1 1.1 Solutions

1. With $S_1 = \{2,3,5,7\}$, $S_2 = \{2,4,5,8,9\}$ and $U = \{1:10\}$ compute $\overline{S_1} \bigcup S_2$.

   Solution: $\overline{S_1} \bigcup S_2 = \{1,2,4,5,6,8,9,10\}$.

2. With $S_1 = \{2,3,5,7\}$ and $S_2 = \{2,4,5,8,9\}$, compute $S_1 \times S_2$ and $S_2 \times S_1$.

   Solution: $S_1 \times S_2 = \{(2,2),(2,4),(2,5),(2,8),(2,9),(3,2),(3,4),(3,5),(3,8),(3,9),$ $(5,2),(5,4),(5,5),(5,8),(5,9),(7,2),(7,4),(7,5),(7,8),(7,9)\}$.

   $S_2 \times S_1 = \{(2,2),(4,2),(5,2),(8,2),(9,2),(2,3),(4,3),(5,3),(8,3),(9,3),$ $(2,5),(4,5),(5,5),(8,5),(9,5),(2,7),(4,7),(5,7),(8,7),(9,7)\}$.

3. For $S = \{2,5,6,8\}$ and $T = \{2,4,6,8\}$, compute $|S \cap T| + |S \cup T|$.

   Solution: $|S \cap T| + |S \cup T| = 3 + 5 = 8$.

4. What relation between two sets S and T must hold so that $|S \cup T| = |S| + |T|$.

   Solution: S and T must be two disjoint sets.

5. ** Show that for all sets S and T, $S - T = S \cap \overline{T}$.

   Solution: Suppose $x \in S - T$. Then $x \in S$ and $x \notin T$, which means $x \in S$ and $x \in \overline{T}$, that is $x \in S \cap \overline{T}$. So $S - T \subseteq S \cap \overline{T}$. Conversely, if $x \in S \cap \overline{T}$, then $x \in S$ and $x \notin T$, which means $x \in S - T$. That is $S \cap \overline{T} \subseteq S - T$. Therefore $S - T = S \cap \overline{T}$.

6. ** Prove De Morgan's laws, Equations (1.2) and (1.3) by showing that an element x is in the set on one side of the equality, then it must also be in the set on the other side.

   Solution: For Equation (1.2), suppose $x \in \overline{S_1 \cup S_2}$. Then $x \notin S_1 \cup S_2$, which means that x cannot be in S_1 or in S_2, that is $x \in \overline{S_1} \cap \overline{S_2}$. So $\overline{S_1 \cup S_2} \subseteq \overline{S_1} \cap \overline{S_2}$. Conversely, if $x \in \overline{S_1} \cap \overline{S_2}$, then x is not in S_1 and x is not in S_2, that is, $x \in \overline{S_1 \cup S_2}$. So $\overline{S_1} \cap \overline{S_2} \subseteq \overline{S_1 \cup S_2}$. Therefore $\overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2}$.

   Similar arguements apply for Equation (1.3).

7. Show that if $S_1 \subseteq S_2$ then $\overline{S_2} \subseteq \overline{S_1}$.

   Solution: Suppose $x \in \overline{S_2}$ which means x is not in S_2. Since S_1 is a subset of S_2, so x cannot be in S_1, that is $x \in \overline{S_1}$. So $\overline{S_2} \subseteq \overline{S_1}$.

8. Show that $S_1 = S_2$ if and only if $S_1 \cup S_2 = S_1 \cap S_2$.

   Solution: If $S_1 = S_2$, then obviously $S_1 \cup S_2 = S_1 \cap S_2 (= S_1 = S_2)$. Conversely, since $S_1 \cap S_2 \subseteq S_1 \subseteq S_1 \cup S_2$, so $S_1 \cup S_2 = S_1 \cap S_2$ means $S_1 = S_1 \cup S_2$. Ihe reasoning gives $S_2 = S_1 \cup S_2$. Therefore $S_1 = S_2$.

9. Use induction on the size of S to show that if S is a finite set, then $\left|2^S\right| = 2^{|S|}$.

   Solution: For $S = \oslash$, $2^S = \{\oslash\}$ which means $|S| = 0$ and $2^{|S|} = 1$. Suppose the conjecture, $\left|2^S\right| = 2^{|S|}$ is true for all sets S with $|S| = 0, 1, ..., n$.

   Let $S_{n+1} = S_n \cup \{x_{n+1}\}$ be a set of length $n + 1$ where $|S_n| = n$ with $2^{S_n} = \{S_{n,1}, ..., S_{n,2^n}\}$. Since the power set $2^{S_{n+1}} = 2^{S_n} \cup (\{S_{n,1} \cup \{x_{n+1}\}, ..., \{S_{n,2^n} \cup \{x_{n+1}\}\})$, so $\left|2^{S_{n+1}}\right| = \left|2^{S_n}\right| + \left|2^{S_n}\right| = 2\left|2^{S_n}\right| = 2(2^n) = 2^{n+1}$. This completes the inductive proof.

10. Show that if S_1 and S_2 are finite sets with $|S_1| = n$ and $|S_2| = m$, then

$$|S_1 \cup S_2| \leq n + m.$$

   Solution: Since $S_1 \cup S_2 = S_1 \cup (S_2 - S_1)$ and $S_1 \cap (S_2 - S_1) = \oslash$, so $|S_1 \cup S_2| = |S_1| + |S_2 - S_1| \leq |S_1| + |S_2| = n + m$.

1 1.2

1. How many substrings aab are in ww^Rw, where $w = aabbab$?

   Solution: Since $ww^Rw = aabbabbabbaaaabbab$, there are only two substrings of aab.

2. ** Use induction on n to show that $|u^n| = n\,|u|$ for all strings u and all n.

   Solution: For $n = 1, |u^1| = |u|$. Assume $|u^k| = k|u|$ holds for $k = 1, 2, ..., n$, then

$$|u^{n+1}| = |u^n u| = |u^n| + |u| = n|u| + |u| = (n+1)|u|.$$

3. The reverse of a string, introduced informally above, can be defined more precisely by the recursive rules

$$\begin{aligned} a^R &= a, \\ (wa)^R &= aw^R, \end{aligned}$$

   for all $a \in \Sigma$, $w \in \Sigma^*$. Use this to prove that

$$(uv)^R = v^R u^R,$$

   for all $u, v \in \Sigma^+$.

   Solution: Many string identities can be proved by induction. Suppose that $(uv)^R = v^R u^R$ for all $u \in \Sigma^*$ and all v of length n. Take now a string of length $n+1$, say $w = va$. Then

$$\begin{aligned} (uw)^R &= (uva)^R \\ &= a\,(uv)^R, \text{ by the definition of the reverse} \\ &= av^R u^R, \text{ by the inductive assumption} \\ &= w^R u^R. \end{aligned}$$

   By induction then, the result holds for all strings.

4. Prove that $\left(w^R\right)^R = w$ for all $w \in \Sigma^*$.

Solution:

Let $w = w_1 w_2 ... w_n$ be an arbitrary string of length n where $w_1, w_2, ..., w_n$ are symbols in Σ. Then apply the result of Exercise 3 repeatedly, we have

$$w^R = (w_1 w_2 ... w_n)^R = w_n (w_1 w_2 ... w_{n-1})^R = w_n w_{n-1} (w_1 w_2 ... w_{n-2})^R$$

$$= ..... = w_n w_{n-1} ... w_1.$$

similarly

$$\left(w^R\right)^R = (w_n w_{n-1} ... w_1)^R = w_1 w_2 ... w_n = w.$$

5. Let $L = \{ab, aa, baa\}$. Which of the following strings are in L^*: $abaabaaabaa$, $aaaabaaaa$, $baaaaabaaaab$, $baaaaabaa$? Which strings are in L^4?

   Solution: Since $abaabaaabaa$ can be decomposed into strings ab, aa, baa, the string is in L^*. Similarly, $aaaabaaaa$ and $baaaaabaa$ are in L^*. However, there is no possible decomposition for $baaaaabaaaab$, so this string is not in L^*. The strings $aaaabaaaa$ and $baaaaabaa$ are in L^4.

6. ** Let $\Sigma = \{a, b\}$ and $L = \{aa, bb\}$. Use set notation to describe $\overline{L}$.

   Solution: $\overline{L} = \{\lambda, a, b, ab, ba\} \cup \{w \in \{a, b\}^+ : |w| \geq 3\}$.