Last Time

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For $\rho(x,t)$

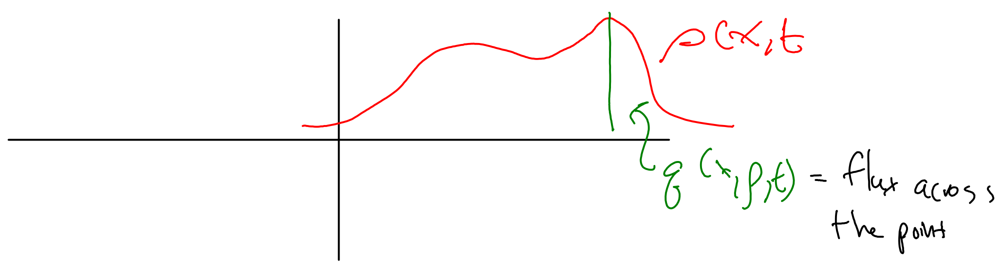
$$\rho_t + q_x = 0$$

where

$q = q(\rho, x, t)$ is a flux equation

(derived method of characteristics)

We also considered traffic flow



For the simple model is

$$q = c\rho \implies \boxed{\rho_t + \frac{\partial}{\partial x}(c\rho) = 0}$$ Traffic eq.

We can rewrite this as (assume $c = c(x,t)$)

$$\underline{\rho_t + c\rho_x + c_x\rho = 0}$$

and this is an example of a first-order linear PDE.

and this is an example of a first-order linear PDE.

Today we learn a method for solving these equations

Method of Characteristics

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(Turn PDE into system of ODEs)

<u>Ex</u> Convection Equation (C = constant = c) for $u(x,t)$

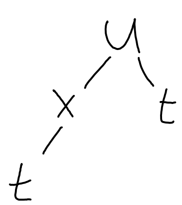
<u>DE</u>: $u_t + C u_x = 0$
<u>I.C</u>: $u(x,0) = f(x)$

Going to do this pedantically

Remember the chain rule.

Look for a curve on which u is a constant

Let $x = x(t)$



If u is a constant on $x(t)$, then

$$\frac{du}{dt} = 0$$

? → c

$$\frac{du}{dt} = \underbrace{\frac{\partial u}{\partial x}}_{u_x} \frac{dx}{dt} + \underbrace{\frac{\partial u}{\partial t}}_{u_t} = 0$$

But $u_t + c u_x = 0$,

this implies $\boxed{\frac{dx}{dt} = c.}$ This ODE defines the characteristic curve $x(t)$.

We can solve this!

$$\boxed{x = \xi + ct}$$

Now going to change variables

$$\xi = x - ct$$

$$(x,t) \to (\xi,\tau)$$

$$\tau = t$$

$$u(x,t) = U(\xi,\tau)$$

$$\frac{\partial u}{\partial t} = \frac{\partial U}{\partial \tau}\frac{\partial \tau}{\partial t} + \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial t}$$

$$\frac{\partial u}{\partial x} = \frac{\partial U}{\partial \tau}\frac{\partial \tau}{\partial x} + \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial x}$$

$$\frac{\partial \tau}{\partial t} = 1, \quad \frac{\partial \tau}{\partial x} = 0, \quad \frac{\partial \xi}{\partial t} = -c, \quad \frac{\partial \xi}{\partial x} = 1$$

$u_t = U_\tau - cU_\xi$

$u_x = U_\xi$

So $\quad u_t + c u_x = \left[U_\tau - cU_\xi \right] + c\left[U_\xi \right] = U_\tau$

$$\Rightarrow \boxed{U_\tau = 0}$$

This has a solution:

$$U(\xi, \tau) = F(\xi) \quad \leftarrow \text{any } F$$

and

$$u(x,t) = A(\xi) = A(x-ct)$$

$$u(x,0) = A(x) = f(x), \text{ so}$$

$$u(x,t) = f(x-ct)$$

Example 2

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Solve for $u(x,t)$

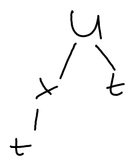
<u>DE:</u> $u_t + x u_x = 0 \qquad -\infty < x < \infty,\ t > 0$

<u>I.C.:</u> $u(x,0) = f(x) \qquad -\infty < x < \infty$

<u>Sol'n:</u>

Look for a characteristic $x(t)$ on which u is constant.

$u(x,t),\quad x(t)$



$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t} = 0$$

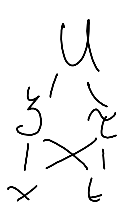
$\frac{\partial x}{\partial t} u_x + u_t = 0$ so

$$\frac{dx}{dt} = x \quad \Rightarrow \quad x(t) = \xi e^{t}$$

We change variables to the characteristic coordinates

$\xi = xe^{-t},\ \tau = t$

$u(x,t) = U(\xi,\tau)$



$$\frac{\partial u}{\partial} = \frac{\partial U}{\partial}\frac{\partial \tau}{\partial} + \frac{\partial U}{\partial}\frac{\partial \xi}{\partial}$$

$$\frac{\partial u}{\partial t} = \frac{\partial U}{\partial \tau}\frac{\partial \tau}{\partial t} + \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial t}$$

$$\frac{\partial u}{\partial x} = \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial U}{\partial \tau}\frac{\partial \tau}{\partial x}$$

$\tau_t = 1$, $\tau_x = 0$, $\xi_t = -xe^{-t}$, $\xi_x = e^{-t}$

$$\frac{\partial u}{\partial t} = U_\tau - xe^{-t}U_\xi$$

$$\frac{\partial u}{\partial x} = U_\xi e^{-t}$$

$$u_t + xu_x = 0$$

$$U_\tau - xe^{-t}U_\xi + xU_\xi e^{-t} = 0$$

$\boxed{U_\tau = 0}$ Same as last time!

$U = A(\xi)$, Changing back to (x,t),

$u(x,t) = A(xe^{-t})$

Add the I.C.

$u(x,0) = A(x) = f(x)$

$u(x,t) = f(xe^{-t})$

Third Time w/ a Twist

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Traffic Equation with $G = x$

So $\rho_t = x\rho_x + \rho = 0$

Or

$$\underline{\text{D.E.}}\ \overbrace{\rho_t + x\rho_x}^{\text{this is a derivative w/ respect to 1 var. under our coordinates}} = -\rho$$

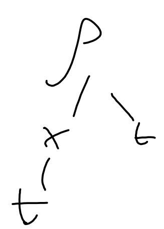
$$\underline{\text{I.C.}}:\ \rho(x,0) = f(x)$$

$$\underline{\rho_t + x\rho_x + \rho = 0}$$

Look for characteristics where the LHS is a perfect derivative

$x(t)$

$\rho(x,t)$



then

$$\rho_t + x\rho_x = \frac{d\rho}{dt}$$

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \frac{dx}{dt}\frac{\partial\rho}{\partial x} = \underbrace{\rho_t + x\rho_x}_{\text{LHS of DE}}$$

So I choose $\boxed{\frac{dx}{dt} = x}$ You can't stop me from making this choice!

Then $x(t) = \xi e^t$ and chose characteristic coordinates

$\xi = xe^{-t}$, $\tau = t$

and

$\rho(x,t) = P(\xi, \tau)$

Now, as before,

$$\rho_t + x\rho_x = P_\tau^{\cancel{\rho}} = -\rho = -P$$

so $\boxed{P_\tau = -P}$

the important thing is that this change of vars yeilds an ODE for $P(\tau)$

Now $\underline{P(\xi,\tau) = A(\xi)e^{-t}}$

and converting back to x, t:

$$\rho(x,t) = A(xe^{-t})e^{-t}$$

Apply I.C.

$\rho(x,0) = A(x) = f(x)$ and

$$\rho(x,t) = \boxed{f(xe^{-t})e^{-t}}$$

Let's Draw Some Characteristics

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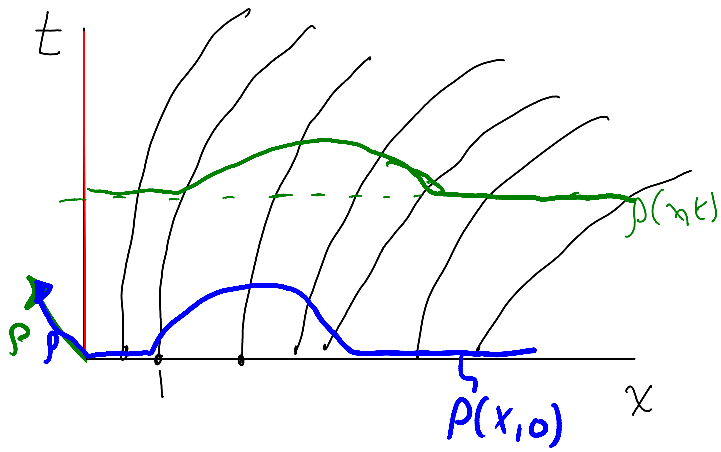

$\xi = xe^{-t}$
$x = \xi e^{t}$

Solution at t_1
Solution at t_2

Exercise: Show that the mass between two characteristics is constant in time (done in homeworks).

Example 4

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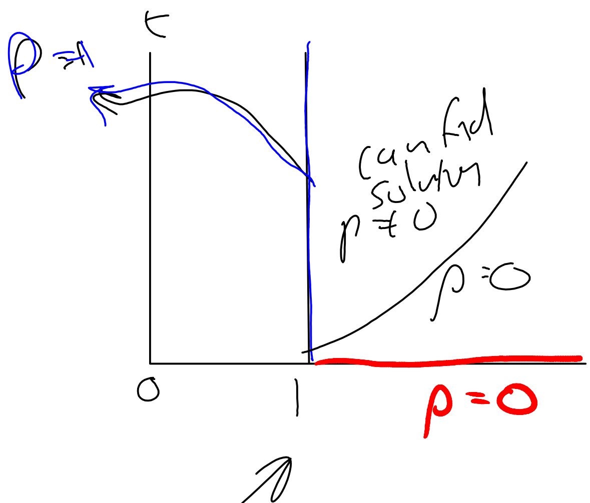
D.E.: $\rho_t + x\rho_x = -\rho$

I.C.: $\rho(x,1) = 0 \qquad x > 1$

B.C.: $\rho(1,t) = 1 \qquad t > 1$



"speed limit is x"

Can I solve this problem? Yes - beginning of next time.

Hint: