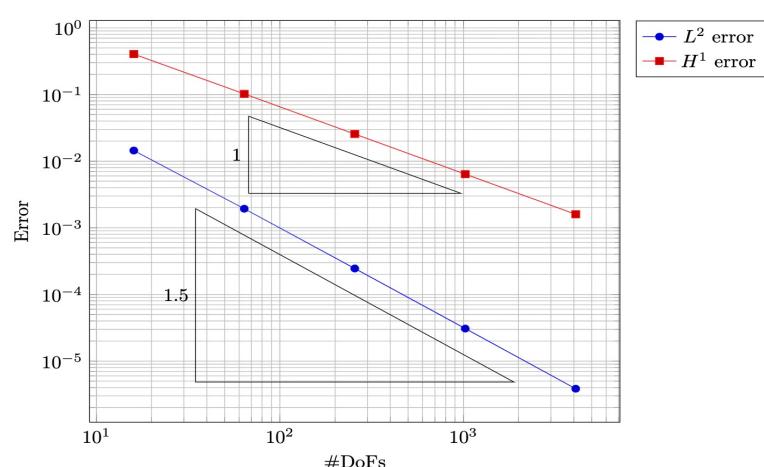
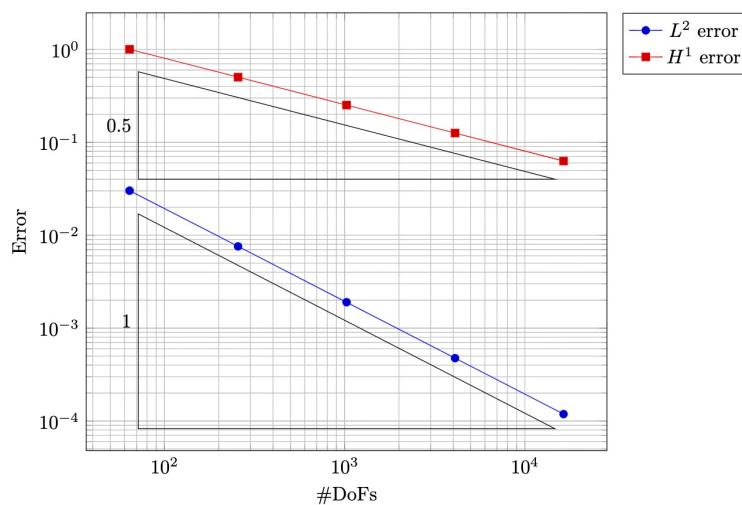


Theory and Practice of Finite Element Methods

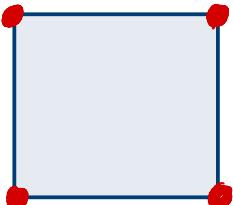
Nitsche's Lemma, Inverse, and Trace inequalities

Luca Heltai <luca.heltai@sissa.it>

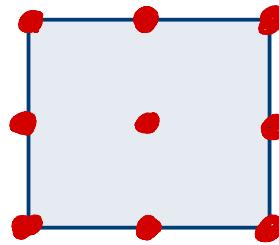
International School for Advanced Studies (www.sissa.it)
 Mathematical Analysis, Modeling, and Applications (math.sissa.it)
 Master in High Performance Computing (www.mhpc.it)
 SISSA mathLab (mathlab.sissa.it)



Q1



Q2



Denis-Lions

$$\|u\|_{k+1, p, T} \lesssim \inf_{q \in P_k(T)} \|u+q\|_{k+1, p, T} \lesssim \|u\|_{k+1, p, T}$$

Scaling Argument

$$\|v\|_{s, p, T_m} \lesssim s^{-s} h_m^{\kappa} \|u\|_{k, p, T_m}$$

Bramble - Hilbert lemma

$$\pi \in L(W^{k+1,p}(\Omega), W^{kp}(\Omega)) \text{ s.t.}$$

$$P_k(\Omega) \subset \ker(\pi)$$

$u - \Pi_k u$ is an example

$$\|\pi(u)\|_{s, p, \Omega} \lesssim \|z\|_* \|u\|_{k+1, p, \Omega}$$

$$\forall f \in V^* \quad V = H_0^1(\Omega)$$

$$Au = f \quad \text{in } V^* \quad \begin{array}{l} \text{admits unique sol} \\ \text{if } A \text{ is coercive} (\alpha) \end{array}$$

$$A u_h = f \quad \text{in } V_h^* \quad V_h \subset V$$

↓

$$\|u - u_h\|_V \lesssim \frac{\|A\|_*}{2} \inf_{v \in V_h} \|u - v\|$$

$$\|u - u_h\|_1 \lesssim \frac{\|A\|_*}{2} \|u - T_h u\|_{1,\Omega}$$

$$\lesssim \frac{\|A\|_*}{2} h^k |u|_{k+1,\Omega}$$

$$h := \max_m \{h_m\}$$

$\exists \delta > 0$ $|S_m| \geq \delta h_m \quad \forall m \quad \text{quasi uniform grid}$

Interpolation error for V_h :

$$I_h: V \rightarrow V_h$$

$$u \rightarrow \tilde{v}^i(u) v_i$$

$$V_h = \text{Span}\{v_i\}_{i=1}^N$$

$$V^T(v_i) = \delta^i_j$$

For $u \in H^{k+1}(\Omega)$, $k \geq 1$ $d=1,2,3$

$$H^{k+1}(\Omega) \subset C^0(\Omega)$$

\tilde{v}^i is H.B.
extension of v^i
on V^*

more regular space: $\tilde{v}^i \equiv v^i$

$$0 \leq s \leq l \leq k+1$$

$$\left(\sum_m \|u - T_h u\|_{s,T_m} \right) \lesssim h^{l-s} |u|_{l,\Omega}$$

$$\|u - u_h\|_1 \lesssim h |u|_{2,\Omega}$$

Assume $V \subset H$ we identify $H^* \equiv H$

$$1) \|u\|_H \lesssim \|u\|_V \quad \forall u \in V$$

$$2) V = \overline{H}^{\|\cdot\|_V}$$

For $f \in H$ we write \tilde{f} Haar-Banach extension of f to V^*

$$\langle \tilde{f}, v \rangle = (f, v)_H \quad \forall v \in V$$

$$|\langle \tilde{f}, v \rangle| = |(f, v)|_H \lesssim \|f\|_H \|v\|_H \lesssim \|f\| \|v\|_V$$

\tilde{f} is injective

Given $f \in H$ find $u \in V$ s.t.

$$\langle Au, v \rangle = \langle \tilde{f}, v \rangle = (f, v)_H \quad \forall v \in V \quad Au = \tilde{f}$$

Observation: given $g \in H$ find $\varphi_g \in V$ s.t.

$$\langle Av, \varphi_g \rangle = (g, v)_H \quad \forall v \in V$$

$$A^T \varphi_g = g$$

$\forall g \in H$, $\exists! \varphi_g \in V$ s.t. $A^T \varphi_g = g$

Theorem (Nitsche's Lemma)

$$\|u - u\|_H \lesssim \|u - u\|_{V^*} \left(\sup_{g \in H} \inf_{\varphi_g \in V} \frac{\|\varphi_g - \varphi_h\|_V}{\|g\|_H} \right)$$

We will show that \rightarrow goes to zero

Proof: $\langle A(u - u_h), v_h \rangle = 0 \quad \forall v_h \in V_h$ Cea's lemma

$$\langle Av, \varphi_g \rangle = (g, v)_H \quad \forall v \in V$$

use as test function $e := u - u_h$

$$\langle A(u - u_h), \varphi_g \rangle = (g, u - u_h)_H \quad \forall v \in V$$

we identify H with its dual, we can also define

$$\|w\|_H := \sup_{g \in H} \frac{(g, w)_H}{\|g\|_H}$$

$$(g, u - u_h)_H = \langle A(u - u_h), \varphi_g - \varphi_h \rangle$$

$$\sup \frac{(g, u - u_h)_H}{\|g\|_H} := \|u - u_h\|_H = \sup_{g \in H} \frac{|\langle A(u - u_h), \varphi_g - \varphi_h \rangle|}{\|g\|_H} \quad \forall \varphi_h \in V_h$$

$$\Rightarrow \|u - u_h\|_H \lesssim \|A\|_* \|u - u_h\|_V \left(\sup_{g \in H} \inf_{\varphi_h \in V_h} \frac{\|\varphi_g - \varphi_h\|_V}{\|g\|_H} \right)$$

Assume:

1) κ is the Sobolev degree of V $\|\cdot\|_V \sim \|\cdot\|_{\kappa, \Omega}$

2) $s(\leq \kappa)$ is the Sobolev degree of H $\|\cdot\|_H \sim \|\cdot\|_{s, \Omega}$

3) A^T is π -regular for ϵ in $[s, s_n]$

That is $\forall g \in H^e \quad s \leq l \leq s_M$

$\exists!$ sol. φ_g to $A^\top \varphi_g = g$ and

$$|\varphi_g|_{e+r} \lesssim \|g\|_e$$

Then $\inf_{\varphi \in V_h} \|\varphi_g - \varphi_h\|_V \lesssim \|\varphi_g - I_h \varphi_g\|_{k, R} \lesssim h^{-k+l+r} |\varphi_g|_{e+r}$

provided that $l+r-k \geq 0$ \uparrow can we

$$\|\varphi_g - I_h \varphi_g\|_{k, R} \lesssim h^{l+r-k} |\varphi_g|_{e+r} \lesssim h^{l+r-k} \|g\|_e$$

$$e=s \Rightarrow \|\varphi_g - I_s \varphi_g\| \lesssim h^{l+r-k} \|g\|_s$$

$$\|\mu - \mu_h\|_H \lesssim \|\mu - \mu_h\|_V h^{s+r-k}$$

Example : Poisson: A on \mathcal{D} smooth $\Rightarrow \mathcal{D} = 2$

$$\|\cdot\|_V = \|\cdot\|_{1,R} \quad \|\cdot\|_H = \|\cdot\|_{0,R}$$

$$\|\mu - \mu_h\|_{1,R} \lesssim h^k |\mu|_{k,R}$$

$$\|\mu - \mu_h\|_{0,R} \lesssim \|\mu - \mu_h\|_{1,R} h^{s+r-k}$$

$\begin{matrix} 0 & r \\ s & \end{matrix} = 1$

$$\|u - u_h\|_{1,2} \lesssim h^k \|u\|_{k+1,2}$$

$$\|u - u_h\|_{0,2} \lesssim h^{k+1} \|u\|_{k+1,2}$$

Inverse Estimates

It's trivial to say that $\|u\|_n \leq \|u\|_s$ $s \geq n$. What if the inverse is possible on T_h for elements of V_h .

We'll show: local inverse Estimates

FE: $\hat{T}, \hat{P}, \hat{\Sigma}$, 1) $\ell \geq 0$ s.t. $\hat{P} \in W^{\ell,\infty}(\hat{T})$

$\exists \delta$ s.t.

$$T_m, P_m \geq \delta h_m$$

$$2) h \leq 1$$

$$3) T_m = F_m(\hat{T}), P_m = \#_{F_m} \hat{P}, \Sigma_m = \#_{F_m} \hat{\Sigma}$$

$$F_m = T_m, P_m, \Sigma_m$$

$$0 \leq k \leq \ell$$

$$1 \leq p \leq +\infty$$

F_m is affine

$$\|v\|_{e,p,T_m} \lesssim h_m^{k-e} \|v\|_{k,p,T_m}$$

Proof 1) all finite dim. norm are equivalent.

$$\forall \hat{v} \in \hat{P} \quad \|\hat{v}\|_{e,p,\hat{T}} \lesssim \|v\|_{0,p,\hat{T}} \quad \textcircled{1}$$

$$v = \hat{v} \circ f_m^{-1} \in P_m$$

(1)

$$\|v\|_{e,p,T_m} \lesssim h_m^{-e} J_m^{\frac{1}{p}} \|\hat{v}\|_{e,p,\hat{T}} = h_m^{-e} J_m^{\frac{1}{p}} \|\hat{v}\|_{0,p,\hat{T}}$$

$$\lesssim h_m^{-e} \|v\|_{0,p,\hat{T}}$$

$$h \leq 1 \Rightarrow \|v\|_{e,p,T_m}^p = \sum_{|m| \leq e} \|v\|_{m,p,T_m}^p$$

insert (2)
 here,
 explain
 $h \leq 1$

$$\|v\|_{e,p,T_m} \lesssim h_m^{-e} \|v\|_{0,p,\hat{T}}$$

$$0 \leq k \leq e, \quad 0 \leq |\alpha| \leq e \quad \text{of multi-index}$$

$$|\alpha| \leq k \quad \|D^\alpha v\|_{0,p,T_m} \leq \|v\|_{e-k,p,T_m} \lesssim h_m^{k-e} \|v\|_{0,p,T_m}$$

$$\lesssim h_m^{k-e} \|v\|_{k,p,T_m}$$

$$e-k \leq |\alpha| \leq e \quad \exists \beta, \gamma \text{ s.t. } \alpha = \beta + \gamma \quad |\beta| = e-k \quad |\gamma| \leq k$$

$$\|D^\alpha\|_{0,p,T_m} = \|D^\beta(D^\gamma)\|_{0,p,T_m} \lesssim \|D^\gamma\|_{e-k,p,T_m}$$

$$\lesssim h^{k-e} \|D^\gamma v\|_{0,p,T_m} \lesssim h^{k-e} \|v\|_{k,p,T_m}$$

If Σ_h is quasi uniform, then $\forall v \in V_h$

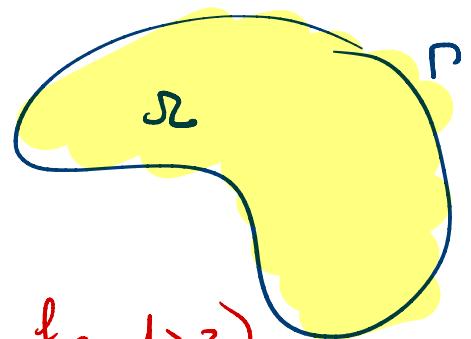
$$\|v\|_{e,\Sigma} \lesssim h^{r-e} \|v\|_{r,\Sigma}$$

is global for $0 \leq r \leq e \leq k$

When $V_h \subset H^k(\Sigma)$

Trace Spaces and Trace inequality

$u \in C^0(\bar{\Omega})$ then $u|_\Gamma$ makes sense point-wise.



What if $u \in H^k \notin C^0(\bar{\Omega})$

($H^k(\Omega)$ for $d \geq 2$)

Trace theorem, for $s \in (\frac{1}{2}, 1]$, Σ Lip. $\exists!$

linear bounded mapping $\delta: H^s(\Sigma) \rightarrow H^{s-\frac{1}{2}}(\Gamma)$

s.t.

①

$$\|\delta v\|_{s-\frac{1}{2}, \Gamma} \lesssim \|v\|_{s, \Sigma}$$

and such that $\forall u \in C^0(\bar{\Omega})$ $\delta u = u|_\Gamma$ pointwise restriction

② δ admits a bounded right inverse E

$$E: H^{s-\frac{1}{2}}(\Gamma) \longrightarrow H^s(\Sigma)$$

$$\delta E v = v$$

$$\forall v \in H^{s-\frac{1}{2}}(\Gamma)$$

Fractional Spaces

1) $\Omega \equiv \mathbb{R}^d$ Fourier Transforms:

$$(\mathcal{F}v)(\xi) := \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d} \exp(-i\xi \cdot x) v(x) dx$$

$$\|v\|_{s,\mathbb{R}^d} := \| |(1 + |\cdot|^2)^{\frac{s}{2}} \hat{v}(\cdot)| \|_{0,\mathbb{R}^d}$$

$$\mathcal{H}^c(\mathbb{R}^d) = \overline{C_c^\infty} \| \cdot \|_{s,\mathbb{R}^d}$$

2) $\Omega \subset \text{Lip}$ $s = m + \lambda$ $m \in \mathbb{N}_0, \lambda \in (0,1)$

$$\|u\|_{s,\Omega}^2 := \|u\|_{m,\Omega}^2 + \|u\|_{\lambda,\Omega}^2$$

$$\|u\|_{\lambda,\Omega}^2 := \sum_{|\alpha|=m} \iint_{\Omega \times \Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{2+2s}} dx dy$$

$$\mathcal{H}_0^s(\Omega) := \overline{C_c^\infty(\Omega)} \| \cdot \|_{s,\Omega}$$

Inverse plus Trace inequalities.

$$u \in P_h(\Gamma) \subset H^s(\kappa) \quad s > \frac{1}{2}$$

$$\|u\|_{s-\frac{1}{2}, \partial\Gamma} \lesssim \|u\|_{s, \Gamma} \lesssim h^{r-s} \|u\|_{r, \Gamma}$$

For example

$$\begin{aligned} \|\nabla u \cdot n\|_{0, \partial\Gamma} &\lesssim \|\nabla u\|_{0, \partial\Gamma} \lesssim \|\nabla u\|_{\frac{1}{2}, \Gamma} \lesssim h^{-\frac{1}{2}} \|\nabla u\|_{0, \Gamma} \\ &\lesssim h^{-\frac{1}{2}} \|u\|_{1, \Gamma} \end{aligned}$$