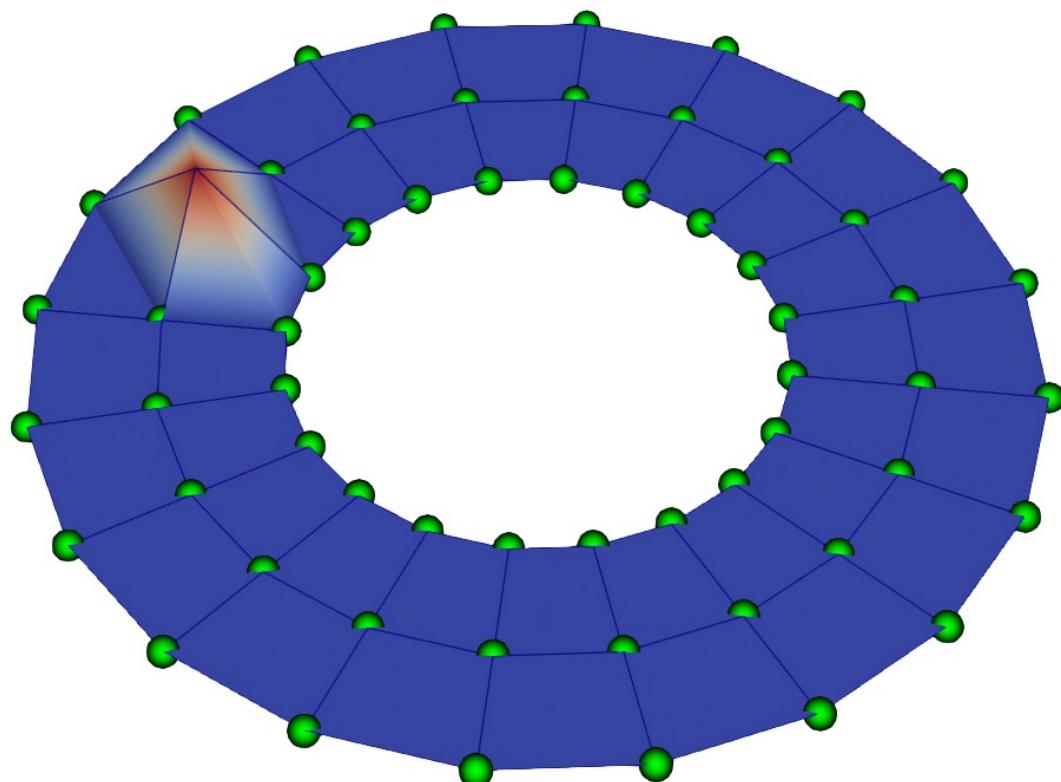


Theory and Practice of Finite Element Methods

Degrees of Freedom and Finite Dimensional Spaces

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Consider cases where $V \in H^1(\Omega)$

$\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$

. Construct $V_h \subset H^1(\Omega)$ according to

- T_h : Triangulation of Ω

- Reference Finite Element $\{\hat{T}, P_{\hat{T}}, \hat{\Sigma}\}$

- Mapping between \hat{T} and T_k $\forall T_k \in T_h$

- Satisfy continuity conditions for $V_h \subset V$

DoF Handler

Red boxes coincide with classes in deal.II

Start with \hat{T} :



Choose $P_{\hat{T}}$ finite dimensional. d -simplex.

• Lagrangian case $P_{\hat{T}} := P^k(\hat{T})$ polynomials of order k

• Construct a basis for $(P_{\hat{T}})^k$

do this through a selection of n_{FE} support points

$$(n_{FE} := \dim(P_{\hat{T}}) = \dim(P^k(\hat{T})))$$

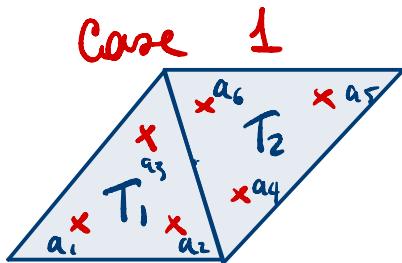
where N_d is dimension

N_0	$= 1$	$\text{if } k=0$	by convention	(points)
N_1	$= k+1$	$\text{if } k>1$		(lines)
N_2	$= \frac{1}{2}(k+1)(k+2)$			(triangles)
N_3	$= \frac{1}{6}(k+1)(k+2)(k+3)$			(tetrahedra)

If we want to have continuity in V_h , then

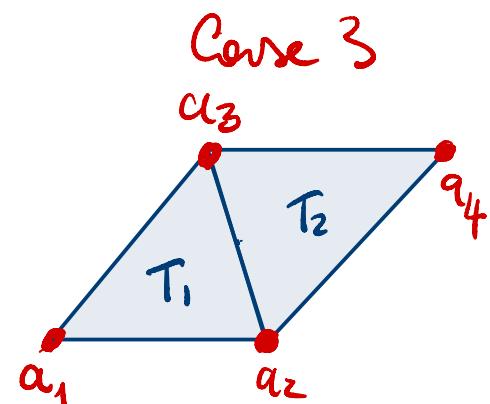
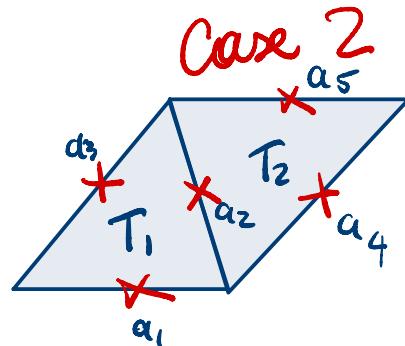
$\forall v_h \in V_h$ if $\overline{T_1} \cap \overline{T_2} \neq \{\emptyset\}$

then $v_h|_{T_1 \cap T_2}$ has to be single valued



$T_1, P^1(T_1), \Sigma(T_1)$

$T_2, P^2(T_2), \Sigma(T_2)$



Support points $\{a_{1i}\}_{i=1}^3, \{a_{2i}\}_{i=1}^3, v^{ki} := \delta(x - a_{ki})$

Case 1: $a_{1i} \neq a_{2j}, \forall i, j \in 1, 2, 3$

(a total of 6 support points) 3 for each triangle

$u_h|_{T_i} \rightarrow \exists! \{u^i\}_{i=1}^3$ s.t. $u_h|_{T_i} = u^i v_{1i} + v^{ki} m_{ki} v_{2i} = \sum_m m_{ki} v_{mi}$

Select 6 real numbers:

$\exists u^{ki} \in \mathbb{R}^{2 \times 3}$ st. $u_h = u^{ki} v_{ki} \notin C^0(\overline{T}_1 \cup \overline{T}_2)$

This is ok if $v_h \in L^2(\Omega)$

Not ok if you want $v_h \in H^1(\Omega)$

dim case 1 : 6 (3×2 triangles)

Case 2 is still not continuous.

dim case 2 : 5 ($1 \times n$ edges)

Local to global indexing : vector of vectors.

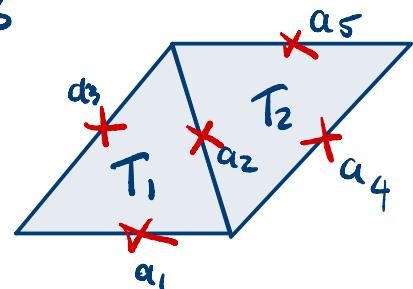
$\ell_{2g}[k][i] \rightsquigarrow$ index of support point
i on element k

Similarly to list of vertices and elements for triangulation.
We need a list of support points, and a list of
indices that refers to them for the definition of
 $FET : \{ T, \mathcal{P}^k(T), \sum_T \}$

$$\ell_{2g}_{ki} \quad k = 1, 2 \quad i = 1, 2, 3$$

$$\ell_{2g}_1 = \{1, 2, 3\}$$

$$\ell_{2g}_2 = \{2, 4, 5\}$$



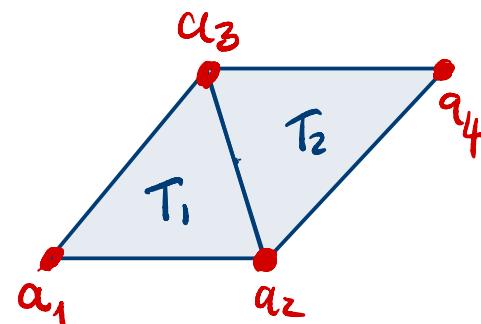
$$\sum_{T_k} := \left\{ \delta(x - a_{\ell_{2g}_{ki}}) \right\}_{i=1}^3$$

for case 1 $\ell_{2g}_1 = \{1, 2, 3\}$ $\ell_{2g}_2 = \{4, 5, 6\}$

Case 3

$$\ell_{2g}_1 = \{1, 2, 3\}$$

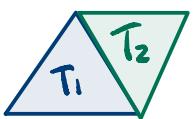
$$\ell_{2g}_2 = \{3, 2, 4\}$$



This space is continuous:

$$M_h = \sum_{k=1}^2 \sum_{i=1}^2 M_h(a_{ki}) v_{ki}$$

$$M_h = \sum_{k=1}^2 \sum_{i=1}^2 M_h(a_{ki}) v_{ki} \quad \text{is cont. only in case 3}$$



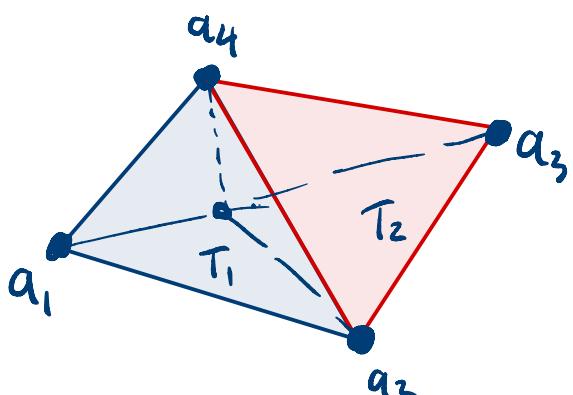
$$M_h(a_{11}) v_{11} + \dots + M_h(a_{21}) v_{21} + \dots +$$

We identify v_i with one single index with the degree of freedom associated to a_i with one single index

If v_{kj} has same support point of v_{mi}

for $k \neq m$ and some pair i, j . Then we identify v_{kj} and v_{mi} with the SAME global basis function, which is $\sqrt{g_{2L}}_{kj} = \sqrt{g_{2L}}_{mi}$

$$\sqrt{g_{2L}}_{kj} \Big|_{T_k} = v_{kj} \quad \leftarrow \text{local basis on } T_k$$



$$\begin{aligned} V_{T_k} &= \text{span} \{ v_{kj} \}_{j=1}^{N_{T_k}} = P_{T_k} \\ \sum_{T_k} &:= \{ V^{kj} \}_{j=1}^{N_k} \\ \langle V^{kj}, v_{mi} \rangle &= \delta_{ij} \end{aligned}$$

$$\sqrt{V_4} \Big|_{T_1} = v_{13} \quad \sqrt{V_4} \Big|_{T_2} = v_{21}$$

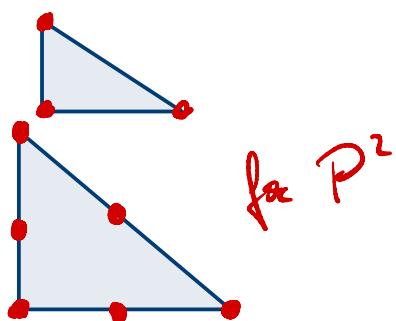
For arbitrary orders, we need to ensure that the number of support points on the subelement X of T_k (faces, edges, vertices) coincides with $\text{Ndim}(\mathbb{P}^k|_X)$

X is manifold of intrinsic dimension d :

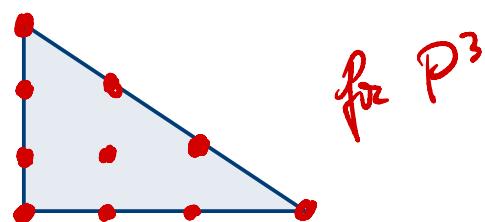
$$N_0 = 0, 1 \quad \text{if } k=0, \quad k \geq 1$$

$$N_d = \binom{d+k}{k}$$

P^1 continuous: need 1 support point per vertex.
 P^2 " " 1 support point per vertex
and 1 " point per line



for P^2



for P^3

DPO for $P^1(T_k)$

$$\text{DPO}_0 = 1 \quad \text{DPO}_i = 0 \quad i > 0$$

Concept of Degrees of freedom per object

dpo

$$\text{DPO} \in \mathbb{N}^{d+1}$$

DPO_i : number of support points on objects of intrinsic dimension i excluding subobjects.

$$\text{DPO}_i \neq N_i$$

$$N_i = \sum_{j=0}^i \text{DPO}_j$$

$$N_{T_k} = \sum_{j=0}^d \text{DPO}_j$$