How to Cook a Spherical Turkey

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First imagine a hot copper ball dropped into a heat bath maintained at 0.

$$\begin{aligned}
&\text{DE: } u_t = k\Delta u \qquad x \in \Omega, t > 0 \\
&\text{IC: } u(x,0) = T_0 \qquad x \in \Omega \\
&\text{BC: } u(x,t) = 0 \qquad x \in \partial\Omega, t > 0
\end{aligned}$$

Where $\Delta = \nabla^2$. Spherical coordinates would be nice here. So make the following assumption of spherical symmetry:

$$u = u(r, \phi, \theta, t) = u(r, t).$$

Note that

$$u = u(x) = u(r) \qquad r = |x|.$$

In this case,

$$\Delta u = u_{x_1 x_1} + u_{x_2 x_2} + \cdots + u_{x_n x_n}$$

$$\begin{aligned}
\frac{\partial u}{\partial x_i} &= u'(r)\frac{\partial r}{\partial x_i} \\
&= u'(r)\frac{x_i}{r} \\
\frac{\partial^2 u}{\partial x_i^2} &= \frac{u'(r)}{r} + x_i \frac{r u''(r)\frac{x_i}{r} - u'(r)\frac{x_i}{r}}{r^2} \\
&= \frac{u'(r)}{r} + \frac{x_i^2}{r^2}[r u'' - u']
\end{aligned}$$

So

$$\begin{aligned}
\Delta u = \nabla^2 &= \sum_{i=1}^{n} u_{x_i x_i} \\
&= n\frac{u'}{r} + \frac{r^2}{r^3}[r u'' - u'] \\
&= u'' + \frac{n-1}{r}u'
\end{aligned}$$

which is the **radial Laplacian**. The PDE now becomes:

$$\begin{aligned}
&\text{DE: } u_t = k(u_{rr} + \frac{2}{r}u_r \qquad 0 < r < \pi, t > 0 \\
&\text{IC: } u(r,0) = T_0 \qquad x \in \Omega u < r < \pi \\
&\text{BC: } u(\pi, t) = 0 \qquad t > 0.
\end{aligned}$$

This is the problem of **spherical cooling**. We can solve this using separation of variables. Assume

$$u(r,t) = T(t)R(r).$$

Then

$$\begin{aligned} T'R &= k(TR'' + \frac{2}{r}TR') \\ \frac{T'}{kT} &= \frac{R'' + \frac{2}{r}R'}{R} = -\lambda. \end{aligned}$$

Our PDEs are thus

$$\begin{aligned} T' &= -\lambda k T \\ R'' + \frac{2}{r}R' + \lambda R &= 0 \end{aligned}$$

We see the t component has solution:

$$T_\lambda(t) = r_\lambda e^{-k\lambda t},$$

which suggests that λ will be positive. This is not a proof, however, so we look at the second problem:

$$\begin{aligned} \text{DE: } & R'' + \frac{2}{r}R' + \lambda R = 0 \\ \text{BC: } & R(\pi) = 0 \qquad (u(\pi,t) = T(t)R(\pi) = 0) \\ \text{HBC: } & R(0) < \infty \qquad (R(0) \text{ bounded}). \end{aligned}$$

Where (HBC) is a hidden boundary condition. Note the sign of λ. Given

$$-\Delta u = \lambda u,$$

(e.g $-\Delta u = \lambda u$):

$$\begin{aligned} (u'' + \frac{n-1}{r}u') &= -\lambda u \\ r^{n-1}u'' + (n-1)r^{n-2}u' + \lambda r^{n-1}u &= 0 \\ \int_0^\pi [r^{n-1}u']'u + \lambda \int_0^p ir^{n-1}u^2 &= 0 \\ r^{n-1}u'u\Big|_0^\pi r^{n-1}(u')^2 i \, dr + \lambda \int_0^\pi rn - 1u^2 \, dr &= 0 \\ \lambda &= \frac{\int_0^\pi r^{n1}u(u')^2 \, dr}{\int_0^\pi r^{n-1}u^2 \, dr} \\ &\geq 0. \end{aligned}$$

Let $\lambda = 0$, then $\int_0^\pi r^{n-1}(u)^2 \, dr = 0$, so u is constant. Recall we are looking at the Sturm Louiville Problem

$$-(R'' + \frac{2}{r}R') = \lambda R,$$

which can be written

$$-\Delta u = \lambda u.$$

Also, note that

$$\Delta u + \lambda u = 0$$

is called the Helmholtz Equation. That asside, a constant u is a problem for our boundary conditions - the function is 0 on the boundaries, so $u = 0$ uniformly in this case! Thus we know that $\lambda > 0$. We thus re-write things again:

$$\begin{aligned}
\lambda &= \mu^2 > 0 \\
\text{IC: } R'' + \frac{2}{r}R' + \mu^2 R &= 0 \\
\text{HBC: } R(0) &\text{ bounded} \\
\text{BC: } R(\pi) &= 0
\end{aligned}$$

Then

$$\begin{aligned}
rR'' + 2R' + \mu^2 rR &= 0 \\
Y'' + \mu^2 Y &= 0 \\
Y(0) &= 0 = Y(\pi) \\
Y_{mu}(r) &= A\cos\mu r + B\sin\mu r \\
&= 0 = \sin\mu\pi \\
\implies \mu &= n \in \mathbb{N} \\
Y_n(r) &= \sin(nr)
\end{aligned}$$

This implies

$$R_n(r) = \frac{\sin(nr)}{r}.$$

Thus the Eigenmodes are

$$\begin{aligned}
u_n(r,t) &= T_n(t)R_n(r) \\
&= e^{-n^2kt}\frac{\sin(nr)}{r}.
\end{aligned}$$

Note that this does not have a singularity at the origin because

$$\lim_{r\to 0}\frac{\sin(nr)}{r} = n.$$

Assume

$$u(r,t) = \sum_{n=1}^{\infty} A_n e^{-n^2kt}\frac{\sin(nr)}{r}$$

Let $t = 0$:

$$u(r,0) = T_0 = \sum_{n=1}^{\infty} A_n \frac{\sin(nr)}{r},$$

so

$$(rT_0) = \sum_{n=1}^{\infty} A_n \sin(nr),$$

$$
\begin{aligned}
A_n &= \frac{2}{\pi}\int_0^\pi T_0 r\sin(nr)\ dr = \frac{2T_0}{\pi}\int_0^\pi r\sin(nr)\ dr \\
&= \frac{2T_0}{\pi}[\frac{-r\cos nr}{n}\Big|_0^r + \int_0^\pi \frac{\cos(nr)}{r}\ dr \\
&= \frac{2T_0}{\pi}\left[\frac{-\pi(-1)^n}{n}\right] \\
&= \frac{2T_0}{n}(-1)^{n+1}
\end{aligned}
$$

Putting everything together

$$u(r,t) = \sum_{n=1}^\infty \frac{(-1)^{n+1}2T_0}{n}e^{-n^2kt}\frac{\sin(nr)}{r}.$$

Note, if $\Omega = B(0,\xi)$, then

$$u(r,t) = 2T_0\sum_{n=1}^\infty (-1)^{n+1}e^{-k(\frac{n\pi}{\xi})^2 t}\frac{\sin(\frac{n\pi}{\xi}r)}{\frac{n\pi}{\xi}r}.$$

Note, we know

$$\int_0^\pi Y_n(r)Y_m(r)\ dr = 0 \qquad m \neq n,$$

so

$$\int_0^\pi R_n(r)R_m(r)r^2\ dr = 0 \qquad m \neq n.$$

We say they are orthogonal with respect to "weight r^2."
Given

$$
\begin{aligned}
\text{DE: } & u_t = k\Delta u \\
\text{IC: } & u(r,0) = u_0(r) \\
\text{BC: } & u(R,t) = f(t)
\end{aligned}
$$

Let us solve **The Turkey Problem**; putting a cold turkey into a heat bath.

$$
\begin{aligned}
\Delta u_s &= 0 \\
u_s &= C.
\end{aligned}
$$

Solving this, we have,

$$
\begin{aligned}
v &= u - u_s \\
v_t &= u_t \\
\Delta v &= \Delta u \\
v(R,t) &= C - C = 0 \\
v(r,0) &= u_0(r) - u_s(r)
\end{aligned}
$$

so the new problem is

$$
\begin{aligned}
\text{DE: } & v_t && = k\Delta v \\
\text{IC: } & v_0(r) \\
\text{BC: } & v(R,t) = 0.
\end{aligned}
$$

We can now solve the turkey problem. Suppose we have turkey of roughly spherical shape that has been defrosted to 75°, and is placed into a 350° oven. Assume $R = 1$, and $k = 0.02$ (this is roughly correct). How long until the temperature at the center is 150°. (Really it should be 165°, but we're living on the edge.)

$$\begin{aligned} &\text{DE: } u_t = k\Delta u \qquad 0 < r < 1, t > 0 \\ &\text{IC: } u(r,0) = 75 \qquad 0 < r < 1 \\ &\text{BC: } u(1,t) = 350 \qquad t > 0. \end{aligned}$$

By our previous work,

$$u = v + w$$

with

$$\begin{cases} \Delta w = 0 \\ w = 350 \end{cases} \implies w = 350$$

$u = v + 350$. Thus

$$\begin{cases} v_t = k\Delta v \\ v(r,0) = -275 \\ v(1,t) = 0 \end{cases} \implies v(r,t) = -550$$

So

$$u = 350 - 550 \sum (\text{modes}).$$

Cut to Mathematica worksheet.