Diffusion of Heat in a Disc

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There are some problems that have very different properties in odd and even dimensions. Diffusion of heat is one such problem.

Consider a disc with an axisymmetric heat distribution.

$$
\begin{aligned}
\text{DE: } & u_t = D\nabla^2 u = D(u_{rr} + \frac{1}{r}u_r) \qquad r < a \\
\text{BC: } & u(a,t) = 0 \\
& u(0,t) \text{ is bounded} \\
\text{IC: } & u(r,0) = f(r)
\end{aligned}
$$

Example distribution drawn on the board. Separate variables.

$$u(r,t) = R(r)T(t)$$

$$\text{DE} : u_t = D\nabla^2 u \implies RT_t = DT(R_{rr} + \frac{1}{r}R_r)$$

Divide by DRT

$$\frac{T_t}{DT} = \frac{R_{rr} + \frac{1}{r}R_n}{R} = -\lambda.$$

First solve the T-equation

$$
\begin{aligned}
T_t + \lambda DT &= 0 \\
T(t) &= e^{-\lambda Dt}
\end{aligned}
$$

and then the R-equation

$$R_{rr} + \frac{1}{r}R_r + \lambda R = 0 \qquad 0 \leq r < a$$

$$
\begin{aligned}
\text{BC's: } & u(a,t) = R(a)T(t) = 0 \implies R(a) = 0 \\
& u(0,t) \text{ is bounded} \implies R(0) \text{ bounded}
\end{aligned}
$$

I'd like a countable set of eigenvalues $\{\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots\}$ and associated eigenfunctions $\{R_n(r)\}$ and an orthogonality condition. From the HW, I can rewrite the DE in S-L form. Multiply by r

$$rR_{rr} + R_r + \lambda rR = 0.$$

Then

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \lambda rR = 0$$

and $R(a) = 0$, $R(0)$ bounded. This is **Bessel's Equation of order zero**.

1 Bessel's Equation

Bessels's Equation is

$$R_{rr} + \frac{1}{r}R_r + (\lambda - \frac{n^2}{r^2})R = 0,$$

where n is the $o$rder - it is most commonly an integer.[1] Our equation is of order zero - that is $n = 0$.

The solution is

$$R(r) = aJ_0(\sqrt{\lambda}r) + b\mathbb{Y}_0(\sqrt{\lambda}r)$$

where a and b are arbitrary constants, and J_0 is a **Bessel function of the first kind**. Cut to Maple Notebook. Note that

$$J_n(x) \sim \left(\frac{x}{2}\right)^n / n!$$

and

$$\mathbb{Y}_n(x) \to -\infty$$

as $x \to 0$.[2]

1.0.1 Boundary Conditions

Note that as $r \downarrow 0$, $\mathbb{Y}_0'(r) \to -\infty$, so $b = 0$. Also

$$R(a) = AJ_0(\sqrt{\lambda}a) = 0$$

So $\sqrt{\lambda}a$ imust be a zero of the Bessel function. Note J_0 has a countable sequence of zeroes $0 < \alpha_1 < \alpha_2 < \alpha_3 \cdots < \alpha_n \cdots$. Set

$$\sqrt{\lambda_n}a = \alpha_n \qquad n = 1, 2, 3, \ldots$$

and

$$\lambda_n = \left(\frac{\alpha_n}{a}\right)^2.$$

From Maple,

$$\alpha_1 = 2.4, \alpha_2 = 5.52, \alpha_3 = 8.65, \ldots$$

as $n \to \infty$, $\alpha_n \approx n\pi$.

So summarizing

$$\lambda_n = \left(\frac{\alpha_n}{a}\right)^2, R_n(r) = J_0(\sqrt{\lambda_n}r) = J_0\left(\frac{\alpha_n r}{a}\right)$$

Now

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-D\lambda_n t} R_n(r)$$

$$u(r,t) = \sum_{n=1}^{\infty} c_n e^{-D(\frac{\alpha_n}{n})^2 t} J_0(\alpha_n \frac{r}{a}).$$

I determine from the I.C.

$$u(r,0) = f(r) = \sum_{n=1}^{\infty} c_n J_0(\frac{\alpha_n r}{a}).$$

[1] When n is not an integer we have Ary functions, which are related to transmission coefficients in physics.

[2] The behavior of how this approaches 0 is highly dependent on n.

I need an orthogonality condition

$$\begin{aligned} \langle R_n(r), R_m(r)\rangle_r &= 0 \qquad n \neq m \\ \int_0^a R_n(r)R_m(r)r\ dr &= 0 \qquad n \neq m. \end{aligned}$$

Here we have used a weighted inner product with weight r coming from the coefficient r in $\lambda r R$ in Bessel's Equation of order zero. We can now use this to find the c_ns.

$$\begin{aligned} \langle R_n(r), (r)\rangle_r &= \sum_{n=1}^{\infty} c_n \langle R_m(r), R_n(r)\rangle_r \\ &= c_m \langle R_m(r), R_m(r)\rangle_r \\ c_m &= \frac{\langle R_m(r), f(r)\rangle}{\langle R_m(r), R_m(r)\rangle} \\ &= \frac{\int_0^a f(r) J_0(\alpha_m \frac{r}{a}) r\ dr}{\int_0^a [J_0(\frac{\alpha_m r}{a})]^2 r\ dr}. \end{aligned}$$

Now we look at solutions in Maple... **IN 3D!**