1 Sturm-Liouville Theory Revisited

Previously we showed that the SLEP

$$\begin{aligned} \text{DE: } & y'' + \lambda y = 0 a < x < b \\ \text{BC: } & y(a) = 0 \quad \text{OR} \quad y'(a) = 0 \\ & y(b) = 0 \quad \text{OR} \quad y'(b) = 0 \end{aligned}$$

has a set of real non-negative eigenvalues

$$0 \le \lambda_1 \le \lambda_2 \le \cdots < \lambda_n \cdots$$

with associated eigenfunctions

$$y_1(x), y_2(x), \ldots, y_n(x), \ldots$$

which are orthogonal in the L^2 inner-product

$$\langle y_n(x), y_m(x) \rangle = \int_{x=a}^{b} y_n(x) y_m(x)\ dx = \begin{cases} 0 & n \neq m \\ c_n n = m \end{cases}.$$

This idea generalizes:

Definition. A **Sturm-Liouville Eigenvalue Problem** (SLEP) for $y(x)$ on $x\ in[a, b]$ is

$$(s(x)y')' + [\lambda p(x) - q(x)]y = 0 \tag{SLEP}$$

subject to

1. $s(x), p(x), q(x)$ are continuous on $x \in [a, b]$.
2. $s(x), p(x) > 0$ on $x \in (a, b)$[1]

Note. Previously, we considered

$$s(x) = 1, p(x) = 1, q(x) = 0.$$

There are many relevant examples

Example. Heat equation with a variable thermal conductivity for $u(x, t)$.

$$\begin{aligned} \text{DE: } & u_t = \frac{\partial}{\partial x}\left(D(x)\frac{\partial u}{\partial x}\right) \qquad a < x < b, t > 0 \\ \text{BC: } & u(a, t) = 0, \quad u(1, t) = 0 \qquad t > 0 \\ \text{IC: } & u(x, 0) = f(x) \qquad a < x < b \end{aligned}$$

Separation of variables suggests solutions of the form

$$u(x, t) = e^{-\lambda t} y(x)$$

which yields

$$\begin{aligned} \text{DE} \to & e^{-\lambda t}[-\lambda y(x)] \\ & = e^{-\lambda t}\left[\frac{\partial}{\partial x}\left(D(x)\frac{\partial y}{\partial x}\right)\right] \end{aligned}$$

[1] There is a very important case where they do vanish at the endpoints.

We cancel $e^{-\lambda t}$ and see

$$\frac{d}{dx}[D(x)\frac{dy}{dx}] + \lambda y(x) = 0$$
$$\text{BC's} \to y(a) = y(b) = 0.$$

If $D(x) > 0$ we find this has an infinite set of eigenfunctions and eigenvalues

$$\lambda_1 < \lambda_2 \cdots < \lambda_n \cdots$$

with λ_n real

$$\langle y_n(x), y_m(x) \rangle = 0$$

if $n \neq m$.

Example. **Schröedinger's Equation and the Harmonic Oscillator**. Note that $\hbar = \frac{h}{2\pi}$, $h =$ Plank's Constant.

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\Psi_{xx} + V(x)\Psi \qquad -\infty < x < \infty, t > 0$$

Suppose $V(x) = \frac{1}{2}mw^2x^2$. I can look for solutions via separation of variables

$$\Psi(x,t) = y(x)e^{-\frac{i\lambda_n}{\hbar}t}$$

when $\lambda_n =$ "Energy". This implies that

$$\lambda_n y_n(x) = -\frac{\hbar^2}{2m}(y_n)_{xx} + \frac{1}{2}mw^2x^2y_n.$$

Re-writing, we see

$$\frac{\hbar^2}{2m}(y_n)_{xx} + (\lambda_n - \frac{1}{2}mw^2x^2)y_n = 0. \qquad x > 0$$

Note that this is the form of a SLEP, $s(x) = \frac{\hbar^2}{2m}, p(x) = 1, q(x) = \frac{1}{2}mw^2x^2$. This has a well known set of eigenfunctions:

$$y_n(x) = \left(\frac{\alpha}{\pi}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n(x)e^{-x^2/2}, \qquad n = 0, 1, 2, \ldots$$

$H_n(x) =$ nth Hermite Polynomial, $\alpha = \frac{mw}{\hbar}$, $\lambda_n = (n + \frac{1}{2})\hbar w$ and

$$\langle y_n(x), y_m(x) \rangle = \int_{-\infty}^{\infty} y_n(x)y_m(x)\, dx = 0 \qquad n \neq m$$

Example. **Axisymmetric Heat Equation** Consider $u(r,t)$ (disk picture on board)

$$\text{DE: } u_t = D\nabla^2 u = D(u_{rr} + \frac{1}{r}u_r) \qquad r < a, t > 0$$
$$\text{BC: } u(a,t) = 0, u(0,t) \text{ bounded}, t > 0$$
$$\text{IC: } u(r,0) = f(r)$$

Separation of variables suggests

$$u_n(r,t) = e^{-\lambda_n D t}y_n(r)$$
$$(y_n)_{rr} + \frac{1}{r}(y_n)_r = -\lambda_n y_n$$

But now multiply by r

$$\begin{aligned} r(y_n)_{rr} + (y_n)_r + \lambda_n^r y_n &= \\ (r(y_n)_r)_r + \lambda_n r y_n &= 0 \qquad 0 < r < a \end{aligned}$$

$y_n(0)$ is bounded, $y_n(a) = 0$.

$$y_n = J_0(\frac{r}{a}\alpha_n) \quad n = 1, 2, 3, \ldots$$

wher J_0 is a Bessel function and α_n is the nth zero of J_0, that is $J_0(\alpha_n) = 0$. AND

$$\begin{aligned} \langle y_n(r), y_m(r) \rangle_r &= \int_0^a y_n(r) y_m(r) r \ dr \\ &= 0 \qquad n \neq m \end{aligned}$$

is a "weighted inner product."

2 Eigenvalues and Eigenfunctions of SLEP

Suppose $g_n(x)$ satisfies SLEP on $a < x < b$ with the BC's $y(a) = 0$ or $y'(a) = 0$ or $s(a) = 0$ and $y(b) = 0$ or $y'(b) = 0$ or $s(b) = 0$.

Theorem. *The eigenvalues are real.*

Proof. Multiply SLEP evaluated at $y = y_n$ by y_n^* and $\int_a^b$

$$\int_{x=a}^b y_n^*(sy_n')' \ dx + \lambda \int_{x=a}^b y_n^* y_n \ dx - \int_{x=a}^b q y_n^* y_n \ dx = 0$$

$$\begin{aligned} \int_{x=a}^b y_n^*(sy_n')' \ dx &= \underbrace{y_n^*}_{=0} \\ &\leq y_n' \Big[_{x=a}^b - \int_{x=a}^b (y_n^*)' y_n' s \ dx + blank - blank \\ &= 0 \end{aligned}$$

So

$$\lambda = \frac{\int_{x=a}^b q|y_n|^2 \ dx + \int_{x=a}^b s y_n'|^2 \ dx}{\int_{x=a}^b p|y_n|^2 \ dx}.$$

This is real. Moreover if $q(x) \geq 0$, $a < x < b$ then $\lambda \geq 0$ □

2.1 Orthagonality

Theorem. *Suppose $y_n(x)$ satisfies SLEP and SLEP-BC's. Then*

$$\langle y_n(x), y_m(x) \rangle_{y(x)} = \int_{x=a}^b y_n(x) y_m(x) p(x) \ dx = 0$$

if $n \neq m$.

Proof. Suppose we consider SLEP with $y(x) = g_n(x)$. Multiply by $y_m(x)$ and integrate from $x = a$ to $x = b$.

$$\int_{x=a}^{b} y_m(sy_n')' \, dx + \int_{x=a}^{b} \lambda_n y_n y_m p(x) \, dx - \int_{x=a}^{x=b} q(x) y_n y_m \, dx = 0$$

$$\int_{x=a}^{b} y_m(sy_n')' \, dx = \underbrace{y_m s}_{\text{0 by ICs}} \; y_n' \Big[_{x=a}^{b} - \int_{x=a}^{b} y_m' - y_n' \, dx$$

$$= \underbrace{-y_m' s y_n}_{=0} \Big[_{x=a}^{b} + \int_{x=a}^{x=b} (y_m' s)' y_n \, dx$$

So $(*)$ becomes

$$\int x = a^b (y_m' s)' y_n \, dx + \lambda_n \langle y_n, y_m \rangle_p - \int_{x=a}^{b} y_n y_m q(x) \, dx.$$

But y_m satisfies

$$(y_m' s)' + [\lambda_m p(x) - q(x)] y_m = 0,$$

so

$$-\int_{x=a}^{b} [\lambda_m p(x) - q(x)] y_m y_n \, dx + \lambda_n \langle y_n, y_m \rangle_p - \int_{x=a}^{b} y_n y_m q(x) \, dx = 0$$

The rightmost term partially cancels the leftmost, so

$$(\lambda_n - \lambda_m) \langle y_n, y_m \rangle_p = 0.$$

Either $\lambda_n = \lambda_m$ or

$$\langle y_n, y_m \rangle_p = 0.$$

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