Define Torque and moment of inertia. Drive a reletion for rotational K.E of body.

first part:-

Moment of Intertia:-

The inability of a body to change its state of rest or of uniform linear motion by itself is called inertia.

In linear motion mass is the measure of inertia. The quantity which plays the same role in rotational motion as mass does in linear motion is called moment of Inertia. It is denoted by I.

Mathematically, $I = \sum_{i=1}^{n} m_i r_i^2$

where,

m_i = mass of individual constituent particles of a rigid body.

r_i = distance of constituent particles to the axis of rotation.

Torque is the vector (cross) product of force and perpendicular distance from the axis of rotation to the line of force. It is denoted by τ.

Mathematically,

$$\vec{\tau} = \vec{r} \times \vec{F}$$ and

$$\tau = I\alpha$$

where, I = moment of Intertia

α = angular acceleration.

⇒ Second Part:-

In linear motion

Small workdone $(dw) = Fdx$

Similarly, In rotational motion (dw) =

Small workdone $(dw) = \tau d\theta$

$\therefore dw = I\alpha d\theta$

Angular acceleration $(\alpha) = \frac{d\omega}{dt}$

$$so, \; dw = I \frac{d\omega}{dt} d\theta$$

Angular velocity $(\omega) = \frac{d\theta}{dt}$

$$\therefore d\theta = \omega dt$$

Now,

$$dw = I \frac{d\omega}{dt} \cdot \omega dt$$

$$\therefore dw = I\omega d\omega$$

Total workdone by a torque (τ) when its angular velocity changes from ω_1 to ω_2.

$$w = \int_{\omega_1}^{\omega_2} dw$$

$$= \int_{\omega_1}^{\omega_2} I\omega d\omega$$

$$= I\left[\frac{\omega^2}{2}\right]_{\omega_1}^{\omega_2}$$

$$= \frac{1}{2} I \left[\omega_2^2 - \omega_1^2\right]$$

$$\therefore W = \frac{1}{2}\left[I\omega_2^2 - I\omega_1^2\right]$$ which is the required expression for rotational Kinetic energy of a body.

2. Define angular ~~vel~~ momentum. Show that the angular momentum is conserved if there is no applied external torque

→ First part:-

Angular momentum is product of moment of inertia and angular velocity. It is denoted by L.

Mathematically,

$$L = I\omega$$

and $$L = r \times p$$

where,

$$p = \text{linear momentum}$$

$$\boxed{\therefore L = r \times mv}$$

⇒ Second part:-

From Newton's second law of motion the force is defined as the rate of change of momentum with time.

$$F = \frac{d(mv)}{dt}$$

Taking cross product of both sides.

$$r \times F = r \times \frac{d(mv)}{dt}$$

we know,

$$\tau = r \times F$$

$$\therefore \tau = r \times \frac{d(mv)}{dt}$$

$$\frac{d(r \times mv)}{dt} = \frac{dr}{dt} \times mv + r \cdot \frac{d(mv)}{dt}$$

The first term on R.H.S is zero because $\frac{dr}{dt}$ and v have same velocity.

$$\therefore r \times \frac{d(mv)}{dt} = \frac{d(r \times mv)}{dt}$$

$$\therefore \tau = \frac{d(r \times mv)}{dt}$$

For linear motion,

$$F = \frac{dp}{dt}$$

Similarly, In rotational motion,

$$\tau = \frac{dL}{dt}$$

where,

$$L = r \times mv$$

Now,

$$\tau = \frac{dL}{dt}$$

if no external torque act on it.

$$\tau = 0$$

$$\frac{dL}{dt} = 0$$

$$\therefore L = \text{constant}$$

$$I\omega = \text{constant}$$

If I_1 and I_2 are initial and final moment of inertia and ω_1 & ω_2 are initial & final angular velocity of a body.

$$\boxed{I_1 \omega_1 = I_2 \omega_2}$$

which is the required expression for conservation of angular momentum.

Hence angular momentum is conserved if no applied external torque.

Define periodic motion. Explain the oscillation of horizontal mass spring system.

The motion which repeats itself after equal interval of time is called periodic motion. The interval of time is called the time period of periodic motion.

For ex:- The motion of pendullum clock of wall, oscillation of mass sucpended from spring, othe motion of planet around the sun, the motion of hands of the clock.

Second part:-

Suppose a body of mass (m) is suspended to a massless spring with spring constant (K) and the body is free to oscillate on a frictionless surface as shown in fig (i).

At rest or equilibrium position, the position co-ordinate is $x=0$.

If the body is pushed to compress the distance x. as shown in fig (ii) and pulled to stretch its distance x as shown in fig(iii).

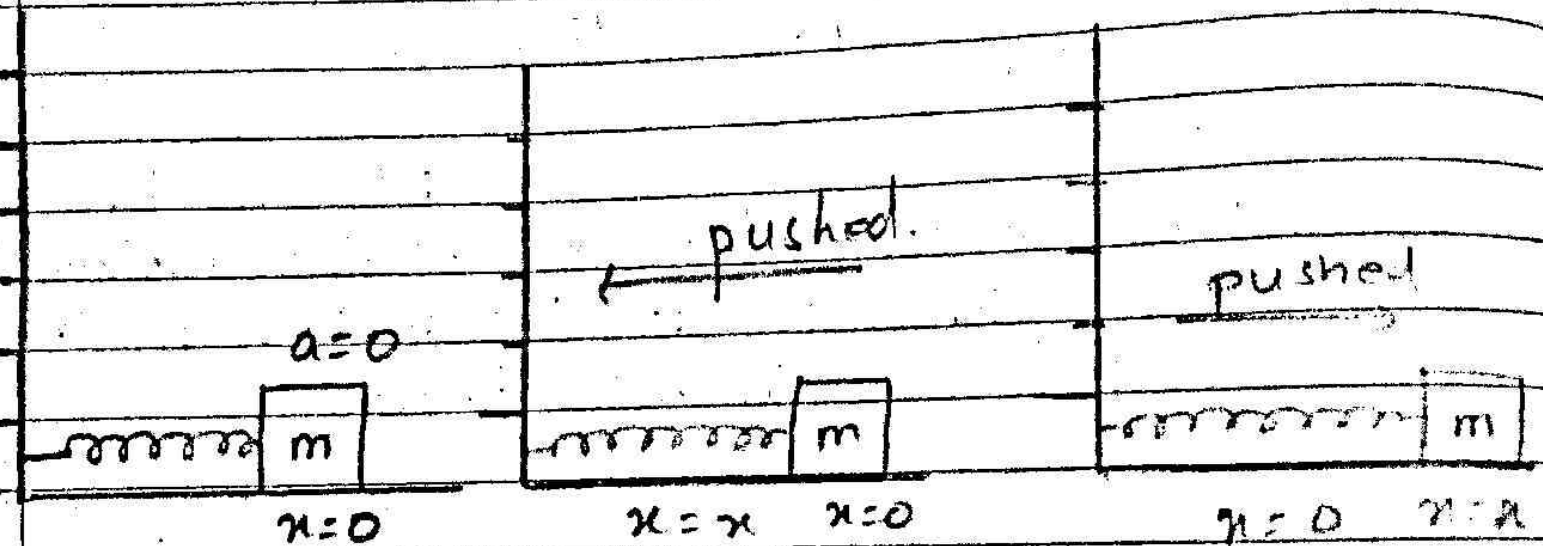


fig (i)   fig ii)   fig iii

Let (F) be the applied force (restoring force) then from Hook's law for a spring :

$$F \propto x \quad \therefore F = -Kx \quad \text{——} (i)$$

the negative sign indicates that the restoring force acts opposite to the displacement. The mass (m) keeps on oscillating horizontally about the mean position $x=0$. If 'a' be the acceleration produced on the mass then from Newton's Second law:

$$f = ma \quad \text{—— (ii)}$$

equating eqn (i) & (ii).

$$ma = -kx$$

$$a = -\left(\frac{k}{m}\right) x \quad \text{—— (iii)}$$

for a mass spring system $\left(\frac{k}{m}\right)$ is constant.

$$\therefore a \propto -x \quad \text{—— (iv)}$$

Eqn (iv) shows that, for a horizontal mass spring system acceleration is directly proportional to the displacement and negative sign shows that it is always directed towards the mean position. So motion of a horizontal mass spring system is simple harmonic motion (SHM). Since for simple harmonic motion,

$$a = -\omega^2 x \quad \text{—— (v)}$$

Comparing (iv) & (v),

$$\omega^2 = \frac{k}{m}$$

$$\text{or, } \omega = \sqrt{\frac{k}{m}}$$

$$\text{or, } \frac{2\pi}{T} = \sqrt{k/m}$$

$$\boxed{\therefore T = 2\pi\sqrt{\frac{k}{m}}} \quad \text{—— (vi)}$$

Eqn (vi) is expression for the time period of oscillation in horizontal mass spring system. It shows that the time period depends upon mass and spring constant (k).

What is simple harmonic motion? Show that in SHM the kinetic energy and potential energy vary time but the total energy remains constant.

A SHM is defined as an oscillatory motion about a fixed position (mean position) in which the restoring force is always directly proportional to the displacement from that point position) and is always directed towards that position.

If x be small displacement from the mean position of the particle of the particle and F be the restoring force acting on it then,

$$F \propto x$$
$$F = -Kx \quad \text{—} (i)$$

where,

K = proportionality constant the negative sign indicates that restoring force F is developed opposite to the displacement from the mean position.

From Newton second law of motion,

$$F = ma \quad \text{—} (ii)$$

where,

m = mass of the particle and

a = acceleration.

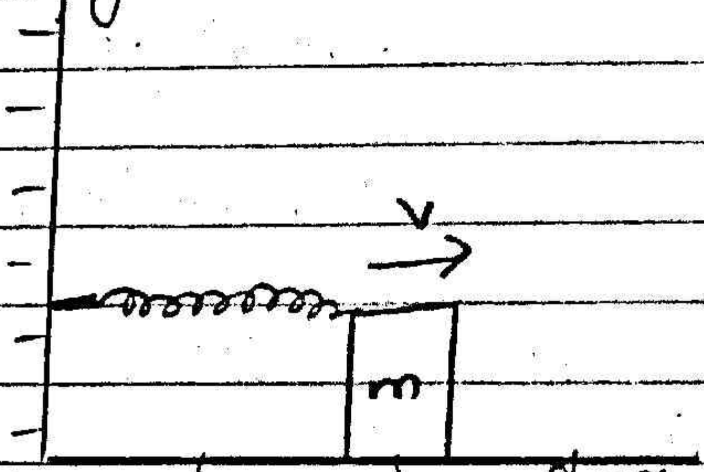
or, $ma = -Kx$.

$\therefore a = \left(-\frac{K}{m}\right)x$.

shows that acceleration & SHM is always directly propotional to the displacement from the mean position and negative sign indicates that it is always directed towards the mean position.

Second part:-

The energy of linear oscillator transfers back and forth between kinetic energy & potential energy. Let us consider a spring block system as shown in fig (i).



Kinetic energy of a system (KE) $= \frac{1}{2} mv^2$

$$= \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 \quad \text{--- (i)}$$

we have, from periodic motion,

$$x = r\sin(\omega t + \phi) \quad \text{--- (ii)}$$

where x = displacement of body.

r = maximum displacement.

ω = angular velocity

t = instataneous times

ϕ = initial phase angle.

Differentiating (ii) w.r.t. 't'

$$\frac{dx}{dt} = \omega r \cos(\omega t + \phi)$$

Now, eqn (i) becomes

$$K.E = \frac{1}{2} m \{\omega r \cos(\omega t + \phi)^2\}$$

$$= \frac{1}{2} m \omega^2 r^2 \cos^2(\omega t + \phi)$$

$$= \frac{1}{2} m \omega^2 r^2 (1 - \sin^2(\omega t - \phi))$$

$$= \frac{1}{2} m \omega^2 \{r^2 - r^2 \sin^2(\omega t - \phi)\}$$

$$= \frac{1}{2} m \omega^2 (r^2 - x^2) \quad \text{(iii)}$$

Eqn (iii) is the expression for kinetic energy of system.

The potential energy of the system is given by amount of workdone to move the system from position 0 to x by applying force F.

$$P.E = -\int_0^x F\,dx$$

$$= -\int_0^x (-kx)\,dx$$

$$= k \int_0^x x\,dx$$

$$= \frac{kx^2}{2}$$

From eqn (iii)

$$P.E = \frac{1}{2}K\{r\sin(\omega t+\phi)\}^2$$

$$= \frac{1}{2}kr^2\sin^2(\omega t+\phi)$$

From SHM,

$$\omega^2 = \frac{K}{m}$$

$$\therefore K = m\omega^2 \quad \text{—— (iv)}$$

eqn (iv) is expression for P.E of the system.

Total mechanical energy(E) = K.E + P.E

$$= \frac{1}{2}m\omega^2(r^2-x^2) + \frac{1}{2}m\omega^2x^2$$

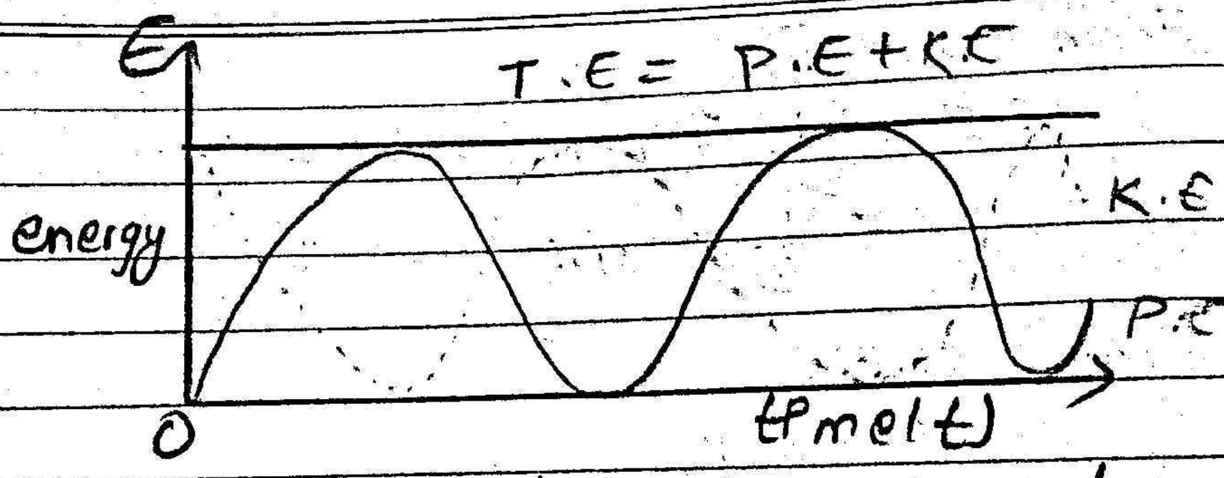
$$= \frac{1}{2}m\omega^2\cancel{x^2} - \frac{1}{2}m\cancel{\omega^2}x^2 +$$

$$\frac{1}{2}m\omega^2r^2$$

$$= \frac{1}{2}m\omega^2r^2 \quad \text{—— (v).}$$

eqn (v) is the expression of total K.E of the system. It is independent of time but potential energy and kinetic energy depends explicitly on time.

eqn (5) shows that total mechanical energy of the system remains constant.



fig(iii) P.E, KE and total mechanical energy as function of time