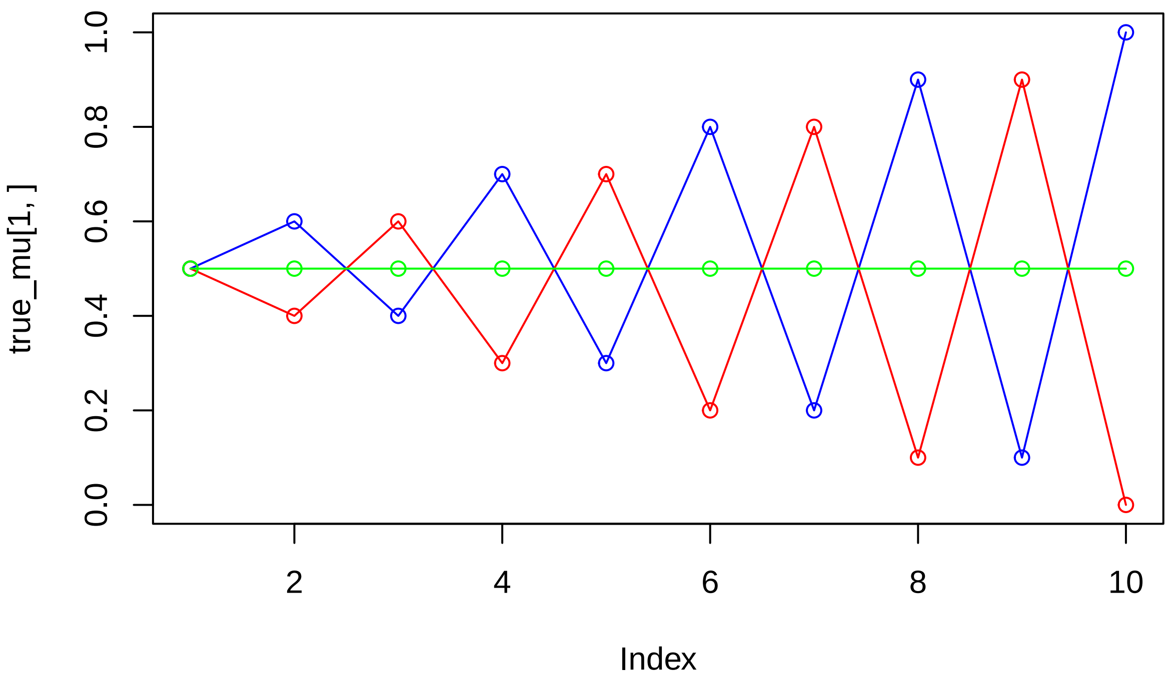
lab block2 Assginment 2

Group A20

2022-12-08

2. Mixture Models

For implementing the EM algorithm for Bernoulli mixture model First we generate a π and μ of a Bernoulli mixture model

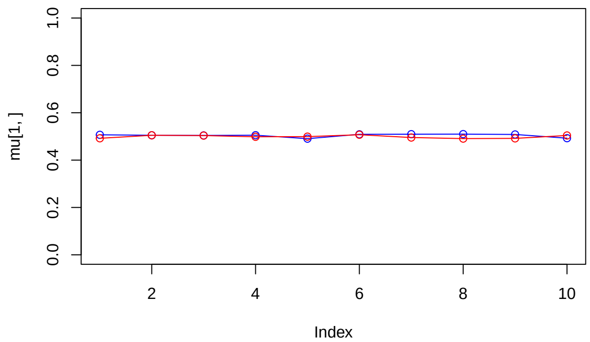


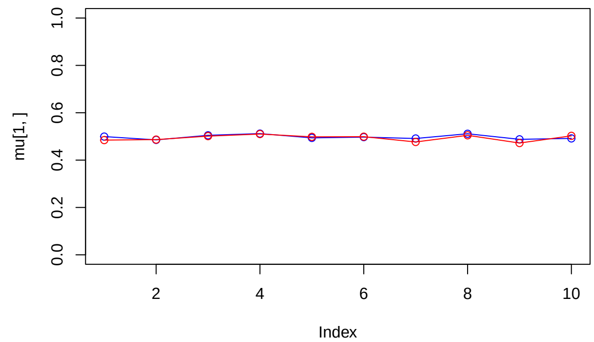
And the plot above shows the 3 μs we are using to generate data set x, with a same $\pi = \frac{1}{3}$ for each μ, and of course with a $Dimension = 10$.

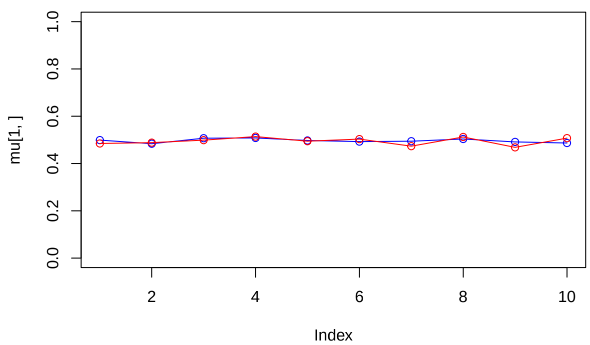
```
## [1] 0.5052178 0.4947822
```

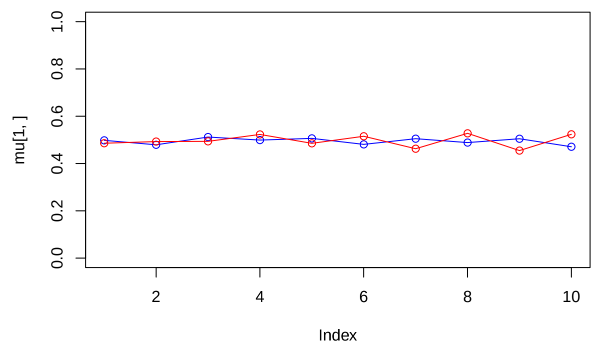
```
##           [,1]      [,2]      [,3]      [,4]      [,5]      [,6]      [,7]
## [1,] 0.5068040 0.5049455 0.5041509 0.5051382 0.4905578 0.5089228 0.509453
## [2,] 0.4921113 0.5048207 0.5036886 0.4985467 0.4991731 0.5071384 0.495380
##           [,8]      [,9]     [,10]
## [1,] 0.5097480 0.5082991 0.4926313
## [2,] 0.4908757 0.4917657 0.5040657
```

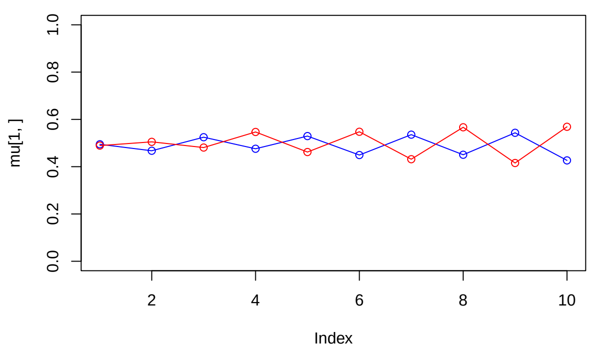
First we start with cluster M = 2. And we generate a π and μ under M = 2 as the start point for our model.

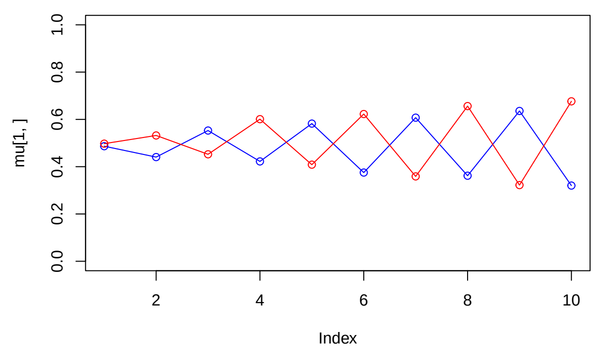


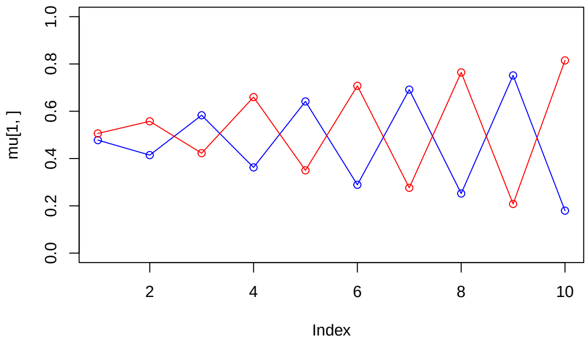


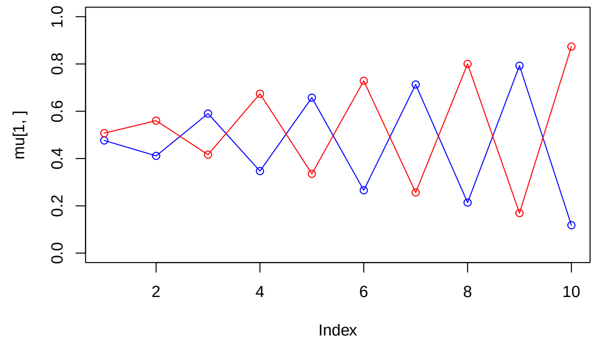


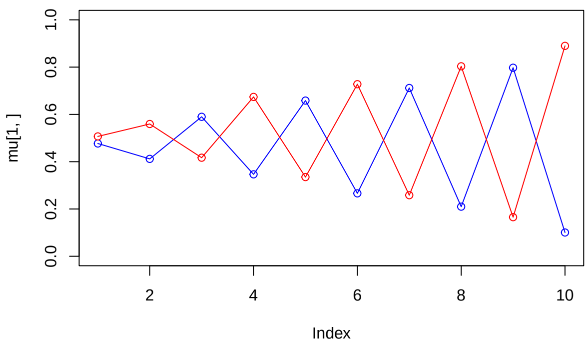


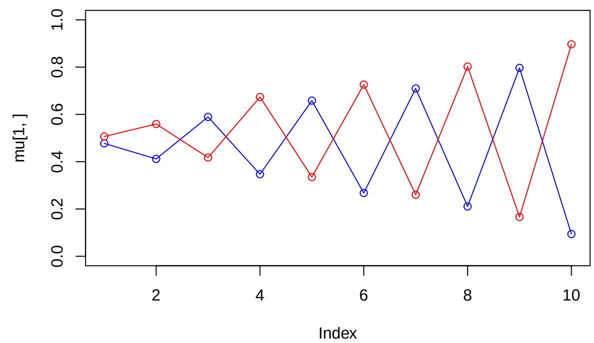


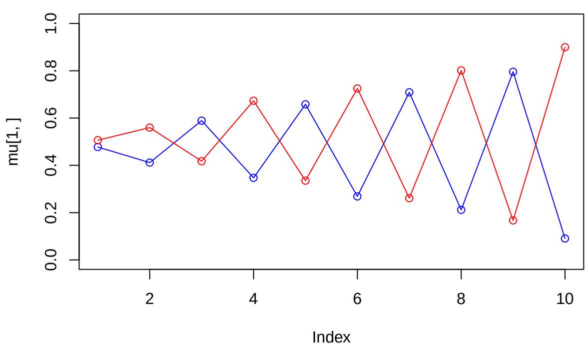


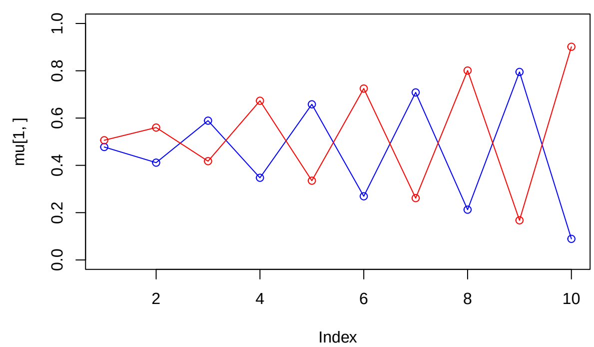








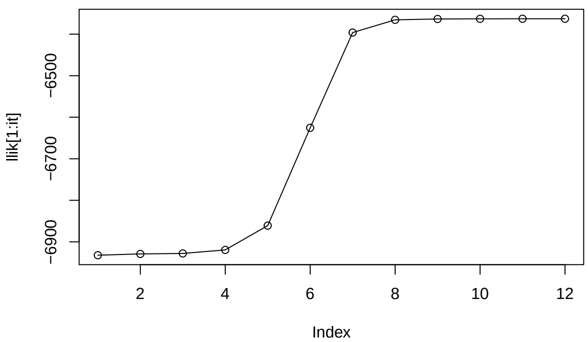





```
##  [1] "iteration:  1 log likelihood:  -6932.1625943363"
##  [2] "iteration:  2 log likelihood:  -6929.09536498845"
##  [3] "iteration:  3 log likelihood:  -6927.94380531483"
##  [4] "iteration:  4 log likelihood:  -6919.43478131895"
##  [5] "iteration:  5 log likelihood:  -6860.93456150249"
##  [6] "iteration:  6 log likelihood:  -6625.43370682142"
##  [7] "iteration:  7 log likelihood:  -6396.10695394879"
##  [8] "iteration:  8 log likelihood:  -6365.47503740155"
##  [9] "iteration:  9 log likelihood:  -6363.41093129657"
## [10] "iteration:  10 log likelihood:  -6363.02364141854"
## [11] "iteration:  11 log likelihood:  -6362.91755252171"
## [12] "iteration:  12 log likelihood:  -6362.88538329492"
```

```
## [1] 0.4979156 0.5020844
```

```
##           [,1]      [,2]      [,3]      [,4]      [,5]      [,6]      [,7]
## [1,] 0.4776009 0.4114577 0.5889962 0.3474863 0.6581537 0.2691781 0.7083954
## [2,] 0.5062796 0.5599234 0.4177178 0.6731561 0.3351926 0.7249220 0.2614677
##           [,8]      [,9]      [,10]
## [1,] 0.2125293 0.7950371 0.08913449
## [2,] 0.8010174 0.1675785 0.90147909
```

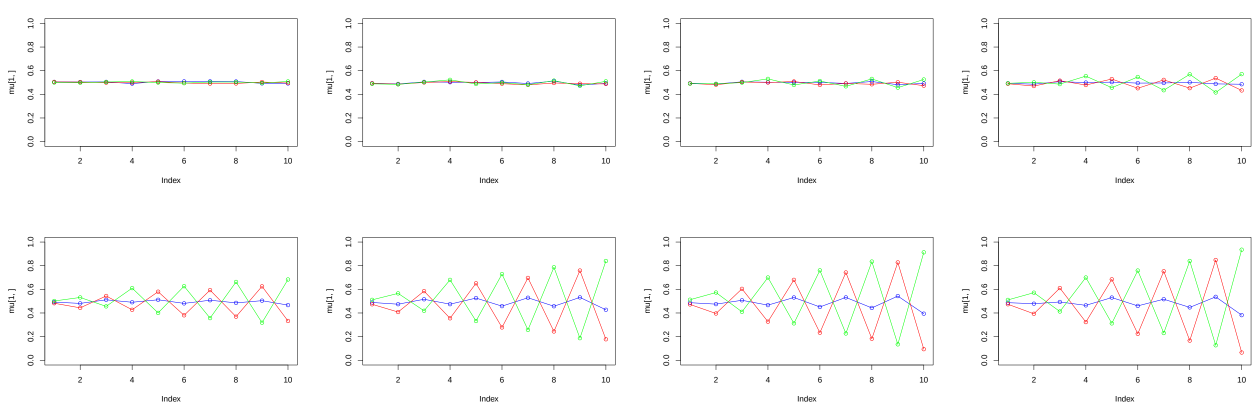


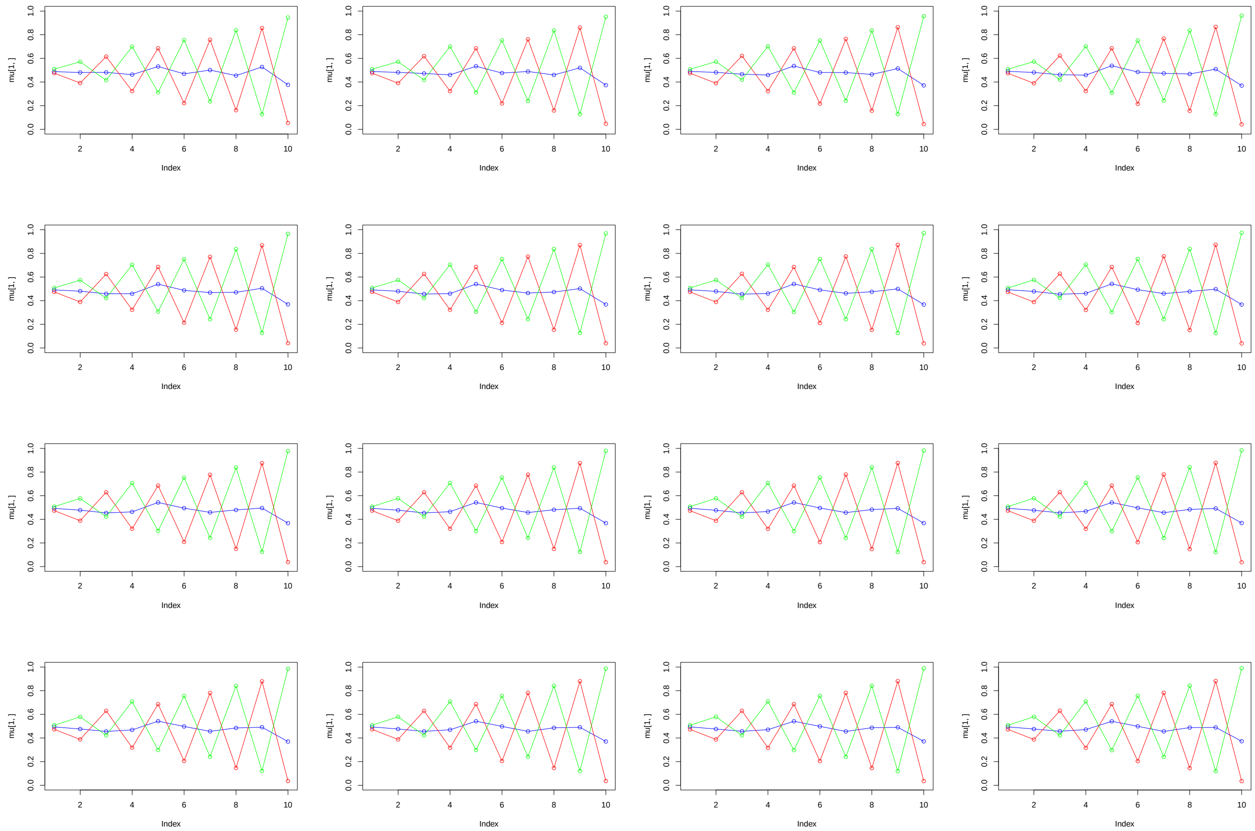
From the plots above. We can see that under this situation, μ which is equal to 0.5 every dimension seems did not being reflect. It might be that the other 2 μs are symmetrical to the 0.5 axis.
And for every iteration the predicted μ is closer and closer to the *true* μ. The rapidly raise of log-likelihood during iteration 5-7 is also reflect by the plot from significant changing of μ in corresponding iterations. The final predicted μ shows above is relatively close to the true value within around 10% differences.

```
## [1] 0.3359578 0.3290183 0.3350239
```

```
##           [,1]      [,2]      [,3]      [,4]      [,5]      [,6]      [,7]
## [1,] 0.5049455 0.5041509 0.5051382 0.4905578 0.5089228 0.5094530 0.5097480
## [2,] 0.5048207 0.5036886 0.4985467 0.4991731 0.5071384 0.4953800 0.4908757
## [3,] 0.4996279 0.4982070 0.5043346 0.5085042 0.4994862 0.4945702 0.5041462
##           [,8]      [,9]     [,10]
## [1,] 0.5082991 0.4926313 0.4921113
## [2,] 0.4917657 0.5040657 0.4956302
## [3,] 0.5040348 0.4955050 0.5088683
```

Then we goes to M = 3, again generate a π and μ under M = 3 as the start point for our model.

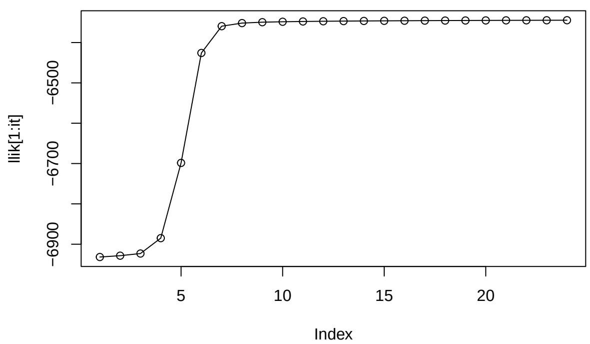




```
##  [1] "iteration:  1 log likelihood:  -6931.75122861544"
##  [2] "iteration:  2 log likelihood:  -6928.46695128332"
##  [3] "iteration:  3 log likelihood:  -6923.13091589768"
##  [4] "iteration:  4 log likelihood:  -6884.94468114899"
##  [5] "iteration:  5 log likelihood:  -6698.4237807576"
##  [6] "iteration:  6 log likelihood:  -6425.806406635"
##  [7] "iteration:  7 log likelihood:  -6359.4223002037"
##  [8] "iteration:  8 log likelihood:  -6351.74320723516"
##  [9] "iteration:  9 log likelihood:  -6349.54335699899"
## [10] "iteration:  10 log likelihood:  -6348.4434579416"
## [11] "iteration:  11 log likelihood:  -6347.73893891381"
## [12] "iteration:  12 log likelihood:  -6347.21907132148"
## [13] "iteration:  13 log likelihood:  -6346.8026280888"
## [14] "iteration:  14 log likelihood:  -6346.45328398564"
## [15] "iteration:  15 log likelihood:  -6346.15273441616"
## [16] "iteration:  16 log likelihood:  -6345.89062787433"
## [17] "iteration:  17 log likelihood:  -6345.66037682444"
## [18] "iteration:  18 log likelihood:  -6345.45729529842"
## [19] "iteration:  19 log likelihood:  -6345.2777404626"
## [20] "iteration:  20 log likelihood:  -6345.11870927259"
## [21] "iteration:  21 log likelihood:  -6344.97764443873"
## [22] "iteration:  22 log likelihood:  -6344.852335347"
## [23] "iteration:  23 log likelihood:  -6344.74086068395"
## [24] "iteration:  24 log likelihood:  -6344.64154880195"
```

```
## [1] 0.2663361 0.3438780 0.3897860
```

```
##             [,1]      [,2]     [,3]      [,4]      [,5]      [,6]      [,7]
## [1,] 0.4938980 0.4758006 0.457040 0.4711376 0.5411178 0.4986316 0.4555200
## [2,] 0.4728756 0.3874569 0.630045 0.3163584 0.6869103 0.2039956 0.7823031
## [3,] 0.5075751 0.5799061 0.422322 0.7099547 0.2967462 0.7569457 0.2402904
##             [,8]     [,9]       [,10]
## [1,] 0.4878804 0.489488 0.37258194
## [2,] 0.1446699 0.881597 0.03501224
## [3,] 0.8422855 0.119219 0.98958938
```

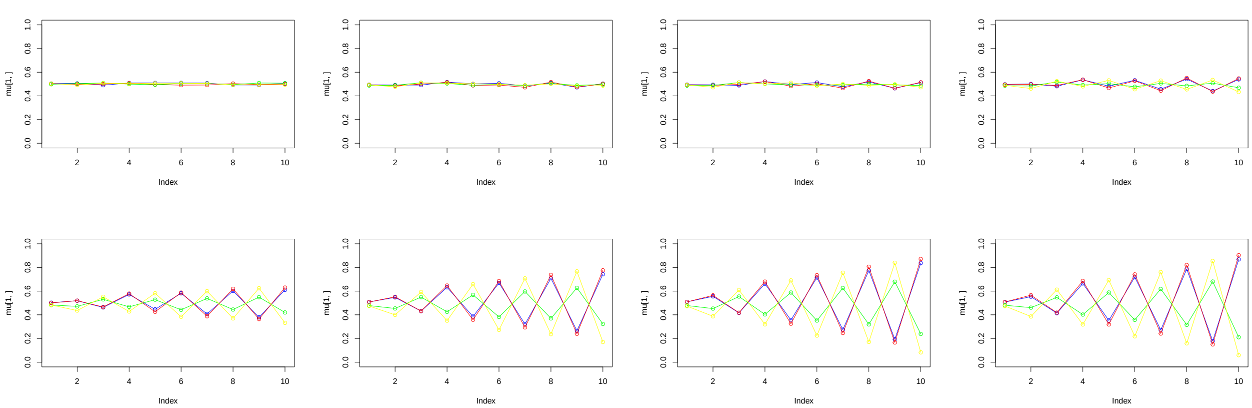


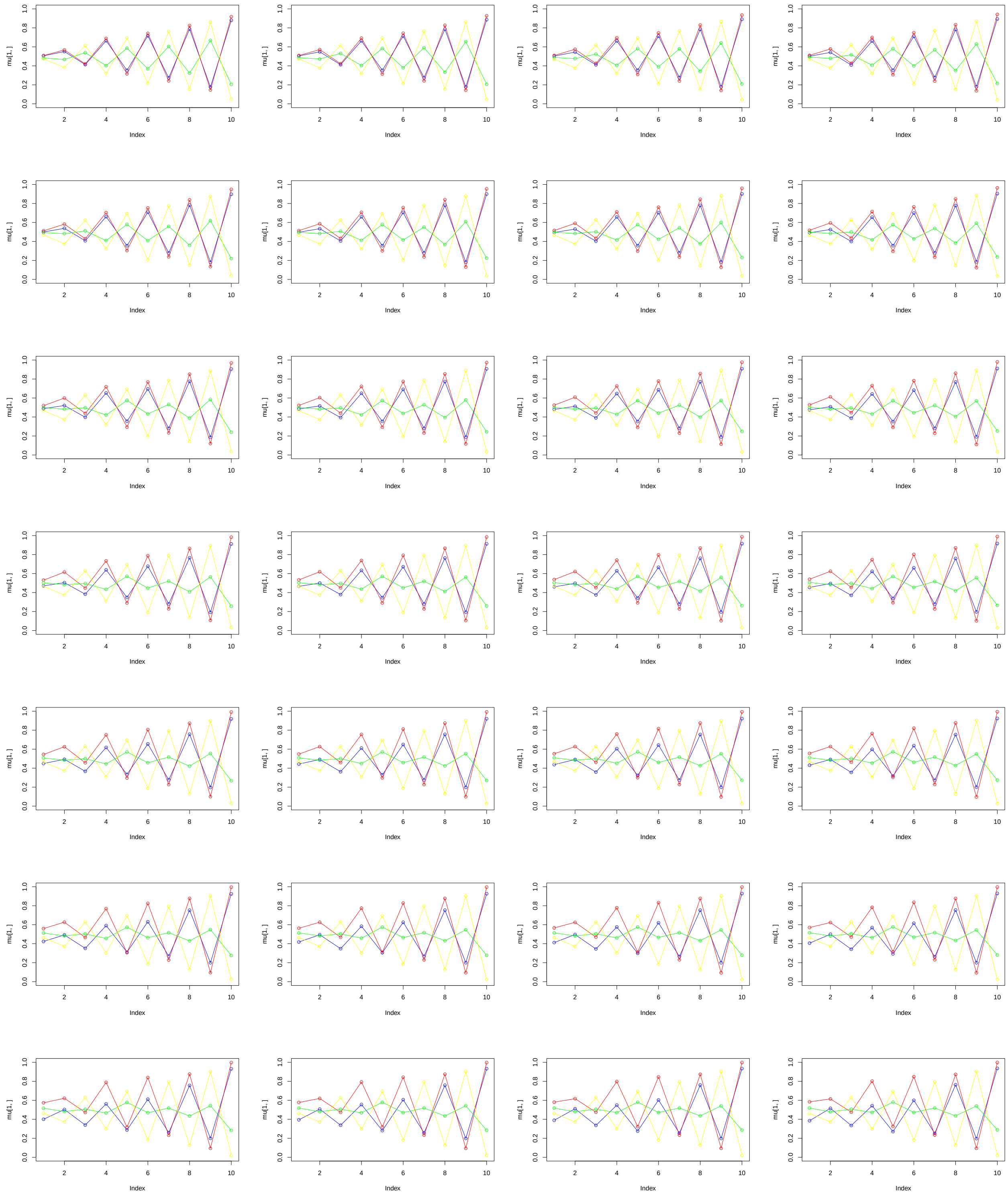
We can see that the log-likelihood rises rapidly during iteration 4-7, its also clearly shown by μ values plot. Then from final result we can see that it predict the true values well, especially in comparison with following M = 4.

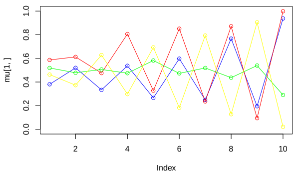
```
## [1] 0.2518811 0.2466783 0.2511809 0.2502598
```

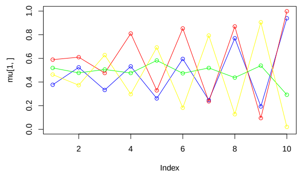
```
##             [,1]      [,2]      [,3]      [,4]      [,5]      [,6]      [,7]
## [1,] 0.5041509 0.5051382 0.4905578 0.5089228 0.5094530 0.5097480 0.5082991
## [2,] 0.5036886 0.4985467 0.4991731 0.5071384 0.4953800 0.4908757 0.4917657
## [3,] 0.4982070 0.5043346 0.5085042 0.4994862 0.4945702 0.5041462 0.5040348
## [4,] 0.5037389 0.4922173 0.5069624 0.5039756 0.5065369 0.5073122 0.5049473
##             [,8]      [,9]     [,10]
## [1,] 0.4926313 0.4921113 0.5048207
## [2,] 0.5040657 0.4956302 0.4996279
## [3,] 0.4955050 0.5088683 0.5072302
## [4,] 0.4943372 0.4951750 0.4940898
```

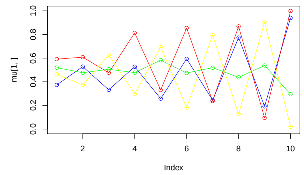
Then we change M to 4, again generate a π and μ under M = 4 as the start point for our model.

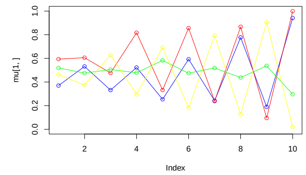


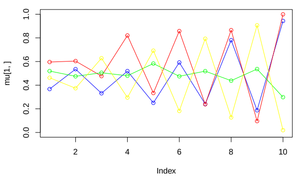

mu[1,]
0.0
0.2
0.4
0.6
0.8
1.0
2
4
6
8
10
Index

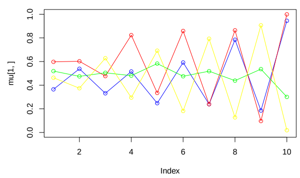


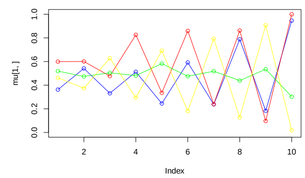










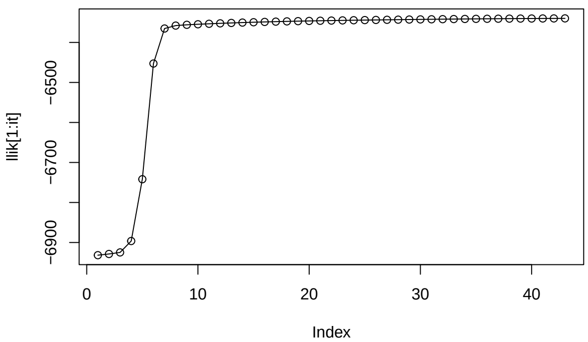



```
##  [1] "iteration:  1 log likelihood:  -6931.66006994864"
##  [2] "iteration:  2 log likelihood:  -6928.66727678128"
##  [3] "iteration:  3 log likelihood:  -6924.77054344575"
##  [4] "iteration:  4 log likelihood:  -6896.29508885115"
##  [5] "iteration:  5 log likelihood:  -6741.85787805056"
##  [6] "iteration:  6 log likelihood:  -6452.71193962151"
##  [7] "iteration:  7 log likelihood:  -6365.22915492218"
##  [8] "iteration:  8 log likelihood:  -6357.90724015069"
##  [9] "iteration:  9 log likelihood:  -6355.94977356183"
## [10] "iteration:  10 log likelihood:  -6354.59564838264"
## [11] "iteration:  11 log likelihood:  -6353.40988750751"
## [12] "iteration:  12 log likelihood:  -6352.31517546734"
## [13] "iteration:  13 log likelihood:  -6351.30092218428"
## [14] "iteration:  14 log likelihood:  -6350.36818071176"
## [15] "iteration:  15 log likelihood:  -6349.51652090179"
## [16] "iteration:  16 log likelihood:  -6348.74223300225"
## [17] "iteration:  17 log likelihood:  -6348.03936943375"
## [18] "iteration:  18 log likelihood:  -6347.40094567819"
## [19] "iteration:  19 log likelihood:  -6346.81968070248"
## [20] "iteration:  20 log likelihood:  -6346.28834868353"
## [21] "iteration:  21 log likelihood:  -6345.79994106786"
## [22] "iteration:  22 log likelihood:  -6345.34778016045"
## [23] "iteration:  23 log likelihood:  -6344.92564619259"
## [24] "iteration:  24 log likelihood:  -6344.52792780162"
## [25] "iteration:  25 log likelihood:  -6344.14978243089"
## [26] "iteration:  26 log likelihood:  -6343.78728731843"
## [27] "iteration:  27 log likelihood:  -6343.43756285041"
## [28] "iteration:  28 log likelihood:  -6343.09885115178"
## [29] "iteration:  29 log likelihood:  -6342.77053143643"
## [30] "iteration:  30 log likelihood:  -6342.4530520589"
## [31] "iteration:  31 log likelihood:  -6342.14776283848"
## [32] "iteration:  32 log likelihood:  -6341.85664504623"
## [33] "iteration:  33 log likelihood:  -6341.58196036309"
## [34] "iteration:  34 log likelihood:  -6341.32586665114"
## [35] "iteration:  35 log likelihood:  -6341.09006544297"
## [36] "iteration:  36 log likelihood:  -6340.87554389922"
## [37] "iteration:  37 log likelihood:  -6340.68245157758"
## [38] "iteration:  38 log likelihood:  -6340.51011833165"
## [39] "iteration:  39 log likelihood:  -6340.35718728741"
## [40] "iteration:  40 log likelihood:  -6340.22181690354"
```

```
## [41] "iteration:  41 log likelihood:  -6340.10190243326"
## [42] "iteration:  42 log likelihood:  -6339.99527661232"
## [43] "iteration:  43 log likelihood:  -6339.89986543919"
```

```
## [1] 0.1838186 0.2308466 0.2874187 0.2979162
```

```
##           [,1]      [,2]      [,3]      [,4]      [,5]      [,6]      [,7]
## [1,] 0.3627106 0.5416076 0.3311785 0.5125500 0.2451860 0.5918027 0.2376350
## [2,] 0.5991155 0.6005747 0.4768083 0.8260669 0.3357255 0.8593412 0.2392769
## [3,] 0.5188276 0.4740528 0.5042778 0.4812818 0.5823257 0.4755847 0.5177702
## [4,] 0.4628904 0.3744350 0.6280787 0.2945785 0.6916641 0.1817551 0.7930593
##           [,8]       [,9]      [,10]
## [1,] 0.7878580 0.18327423 0.94572455
## [2,] 0.8624625 0.09719753 0.99920005
## [3,] 0.4387190 0.53515949 0.30265960
## [4,] 0.1275007 0.90649071 0.01848283
```



We can see that after the rapid rising of log-likelihood, the iteration 8-15 seems still have a reasonable μ value, but then it falls into overfitting and the result μ and π values seems not so favorable.

```
# Code Appendix

set.seed(1234567890)
max_it <- 100 # max number of EM iterations
min_change <- 0.1 # min change in log lik between two consecutive iterations
n=1000 # number of training points
D=10 # number of dimensions
x <- matrix(nrow=n, ncol=D) # training data
true_pi <- vector(length = 3) # true mixing coefficients ; p(y)
true_mu <- matrix(nrow=3, ncol=D) # true conditional distributions ; p(xcol | y = Row)
true_pi=c(1/3, 1/3, 1/3)
true_mu[1,]=c(0.5,0.6,0.4,0.7,0.3,0.8,0.2,0.9,0.1,1)
true_mu[2,]=c(0.5,0.4,0.6,0.3,0.7,0.2,0.8,0.1,0.9,0)
true_mu[3,]=c(0.5,0.5,0.5,0.5,0.5,0.5,0.5,0.5,0.5,0.5)
plot(true_mu[1,], type="o", col="blue", ylim=c(0,1))
points(true_mu[2,], type="o", col="red")
points(true_mu[3,], type="o", col="green")
# Producing the training data x
for(i in 1:n) {
  m <- sample(1:3,1,prob=true_pi)
  for(d in 1:D) {
    x[i,d] <- rbinom(1,1,true_mu[m,d])
```

```r
  } }

set.seed(1234567890)
M <- 2 # number of clusters
w <- matrix(nrow=n, ncol=M) # weights ; p(y = m | xi,thetaHat)
pi <- vector(length = M) # mixing coefficients
mu <- matrix(nrow=M, ncol=D) # conditional distributions
llik <- vector(length = max_it) # log likelihood of the EM iterations
# Random initialization of the parameters
pi <- runif(M,0.49,0.51)
pi <- pi / sum(pi)
for(m in 1:M) {
  mu[m,] <- runif(D,0.49,0.51)
}
pi
mu

set.seed(1234567890)
iterLog <- vector(length = max_it)
for(it in 1:max_it) {
  plot(mu[1,], type="o", col="blue", ylim=c(0,1))
  points(mu[2,], type="o", col="red")
  # points(mu[3,], type="o", col="green")
  #points(mu[4,], type="o", col="yellow")
  Sys.sleep(0.5)
  # E-step: Computation of the weights

  for (i in 1:n) {
    # px <- 0
    pxi <- 0
    for (m in 1:M) {
      bernXMum <- 1
      for (d in 1:D) {
        bernXMum <- bernXMum * (mu[m,d]^x[i,d]) * (1-mu[m,d])^(1-x[i,d])
        # print(paste(i,m,d,bernXMum))
      }
      pxi <- pxi + pi[m] * bernXMum
      # print(paste(i,m,d,pxi))
    }
    for (m in 1:M) {
      bernXMum <- 1
      for (d in 1:D) {
        bernXMum <- bernXMum * (mu[m,d]^x[i,d]) * (1-mu[m,d])^(1-x[i,d])
        # print(paste(i,m,d,bernXMum))
      }
      w[i,m] <- (bernXMum * pi[m]) / pxi
    }
    w[i, ] <- w[i,] /sum(w[i,])
  }


  # Your code here
  #Log likelihood computation.
```

```
  llik[it] <- 0
  for (i in 1:n) {
    pxi <- 0
    for (m in 1:M) {
      bernXMum <- 1
      for (d in 1:D) {
        bernXMum <- bernXMum * (mu[m,d]^x[i,d]) * (1-mu[m,d])^(1-x[i,d])
      }
      pxi <- pxi + pi[m] * bernXMum
    }
    llik[it] <- llik[it] + log(pxi)
  }


  # Your code here
  iterLog[it] <- paste("iteration: ", it, "log likelihood: ", llik[it])
  flush.console()
  # Stop if the lok likelihood has not changed significantly
  stopFlag <- it > 1 && (llik[it] - llik[it - 1]) < min_change
  if(stopFlag) break
  #M-step: ML parameter estimation from the data and weights
  # pi mu
  pi <- apply(w, 2, mean)
  mu <- t(w) %*% x / colSums(w)
  # Your code here
}
print(iterLog[1:it])
pi
mu
plot(llik[1:it], type="o")

set.seed(1234567890)
M <- 3 # number of clusters
w <- matrix(nrow=n, ncol=M) # weights ; p(y = m | xi,thetaHat)
pi <- vector(length = M) # mixing coefficients
mu <- matrix(nrow=M, ncol=D) # conditional distributions
llik <- vector(length = max_it) # log likelihood of the EM iterations
# Random initialization of the parameters
pi <- runif(M,0.49,0.51)
pi <- pi / sum(pi)
for(m in 1:M) {
  mu[m,] <- runif(D,0.49,0.51)
}
pi
mu


iterLog <- vector(length = max_it)
for(it in 1:max_it) {
    # plotChoose <- c(1, 5, 6, 12, 24)
  # if (it == any(plotChoose)) {
    plot(mu[1,], type="o", col="blue", ylim=c(0,1))
    points(mu[2,], type="o", col="red")
```

```r
  points(mu[3,], type="o", col="green")
  # points(mu[4,], type="o", col="yellow")
# }
Sys.sleep(0.5)
# E-step: Computation of the weights

for (i in 1:n) {
  # px <- 0
  pxi <- 0
  for (m in 1:M) {
    bernXMum <- 1
    for (d in 1:D) {
      bernXMum <- bernXMum * (mu[m,d]^x[i,d]) * (1-mu[m,d])^(1-x[i,d])
      # print(paste(i,m,d,bernXMum))
    }
    pxi <- pxi + pi[m] * bernXMum
    # print(paste(i,m,d,pxi))
  }
  for (m in 1:M) {
    bernXMum <- 1
    for (d in 1:D) {
      bernXMum <- bernXMum * (mu[m,d]^x[i,d]) * (1-mu[m,d])^(1-x[i,d])
      # print(paste(i,m,d,bernXMum))
    }
    w[i,m] <- (bernXMum * pi[m]) / pxi
  }
  w[i, ] <- w[i,] /sum(w[i,])
}


# Your code here
#Log likelihood computation.
llik[it] <- 0
for (i in 1:n) {
  pxi <- 0
  for (m in 1:M) {
    bernXMum <- 1
    for (d in 1:D) {
      bernXMum <- bernXMum * (mu[m,d]^x[i,d]) * (1-mu[m,d])^(1-x[i,d])
    }
    pxi <- pxi + pi[m] * bernXMum
  }
  llik[it] <- llik[it] + log(pxi)
}


# Your code here
iterLog[it] <- paste("iteration: ", it, "log likelihood: ", llik[it])
flush.console()
# Stop if the lok likelihood has not changed significantly
stopFlag <- it > 1 && (llik[it] - llik[it - 1]) < min_change
if(stopFlag) break
#M-step: ML parameter estimation from the data and weights
```

```
  # pi mu
  pi <- apply(w, 2, mean)
  mu <- t(w) %*% x / colSums(w)
  # Your code here
}

print(iterLog[1:it])
pi
mu
plot(llik[1:it], type="o")

set.seed(1234567890)
M <- 4 # number of clusters
w <- matrix(nrow=n, ncol=M) # weights ; p(y = m | xi,thetaHat)
pi <- vector(length = M) # mixing coefficients
mu <- matrix(nrow=M, ncol=D) # conditional distributions
llik <- vector(length = max_it) # log likelihood of the EM iterations
# Random initialization of the parameters
pi <- runif(M,0.49,0.51)
pi <- pi / sum(pi)
for(m in 1:M) {
  mu[m,] <- runif(D,0.49,0.51)
}
pi
mu


iterLog <- vector(length = max_it)
for(it in 1:max_it) {
  # plotChoose <- c(1, 15, 43)
  # if (it == any(plotChoose)) {
    plot(mu[1,], type="o", col="blue", ylim=c(0,1))
    points(mu[2,], type="o", col="red")
    points(mu[3,], type="o", col="green")
    points(mu[4,], type="o", col="yellow")
  # }
  Sys.sleep(0.5)
  # E-step: Computation of the weights

  for (i in 1:n) {
    # px <- 0
    pxi <- 0
    for (m in 1:M) {
      bernXMum <- 1
      for (d in 1:D) {
        bernXMum <- bernXMum * (mu[m,d]^x[i,d]) * (1-mu[m,d])^(1-x[i,d])
        # print(paste(i,m,d,bernXMum))
      }
      pxi <- pxi + pi[m] * bernXMum
      # print(paste(i,m,d,pxi))
    }
    for (m in 1:M) {
      bernXMum <- 1
```

```r
      for (d in 1:D) {
        bernXMum <- bernXMum * (mu[m,d]^x[i,d]) * (1-mu[m,d])^(1-x[i,d])
        # print(paste(i,m,d,bernXMum))
      }
      w[i,m] <- (bernXMum * pi[m]) / pxi
    }
    w[i, ] <- w[i,] /sum(w[i,])
  }


  # Your code here
  #Log likelihood computation.
  llik[it] <- 0
  for (i in 1:n) {
    pxi <- 0
    for (m in 1:M) {
      bernXMum <- 1
      for (d in 1:D) {
        bernXMum <- bernXMum * (mu[m,d]^x[i,d]) * (1-mu[m,d])^(1-x[i,d])
      }
      pxi <- pxi + pi[m] * bernXMum
    }
    llik[it] <- llik[it] + log(pxi)
  }


  # Your code here
  iterLog[it] <- paste("iteration: ", it, "log likelihood: ", llik[it])
  flush.console()
  # Stop if the lok likelihood has not changed significantly
  stopFlag <- it > 1 && (llik[it] - llik[it - 1]) < min_change
  if(stopFlag) break
  #M-step: ML parameter estimation from the data and weights
  # pi mu
  pi <- apply(w, 2, mean)
  mu <- t(w) %*% x / colSums(w)
  # Your code here
}

print(iterLog[1:it])
pi
mu
plot(llik[1:it], type="o")
```