1 Well Posed Problems

Heat Equation:

Dirchlet Problem

$$\begin{aligned} \text{DE: } & u_t = Du_{xx}, 0 < x < l, t > 0 \\ \text{IC: } & u(x,0) = f(x) \\ \text{BC: } & u(0,t) = g(x) \end{aligned} \tag{D}$$

Neumann Problem

$$\begin{aligned} \text{DE: } & u_t = Du_{xx}, 0 < x < l, t > 0 \\ \text{IC: } & u(x,0) = f(x) \\ \text{BC: } & u_x(0,t) = a(t), u_x(l,t) = b(t) \end{aligned} \tag{N}$$

Wave Equation:

$$\begin{aligned} \text{DE: } & u_t t = c^2 u_x x, 0 < t, 0 < x < l \\ \text{BC: } & u(0,t) = 0, u(l,t) = 0 \\ \text{IC: } & u(x,0) = f(x), u_t(x,0) = y(x) \end{aligned} \tag{W}$$

Why these three problem? They are examples of well posed problems.
A well posed problem has 3 characteristics

1) Existence: A solution exists to the problem.
2) Uniqueness: The solution is unique.
3) Stability: If a small change is made in the initial condition or boundary condition, the solution changes by only a small amount.

We need to also talk about regularity. Solutions "live" in a function space. For example, for the heat equation, it is natural to talk about $u(x,t) \in C_x^2[0,l]$ - that is $u(x,t)$, $u_x(x,t)$, and the second derivative are continuous, and $u(x,t) \in C_t^1[0,\infty)$ - that is u(x,t) and $u_t(x,t)$ are continuous.

1.1 Existence

Existence usually (for this course) is demonstrated by an explicit solution. For example,

$$u(x,t) = \sum_{n=1}^{N} e^{-D(n^2\pi^2/L^2)t} \sin(\frac{n\pi x}{L})$$

is a solution to (D) (Dirchlet problem) with $a(t) = b(t) = 0$ and $f(x) = \sum_{n=1}^{N} a_n \sin(\frac{n\pi x}{L})$.

There is another method, called fixed point analysis, in which you bound the solution by a strictly decreasing sequence of sets in function space that converges to a point.

To show that a solution does not exist, one must usually derive a contradiction.

1.2 Uniqueness

To show uniqueness for linear problems, one almost always starts the same way. Proof by contradiction: Suppose $u_1(x,t)$ and $u_2(x,t)$ are two solutions of the Dirchlet problem

Let $v = u_1(x,t) - u_2(x,t)$. Note

$$v_t - Dv_x x = (u_1)_t s D(u_1)_x x) - ((u_2)_t - D(u_2)_x x) = 0;$$

So the DE for v is

$$\begin{aligned} \text{DE: } & v_t = Dv_x x \quad 0 < x < l \\ \text{IC: } & v(x,0) = 0 \quad 0 < x < l^1 \\ \text{BC: } & v(0,t) = 0, v(l,t) = 0, \quad t \geq 0. \end{aligned} \qquad (☺)$$

I need to show $v = 0$ is the only solution.

1.2.1 Energy Methods

Let $E[v] = \int_0^l \frac{v^2}{2}\, dx$. Note $E[v]$ is a function of time only. What is $\frac{dE}{dt}$, assuming v satisfies (☺)?

$$\begin{aligned} \frac{dE[v]}{dt} &= \frac{d}{dt}\int_0^l \frac{v^2}{2} dx \\ &= \int_0^l \frac{d}{dt}\frac{v^2}{2} dx \\ &= \int_0^l v v_t dx \\ &= \int_0^l v(Dv_x x)\ dx. \end{aligned}$$

Note that since the boundaries of the integral do not depend on time, the derivative with respect to time can move inside. Integrate by parts

$$\begin{aligned} p &= v, & dp &= v_x dx \\ dq &= v_x x dx, & q &= v_x \end{aligned}$$

$$\frac{dE}{dt} = D\Big[pq[_0^l - \int qdp]$$

So

$$\frac{dE}{dt} = D[vv_x[_0^l - \int_0^l v_x v_x dx].$$

Note that v v_x vanishes by the (☺) boundary conditions, and

$$\frac{dE}{dt} = -D\int_0^l (v_x)^2 dx.$$

[1] For example, $v(x,0) = u_1(x,0) - u_2(x,0) = f(x) - f(x) = 0$.

Note, $\frac{dE}{dt} \leq 0$ which implies that E is non-increasing. Also,

$$E = \int_0^l \frac{v^2}{2}\ dx \geq 0,$$

and

$$E[v(0)] = \int_0^l \frac{0^2}{2}\ dx = 0,$$

so E is initially 0, always non-negative, and non-increasing. Thus $E = 0$ for all $t > 0$. Note that this implicitly used the continuity of u in time. If $E \geq 0$ and continuous, then $v = 0$ for all $t > 0$; therefore $u_1 = u_2$ and the solution is unique.

1.3 Stability

"Can a butterfly flapping its wings in Beiking alter the weather in San Francisco?" - Paraphrase of Ed Lorenz

If a system is stable, and you make a small change, things don't change much. If a system is not stable, small changes make a huge difference.

An example of instability is the backwards heat equation. Recall that in the heat equation we assume D is positive. Suppose in the Dirchlet problem that $D < 0$. Then heat flows from cold to hot. Note that in our previous derivation of the solution, we did not make any use of the sign of D.

$$\begin{aligned} \text{DE:} & u_t = D u_{xx} \quad 0 < x < l, \quad D < 0 \\ \text{DC:} & u(0,t) = 0, u(l,t) = 0, \quad t > 0 \\ \text{IC:} & u(x,0) = \frac{1}{n}\sin(\frac{n\pi x}{l}) \quad 0 < x < l \end{aligned}$$

The solution is

$$u(x,t) = \frac{1}{n}\sin(\frac{n\pi x}{l} e^{-D\frac{n^2\pi^2}{l^2}t}.$$

Note that

$$\max u(x,0) = \frac{1}{n} \qquad 0 < x < l$$

but

$$\max_{0<x<l} u(x,t) = \frac{1}{n} e^{-D\frac{n^2\pi^2}{l^2}t}.$$

So given any $\delta > 0$, I can choose n such that $\frac{1}{n} < \delta$ and $|u(x,0)| < \delta$. But at $t = 1$

$$\max_{0<x<l} u(x,t) = \frac{1}{n} e^{-D\frac{n^2\pi^2}{l^2}}$$

and as $n \to \infty$, this max tends towards infinity. It turns out that for a generic initial condition, the temperature goes to infinity in a finite amount of time.

In fact, the forward heat equation is stable. As a handwaved argument, let

$$|u_1(x,0) - u_2(x,))| \leq \delta,$$

so in the energy derivation

$$0 \leq \int_0^l \frac{v^2}{2}\ dx \leq \delta^2,$$

so we have convergence in the L^2 norm.