# Werkcollege 13

In these assignments you practice your ability to prove properties related to algorithms and to write down these proofs.

13.1. Consider a town with $n$ men and $n$ women seeking to get married to one another. Each man has a preference list that ranks all the women, and each woman has a preference list that ranks all the men.

The set of all $2n$ people is divided into two categories: *good* people and *bad* people. Suppose that for some number $k$, $1 \leq k \leq n-1$, there are $k$ good men and $k$ good women; thus there are $n-k$ bad men and $n-k$ bad women.

Everyone would rather marry any good person than any bad person. Formally, each preference list has the property that it ranks each good person of the opposite gender higher than each bad person of the opposite gender: its first $k$ entries are the good people (of the opposite gender) in some order, and its next $n-k$ are the bad people (of the opposite gender) in some order.

**(a)** Show that there **exists** a stable matching in which every good man is married to a good woman.

**(b)** Show that in **every** stable matching, every good man is married to a good woman.

13.2. We have a connected graph $G = (V, E)$, and a specific vertex $u \in V$. Suppose we compute a depth-first search tree rooted at $u$, and obtain the spanning tree $T$. Suppose we then compute a breadth-first search tree rooted at $u$, and obtain the same spanning tree $T$. Prove that $G = T$. (In other words, if $T$ is both a depth-first search tree and a breadth-first search tree rooted at $u$, then $G$ cannot contain any edges that do not belong to $T$.)

13.3. Some friends of yours work on wireless networks, and they're currently studying the properties of a network of $n$ mobile devices. As the devices move around (really, as their human owners move around), they define a graph at any point in time as follows: there is a node representing each of the $n$ devices, and there is an edge between device $i$ and device $j$ if the physical locations of $i$ and $j$ are no more than 500 meters apart. (If so, we say that $i$ and $j$ are "in range" of each other.)

They'd like it to be the case that the network of devices is connected at all times, and so they've constrained the motion of the devices to satisfy the following property: at all times, each device $i$ is within 500 meters of at least $n/2$ of the other devices. (We'll assume $n$ is an even number.) What they'd like to know is: Does this property by itself guarantee that the network will remain connected?

Here's a concrete way to formulate the question as a claim about graphs:

> *Claim: Let $G$ be a graph on $n$ nodes, where $n$ is an even number. If every node of $G$ has degree at least $n/2$, then $G$ is connected.*

Decide whether you think the claim is true or false, and give a proof of either the claim or its negation.

13.4. Suppose you're running a lightweight consulting business — just you, two associates, and some rented equipment. Your clients are distributed between the East Coast and the West Coast, and this leads to the following question.

Each month, you can either run your business from an office in New York (NY), or from an office in San Francisco (SF). In month $i$, you'll incur an *operating cost* of $N_i$ if you run the business out of NY; you'll incur an operating cost of $S_i$ if you run the business out of SF. (It depends on the distribution of client demands for that month.)

However, if you run the business out of one city in month $i$, and then out of the other city in month $i+1$, then you incur a fixed *moving cost* of $M$ to switch base offices.

Given a sequence of $n$ months, a *plan* is a sequence of $n$ locations — each one equal to either NY or SF — such that the $i^{\text{th}}$ location indicates the city in which you will be based in the $i^{\text{th}}$ month. The *cost* of a plan is the sum of the operating costs for each of the $n$ months, plus a moving cost of $M$ for each time you switch cities. The plan can begin in either city.

**The problem is:** Given a value for the moving cost $M$, and sequences of operating costs $N_1, \ldots, N_n$ and $S_1, \ldots, S_n$, find a plan of minimum cost. (Such a plan will be called *optimal.*)

**Example.** Suppose $n = 4$, $M = 10$, and the operating costs are given by the following table.

| | Month 1 | Month 2 | Month 3 | Month 4 |
|---|---|---|---|---|
| NY | 1 | 3 | 20 | 30 |
| SF | 50 | 20 | 2 | 4 |

Then the plan of minimum cost would be the sequence of locations

$$[NY, NY, SF, SF],$$

with a total cost of $1 + 3 + 2 + 4 + 10 = 20$, where the final term of 10 arises because you change locations once.

**(a)** Show that the following algorithm does not correctly solve this problem, by giving an instance on which it does not return the correct answer.

> For $i = 1$ to $n$ If $N_i < S_i$ then Output "NY in Month $i$"
> Else Output "SF in Month $i$" End

In your example, say what the correct answer is and also what the above algorithm finds.

**(b)** Give an example of an instance in which every optimal plan must move (i.e. change locations) at least three times.

Provide an explanation, of at most three sentences, saying why your example has this property.

**(c)** Give an algorithm that takes values for $n$, $M$, and sequences of operating costs $N_1, \ldots, N_n$ and $S_1, \ldots, S_n$, and returns the *cost* of an optimal plan.

The running time of your algorithm should be polynomial in $n$. You should prove that your algorithm works correctly, and include a brief analysis of the running time.