Ejercicios - Modelado y simulación

2.1 En los siguientes tres ejercicios, interprete $\dot{x} = \sin x$ como un flujo sobre la línea

2.1.1 Encontrar todos los puntos fijos de el flujo.

$$0 = \sin x$$

$$\arcsin(0) = x$$

$$n\pi = x \qquad n \in \mathbb{Z}$$

2.1.2 ¿En qué puntos x el flujo tiene mayor velocidad hacia la derecha?

$$x = \frac{(2n+1)\pi}{2}$$

$$\dot{x} = \pm 1$$

2.1.3

a) Encuentre los flujos de aceleración $\ddot{x}$ como una función de x

$$\ddot{x} = \frac{d(\dot{x})}{dx} = \frac{d(\sin x)}{dx} = \cos x \cdot \dot{x} = \cos x \sin x$$

b) Encuentra los puntos donde el flujo tiene la aceleración máxima positiva

$$\frac{d\ddot{x}}{dx} = \frac{d(\cos x \sin x)}{dx} = \cos 2x = 0 \quad 2x = \frac{(2n+1)\pi}{2} \Rightarrow x = \frac{(2n+1)\pi}{4}$$

$$\frac{d(\cos 2x)}{dx} = -\sin(2x)\cdot 2 \qquad -\sin(2x)\cdot 2 > 0 \quad = \left\{\frac{\pi}{4}, \frac{5}{2}\pi, \dots\right\}$$

2.1.4

$$\dot{x} = \sin x$$

$$t = \ln |(\csc x_0 + \cot x_0)/(\csc x + \cot x)|$$

a) $x_0 = \frac{\pi}{4}$

$$t = \ln \left| \frac{\csc \frac{\pi}{4} + \cot \frac{\pi}{4}}{\csc x + \cot x} \right| = \ln \left| \frac{\sqrt{2} + 1}{\csc x + \cot x} \right|$$

$$e^t = \frac{1+\sqrt{2}}{\csc x + \cot x}$$

$$e^t = \frac{1+\sqrt{2}}{}$$

$$\frac{e^t}{1+\sqrt{2}} = \frac{1}{\csc x + \cot x}$$

$$\frac{e^t}{1+\sqrt{2}} = \tan\left(\frac{x}{2}\right)$$

$$\tan^{-1}\left(\frac{e^t}{1+\sqrt{2}}\right) = \frac{x}{2}$$

$$x = 2\tan^{-1}\left(\frac{e^t}{1+\sqrt{2}}\right)$$

b) $x_0 = \frac{\pi}{3}$

$$t = \ln\left|\frac{\csc\frac{\pi}{3} + \cot\frac{\pi}{3}}{\csc x + \cot x}\right|$$

$$t = \ln\left|\frac{\frac{2\sqrt{3}}{3} + \frac{\sqrt{3}}{3}}{\csc x + \cot x}\right|$$

$$e^t = \frac{\sqrt{3}}{\csc x + \cot x}$$

$$\frac{e^t}{\sqrt{3}} = \tan\left(\frac{x}{2}\right)$$

$$\tan^{-1}\left(\frac{e^t}{\sqrt{3}}\right) = \frac{x}{2}$$

$$x = 2\tan^{-1}\left(\frac{e^t}{\sqrt{3}}\right)$$

2.1.5

$\dot{x} = \sin x$

$\sin x = 0$

$x = \arcsin(0)$

$x = n\pi$

$f_x = \cos x$

$f_x\big|_{x=0} = 1 \rightarrow$ Inestable en $x^* = 0$

$f_x\big|_{x=\pi} = -1 \rightarrow$ Estable en $x^* = \pi$

2.2 Analizando las siguientes ecuaciones gráficamente. En cada caso, dibuje el campo vectorial en la recta real, encuentre todos los puntos fijos, clasifique su estabilidad y grafica de $x(t)$ para diferentes condiciones iniciales. Luego intente unos minutos para obtener la solución analítica para $x(t)$; si te quedas atascado, no lo intentes durante mucho tiempo, ya que en varios casos es imposible resolver la ecuación en forma cerrada!

2.2.1 $\dot{x} = 4x^2 - 16$

$$f_x = \frac{df}{dx} = \frac{d(4x^2-16)}{dx} = 8x$$

$$0 = 4x^2 - 16$$

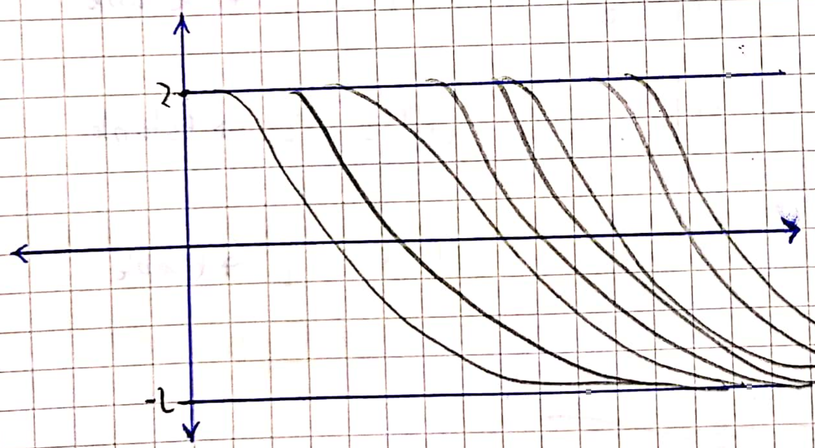
$$0 = x^2 - 4$$

$$0 = (x+2)(x-2)$$

$$x^*_{1,2} = \pm 2$$

$f_x|_{x=2} = 16 \rightarrow$ Inestable

$f_x|_{x=-2} = -16 \rightarrow$ Estable



2.2.2 $\dot{x} = 1 - x^{14}$

$$f_x = \frac{df}{dx} = \frac{d(1-x^{14})}{dx} = -14x$$

$$0 = 1 - x^{14}$$

$$x^{14} = 1$$

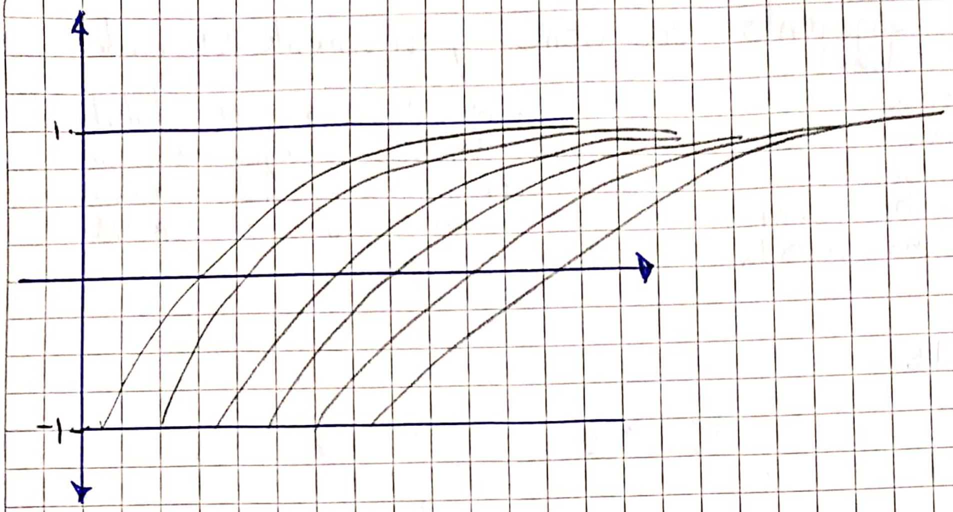
$$x = \sqrt[14]{1} \quad ; \quad x = -\sqrt[14]{1}$$

$$x^*_1 = 1 \quad , \quad x^*_2 = -1$$

$f_x|_{x=1} = -14 \rightarrow$ Estable

$f_x|_{x=-1} = 14 \rightarrow$ Inestable

Escaneado con CamScanner



2.2.3 $\dot{x} = x - x^3$

$0 = x - x^3$

$0 = x(1 - x^2)$

$x_1^* = 0$

$x^2 = 1$

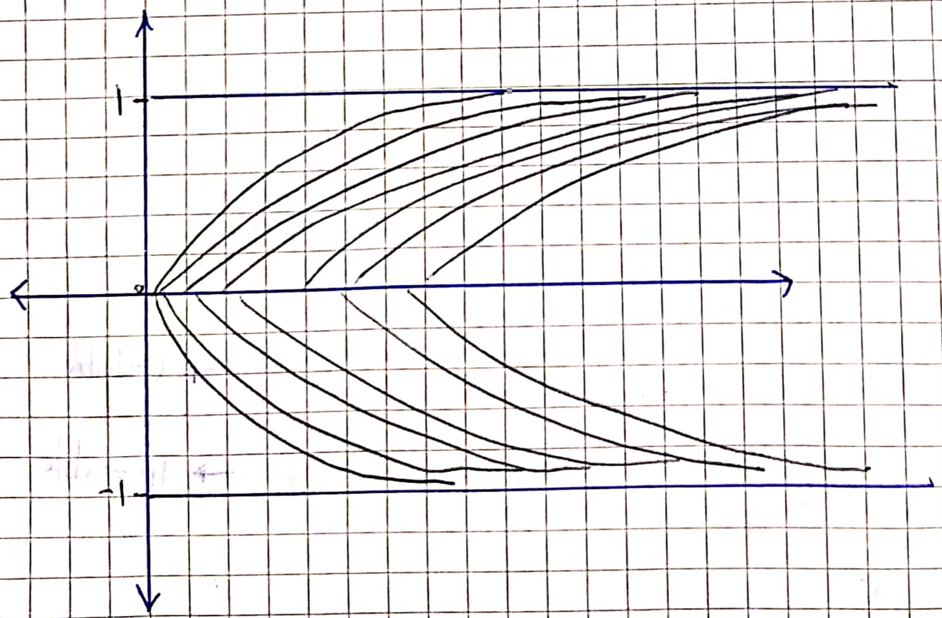
$x_{2,3}^* = \pm 1$

$f_x = \frac{df}{dx} = \frac{d(x - x^3)}{dx} = 1 - 3x^2$

$f_x|_{x=0} = 1 - 3(0)^2 = 1 \rightarrow$ inestable

$f_x|_{x=1} = 1 - 3(1)^2 = -2 \rightarrow$ Estable

$f_x|_{x=-1} = 1 - 3(-1)^2 = -2 \rightarrow$ Estable



2.2.4 $\dot{x} = e^{-x} \sin x$

$0 = e^{-x} \sin x$

$e^{-x} = 0$

$\sin x = 0$

$x_2^* = n\pi$

$x_1^* = \infty$

$f_x = \frac{df}{dx} = \frac{d(e^{-x} \sin x)}{dx} = e^{-x} \sin x + e^{-x} \cos x$

$f_x \Big|_{x=n\pi} = e^{-n\pi} \sin n\pi + e^{-n\pi} \cos n\pi = (-1)^n e^{\pi n}$

si n es par es inestable, si n es impar es estable

2.2.5 $\dot{x} = 1 + \frac{1}{2} \cos x$

$0 = 1 + \frac{1}{2} \cos x$

$-1 = \frac{1}{2} \cos x$

$-2 = \cos x$

$x = \arccos(-2)$

No hay equlibrios

2.2.6 $\dot{x} = 1 - 2\cos x$

$0 = 1 - 2\cos x$

$-1 = -2\cos x$

$\frac{1}{2} = \cos x$

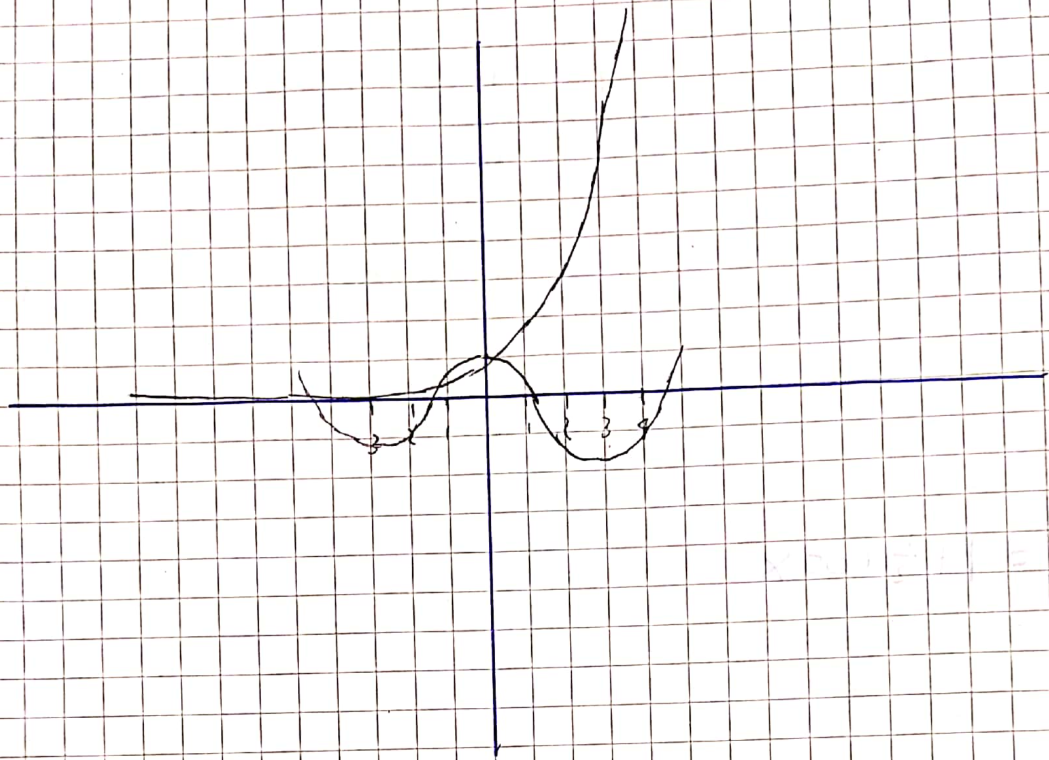
$x = \arccos\left(\frac{1}{2}\right)$

$x = \frac{\pi}{3}$

$f_x = \frac{df}{dx} = \frac{d(1-2\cos x)}{dx} = 2\sin x$

$f_x \Big|_{x=\frac{\pi}{3}} = 2\sin\left(\frac{\pi}{3}\right) = \sqrt{3}$ Inestable

2.2.7 $\dot{x} = e^{x} - \cos x$



2.2.8 Raices = -1 , 0 , 2

$(x+1)(x-2)(x) = x^3 - x^2 - 2x$

$f_x = 3x^2 - 2x - 2$

$f_x\big|_{x=-1} = 3$ Inestable $x^* = -1$

$f_x\big|_{x=0} = -2$ Estable $x^* = 0$

$f_x\big|_{x=2} = 6$ Inestable $x^* = 2$

Escaneado con CamScanner

2.2.9

Raíces: 1, 0, -1

$$(x-1)(x+1)x = x^3 - x$$

$$f_x = 3x^2 - 1$$

$f_x|_{x=1} = 2$ Inestable $x^* = 1$

$f_x|_{x=0} = -1$ Estable $x^* = 0$

$f_x|_{x=-1} = 2$ Inestable $x^* = -1$

2.2.10

a) $\dot{x} = 0$, puntos fijos infinitos

b) $\dot{x} = \sin \pi x$, puntos fijos en enteros

c) No puede haber dos puntos fijos, uno de ellos debe ser inestable.

d) $\dot{x} = 1$, Ya que ningun punto pasara por 0, no hay equilibrios.

e)

2.2.11

$$\dot{Q} = \frac{V_0}{R} - \frac{Q}{RC} \qquad Q(0) = 0$$

$$Q(t) = C + e^{-\frac{1}{CR}} C_1 \qquad -C - e^{-\frac{1}{CR}} = C_1$$

$$0 = C + e^{-\frac{1}{CR}} C_1$$

$Q(0) = c + e^{-\frac{1}{cx}}\left(-c - e^{-\frac{1}{cx}}\right)$

$Q(0) = c - ce^{-\frac{1}{cx}} - e^{-2\frac{1}{cx}}$

2.3.2

$\dot{x} = k_1 a x - k_{-1} x^2$

a) $0 = x(k_1 a - k_{-1} x)$

$f_x = k_1 a - 2k_{-1} x$

$x_1^* = 0$

$f_x\big|_{x=0} = k_1 a$ instable en $x_1^* = 0$

$x_2^* = \dfrac{k_1 a}{k_{-1}}$

$f_x\big|_{x=\frac{k_1 a}{k_{-1}}} = k_1 a - 2k_{-1}\left(\dfrac{k_1 a}{k_{-1}}\right) = \ldots$

$\ldots = k_1 a - 2k_1 a = -k_1 a$

$\downarrow$

Estable

$x_2^* = -k_1 a$

2.4.3 $\dot{x} = \tan x$

$0 = \tan x$

$x = \arctan(0)$

$x = 0$

$f_x = \sec^2(x)$

$f_x\big|_{x=0} = \sec^2(0) = 1$ Inestable $x^* = 0$

2.4.4 $\dot{x} = x^2(6-x)$

$0 = x^2(6-x)$

$6 - x = 0$

$x_1^* = 6$

$x^2 = 0$

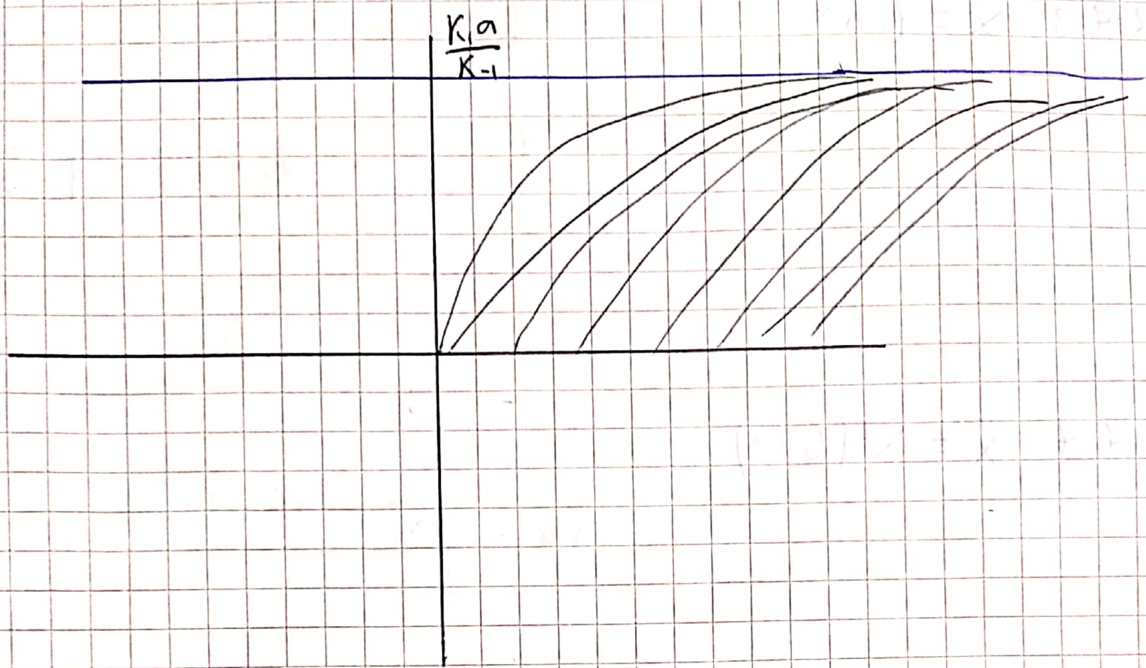
$x_2^* = 0$

$f_x = \dfrac{d(-x^3+6x^2)}{dx} = -3x^2 + 12x$

$f_x\big|_{x=6} = -36$ Estable $x^* = 6$

$f_x\big|_{x=0} = 0$

b)



2.4

2.4.1 $\dot{x} = x(1-x)$

$$f_x = \frac{d(x - x^2)}{dx} = 1 - 2x$$

$$0 = x(1-x)$$

$$x_1^* = 0$$

$$x_2^* = 1$$

$f_x|_{x=0} = 1$ Inestable $x^* = 0$

$f_x|_{x=1} = -1$ Estable $x^* = 1$

2.4.2 $\dot{x} = x(1-x)(2-x)$

$$f_x = \frac{d(x^3 - 3x^2 + 2x)}{dx} = 3x^2 - 6x + 2$$

$$0 = x(1-x)(2-x)$$

$$x_1^* = 0$$

$$x_2^* = 1$$

$$x_3^* = 2$$

$f_x|_{x=0} = 2$ Inestable $x^* = 0$

$f_x|_{x=1} = -1$ Estable $x^* = 1$

$f_x|_{x=2} = 2$ Inestable $x^* = 2$

2.4.5 $\dot{x} = 1 - e^{-x^2}$

$0 = 1 - e^{-x^2}$

$-1 = -e^{-x^2}$

$1 = e^{-x^2}$

$\ln|1| = -x^2$

$0 = -x^2$

$0 = x$

$f_x = 2e^{-x^2}x$

$f_x|_{x=0} = 0$ Semi estable $x^* = 0$

2.4.6 $\dot{x} = \ln x$

$0 = \ln x$

$x = 1$

$f_x = \frac{1}{x}$

$f_x|_{x=1} = 1$ Inestable $x^* = 1$

2.4.7 $\dot{x} = ax - x^3$

$0 = ax - x^3$

$0 = x(a - x^3)$

$x_1 = 0$

x_2 cuando a es positivo

$0 = a - x^2$ $\qquad$ $f_x = a - 3x^2$

$a = -x^2$ $\qquad$ $f_x\big|_{x=0} = a$ Como es positivo

Inestable $x^* = 0$

$\sqrt{-a} = x_2$

No existe, es imaginario

x_2 cuando a es negativo

$0 = -a - x^2$ $\qquad$ $f_x = -a - 3x^2$

$-a = -x^2$ $\qquad$ $f_x\big|_{x=0} = -a$ Como es negativo

Estable $x^* = 0$

$\sqrt{a} = x$ $\qquad$ $f_x\big|_{x=\sqrt{a}} = -a - 3a = -4a$ Estable $x^* = \sqrt{a}$

$f_x\big|_{x=-\sqrt{a}} = -a - 3a = -4a$ Estable $x^* = -\sqrt{a}$

$\pm\sqrt{a} = x$ $\quad b < a$

x_1 cuando a es 0

$0 = -x^2$ $\qquad$ $f_x\big|_{x=0} = -3x^2 = 0$

$\sqrt{0} = x$

$0 = x$

2.4.8 $\dot{N} = -aN\ln(bN)$

$0 = -aN\ln(bN)$

$0 = \ln(bN)$

$bN = 1$

$N = \frac{1}{b}$

$f_N = -a\ln(bN) + (-aN)\frac{1}{bN}b$

$f_N = -a\ln(bN) - aN\frac{1}{N}$

$f_N = -a\ln(bN) - a$

$f_N|_{N=\frac{1}{b}} = -a\ln\left(b\frac{1}{a}\right) - a$

$= -a\left(\ln\left(b\frac{1}{a}\right) + 1\right)$

Estable $N^* = \frac{1}{b}$

2.4.9 $\dot{x} = -x^3$ $\quad x(1) = 1$

a) $\frac{dx}{dt} = -x^3$

$dx = -x^3\,dt$

$-x^{-3}dx = dt$

$-\int x^{-3}dx = \int dt$

$\frac{1}{2x^2} + C = t + C$

$\frac{1}{2x^2} = t - C_1 \Rightarrow \sqrt{\frac{1}{2t-2C_1}} = x \Rightarrow x(t) = \pm\sqrt{\frac{1}{2t-2C_1}}$

$$1 = \sqrt{\frac{1}{2(1) - 2C_1}}$$

$$1 = \frac{1}{2 - 2C_1}$$

$$2 - 2C_1 = 1$$

$$-2C_1 = 1 - 2$$

$$-2C_1 = -1$$

$$2C_1 = 1$$

$$C_1 = \frac{1}{2}$$

$$x(t) = \pm \frac{1}{\sqrt{2t - 1}}$$

$$\lim_{t \to \infty} \pm \frac{1}{\sqrt{2t - 1}} = 0$$

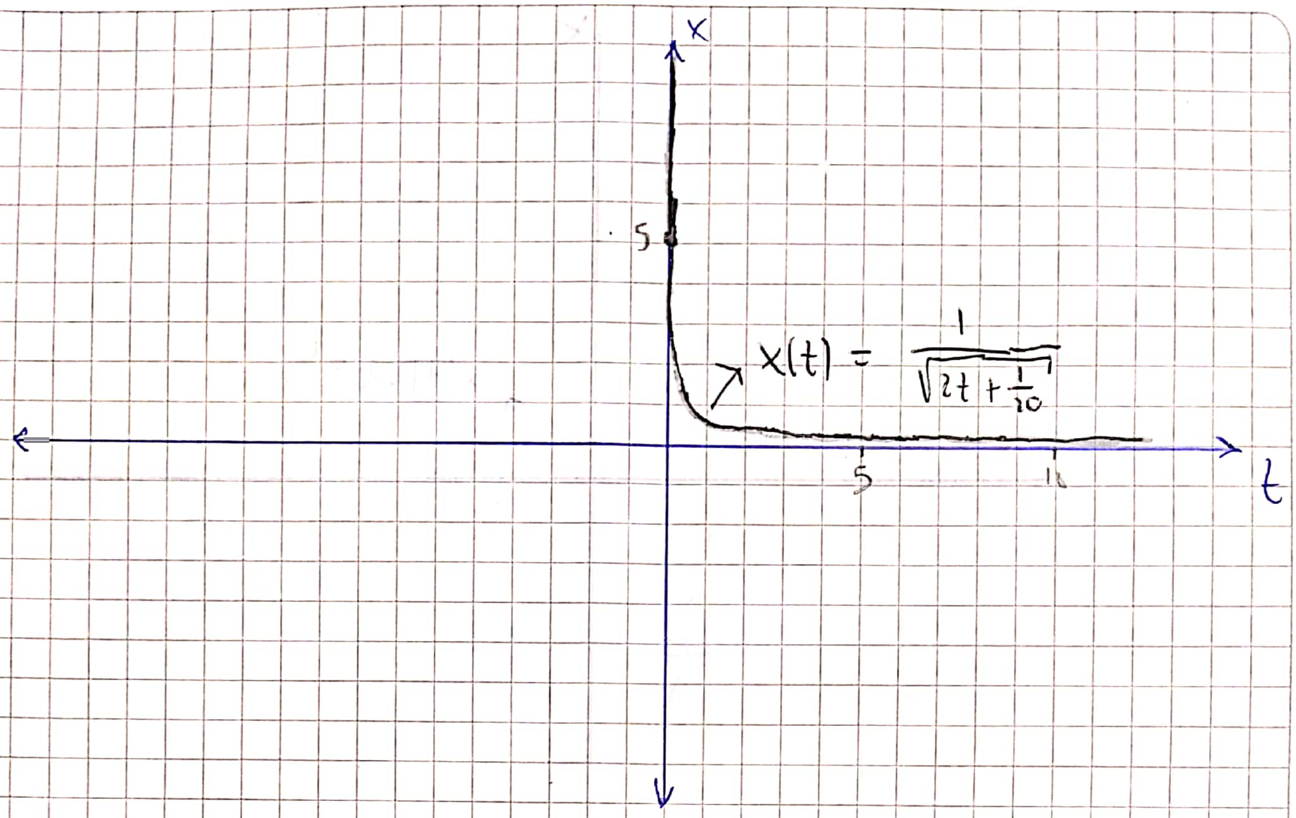
b.)

$$10 = \frac{1}{-2C_1}$$

$$C_1 = \frac{-1}{20}$$

$$x(t) = \frac{1}{\sqrt{2t + \frac{1}{20}}}$$

Escaneado con CamScanner



$\dot{x} = -x$

$\frac{dx}{dt} = -x$

$dx = -x\,dt$

$-x^{-1}dx = dt$

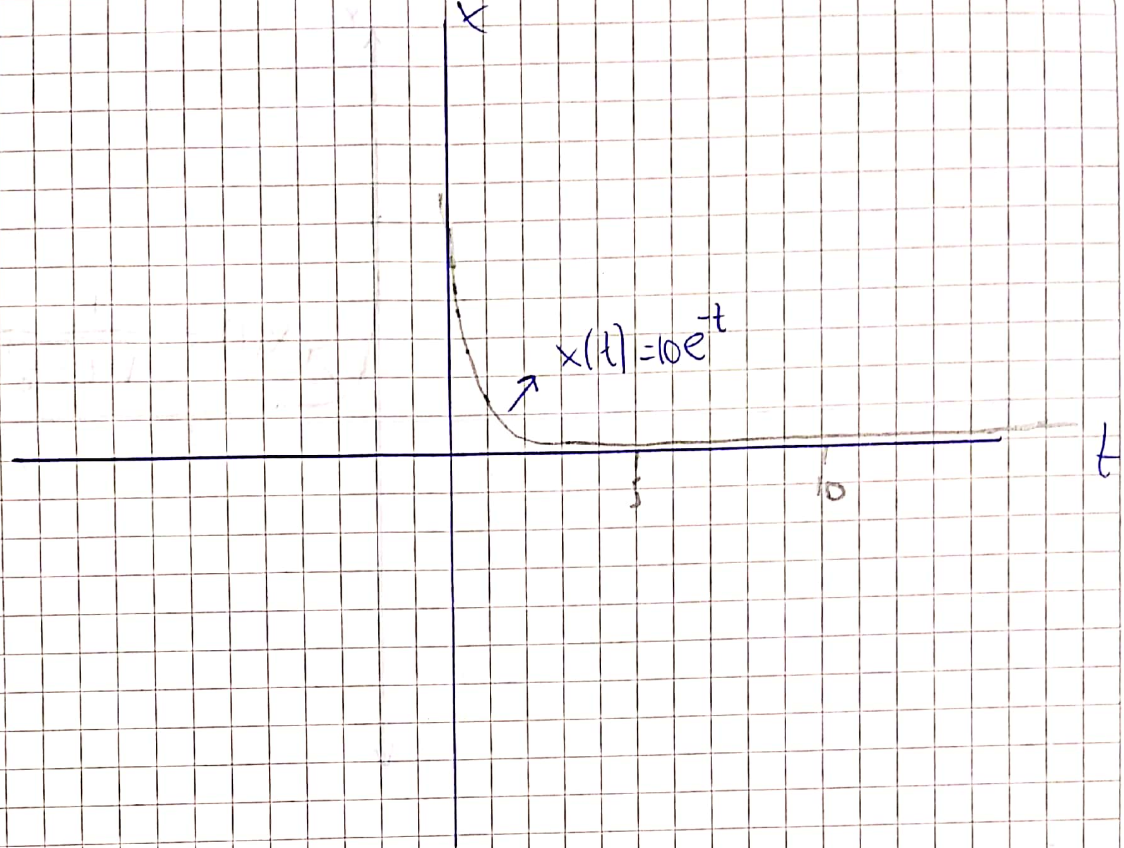
$-\int x^{-1}dx = \int dt$

$-\ln|x| = t + c$

$x(t) = e^{-t}c_1$

$10 = e^{-0}c_1 \qquad c_1 = 10$

$\boxed{x(t) = 10e^{-t}}$



x
$x(t) = 10e^{-t}$
t
5
10

2.8

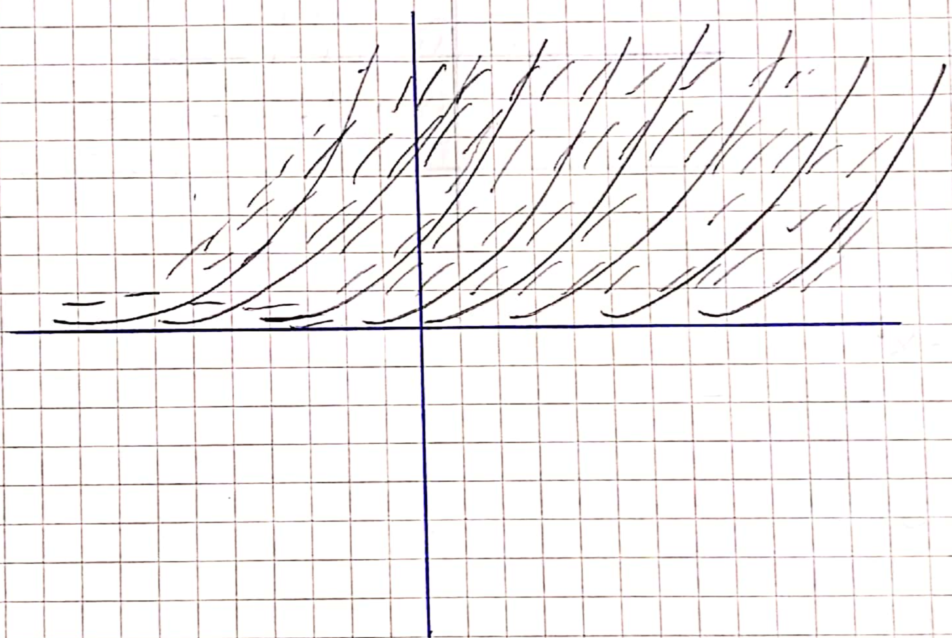
2.8.1

R/ La pendiente es constante en las lineas horizontales, ya que su derivada será 0 en esos puntos que se acercan al punto estable.

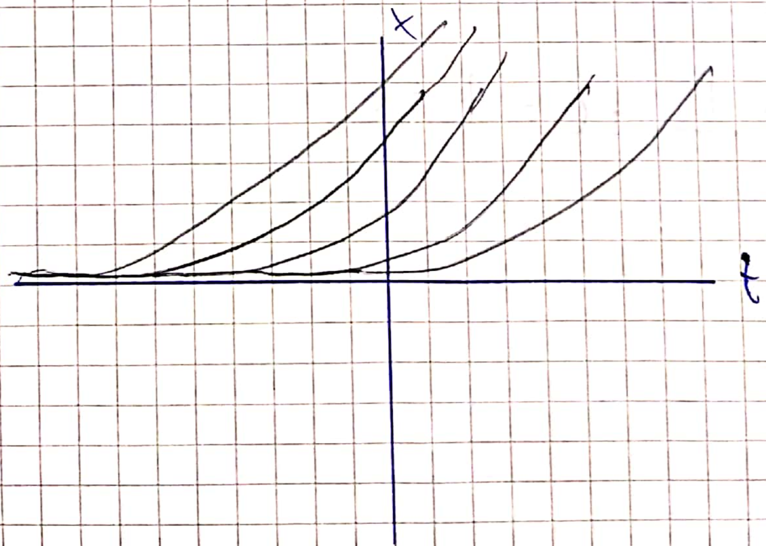
2.8.2

a) $\dot{x} = x$ $\quad f_x = 1$ Inestable

$0 = x$



$x(t) = e^{x} C_1$



b) $\dot{x} = 1 - x^2$

$0 = 1 - x^2$

$-1 = -x^2$

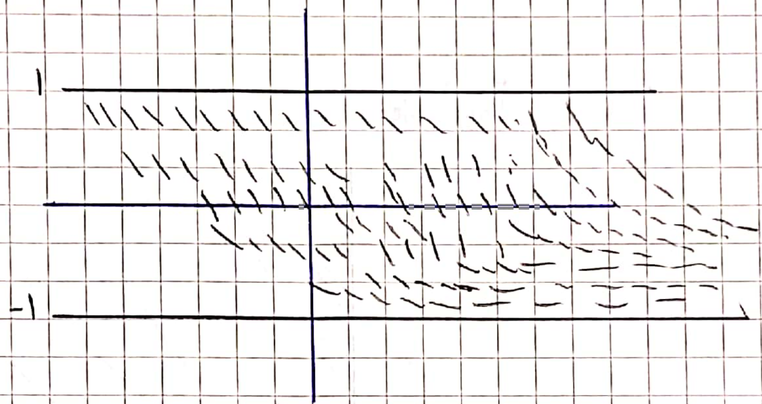
$1 = x^2$

$x = \pm 1$

$f_x = -2x$

$f_x|_{x=1} = -2$ Estable $x^* = 1$

$f_x|_{x=-1} = 2$ Inestable $x^* = -1$



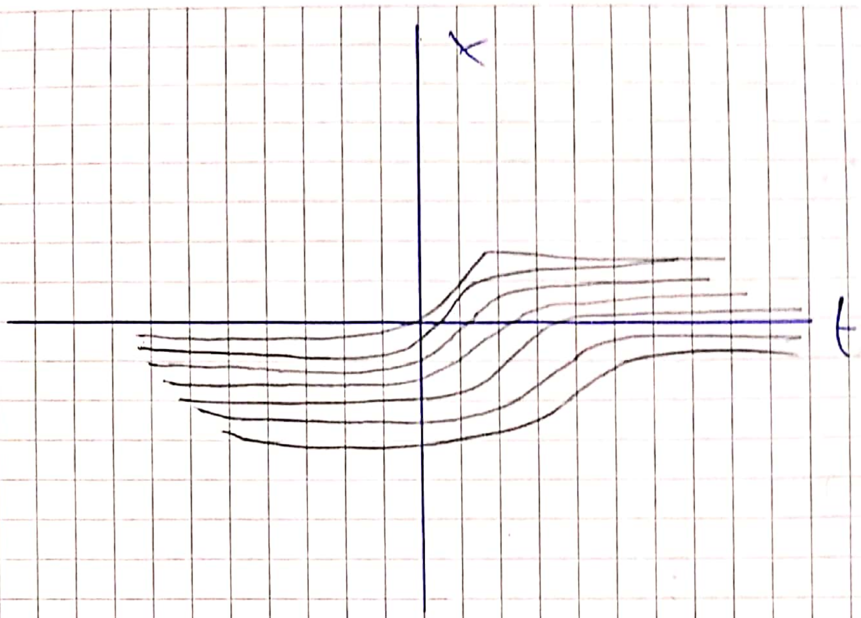
$$\frac{dx}{dt} = 1 - x^2$$

$$\frac{dx}{1-x^2} = dt$$

$$\int \frac{dx}{1-x^2} = \int dt$$

$$\frac{\ln|x+1|}{2} - \frac{\ln|x-1|}{2} + C = t + C$$

$$x(t) = \frac{e^{2t} - e^{2C_1}}{e^{2t} + e^{2C_1}}$$



c) $\dot{x} = 1 - 4x(1-x)$

$0 = 4x^2 - 4x$

$0 = 4x(x-4)$

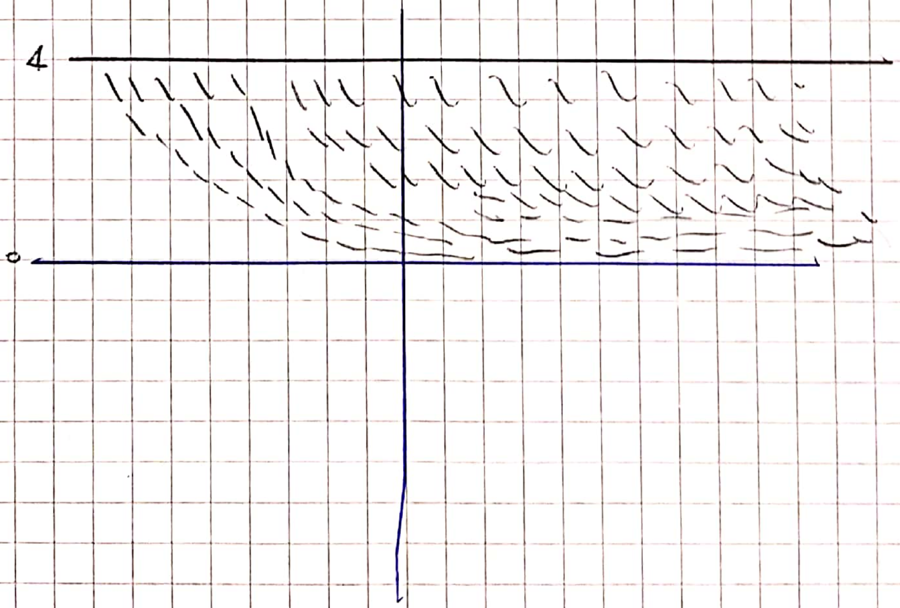
$x_1 = 0$

$x_2 = 4$

$f_x = 8x - 4$

$f_x|_{x=0} = -4$ Estable $x^* = 0$

$f_x|_{x=4} = 28$ Inestable $x^* = 4$



$\dot{x} = 4x^2 - 4x$

$\frac{dx}{dt} = 4x^2 - 4x$

$\frac{dx}{4x^2 - 4x} = dt$

$\int \frac{dx}{4x^2 - 4x} = \int dt$

$x(t) = \frac{1}{1 + e^{4t + c_1}}$