_n(x)\}$ are an orthogonal sequence.

This choice of c_n minimizes the L^2 error, E_N

$$E_N^2 = \|F(x) - F_N(x)\|^2$$

Cor #1: Bessel's Inequality

$$\|F(x)\|^2 \geq \sum_{n=1}^{N} c_n^2 \|f_n(x)\|^2$$

Cor #2: If $\lim_{N\to\infty} E_N = 0$ (converges in L^2)

Parseval's Thm:

$$\|F(x)\|^2 = \sum_{n=1}^{\infty} c_n^2 \|f_n(x)\|^2$$

Fourier Series

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Remember when we did separation of varables we found that the IC yielded

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

We wanted to

1) Know what a_n is. $\leftarrow$ easy

2) Does this converge? $\leftarrow$ hard, possibly one of the hardest problems in analysis

Carlson: Converges for anything in L^2.

What is a_n?

Note if

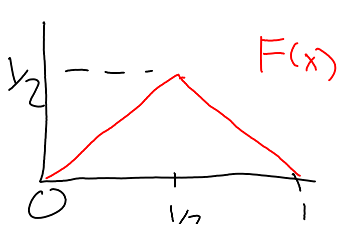
$$(f,g) = \int_0^L f(x)g(x)\,dx$$

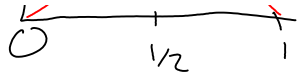
then $\{f_n(x)\}$ where $f_n(x) = \sin\left(\frac{n\pi x}{L}\right)$

is orthogonal set. And $a_n = c_n$

Ex. See Maple worksheet.

$$F(x) = \frac{1}{2} - \left|x - \frac{1}{2}\right| \qquad 0 < x < 1$$





$$F(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

$$a_n = \frac{(\sin(n\pi x), F(x))}{\|F(x)\|^2} = 2\int_0^1 \sin(n\pi x)\,dx = \begin{cases} 0 & n \text{ even} \\ \frac{4}{\pi^2}(-1)^{\frac{n-1}{2}} \frac{1}{n^2} & n \text{ odd} \end{cases}$$

Bessel's Inequality:

$$\|F(x)\|^2 = \int_0^1 (F(x))^2\,dx = \frac{1}{12} \approx 0.08\overline{33}$$

which is what the sequences of norms converge to

Note

$$\|f_n(x)\|^2 = \int_0^1 \sin^2(n\pi x)\,dx = \frac{1}{2}$$

$\sin^2 + \cos^2 = 1$

But Parseval's Thm says

$$\|F(x)\|^2 = \sum_{n=1}^{\infty} \|f_n(x)\|^2 c_n^2$$

$$\frac{1}{12} = \underbrace{\left(\frac{1}{2}\right)\cdot\left(\frac{4}{\pi^2}\right)^2}_{\|f_1\|^2 c_1^2} + \underbrace{0}_{\|f_2\|^2 c_2^2} + \underbrace{\left(\frac{1}{2}\right)\left(\frac{4}{\pi^2}\frac{1}{9}\right)^2}_{\|f_3\|^2 c_3^2} + \cdots$$

$$+ \frac{1}{2}\left(\frac{4}{\pi^2}\right)^2 \frac{1}{(2n-1)^2}$$

So

$$\frac{1}{12} = \frac{16}{\pi^4}\frac{1}{2}\left[1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right]$$

$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots$$

(fun fact for the day)

Sturm-Liouville Eigenvalue Problem (SLEP)

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Consider the following problem for y(x)

DE
$$y'' + \lambdy y = 0 \qquad a \leq x \leq b$$

BC
$$ y(a) = y(b) = 0$$

Let's write this n the eigenvalue form

$$\L y = \lambda y$$
Where $\L = - \frac{d^2}{dx^2}$.

We say $\{y_n, \lambda_n\}$ are an eigenfunction and eigenvalue if
$$\L y_n = \lambda_n y_n.$$

Define: We say $\L$ is a self-adjoint operator if $$\ip{\L u,v} = \ip{u,\L v}$$.

Claim: $\L$ is self-adjoint for functions u,v that satisfy the boundary conditions.

Proof:
$$\ip{\L u, v} = \int_0^L -u_{xx} v dx = \int_a^b u_x v dx - u_x v |_a^b$$
Argument follows as in quantum.

Theorem: If $\L$ is a self-adjoint operator, $\lamda$ is real.

Proof:
Suppose $\L y_n = \lamda_n y_n$, then $\L y_n^{\star} = \lamda_n^{\star} y_n^{\star}$.
Consider $\ip{ …} \ldots$ see notes.

Theorem Consider $\{y_n, \lambda_n\}$ and $\{y_m, \lambda_m\}$.
If $\lambda_n \neq \lambda_m$, $\ip{y_n,y_m} = 0$. The functions are orthogonal.

Proof: Orthogonal eigenspaces (or book, take your pick).

Neumann Problem

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DE: (heat equation) on $0 < x < L$, $t > 0$

BC:
$u_x(0,t) = u_x(L,t) = 0$

IC:
$u(x,0) = f(x)$

Sol'n:

Use separation of variables