Master in High Performance Computing

Theory and Practice of Finite Element Methods

A posteriori error estimates and adaptive meshes — deal.II implementation —

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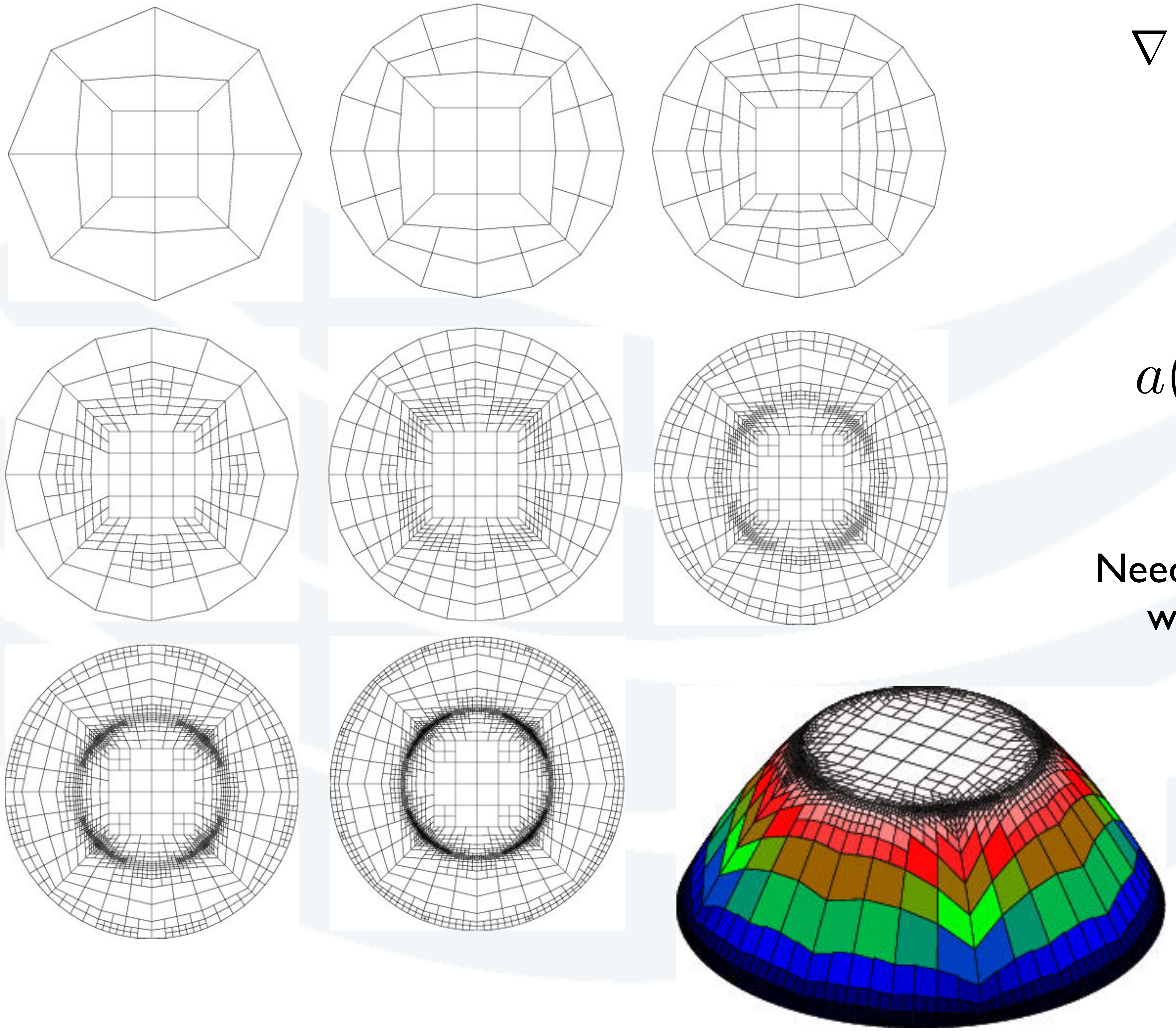
Adaptive mesh refinement

SOLVE — ESTIMATE — MARK — REFINE



$$\nabla \cdot a(\boldsymbol{x}) \nabla u(\boldsymbol{x}) = 1 \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

$$a(\boldsymbol{x}) = \begin{cases} 20 & \text{if } |\boldsymbol{x}| < 0.5 \\ 1 & \text{otherwise} \end{cases}$$

Need an error indicator η_K on each cell without knowing the exact solution.

Adaptive mesh refinement

$$\|u\|^2_{H^2(K)} := \int_K u^2 + |\nabla u|^2 + |\nabla^2 u|^2$$

$$|u|^2_{H^2(K)} := \int_K |\nabla^2 u|^2$$

Error estimate for Q1/P1 elements applied to Laplace problem:

$$\|u - u^h\|_{H^1} \equiv \|e\|_{H^1} \leq C\, h_{max} \|u\|_{H^2}$$

this error depends on the largest element size and the global norm of the solution.
To reduce error (increase accuracy) one can refine the mesh size.

more precisely...

$$\|e\|^2_{H^1} \leq C^2 \sum_K h_K^2 |u|^2_{H^2(K)}$$

Thus one needs to make mesh finer where the local H^2 semi-norm is large.

But apart from some special cases we don't know the exact solution u !

Thus we need to create meshes iteratively (adaptively).

Optimal strategy is to equilibrate the error $e_K := C\, h_K |u|_{H^2(K)}$

That is, we want to choose $h_K \sim \dfrac{1}{|u|_{H^2(K)}}$

a-posteriori error estimation

$$||e||^2_{H^1(\Omega)} \leq C \sum_K e_K^2$$

$$e_K = h_K ||\nabla^2 u||_K$$

cell-wise error indicators

(wrong) idea:

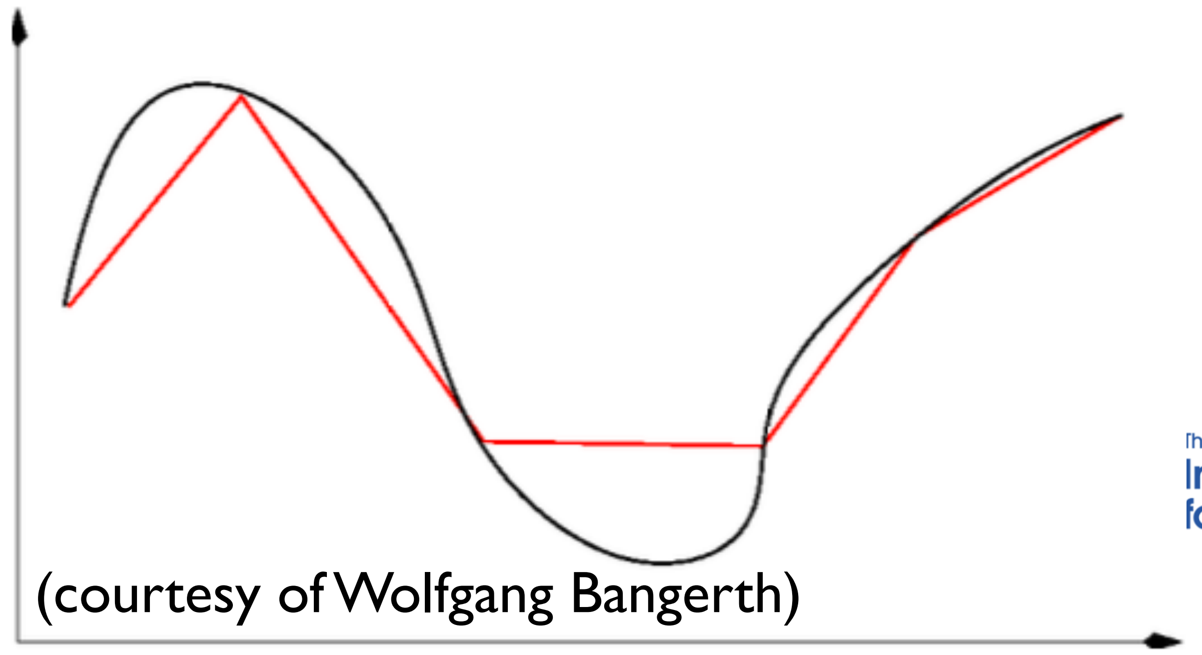
$$e_K \approx h_K ||\nabla^2 u^h||_K$$

will not work as linear elements have zero second derivates within the element and first derivatives have jumps on the interfaces

a better idea (in 1D) to approximate second derivatives at interface i:

$$\nabla^2 u \approx \frac{\nabla u^h(x^+) - \nabla u^h(x^-)}{h} =: \frac{[\![\nabla u^h]\!]_i}{h}$$

use jump in gradient as an indicator of the second derivative at vertices



(courtesy of Wolfgang Bangerth)

can generalize to:

$$||\nabla^2 u||^2_K \approx \sum_{i \in \partial K} \frac{[\![\nabla u^h]\!]_i^2}{h}$$

a-posteriori error estimation

As a result, the simplest and most widely used Kelly error **indicator** in 2D/3D follows:

$$e_K^2 = h_K^2 ||\nabla^2 u||_K^2 \approx h_K \int_{\partial K} |[\![\boldsymbol{\nabla} u \cdot \boldsymbol{n}]\!]|^2 \mathrm{ds} =: \eta_{\mathrm{K}}^2$$

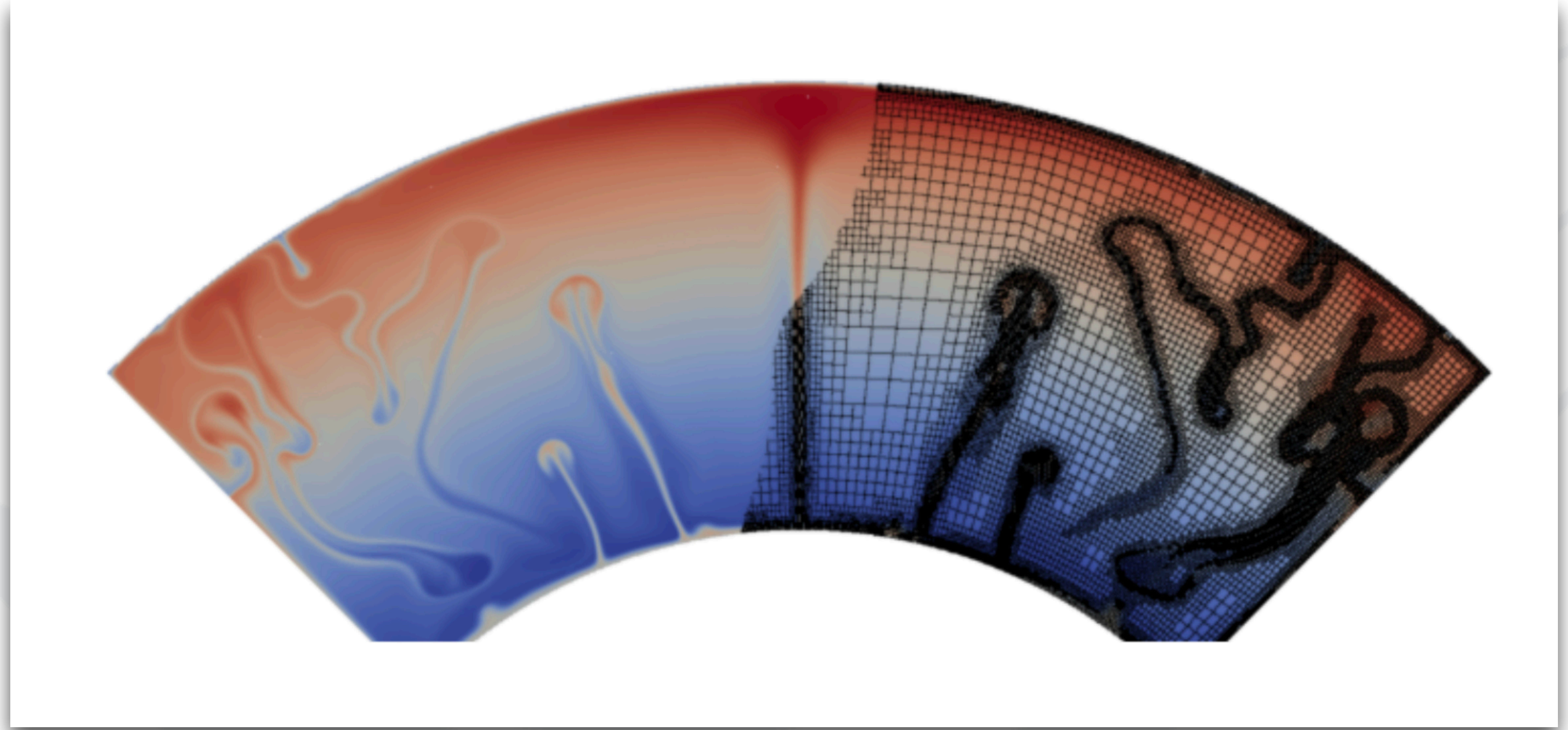
For the Laplace equation, **Kelly, de Gago, Zienkiewicz, Babushka (1983)** proved that

$$||\nabla \left[u - u^h\right]||^2 \leq C \sum_K \eta_K^2 \qquad \text{a-posteriori error estimator} \qquad (*)$$

Note 1:

"**estimator**" is always a <u>proven upper bound of error</u> (*), whereas "**indicator**" is our best guess of error per cell which may not be an upper bound in the sense (*), but may still work well for considered equations and/or FE space.

Adaptive mesh refinement



Refine where things are happening!

Basic AFEM algorithm

- SOLVE-ESTIMATE-MARK-REFINE
  - On the current mesh, solve the problem
  - Estimate the error per cell (Exact, Kelly, Residual, etc.)
  - Mark cells according to given criterion (estimator is greater than a tolerance, or fraction of cells with largest error, or …)
  - Refine the marked cells
- Repeat until tolerance met, or max number of cycles

deal.II classes

- Error estimate is problem dependent:
  - Approximate gradient jumps: `KellyErrorEstimator` class
  - Approximate local norm of gradient: `DerivativeApproximation` class
  - … or something else
- Cell marking strategy:
  - `GridRefinement::refine_and_coarsen_fixed_number(...)`
  - `GridRefinement::refine_and_coarsen_fixed_fraction(...)`
  - `GridRefinement::refine_and_coarsen_optimize(...)`
- Refine/coarsen grid: `triangulation.execute_coarsening_and_refinement ()`
- Transferring the solution: `SolutionTransfer` class (discussed later)