Convergence of Fourier Series

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Remember from last time that given a function

$f(x)$ on the interval $-L < x < L$ define the fourier

series of $f(x)$ as

$$FS[f(x)] = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

when $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

Remember $e^{i\theta} = \cos\theta + i\sin\theta$

So let the <u>complex Fourier Series</u>

$$G[f(x)] = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

$c_n = ?$

Is $\{e^{in\pi x/L}\}$ an orthogonal sequence?

A: Well, $\left(e^{-i\pi x/L}, e^{i\pi x/L}\right) = \int_{-L}^{L} e^{-\frac{i\pi x}{L}} e^{i\pi x/L} dx$

$= \int_{-L}^{L} 1\,dx = 2L \neq 0$

which is a problem. There is a solution. Define

$$\left[f(x), g(x)\right] = \int_{-L}^{L} f^*(x)\, g(x)\, dx$$

Then

$$\left[e^{in\pi x/L}, e^{im\pi x/L}\right] = \int_{-L}^{L} e^{-in\pi x/L} e^{im\pi x/L}$$

$$= \int_{-L}^{L} e^{i(m-n)\pi x/L}\, dx$$

$$= \int_{-L}^{L} \cos\left(\frac{n-m\ \pi x}{L}\right) + i\sin\left(\frac{n-m\ \pi x}{L}\right) dx$$

$$= \begin{cases} 0 & n \neq m \\ 2L & n = m \end{cases}.$$

So

$$\left[e^{im\pi x/L}, f(x)\right] = \sum_{n=1}^{\infty} c_n \underbrace{\left[e^{im\pi x/L}, e^{in\pi x/L}\right]}_{\text{0 unless } n=m}$$

$$= c_m\left[e^{im\pi x/L}, e^{im\pi x/L}\right]$$

$$= 2L c_m$$

So

$$c_m = \frac{1}{2L}\left[e^{\frac{im\pi x}{L}}, f(x)\right]$$

$$\boxed{c_m = \frac{1}{2L}\int_{-L}^{L} e^{-\frac{im\pi x}{L}} f(x)\, dx}$$

(Remember $(e^{i\theta})^* = e^{i\theta}$

Relationship between real & complex Fourier series

$$C_m = \frac{1}{2L}\int_{-L}^{L} e^{-\frac{im\pi x}{L}} f(x)\,dx = \frac{1}{2L}\left[\int_{-L}^{L}\left\{\cos\left(\frac{m\pi x}{L}\right) - i\sin\left(\frac{m\pi x}{L}\right)\right\} f(x)\,dx\right.$$

$$C_m = \frac{1}{2}(a_m - i b_m)$$

Folding over $\Rightarrow$ extra term for c_0

Claim:

$$G[f(x)] = c_0 1 + \sum_{n=1}^{\infty} c_n e^{\frac{in\pi x}{L}} + c_{-n} e^{-\frac{in\pi x}{L}}$$

$$= c_0 + \sum_{n=1}^{\infty} c_n\left[\cos\left(\frac{n\pi x}{L}\right) + i\sin\left(\frac{n\pi x}{L}\right)\right]$$

$$+ c_{-n}\left[\cos\left(\frac{n\pi x}{L}\right) - i\sin\left(\frac{n\pi x}{L}\right)\right]$$

$$= c_0 + \sum_{n=1}^{\infty} (c_n - c_{-n}$$

$$= \frac{a_0}{2} + \sum \ldots$$ see notes

$$C_m + C_{-m} = a_m$$

$$C_m - C_{-m} = -i b_m$$

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Choose $L = \pi$

Define

$$S_N(x) = \sum_{n=-N}^{N} c_n e^{inx}$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

$$= \frac{1}{2\pi} \oint f(x) e^{inx} dx$$

<u>Claim:</u> Suppose $f(x)$ is a cont. function on $-\pi \leq x \leq \pi$

and $f(-\pi) = f(\pi)$

and $f(x)$ is piecewise differentiable

Then $S_N(x)$ converges uniformly to $\tilde{f}(x)$, the 2π-periodic extension of $f(x)$