Master in
High Performance Computing

Theory and Practice of Finite Element Methods

Lagrangian Finite Element Spaces

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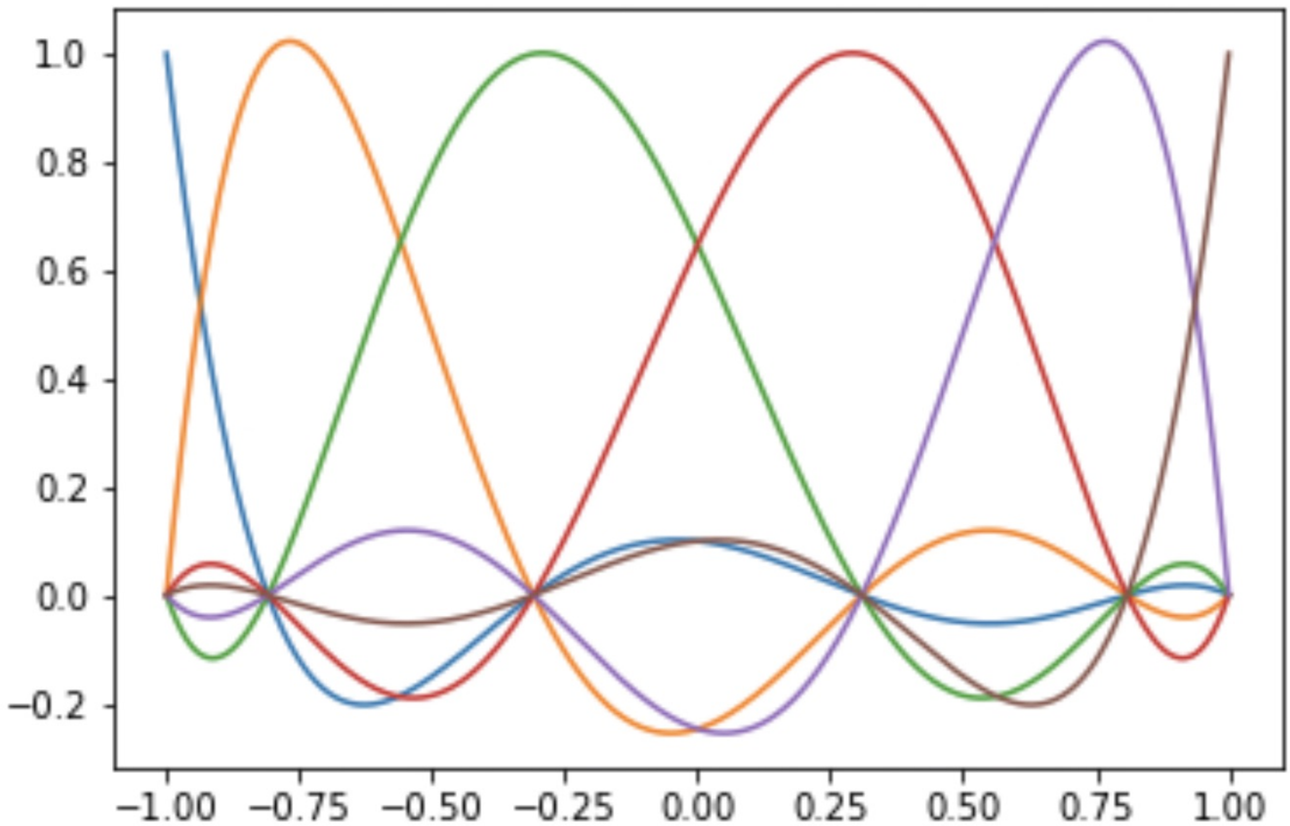
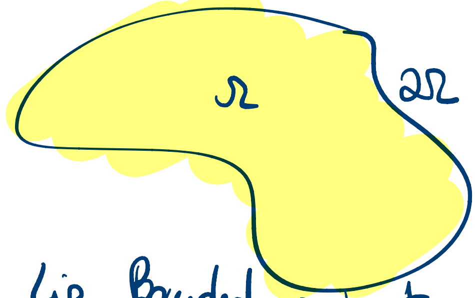


Prototypical Problem

$$
1)\ \boxed{\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}}
$$



Ω, Lip. Bounded subset of $\mathbb{R}^d$

$d = 1, 2, 3$

Set the problem in a Hilbert Space V

$$
V = H_0^1(\Omega) := \{ v \in L^2(\Omega),\ \nabla v \in L^2(\Omega),\ v|_{\partial\Omega} = 0 \}
$$

Multiply 1) by v, integrate by parts:

$$
\int_\Omega -\Delta u\, v = \int_\Omega f v \qquad \forall v \in V \equiv H_0^1(\Omega)
$$

$$
\underbrace{\boxed{\int_\Omega \nabla u \nabla v}}_{a(u,v)} - \underbrace{\cancel{\int_{\partial\Omega} \frac{\partial u}{\partial n} v}}_{\to 0} \underbrace{\boxed{= \int_\Omega f v}}_{\langle F, v\rangle} \qquad \forall v \in V \equiv H_0^1(\Omega)
$$

$a(u,v)$ bylinear on $V \times V$

$\langle F, v\rangle := \int_\Omega f v$ F is linear on V $\mathcal{L}(V, \mathbb{R})$

$F \in V'$

$$
\forall a : V \times V \to \mathbb{R} \qquad \exists!\ A : V \longrightarrow V'
$$

$$
a(u,v) = \langle Au, v\rangle
$$

Variational formulation: Given $F \in V'$, find $u \in V$ s.t.

$$
\boxed{a(u,v) = \langle F, v\rangle \quad \forall v \in V} \iff \boxed{Au = F \quad \text{in } V'}
$$

if a is coercive (and bounded) then

$\forall F \in V'$, $\exists! u \in V$ s.t. $Au = F$ and

$$\|u\|_V \leq \frac{1}{\alpha} \|F\|_{V'}$$

recall: a coercive means: $\exists\ \alpha > 0$ s.t.

$$a(u,u) \geq \alpha \|u\|^2 \qquad \forall u \in V$$

Construct V_h finite dimensional, s.t. $V_h \subset V$

Search for $u_h \in V_h$ s.t.

$$(2)\quad \boxed{a(u_h, v_h) = \langle F, v_h \rangle \qquad \forall v_h \in V_h}$$

$$A u_h = F \qquad \text{in } V_h'$$

By Lax Milgram, $\exists!\ u_h$ that satisfies (2)

$$\Rightarrow \boxed{a(u - u_h, v_h) = 0 \qquad \forall v_h \in V_h}$$

orthogonality of error

$$\alpha \|u - u_h\|^2 \leq a(u - u_h, u - u_h) = a(u - u_h, u - v_h)$$

$$\leq \|A\| \|u - u_h\| \|u - v_h\| \qquad \forall v_h \in V_h$$

$$\boxed{\|u - u_h\| \leq \frac{\|A\|}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|}$$

Cea's Lemma

How to construct V_h ?

$$V_h = \text{span}\{v_i\}_{i=1}^{n} \quad \text{where } n = \text{dimension}(V_h)$$

$$\Rightarrow \forall u_h \in V_h \quad \exists! \{u^i\}_{i=1}^{n} \in \mathbb{R}^n \text{ s.t.}$$

$$u_h(x) = u^i v_i(x)$$ sum is implied

$\exists$ n dual basis functions $\{v^i\}_{i=1}^{n} \in V'$ s.t.

$$\forall u_h(x) \quad \langle v^i, u_h \rangle = u^i \quad \forall i = 1, \dots, n$$

$$\Leftrightarrow \quad v^i(v_j) = \delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

think of $\{v^i\}_{i=1}^{n}$ as a "basis for V_h'"

$$\Rightarrow A\underline{u} = \mathbb{F} \qquad A \in \mathbb{R}^{n \times n} \qquad \mathbb{F} \in \mathbb{R}^n$$

$$\underline{u} = \{u^i\}_{i=1}^{n} \in \mathbb{R}^n$$

$$A_{ij} u^j = \mathbb{F}_i \qquad A_{ij} := \int_\Omega \nabla v_j \nabla v_i = \langle \underbrace{A v_j}_{\in V'}, v_i \rangle$$

$$\mathbb{F}_i := \int_\Omega f v_i = \langle \underbrace{F}_{\in V'}, v_i \rangle$$

If I know $\{v^i\}_{i=1}^{n}$ then I can define

$$\Pi_{V_h} : V \longrightarrow V_h$$

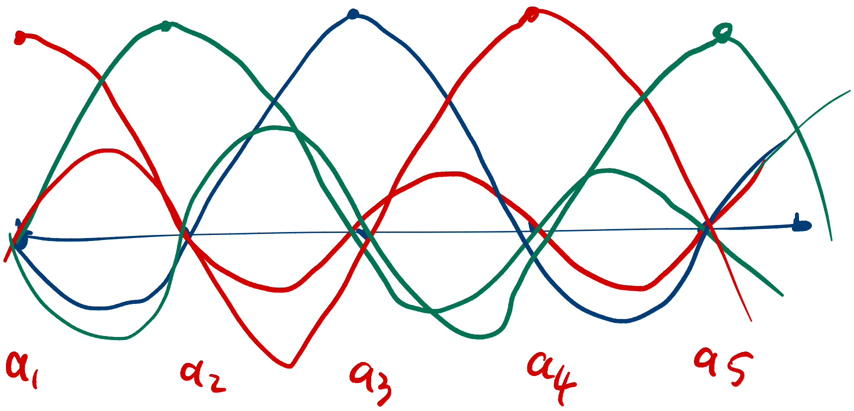
$$u \longrightarrow \underbrace{\langle v^i, u \rangle}_{u^i} v_i$$

<u>1D</u>: $\Omega = [0,1]$ $\qquad V_h = \mathbb{P}^{n-1}([0,1])$

- Select a set of support points $\{a_i\}_{i=1}^n$ $\quad a_1 < a_2 < \ldots < a_n$
- Define $\quad v^i(u) = u(a_i) \qquad \forall u \in H^1(0,1)$
- Construct $\quad v_i(x) \quad$ s.t. $\quad v_i(x) \in \mathbb{P}^{n-1}$, and

$$v^j(v_i) = v_i(a_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$v_i \in \mathbb{P}^{n-1}$



Lagrange Polynomials for $\{a_i\}_{i=1}^n$

$$v_i := \prod_{\substack{i \neq j \\ j=1}}^{n} \frac{(x - a_j)}{(a_i - a_j)}$$

The set $\{v^i\}_{i=1}^n$ is called "nodal basis functions"

Lagrange basis functions:

$$\boxed{v_i = \prod_{\substack{j \neq i \\ j=1}}^{n} \frac{(x - a_j)}{(a_i - a_j)}}$$

$$\boxed{v^i := \delta(x - a_i)} \Rightarrow \langle v^i, u \rangle = \int_\Omega v^i u = u(a_i)$$

$$\Pi_{V_h} : V \longrightarrow V_h$$

$$u \longrightarrow \langle v^i, u \rangle v_i = u^i v_i = \sum_{i=1}^{n} u(a_i) v_i$$

$\{v^i\}$ Allows you to put V_h in one to one relation with $\mathbb{R}^n$

For $H_0^1([0,1])$ we choose $\{a_i\}_{i=0}^{n+1}$, $\dim \equiv n$

We set $\{v^i\}_{i=1}^{n} = \delta(x - a_i)$ $\qquad a_0 = 0, \; a_{n+1} = 1$

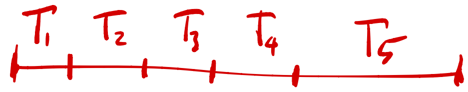
$$v_i = \prod_{\substack{j=0 \\ i \neq j}}^{n+1} \frac{(x - a_j)}{(a_i - a_j)} \in \mathcal{P}^{n+1}$$

and define $\Pi_{V_h} u : v^i(u)\, v_i$

$\boxed{\text{sum is between } 1 \text{ and } n}$

Split the domain into $\mathcal{T}_h$ which is a collection of simple domains $\boxed{\text{Triangulation}}$ *

$$\mathcal{T}_h = \overset{\circ}{\overline{\left(\bigcup_{k=1}^{m} \overline{T_k}\right)}}$$

1D 

choose $\{x_i\}_{i=1}^{m+1}$ s.t. $x_i \neq x_j$ if $i \neq j$

define $T_i = (x_i, x_{i+1})$ $\quad i = 1, \dots, m$

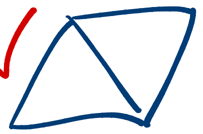
$$\mathcal{T}_h \equiv (0,1)$$

2D Assume that Ω is polygonal

$$\mathcal{T}_h = \overset{\circ}{\overline{\left(\bigcup_{k=1}^{m} \overline{T_k}\right)}}$$

T_k is triangle with vertices V_{ki} $\quad k = 1, \dots, m, \quad i = 1, \dots, 3$

- $\overline{T_i} \cap \overline{T_j} = \{\phi\}$
  $= \{\text{edge of triangle}\}$
  $= \{\text{common vertex}\}$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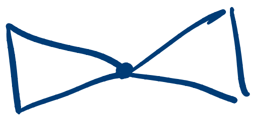
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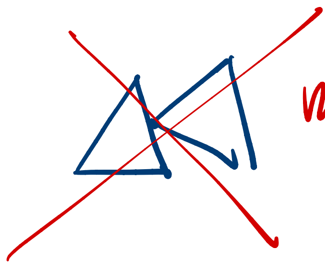
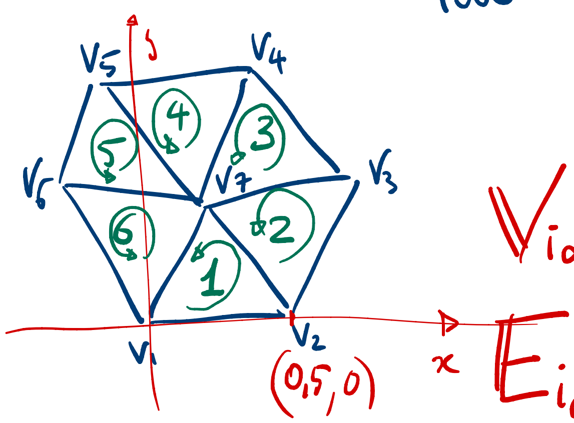

not ok

In higher dimensions the same

Two lists:
- vertices $V \in \mathbb{R}^{N_v \times d}$
- elements $E \in \mathbb{N}^{m \times n_v(d)}$ $\quad n_v(d) = d+1$



$V_{i\alpha} :=$ coordinate α of vertex i

$E_{i\alpha} :=$ index of vertex α that belongs to element i

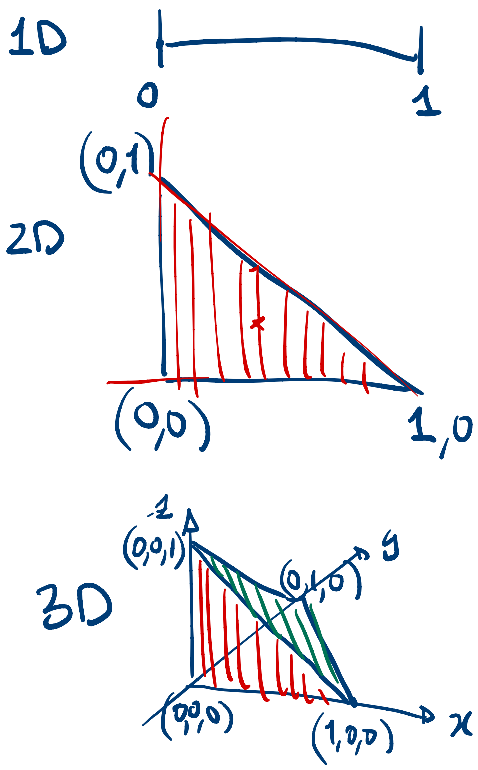
For simplices $\quad n_v(d) = d+1$

$V_{11} = 0 \qquad V_{12} = 0 \qquad V_{21} = 0.5 \qquad V_{22} = 0$

$E_{11} = 1 \qquad E_{12} = 2 \qquad E_{13} = 7$

For every T_k, we create a local basis.

$\boxed{\text{How?}}$ we start with $\boxed{\text{"Reference Element"}}$ *



Reference element $\hat{T}$

$$\hat{T} := \left\{ \hat{x} \;\middle|\; 0 < \sum_{i=1}^{d} \hat{x}_i < 1 \right\}$$

Finite dimensional space on $\hat{T}$:

Polynomials in $\mathbb{R}^d$ of order k

$$\dim(P^k(\hat{T})) = \binom{d+k}{k} = \hat{n}_k$$

		$k=1$
$d=1 \Rightarrow$	$\hat{n}_k = k+1$	2
$d=2 \Rightarrow$	$\hat{n}_k = \frac{1}{2}(k+1)(k+2)$	3
$d=3 \Rightarrow$	$\hat{n}_k = \frac{1}{6}(k+1)(k+2)(k+3)$	4

How do we choose a basis for $P^k(\hat{T})$

1) basis on $P^k(\hat{T})\Big|_{\text{Face of } \hat{T}\ F}$ to be a basis for $P^k(F)$

2) define the basis in terms of support points that

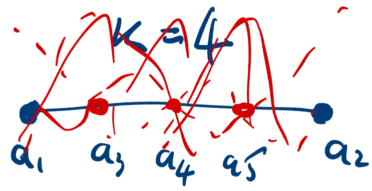
Give as much as possible on ∂T

Start with 1D: we need $k+1$ support points

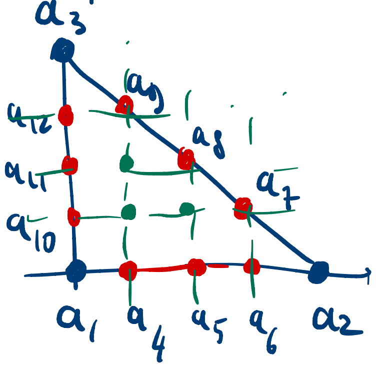
→ •) 2 support points are taken by vertices

••) the rest are internal

•••) Lagrangian basis on a_i



Go up to 2D Remember $\hat{n}_k = \frac{1}{2}(k+1)(k+2)$



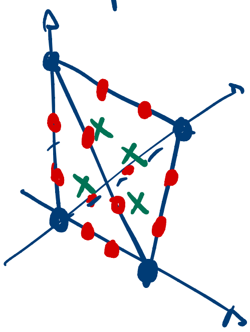
1 per vertex (3)

3 per edge (9)

3 per triangle

$k=4$
$\Downarrow$
$\underline{\underline{15}}$

Go up to 3D (take $k=3$)



n_k 1D $= 4$

n_k 2D $= 10$

n_k 3D $= 20$ → 4 vertices + 2 edges × 6 + $1_{\text{Triangle}} \times 4$

no support point inside Tetraedra

Now define $\hat{v}_i$ s.t. $\hat{v}_i(\hat{a}_j) = \delta_{ij}$ $\hat{v}_i \in \mathbb{P}^k(\hat{T})$

Formal definition of $\boxed{\text{Finite Element}}$ *

Triple $\{\hat{T}, \hat{P}, \hat{\Sigma}\}$ $\hat{\Sigma} = \{\hat{v}^i\}_{i=1}^{n_p}$

i) • $\hat{T}$: Reference cell

ii) $P(\equiv \mathbb{P}^k(\hat{T}))$: Finite dimensional space (Polynomial) ii). $\hat{\Sigma}$: a basis for $(\hat{\mathcal{P}})'$ (dual basis of nodal basis)

iii) equivalent to saying that $\{\hat{v}^i(p)\}_{i=1}^{n_p} \in \mathbb{R}^n$ is a bijective operation on $\hat{\mathcal{P}}$

$\boxed{\hat{\Sigma} \text{ is also called "set of degrees of freedom"}}$

iii) $\Rightarrow \quad \exists! \ \{v_i\}_{i=1}^{n_p}$ s.t. $v^j(v_i) = \delta^j{}_i$

s.t. $\mathcal{P} = \operatorname{span}\{v_i\}_{i=1}^{n_p}$

$\boxed{\{T, P, \Sigma\} \text{ is a Lagrangian FE space}}$ *

if $\exists \ \{a_i\}_{i=1}^{n_p}$ s.t. $\Sigma = \{\delta(x - \hat{a}_i)\}_{i=1}^{p}$

a_i are "support points"

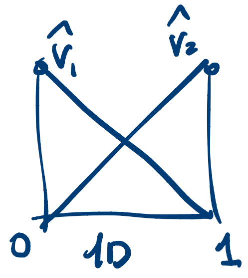
$$\Rightarrow \Pi_p : C^0(\hat{T}) \longrightarrow P$$

$$u \longrightarrow \langle v^i, u \rangle v_i = u(\hat{a}_i) v_i$$

is called $\boxed{\text{Lagrange interpolation}}$ *

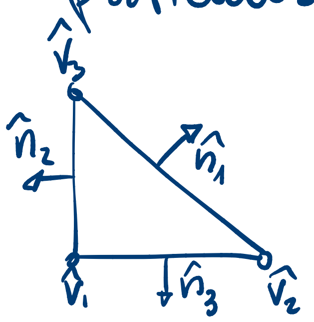
General case : particular basis: $P^1(\hat{T})$

$\hat{a}_i = \hat{v}_i$



$\hat{v}_1 = \hat{x}$

$\hat{v}_2 = 1 - \hat{x}$



$$v_i = 1 - \frac{(\hat{x} - \hat{a}_i)\,\hat{n}_i}{(\hat{a}_j - \hat{a}_i)\cdot n_j}$$

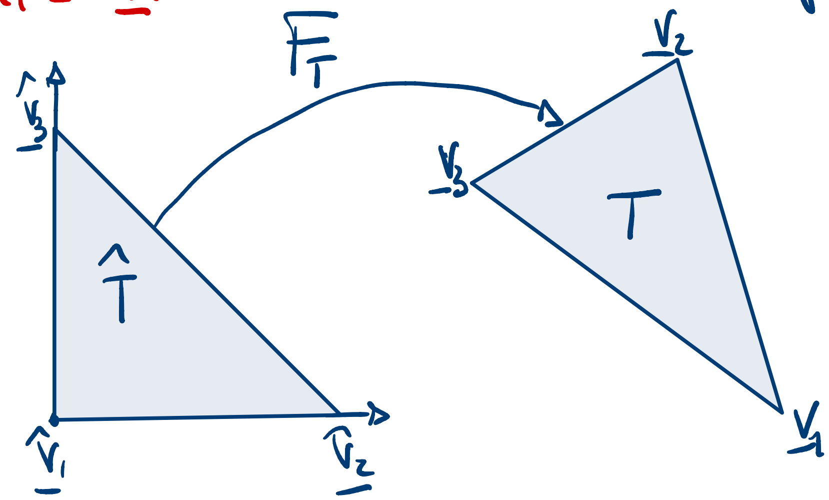
any $j \neq i$

2D

$$\boxed{\begin{aligned} v_1 &= 1 - x - y \\ v_2 &= x \\ v_3 &= y \end{aligned}}$$

$\hat{a}_i \equiv \hat{\underline{v}}_i$

Any triangle $T = \underline{v}_1, \underline{v}_2, \underline{v}_3$ can be written as a push forward of $\hat{T}$:



$$F_T : \hat{K} \longrightarrow K$$

$$F_T(\hat{x}) = \sum_{i=1}^{n_v} \underline{v}_i \; v_i(\hat{x})$$

$$F_T(\hat{\underline{v}}_i) = \underline{v}_i$$

Given $\{\hat{T}, \hat{P}, \hat{\Sigma}\}$ we can write $\forall T$

$\{T, P, \Sigma\}$ as the push forward

through the mapping F_T of $\{\hat{T}, \hat{P}, \hat{\Sigma}\}$

$$\Rightarrow \qquad v_i \circ F_T := \hat{v}_i$$

$$T = F_T(\hat{T})$$

$$\Sigma := \{\delta(x - F_T(\hat{a}_i))\}_{i=1}^{n_k}$$