Course Website

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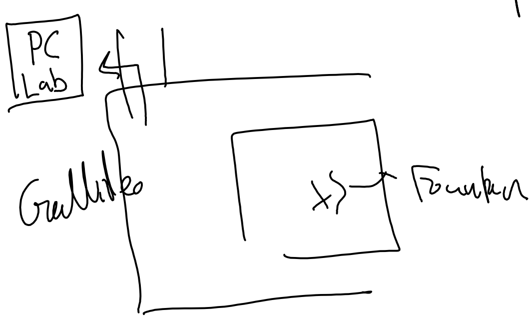
math.hmc.edu/~ajb/m180

- Explore Webpage
  - 2:45 – 4:00 pm in Beckman B134
  - No textbook, but lots of written notes
    - PC Lab for Maple practice



- Get Maple!
- Don't circulate materials outside Claremont Colleges
  - on-campus restricted access
    - mostly HW solutions, etc.

Homework

- Due Every Tuesday
  - Homework 35%
  - Midterm 30%

- Final 35%

Schedule on Webpage Subject to Change

- Always interesting links
- Lecture Notes on webpage (but not identical)

(HW on site is currently from 115, will be changed soon)

What is a PDE?

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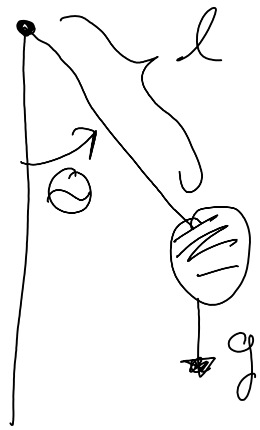
What is a partial differential equation?

Remember: (from ODEs)



Derived: $\boxed{\text{The pendulum Equation}}$

$$\Theta_{tt} + \frac{g}{\ell}\sin\Theta = 0$$

$\Theta(t)$ - is dependent variable

t - is the independent variable

(*) a second order, nonlinear ODE

Real Life is more complicated
- need to consider functions of multiple variables and their partial derivatives.

Define: A partial differential equation is a relationship

Define: A partial differential equation is a relationship between a function and its partial derivatives.

Examples:

Laplace's Equation

$$\nabla^2 \Phi \equiv \Phi_{xx} + \Phi_{yy} = 0 \qquad \boxed{1}$$

This describes an electrostatic potential in free space.

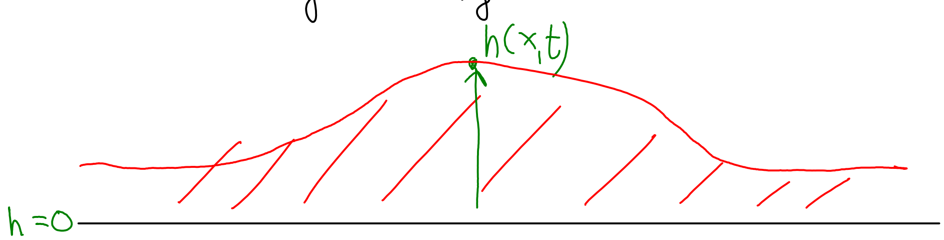
Poisson's Equation for $\Phi(x,y)$ and $\rho(x,y)$ known

$$\Phi_{xx} + \Phi_{yy} = \frac{\rho}{\varepsilon_0} \qquad \boxed{2}$$

permitivity of free space

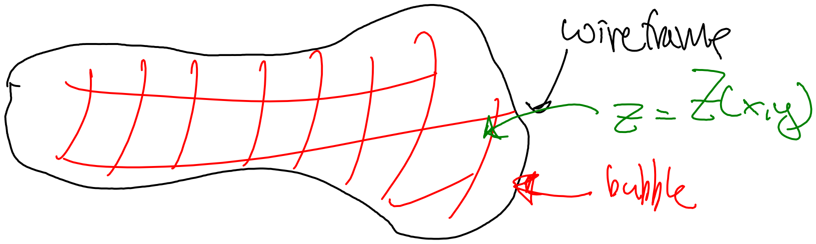
Korteweg-de Vries Equation

Describes long-wavelength fluid waves (think tsunamis)



$$h_t + 6h\,h_x = h_{xxx} \qquad \boxed{3}$$

Minimal Surface Equations



$$(1+z_x^2) z_{yy} - 2 z_x z_y z_{xy} + (1+z_y^2) z_{xx} = 0 \qquad \boxed{4}$$

(Take Diff-Geo to see where this comes from)

Terminology

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Define: The $\boxed{\text{order}}$ of a differential equation is the highest partial derivative.

In examples

$\boxed{1}$ - 2nd

$\boxed{2}$ - 2nd

$\boxed{3}$ - 3rd

$\boxed{4}$ - 2nd

Define: We say an equation is $\boxed{\text{linear}}$ if the independent variable, and its partial derivatives, appear is terms of degree at most one.

Define: The $\boxed{\text{degree}}$ of a term is the number of times the dependant variable appears. (Defined better in notes). (scaling properties)

Ex. Φ^2 is deg. 2, $\Phi \Phi_x$ is deg. 2

Example: Find the most general linear first order equation that $u(x,y)$ satisfies.

u_x, u_y, u_t are the first order derivatives

$$\boxed{a(x,y,t)u_t + b(x,y,t)u_x + c(x,y,t)u_y + d(x,y,t)u = e(x,y,t)}$$

$$\underbrace{a(x,y,t)u_t + b(x,y,t)u_x + c(x,y,t)u_y + d(x,y,t)u = e(x,y,t)}_{L}$$

Anything that isn't linear is $\boxed{\text{non-linear}}$.

From examples

linear: $\boxed{1}$, $\boxed{2}$

nonlinear: $\boxed{3}$, $\boxed{4}$

Define: Homogeneous vs. non/in-homogeneous Linear PDEs

A $\boxed{\text{homogeneous}}$ PDE has terms of degree exactly 1. All others are $\boxed{\text{non-homogeneous}}$.

In examples

Homogeneous $\boxed{1}$

NonHomogeneous $\boxed{2}$

Other two are non-linear so the question makes no sense.

In most general case

homogeneous $\longleftrightarrow$ $e=0$

non-homogeneous $\longleftrightarrow$ $e \neq 0$

Solutions

Define: A $\boxed{\text{solution}}$ is a particular function that

Define: A $\boxed{\text{solution}}$ is a particular function that satisfies a partial differential equation.

Example: Laplace's Equation

Show that

a) $\Phi = x$ is a solution

b) $\Phi = x^2 - y^2$ is a solution

$(x)_{xx} + (x)_{yy} = 0 + 0 = 0$ ✓

Notation: $\Phi_{xx} = \dfrac{\partial^2 \Phi}{\partial x^2}$, etc.

$(x^2 - y^2)_{xx} + (x^2 - y^2)_{yy} = 2 + (-2) = 0$ ✓

c) Show any linear combination of these two soln's is a solution:

In general:

Suppose Φ_1 and Φ_2 satisfy Laplace's Equation:

Let $\Phi = a\Phi_1 + b\Phi_2$. Now

show this

$\nabla^2 \Phi = \nabla^2 (a\Phi_1 + b\Phi_2) = a\nabla^2 \Phi_1 + b\nabla^2 \Phi_2$

$= a(0) + b(0) = 0$ ✓

Corrolary — IMPORTANT

The solutions to a linear homogeneous PDE form a vector space.

affine sol'n to laplace's equation

HW - Find most general, quadr

Linear PDEs

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There is an enormous amount of theory

Non-Linear PDEs

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A really hard problem!

From Ex-3

Show $h(x,t) = 2\,\mathrm{sech}^2(x-4t)$ is a solution to

$$h_t + 6hh_x = h_{xxx}$$

Don't do all the work! Let computers do it.

(see maple example, worksheet online)

PDE := sum of partials

simplify(PDE),

Plot it!

Our solution was a traveling wave!
(a non-linear wave)
(John Russel - crazy wave chasing)

All solutions can be found by using inverse scattering (part of integrable systems). Very rare for PDES

$\boxed{4}$ has many known solutions

- You can find solution! (This will be a problem in the future)
  - this solution only exists for the box!

- Many PDEs (and ODEs) have solutions that are only defined within certain boundaries

- Soap Films (min-surface)
  look for surface that goes through boundary (boundary condition)
  and minimizes area (any local perturbation increases area).

Fly Paper

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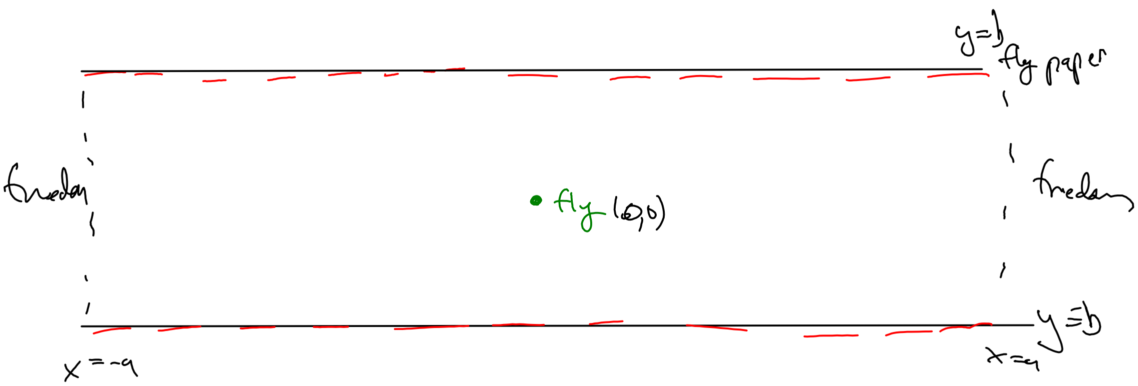
Why do you care about PDEs?

Suppose you're a fly at the origin



$p(x,y)$ = probability that fly starting at (x,y) escapes

One can show

$$\nabla^2 p = 0 = p_{xx} + p_{yy} \qquad \begin{array}{c} -a < x < a \\ -b < y < b \end{array}$$

also

$$p(x,b) = p(x,-b) = 0 \qquad -a < x < a$$

$$p(a,y) = p(-a,y) = 1 \qquad -b \le y \le b$$

Fly choose a random path.

The solution is an ugly infinite sum.

Don't know if it even converges.

Don't know if it even converges.

For $a=b=1$ (looks hyperbolic)/saddle

$$p(0,0) = \frac{1}{2}$$, as expected

For $a=\sqrt{3}, b=1$

$$p(0,0) = 1/6$$ ← try and show this using geometry

For $a=5, b=1$

$$p(0,0) = \sim 10^{-3}$$

For $a \gg b$

$$p(0,0) \sim \frac{8e^{-a\pi/2b}}{\pi}$$

Related to mixing, escape probabilities, etc.

A lot of what we will be doing is series work.