Duhamel's Principle

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The inverse transform of the integral is the integral of the inverse transform. - Duhamel

We've talked about arbirtary forcing in the spatial variable— What about arbitrary forcing in time?

Example.

$$u_e + u_x = f(t)\delta(x) \qquad -\infty < x < \infty, t > 0$$
$$u(x,0) = 0 \qquad -\infty < x < \infty.$$

Imagine wind blowing with unit speed (diagram on board).

$$u_t + u_x = 0 \longrightarrow u(x,t) = f(x-t).$$

To solve, FT in x

$$\mathscr{F}\{u(x,t)\} = \hat{u}(k,t)$$

and

$$\mathscr{F}\{u_x(x,t)\} = ik\hat{u}(k,t)$$
$$\mathscr{F}\{u_t(x,t)\} = \hat{u}_t(k,t)$$

so

$$\text{DE: } \hat{u}_t ik\hat{u} = f(t)$$
$$\text{IC: } \hat{u}(k,0) = 0$$

Solve with the method of integrating factors:

$$\mu(t) = e^{\int ik \ dt}$$
$$= e^{ikt}$$

So

$$e^{ikt}(\hat{u}_t + ik\hat{u}) = f(t)e^{ikt}$$
$$\frac{d}{dt}(\hat{u}e^{ikt}) = f(t)e^{ikt}$$

Integrate from 0 to t, because the IC is at $t = 0$.

$$\begin{aligned}
\int_0^t \frac{d}{dt}(\hat{u}e^{ikt})\ dt &= \int_0^t f(s)e^{iks}\ ds \\
\hat{u}e^{ikt}\Big[_0^t &= \int_0^t f(s)e^{iks}\ ds \\
\hat{u}(k,t)e^{ikt} - \underbrace{\hat{u}(k,0)}_{=0}1 &= \int_0^t f(s)e^{iks}\ ds
\end{aligned}$$

and

$$\begin{aligned}
\hat{u}(k,t) &= e^{-ikt} int_0^t f(s)e^{iks}\ ds \\
\hat{u}(k,t) &= \int_0^t f(s)e^{ik(s-t)}\ ds
\end{aligned}$$

Recall. The method of integrating factors: Suppose

$$y'a(t)y = b(t), y(0) = 0.$$

To solve, multiply by the integrating factor that makes the left hand side an exact derivative:

$$\mu(t) = e^{\int a(s)\ ds}$$

Note

$$\begin{aligned}
(y(t)e^{\int a(s)\ ds})' &= y'e^{\int a(s)\ ds} + ye^{\int a(s)\ ds} \\
&= (y' + a(t)y)e^{\int a(s)\ ds}
\end{aligned}$$

So multiplying the DE by $\mu(t)$ yields

$$\begin{aligned}
(y' + a(t)y)\mu(t) &= b(t)\mu(t) \\
(y\mu(t))' &= b(t)\mu(t) \\
\int_0^t (y(s)\mu(s))'\ ds &= \int_0^t b(t)\mu(t)\ dt \\
y(s)\mu(s)\Big[_0^t &= \int_0^t b(t)\mu(t)\ dt \\
y(t) &= \frac{1}{\mu(t)}\int_0^t b(s)\mu(s)\ ds
\end{aligned}$$

where

$$\mu(t) = e^{\int_0^s a(s)\ ds}.$$

Definition. A *t-convolution* is the following

$$f \star g = \int_0^t f(s)(t-s)\ ds$$

This comes up often when studying Laplace Transforms.

$$u(x,t) = \mathscr{F}^{-1}\left\{\int_0^t f(s)e^{ik(s-t)}\ ds\right\}$$
$$\int_0^t f(s)\mathscr{F}^{-1}e^{ik(s-t)}\ ds$$

This is *Duhamel's Principle* in action - as long as the integrals converge absolutely, I can exchange their order.

$$\begin{aligned}\mathscr{F}^{-1}e^{ik(s-t)} &= \delta(x-(s+s)) \\ &= \delta((x-t)+s)\end{aligned}$$

(this is likely incorrect) so

$$\begin{aligned}\int_0^t f(s)\mathscr{F}^{-1}e^{ik(s-t)}\ ds = & \\ &= \int_0^t f(s)\delta(x-(s-t)) \\ &= \begin{cases} f(t-x) & 0<x<t \\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

Graph on board.

1 Diffusion of a variable point source

$$\begin{aligned}\text{DE: } & u_t = Du_{xx} + q(t)\delta(x) \\ \text{IC: } & u(x,0) = 0\end{aligned}$$

Diagram on board, something something blowtorch at origin at $t = 0$; things are getting HOT (because blow-torches don't tend to blow cold). As time increases, this heat distribution grows and spreads.

To solve, FT:

$$\begin{aligned}\mathscr{F}\{u_t\} &= \hat{u}_t \\ \mathscr{F}\{u_{xx} &= (ik)^2\hat{u} = -k^2\hat{u}, \mathscr{F}\{\delta(x)\} = 1\end{aligned}$$

$$\begin{aligned}\text{DE: } & \hat{u}_t = -Dk^2\hat{u} + q(t) \\ \text{IC: } & \hat{u}(k,0) = 0\end{aligned}$$

So

$$\hat{u}_t + Dk^2\hat{u} = q(t)$$

The EF, $\mu(t) = e^{\int Dk^2\ dt} = e^{Dk^2t}$ and

$$\begin{aligned}e^{Dk^2t}(\hat{u}_t + Dk^2\hat{u}) &= e^{Dk^2t}q(t) \\ \frac{d}{dt}\left(\hat{u}e^{Dk^2t}\right) &= e^{Dk^2t}q(t)\end{aligned}$$

Integrate w.r.t. t from 0 to T

$$
\begin{aligned}
int_0^T \frac{d}{dt}\left(\hat{u}e^{Dk^2t}\right)\ dt &= \int_{t=0}^{T} e^{Dk^2t}q(t) \\
\hat{u}e^{Dk^2t}\Big[_{t=0}^{T} &= \int_{t=0}^{T} e^{Dk^2t}q(t) \\
\hat{u}(k,t)e^{Dk^2T} - \underbrace{\hat{u}(k,0)}_{=0}1 &= \int_0^T q(t)e^{Dk^2t}\ dt \\
\hat{u}(k,t) &= e^{-Dk^2T}\int_0^T q(t)e^{Dk^2t}\ dt \\
\hat{u}(k,t) &= \int_0^T q(t)e^{-Dk^2(T-t)}\ dt
\end{aligned}
$$

or

$$\hat{u}(k,T) = q(T) \star e^{-Dk^2T}.$$

Need to invert the FT

$$
\begin{aligned}
u(x,t) &= \mathscr{F}^{-1}\{\int_0^T q(t)e^{-Dk^2(T-t)}\ dt\} \\
&= \int_0^T q(t)\mathscr{F}^{-1}\{e^{-Dk^2(T-t)}\ dt\}
\end{aligned}
$$

Remember

$$
\begin{aligned}
\mathscr{F}^{-1}\{e^{-bk^2}\} &= \frac{1}{2\sqrt{\pi b}}e^{-x^2/4b} \qquad b = D(T-t) \\
u(x,t) &= \int_0^T q(t)\frac{e^{-x^2/4D(T-t)}}{2\sqrt{\pi D(T-t)}}\ dt \\
&= q(T) \star G(x,T)
\end{aligned}
$$

where

$$G(x,T) = \frac{e^{-x^2/4Dt}}{2\sqrt{\pi DT}}.$$

Superimpose a $G(k,t)$ [corresponding to a δ-function IC at the origin] with stregth $q(t)$ for every time $0 < t < T$. This formula is complex, but is simple if we only consider the temperature at $x = 0$.

$$u(0,T) = \frac{1}{2\sqrt{\pi D}}\int_0^T \frac{q(t)}{\sqrt{T-t}}\ dt$$

Example.

$$q(t) = 1 \implies u(0,T) = \frac{1}{2\sqrt{\pi D}}\int_0^T \frac{1}{\sqrt{T-t}}\ dt = \frac{T^{1/2}}{\sqrt{\pi D}}.$$

Diagram on board.

Example.

$$q(t) = t^{\beta}$$

$$U(0,T) = \frac{1}{2\sqrt{\pi D}} \int_0^T t^\beta (T-t)^{-1/2}\ dt \qquad (\beta\text{eta Function})$$

$$U(0,T) = \frac{1}{\sqrt{\pi D}} t^{\beta t \frac{1}{2}} \frac{(\beta!)^2 2^{2\beta}}{(2\beta+1)!}$$