# Computer Lab 3

Shipeng Liu (shili506) Dongwei Ni (donni508)

## Question 1: Stable distribution

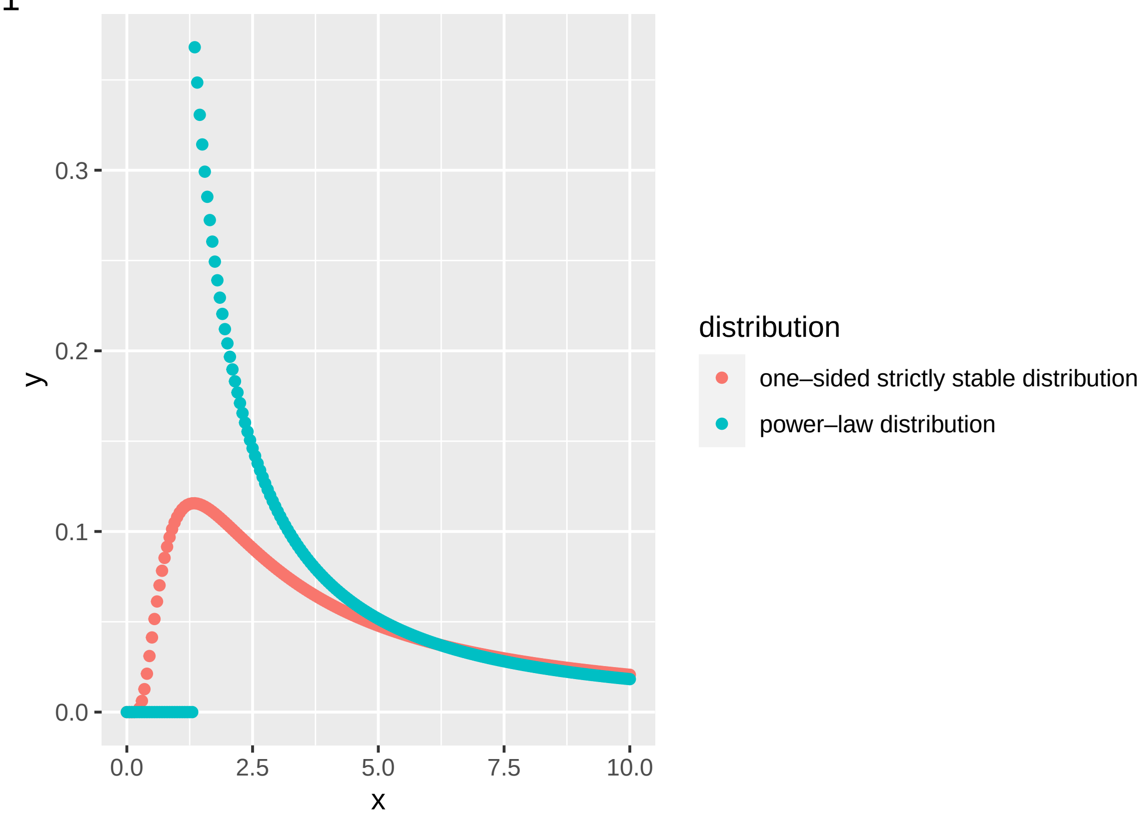
### Task 1

**Plot $f(x)$ and $fp(x)$ together**

```
x=seq(0,10,0.05)

#one-sided strictly stable distribution of order 1/2 where c=2
f=function(x){
  c=2
  sapply(x,function(y){
    if(y>0){
      return(c*sqrt(2*pi)^(-1)*exp(-(c^2)/(2*y))*(y^(-3/2)))
    }else{
      return(0)
    }
  })
}

y_f=f(x)

#power-law distribution where a=1.5 and t_min=4/3
fp=function(x){
  a=1.5;t_min=4/3
  sapply(x,function(y){
    if(y>t_min){
      return(((a-1)/t_min)*(y/t_min)^(-a))
    }else{
      return(0)
    }
  })
}
y_fp=fp(x)

df=rbind(data.frame("x"=x,"y"=y_f,"class"="f"),data.frame("x"=x,"y"=y_fp,"class"="fp"))



p1=ggplot(data=df,aes(x=x,y=y))+geom_point(aes(color=class))+
  labs(tag="Fig. 1")+
  scale_color_discrete(
    name="distribution",
    labels=c("one-sided strictly stable distribution","power-law distribution")
```

```
  )
p1
```

Fig. 1




The power–law distribution can not be used just by itself,because the two distribution don't have the same support. In $(0, t_{min})$,the density of the power–law distribution is 0,but for one–sided strictly stable distribution of order 1/2 it isn't.The power–law distribution can't generate the proper samples in accept-reject method in this interval. We can apply a uniform distribution on the interval $(0, t_{min})$ as a majorizing function

For optimal Tmin: take the derivative of target density function

$$\frac{df(x)}{dx} = (\sqrt{2\pi}^{-1} e^{-\frac{c^2}{2x}} (\frac{c^2}{2x} x^{-3/2} - \frac{3}{2} x^{-5/2}))$$

take the root, we got

$$max f(x) = f(\frac{1}{3}c^2)$$

so the optimal $T_{min} = \frac{1}{3}c^2$

For optimal $\alpha$: we need to make sure $\forall_x a f_p(x) \geq f(x)$ (a is a constant) hence we should take an appropriate value of a
s.t. the difference between $f(x)$ and $f_p(x)$ will not be large in the interval$(T_{min}, \infty)$ according to the plot, we set $\alpha = 1.5$

## Task 2

**Implement an acceptance-rejection algorithm for sampling from the one-sided strictly stable distribution of order 1/2 with the proposal distribution built around the power-law distribution.**

```
sampling=function(samp_x,constant,c,t_min,alpha){
  a=sapply(samp_x,function(samp_x){
    x=NA
    while(is.na(x)){
      u=runif(1)
      if(samp_x>=0 && samp_x<=t_min+0.5){
        y=runif(1)
        if(u<=f(y)/constant*dunif(y)){x<-y}
      }else{
        y=rplcon(1,t_min,alpha)
        if(u<=f(y)/constant*dplcon(y,t_min,alpha)){x<-y}
      }
    }
    return(x)
  })
  return(a)
}

t_min=4/3
alpha=1.5
c=2
constant=1

samp_x=runif(2000,0,10)
samp=sampling(samp_x,constant,c,t_min,alpha)
p2=ggplot(data.frame("x"=samp),aes(x=x))+geom_density()+xlim(c(0,10))+
  labs(title="The sampler density",tag="Fig. 2")
df_f=df%>%filter(class=="f")
p3=ggplot(data=df_f,aes(x=x,y=y))+geom_line()+
  labs(title="The target density",tag="Fig. 3")
plot_comb=grid.arrange(p2,p3)
```

Fig. 2

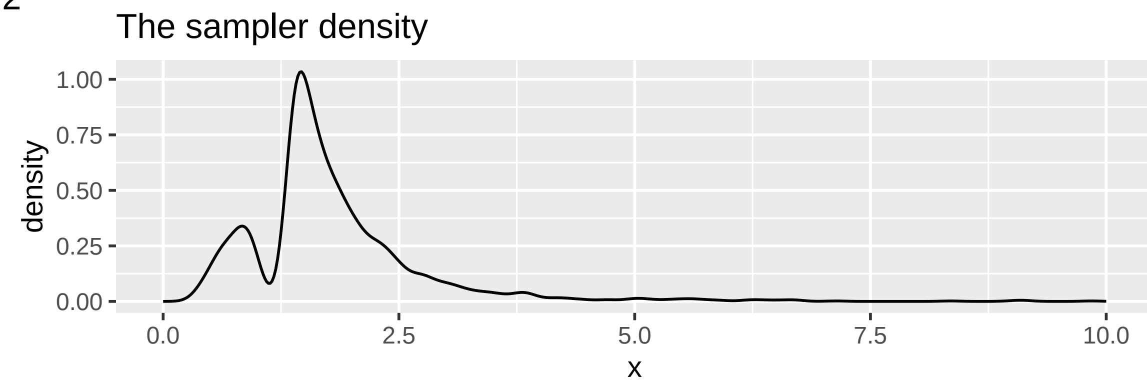
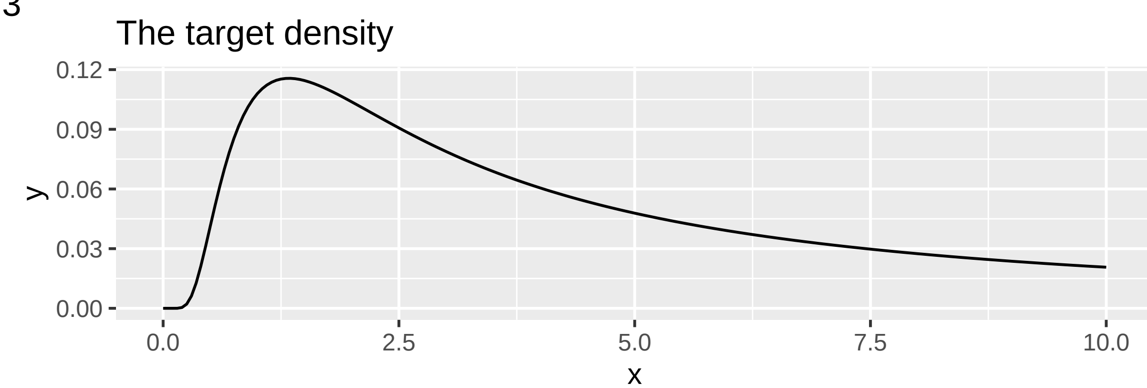



Fig. 3




```
plot_comb
```

```
## TableGrob (2 x 1) "arrange": 2 grobs
##   z     cells    name           grob
## 1 1 (1-1,1-1) arrange gtable[layout]
## 2 2 (2-2,1-1) arrange gtable[layout]
```

### Task 3

## Question 2: Laplace distribution

### Task 1

**Write a code generating double exponential distribution DE(0, 1) from Unif(0, 1) by using the inverse CDF method.**

The PDF of the Laplace distribution is given:

$$f(x) = \frac{\alpha}{2} exp(-\alpha \mid x - \mu \mid)$$

And we calculate the CDF:

$$F(x) = \int_{-\infty}^{x} f(u)du = \begin{cases} \frac{1}{2}\mathrm{e}^{(\frac{\mathrm{x}-\mu}{\alpha})} & \text{if } \mathrm{x} < \mu \\ 1 - \frac{1}{2}\mathrm{e}^{(\frac{\mathrm{x}-\mu}{\alpha})} & \text{if } \mathrm{x} \geq \mu \end{cases}$$

The inverse of CDF:

$$F^{-1}(x) = \begin{cases} -\frac{1}{\alpha}\ln 2(1 - F(x)) + \mu & \text{if } \mathrm{x} \geq \mu \\ \frac{ln(2F(x))}{\alpha} + \mu & \text{if } \mathrm{x} < \mu \end{cases}$$

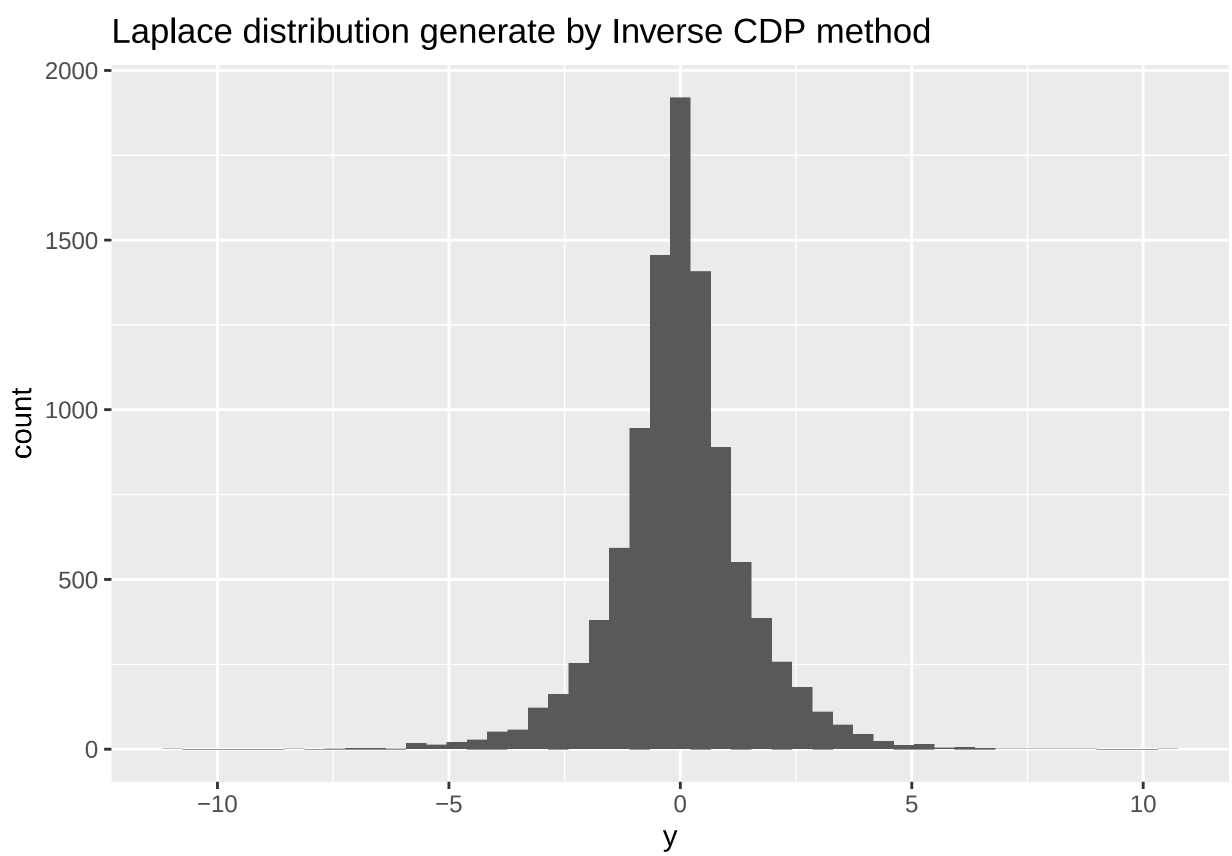
when $x=\mu$,$F(x)=\frac{1}{2}$ hence The inverse of CDF:

$$F^{-1}(x)=\begin{cases} -\frac{1}{\alpha}\ln 2(1-F(x))+\mu & \text{if } \mathrm{F(x)}\geq\frac{1}{2} \\ \frac{ln(2F(x))}{\alpha}+\mu & \text{if } \mathrm{F(x)}<\frac{1}{2} \end{cases}$$

```
r_laplace=function(gen_num,mu,alpha){
  unif=runif(gen_num,0,1)
  temp=sapply(unif,function(unif){
    if(unif>=0.5){
      return((-1/alpha)*log(2*(1-unif))+mu)
    }else{
      return((log(2*unif)/alpha)+mu)
    }
  })
  return(temp)
}
```

**Generate 10000 random numbers from this distribution, plot the histogram and comment whether the result looks reasonable.**

```
p4=ggplot(data=data.frame("y"=r_laplace(10000,0,1)),aes(x=y))+
  geom_histogram(bins=50)+
  labs(title="Laplace distribution generate by Inverse CDP method")
p4
```




Compare to the plot of Laplace distribution probability density function(it can be thought of as two exponential distributions spliced together along the abscissa),the result seems resonable.

## Task 2

**Use the Acceptance/rejection method with DE(0, 1) as a majorizing density to generate N (0,1) variables.**

```
ar_norm<-function(c){
  x<-NA
  num.reject<-0
  while (is.na(x)){
    y<-r_laplace(1,0,1)
    u<-runif(1)
    if (u<=dnorm(y,0,1)/(c*(exp(-1*abs(y))/2))){x<-y}
    else{num.reject<-num.reject+1}
  }
  c(x,num.reject)
}
```

We sample Y from the majorizing distribution, sample U from the uniform distribution, and then filter out samples that satisfy $U \leq \frac{f_X(Y)}{cf_Y(Y)}$, and resample if not satisfied , until a sample that satisfies the condition is obtained.

### majorizing constant c

We need to choose majorizing constant c such that:

$$\forall_x c f_Y(x) \geq f_X(x)$$

where $f_Y(x)$ is PDF of Laplace distribution,$f_X(x)$ is PDF of Normal distribution. c mush be large enough,but too large c may cause large rejection rates,we need to choose carefully.

$$\frac{c}{2}e^{-|x|} \geq \frac{1}{\sqrt{2\pi}}e^{|x|-\frac{x^2}{2}}$$

solve it and we'll get:

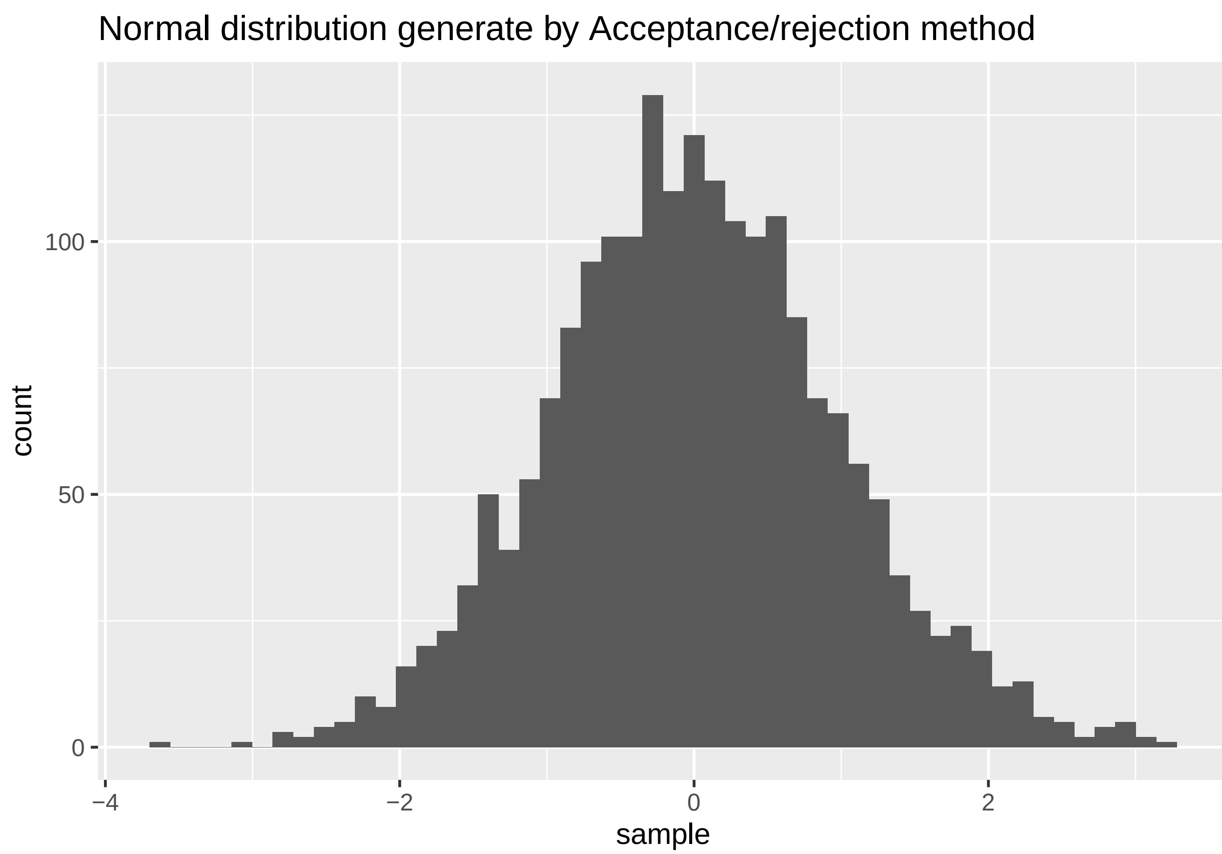
$$c \geq \sqrt{\frac{2}{\pi}}e^{(|x|-\frac{x^2}{2})}$$

when$x = 1$,we get the maximum $c$:

$$c = \sqrt{\frac{2e}{\pi}}$$

**Generate 2000 random numbers N (0, 1) using your code and plot the histogram.**

```
df_norm=data.frame(t(data.frame(sapply(rep(sqrt((2*exp(1))/pi),2000),ar_norm))))
colnames(df_norm)=c("sample","reject")
p4=ggplot(data=df_norm,aes(x=sample))+
  geom_histogram(bins=50)+
  labs(title="Normal distribution generate by Acceptance/rejection method")
p4
```




**Rejection rate**

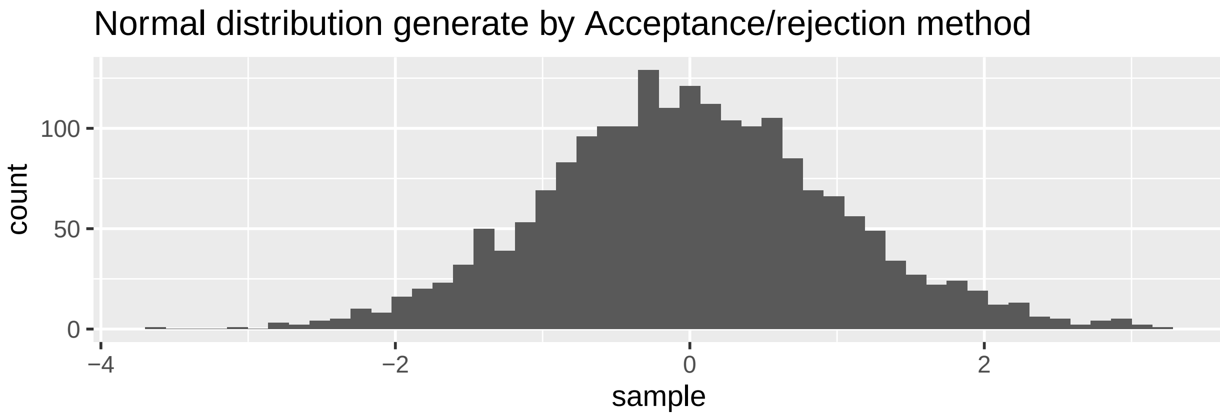
```
mean_r=sum(df_norm[,2])/(2000+sum(df_norm[,2]))
ER=1-(1/sqrt((2*exp(1))/pi))

cat("The average rejection rate R: ",mean_r,"\nThe expected rejection rate ER: ",
    ER,"\nThe difference:",abs(ER-mean_r))
```

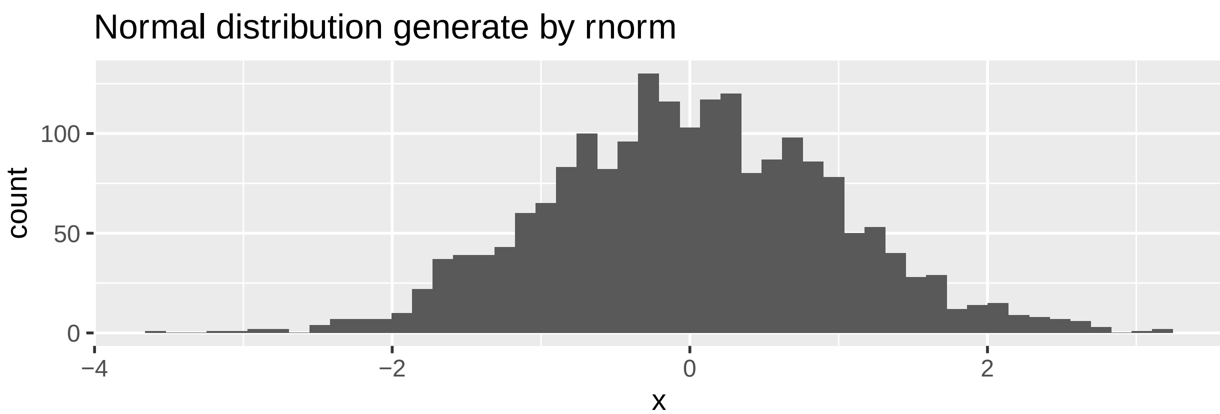
```
## The average rejection rate R:  0.2337165
## The expected rejection rate ER:  0.2398265
## The difference: 0.006110074
```

The R and ER are very close to each other.

**Generate 2000 numbers from N (0, 1) using standard rnorm() procedure**

```
p5=ggplot(data=data.frame("x"=rnorm(2000,0,1)),aes(x=x))+
  geom_histogram(bins=50)+
  labs(title="Normal distribution generate by rnorm")
plot_comb1=grid.arrange(p4,p5)
```




The obtained two histograms has similar distribution.The number of samples concentrated around 0 is the largest, and the number of samples decreases as the absolute value of the sample increases.