Master in High Performance Computing

# Theory and Practice of Finite Element Methods

## Denis-Lions lemma and Bramble-Hilbert lemma

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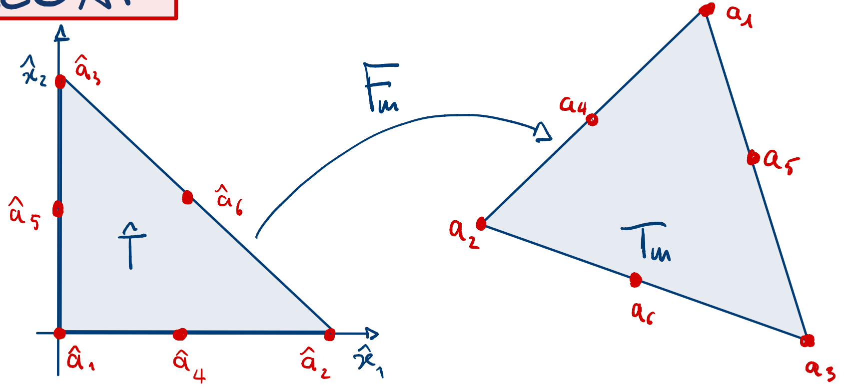
Lagrange:

$$\hat{\Sigma} = \mathrm{span}\{\hat{v}^i\}$$

$$\hat{v}^i := \delta(\hat{x} - \hat{a}_i)$$

$$\hat{v}^i(u) = \int_\Omega \delta(\hat{x} - \hat{a}_i)\, u(\hat{x})\, d\hat{x} = u(\hat{a}_i)$$




Reference FE

$$\hat{FE} = \{\hat{T}, \hat{P}, \hat{\Sigma}\}$$

$$\hat{P} = \mathrm{span}\{\hat{v}_i\}_{i=1}^{\hat{N}}$$

$$\hat{\Sigma} = \mathrm{span}\{\hat{v}^i\}_{i=1}^{\hat{N}}$$

$$\hat{v}^i(\hat{v}_J) = \delta^i_J$$

$$\Omega = \mathring{\overline{\mathcal{T}_h}}$$

$$\mathcal{T}_h = \left(\bigcup_{m=1}^{M} \overline{T_m}\right)$$

Local FE

$$FE_m = \{T_m, P_m, \Sigma_m\}$$

$$T_m = F_m(\hat{T})$$

$$v_i = \hat{v}_i \circ F_m^{-1}$$

$$v^i = \hat{v}^i \circ F_m^{-1}$$

Global Space

$\tilde{\cdot}$ := extension by zero

$$P_{mIJ} := \begin{cases} 1 & \text{if } v_I|_{T_m} = \hat{v}_J \circ F_m^{-1} \\ 0 & \text{otherwise} \end{cases}$$

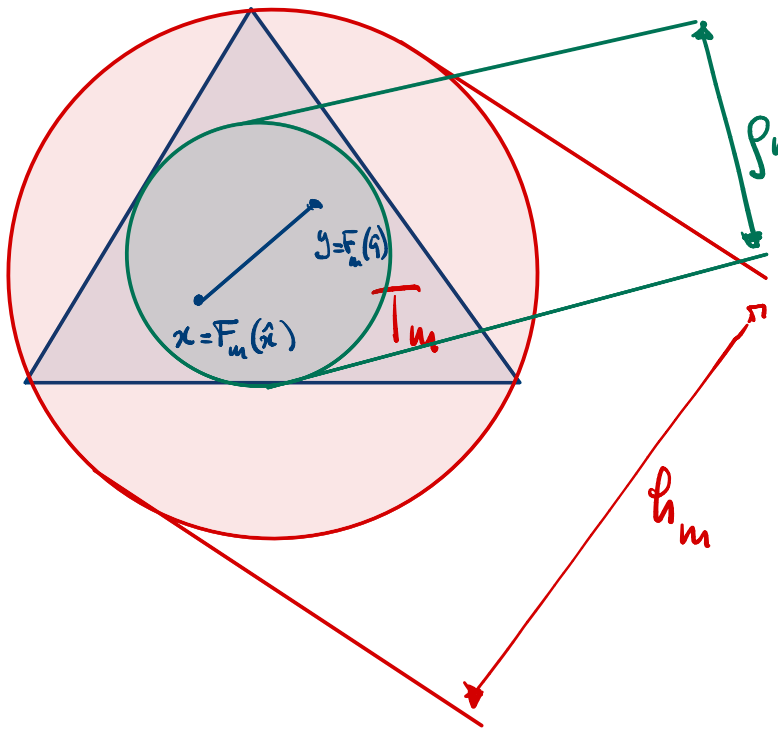
$$V_h = \mathrm{Span}\{v_I\}_{I=1}^{N_{dofs}}, \quad v_I \in V,$$

$$V_I = \sum_m \sum_J P_{mIJ}\, \widetilde{\hat{v}_J \circ F_m^{-1}}$$

For Affine mappings. Given Lagrangian $P^1$

- $F_m = B_m \hat{x} + b_m = \sum_{i=1}^{3} \hat{v}_i^{P^1}(\hat{x}) a_i$

$$P^1(\hat{T}) := \operatorname{span}\{\hat{v}_i^{P^1}\}_{i=1}^{\mathrm{dim}+1}$$




$$\xi = y - x = F_m(\hat{y}) - F_m(\hat{x}) = B_m \hat{\xi}$$

$$\hat{\xi} = \hat{y} - \hat{x} = F_m^{-1}(y) - F_m^{-1}(x) = B^{-1} \xi$$

$$\|B\| := \sup_{|\hat{\xi}| = \hat{\rho}} \frac{|B\hat{\xi}|}{|\hat{\xi}|} \leq \frac{h_m}{\hat{\rho}}$$

$$\|B^{-1}\| := \sup_{|\xi| = \rho_m} \frac{|B^{-1}\xi|}{|\xi|} = \frac{\hat{h}}{\rho_m}$$

$$\boxed{\text{Transformation of Sobolev norms under affine mappings}}$$

$$|\hat{v} \circ F_m^{-1}|^p_{k,p,T_m} := \int_{T_m} |D^k(\hat{v} \circ F_m^{-1})|^p \, dx$$

$$= \int_{T_m} \left| \left[ (\hat{D}^k \hat{v}) \circ F_m^{-1} \right] \cdot B_m^{-k} \right|^p dx$$

$$= \int_{\hat{T}} \left| \hat{D}^k \hat{v} \, B_m^{-k} \right|^p J_m \, d\hat{x}$$

$$\leq \|B_m\|^{-kp} J \, |\hat{v}|^p_{k,p,\hat{T}}$$

Notation: $\boxed{\begin{array}{l} a \lesssim b \;:=\; \exists\, c, \text{ ind. of } a, b, \text{ s.t.} \\ a \leq c\, b \quad , \quad a \sim b \Leftrightarrow a \lesssim b,\, b \lesssim a \end{array}}$

$$|\hat{v} \circ F_m^{-1}|_{k,p,\hat{T}} \lesssim \rho_m^{-k} J^{\frac{1}{p}} |\hat{v}|_{k,p,\hat{T}}$$

$$|\hat{v}|_{k,p,\hat{T}} \lesssim h_m^{k} J^{-\frac{1}{p}} |\hat{v} \circ F_m^{-1}|_{k,p,T}$$

$$\boxed{\begin{array}{c} \text{for} \quad 0 \leq k, e \leq q \\ |v|_{k,p,T_m} \lesssim \rho_m^{-k} h_m^{e} |v|_{e,p,T_m} \quad \forall v \in \mathbb{P}^q(T_m) \end{array}}$$

## Bramble-Hilbert

Let $\tau \in \mathcal{L}(W^{k+1,p}(\Omega), W^{s,p}(\Omega))$, $s \leq k$

If $\mathbb{P}^k \subset \ker(\tau)$ then

$$\|\tau(u)\|_{s,p,\Omega} \lesssim \|\tau\|_{\mathcal{L}(W^{k+1,p}(\Omega), W^{s,p}(\Omega))} |u|_{k+1,p,\Omega}$$

Think of $\tau$: $I - \Pi^k$

$$\|u - \Pi^k u\|_{s,p,\Omega} \lesssim \|\tau\|_* \, |u|_{k+1,p,\Omega}$$

Proof: step 1: show that $\forall\, \tau \in \mathcal{L}(V,H)$, s.t. $Q = \ker \tau$ we have

$$\|\tau(u)\|_H = \|\tau(u+p)\|_H \leq \|\tau\|_* \|u+p\|_V \quad \forall\, p \in Q$$

$$\|\tau(u)\|_H \leq \|\tau\|_* \inf_{p \in Q} \|u+p\|_V \quad \forall u \in V$$

Step 2: (Denis - Lions Lemma)

for $u \in W^{k+1,p}(\Omega)$

$$|u|_{k+1,p,\Omega} \lesssim \inf_{q \in \mathbb{P}^k(\Omega)} \|u+q\|_{k+1,p,\Omega} \lesssim |u|_{k+1,p,\Omega}$$

1st ineq.:

$$|u|_{k+1,p,\Omega} \lesssim \inf_{q \in \mathbb{P}^k(\Omega)} \|u+q\|_{k+1,p,\Omega}$$

$$|u+q|_{k+1,p,\Omega} = |u|_{k+1,p,\Omega} \quad \forall\, q \in \mathbb{P}^k$$

because $|q|_{k+1,p,\Omega} = 0 \quad \forall q \in \mathbb{P}^k$

2nd ineq:

we show that 2nd ineq. is equivalent to proving the following result:

$\mathbb{P}^k = \text{span}\{v_i\}_{i=1}^{N_{p^k}}$ , $v^j \in (\mathbb{P}^k)^*$ s.t. $v^j(v_i) = \delta^j_i$

By Hahn-Banach extension theorem,

$\exists \ \tilde{v}^i \in (W^{k+1,p}(\Omega))^*$ s.t. $\tilde{v}^i(q) = v^i(q) \ \forall q \in \mathbb{P}^k$

$$\Pi^k : W^{k+1,p}(\Omega) \longrightarrow \mathbb{P}^k \qquad (\Pi^k)^2 = \Pi^k$$

$$u \longrightarrow \sum_i \tilde{v}^i(u)\, v_i$$

2nd ineq. follows from proving that

1) $$\|u\|^p_{k+1,p,\Omega} \lesssim |u|^p_{k+1,p,\Omega} + \sum_i |\tilde{v}^i(u)|^p$$

1) implies that $\exists$ c. s.t. $\forall u \in W^{k+1,p}(\Omega)$ we have $\|u\|^p_{k+1,p,\Omega} \lesssim |u|^p_{k+1,p,\Omega} + \sum_i |\tilde{v}^i(u)|^p$

$\Rightarrow$ prove by contradiction:

$\forall c$, $\exists \ w_c \in W^{k+1,p}(\Omega)$ s.t. $\|w_c\|_{k+1,p,\Omega} = 1$,

$$c\left(|w_c|^p_{k+1,p,\Omega} + \sum_i |\tilde{v}^i(w_c)|^p\right) < \|w_c\|_{k+1,p,\Omega}$$

Take $c = i$ in $[0, 1, \dots N \dots)$

$\Rightarrow$ There exists a sequence $w_i$ s.t.

$\|w_i\|_{k+1,p,\Omega} = 1$ and $\lim_{i\to\infty}\left(|w_i|^p_{k+1,p,\Omega} + \sum_{j=1}^{N} |\tilde{v}^j(w_i)|^p\right) = 0$

- $W^{k+1,p}(\Omega) \hookrightarrow W^{k,p}(\Omega)$

  Bounded sequence in $W^{k+1,p}(\Omega) \Rightarrow \exists$ strongly convergent subsequence in $W^{k,p}(\Omega)$

a) $\exists\ w \in W^{k,p}(\Omega)$ s.t.

$$\| w_i - w \|_{k,p,(\Omega)} \longrightarrow 0$$

b) for hypothesis, $\quad |w_i|_{k+1,p} \longrightarrow 0$

a) + b) $\longrightarrow \quad w \in W^{k+1,p}(\Omega) \quad \|w_i - w\|_{k+1,p} \longrightarrow 0$

b) implies that $w \in \mathbb{P}^k(\Omega)$ ($\Omega$ is connected)

$$|w|_{k+1,p} = 0 \Rightarrow w \in \mathbb{P}^k(\Omega)$$

$\|w\|_{k+1,p} = 1$ for continuity of the norm

$(\ \|w_i\|_{k+1,p} = 1 \quad \forall i\ )$

$\|w_i\|_{k,p}$ norm is zero because

$$|w|^p_{k+1,p,\Omega} + \sum_J \left| \tilde{V}^J(w) \right|^p = 0$$

$$\Rightarrow \quad \tilde{V}^J(w) = V^J(w) = 0 \quad \forall J \Rightarrow w = 0$$

$$\Rightarrow \quad \|w\|_{k,p} = 0, \quad |w|_{k+1,p} = 0, \quad \|w\|_{k+1,p} = 1$$

impossible

$$\|u\|^p_{k+1,p,\Omega} \lesssim |u|^p_{k+1,p,\Omega} + \sum_i |\tilde{V}^i(u)|^p$$

$$\Downarrow$$

$$\inf_{p\in P^k} \|u+p\|^p_{k+1,p,\Omega} \leq \|u-\Pi^k u\|^p_{k+1,p,\Omega}$$

$$\lesssim |u|^p_{k+1,p,\Omega} + \sum_i \left|\tilde{V}^i(u-\Pi^k u)\right|^p$$

by construction

$$\tilde{V}^i(u-\Pi^k(u)) = \tilde{V}^i(u) - \tilde{V}^J(u)\,\underbrace{\tilde{V}^i(V_J)}_{\delta^i_J}$$

$$= \tilde{V}^i(u) - \tilde{V}^i(u) = 0$$

Take $\Omega$ as a single element of our triang. $T$

$$\|u-\Pi u\|_{s,p,T} \lesssim \|I-\Pi\|_* \, |u|_{k+1,p,T} \qquad T = F(\hat{T})$$

$$\hat{u} = u\circ F_T \quad , \quad \hat{\Pi}\hat{u} = (\Pi u)\circ F$$

$$\|u-\Pi u\|_{s,p,T} \lesssim \rho_T^{-s} J^{\frac{1}{p}} \|\hat{u}-\hat{\Pi}\hat{u}\|_{s,p,\hat{T}} \lesssim \rho_T^{s} J^{\frac{1}{p}} |\hat{u}|_{k+1,p,\hat{T}}$$

$$\boxed{\|u - \Pi u\|_{s,p,T} \lesssim \rho_T^{-s}\, h_T^{k+1}\, |u|_{k+1,p,T}}$$

For quasi uniform Triangulations:

$$h := \max_m h_m \quad , \quad \rho = \min_m \rho_m$$

$$\forall m \Rightarrow \rho_m \geq \sigma h_m \qquad \sigma \in (0,1)$$

$$\boxed{\|u - \Pi u\|_{s,p,T} \lesssim \sigma^{-s}\, h^{k+1-s}\, |u|_{k+1,p,T}}$$

For example:

$V = H_0^1(\Omega)$

$$\boxed{\begin{aligned} \langle A u, v\rangle &:= \int_\Omega \nabla u \nabla v \, d\Omega \\ \langle F, v\rangle &:= \int_\Omega f v \, d\Omega \end{aligned}} \quad *$$

$f \in L^2(\Omega)$, $\underline{\Omega \text{ Lip.}}$ $\underline{\text{convex}}$ $\Rightarrow$ $\underline{u \in H^2(\Omega) \cap H_0^1(\Omega)}$

2-regular: $f \in H^s \Rightarrow u \in H^{s+2}$

$$* \Rightarrow \langle A(u - u_h), v_h\rangle = 0$$

$$\Rightarrow \|u - u_h\|_{1,\Omega} \leq \overbrace{\frac{\|A\|_*}{\alpha}}^{\equiv 1} \inf_{v_h \in V_h} \|u - v_h\|_{1,\Omega}$$

$$k+1-s = 1$$

$$\leq \frac{\|A\|_*}{\alpha} \|u - \Pi u\|_{1,\Omega} \leq \frac{\|A\|_*}{\alpha} \sigma^{-1} h\, |u|_{2,\Omega}$$

$\Pi u \in V_h \rightsquigarrow$

Hermbaual Extension — extension by zero

$$\Pi u = \sum_m \sum_I \sum_J P_{mIJ} \, \tilde{\hat{v}}^J (u \circ F_m) \, \overbrace{\hat{v}_J \circ F_m^{-1}}^{\text{extension by zero}}$$

$$\sum_m \| u - \Pi u \|_{1, T_m} \leq \sigma^{-1} h^{1} \sum_m |u|_{2, T_m}$$

In general you can say

$$\left( \sum_m \| u - \Pi u \|_{s, T_m} \right) \leq \sigma^{-s} h^{k+1-s} |u|_{k+1, \Omega} \quad \forall u \in H^{k+1}(\Omega)$$

$$\forall s \leq k$$

in general $\sum_m \| u - \Pi u \|_{s, T_m} \neq \| u - \Pi u \|_{s, \Omega}$

The equality holds for all $H^s$ s.t. $V_h \subset H^s$

For example: $V_h \subset H^1_0(\Omega)$, then

$$\| u - \Pi u \|_{s, \Omega} \leq \sigma^{-s} h^{k+1-s} |u|_{k+1, \Omega}$$

for $s = 0, 1$, $k \geq 1$ arbitrary