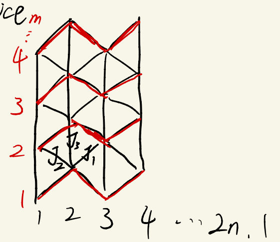
2D-Ising in

Lattice



$$H = -\sum_{i=1}^{m}\sum_{j=1}^{n}\Big[J_1\left(\sigma^z_{i,2j}\sigma^z_{i+1,2j+1} + \sigma^z_{i,2j-1}\sigma^z_{i,2j}\right) + J_2\left(\sigma^z_{i+1,2j-1}\sigma^z_{i,2j} + \sigma^z_{i,2j}\sigma^z_{i,2j+1}\right) + J_3\left(\sigma^z_{i,2j}\sigma^z_{i+1,2j} + \sigma^z_{i,2j-1}\sigma^z_{i,2j-1}\right)\Big]$$

$$Z = \sum_{\{\sigma^z\}} e^{-\beta H} \quad (\text{set } K_i = \beta J_i)$$

$$= \sum_{\{\sigma^z\}} \langle 1| e^{\sum_{j=1}^{n}\left[K_1(\sigma^z_{1,2j}\sigma^z_{2,2j+1} + \sigma^z_{1,2j-1}\sigma^z_{1,2j}) + K_2(\sigma^z_{1,2j}\sigma^z_{2,2j-1} + \sigma^z_{1,2j}\sigma^z_{1,2j+1}) + K_3(\sigma^z_{1,2j}\sigma^z_{2,2j} + \sigma^z_{1,2j-1}\sigma^z_{2,2j-1})\right]}|2\rangle$$
$$\langle 2|\cdots|3\rangle\langle 3|\cdots|m\rangle\langle m|\cdots|1\rangle$$

$$\begin{cases} V_1 = \langle 1| e^{\sum_{j=1}^{n}\left[K_1\sigma^z_{2j-1}\sigma^z_{2j} + K_2\sigma^z_{2j}\sigma^z_{2j+1}\right]}|1'\rangle \\ V_2 = \langle 1'| e^{\sum_{j=1}^{n}\left[K_3\sigma^z_{2j-1}(\sigma^z_{2j-1})'\right]}|2'\rangle \\ V_3 = \langle 2'| e^{\sum_{j=1}^{n}\left[K_1\sigma^z_{2j}\sigma^z_{2j+1} + K_2\sigma^z_{2j}\sigma^z_{2j-1}\right]}|1''\rangle \\ V_4 = \langle 1''| e^{\sum_{j=1}^{n}\left[K_3\sigma^z_{2j}(\sigma^z_{2j})'\right]}|2\rangle \end{cases}$$

$$\therefore Z = Tr\left((V_1V_2V_3V_4)^m\right) = \sum_{i=1}^{2^n}\lambda_i^m \sim \lambda_{max}^m$$

Here, use $e^{K\sigma_j^z(\sigma_j^z)'} = A e^{K^*\sigma_j^x}$

$$\begin{cases} A = \sqrt{2\sinh(2K)} \\ \tanh(K^*) = e^{-2K} \end{cases}$$

$$H_1 = \sum_{j=1}^{n}\left(K_1\sigma^z_{2j-1}\sigma^z_{2j} + K_2\sigma^z_{2j}\sigma^z_{2j+1}\right)$$

$$H_2 = \sum_{j=1}^{n} K_3^*\sigma^x_{2j-1}$$

$$H_3 = \sum_{j=1}^{n}\left(K_1\sigma^z_{2j}\sigma^z_{2j+1} + K_2\sigma^z_{2j}\sigma^z_{2j-1}\right)$$

$$H_4 = \sum_{j=1}^{n} K_3^*\sigma^x_{2j}$$

$$\boxed{Z = Tr\left(\left((2\sinh(2K_3))^n e^{H_1}e^{H_2}e^{H_3}e^{H_4}\right)^m\right)}$$

J-W transformation: $S^- = \begin{pmatrix}0&0\\1&0\end{pmatrix}$ $S^+ = \begin{pmatrix}0&1\\0&0\end{pmatrix}$

$$\begin{cases} C_i^+ = \prod_{k<i}\sigma_k^x S_i^+ \\ C_j = \prod_{k<j}\sigma_k^x S_j^- \end{cases} \qquad \begin{cases} \sigma^x = 1 - 2S^-S^+ \\ \sigma^z = S^- + S^+ \end{cases}$$

$$\sigma_i^z\sigma_{i+1}^z$$
$$= (S_i^- + S_i^+)(S_{i+1}^- + S_{i+1}^+)$$
$$= -(2C_i^+C_i - 1)(C_i + C_i^+)(C_{i+1} + C_{i+1}^+)$$
$$= -C_i(C_{i+1} + C_{i+1}^+) + C_i^+(C_{i+1} + C_{i+1}^+)$$
$$= C_{i+1}^+C_i + C_i^+C_{i+1} + C_i^+C_{i+1}^+ + C_{i+1}C_i$$

$$\sigma_i^x = 1 - 2C_iC_i^+ = 2C_i^+C_i - 1$$

Fourier Transformation

$$\begin{cases} C_{2i-1} = \frac{1}{\sqrt{n}}\sum_k e^{ika}C_{k,A} \\ C_{2i} = \frac{1}{\sqrt{n}}\sum_k e^{ika}C_{k,B} \\ C_{2i-1}^+ = \frac{1}{\sqrt{n}}\sum_k e^{-ika}C_{k,A}^+ \\ C_{2i}^+ = \frac{1}{\sqrt{n}}\sum_k e^{-ika}C_{k,B}^+ \end{cases}$$

By J-W transformation:

$$H_1 = \sum_{i=1}^{n} -K_1\left(C_{2i-1}^+C_{2i} + C_{2i}^+C_{2i-1} + C_{2i-1}^+C_{2i}^+ + C_{2i}C_{2i-1}\right) - K_2\left(C_{2i}^+C_{2i+1} + C_{2i+1}^+C_{2i} + C_{2i}^+C_{2i+1}^+ + C_{2i+1}C_{2i}\right)$$

$$H_2 = \sum_{i=1}^{n} K_3^*\left(2C_{2i-1}^+C_{2i-1} - 1\right)$$

$$H_3 = \sum_{i=1}^{n} -K_1\left(C_{2i}^+C_{2i+1} + C_{2i+1}^+C_{2i} + C_{2i}^+C_{2i+1}^+ + C_{2i+1}C_{2i}\right) - K_2\left(C_{2i-1}^+C_{2i} + C_{2i}^+C_{2i-1} + C_{2i-1}^+C_{2i}^+ + C_{2i}C_{2i-1}\right)$$

$$H_4 = \sum_{i=1}^{n} K_3^*\left(2C_{2i}^+C_{2i} - 1\right)$$

By Fourier Transformation:

$$H_1(|k|) = -K_1\big(C^+_{k,A}C_{k,B} + C^+_{k,B}C_{k,A} + C^+_{k,A}C^+_{-k,B} + C_{-k,B}C_{k,A}$$
$$+ C^+_{-k,A}C_{-k,B} + C^+_{-k,B}C_{-k,A} + C^+_{-k,A}C^+_{k,B} + C_{k,B}C_{-k,A}\big)$$
$$-K_2\big(C^+_{k,B}C_{k,A}e^{ik} + C^+_{k,A}C_{k,B}e^{-ik} + C^+_{k,B}C^+_{-k,A}e^{ik} + C_{k,A}C_{k,B}e^{-ik}$$
$$+ C^+_{-k,B}C_{-k,A}e^{-ik} + C^+_{-k,A}C_{-k,B}e^{-ik} + C^+_{-k,B}C^+_{k,A}e^{-ik} + C_{k,A}C_{-k,B}e^{ik}\big)$$

$$H_3(|k|) = H_1(|k|)[K_1 \leftrightarrow K_2]$$

$$H_2(|k|) = K_3^*\left(2C^+_{k,A}C_{k,A} + 2C^+_{-k,A}C_{-k,A} - 2\right)$$

$$H_4(|k|) = H_2(|k|)[A \to B]$$

By choosing basis $[C^+_{k,A}, C^+_{k,B}, C_{-k,A}, C_{-k,B}]\left[\; H_i(|k|) \;\right]\begin{bmatrix} C_{k,A} \\ C_{k,B} \\ C^+_{-k,A} \\ C^+_{-k,B} \end{bmatrix}$, $H_i(|k|)$ in matrix form:

$$H_1(|k|) = \left(\begin{array}{cc|cc} 0 & -K_1-K_2e^{-ik} & 0 & -K_1+K_2e^{-ik} \\ -K_1-K_2e^{ik} & 0 & K_1-K_2e^{ik} & 0 \\ \hline 0 & K_1-K_2e^{-ik} & 0 & K_1+K_2e^{-ik} \\ -K_1+K_2e^{ik} & 0 & K_1+K_2e^{ik} & 0 \end{array}\right)$$

$$H_2(|k|) = \left(\begin{array}{cc|cc} 2K_3^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -2K_3^* & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$H_3(|k|) = H_1(|k|)[K_1 \leftrightarrow K_2] \qquad H_4(|k|) = H_2(|k|)[A \to B]$$

By mathematica, the eigenfunction of $\Gamma_{|k|} = e^{H_1(|k|)}e^{H_2(|k|)}e^{H_3(|k|)}e^{H_4(|k|)}$ is

$$1 + E^4 + XE^2 + (Y_r - iY_i)E + (Y_r + iY_i)E^3 = 0$$

① It is obvious that if E_1 is one root, then $\frac{1}{E_1^*}$ must be another root.

② $\det(\Gamma_{|k|}) = \prod_i e^{Tr(H_i(|k|))} = 1 \quad \therefore E_1 \cdot E_2 \cdot E_3 \cdot E_4 = 1$ (here we set $E_1 = e^{\eta_{k,1}}, E_2 = e^{-\eta^*_{k,1}}, E_3 = e^{\eta_{k,2}}, E_4 = e^{-\eta^*_{k,2}}$)

$$Z = \lim_{m\to\infty} \lambda_{max}^m$$
$$= \lim_{m\to\infty}\left[(2\sinh(2K_3))^n e^{Re\left[\sum_{|k|}(\eta_{k,1}+\eta_{k,2})\right]}\right]^m$$
$$= \lim_{m,n\to\infty}\left((2\sinh(2K_3))^n e^{Re\left[\frac{n}{4\pi}\int_0^{2\pi}(\eta_{k,1}+\eta_{k,2})dk\right]}\right)^m$$

$$\eta_k = \frac{1}{2\pi}\int_0^{2\pi} \ln\left(e^{\eta_k} + e^{-\eta_k} + 2\cos(w)\right)dw$$

$$F = -kT\frac{\ln(Z)}{2nm}$$
$$= -kT\left(\ln(2\sinh(2K_3))^{\frac{1}{2}} + \frac{1}{16\pi}\int_0^{2\pi}(\eta_{k,1} + \eta^*_{k,1} + \eta_{k,2} + \eta^*_{k_2})\,dk\right)$$
$$= -kT\left(\ln(2\sinh(2K_3))^{\frac{1}{2}} + \frac{1}{32\pi^2}\int_0^{2\pi}dk\int_0^{2\pi}\ln\left(\prod_{i=1}^{4}\left(E_i + \frac{1}{E_i} + 2\cos(w)\right)\right)dw\right)$$

Here, we write down X, Y_r, Y_i exactly:

$$X = -4\sinh(2K_1)\sinh(2K_2)\sinh^2(2K_3^*)\cos(k)$$
$$+8\cosh(2K_1)\cosh(2K_2)\cosh(2K_3^*)\sinh(2K_1)\sinh(2K_2)$$
$$+2\cosh^2(2K_3^*)$$
$$+2(\cosh^2(2K_1)\cosh^2(2K_2)+\sinh^2(2K_1)\sinh^2(2K_2))(1+\cosh^2(2K_3^*))$$

$$Y_i = \sinh^2(2K_3^*)\sin(k)(\sinh^2(2K_1)-\sinh^2(2K_2))$$

$$Y_r = -4\cosh(2K_1)\cosh(2K_2)\cosh(2K_3^*) - 2\sinh(2K_1)\sinh(2K_2)(1+\cosh^2(2K_3^*))$$
$$-(\sinh^2(2K_1)+\sinh^2(2K_2))\sinh^2(2K_3^*)\cos(k)$$

Make eigenfunction a reciprocal 8th degree equation whose roots are $E_1, \frac{1}{E_1^*}, E_2, \frac{1}{E_2^*}, \frac{1}{E_1}, E_1^*, \frac{1}{E_2}, E_2^*$

$$E^8+2Y_rE^7+(2X+Y_r^2+Y_i^2)E^6+2Y_r(1+X)E^5+(2+2Y_r^2-2Y_i^2+X^2)E^4$$
$$+2Y_r(1+X)E^3+(2X+Y_r^2+Y_i^2)E^2+2Y_rE+1=0 \quad (E^8+aE^7+bE^6+cE^5+dE^4+cE^3+bE^2+aE+1=0)$$

By replacing $E+\frac{1}{E}=y$, we have

$$y^4+ay^3+(b-4)y^2+(c-3a)y+d-2b+2=0$$

By Vieta theorem,
$$\begin{cases} y_1+y_2+y_3+y_4=-a \\ y_1y_2+y_1y_3+y_1y_4+y_2y_3+y_2y_4+y_3y_4=b-4 \\ y_1y_2y_3+y_1y_2y_4+y_1y_3y_4+y_2y_3y_4=3a-c \\ y_1y_2y_3y_4=d-2b+2 \end{cases}$$

$$X=(y_1-m)(y_2-m)(y_3-m)(y_4-m)=d-2b+2-m(3a-c)+m^2(b-4)+m^3a+m^4$$

$$\therefore F=-kT\left(\ln(2\sinh(2K_3))^{\frac{1}{2}}+\frac{1}{32\pi^2}\int_0^{2\pi}dk\int_0^{2\pi}\ln(X)\,d\omega\right)$$
$$=-kT\Big(\ln(2)+\frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi}\ln[\cosh(2K_1)\cosh(2K_2)\cosh(2K_3)+\sinh(2K_1)\sinh(2K_2)\sinh(2K_3)$$
$$-\sinh(2K_1)\cos(\omega_1)-\sinh(2K_2)\cos(\omega_2)-\sinh(2K_3)\cos(\omega_1+\omega_2)]\,d\omega_1d\omega_2\Big)$$

Phase transition point: $\ln() \to -\infty$ $\quad \cosh(2K_1)\cosh(2K_2)\cosh(2K_3)+\sinh(2K_1)\sinh(2K_2)\sinh(2K_3)$
$(J_1+J_2>0 \,\&\, J_2+J_3>0 \,\&\, J_3+J_1>0)$ $\quad =\sinh(2K_1)+\sinh(2K_2)+\sinh(2K_3)$

$$\Rightarrow \sinh(2K_1)\sinh(2K_2)+\sinh(2K_2)\sinh(2K_3)+\sinh(2K_3)\sinh(2K_1)=1$$