1 Shifting Gears

This week:

1. Wave equation (Cauchy Problem)
2. d'Alembert's Solution
3. Fourier Transform

This will take next 3 weeks. Basically, we are now studying problems on $-\infty < x < \infty$, i.e., the real line. Problems on the whole real line are generally called Cauchy Problems.

2 Cauchy Problem for the Wave Equation

$$\begin{aligned} \text{DE: } u_{tt} &= c^2 u_{xx} \qquad -\infty < x < \infty \\ \text{IC: } u(x,0) &= f(x) \\ u(x,0) &= g(x) \end{aligned} \tag{c}$$

Think waterwaves: $u(x,t)$ = displacement from the mean (picture in one-note). We can solve this via a change of variables, similar to the method of characteristics.

2.1 Solution 1: Factoring

Write this problem as

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} &= 0 \\ \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u &= 0 \\ \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u &= 0 \end{aligned}$$

So if

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = u_t + cu_x = 0$$

then u satisfies (c). But we know the solution of

$$u_t + cu_x = 0$$

to be

$$u(x,t) = A(x - ct),$$

where A is an arbitrary function. Similarly, if

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u = 0$$

then u satisfies (c)

$$u_t - cu_x = 0$$

which mean that

$$u(x,t) = B(x + ct),$$

where B is an arbitrary function. By linearity,

$$u(x,t) = A(x-ct) + B(x+ct)$$

is a solution.
This is not very rigorous, however; we can do better.

2.2 Solution 2: Change of Variables

Let

$$\xi = x - ct$$

and

$$\eta = x + ct.$$

So (diagram in one-note). Now we change variables from $u(x,t)$ to $u(\xi,\eta)$:

$$\begin{aligned} u_t &= u_\xi \xi_t + u_\eta \eta_t = -cu_\xi + cu_\eta \\ u_x &= u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta. \end{aligned}$$

Remember $u_\xi(\xi,\eta)$, $u_\eta(\xi,\eta)$

$$\begin{aligned} u_{tt} &= -cu_{\xi\xi}\xi_t - cu_{\xi\eta}\eta_t + cu_{\eta\xi}\xi_t + cu_{\eta\eta}\eta_t \\ &= c^2 u_{\xi\xi} - c^2 u_{\xi\eta} - c^2 u_{\eta\xi} + c^2 u_{\eta\eta} \\ &= c^2(u_{\xi\xi} + u_{\eta\eta} - 2u_{\eta\xi}) \end{aligned}$$

where we have assumed twice differentiability. Also

$$\begin{aligned} u_{nn} &= u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x + u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x \\ &= u_{\xi\xi} + u_{\xi\eta} + u_{\eta\xi} + u_{\eta\eta} \\ &= u_{\xi\xi} + 2u_{\eta\xi} + u_{\eta\eta}. \end{aligned}$$

now

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= c^2[u_{\xi\xi} + u_{\eta\eta} - 2u_{\eta\xi}] - c^2[u_{\xi\xi} + u_{\eta\eta} + 2u_{\eta\xi}] \\ &= -4c^2 u_{\eta\xi} \end{aligned}$$

so $u(\eta,\xi)$, $u_{\eta\xi} = 0$. Note

$$(u_\eta)_\xi = 0,$$

which implies $u_n = C(n)$. Now integrate with respect to η:

$$\begin{aligned} u(\eta,\xi) &= \int^\eta C(\eta')d\eta' + D(\xi) \\ &= E(\eta) + D(\xi). \end{aligned}$$

Reversing the change of variables;

$$u(x,t) = E(x+ct) + D(x-ct).$$

Note that we have not yet used the initial conditions.

2.2.1 Initial Condition

To apply the IC's to

$$u(x,t) = A(x-ct) + B(x+ct).$$

Note

$$u(x,0) = A(x) + B(x) = f(x). \tag{*}$$

Also

$$\begin{aligned} u_t(x,t) &= A'(x-ct)(-c) + B'(x+ct)(c) \\ &= -cA'(x-ct) + cB'(x+ct) \end{aligned}$$

[1] So

$$u_t(x,0) = -cA'(x) + cB'(x) = g(x).$$

Differentiate (*):

$$A'(x) + B'(x) = f'(x).$$

So the solution is

$$\begin{aligned} B'(x) &= \frac{1}{2}[f'(x) + \frac{1}{c}g(x)] \\ A'(x) &= \frac{1}{2}[f'(x) - \frac{1}{c}g(x)]. \end{aligned}$$

To solve, integrate each from $x = 0$ to $x = z$.

$$\int_0^z B'(x)\ dx = \frac{1}{2}\int_0^z f'(x)\ dx + \frac{1}{2c}\int_0^z g(x)\ dx,$$

which by the fundamental theorem of calculus tells me

$$B(z) - B(0) = \frac{1}{2}[f(z) - f(0)] + \frac{1}{2c}\left[\int_0^z g(x)\ dx\right].$$

Repeat this procedure for A:

$$\begin{aligned} \int_0^z A'(x)\ dx &= \frac{1}{2}\left[\int_0^z f'(x)\ dx - \frac{1}{c}\int_0^z g(x)\ dx\right] \\ A(z) - A(0) &= \frac{1}{2}\left[f(z) - f(0) - \frac{1}{c}\int_0^z g(x)\ dx\right] \\ A(z) &= \frac{1}{2}\left[f(z) - f(0) - \frac{1}{c}\int_0^z g(x)\ dx\right] + A(0). \end{aligned}$$

We want to combine these two, so re-arrange to solve for $B(z)$,

$$B(z) = \frac{1}{2}[f(z) - f(0)] + \frac{1}{2c}\left[\int_0^z g(x)\ dx\right] + B(0).$$

Also, insert dummy variables to yield

$$\begin{aligned} A(z) &= \frac{1}{2}\left[f(z) - f(0) - \frac{1}{c}\int_0^z g(y)\ dy\right] + A(0) \\ B(z) &= \frac{1}{2}[f(z) - f(0)] + \frac{1}{2c}\left[\int_0^z g(y)\ dy\right] + B(0). \end{aligned}$$

[1] Suppose $A = A(\xi)$ and $\xi = x - ct$, then $A_t = A_\xi \xi_t = A_\xi(-c) = -cA'$. The use of " ′ " denotes the derivative with respect to the argument.

Then

$$
\begin{aligned}
u(x,t) &= A(x-ct)+B(x+ct) \\
&= \frac{1}{2}[f(x-ct)-f(0)-\frac{1}{c}\int_{y=0}^{x-ct} g(y)\ dy] + \frac{1}{2}[f(x+ct)-f(0)+\frac{1}{c}\int_{y=0}^{x+ct} g(y)\ dy] + A(0)+B(0) \\
&= \frac{1}{2}[f(x-ct)+f(x+ct)] + \frac{1}{2c}[\int_{0}^{x+ct} g(y)\ dy - \int_{0}^{x-ct} g(y)\ dy] + A(0)+B(0)-f(0)
\end{aligned}
$$

But $(*)|_{x=0}$ implies that $A(0)+B(0)=f(0)$, so

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}[f(x-ct)+f(x+ct)] + \frac{1}{2c}[\int_{0}^{x+ct} g(y)\ dy - \int_{0}^{x-ct} g(y)\ dy] \\
&= \frac{1}{2}[f(x-ct)+f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\ dy.
\end{aligned}
$$

This is the final solution, called *d'Alembert's Solution*. Note that the prescence of $x-ct$ and $x+ct$ indicates traveling waves propogating in opposite directions.

2.3 Example 1:

Suppose $g(x)=0$, and $f(x)=e^{-x^2}$. Then

$$u(x,t) = \frac{1}{2}[e^{-(x-ct)^2} + e^{-(x+ct)^2}].$$

Graphs in one-note. This is one solution. There is another solution, and we have actually seen it before.

2.4 Example 2:

Suppose $g(x)=0$ and $f(x)$ is the odd periodic continuation of (graph in one note). By d'Alembert,

$$u(x,t) = \frac{1}{2}[f(x-ct)+f(x+ct)].$$

By symmetry,

$$u(0,t)=0$$

and

$$u(l,t)=0,$$

so the solution solves the Dirichlet problem with $u(x,0) =$ triangle initial condition and $u_t(x,0)=0$. The graph of this solution in time is (graph in one-note).