Diffusion of Heat

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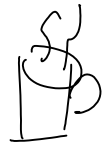
Recall: Newton's Law of Cooling



$U(t)$ = temp of coffee
$\bar{U}$ = temp of air.

Remember Newton says:

$$(\text{Heat Flux}) \propto \text{Temperature Difference}$$

HW - assume wave speed is positive

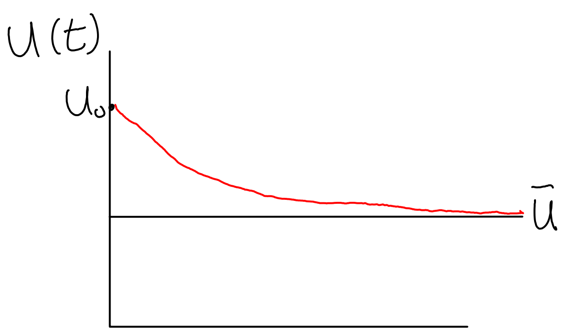
D.E.

$$\frac{dU}{dt} = -k(U - \bar{U}) \qquad k > 0.$$

I.C. $U(0) = U_0$

The solution is

$$\boxed{U(t) = \bar{U} + (U_0 - \bar{U})e^{-kt}}.$$

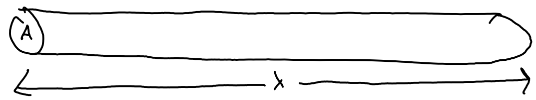


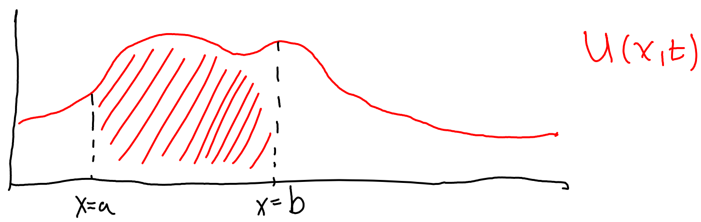
Assumption

Temp. in coffee is uniform! Not always the case.

In real life, temperature can be a function of position also

Let's consider a metal bar whose temperature varies along its length.





Assumptions: temp. always varies w/ length, no loss to atmosphere

$\mathcal{E}(t)$ = thermal energy in bar in region $a<x<b$

$$= \rho c_v \int_a^b A\, u(x,t)\, dx$$

degrees

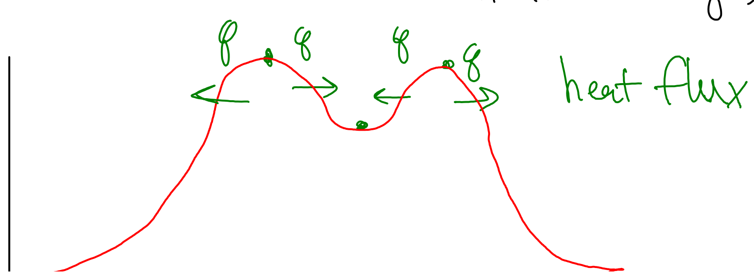
A = area = length^2

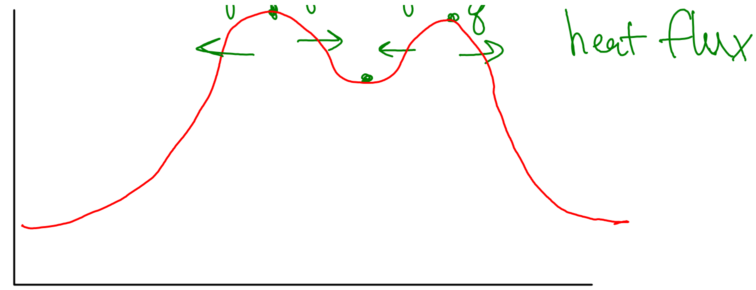
ρ = mass/length^3 (density)

c_v = heat capacity = $\frac{\text{energy}}{\text{degree mass}}$ energy to raise 1 mass by 1 temp.

Claim:

$$\frac{d\mathcal{E}}{dt} = -\left[q(b) - q(a)\right]$$ where $q(x)$ is the heat flux through x (to the right)





"phlogisten" - thought to be in material (heat)

Fourier's Law of Cooling

heat flux is proportional to the gradient of temperature

$$q = -K u_x$$

$K > 0$ is thermal conductivity.
(energy length / temp.time)

So

$$\frac{d\mathcal{E}}{dt} = -\left[q(b) - q(a)\right] = K\left[\underbrace{u_x(b) - u_x(a)}_{\text{limits of an integral}}\right]$$

but

$$\frac{d\mathcal{E}}{dt} = \frac{d}{dt}\left[\rho c_v A \int_{x=a}^{b} u\,dx\right]$$

$$= \rho c_v A \int_{x=a}^{b} \frac{\partial u}{\partial t}\,dx$$

So

$$\rho c_v A \int_a^b \frac{\partial u}{\partial t}\,dx = K\left[\int_a^b \frac{\partial^2}{\partial x^2} u\,dx\right]$$

Rearanging

$$\rho c_v A \int^b \frac{\partial u}{\partial t} - D \frac{\partial^2 u}{} \, dx = 0$$

$$\rho c_v A \int_{x=a}^{b} \frac{\partial u}{\partial t} - D \frac{\partial^2 u}{\partial x^2} \, dx = 0$$

where $D = \dfrac{K}{\rho c_v A}$ is the diffusion constant. $\left(\frac{\text{length}^2}{\text{time}}\right)$

a and b are arbitrary, so integral vanishes over every interval, and assuming sufficient smoothness

$$\boxed{\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}}$$

Heat / diffusion equation

All we need to do is solve it.

Dirchlet Problem for the Heat Equation

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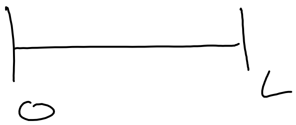
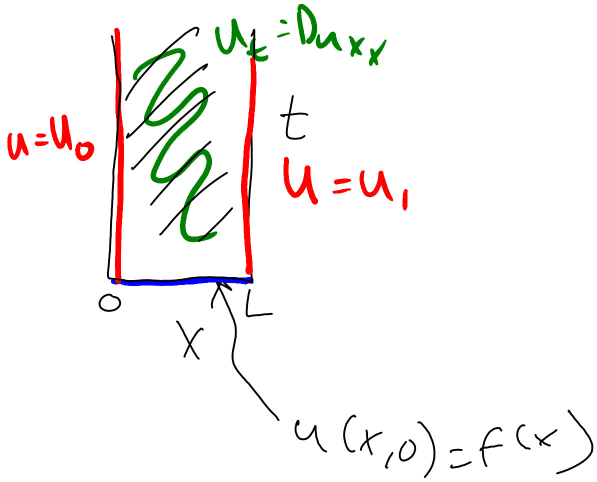


Can specify flux of temps at endpoints.
(Nueman or Dirchet)

D.E.: $u_t = D u_{xx} \quad 0 \le x < L \quad , t > 0$

B.C.: $u(0,t) = u_0 \quad , u(L,t) = u_1 \quad t > 0$

I.C.: $u(x,0) = f(x) \quad 0 < x < L$



This is a well specified problem (we can prove in a few weeks).

This problem has unique solution

Separation of Variables

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Today we will solve a special case $u_0 = u_1 = 0$ using method of separation of variables.

Guess a solution of the form

$$u(x,t) = X(x)\,T(t)$$

Substitute into DE.

<u>D.E.</u>

$$u_t = D\,u_{xx} \rightsquigarrow$$

$$X\,T_t = D\,X_{xx}\,T$$

Separate variables

Divide by $X \cdot T \cdot D$:

$$\underbrace{\frac{T_t}{DT}}_{\text{fn of } t} = \underbrace{\frac{X_{xx}}{X}}_{\text{fn of } x} = -\lambda$$

a fn of x which is also a function of t. Called the separation constant.

We now have two ODEs:

We now have two ODEs:

$$T_t + \lambda D T = 0 \rightarrow \boxed{T(t) = Ae^{-\lambda D t}}$$

$$X_{xx} + \lambda X = 0 \qquad 0 < x < L$$

Can separate the BC's also

note $0 = u(0,t) = \underline{X}(0)\,T(t)$

$$\Longrightarrow X(0) = 0 \quad \text{or} \quad T(t) = 0$$

If $T(t) = 0$, $u = (x,t) = 0 \Rightarrow u(x,t) = 0.$

true, but boring. $\boxed{\text{TRIVIAL SOLUTION}}$

Choose $X(0) = 0$.

Other boundary condition

$$u(L,t) = X(L)\,T(t) = 0$$

$\underline{X(L) = 0}$ or $T(t) = 0$ (trivial)

Yeilds $\boxed{\text{BVP}}$ (boundary value problem)

$$X_{xx} + \lambda X = 0$$

$\underline{B.C.}$: $\quad X(0) = X(L) = 0.$

Assume $\lambda > 0$ (don't even know real

* (we'll prove that if $\lambda \not> 0$, then $X=0$ is only soln).

D.E.:

$$X_{xx} + \lambda X = 0$$

$$X(x) = A\cos(\sqrt{\lambda x}) + B\sin(\sqrt{\lambda x}),$$

but $X(0) = A = 0$, so $A=0$.

and

$$X(L) = B\sin\sqrt{\lambda L} = 0$$

so $\sqrt{\lambda}\, L = n\pi$ for $n = 1, 2, 3 \ldots$

so, $\lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2$

$$X = X_n = \underline{B\sin\left(\frac{n\pi x}{L}\right)}$$

Combining these yield

$$u_n(x,t) = T_n(t)\, X_n(t) = \tilde{A}\, e^{-\lambda_n D t} \sin\left(\frac{n\pi x}{L}\right)$$

$$u_n(x,t) = C\, e^{-\frac{n^2\pi^2}{L^2} D t} \sin\left(\frac{n\pi x}{L}\right) \quad n = 1, 2, 3, \ldots$$

Check solution (Maple WS online)

More General Solution

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The problem is homogeneous, so the most general solution is a linear combination

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2}{L^2} D t}}$$

Need to solve I.C.

Don't know c_n, haven't use I.C.

$$\underline{u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) = f(x)}$$

At this point I remember the orthogonality condition Let

$$\langle g(x), h(x) \rangle = \int_0^L g(x) h(x)\, dx.$$

$$\left\langle \sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{m\pi x}{L}\right) \right\rangle = \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx$$

$$\left\{ = 0 \text{ if } n \neq m, \ = \frac{L}{2} \text{ if } n = m \right\}$$

So

$$\left\langle f(x), \sin\left(\frac{m\pi x}{L}\right) \right\rangle = \left\langle \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{m\pi x}{L}\right) \right\rangle$$

$$= \sum_{n=1}^{\infty} c_n \left\langle \sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{m\pi x}{L}\right) \right\rangle = c_m \frac{L}{2}$$

projecting onto n^{th} vector

so $c_m = \frac{2}{L}\left\langle f(x), \sin\left(\frac{m\pi x}{L}\right)\right\rangle$

$$\boxed{= \frac{2}{L}\int_0^L f(x)\,\sin\left(\frac{m\pi x}{L}\right)dx}$$

Fourier Sine Series

If $f(x) = \left|x - \frac{L}{2}\right|$

I get the series in the MAPLE worksheet.