Prove inf sup for Stokes / Poisson Mixed form

Stokes	Poisson
$V := H_0^1(\Omega)^d$	$V := H_0^{div}(\Omega)$ $\qquad \|v\|_0 + \|div\, v\|_0$
$Q := L_0^2(\Omega)$	$Q := L_0^2(\Omega)$
$B : V \longrightarrow Q' \equiv Q$	

$$\boxed{\begin{aligned} &\langle \nabla u \nabla v\rangle \quad -(div\, v, p) = \langle f, v\rangle \\ &\langle div\, u, q\rangle \qquad\qquad = 0 \end{aligned}} \qquad \boxed{\begin{aligned} &\langle u, v\rangle - \langle div\, v, p\rangle = 0 \\ &\langle div\, u, q\rangle \qquad\qquad = \langle g, q\rangle \end{aligned}}$$

$$\langle Bv, q\rangle = \int_\Omega div\, v\, q$$

A is V-elliptic $\Rightarrow$	A is Z-elliptic
$\inf\sup_\alpha$ is satisfied $\forall v \in V$	infsup satisfied only in $\ker B$
	$\ker B : \; v \in V \mid div\, v = 0$

On B:

$$\inf_{q \in Q} \sup_{v \in V} \frac{b(v,q)}{\|v\| \|q\|} \geq \beta$$

$\forall q \in L^2$ we'd like to find $v \in V$ s.t.

$$div\, v_q = q \quad \Rightarrow \quad b(v_q, q) = \|q\|^2$$

$q \in L^2_0(\Omega) \quad : \quad q \in L^2(\Omega) \quad , \quad (q,1) = 0$

$-\Delta \psi_q = q \, , \quad \nabla \psi_q \cdot n \big|_{\partial\Omega} = 0 \qquad \psi_q \in H^1(\Omega)$ with zero average

$-\text{div}(\underbrace{\nabla \psi_q}_{:= v_q}) = q \qquad \text{div}(v_q) = q \quad \in L^2$

$\psi_q \in H^1 \Rightarrow \nabla \psi_q \in L^2(\Omega)^d$

$\|\psi_q\|_1 \lesssim \frac{1}{\alpha} \|q\|_{-1} \qquad$ if Ω - Lip. $\Rightarrow \psi_q \in H^2(\Omega)$

$\|\psi_q\|_0 + \|v_q\|_0 + \|\nabla v_q\|_0 = \|\psi_q\|_2 \lesssim \|q\|_0$

$v_q \quad \sup_{v \in V} \frac{\int \text{div}\, v \, q}{\|v\|_1 \|q\|_0} \geq \frac{\int_\Omega \text{div}\, v_q \, q}{\|q\|_0 \|v_q\|_1} = \frac{\|q\|^2}{\|q\|_0 \|v_q\|_1} \geq \frac{1}{c}$

$\boxed{\|v_q\|_1 \lesssim c \|q\|}$

Assuming inf sup on B is satisfied

FORTIN'S TRICK

if $\exists \; \Pi \in \mathcal{L}(V, V_h)$ s.t. $\exists \, c_\Pi$

1) $\|\Pi v\| \leq c_\Pi \|v\| \qquad \forall \, v \in V$

2) $b(\Pi v - v, q_h) = 0 \qquad \forall q_h \in Q_h$

Then $\boxed{\inf \sup_{h}}$ is satisfied

inf sup for B_h

$$\begin{tikzcd} V \arrow[r, "B"] \arrow[d, "\Pi"'] & Q' \arrow[d, "P_h'"] \\ V_h \arrow[r, "B_h"] & Q_h' \end{tikzcd}$$

$\forall v \in V, \forall q_h \in Q_h$

$b(\Pi v, q_h) = b(v, q_h)$

Proof

$$\sup_{v_h \in V_h} b(v_h, q_h) \geq \sup_{v \in V} b(\Pi v, q_h) = \sup_{v \in V} b(v, q_h) \geq \beta \|q_h\|$$

$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\| \, \|q_h\|} \geq \frac{\beta}{c_\Pi}$$

II Fortin's trick

Assume that inf sup is OK, and

that $V_h \supset P^{1,0}$ (piecewise linear, cont. functions)

Then if $\exists\, \Pi_2$ s.t. $\boxed{b(\Pi_2 v - v, q_h) = 0}$ (1)

and $\boxed{\|\Pi_2 v\|_0 \lesssim \|v\|_0 + h |v|_1}$ (2)

Then the operator

$$\Pi = \Pi_2 (I - \Pi_{SZ}) + \Pi_{SZ}$$

satisfies $\boxed{\text{FORTIN'S TRICK}}$

Recall that
$$\|\Pi_{SZ} v\|_1 \lesssim \|v\|_1$$
$$\|\Pi_{SZ} v - v\|_0 \lesssim h \|v\|_1$$

$$\boxed{\|\Pi_{SZ} v\|_1 + h^{-1} \|\Pi_{SZ} v - v\|_0 \lesssim \|v\|_1}$$

$$b(\Pi v - v, q_h) = b\left(\Pi_2 (I - \Pi_{SZ}) v + \Pi_{SZ} v - v, q_h\right)$$
$$= b\left((I - \Pi_{SZ}) v + \Pi_{SZ} v - v, q_h\right) = 0$$

$$\|\Pi v\|_1 = \|\Pi_2 (I - \Pi_{SZ}) v + \Pi_{SZ} v\|_1$$

$$\lesssim \|\underbrace{\Pi_2 (I - \Pi_{SZ}) v}_{\in V_h}\|_1 + \|\Pi_{SZ} v\|_1$$

$$\lesssim h^{-1} \|\Pi_2 (I - \Pi_{SZ}) v\|_0 + \|v\|_1$$

by 2

$$\lesssim h^{-1} \left(\|(I - \Pi_{SZ}) v\|_0 + h \|(I - \Pi_{SZ}) v\|_1 \right)$$

$$\lesssim h^{-1} \left(h \|v\|_1 + h \, 2 \|v\|_1 \right) + \|v\|_1 \lesssim C_\Pi \|v\|_1$$

Trivial considerations. $\dim(V_h) \geq \dim(Q_h)$

If not then $\ker(B_h) = \{0\} \Rightarrow$ only solution to $Au + B^T p = f$ is $u = 0$

Apply to $\boxed{P^1 - P^0 \text{ on triangles}}$ in 2D

$\dim(V_h) = 2N_v$ # of vertices

$\dim(Q_h) = N_T$ # of triangles

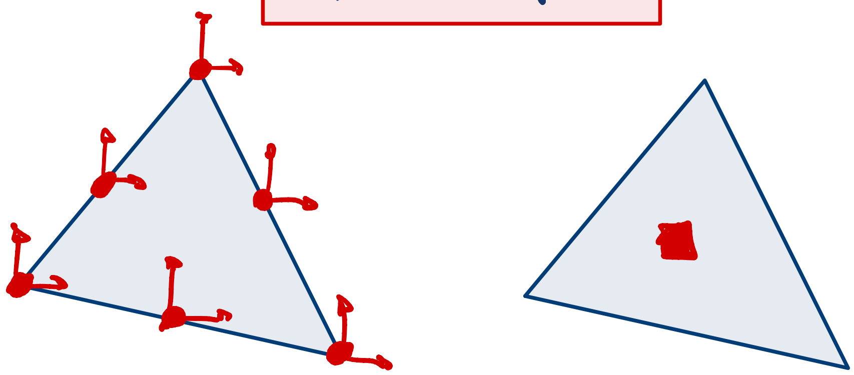
Euler law: N_{V_i} := # internal vertices
N_{V_b} := # boundary vertices

$$\boxed{N_T = 2N_{V_i} + N_{V_b} - 2}$$

$N_T > 4 \Rightarrow 2N_v < N_T$

$\Rightarrow$ $\boxed{\text{locking}}$

$$\boxed{P^2 - P^0}$$



$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{\int_\Omega \operatorname{div} v_h \, q_h}{\|v_h\| \, \|q_h\|} \geq \beta_h$$

$$Q_h = \mathbb{P}^0$$

$$\Longrightarrow \quad \sup_{v_h \in V_h} \frac{\int_K \operatorname{div} v_h}{\|v_h\|} \geq \beta_h$$

$\boxed{\text{Fortin trick}}$

Let's build Π s.t.

$$\int_K \operatorname{div}(\Pi v - v) = 0 \qquad \forall\, v \in H^1(K)$$

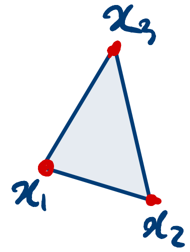
$$\sum_e \int_e (\Pi v - v) \cdot n = 0$$

Π should be s.t. $\quad \forall\, v \in (\mathbb{P}^2)^2$

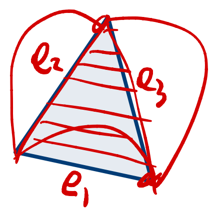
$$\Pi v = v$$

Moreover

- $(\Pi v)(x_i) = \text{"}v(x_i)\text{"} \quad \forall\, x_i$ vertex of K_i



- $\int_{e_J} \Pi v = \int_{e_J} v \quad \forall\, e_J$ edge of K_i



$b_i :=$ bubble functions on edges s.t. $\left(b_i(x_J) = 0 \;\; \forall\, x_J \text{ vertex}\right)$

$$b^J(b_i) = \delta^J_i \qquad b^J(v) := \int_{e_J} v$$

$$\Pi_2 v := b^J(v)\, b_J \qquad \Pi_2 v \in P^2$$

$$= \left(\int_{e_i} v\right) b_i$$

$$\Rightarrow \quad \int_e \Pi_2 v = \int_e v \quad \Rightarrow \int_K \operatorname{div}(\Pi v) = \int_K \operatorname{div} v$$

$$\int_{\partial K} \Pi v \cdot n = \int_{\partial K} v \cdot n$$

$$\Pi := \Pi_2 (I - SZ) + SZ$$

$$\Rightarrow \quad b(\Pi v - v, q_h) = 0$$

$$\|\Pi_2 v\|_0 \lesssim \|v\| + h\, |v|_1$$

by trace and inverse ineq.

second way:

$\boxed{\text{Macro Elements}}$ Stenberg.

Split the domain in Macroelements.

(Finite union of adjacent elements)

M is isomorphic to $\hat{M}$ if $\exists$ a map

F_M cont, invertible, and $F(\hat{M}) = M$

$F_M|_{K_J}$ is affine $\forall K_J$

Macrospaces are

$$V_{0,M} := \{ v \in (H_0^1(M))^d \quad v = w|_M ,\ w \in V_h \}$$

$$Q_{0,M} := \{ q \in L^2(M) \text{ s.t. } (q,1)_M = 0 ,\ q \in Q_h \}$$

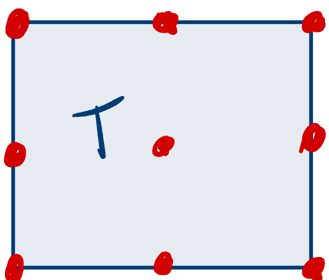
$$K_M := \ker B_h^T \text{ on } M$$

$$= \{ p \in Q_{0,M} \text{ s.t. } B_h^T p = 0 \}$$

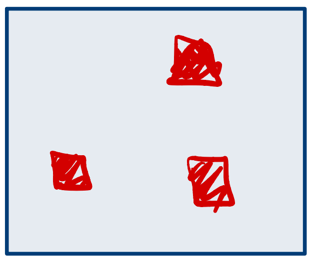
INF-SUP$_h$ is satisfied if

1) $V_h - P^0_M$ is satisfied

2) $K_M = \{0\}$



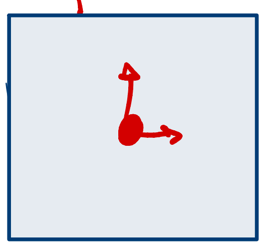
$(Q^2)^2$

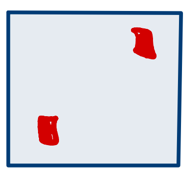


$(Q^2)^2 - P^1$

$P^1 = \text{span}\{x, y, 1\}$

$$\text{span}\{\underbrace{x, y, xy, 1}_{Q^1}, x^2, y^2, xy^2, y^2x, x^2y^2\}^2$$

Use $T \equiv M \rightsquigarrow V_{M,0} :=$ 

$Q_{M,0} :=$  $= \text{span}\{x - \bar{x}, y - \bar{y}\}$

$(Q^2)^2 - P^0$ satisfies inf sup (Fortin's trick)

$\ker B_h^T = \{0\} \iff B_h$ is non singular (if square)

has maximum rank

$B_{h,H} :=$

$\underline{b_1}, \underline{b_2}$: bubble functions $\begin{pmatrix} b_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b_2 \end{pmatrix}$

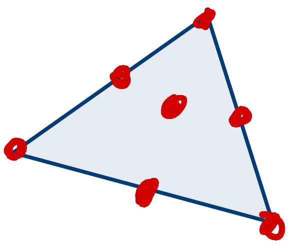
$q_1 = x - \bar{x}$

$q_2 = y - \bar{y}$

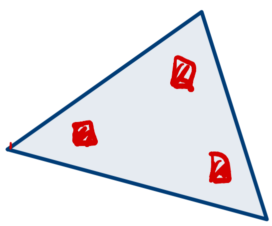
$$(B b_i, q_j) = \begin{pmatrix} -\int_T \operatorname{div} b_1 \, q_1 & -\int_T \operatorname{div} b_1 \, q_2 \\ -\int_T \operatorname{div} b_2 \, q_1 & -\int_T \operatorname{div} b_2 \, q_2 \end{pmatrix}$$

$$= \begin{pmatrix} \int_T b_1 & 0 \\ 0 & \int_T b_2 \end{pmatrix}$$

$\boxed{\text{Invertible}}$

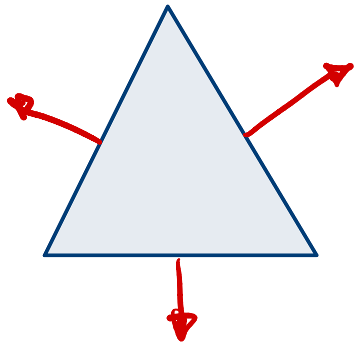


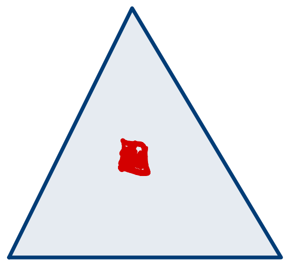
$P^2 \oplus b^3$



$P^{1,-1}$

Raviart - Thomas Space





$RT^1 - P^0$

infsup stable for Mixed Poisson

RT^1 : subspace of $(P^1)^2$ of dimension $\underline{\underline{3}}$

$$N^i(u) := \int_{e_i} u \cdot n \qquad e_i : \text{edge}$$

$$N^j(N_i) = \delta_i^j$$