The final, low-entropy sum is the famous Basel sum (high-entropy results are not often famous). Its value is B $= \pi^2/6$ (Problem 6.22).

▷ *How does knowing* B $= \pi^2/6$ *help evaluate the original sum* $\sum_1^\infty (2n+1)^{-2}$*?*

The major modification from the original sum was to include the even squared reciprocals. Their sum is B/4.

$$\sum_1^\infty \frac{1}{(2n)^2} = \frac{1}{4}\sum_1^\infty \frac{1}{n^2} \tag{6.26}$$

The second modification was to include the n $= 0$ term. Thus, to obtain $\sum_1^\infty (2n+1)^{-2}$, adjust the Basel value B by subtracting B/4 and then the n $= 0$ term. The result, after substituting B $= \pi^2/6$, is

$$\sum_1^\infty \frac{1}{(2n+1)^2} = B - \frac{1}{4}B - 1 = \frac{\pi^2}{8} - 1 \tag{6.27}$$

This exact sum, based on the asymptote approximation for x_n, produces the following estimate of S.

$$S \approx \frac{4}{\pi^2}\sum_1^\infty \frac{1}{(2n+1)^2} = \frac{4}{\pi^2}\left(\frac{\pi^2}{8} - 1\right) \tag{6.28}$$

Simplifying by expanding the product gives

$$S \approx \frac{1}{2} - \frac{4}{\pi^2} = 0.094715\ldots \tag{6.29}$$

Problem 6.25 Check the earlier reasoning

Check the earlier pictorial reasoning (Problem 6.24) that $1/6 + 1/18 = 2/9$ underestimates $\sum_1^\infty (2n+1)^{-2}$. How accurate was that estimate?

This estimate of S is the third that uses the asymptote approximation $x_n \approx (n+0.5)\pi$. Assembled together, the estimates are

$$S \approx \begin{cases} 0.067547 & \text{(integral approximation to } \sum_1^\infty (2n+1)^{-2}), \\ 0.090063 & \text{integral approximation and triangular overshoots)}, \\ 0.094715 & \text{exact sum of } \sum_1^\infty (2n+1)^{-2}), \end{cases}$$

Because the third estimate incorporated the exact value of $\sum_1^\infty (2n+1)^{-2}$, any remaining error in the estimate of S must belong to the asymptote approximation itself.

▷ *For which term of $\sum x_n^{-2}$ is the asymptote approximation most inaccurate?*

As x grows, the graphs of x and tan x intersect ever closer to the vertical asymptote. Thus, the asymptote approximation makes its largest absolute error when n = 1. Because x_1 is the smallest root, the fractional error in x_n is, relative to the absolute error in x_n, even more concentrated at n = 1. The fractional error in x_n^{-2} being −2 times the fractional error in x_n (Section 5.3), is equally concentrated at n = 1. Because x_n^{-2} is the largest at n = 1, the absolute error in x_n^{-2} (the fractional error times x_n^{-2} itself) is, by far, the largest at n = 1.

Problem 6.26 Absolute error in the early terms

Estimate, as a function of n, the absolute error in x_n^{-2} that is produced by the asymptote approximation.

With the error so concentrated at n = 1, the greatest improvement in the estimate of S comes from replacing the approximation $x_1 = (n + 0.5)\pi$ with a more accurate value. A simple numerical approach is successive approximation using the Newton–Raphson method (Problem 4.38). To find a root with this method, make a starting guess x and repeatedly improve it using the replacement

$$x \longrightarrow x - \frac{\tan x - x}{\sec^2 x - 1} \tag{6.30}$$

When the starting guess for x is slightly below the first asymptote at 1.5π, the procedure rapidly converges to $x_1 = 4.4934\ldots$

Therefore, to improve the estimate $S \approx 0.094715$, which was based on the asymptote approximation, subtract its approximate first term (its big part) and add the corrected first term.

$$S \approx S_{old} - \frac{1}{(1.5\pi)^2} + \frac{1}{4.4934^2} \approx 0.09921. \tag{6.31}$$

Using the Newton–Raphson method to refine, in addition, the $1/x_2^2$ term gives $S \approx 0.09978$ (Problem 6.27). Therefore, a highly educated guess is

$$S = \frac{1}{10} \tag{6.32}$$

The infinite sum of unknown transcendental numbers seems to be neither transcendental nor irrational! This simple and surprising rational number deserves a simple explanation.

Problem 6.27 Continuing the corrections
Choose a small N, say 4. Then use the Newton–Raphson method to compute accurate values of x_n for n=1 ... N; and use those values to refine the estimate of S. As you extend the computation to larger values of N, do the refined estimates of S approach our educated guess of 1/10?

6.4.3 Analogy with polynomials

If only the equation $\tan x - x = 0$ had just a few closed-form solutions! Then the sum S would be easy to compute. That wish is fulfilled by replacing $\tan x - x$ with a polynomial equation with simple roots. The simplest interesting polynomial is the quadratic, so experiment with a simple quadratic — for example, $x^2 - 3x + 2$.

This polynomial has two roots, $x_1 = 1$ and $x_2 = 2$; therefore $\sum x_n^{-2}$, the polynomial-root sum analog of the tangent-root sum, has two terms.

$$\sum x_n^{-2} = \frac{1}{1^2} + \frac{1}{2^2} = \frac{5}{4}. \tag{6.33}$$

This brute-force method for computing the root sum requires a solution to the quadratic equation. However, a method that can transfer to the equation $\tan x - x = 0$, which has no closed-form solution, cannot use the roots themselves. It must use only surface features of the quadratic—namely, its two coefficients 2 and −3. Unfortunately, no plausible method of combining 2 and −3 predicts that $\sum x_n^{-2} = 5/4$.

▷ *Where did the polynomial analogy go wrong?*

The problem is that the quadratic $x^2 - 3x + 2$ is not sufficiently similar to $\tan x - x$. The quadratic has only positive roots; however, $\tan x - x$, an odd function, has symmetric positive and negative roots and has a root at x = 0. Indeed, the Taylor series for $\tan x$ is $x + x^3/3 + 2x^5/15 + \ldots$ (Problem 6.28); therefore,

$$\tan x - x = \frac{x^3}{3} + \frac{2x^5}{15} + \ldots \tag{6.34}$$

The common factor of x^3 means that $\tan x - x$ has a triple root at x = 0.An analogous polynomial — here, one with a triple root at x = 0, a positive root, and a symmetric negative root—is $(x+2)x^3(x-2)$ or, after expansion, $x^5 - 4x^3$. The sum $\sum x_n^{-2}$(using the positive root) contains only one term and is simply 1/4. This value could plausibly arise as the (negative) ratio of the last two coefficients of the polynomial.

To decide whether that pattern is a coincidence, try a richer polynomial: one with roots at −2, −1, 0 (threefold), 1, and 2. One such polynomial is

$$(x+2)(x+1)x^3(x-1)(x-2) = x^7 - 5x^5 + 4x^3. \tag{6.35}$$

The polynomial-root sum uses only the two positive roots 1 and 2 and is $1/1^2 + 1/2^2$, which is 5/4 — the the (negative) ratio of the last two coefficients. As a final test of this pattern, include −3 and 3 among the roots. The resulting polynomial is

$$(x^7 - 5x^5 + 4x^3)(x+3)(x-3) = x^9 - 14x^7 + 49x^5 - 36x^3. \tag{6.36}$$

The polynomial-root sum uses the three positive roots 1, 2, and 3 and is $1/1^2 + 1/2^2 + 1/3^2$, which is 49/36 — again the (negative) ratio of the last two coefficients in the expanded polynomial.

▷ *What is the origin of the pattern, and how can it be extended to* $\tan x - x$*?*

To explain the pattern, tidy the polynomial as follows:

$$x^9 - 14x^7 + 49x^5 - 36x^3 = -36x^3\left(1 - \frac{49}{36}x^2 + \frac{14}{36}x^4 - \frac{1}{36}x^6\right). \tag{6.37}$$

In this arrangement, the sum 49/36 appears as the negative of the first interesting coefficient. Let's generalize. Placing k roots at x = 0 and single roots at $\pm x_1, \pm x_2, \ldots, \pm x_n$ gives the polynomial

$$Ax^k\left(1 - \frac{x^2}{x_1^2}\right)\left(1 - \frac{x^2}{x_2^2}\right)\left(1 - \frac{x^2}{x_3^2}\right)\cdots\left(1 - \frac{x^2}{x_n^2}\right), \tag{6.38}$$

where A is a constant. When expanding the product of the factors in parentheses, the coefficient of the x^2 term in the expansion receives one contribution from each x^2/x_k^2 term in a factor. Thus, the expansion begins

$$Ax^k\left[1 - \left(\frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2} + \ldots + \frac{1}{x_n^2}\right)x^2 + \ldots\right]. \tag{6.39}$$

The coefficient of x^2 in parentheses is $\sum x_n^{-2}$, which is the polynomial analog of the tangent-root sum.

Let's apply this method to $\tan x - x$. Although it is not a polynomial, its Taylor series is like an infinite-degree polynomial. The Taylor series is

$$\frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \ldots = \frac{x^3}{3}\left(1 + \frac{2}{5}x^2 + \frac{17}{105}x^4 + \ldots\right). \tag{6.40}$$