Assignment week 2 - Big O Notation

Assume that T(1) = 1

1. $T(n) = T(\frac{n}{2}) + 1$

   - Assume that: $2^k = n \to 2^{-k}n = 1 \to k = \log_2 n$
     $T(n) = T(\frac{n}{2}) + 1 = (T(\frac{n}{4}) + 1) + 1 = T(2^{-2}n) + 2 = T(2^{-3}n) + 3 = ... = T(2^{-k}n) + k = T(1) + \log_2 n = 1 + \log_2 n$
   - With $C = 1 + \log_2 e,\ f(n) = \log n,\ n_0 = e \to T(n) \leq C.f(n),\ \forall n \geq n_0$ because of:
     - $1 = \log e \leq \log n\ \forall n \geq e$
     - $\log_2 n = \log_2 e. \log n$

     $\to T(n) = 1 + \log_2 n \leq \log n + \log_2 e. \log n = (1 + \log_2 e) \log n,\ \forall n \geq e$

   So, $T(n) = O(\log n)$

2. $T(n) = 2T(\frac{n}{2}) + n$

   - Assume that: $2^k = n \to 2^{-k}n = 1 \to k = \log_2 n$
     $T(n) = 2T(\frac{n}{2}) + n = 2(2T(\frac{n}{4}) + \frac{n}{2}) + n = 2^2 T(2^{-2}n) + 2n = 2^3 T(2^{-3}n) + 3n = ... = 2^k T(2^{-k}n) + kn = nT(1) + n \log_2 n = n + n \log_2 n$
   - With $C = 1 + \log_2 e,\ f(n) = n \log n,\ n_0 = e \to T(n) \leq C.f(n),\ \forall n \geq n_0$ because of:
     - $n = n.1 \leq n \log n,\ \forall n \geq e$
     - $\log_2 n = \log_2 e. \log n$

     $\to T(n) = n + n \log_2 n \leq n \log n + n \log_2 e. \log n = (1 + \log_2 e) n \log n,\ \forall n \geq e$

   So, $T(n) = O(n \log n)$

3. $T(n) = 3T(\frac{n}{2}) + \log n$

   - Asumme that: $2^k = n \to 2^{-k}n = 1 \to k = \log_2 n$
     $T(n) = 3T(\frac{n}{2}) + \log n = 3(3T(\frac{n}{4}) + \log \frac{n}{2} + \log n = 3^2 T(2^{-2}n) + \log n + 3 \log 2^{-1} n = ... = 3^k T(2^{-k}n) + \sum_{i=0}^{k-1} 3^i \log 2^{-i} n \leq 3^k T(2^{-k}n) + \sum_{i=0}^{k} 3^i \log 2^{-i} n$
   - We have $2^{-k}n = 1$ and $k = \log_2 n$, and $\log x < x,\ \forall x > 0$ so:
     $T(n) \leq 3^{\log_2 n} T(1) + \sum_{i=0}^{\log_2 n} 3^i 2^{-i} n \leq n^{\log_2 3} + \sum_{i=0}^{\log_2 n} 3^i (2^{-i} n)^{\log_2 3 - 0.5}$
     $= n^{\log_2 3} + n^{\log_2 3 - 0.5} \sum_{i=0}^{\log_2 n} 3^i 2^{-i \log_2 3} 2^{0.5i} = n^{\log_2 3} + n^{\log_2 3 - 0.5} \sum_{i=0}^{\log_2 n} 3^i 3^{-i} 2^{0.5i}$
     $= n^{\log_2 3} + n^{\log_2 3 - 0.5} \sum_{i=0}^{\log_2 n} 2^{0.5i} = n^{\log_2 3} + n^{\log_2 3 - 0.5} \frac{2^{0.5(\log_2 n + 1)} - 1}{2^{0.5} - 1} \leq n^{\log_2 3} + n^{\log_2 3 - 0.5} \frac{2^{0.5(\log_2 n + 1)}}{2^{0.5} - 1}$
     $= n^{\log_2 3} + n^{\log_2 3 - 0.5} \frac{n^{0.5} 2^{0.5}}{2^{0.5} - 1}$
     $= n^{\log_2 3} + \frac{\sqrt{2}}{\sqrt{2} - 1} n^{\log_2 3} = \frac{2\sqrt{2} - 1}{\sqrt{2} - 1} n^{\log_2 3}$
   - With $C = \frac{2\sqrt{2} - 1}{\sqrt{2} - 1},\ f(n) = n^{\log_2 3},\ n_0 = 1 \to T(n) \leq C.f(n),\ \forall n \geq n_0$

   So, $T(n) = O(n^{\log_2 3})$