Master in High Performance Computing

Theory and Practice of Finite Element Methods

BNB condition, saddle point problems

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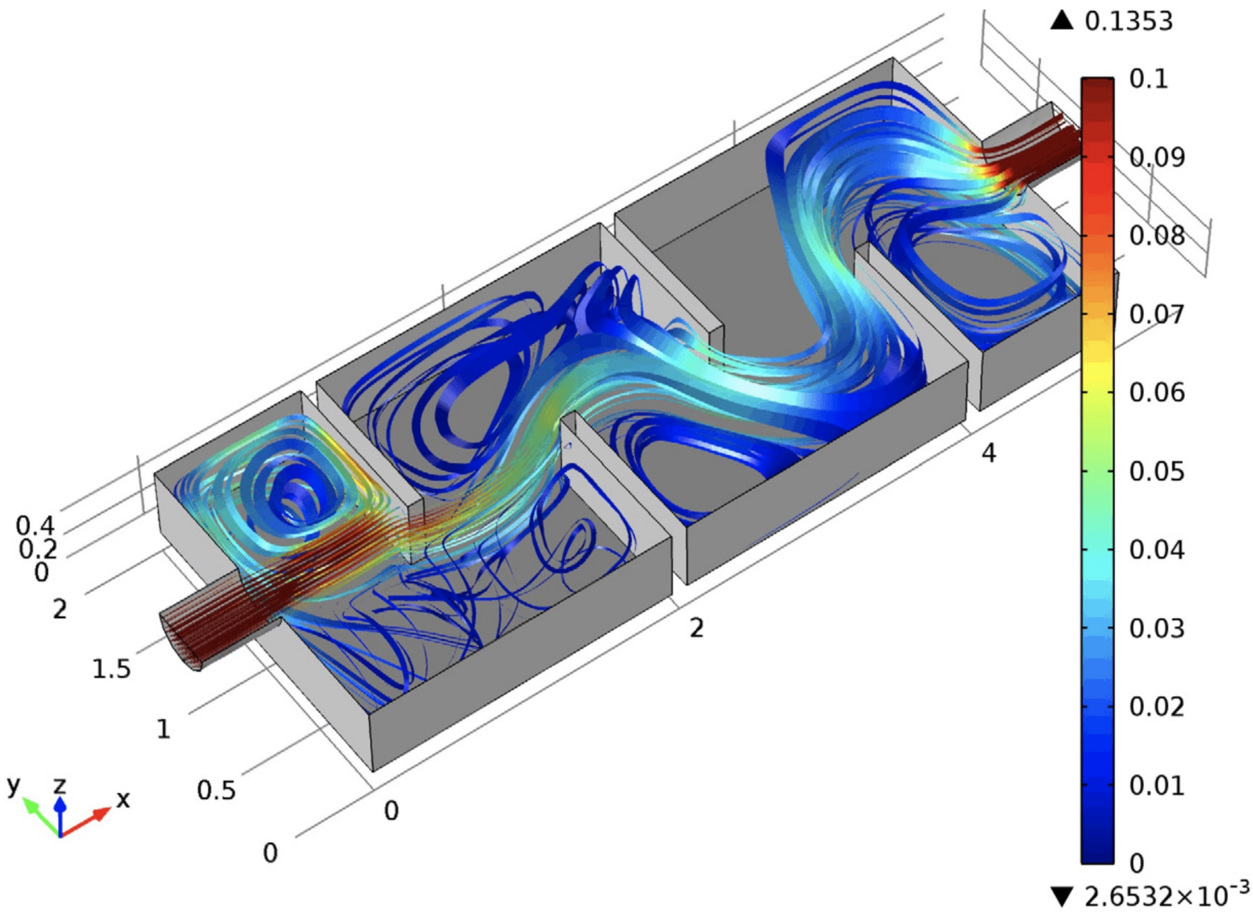


General setting: Banach Spaces

V Banach

W Banach and reflexive

$A \in \mathcal{L}(V, W')$ $\quad a \in \mathcal{L}(V \times W; \mathbb{R})$

$$\boxed{\begin{aligned} A : V &\longrightarrow W' \\ u &\longmapsto Au \in W' \end{aligned}} \iff \begin{aligned} a : V \times W &\longrightarrow \mathbb{R} \\ u, v &\longmapsto a(u,v) \end{aligned}$$

$\boxed{\text{Pb}}$ Given $f \in W'$, find $u \in V$ s.t.

$$Au = f \text{ in } W' \iff a(u,v) = \langle f, v \rangle \quad \forall v \in W$$

$\boxed{\text{Well posedness (Hadamard)}}$

$\exists\, \alpha$ s.t. $\forall f \in W'$ [1], $\exists!\, u$ [2] $\in V$ s.t.

$$\|u\|_V \leq \frac{1}{\alpha} \|f\|_{W'} = \frac{1}{\alpha} \|Au\|_{W'}$$

1) A has to be surjective
2) A has to be injective
3) A has to be bounding

1, 2, 3 are different faces of the same medal

- Closed Range Theorem
- Open Map theorem

$$A : V \longrightarrow W' \qquad A \in \mathcal{L}(V, W')$$

$$A^T : W \longrightarrow V' \qquad a \in \mathcal{L}(V \times W, \mathbb{R})$$

$$K = \ker(A) = \{ v \in V \ \text{s.t.} \ \langle Av, w \rangle = 0 \quad \forall w \in W \}$$

$$H = \ker(A^T) = \{ w \in W \ \text{s.t.} \ \langle v, A^T w \rangle = 0 \quad \forall v \in V \}$$

$Z \subset H$ we call "polar of Z" or "annihilator" of Z

$$Z^\circ := \{ f \in H' \mid \langle f, z \rangle = 0 \quad \forall z \in Z \}$$

By construction and continuity of $\langle \cdot, \cdot \rangle$ we have that

$$\boxed{Z^\circ = \overline{Z^\circ}} \qquad \boxed{\ker(A) = \overline{\ker(A)}}$$

$$\boxed{\begin{aligned} \ker(A) &= \operatorname{Im}(A^T)^\circ \\ \ker(A^T) &= \operatorname{Im}(A)^\circ \end{aligned}}$$

$$0 = \langle Au, w \rangle = \langle u, A^T w \rangle \quad \forall v \in K$$

$$0 = \langle u, A^T w \rangle = \langle Au, w \rangle \quad \forall w \in H$$

Closed Range Theorem

- $(Z^\circ)^\circ = Z \iff Z = \overline{Z}$
- $\boxed{\ker(A)^\circ = \operatorname{Im}(A^T)} \iff \operatorname{Im}(A^T) = \overline{\operatorname{Im}(A^T)}$
- $\boxed{\ker(A^T)^\circ = \operatorname{Im}(A)} \iff \operatorname{Im}(A) = \overline{\operatorname{Im}(A)}$

Simple and trivial example

$$A : H^1_0(\Omega) \equiv V \longrightarrow L^2(\Omega) \equiv W' \quad (= L^2)$$

$$Av = v \qquad \forall v \in H^1_0(\Omega)$$

A is not surjective $\quad (\exists\, q \in L^2$ s.t. $\nabla q \notin L^2 \quad ($ ⊣ $))$

$H^1_0(\Omega) \equiv \mathrm{Im}(A)$ is dense in $L^2(\Omega)$

But $\mathrm{Im}(A)$ is NOT closed in L^2

$$\ker(A) = \{0\}$$

$$A^T : L^2 \longrightarrow H^{-1}(\Omega) = (H^1_0(\Omega))' = V'$$

$$\forall q \in L^2 \longrightarrow (A^T q)(u) = \int_\Omega u q = \langle u, A^T q \rangle = \langle u, q \rangle$$

$$(u,q) = \langle u, q \rangle = 0 \quad \forall q \in L^2 \Rightarrow u = 0$$

$$\ker(A^T) = \{0\}$$

In finite dimensions: $\ker(A) = \{0\}$ $\ker(A^T) = \{0\}$

$\Rightarrow$ A is invertible

In Banach Spaces it is not enough

$\forall\, w_n \in L^2$ cauchy s.t. $\quad w_n \to w \notin V \equiv H^1_0(\Omega)$

and $w_n \in \mathrm{Im}(A)$, $\Rightarrow \exists\, v_n \in V$ s.t. $Av_n = w_n$

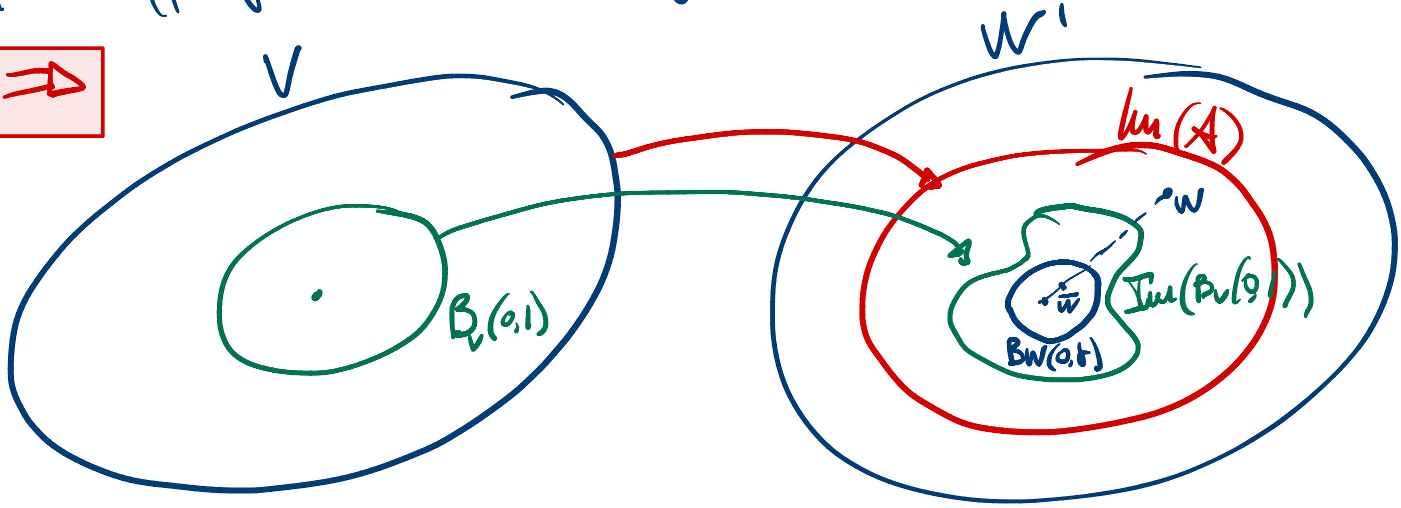
but $\quad v_n \not\to v \in V \qquad \mathrm{Im}(A) \neq \overline{\mathrm{Im}(A)}$

Theo (Open Mapping and the Closed Range Theorems)

$$\boxed{\mathrm{Im}(A) = \overline{\mathrm{Im}(A)}} \iff \boxed{\begin{array}{l}\exists \alpha \mid \forall w' \in \mathrm{Im}(A)\ \exists u \text{ s.t. } Au = w' \\ \|Au\|_{W'} = \|w'\|_{W'} \geq \alpha \|u\|_V\end{array}}$$

Open Mapping theo: A is surjective, $A(U)$ is open $\forall U$ open

$\boxed{\Rightarrow}$



i) If $\mathrm{Im}(A) = \overline{\mathrm{Im}(A)}$ then $\overline{\mathrm{Im}(A)}$ is a linear subspace

$\Rightarrow$ we can apply open map theorem with $\mathrm{Im}(A)$ as target space. A: surjective between

$V \longrightarrow \mathrm{Im}(A) \equiv \tilde{W}'$

ii) $B_V(0,1)$: unit ball in $V := \{ v \in V \text{ s.t. } \|v\|_V < 1 \}$ (open set)

iii) $A(B_V(0,1)) \supset O_W \Rightarrow \exists \gamma \text{ s.t. } B_W(0,\gamma) \subset \mathrm{Im}(B_V(0,1))$

iv) $\forall 0 \neq w \in \mathrm{Im}(A) \quad \overline{w} = \frac{\gamma}{2} \frac{w}{\|w\|_{W'}} \in B_W(0,\gamma)$

$\Rightarrow \exists z \in B_V(0,1) \text{ s.t. } Az = \overline{w} \quad \Rightarrow \|z\| \leq 1$

v) $v = z \cdot \frac{2\|w\|_{W'}}{\gamma} \Rightarrow Av = w$

$$\Rightarrow \|v\| \leq \|z\| \cdot \frac{2}{\gamma}\|w\|_{W'} \leq 1 \cdot \|Av\| \frac{2}{\gamma}$$

$$\|Av\| \geq \frac{2}{\gamma}\|v\| \qquad \underline{\text{QED}}$$

$\boxed{\Leftarrow}$

$\|Av\| \geq \alpha\|v\| \implies \operatorname{Im}(A) = \overline{\operatorname{Im}(A)}$

1) $w_n \in \operatorname{Im}(A)$ Cauchy $\Rightarrow \exists w$ s.t. $w_n \to w \in W'$

2) $\forall w_n \in \operatorname{Im}(A)\ \exists v_n$ s.t. $Av_n = w_n$

$$\|w_n\|_{W'} = \|Av_n\|_{W'} \geq \alpha\|v_n\|_V$$

w_n Cauchy $\Rightarrow$ v_n is also cauchy

$\Rightarrow \exists v \in V$ s.t. $v_n \to v$

A is continuous $\Rightarrow Av_n \longrightarrow Av \in \operatorname{Im}(A)$

$\Rightarrow \operatorname{Im}(A)$ is closed

Equivalent statements

same for A^T

i) A^T is surjective

ii) A is injective and $\mathrm{Im}(A) = \overline{\mathrm{Im}(A)}$

iii) A is bounding ($\exists \alpha \mid \|Au\|_{W'} \geq \alpha \|u\|_V$)

iv) the inf sup condition is satisfied

$$\exists \alpha \quad \text{s.t.}$$

$$\inf_{v \in V} \sup_{w \in W} \frac{\langle Av, w \rangle}{\|u\|_V \|w\|_W} \geq \alpha$$

A surjective $\quad V \longrightarrow W' \quad \alpha > 0$

P1) $\forall w \in \mathrm{Im}(A) \quad \exists v_w \in V$ s.t.

$$Av_w = w \quad \text{and}$$

$$\|A v_w\|_{W'} \geq \alpha \|w\|_V$$

Implies

P2) $$\|A^T w\|_{V'} \geq \alpha \|w\|_W \qquad \forall w \in W$$

P2) $\Rightarrow$ P1) $\boxed{\text{if } V \text{ is Reflexive}}$

Hilbert Case and $\boxed{\text{BNB}}$

$$A : V \longrightarrow W' \qquad V, W \text{ Hilbert spaces.}$$

$$\exists \alpha \mid \forall f \in W' \quad \exists!\, u \in V \quad \text{s.t. } Au = f$$

$$\|u\| \leq \frac{1}{\alpha}\|f\| = \frac{1}{\alpha}\|Au\|$$

$$\Longleftrightarrow$$

BNB for Hilbert spaces

$$\boxed{\begin{array}{l} \exists\, \alpha > 0 \quad \text{s.t.} \\ \text{i)} \quad \inf\limits_{u \in V} \sup\limits_{w \in W} \dfrac{\langle Au, w \rangle}{\|u\|_V \|w\|_W} = \alpha \\ \text{ii)} \quad \inf\limits_{w \in W} \sup\limits_{u \in V} \dfrac{\langle Au, w \rangle}{\|u\|_V \|w\|_W} = \alpha \end{array}}$$

For Banach, ii) becomes $\ker(A^T) = \{0\}$

LAX MILGRAM : $W \equiv V$,

$\Longrightarrow$ BNB (the converse is false)

$$\langle Au, u \rangle \geq \alpha \|u\|^2 \quad \Rightarrow \quad \frac{\langle Au, u \rangle}{\|u\| \, \|u\|} \geq \alpha \quad \forall u \in V$$

Mixed Problems

Two Hilbert spaces: V, Q, and two operators:

$$A : V \longrightarrow V' \qquad A \in \mathcal{L}(V, V')$$

$$B : V \longrightarrow Q' \qquad B \in \mathcal{L}(V, Q')$$

Given $f \in V'$, $g \in Q'$ find (u,p) in $V \times Q$ s.t.

$$\begin{cases} Au + B^T p = f \\ Bu = g \end{cases}$$

1) $g \in \mathrm{Im}(B) \Rightarrow \exists\, u_g$ s.t. $Bu_g = g$

2) $Z = \ker(B)$ $\qquad u = u_0 + u_g$, $u_0 \in Z$

$$\Rightarrow \begin{cases} Au_0 + B^T p = f - Au_g = \tilde{f} & \text{in } V' \\ Bu_0 = 0 \end{cases}$$

Restrict our analysis to $g = 0$ $\quad (\tilde{f} = f - Au_g)$

$$\langle Au_0, v_0 \rangle + \langle B^T p, v_0 \rangle = \langle \tilde{f}, v_0 \rangle \quad \forall v_0 \in Z$$

$\langle B^T p, v_0 \rangle = 0 \quad \forall v_0 \in Z$

$\boxed{\begin{array}{l} u\text{-problem} \\ \langle A u_0, v_0 \rangle = \langle \tilde{f}, v_0 \rangle \end{array}}$ $\iff$ BNB is satisfied

1)
$$\boxed{\inf_{u_0\in Z}\sup_{v_0\in Z}\frac{\langle Au_0, v_0\rangle}{\|u\|_V\|v\|_V}=\alpha \qquad \inf_{v_0\in Z}\sup_{u_0\in Z}\frac{\langle Au_0, v_0\rangle}{\|u\|_V\|v\|_V}=\alpha}$$

ELL-KER on A

Then $\exists!\ u_0\in Z$ s.t.

$$\boxed{\|u_0\|_V\leq\frac{1}{\alpha}\|\tilde f\|_{V'}\leq\frac{1}{\alpha}\|f\|_{V'}+\frac{1}{\alpha}\|A\|_{V'}\|u_g\|_V}$$

Given u_0 solution to $\boxed{u\text{-problem}}$

Find p s.t. $(u=u_0+u_g)$

p-problem
$$\boxed{\langle B^Tp, v\rangle = -\langle Au, v\rangle+\langle f, v\rangle}$$
$$= -\langle Au_0, v\rangle+\langle\tilde f, v\rangle \qquad \forall v\in V$$

BNB$_1$ + BNB$_2$ $\rightarrow$ $\ker(B^T)=0$ $\quad \operatorname{Im}(B^T)=\overline{\operatorname{Im}(B^T)}$

B is surjective

$\exists\ \beta>0 \quad \|B^Tp\|_{V'}\geq\beta\|p\|_Q \quad \forall\ p\in Q$

INF-SUP on B

$$\Leftrightarrow \boxed{\exists\beta>0 \text{ s.t. } \inf_{p\in Q}\sup_{v\in V}\frac{\langle Bv, p\rangle}{\|v\|_V\|p\|_Q}=\beta}$$

Notice that $Au - f =: L \in V'$

$\langle L, v_0 \rangle = 0 \quad \forall \; v_0 \in Z \Rightarrow L \in Z^{\circ}$

Summary: $\forall (f, g) \in V', Q' \quad \exists! \, (u, p) \in V, Q$ s.t.

$$\boxed{\begin{cases} Au + B^T p = f \\ Bu = g \end{cases}} \qquad Z := \ker(B)$$

If and only if $\exists \, \alpha, \beta$ s.t.

$$\boxed{\begin{array}{ll} 1) & \inf_{u \in Z} \sup_{v \in Z} \dfrac{\langle Au, v \rangle}{\|u\|_V \|v\|_V} = \alpha > 0 \\[2ex] 1) & \inf_{v \in Z} \sup_{u \in Z} \dfrac{\langle Au, v \rangle}{\|u\|_V \|v\|_V} = \alpha > 0 \\[2ex] 3 & \inf_{q \in Q} \sup_{u \in V} \dfrac{\langle Bu, q \rangle}{\|u\|_V \, \|q\|} = \beta > 0 \end{array}}$$

$$\boxed{\|u_0\|_V \leq \frac{1}{\alpha} \|\tilde{f}\|_{V'} \leq \frac{1}{\alpha} \|f\|_{V'} + \frac{1}{\alpha} \|A\|_{V'} \|u_g\|_V}$$

$$\|g\|_{Q'} = \|Bu_g\|_{V'} \geq \beta \|u_g\|_V$$

$$\boxed{\|u_0\|_V \leq \frac{1}{\alpha} \|f\|_{V'} + \frac{\|A\|_{V'}}{\alpha \beta} \|g\|_{Q'}}$$

$$\|u\|_V \leq \frac{1}{\alpha}\|f\|_{V'} + \frac{\|A\|_{V'}}{\alpha\beta}\|g\|_{Q'} + \frac{1}{\beta}\|g\|_{Q'}$$

$$\boxed{\|u\|_V \leq \frac{1}{\alpha}\|f\|_{V'} + \left(\frac{\|A\|_{V'}+\alpha}{\alpha\beta}\right)\|g\|_{Q'}}$$

$$\|p\|_Q \leq \frac{1}{\beta}\|L\|_{V'} \leq \frac{1}{\beta}\left(\|A\|\,\|u\| + \|f\|_{V'}\right)$$

$$\|p\|_Q \leq \frac{\|A\|}{\beta}\left(\|f\|_{V'} + \left(\frac{\|A\|_{V'}+\alpha}{\alpha\beta}\right)\right)\|g\|_{Q'} + \frac{1}{\beta}\|f\|_{V'}$$

$$\boxed{\|p\|_Q \leq \frac{(\|A\|+1)}{\beta}\|f\|_{V'} + \left(\frac{\|A\|_{V'}+\alpha}{\alpha\beta}\right)\frac{\|A\|}{\beta}\|g\|_{Q'}}$$

1), 2), 3) $\Longleftrightarrow$ a), b) $\quad V \times Q = \mathbb{V}$

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} = \mathbb{A} \qquad \mathbb{A} : \mathcal{L}(V \times Q \to V' \times Q')$$

equivalent to 1,2,3

$\exists\ \bar{\alpha}$ s.t.

a) $$\inf_{\psi\in V}\ \sup_{\theta\in V}\ \frac{\langle \mathbb{A}\psi,\theta\rangle}{\|\psi\|\,\|\theta\|}=\bar{\alpha}$$

b) $$\inf_{\theta\in V}\ \sup_{\psi\in V}\ \frac{\langle \mathbb{A}\psi,\theta\rangle}{\|\psi\|\,\|\theta\|}=\bar{\alpha}$$