Conclusion to Last Lecture

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In notes.

Non-Linear Conservation Laws

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Remember the transport equation for $\rho(x,t)$

for $-\infty < x < \infty$, $t > 0$

$$\rho_t + q_x = 0$$

where q = flux function.

For the traffic equation

$$q = c\rho$$

$$\frac{\text{mass}}{\text{time}} = \frac{\text{length}}{\text{time}} \cdot \frac{\text{mass}}{\text{length}}$$

Previously we set $c = c(x,t)$ which could be thought of as the speed limit.

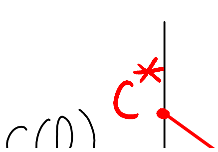
A much more realistic model is

$$c = c(\rho)$$

This will yeild a non-linear DE for the density.

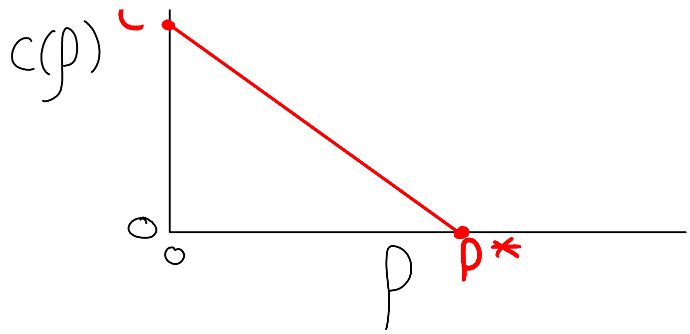
Choose simplest model: a linear model for $c(\rho)$

c linear $\rightarrow$ q quadratic $\Rightarrow$ non-linear



c^* = open highway speed

ρ^* = jam density

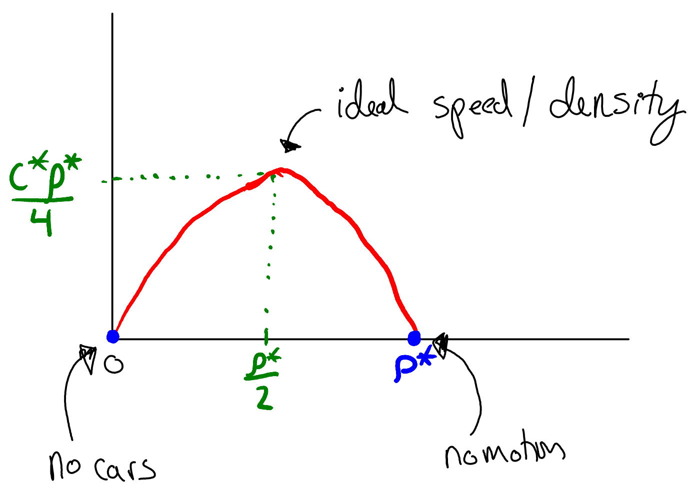


$\rho^* =$ jam density

We will limit density $0 \le \rho \le \rho^*$

$$c(\rho) = c^*\left(1 - \frac{\rho}{\rho^*}\right)$$

So $$q(\rho) = c\rho = c^*\left(1 - \frac{\rho}{\rho^*}\right)\rho$$



Our traffic equation becomes

$$\rho_t + [q(\rho)]_x = 0$$

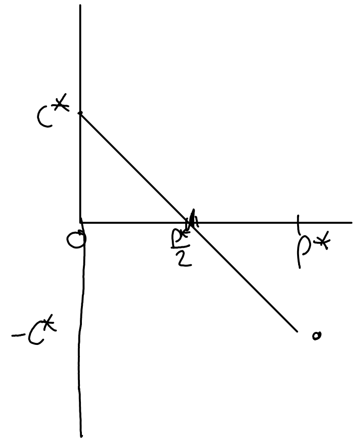
This yeilds

$$\rho_t + q'(\rho)\rho_x = 0$$

Let $q'(\rho) = V(\rho) = c^*\left(1 - 2\frac{\rho}{\rho^*}\right)$

Let $q'(\rho) = V(\rho) = c^*\left(1 - 2\frac{\rho}{\rho^*}\right)$

$$\boxed{\rho_t + V(\rho)\rho_x = 0}$$



$c \equiv$ phase velocity (speed of a car)

$V \equiv$ group velocity (speed of a line of constant density)

can be negative, related to fact that if car puts on its breaks, break lights propogate backwards.

Let's simplify our lives,

set $c^* = 1$, $\rho^* = 1$ (like gaussian units)

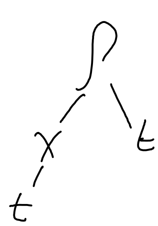
<u>D.E.</u>: $\rho_t + V(\rho)\rho_x = 0 \qquad -\infty < x < \infty$

$V(\rho) = 1 - 2\rho \qquad 0 < t$

<u>I.C.</u>: $\rho(x,0) = f(x) \qquad -\infty < x < \infty$

Know initial condition

Solve using characteristics. Look for $x(t)$, a characteristic curve on which ρ is constant.



$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{dx}{dt}\frac{\partial \rho}{\partial x} = 0$$

But from traffic equation

$$\rho_t + V(\rho)\rho_x = 0,$$

This implies

$$\boxed{\frac{dx}{dt} = V(\rho)}$$ and also on this curve $$\boxed{\frac{d\rho}{dt} = 0}$$

a function of ρ and x on right would go here.

$\Downarrow$

$\rho = $ const. on characteristics

The characteristics are: the solution to

$$\frac{dx}{dt} = V(\rho)$$

can do this because ρ is constant on characteristic

$$x = V(\rho)t + \xi$$

$$\xi = x - V(\rho)t$$

Now

$$\rho(\xi) = A(\xi)$$

So converting back to

$$\rho(x,t) = A(x - V(\rho)t) \quad \text{but}$$

$$\rho(x,0) = A(x) = f(x)$$, so $A = f$ and

$$\boxed{\rho(x,t) = f(x - V(\rho)t)}$$

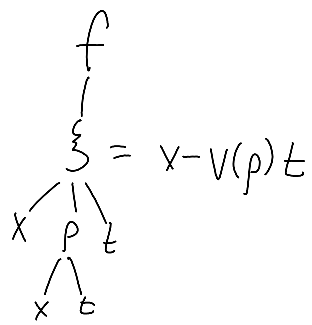
__Good News:__ I have a solution.

__Bad News:__ The solution is implicit.

Let's check this. Certainly

__I.C.__ $\rho(x,0) = f(x)$ ✓

What about the D.E.?



$$\rho(x,t) = f(\xi) = f(x - v(\rho)t)$$

$$\frac{\partial \rho}{\partial x} = \frac{\partial}{\partial x}\left[\left(f_{\xi}\right)\right] = f'(\xi)\,\xi_x \quad \text{but } \xi_x = \frac{\partial}{\partial x}(x - v(\rho)t)$$

$$= 1 - v'(\rho)\rho_x t$$

so

$$\rho_x = f'(\xi)\left(1 - v'(\rho)\rho_x t\right)$$

and

$$\rho_x\left(1 - f'(\xi)v'(\rho)t\right] = f'(\xi)$$

$$\boxed{\rho_x = \frac{f'(\xi)}{1 + f'(\xi)\,v'(\rho)\,t}}$$

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t}\left[(f\xi)\right] = f'(\xi)\,\xi_t$$

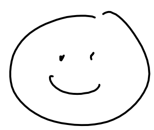
but $\xi_t = \frac{\partial}{\partial t}\left(x - v(\rho)t\right) = 0 - v'(\rho)\rho_t t - v(\rho)$

$$\rho_t = f'(\xi)\left(-v'(\rho)\rho_t t - v(\rho)\right)$$

$$\rho_t\left(1 + f'(\xi)\,v'(\rho)\,t\right) = -v(\rho)$$

$$\boxed{\rho_t = \frac{-f'(\xi)v(\rho)}{1 + f'(\xi)\,v'(\rho)\,t} = -v(\rho)\rho_x}$$

$$\boxed{\rho_t + v(\rho)\rho_x = 0}$$



Example Solutions

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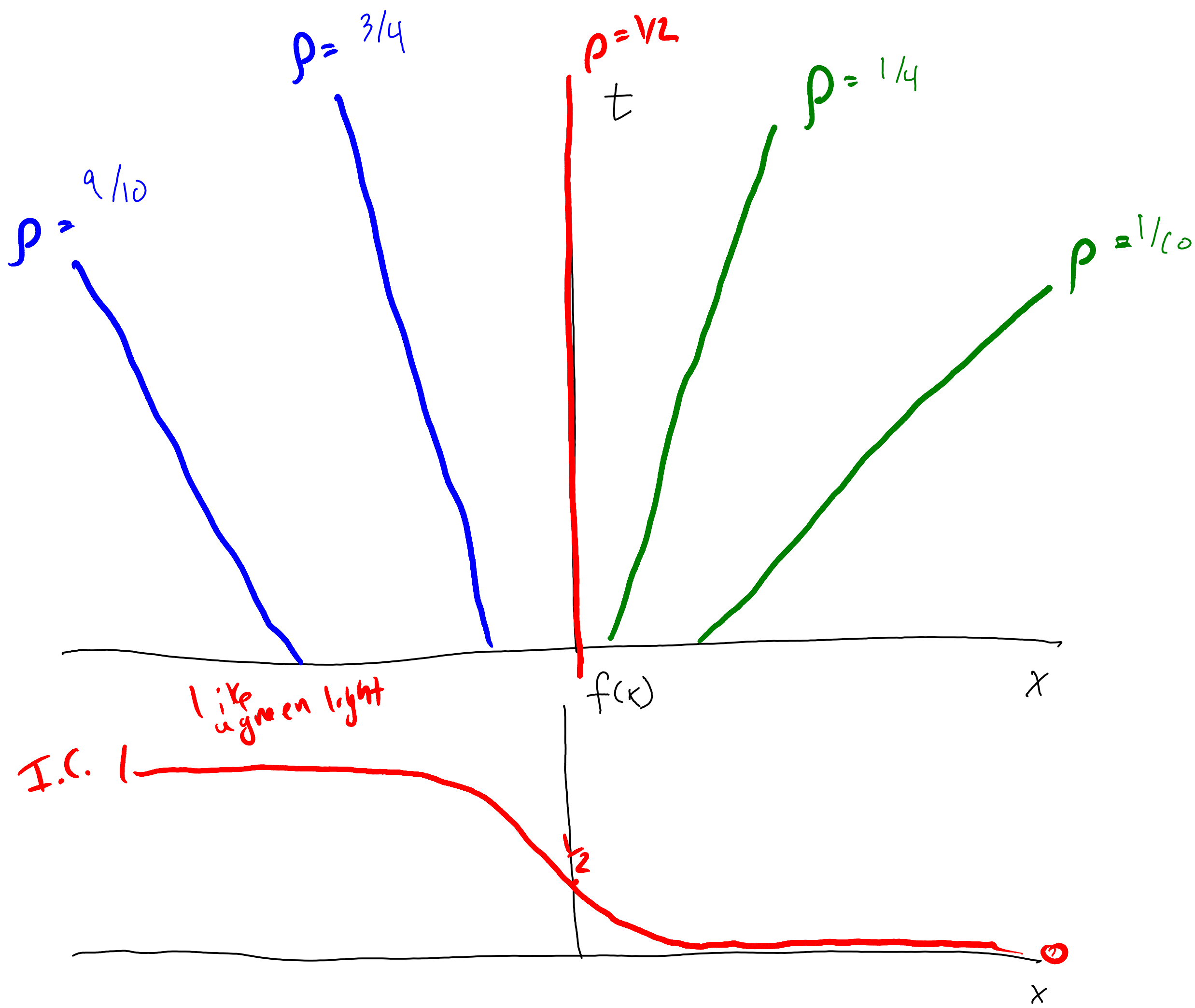
Maple Worksheet online

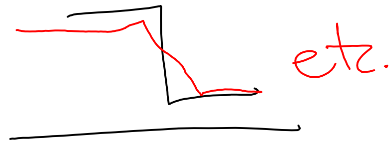
"The Green Light Problem"

Characteristics (lines of constant density)



Spreads out as time increases, → linear
constant (in HW)

Called a rarifaction



Density is spreading,

Example 2

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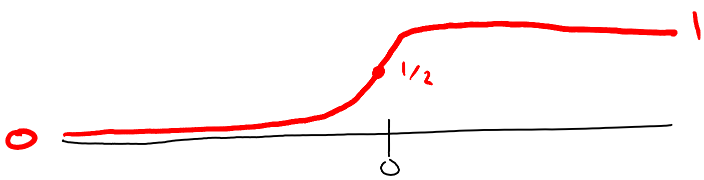
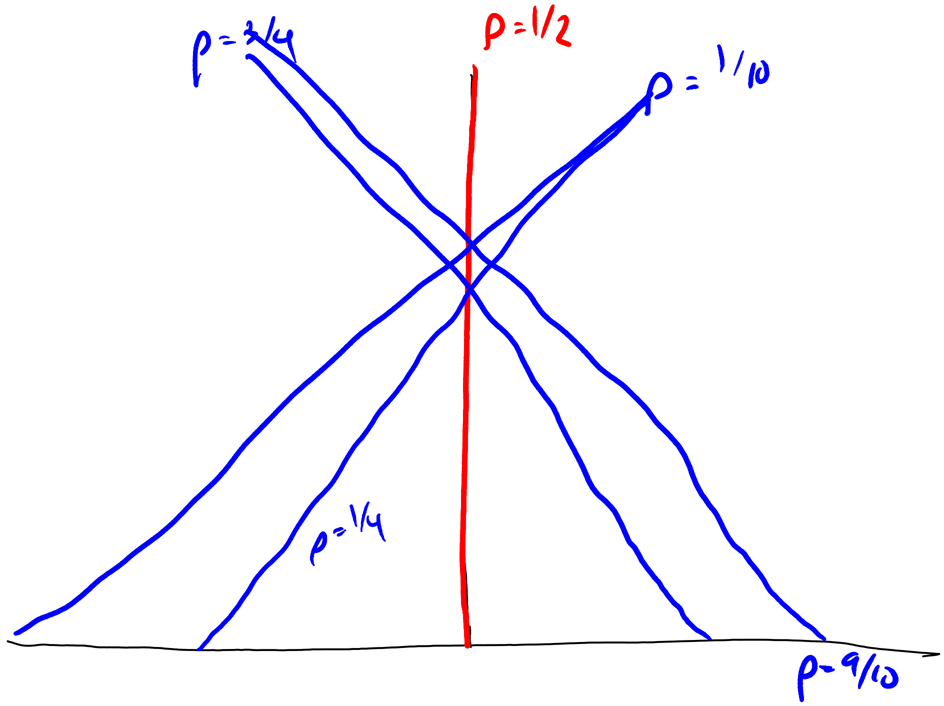
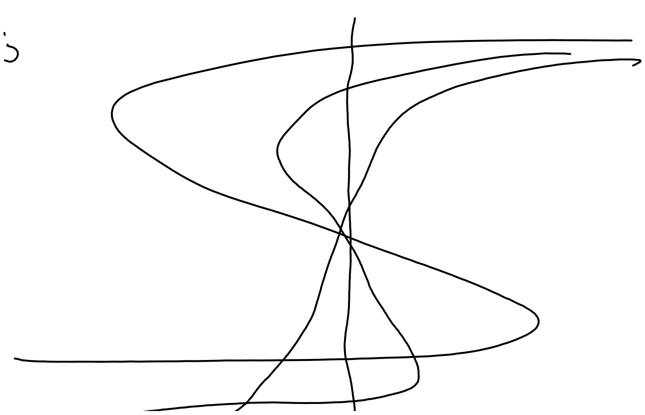


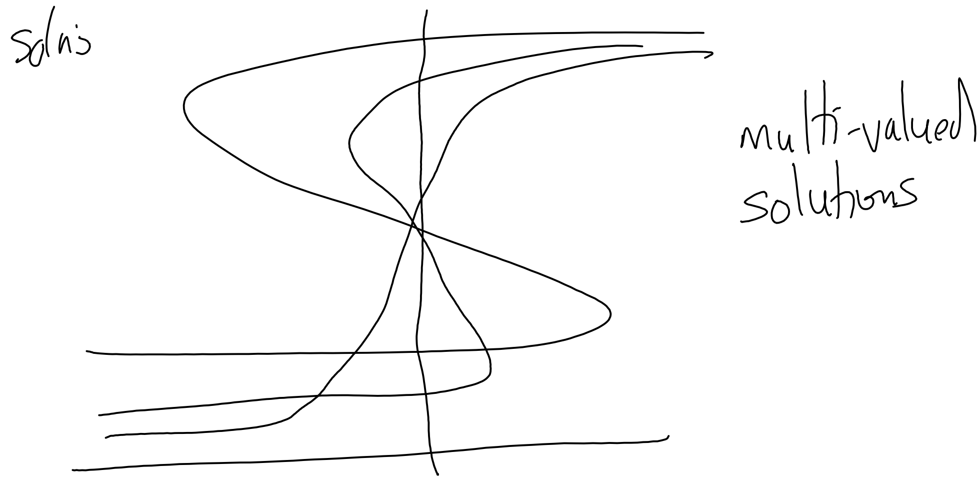


Characteristics cross?

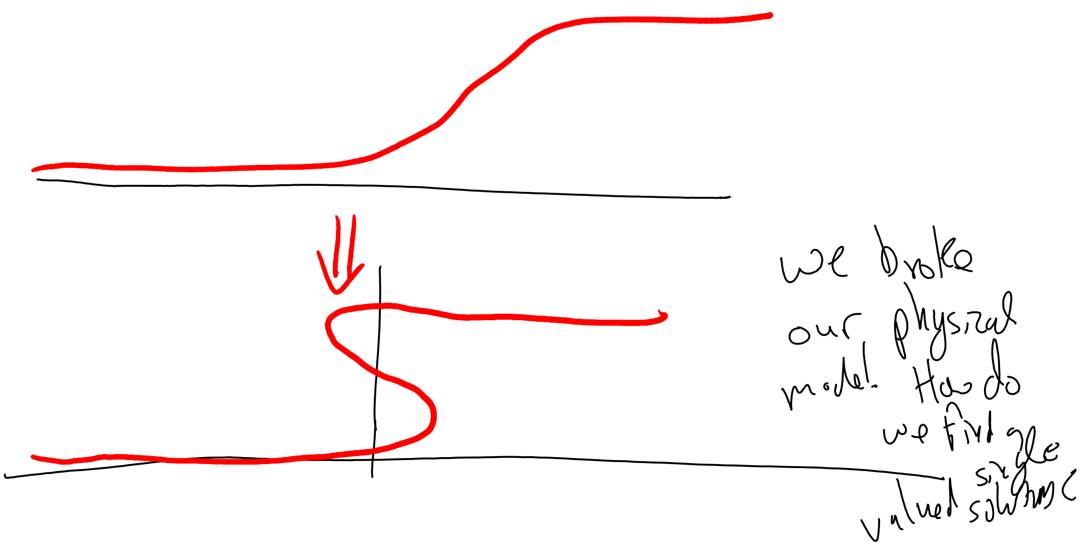
sol'ns



multi-valued solutions



This is compression – solutions become multi-valued. (huh?)



How do you find a single valued solution?

- Look for the unique solution w/ a jump that conserves mass.

This is called a SHOCK

Jump is called a **SHOCK**.

How to do this?

Shock Example

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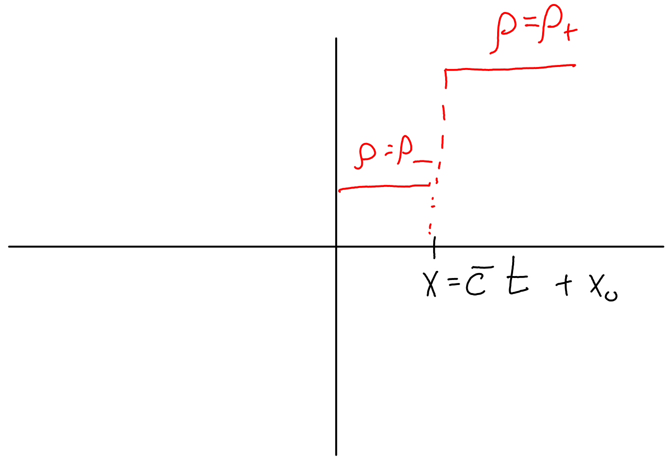
Ex. Look for a piecewise constant solution



(look for a traveling wave solution)

Let $\rho = \bar{\rho}(x - \bar{c}t) = \bar{\rho}(\xi)$

$$\rho_t + \frac{\partial}{\partial x}\left(Q(\rho)\right) = 0$$

Note $\rho_t = \bar{\rho}_\xi \dot{\xi}_t = -c\bar{\rho}_\xi$

and $$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{\partial}{\partial \xi}$$

The equation for $\bar{\rho}$ is

$$-c\bar{\rho}_\xi + \frac{\partial}{\partial \xi}\left[Q(\rho)\right] = 0$$

$$\bar{c}\rho_\xi = \frac{\partial}{\partial \xi}\left[Q(\rho)\right]$$

Integrate from $\int_{\xi=-\varepsilon}^{\xi=\varepsilon}$

$$\int_{-\varepsilon}^{\varepsilon} \bar{c}\,\bar{\rho}_\xi \, d\xi = \int_{-\varepsilon}^{\varepsilon} \frac{\partial}{\partial \xi} Q(\rho)\, d\xi$$

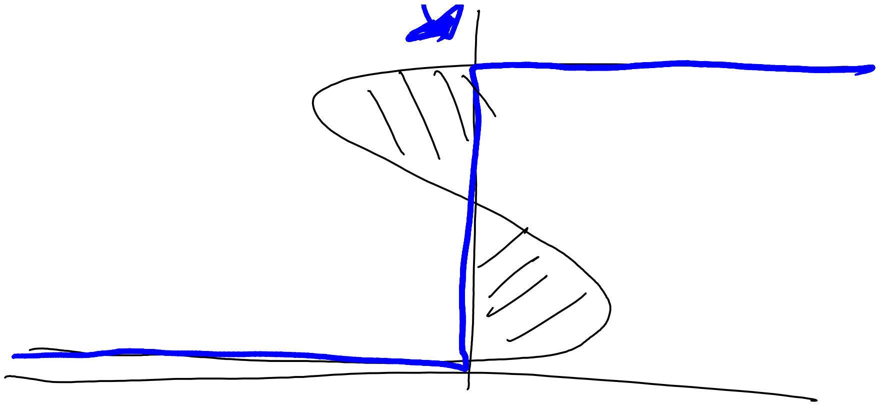
$$\bar{c}\,\bar{\rho}\Big|_{-\varepsilon}^{\varepsilon} = Q(\bar{\rho})\Big|_{-\varepsilon}^{\varepsilon}$$

$$\bar{c}\left[\rho_1 - \rho_2\right] = Q(\rho_+) - Q(\rho_-)$$

$$\boxed{\bar{c} = \frac{Q(\rho^+) - Q(\rho^-)}{\rho^+ - \rho^-}}$$ shock speed

Unique speed at which you can move at which the shock can move at.

From last prob



equal area rule

Choose the shock that cuts off equal areas.

Moral of the story:

non-linear systems are nasty

Sometimes you've got to add more physics in

One non-linear prob. in HW, no shocks.