Master in High Performance Computing

Theory and Practice of Finite Element Methods

Discretisation of Petrov Galerkin Methods and Mixed Problems

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Given $g \in W'$ find $u \in V$ s.t.

$$Au = g \quad \text{in } W' \iff \langle Au, v \rangle = \langle g, v \rangle \ \forall v \in W$$

$$A : V \longrightarrow W'$$

$\exists!\, u$ and $\|u\| \le \frac{1}{\alpha}\|g\|$

$\iff$

Hilbert Spaces

1) $\mathrm{Im}\, A = \overline{\mathrm{Im}\, A}$

2) $\ker A^T = \{0\}$

$\iff$

1) $\inf_{v \in W} \sup_{u \in V} \frac{\langle Au, v \rangle}{\|u\| \|v\|} \geq \alpha$

2) $\inf_{u \in V} \sup_{v \in W} \frac{\langle Au, v \rangle}{\|u\| \|v\|} \geq \alpha$

Cea's lemma for Petrov-Galerkin.

$$V_h \subset V \quad , \quad Q_h \subset Q$$

$$\Pi_h : V_h \longrightarrow V \qquad P_h : W_h \longrightarrow W$$

$$\boxed{\begin{aligned} \langle \Pi_h v, v_h \rangle &= \langle v, v_h \rangle \quad \forall\, v_h \in V_h \\ \langle P_h p, q_h \rangle &= \langle p, q_h \rangle \quad \forall\, q_h \in Q_h \end{aligned}}$$

$$\Pi_h^T : V' \longrightarrow V_h'$$

$$P_h^T : W' \longrightarrow W_h'$$

Given $f \in W'$, find $u \in V$ s.t.

$$a(u,v) = \langle f, v \rangle \qquad \langle Au, v \rangle = \langle f, v \rangle \qquad \forall v \in W$$

Given $f \in W'$, find $u \in V_h$ s.t.

$$a(u_h, v_h) = \langle f, v_h \rangle \qquad \forall\, v_h \in W_h$$

$$a(u - u_h, v_h) = 0 \qquad \forall\, v_h \in W_h$$

Discrete inf sup: $\exists\, \alpha_h$ (independent on h)

1. $$\inf_{u_h \in V_h} \sup_{v_h \in W_h} \frac{a(u_h, v_h)}{\|u_h\| \, \|v_h\|} \geq \alpha_h$$

2. $$\inf_{v_h \in W_h} \sup_{u_h \in V_h} \frac{a(u_h, v_h)}{\|u_h\| \, \|v_h\|} \geq \alpha_h$$

$$\alpha_h \leq \alpha$$

$$\|u - u_h\| \leq \|u - v_h\| + \|v_h - u_h\|$$

$$\leq \|u - v_h\| + \frac{1}{\alpha} \|A(v_h - \underline{\underline{u_h}})\|_{*, W_h}$$

$$= \|u - v_h\| + \frac{1}{\alpha} \|A(v_h - \underline{\underline{u}})\|_{*, W_h}$$

$$\|u - u_h\| \leq \left(\underset{=}{1} + \frac{\|A\|}{\alpha}\right) \|u - v_h\| \qquad \forall\, v_h \in V_h$$

$$\Longleftrightarrow$$

$$\|u - u_h\| \leq \left(1 + \frac{\|A\|}{\alpha_h}\right) \inf_{v_h \in V_h} \|u - v_h\|$$

$$\frac{1}{\alpha_h} = \|A_h^{-1}\| \implies \|u - u_h\| \leq \left(1 + \|A_h^{-1}\| \|A\|\right) \inf_{v_h \in V_h} \|u - v_h\|$$

$$\|A^{-1} f\| = \|u\| \qquad \|u\| \leq \|A^{-1}\| \|f\| \qquad = \|A^{-1}\|$$

$$\Uparrow$$

$$\|f\| \leq \alpha \|u\| \implies \|u\| \leq \frac{1}{\alpha} \|f\|$$

Who is A_h ?

$$\langle A_h u_h, v_h \rangle = \langle A u_h, v_h \rangle \quad \forall u_h \in V_h, \ \forall v_h \in W_h$$

$$\langle A_h \Pi_h u_h, P_h v_h \rangle = \langle A \Pi_h u_h, P_h v_h \rangle$$

$$\implies A_h = P_h^T A \Pi_h$$

$$A_h : V \longrightarrow W'$$

$$u \xrightarrow{\Pi_h} \Pi_h u \xrightarrow{A} A\Pi_h u \xrightarrow{P_h^T} P_h^T A \Pi_h u$$

$$V \qquad V_h \qquad W' \qquad W_h'$$

$$\langle P_h^T A \Pi_h u, w \rangle = \langle A\Pi_h u, P_h w \rangle$$

$$\forall v_h \in V_h \qquad \Pi_h v_h = v_h \Rightarrow$$

$$\underbrace{P_h^T A \Pi_h}_{A_h} v_h = P_h^T A v_h \qquad \forall v_h \in V_h$$

$$v_h = A_h^{-1} P_h^T A v_h \qquad \forall v_h \in V_h$$

$$\|u - u_h\| = \|u - v_h + v_h - u_h\|$$

$$= \|u - v_h + A_h^{-1} P_h^T A (v_h - u_h)\|$$

By construction $P_h^T A u = P_h^T f$ $\qquad P_h^T A u = P_h^T A_h u_h$

$$P_h^T A_h u_h = P_h^T f$$

$$= \|u - v_h + A_h^{-1} P_h A (v_h - u)\|$$

$$\|u - u_h\| \leq \left(1 + \|A_h\|^{-1} \|A\|\right) \|v_h - u\| \qquad \forall v_h \in V_h$$

Mixed pb: given $q \in Q'$, $f \in V'$

Find (u,p) in $V \times Q$ s.t.

$$\begin{cases} Au + B^T p = f \\ Bu = g \end{cases}$$

1) inf sup / sup inf of A on $\mathcal{Z} \equiv \ker B$

2) inf sup of B^T : $\exists \beta$ s.t. $\inf_{q\in Q} \sup_{v \in V} \frac{\langle Bv, q\rangle}{\|v\|\|q\|} \geq \beta$

Discrete version

$A : V \longrightarrow V'$ $\qquad$ $\Pi_h : V \longrightarrow V_h \subset V$

$B : V \longrightarrow Q'$ $\qquad$ $Q_h : Q \longrightarrow Q_h \subset Q$

$A_h : \Pi_h^T A \Pi_h$ $\qquad$ $B_h : \Pi_h^T B P_h$

$$\begin{tikzcd} V \arrow[r, "B"] \arrow[d, "\Pi_h"'] & Q' \arrow[d, "P_h^T"] \\ V_h \arrow[r, dashed, "B_h"] & Q'_h \end{tikzcd}$$

Trivial if $\ker B_h \subset \ker(B)$

Assume α_h 1) ELL-KER (2 infsup cond on A_h)

β_h 2) INF-SUP (1 infsup cond on B_h)

$$1)\quad \begin{cases} Au + B^T p = f \\ Bu = g \end{cases}$$

$$2)\quad \begin{cases} A_h u_h + B_h^T p_h = \Pi_h^T f \\ B_h u_h = P_h^T g \end{cases}$$

$u, p \in V \times Q$

$u_h, p_h \in V_h \times Q_h$

$$3)\quad \begin{cases} \Pi_h^T A u + \Pi_h^T B^T p = \Pi_h^T f \\ P_h^T B u = P_h^T g \end{cases}$$

$\underline{3-2}$

$$\Pi_h^T A(u - u_h) + \Pi_h^T B^T (p - p_h) = 0$$

$$P_h^T B(u - u_h) = 0$$

Cont: restrict $v \in Z = \ker B \Rightarrow \|Au_0\|_{*,Z} \geq \alpha \|u_0\|$

$\exists!\, u_0 \Rightarrow \|u_0\| \leq \frac{1}{\alpha}\|f\|$

$\Rightarrow \exists!\, p \quad \|\underbrace{B^T p}\| \geq \beta \|p\|$

$\|Au - f\| = \|B^T p\| \geq \beta \|p\|$

$Z_h \not\subset Z$ in generale

1) ELL-Ker in $Z_h \Rightarrow \exists \alpha_h$ s.t. $\inf\sup$ on Z_h

2) INF SUP on V_h, Q_h for B_h

$$Z_h := \{ v_h \in V_h \text{ s.t. } b(v_h, q_h) = 0 \ \forall q_h \in Q_h \}$$

$$\begin{array}{l} \ker B \subset V \\ V_h \subset V \end{array} \not\Rightarrow \ker B \cap V_h \neq \{\emptyset\}$$

$$\ker B_h \not\subset \ker B$$

$$a(u - u_h, v_h) = -b(p - p_h, v_h) \qquad \forall v_h \in V_h$$

Restrict to $w_h \in Z_h$

$$a(u - u_h, w_h) = -b(p, w_h) \qquad \forall w_h \in Z_h$$

$$= -b(p - q_h, w_h) \qquad \forall w_h \in Z_h$$

$$\langle A u_h, w_h \rangle = \langle A u, w_h \rangle + \langle B^T(q_h - p), w_h \rangle \ \forall q_h \in Q_h$$

$$\|u - u_h\| \leq \|u - v_h + v_h - u_h\|$$

$$\leq \|u - v_h\| + \frac{1}{\alpha_h} \|A(v_h - u_h)\|_{*, Z_h}$$

$$\|u - u_h\| \leq \|u - v_h\| + \frac{\|A\|}{\alpha_h} \|u - v_h\| + \frac{1}{\alpha_h} \|B\| \|q_h - p\| \quad \begin{array}{l} \forall v_h \in V_h \\ \forall q_h \in Q_h \end{array}$$

$$\boxed{\|u-u_h\| \leq \left(1+\frac{\|A\|}{\alpha_h}\right) \inf_{v_h \in Z_h} \|u-v_h\| + \frac{\|B\|}{\alpha_h} \inf_{q_h \in Q_h} \|q_h - p\|}$$

$$B : V \longrightarrow Q'$$

$$B^T : Q \longrightarrow V'$$

$$\|p-p_h\| \leq \|p-q_h+q_h-p_h\|$$

$$\leq \|p-q_h\| + \frac{1}{\beta_h}\|B_h^T(q_h-p_h)\|_{*,V_h}$$

$$\|B_h^T q_h\|_{*,V_h} \geq \beta \|q_h\|$$

$$\langle B^T(q_h-p), v_h \rangle = \langle A(u_h-u), v_h \rangle$$

$$\leq \|p-q_h\| + \frac{\|A\|}{\beta_h}\|u_h-u\| + \frac{\|B\|}{\beta_h}\|p-q_h\|$$

$$\boxed{\|p-p_h\| \leq \left(1+\frac{\|B\|}{\beta_h}\right) \inf_{q_h \in Q_h} \|p-q_h\| + \frac{\|A\|}{\beta_h}\|u-u_h\|}$$

$$\|p-p_h\| \leq \left(1+\frac{\|B\|}{\beta_h}+\frac{\|A\|\,\|B\|}{\alpha_h \beta_h}\right) \inf_{q_h \in Q_h} \|p-q_h\| +$$

$$\frac{\|A\|}{\beta_h}\left(1+\frac{\|A\|}{\alpha_h}\right) \inf_{v_h \in Z_h} \|u-v_h\|$$

Two concrete examples :

Mixed Poisson with nat. b.c.

$$-\Delta p = g \qquad -\nabla p = u$$
$$\Downarrow \qquad \nabla p \cdot n = 0$$
$$v \cdot n = 0$$

$$\begin{cases} u + \nabla p = 0 \\ \operatorname{div} u = g \end{cases}$$

$$(u,v) - (\operatorname{div} v, p) = 0 \quad \forall v \in H^{div}_0(\Omega)$$

$$(\operatorname{div} u, q) = (g,q) \quad \forall q \in L^2(\Omega)$$

Stokes

$$-\Delta u + \nabla p = f$$
$$\operatorname{div} u = 0$$
$$\Downarrow$$
$$(\nabla u, \nabla v) - (\operatorname{div} v, p) = (f,v) \quad \forall v \in H^1_0(\Omega)^d$$
$$(\operatorname{div} u, q) = 0 \quad \forall q \in L^2_0(\Omega)$$

$$\langle Bv, q\rangle := \int_\Omega \operatorname{div} v \, q \qquad \forall v \in H^1(\Omega) \subset H^{div}(\Omega)$$
$$\forall q \in L^2(\Omega)$$

$$H^{div} := \{ v \in (L^2)^d \text{ s.t. } \operatorname{div} v \in L^2 \} \qquad \|u\|^2_{div} := \|u\|^2_0 + \|\operatorname{div} u\|^2_0$$

$$H^{div}_0 := \{ v \in H^{div} \text{ s.t. } v \cdot n = 0 \text{ on } \gamma \}$$

$$M : H^{div}_0 \longrightarrow (H^{div}_0)'$$
$$\langle Mu, v\rangle \longrightarrow \int uv$$

$$A : (H^1_0(\Omega))^d \longrightarrow ((H^1_0(\Omega))^d)'$$
$$\langle Au, v\rangle \longrightarrow \int_\Omega \nabla u : \nabla v$$
$$\sum_{i,j} u_{i,j} v_{i,j}$$

M satisfies ELL-KER ?

$\langle Mu, u\rangle = \|u\|_0^2$ ~ No control on div / ∇.

M is elliptic on $\underline{\ker B}$

$\ker B := \{ v \in H_0^{div}(\Omega)$ s.t. $\operatorname{div} v = 0 \}$

$\forall v \in \ker B \qquad \|v\|_{div} = \|v\|_0 + \|\operatorname{div} v\|_0$, $\|\operatorname{div} v\|_0 \to 0$

$\Rightarrow$ M is coercive on $\ker B$ $\quad \alpha = 1$

$H_0^1(\Omega)^d$, $\langle Au, v\rangle := \int_\Omega \nabla u : \nabla v$

A is coercive on $H_0^1(\Omega)^d$ not just on $\ker B$

$\langle Au, u\rangle \geq \|u\|_1$ By poincaré

Simpler case : Stokes : $B : H_0^1(\Omega)^d \to L_0^2(\Omega)$

INF SUP of B ?

$$\inf_{q \in L_0^2(\Omega)} \sup_{v \in H_0^1(\Omega)^d} \frac{\int \operatorname{div} v \, q}{\|v\| \, \|q\|} \geq \beta \quad ?$$

If $\forall q \in L^2_0(\Omega)$ $\quad \exists \; v_q \in H^1_0(\Omega)$ s.t.

$$\operatorname{div} v_q = q \quad \Longrightarrow \quad \int \operatorname{div} v_q \, q = \int q^2 = \|q\|^2$$

$$\longrightarrow \quad \inf_{q \in L^2_0(\Omega)} \; \sup_{v \in H^1_0(\Omega)^d} \frac{\int \operatorname{div} v \, q}{\|v\| \, \|q\|} \geq \inf_{q \in L^2_0(\Omega)} \frac{\|q\|^{\cancel{2}}}{\|v_q\| \cancel{\|q\|}}$$