Master in High Performance Computing

Theory and Practice of Finite Element Methods

A posteriori error estimates and adaptive meshes — optimality —

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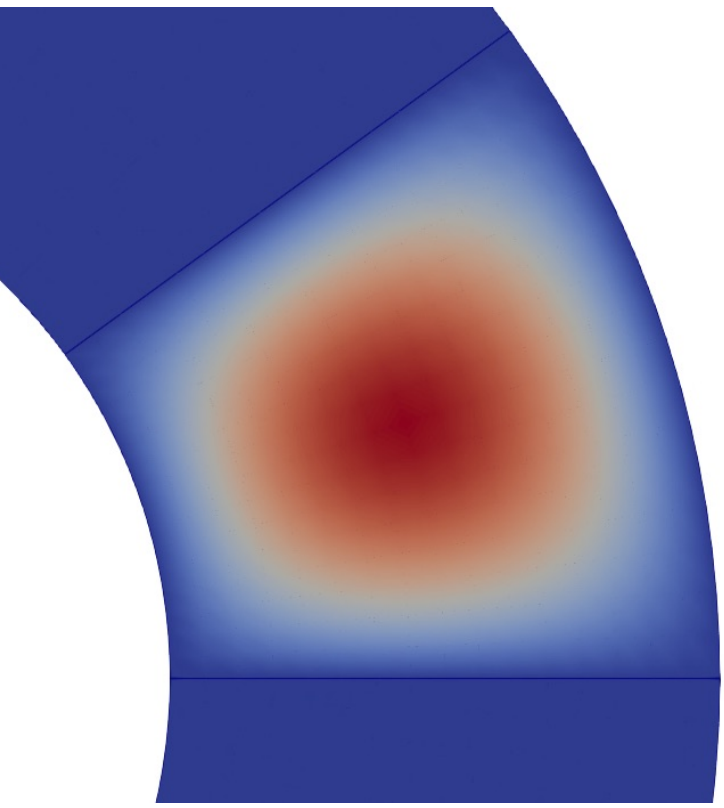
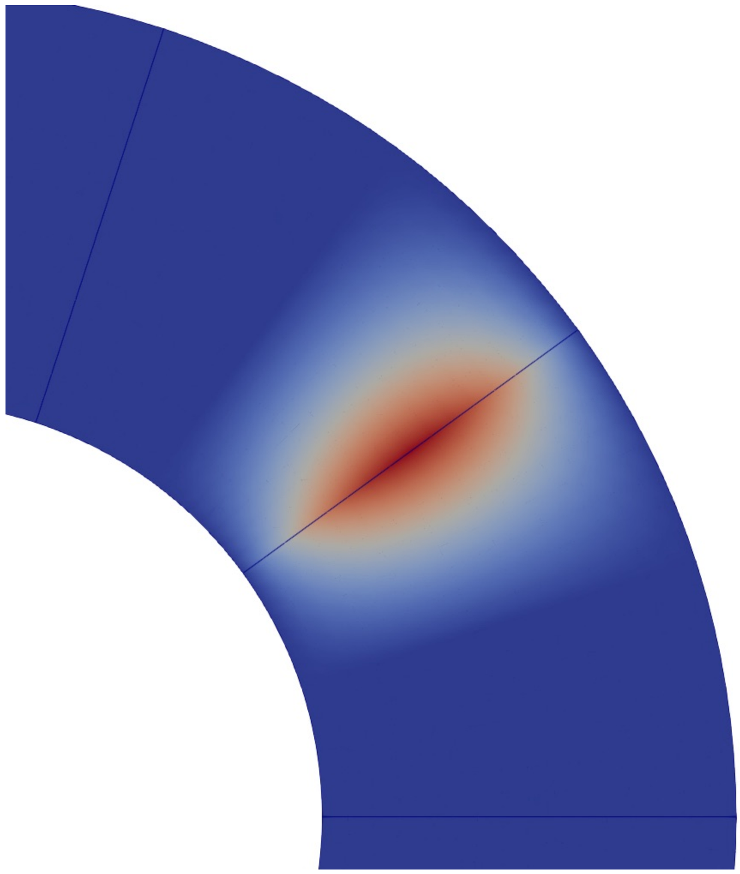


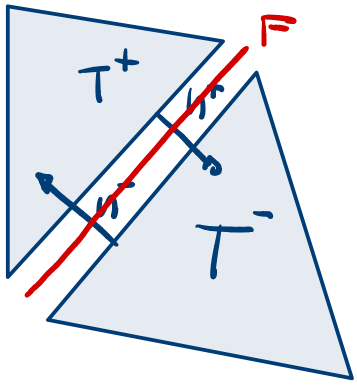
b_K



b_F

Definition of Jump

$$\boxed{[\![a]\!]\Big|_F := \begin{cases} a^+ n^+ + a^- n^- & \text{if } F \subset \mathring{\mathcal{E}}_h \\ a n - g_N & \text{if } F \subset \partial\Omega_N \\ 0 & \text{if } F \subset \partial\Omega_D \end{cases}}$$



Local error estimator

$$\eta_T := h_T \|f + \Delta u_h\|_{0,T} + \sum_{F \subset \partial T} \frac{1}{2} h_F^{\frac{1}{2}} \|[\![\nabla u_h]\!]\|_{0,F}$$

IDEALLY (GLOBAL PROPERTIES)

$$\sum_{T \in \mathcal{T}} \eta_T^2 \lesssim |u - u_h|^2_{1,\Omega} \lesssim \sum_{T \in \mathcal{T}} \eta_T^2$$

(LOCAL PROPERTIES)

$$\overbrace{\eta_T^2 \lesssim |u - u_h|^2_{1,T}}^{\text{OPTIMALITY}} \lesssim \eta_T^2$$

$$|u - u_h|^2_{1,T} \lesssim \eta_T^2 \quad \text{(RELIABILITY)}$$

FALSE IN GENERAL

THEOREM : LOCAL OPTIMALITY

$$\eta_T := h_T \|f - \Delta u_h\| + \sum_{F \subset \partial T} \frac{1}{2} h_F^{\frac{1}{2}} \| [\![\nabla u_h]\!] \|_{0,F}$$

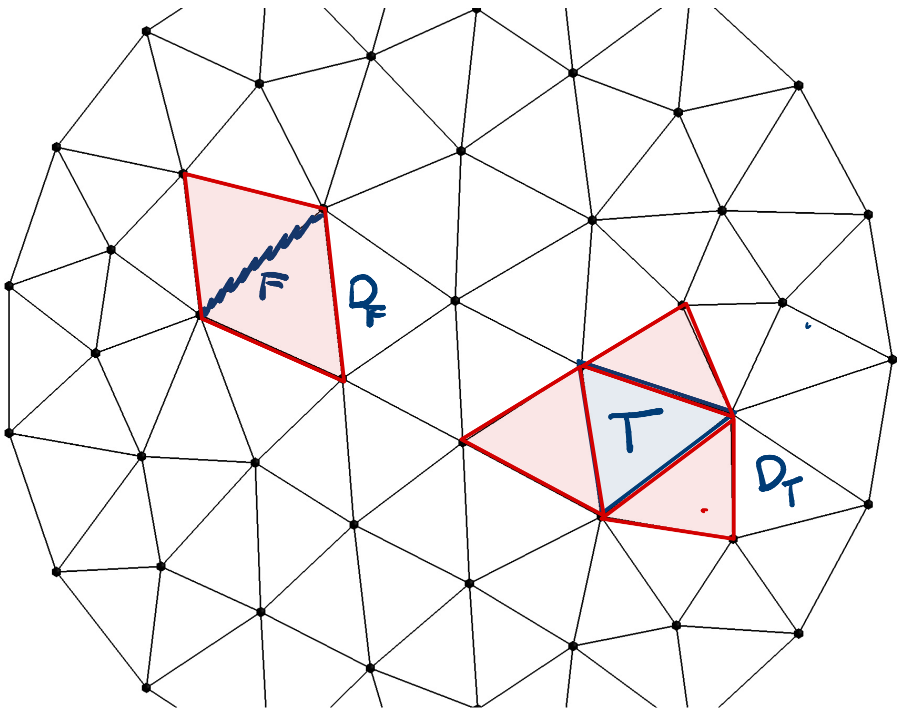
$$\lesssim$$

$$\eta_T \lesssim |u - u_h|_{1,D_T} + h_T \inf_{v_h \in V_h} \|f - v_h\|_{0,D_T}$$

$$D_T := \left\{ \overset{\circ}{\overline{\bigcup_{T_m} \overline{T}_m}} \text{ s.t. } \overline{T}_m \cap \overline{T} = F \neq \{\phi\} \text{ where } \dim(F) = d-1 \right\}$$

All elements that share a face with T

$$D_F := \left\{ \overset{\circ}{\overline{\bigcup_{T_m} \overline{T}_m}} \text{ s.t. } \overline{T}_{m_1} \cap \overline{T}_{m_2} = F \right\}$$



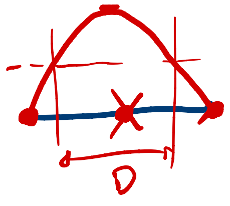
$\boxed{\text{Second Ingredient}}$ Bubble functions on T, or on F

b_T is a bubble function if:

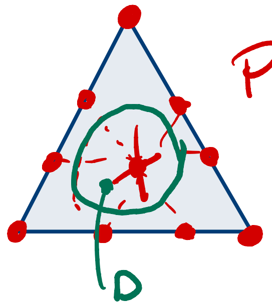
a): $0 \leq b_T \leq 1$ in $\overline{T}$

b) $b_T \in H_0^1(T)$

c) $\exists D \subset T$ s.t. $|D| > 0$ s.t. $b_T|_D \geq \frac{1}{2}$



$\mathbb{P}^2(E)$



$\mathbb{P}^3(T)$

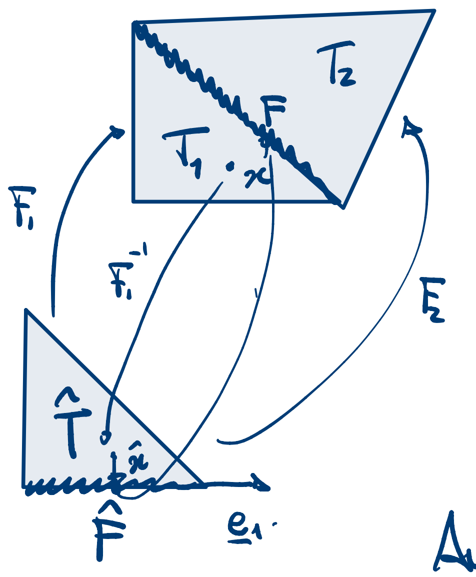
Valid for $T \in \mathbb{R}^d$ simplex or hyper cube

with a), b), c) as properties then

i) $\| b_T \phi \|_{0,T} \lesssim \| \phi \|_{0,T} \lesssim \| b_T^{\frac{1}{2}} \phi \|_{0,T}$

ii) $| b_T \phi |_{1,T} \lesssim h_T^{-1} \| \phi \|_{0,T} \quad \forall v \in \mathbb{P}^k(T)$ $k \geq 1$

$\boxed{\text{LIFTING OPERATOR}}$ $RE : \mathbb{P}^k(F) \longrightarrow \mathbb{P}^k(D_F)$ element wise



$$T_1 = F_1(\hat{T})$$
$$T_2 = F_2(\hat{T})$$

Assume that $\hat{F}$ is such that

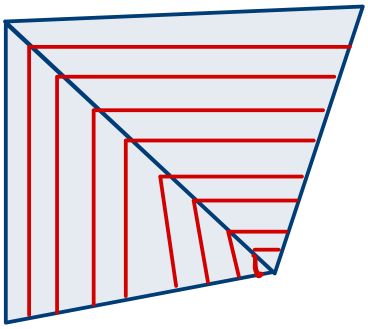
$$F_1(\hat{F}) = F_2(\hat{F}) = F$$

And assume that $\hat{F}$ is aligned with $\hat{x}_1$

(simplifying assumption, but not necessary)

$$RE(v)(x)\Big|_{T_{1/2}} := v\left(F_1\left(\hat{\Pi}_{\hat{F}}\left(F_1^{-1}(x)\right)\right)\right) = v \circ F_{1/2} \circ \hat{\Pi}_{\hat{F}} \circ F_{1/2}^{-1}$$

$$\hat{\Pi}_{\hat{F}}(\hat{x}) := e_1 \otimes e_1 \; \hat{x} = (\hat{x} \cdot e_i)\, e_i$$ orthogonal projection of $\hat{x}$ onto $\hat{F}$



Contour plots of $RE(v)$ for $v \in P^1(F)$

Define a bubble function on F <u>and</u> on D_F

(It is a bubble function for both F as $d-1$ manifold and for D_F as d-manifold)

a) $0 \le b_F \le 1$

b) $b_F\big|_F \in H_0^1(F)$, $\quad b_F \in H_0^1(D_F)$

c) $\exists\ \tilde{D}_F \subset F$ s.t. $|\tilde{D}_F| > 0$ (measure of dim. $(d-1)$)
when $b_F\big|_{\tilde{D}_F} \ge \frac{1}{2}$

d) $\exists\ \tilde{D}_{D_F} \subset D_F$ s.t. $|\tilde{D}_{D_F}| > 0$ (measure of d)

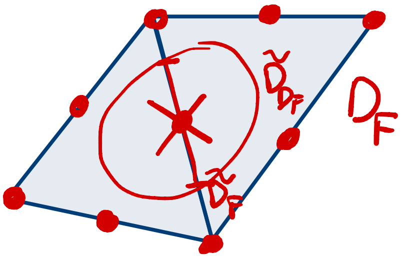
when $b_F\big|_{\tilde{D}_{D_F}} \geqslant \frac{1}{2}$

Then

$$\boxed{\begin{array}{ll} \text{i)} & \| b_F \phi \|_{0,F} \lesssim \| \phi \|_{0,F} \lesssim \| b_F^{\frac{1}{2}} \phi \|_{0,F} \quad \forall \phi \in P^k(F) \\ \text{ii)} & h_F^{\frac{1}{2}} \| \phi \|_{0,F} \lesssim \| b_F RE(\phi) \|_{0,D_F} \lesssim h_F^{\frac{1}{2}} \| \phi \|_{0,F} \\ \text{ii)} & | b_F RE(\phi) |_{1,D_F} \lesssim h_F^{-\frac{1}{2}} \| \phi \|_{0,F} \end{array}}$$

Possible b_F:

bar's function of $\mathbb{P}^2(T)$ associated with center of F.



THEO:

$$\eta_T \lesssim |u-u_h|_{1,D_T} + h_T \inf_{v_h \in V_h} \|f-v_h\|_{0,D_T}$$

$$\eta_T := \underbrace{h_T \|f+\Delta u_h\|_{0,T}}_{1} + \underbrace{\sum_{F\subset \partial T} \frac{1}{2} h_F^{\frac{1}{2}} \|[\![\nabla u_h]\!]\|_{0,F}}_{2}$$

1: $$\|f+\Delta u_h\|_{0,T} \lesssim \|f-v_h\|_{0,T} + \overbrace{\|v_h+\Delta u_h\|_{0,T}}^{3}$$

3: $$\|v_h+\Delta u_h\|_{0,T}^2 \lesssim \|b_T^{\frac{1}{2}}(v_h+\Delta u_h)\|_{0,T}^2$$

$$\lesssim \int_T (v_h \overbrace{-f-\Delta u}^{0}+\Delta u_h)\, b_T\,(v_h+\Delta u_h)\, dT$$

$b_T|_{\partial T} = 0$ by construction

$$\lesssim \int_T (v_h-f)\, b_T(v_h+\Delta u_h)\,dT + \int_T \nabla(u-u_h)\nabla\big(b_T(v_h+\Delta u_h)\big)\,dT$$

$$\lesssim \|v_h-f\|_{0,T}\, \overbrace{\|b_T(v_h+\Delta u_h)\|_{0,T}}^{\lesssim \|v_h+\Delta u_h\|_{0,T}} + |u-u_h|_1\, \overbrace{|b_T(v_h+\Delta u_h)|_1}^{\lesssim h_T^{-1}\|v_h+\Delta u_h\|_{0,T}}$$

$$\|v_h + \Delta u_h\|_{0,T}^{\cancel{2}} \lesssim \cancel{\|v_h + \Delta u_h\|_{0,T}} \left(|u - u_h|_1 h_T^{-1} + \|v_h - f\|_{0,T} \right)$$

$$\forall v_h \in V_h$$

(1) $\Rightarrow$ $$h_T \|v_h + \Delta u_h\|_{0,T} \lesssim |u - u_h|_{1,T} + h_T \|v_h - f\|_{0,T}$$

$$h_T \|f + \Delta u_h\| \lesssim h_T \|f - v_h\| + |u - u_h|_{1,T}$$

(2) $$\|[\![\nabla u_h]\!]\|_{0,F}^2 \lesssim \|b_F^{\frac{1}{2}} [\![\nabla u_h]\!]\|_0^2$$

$$\lesssim \int_F [\![\nabla u_h - \underbrace{\nabla u}_{\equiv 0}]\!] \, b_F [\![\nabla u_h]\!] \, dF$$

$$\sum_{T \in D_F} \int_T -\Delta u v = \sum_{T \in D_F} \left(\int_T \nabla u \nabla v - \int_{\partial T} n \cdot \nabla u \, v \right) \qquad \forall v, u \text{ smooth enough}$$

$$\sum_{T \in D_F} \int_T -\Delta u v = \int_{D_F} \nabla u \nabla v - \int_F [\![\nabla u]\!] v - \int_{\partial D_F} n \cdot \nabla u \cdot v \qquad \forall v, u \text{ smooth enough}$$

If $v \big|_{\partial D_F} = 0 \implies \int_F [\![\nabla u]\!] v = \int_{D_F} \Delta u v + \int_{D_F} \nabla u \nabla v$

Apply this to $\int_F \nabla u \, b_F [\![\nabla u_h]\!]$

$$\|[\![\nabla u_h]\!]\|_{0,F}^2 \lesssim \int_F [\![\nabla u_h - \nabla u]\!] \, b_F \, \underbrace{RE([\![\nabla u_h]\!])}_{\equiv [\![\nabla u_h]\!] \text{ on } F} dF$$

$$\lesssim \sum_{T\in D_F} \int_T (\Delta u_h - \Delta u)\, b_F\, RE(\llbracket \nabla u_h \rrbracket)\, dT$$

$$+ \int_{D_F} (\nabla u_h - \nabla u) \nabla \left(b_F\, RE(\llbracket \nabla u_h \rrbracket)\right) dT$$

$$\lesssim \sum_{T\in D_F} \left(\| \Delta u_h + f \|_{0,T}\ h_F^{\frac{1}{2}} \| \llbracket \nabla u_h \rrbracket \|_{0,F} \right)$$

$$+ |u - u_h|_{1,D_F}\ \underbrace{|b_F\, RE(\llbracket \nabla u_h \rrbracket)|_{1,D_F}}_{\lesssim h_F^{-\frac{1}{2}} \| \llbracket \nabla u_h \rrbracket \|_{0,F}}$$

$$\| \llbracket \nabla u_h \rrbracket \|_{0,F} \lesssim \left(|u - u_h|_{1,D_F} h_F^{-\frac{1}{2}} + \| \Delta u_h + f \|_{0,D_F} h_F^{\frac{1}{2}} \right)$$

$h_F \sim h_T$ for shape regular $\mathcal{T}$

(2) $$\frac{1}{2} h_F^{\frac{1}{2}} \| \llbracket \nabla u_h \rrbracket \|_{0,F} \lesssim |u - u_h|_{1,D_F} + h_T \| \Delta u_h + f \|_{0,D_F}$$

Sum (1) and (2), take inf on v_h:

$$\boxed{\eta_T \lesssim |u - u_h|_{1,D_T} + h_T \inf_{v_h \in V_h} \| f - v_h \|_{0,D_T}}$$

Constructing a new mesh

Error balancing : Reliability implies that

$$|u-u_e|_1 \lesssim \left(\sum_K \eta_K^2\right)^{\frac{1}{2}} \lesssim Tol$$

if $\left(\sum_K \eta_K^2\right) \lesssim tol \Rightarrow |u-u_e| \lesssim tol.$

$M^i := \# \mathcal{T}^i$ number of elements of $\mathcal{T}^i$

$\mathcal{T}^{i+1}$ is such that $\forall\, T^{i+1} \subset \mathcal{T}^{i+1}$

$$\eta_{T^{i+1}}^2 \lesssim \frac{tol^2}{M^{i+1}}$$

Bulk chasing algorithm, or Dörfler algorithm or "fixed fraction" algorithm, bulk criterion —

→ Given $0 < \theta < 1$ Define

$$E_{TOT} = \left(\sum_{K^i \in \mathcal{T}^i} \eta_{K^i}^2\right)^{\frac{1}{2}}$$

Mark for refinement smallest subset $M \subset \mathcal{T}^i$ s.t.

$$\theta E_{TOT} \leq \left(\sum_{K^i \in M} \eta_{K^i}^2\right)^{\frac{1}{2}}$$

1. order T_k^i s.t. $\eta_{T_k^i} \leq \eta_{T_j^i}$ when $j \leq k$

2. compute $E_{TOT} = \left(\sum_{T \subset \mathcal{T}^i} (\eta_T^2) \right)^{\frac{1}{2}}$

3. Mark for refinement up to element $\bar{k}$ s.t.

$$\left(\sum_{j=1}^{\bar{k}} \eta_{T_j}^2 \right)^{\frac{1}{2}} \geq \theta E_{TOT}$$

and

$$\left(\sum_{j=1}^{\bar{k}-1} \eta_{T_j}^2 \right)^{\frac{1}{2}} < \theta E_{TOT}$$

Quasi uniform partition

$$\min_{T \in \mathcal{T}} h_T \lesssim \max_{T \in \mathcal{T}} h_T \lesssim \max_{T \in \mathcal{T}} h_T^{-1} \lesssim \min_{T \in \mathcal{T}} h_T^{-1}$$

$$\boxed{\frac{|\Omega|}{M} \sim h_T^d} \qquad \boxed{h_T \sim M^{-\frac{1}{d}}}$$

$$\|u - u_h\|_{m,\Omega} \lesssim \left(\sum_{T \subset \mathcal{T}_h} h_T^{2(\ell - m)} |u|_\ell^2 \right)^{\frac{1}{2}}$$

$$\|u - u_h\|_{m,\Omega} \lesssim h_T^{\ell-m} |u|_\ell$$

$m = 0, 1$ for $\ell \le k+1$

$$\boxed{\|u - u_h\|_{m,\Omega} \lesssim M^{-(\ell-m)/d} |u|_\ell}$$

$k = 1 \leadsto u \in H^2 \cap H^1_0$

$$\|u - u_h\|_{m,\Omega} \lesssim M^{-(2-m)/d} |u|_2$$

what happens if $u \notin H^2$?

If you equidistribute error ($\eta_K \sim \delta \;\; \forall K$)

$$\Rightarrow \|u - u_h\|_{m,\Omega} \lesssim \underline{\underline{M^{-(2-m)/d}}} |u|_1$$

no longer true that $\min h_T \sim \max h_T$

but it is true that

$$\eta_K \sim \delta \qquad \forall K$$

$\Rightarrow$ locally error is constant