1 Midterm Exam Discussion

Last material for midterm is today.
Exam materials - 6 pages of notes. Perhaps can use Maple or Mathematica.

2 Well Posed Problems

Well posed problems have three characteristics

1) Existence - A solution exists.

2) Uniqueness - There is only one solution.

3) Stability - Small changes in initial data only change the solution slightly.

3 Stability

Stability has similar observations as Fourier convergence.
Our first tool is Energy.
Example:
Suppose $u(x,t)$ satisfies

$$\begin{aligned} \text{DE: } & u_t = u_{xx} \quad -l \le x \le l \quad t > 0 \\ \text{BC: } & u(-l,t) = 0, \quad u(l,t) = 0, \quad t > 0 \\ \text{IC: } & u(x,0) = f(x), \quad -l < x < l \end{aligned} \tag{D}$$

Show that $u(x,t)$ is stable blank to perturbations in the IC, $f(x)$.
Solution: Suppose $u_1(x,t)$ and $u_2(x,t)$[1] satisfy (D) with $f(x) = f_1(x)$ and $f(x) = f_2(x)$ respectively.
Suppose also

$$\max_{-l<x<l} |f_1(x) - f_2(x)| < \epsilon.$$

We wish to show

$$|u_1(x,t) - u_2(x,t)| < \delta$$

where as $\epsilon \to 0$, $\delta \to 0$. Let

$$E[u] = \int_{-l}^{l} \frac{u^2}{2}\, dx.$$

Previously we showed

$$\frac{dE}{dt} \le 0.$$

Let $v = u_1 - u_2$. Then

$$v(x,0) = u_1(x,0) - u_2(x,0) = f_1(x) - f_2(x).$$

Now v satisfies (D) with

$$v(x,0) = f_1 - f_2.$$

Note

$$\begin{aligned} E[v(0)] &= \frac{1}{2}\int_{-l}^{l} [f_1(x,0) - f_2(x,0)]^2\, dx \\ &\le \frac{1}{2} \cdot 2l \cdot \epsilon^2 \\ &= \epsilon^2 l. \end{aligned}$$

[1] $u_i(x,t) \in C_x^2[-l,l], C_t^1[0,t]$

But

$$E[v(x,t)] \leq E[v(x,0)].$$

So

$$\begin{aligned}
\epsilon^2 l &\geq \int_{-l}^{l} \frac{v^2}{2}\ dx \\
&= \frac{1}{2}\int_{-l}^{l} [u_1 - u_2]^2\ dx \\
\epsilon^2 2l &\geq \int_{-l}^{l} [u_1 - u_2]^2\ dx \\
\epsilon\sqrt{2l} &\geq \sqrt{\int_{-l}^{l} [u_1 - u_2]^2\ dx} \\
&= ||u_1 - u_2||
\end{aligned}$$

This is L^2-stability. In fact this says nothing about

$$\max_{-l<x<l} |u_1(x,t) - u_2(x,t)|.$$

We need something stronger to show *pointwise stability.*

3.1 The Maximum Principle

Theorem (The Maximum Principle)**.** *If $u(x,t)$ satisfies $u_t = Du_{xx}$ in a rectangle in spacetime, (say $-l < x < l, 0 < t < T$), then the maximum of $u(x,t)$ occurs initially (on $u(x,0)$ for $-l \leq x \leq l$) or on the lateral boundaries ($u(l,t)$ or $u(-l,t)$ for $0 \leq t \leq T$.)*[2] *This is what is called the weak maximum principle; that the function assumes its maximum on the boundary. There exists a* strong maximum principle*, which states that it only assumes its maximum on the boundary, unless u is constant.*

Corollary (Minimum Principle)**.** *MP is true if "maximum" is replaced by "minimum".*
Proof: Replace $u(x,t)$ by $-u(x,t)$.

Back to stability for a second. If

$$\max |u_1(x,0) - u_2(x,0)| = \max |f_1(x) - f_2(x)| < \delta,$$

then initially $|v(x,0)| < \delta$ and also $v(-l,t) = v(l,t) = 0$. This implies

$$\max_{-l<x<l} |u_1(x,t) - u_2(x,t)| < \delta$$

for all t, $0 < t < T$. This is *pointwise stability.*

3.1.1 Proof of the Maximum Principle

Motivation: Suppose we have a maximum in the interior of a region $R = [-l,l] \times [0,T]$ - call it $(\overline{x}, \overline{t})$. Then $u_t(\overline{x}, \overline{t}) = u_x(\overline{x}, \overline{t}) = 0$. Well if it's a maximum, we might guess $u_x x < 0$. But, $u_t = Du_{xx} < 0$ then. This is a contradiction.
Problem: Suppose $u_x x = 0$.
Solution: We lift the function. Let

$$M_1 = \max_R [u(x,t)]$$

[2] $u(x,t) \in C_x^2[-l,l], C_t[0,T]$

and

$$M_2 = \max_{t=0 \cup x=l \cup x=-l \in R} [u(x,t)].$$

He now waves his hands: these are both compact sets, thus they achieve their maximums, so these exist. Suppose $M_1 > M_2$. Let $M_1 - M_2 = \epsilon$. Let $\tilde{u}(x,t) = u(x,t) + \frac{\epsilon}{2}\frac{x^2}{l^2}$. For $\tilde{u}(x,t)$,

$$\tilde{M_1} = \max_R [\tilde{u}(x,t)] = M_1 - \frac{\epsilon}{2} > M_2$$
$$\tilde{M_2} = \max_{t=0 \cup x=l \cup x=-l \in R} [\tilde{u}(x,t)] \leq M_2.$$

So $\tilde{M_1} > \tilde{M_2}$. But

$$\begin{aligned} \tilde{u}_t &= u_t + \frac{\partial}{\partial t}[\frac{\epsilon}{2}(\frac{x^2}{l^2} - 1)] \\ &= u_t \\ \tilde{u}_{xx} &= u_{xx} + \frac{\partial^2}{\partial x^2}[\frac{\epsilon}{2}(\frac{x^2}{l^2} - 1)] \\ &= u_{xx} + \frac{\epsilon}{l^2}. \end{aligned}$$

So

$$\tilde{u}_t - D\tilde{u}_{xx} = \underbrace{u_t - Du_{xx}}_{=0} - \frac{\epsilon D}{l^2}$$

and

$$\tilde{u}_t = D\tilde{u}_{xx} - \frac{\epsilon D}{l^2}.$$

If $\tilde{u}_t = 0$, this implies $D\tilde{u}_{xx} = \frac{\epsilon D}{l^2} > 0$ any point in the interior that in an extrenum ($u_t = u_x = 0$) has $\tilde{u}_{xx} = \frac{\epsilon}{l^2} > 0$. Therefore it is not a maximum.
What about the top boundary? If the maximum occurs on $t = T$, then $\tilde{u}_t(x,T) \geq 0$, which implies

$$D\tilde{u}_t(x,t) - \frac{\epsilon D}{l^2} \geq 0,$$

or

$$\tilde{u}_{xx}(x,T) \geq \frac{\epsilon}{l^2} > 0.$$

So the upper boundary is convex up. ☺