1 Last Time

Fourier Transform: We had annoying problems with ξ notation, so now

$$\mathscr{F}\{f(x)\} = \hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\ dx$$
$$\mathscr{F}^{-1}\{\hat{f}(k)\} = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}\ dx$$

2 Properties of the Fourier Transform

1. Existence: If $f(x)$ is absolutely integrable, that is

$$\int_{-\infty}^{\infty} |f(x)|\ dx = M < \infty$$

   then $\hat{f}(k)$ exists.

2. Uniqueness: Suppose $f(x)$ and $g(x)$ are absolutely integrable and

$$\hat{f}(k) = \hat{g}(k).$$

   Then

$$\int_{-\infty}^{\infty} |f(x) - g(x)|\ dx = 0.$$

   If in addition, f and g are continuous, then $f(x) = g(x)$ almost everywhere.

3. Derivatives: Suppsoe $f(x)$ is absolutely integrable and

$$\mathscr{F}\{f(x)\} = \hat{f}(k).$$

   Then

$$\mathscr{F}\{f'(x)\} = ik\hat{f}(k).$$

   Note

$$\begin{aligned}\mathscr{F}\{f''(x)\} &= \mathscr{F}\{(f'(x))'\} \\ &= ik\mathscr{F}\{f'(x)\} \\ &= (ik)^2\hat{f}(k)\end{aligned}$$

4. Some Examples

   (a) Top Hat:

$$f(x) = \begin{cases} 1 & |x| < L \\ 0 & |x| \leq L \end{cases}$$

   then

$$\mathscr{F}\{f(x)\} = \int_{-L}^{L} 1e^{-ikx}\ dx = \frac{2\sin kt}{k}.$$

(b) Heaviside Funtion (Include Graph)

$$H(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases}$$

Think about this as a switch - if you multiply a function by $H(x)$, it switches on at $x = 0$.

(c) $f(x) = H(x)e^{-ax} \quad \Re\{a\} > 0$

$$\begin{aligned}
\mathscr{F}\{f(x)\} &= \int_{-\infty}^{\infty} H(x)e^{-ax}e^{-ikx}\ dx \\
&= \int_{0}^{\infty} e^{-(a+ik)x}\ dx \\
&= \frac{1}{a+ik}
\end{aligned}$$

5. Linearity:

$$\mathscr{F}\{af(x) + bg(x)\} = a\hat{f}((k) + b\hat{g}(k).$$

6. Reflection:

$$\begin{aligned}
\mathscr{F}\{f(-x)\} &= \int_{-\infty}^{\infty} f(-x)e^{-ikx}\ dx \qquad & z = -x \\
&= \int_{-\infty}^{\infty} f(z)e^{ikz}(-dz) \qquad & dx = -dz \\
&= \int_{z=-\infty}^{\infty} f(z)e^{-i(-k)z}\ dz \\
&= \hat{f}(-k)
\end{aligned}$$

7. Compute for $\Re\{a\} > 0$

$$\begin{aligned}
\mathscr{F}\{e^{-a|x|}\} &= \mathscr{F}\{H(x)e^{-ax} + H(-x)e^{ax}\} \\
&= \mathscr{F}\{H(x)e^{-ax}\} + \mathscr{F}\{H(-x)e^{ax}\} \\
&= \frac{1}{a+ik} + \frac{1}{a-ik} \\
&= \frac{2a}{a^2+k^2}
\end{aligned}$$

8. Shifting:

$$\begin{aligned}
\mathscr{F}\{f(x+a)\} &= \int_{-\infty}^{\infty} f(x+a)e^{-ikx} \qquad & z = a + x \\
&= \int_{-\infty}^{\infty} f(z)e^{-ik(z-a)}\ dz \qquad & dz = dx \\
&= e^{ika}\int_{-\infty}^{\infty} f(z)e^{-ikz}\ dz \\
&= e^{ika}\hat{f}(k)
\end{aligned}$$

$f(x)$	$\hat{f}(k)$
$f'(x)$	$ik\hat{f}(k)$
$f^{(n)}(x)$	$(ik)^n\hat{f}(k)$
$f(x) = \begin{cases} 1 & \lvert x\rvert < L \\ 0 & \lvert x\rvert \geq L \end{cases}$	$\frac{2\sin(kL)}{k}$
$H(x)e^{-ax}$	$\frac{1}{a+ik}$
$f(-x)$	$\hat{f}(-k)$
$e^{-a\lvert x\rvert}$	$\frac{2a}{a^2+k^2}$
$f(x+a)$	$e^{ika}\hat{f}(k)$
$\mathscr{F}\{f(bx)\}$	$\frac{1}{b}\hat{f}\left(\frac{h}{b}\right)$

Table 1: Table of Fourier Transform Identities Derived in Notes.

3 Uses of the Fourier Transform

Consider the following PDE:

$$\begin{aligned} \text{DE: } u_t + cu_x = 0 &\qquad -\infty < x < \infty, t > 0 \\ \text{IC: } u(x,0) = F(x) &\qquad -\infty < x < \infty \end{aligned}$$

Solution: Note, if we Fourier Transform in x

$$\begin{aligned} \mathscr{F}_x\{u(x,t)\} &= \int_{-\infty}^{\infty} u(x,t)e^{-ikx}\ dx \equiv \hat{u}(k,t) \\ \mathscr{F}_x\{u_t(x,t)\} &= \int_{-\infty}^{\infty} u_t(x,t)e^{-ikx}\ dx = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} u(x,t)e^{-ikx}\ dx \\ &= \frac{\partial \hat{u}}{\partial t}(k,t) \end{aligned}$$

Also

$$\mathscr{F}\{u_x(x,t)\} = uk\hat{u}(k,t).$$

Fourier the Transform of the DE

$$\mathscr{F}\{u_t + cu_x\} = \hat{u}_t + ikc\hat{u} = 0. \qquad (☺)$$

This is an ODE in t. We can solve this:

$$\hat{u}(k,t) = A(k)e^{-ikct}.$$

To solve for $A(k)$, FT the IC

$$\mathscr{F}\{u(x,0)\} = \hat{u}(k,0) = \mathscr{F}\{F(x)\} = \hat{F}(k).$$

Now

$$\hat{u}(k,0) \overset{☺}{=} A(k) = (IC)\hat{F}(k)$$

and

$$\hat{u}(k,t) = \hat{F}(k)e^{-ikct}.$$

Now by the shifting formula,

$$u(x,t) = F(x-ct).$$

Note the pattern

$$\begin{tikzcd} u(x,t) \arrow[r, "\mathscr{F}"] \arrow[d] & \text{DE for } \hat{u}(k,t) \arrow[d, "\text{solve in t}"] \\ u(x,t) & \hat{u}(k,t) \arrow[l] \end{tikzcd}$$

4 Scaling

Suppose I want to compute

$$\begin{aligned} \mathscr{F}\{f(bx)\} &= \int_{-\infty}^{\infty} f(bx)e^{-ikx}\ dx \qquad z = bx \quad dz = bdx \\ &= \int_{-\infty}^{\infty} f(z)e^{-ik\frac{b}{z}}\frac{dz}{b} \\ &= \frac{1}{b}\hat{f}\left(\frac{k}{b}\right). \end{aligned}$$

So

$$\mathscr{F}\{f(bx)\} = \frac{1}{b}\hat{f}\left(\frac{h}{b}\right).$$

5 δ-sequences and the Transform of a δ-function

A δ-sequence satisfies

$$\int_{-\infty}^{\infty} \delta_\epsilon(x)\ dx = 1 \quad \text{ and } \lim_{\epsilon\to 0}\delta_\epsilon(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases} .$$

Given any function $f(x)$ that is $f(x) > 0$, $f(x)$ is continuous,

$$\int_{-\infty}^{\infty} f(x)\ dx = 1,$$

I claim that

$$f_\epsilon(x) = \frac{1}{\epsilon}f(\frac{x}{\epsilon})$$

is a δ-sequence.

Proof: Note

$$\begin{aligned} \int_{-\infty}^{\infty} f_\epsilon(x)\ dx &= \int_{-\infty}^{\infty} f(\frac{x}{\epsilon}) \qquad \frac{x}{\epsilon} = z \qquad \frac{dx}{\epsilon} = dz \\ &= \int_{-\infty}^{\infty} f(z)\ dz \\ &= 1 \end{aligned}$$

Note this rescaling as $\epsilon \to 0$: (diagram of gaussians getting thinner and taller.)

Consider the Fourier Transform:

$$\mathscr{F}\{f_\epsilon(x)\} = \hat{f}_\epsilon(k).$$

First note

$$\hat{f}_\epsilon(0) = \int_{-\infty}^{\infty} f_\epsilon(x) e^{-i0x}\ dx = 1.$$

Also,

$$\begin{aligned}
\mathscr{F}\{f_\epsilon\} &= \mathscr{F}\{\frac{1}{\epsilon} f\left(\frac{x}{\epsilon}\right)\} \\
&= \frac{1}{\epsilon}\mathscr{F}\{f\left(\frac{x}{\epsilon}\right)\} \\
&= \frac{1}{\epsilon}\mathscr{F}\{f\left(\frac{x}{\epsilon}\right)\} \\
&= \frac{1}{\epsilon}[\epsilon \hat{f}(k\epsilon)] \\
&= \hat{f}(k\epsilon).
\end{aligned}$$

Now

$$\lim_{\epsilon\to 0} \hat{f}_\epsilon(k) = \lim_{\epsilon\to 0} \hat{f}(k\epsilon)$$

and for any fixed k,[1]

$$\lim_{\epsilon\to 0} \hat{f}_\epsilon(k) = \hat{f}(0) = 1.$$

This is the Fourier Transform

$$\mathscr{F}\{\delta(x)\} = 1.$$

[1]Note that we assume the Fourier Transform is continuous, which can be proven.