

Theory and Practice of Finite Element Methods

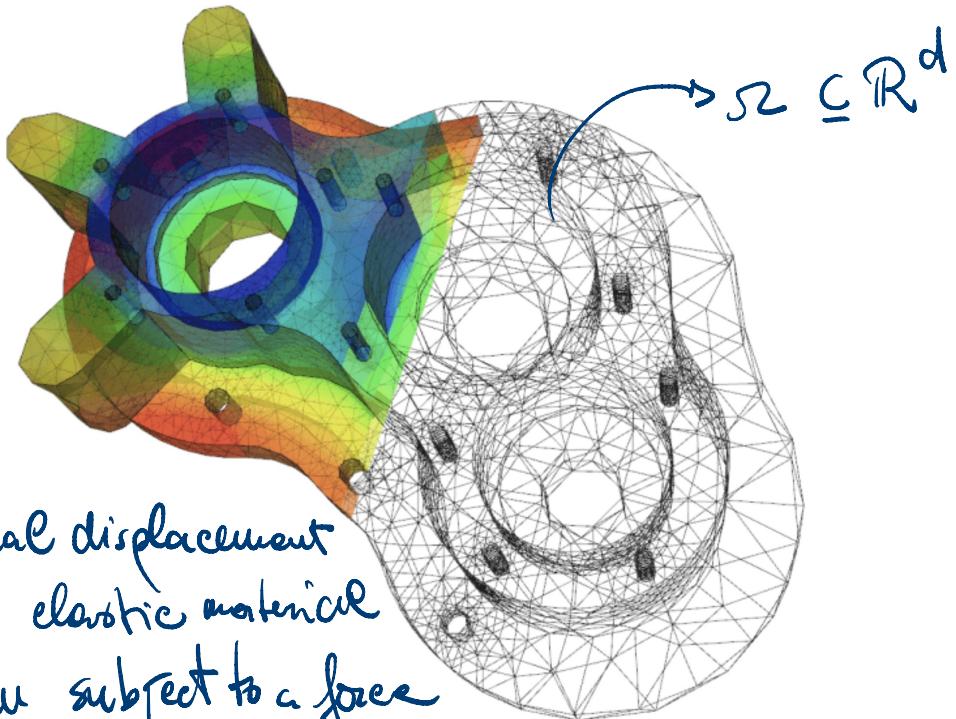
Introduction

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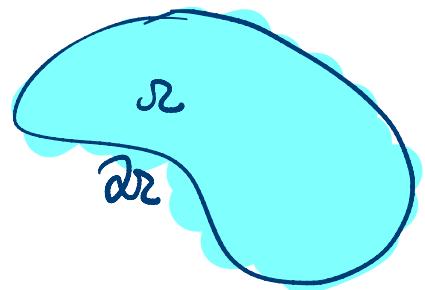


u : vertical displacement
of a elastic material
when subject to a force

f : delta

Prototypical problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$



Is formulated in **Weak form**

Ω is Lipschitz bounded subset of \mathbb{R}^d

$$d = 1, 2, 3 (4, 5, 6)$$

" $-\Delta$ " operator is seen as

a Weak derivative operator on a Hilbert space

Algebraically, $-\Delta u := -\sum_{i=1}^d \frac{\partial^2 u}{\partial x_i^2}$

" $-\Delta$ " is interpreted weakly:

$$\begin{aligned} -\Delta : A : V &\longrightarrow V' & V \text{ is Hilbert} \\ u &\longrightarrow Au \end{aligned}$$

$$Au = F \quad \text{in } V'$$

\Rightarrow same thing as

$$\langle Au, v \rangle = \langle F, v \rangle \quad \forall v \in V$$

$$\langle Au, v \rangle := \int_{\Omega} Au v$$

$$\underline{\underline{Au(v) \in \mathbb{R}}}$$

\downarrow
Au is linear functional from
 $V \rightarrow \mathbb{R}$

$$\langle \Delta u, v \rangle := \int_{\Omega} \nabla u \cdot \nabla v \, dx$$

$$-\Delta u = -\operatorname{div}(\nabla u)$$

$$\int_{\Omega} -\Delta u \cdot v \, dx = \int_{\Omega} \nabla u \cdot \nabla v - \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, dS$$

Typical procedure in FEM:

• Replace the space V (infinite dimensional, Hilbert) with V_h , finite dimensional, and $V_h \subset V$

$$Au = F \quad \text{in } V$$

\Downarrow

$$A u_h = F \quad \text{in } V_h$$

$$V_h = \text{span} \{v_i\}_{i=1}^n$$

\Downarrow

$$u_h \quad \exists! \quad \{u^i\}_{i=1}^n \in \mathbb{R}^n$$

s.t. $u_h(x) = \sum_{i=1}^n u^i v_i(x)$

$$\langle A u_h, v_h \rangle = \langle F, v_h \rangle \quad \forall v_h \in V_h$$

$$\langle A v_j u^i, v_i \rangle = \langle F, v_i \rangle \quad i=1 \dots n$$

sum is implied
when repeating
indices in "opposite"
positions.

$$A_M = F$$

$$A_{ij} \underset{M}{=} F_i$$

$$A_{ij} := \langle Av_j, v_i \rangle$$

$$F_i := \langle F, v_i \rangle$$

linear system of equations.

LAX-MILGRAM LEMMA

Assume V is Hilbert with norm $\|\cdot\|$ and assume that a is a bilinear operator on V : $a: V \times V \rightarrow \mathbb{R}$

$$a(u, v) := \langle Au, v \rangle$$

and assume that there exist $\alpha \in \mathbb{R}, \alpha > 0$ s.t.

→ bounded $a(u, v) \leq \|A\| \|u\| \|v\| \quad \forall u, v \in V$

→ coercive $a(u, u) \geq \alpha \|u\|^2$

Then $\nexists F \in V'$, $\exists! u \in V$ s.t.

$$\langle Au, v \rangle = a(u, v) = \langle F, v \rangle = F(v) \quad \forall v \in V$$

$$\|u\| \leq \frac{1}{\alpha} \|F\|_*$$

$$\|F\|_* := \sup_{0 \neq v \in V} \frac{|F(v)|}{\|v\|} = \sup_{0 \neq v \in V} \frac{|F(v)|}{\|v\|}.$$

Intermediate result: Rietz representation theorem

$\forall F \in V' \exists! f \in V$ st.

$$(f, v)_V = \langle F, v \rangle \quad \forall v \in V$$

$$\|f\| = \|F\|_*$$

we call $\pi: V' \longrightarrow V$ the Rietz operator that maps elements of V' to their representation in V

Useful to define a fixed point problem in V .

$$T_g(u) = u \iff u \text{ is solution to}$$

$$T_g: V \longrightarrow V$$

$$v \in V \quad v \in V$$

$$Av = F \text{ in } V' \\ \langle Av, v \rangle = \langle F, v \rangle \quad \forall v \in V$$

$$-\underbrace{g\tau(Au - F)}_{\text{By construction}} + u = u$$

$$T_g(u) := u - g\tau(Au - F)$$

if u is s.t.

$$T_g(u) = u$$

then

$$Au = F$$

Search for g st. T_g is a contraction ($\exists L < 1$)

$$\|T_g(u_1) - T_g(u_2)\| \leq L \|u_1 - u_2\|$$

\Rightarrow start from u_0 arbitrary, and define

$$u_{k+1} = T_g(u_k)$$

$$\|u_{k+1} - u_k\| = \|T_g(u_k) - T_g(u_{k-1})\| \leq L \|u_k - u_{k-1}\|$$

$$\underbrace{\|u_{k+1} - u_k\|} \leq \underbrace{L^{k-1}} \|T_g(u_0) - u_0\|$$

then $\exists u$ st. $T_g(u) = u$

It is unique. If not then

$$\|\tilde{u} - u\| \leq \|T(\tilde{u}) - T(u)\| \leq L \|\tilde{u} - u\|$$

contradiction

$$\|\tilde{u} - u\| < \|\tilde{u} - u\|$$

$\Rightarrow u$ is unique

$$\|T_g(u_1) - T_g(u_2)\|^2 = \|(u_1 - u_2) - g^2 A(u_1 - u_2)\|^2$$

$$\|u_1 - u_2\|^2 + g^2 \|z A(u_1 - u_2)\|^2 - 2g(z A(u_1 - u_2))(u_1 - u_2) \\ \leq L^2 \|u_1 - u_2\|^2 \quad \text{with } L < 1$$

$$-2g(z A(u_1 - u_2), (u_1 - u_2)) = -2g \underbrace{z A(u_1 - u_2)(u_1 - u_2)}_{\geq \alpha \|u_1 - u_2\|^2}$$

$$\leq \|u_1 - u_2\|^2 + g^2 \|A\|_*^2 \|u_1 - u_2\|^2 - \frac{2g\alpha}{L^2} \|u_1 - u_2\|^2$$

$$\|T_g(u_1) - T_g(u_2)\|^2 \leq (1 + g^2 \|A\|_*^2 - 2g\alpha) \|u_1 - u_2\|^2$$

$$L < 1 \Rightarrow -1 \leq 1 + g^2 \|A\|_*^2 - 2g\alpha \leq 1$$

\rightarrow if $g < \frac{2\alpha}{\|A\|_*^2}$ then $L < 1 \Rightarrow \exists!$ sol. to fixed point.

$$\alpha \|u\|^2 \leq a(u, u) = \langle F, u \rangle \leq \|F\|_* \|u\|$$

$$\Rightarrow \|u\| \leq \frac{1}{\alpha} \|F\|_*$$

Galerkin Method

Given $V_h \subset V$

Find u_h s.t.

$$\langle A u_h, v_h \rangle = \langle F, v_h \rangle \quad \forall v_h \in V_h$$

$$\langle A u, v_h \rangle = \langle F, v_h \rangle \quad \forall v_h \in V_h \subset V$$

$$\langle A(u - u_h), v_h \rangle = 0 \quad \forall v_h \in V_h$$

the error is A -orthogonal to V_h

Cea's lemma

$$\langle A(u - u_h), u_h \rangle = 0$$

$$\alpha \|u - u_h\|^2 \leq \langle A(u - u_h), (u - u_h) \rangle \stackrel{\approx 0}{\approx} 0$$

$$\leq \langle A(u - u_h), (u - v_h) \rangle \stackrel{\approx 0}{\approx} 0$$

$$\leq \|A\|_* \|u - u_h\| \|u - v_h\| \quad \forall v_h \in V_h$$

$$\|u - u_h\| \leq \frac{\|A\|_*}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|$$

Main idea and goal of the course:

- construct V_h s.t. \rightsquigarrow A-priori (Worst case)

$$\rightarrow \text{dist}(V, V_h) := \sup_{u \in V} \inf_{v_h \in V_h} \|u - v_h\| \quad \text{is "small"} \\ \text{(typically of order } h^k)$$

- measure $\text{dist}(V, V_h)$, estimate $\|u - v_h\|$

- given F, u , find the best possible V_h \rightsquigarrow A-posterior p.b.

— How to do this in 1D —

$$\Omega = (0, 1), \partial\Omega := \{0, 1\}$$

$$\begin{cases} -u'' = f & \text{in } (0, 1) \\ u(0) = u(1) = 0 \end{cases} \quad \text{Weak form} \\ \rightsquigarrow \int_0^1 u' v' = \int_0^1 f v + v \in H_0^1([0, 1])$$

$$H_0^1([0, 1]) := \left\{ v \in L^2([0, 1]) \text{ s.t. } v' \in L^2([0, 1]), v(0) = v(1) = 0 \right\}$$

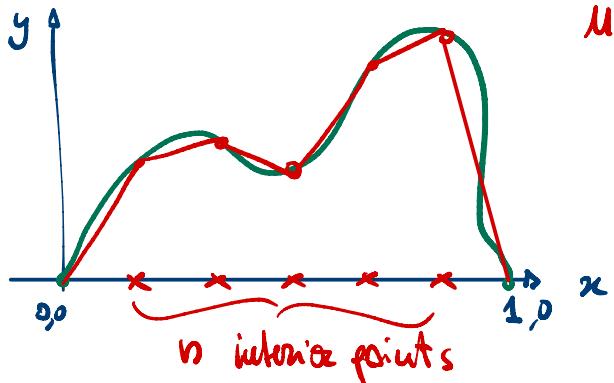
$$H_0^1([0, 1]) \subset C^0([0, 1])$$

$$H^s \subset C^0 \quad s > \frac{d}{2}$$

How to define V_h ?

$$h = \left(\frac{b-a}{n+1} \right) = \frac{1}{n+1}$$

$$x_i := ih \quad i = 1, \dots, n$$



$$M_h, m$$

$$M_h = \sum_{i=1}^n m^i v_i$$

$$V_h \subset V \equiv H_0^1([0,1])$$

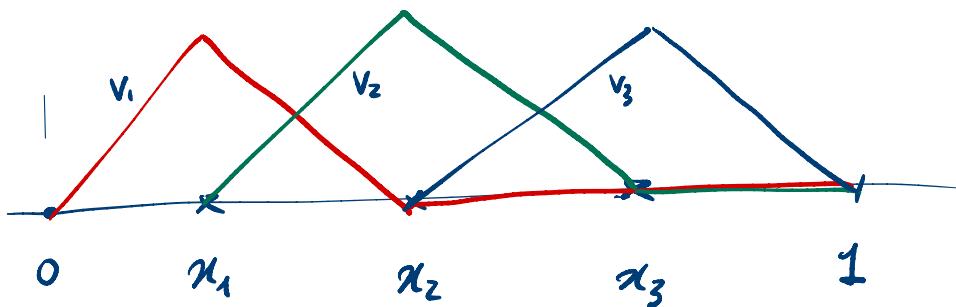
$$V_h = \left\{ v \in C^0([0,1]), v|_{(x_i, x_{i+1})} \in P^1(x_i, x_{i+1}) \text{ for } i=0, \dots, n, v(0) = v(1) = 0 \right\}$$

We know the dimension of V_h . ($\equiv n$).

- Choose n functionals in V_h' , called nodal functionals, s.t. they are linearly independent. (choose a base for V_h'), call them $v^i \in V_h'$
- construct v_i such that $v_i \in V_h$, $v^j(v_i) = \langle v^j, v_i \rangle = \delta^j{}_i = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$
- Simplest choice: $v^i(x) := \delta(x-x_i) \quad i=1, \dots, n$

$$v^i(\mu) = \mu(x_i) = \int_0^1 \delta(x-x_i) \mu(x) dx$$

- $v_i(x)$ are basis functions:



$$\mathcal{I}: V \longrightarrow V_h$$

$$\mu \longrightarrow v^i(\mu) \quad v_i = \sum_{i=1}^n \mu(x_i) v_i$$

