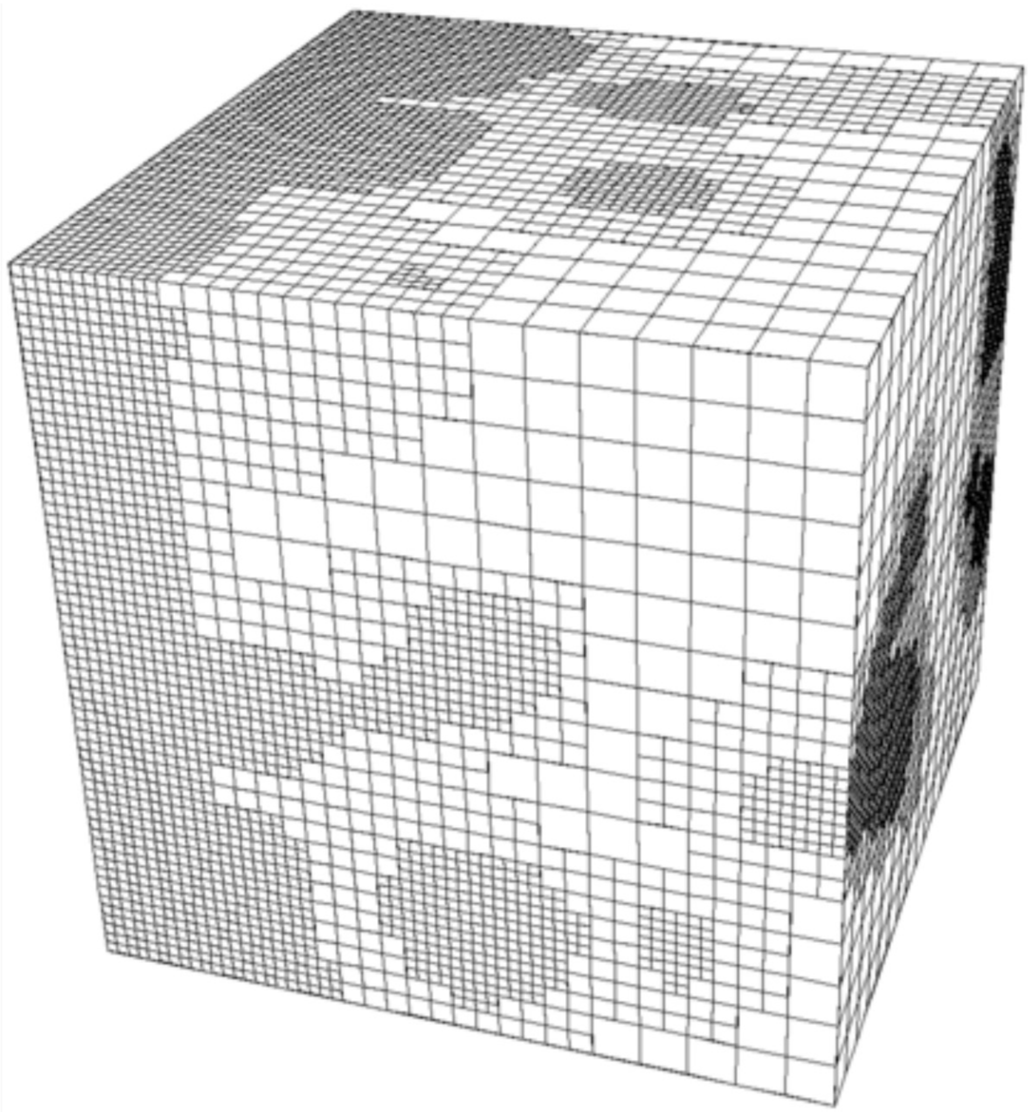
Master in High Performance Computing

Theory and Practice of Finite Element Methods

A posteriori error estimates and adaptive meshes

Luca Heltai <luca.heltai@sissa.it>

International School for Advanced Studies (www.sissa.it)
Mathematical Analysis, Modeling, and Applications (math.sissa.it)
Master in High Performance Computing (www.mhpc.it)
SISSA mathLab (mathlab.sissa.it)



Short Recap

Inverse Inequalities

on quasi uniform grids

$\forall v \in P_T \subset P_k(T)$, $0 \leq m, \ell \leq k$ real m, ℓ

$$\|v\|_{m,T} \lesssim h_T^{\ell-m} \|v\|_{\ell,T}$$

Trace theorems

Ω Lip. $H^s(\Omega) \longrightarrow H^{s-\frac{1}{2}}(\Gamma)$

For $s \in (\frac{1}{p}, 1]$, $\exists\ \gamma : W^{s,p}(\Omega) \longrightarrow W^{s-\frac{1}{p},p}(\Gamma)$

bounded and linear. Hahn Banach extension of cont. restriction $\gamma u = u$ for $u \in C^0(\bar{\Omega})$

1) $$\|\gamma v\|_{s-\frac{1}{p},p,\Gamma} \lesssim \|v\|_{s,p,\Omega}$$

γ has bounded right inverse $\boxed{E}$ $\boxed{\ker \gamma := W_0^{s,p}(\Omega)}$

2) $$\|Eg\|_{s,p,\Omega} \lesssim \|g\|_{s-\frac{1}{p},p,\Gamma}, \quad \gamma E g = g \quad \forall g \in W^{s-\frac{1}{p},p}(\Gamma)$$

3) For $v \in W^{1,p}(\Omega)$, Ω Lip, $1 \leq p \leq \infty$

$$\boxed{\|\gamma v\|_{0,p,\partial\Omega}^{p} \lesssim \|v\|_{0,p,\Omega}^{p-1} \|v\|_{1,p,\Omega}^{1} \quad \forall v \in W^{1,p}(\Omega)}$$

$$\boxed{\|\gamma v\|_{0,\partial\Omega}^{2} \lesssim \|v\|_{0,\Omega} \|v\|_{1,\Omega} \quad \forall v \in H^1(\Omega)}$$

Interpolation of non smooth functions

Scott-Zhang interpolation

$SZ : H^1(\Omega) \longrightarrow V_h \equiv P^{k,0}(\Omega)$

k polynomials on $T \subset T_h$, $V_h \subset H^1$

1) $\forall v \in H^1_0(\Omega) \qquad SZv \in H^1_0(\Omega)$

2) $SZ u_h = u_h \quad \forall u_h \in V_h$

in general $\tilde{\phi}^i$ is not unique. We construct S^i s.t. $S^i(v_h) = \phi^i(v_h)$

$SZ u \neq \Pi u := \underline{\underline{\tilde{\phi}^i}}(u)\, \phi_i \qquad V_h = \operatorname{span}\{\phi_i\}_{i=1}^N$

$\phi^j(\phi_i) = \delta_{ij}$

$SZ u := \underline{S^i(u)}\, \phi_i \qquad S^i(u) \neq \tilde{\phi}^i(u)$

Make a choice: $\forall\, a_i$ support point of V_h

V_h: piecewise polynomial of order k on each T,

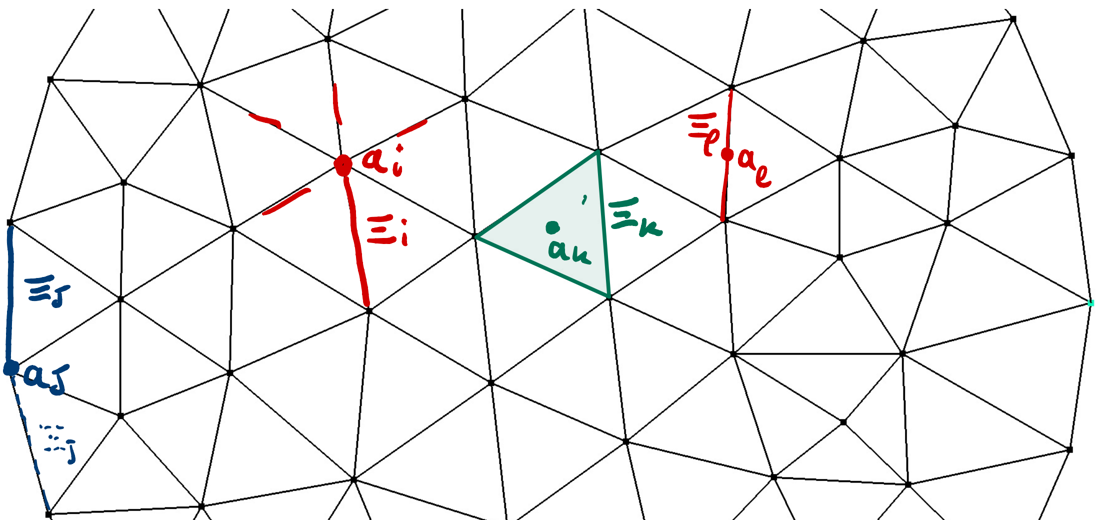
$\phi^i(u) = u(a_i) \quad \forall u \in C^0(\bar{\Omega})$

Choose one Ξ_i for each a_i

case 1: $a_i \in \mathring{T}_m \Rightarrow \Xi_i := T_m$

case 2: $a_i \in \partial T_m \setminus \partial\Omega \Rightarrow \Xi_i := F_i \in \partial T_m$ one of the faces not on $\partial\Omega$

case 3: $a_i \in \partial T_m \cap \partial\Omega \Rightarrow \Xi_i := F_i \in \partial T_m \cap \partial\Omega$



Ξ_i is a d-simplex if $a_i \in \mathring{T}_m$, otherwise it is a $(d-1)$-simplex

- n_i is number of support points that belong to Ξ_i
- $\{\phi_{i,q}\}_{q=1}^{n_i}$ are the basis functions associated to the support points in Ξ_i $\quad \phi_{i,1} \equiv \phi_i$
- $\{\gamma_{i,q}\}_{q=1}^{n_i} \in \operatorname{span}\{\phi_{i,q}\}_{q=1}^{n_i}$ s.t.

$$\int_{\Xi_i} \gamma_{iq}\, \phi_{i,r} = \delta_{qr}$$

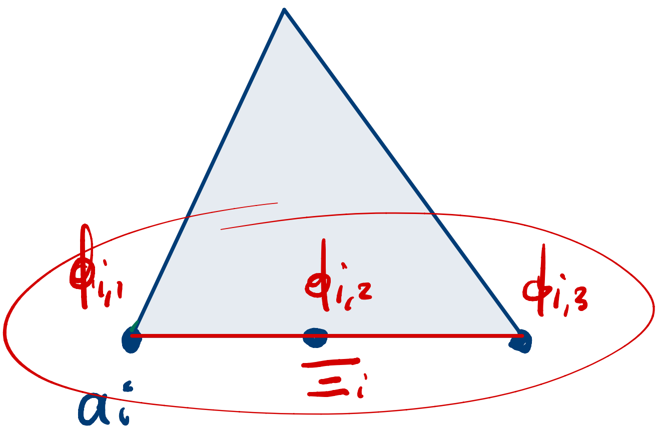
$$S^i(u) := \int_{\Xi_i} \gamma_{i,1}\, u \qquad SZ_u := \sum_{i=1}^{N} \phi_i \int_{\Xi_i} \gamma_{i,1} u$$

$$= S^i(u)\, \phi_i$$

$\forall u \in V_h$ s.t. $u = u^J \phi_J$

$$SZ_u = S^i(u^J \phi_J)\, \phi_i = \sum_{i=1}^{N} \phi_i \underbrace{\int_{\Xi_i} \gamma_{i,1}\, u^J \phi_J}_{\delta_{iJ}}$$

$$= u^i \phi_i = u$$



By construction $\quad \gamma SZ_u = 0 \quad \forall u \in H_0^1(\Omega)$

Properties of Scott-Zhang

$1 \leq p \leq +\infty \qquad \ell \geq 1$ if $p = 1 \qquad \ell > \frac{1}{p}$ otherwise

1) $\forall h, \forall v \in W^{\ell,p}(\Omega)$ $\quad \boxed{\| SZ u \|_{\ell,p,\Omega} \lesssim \| u \|_{\ell,p,\Omega}}$

stability property

polynomial degree of V_h

2) $\ell \leq k+1 \qquad 0 \leq m \leq \ell$

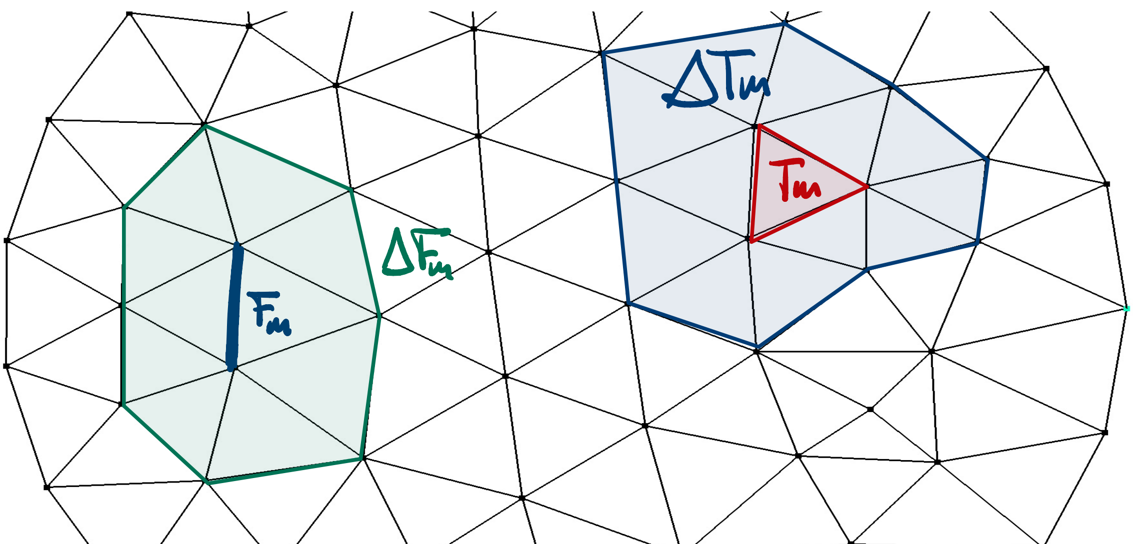
$$\forall h, \forall T \in \mathcal{T}_h, \forall v \in W^{\ell,p}(\Delta T)$$

$$\| v - SZ v \|_{m,p,T} \lesssim h_T^{\ell - m} |v|_{\ell,p,\Delta T}$$

3) $$\| v - SZ v \|_{m,p,F} \lesssim h_F^{\ell - m} |v|_{\ell,p,\Delta F}$$

$\Delta T_m : \overline{\left(\bigcup_J T_J \right)}^{\circ}$ s.t. $\exists\, i_J, k$ s.t. $a_{mi} = a_{Jk}$

$\Delta F_m : \overline{\left(\bigcup_J T_J \right)}^{\circ}$ s.t. $\exists\, i, J, k$ s.t. $a_{mi} = a_{Jk}$



Orthonormal projections

$$\Pi^0 : L^2(\Omega) \longrightarrow V_h \subset H^1(\Omega)$$ ← L^2 projection

$$\Pi^1 : H^1(\Omega) \longrightarrow V_h \subset H^1(\Omega)$$ ← Rietz Projection
Elliptic projection

$$(\Pi^0 u, v)_0 = (u, v)_0 \quad \forall \; v \in V_h$$

$$(\Pi^1 u, v)_1 = (u, v)_1 \quad \forall \; v \in V_h$$

$$(u,v)_0 := \int_\Omega u \, v \, d\Omega$$

$$(u,v)_1 := \int_\Omega (uv + \nabla u \cdot \nabla v) \, d\Omega$$

$$M_{ij} u^j = U_i^0 \qquad U_i^0 := (u, v_i)_0$$

$$M_{ij} := (v_j, v_i)_0$$

$$K_{ij} u^j = U_i^1 \qquad U_i^1 := (u, v_i)_1$$

$$K_{ij} := (v_j, v_i)_1$$

$$M^{ij} M_{jk} = \delta^i_k$$

↑ Inverse matrix

$$u^j = M^{jk} U_k^0 \qquad u^j = K^{jk} U_k^0$$

$$\Pi^0 u := M^{JK}(u, v_K)_0 \, v_J$$

$$\Pi^1 u := K^{JK}(u, v_K)_1 \, v_J$$

$$\|\Pi^0 v\|_{0,\Omega} \lesssim \|v\|_{0,\Omega} \quad \forall v \in L^2(\Omega)$$

$$\|\Pi^1 v\|_{1,\Omega} \lesssim \|v\|_{1,\Omega} \quad \forall v \in H^1(\Omega)$$

If $\mathcal{T}_h$ is quasi uniform:

$\|\Pi^0 v\|_{1,\Omega} \lesssim \|v\|_{1,\Omega}$ is also stable

For "smooth" functions: $k \geq 1 \qquad 1 \leq e \leq k$

$$\|v - \Pi^0 v\|_{0,\Omega} \lesssim h^{e+1} |v|_{e+1,\Omega}$$

$$\|v - \Pi^1 v\|_{1,\Omega} \lesssim h^{e} |v|_{e+1,\Omega}$$

If $\mathcal{T}_h$ is quasi uniform

$$\|v - \Pi^0 v\|_{1,\Omega} \lesssim h^{e} |v|_{e+1,\Omega}$$

A posteriori error analysis

$\begin{cases} -\Delta u = f & \text{with } H^{-1} \\ u = 0 & \text{on } \partial \end{cases}$

$$(\nabla u, \nabla v) = (f, v) \quad \forall\, v \in H^1_0(\Omega)$$

$$(\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall\, v \in V_h \subset H^1_0(\Omega)$$

Instead of exploiting coercivity, we exploit infsup.

$$\langle Au, v \rangle = \langle f, v \rangle \qquad A : \underset{H^1_0(\Omega)}{V} \longrightarrow \underset{H^{-1}(\Omega)}{V^*}$$

infsup: A is bounding: $\|Au\|_{V^*} \geq \alpha \|u\|_V$

$\boxed{\text{Ideally}}$ we'd like to be able to build $\boxed{e_T}$ s.t.

$$\boxed{e_T(u_h, f) \lesssim \|u - u_h\|_{1,T} \lesssim e_T(u_h, f)}$$

FALSE IN GENERAL

Then you could ask

1) $\left(\sum_{T \in \mathcal{T}_h} e_T^2 \right)^{\frac{1}{2}} \leq tol$

2) e_T is balanced over all $T \in \mathcal{T}_h$

$$\frac{tol}{M} \lesssim e_T \lesssim \frac{tol}{M} \qquad M: \text{number of elements}$$

Easy part: Global upper bound

$$\boxed{\sum_T \|u - u_h\|^2_{1,T} \lesssim \sum_T e_h^2}$$

Difficult part: "Almost local" lower bound

$$\boxed{e_T(u_h, f) \lesssim \|u - u_h\|_{1,\Delta T} + \Pi(h_T, \Delta T, f)}$$

ΔT: patch of all elements around T

Π: perturbation negligible or same order of error $\|u - u_h\|$

Easy part

$$\|u - u_h\|_{1,\Omega} \lesssim \frac{\|A\|}{\alpha} \inf_{V_h \in V_h} \|u - V_h\| \lesssim \left(\sum_{T \subset \mathcal{T}_h} h_T^{2k} |u|^2_{k+1,T} \right)^{\frac{1}{2}}$$

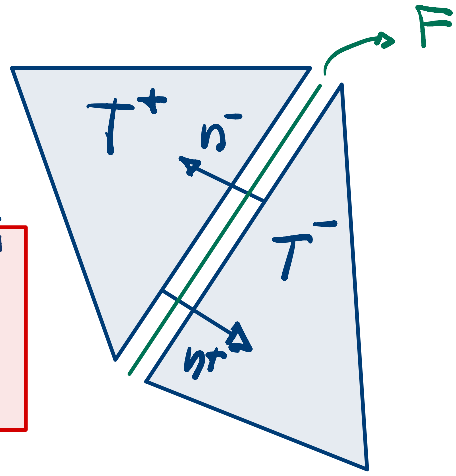
Simplest a posteriori estimate

$$\alpha \|u - u_h\|_{1,\Omega} \lesssim \|A(u - u_h)\|_{-1,\Omega} \lesssim \underline{\|f + \Delta u_h\|_{-1,\Omega}}$$

issue: $\|\cdot\|_{-1,\Omega}$ is NOT a local norm

$$[\![a]\!]\Big|_F := a^+ n^+ + a^- n^-$$

$\forall F \in$ $\boxed{\bigcup_{T \subset \mathcal{T}_h} \partial T \setminus \partial\Omega := \mathcal{E}^i}$



- Exploit definition of $\|\cdot\|_{-1,\Omega}$
- " the orthogonality of error

- Estimate results using SZ int. properties.

1

$$\|u-u_h\|_{1,\Omega} \lesssim \frac{1}{\alpha}\|A(u-u_h)\|_{-1,\Omega} = \frac{1}{\alpha}\sup_{v\in V}\frac{|\langle A(u-u_h),v\rangle|}{\|v\|_{1,\Omega}}$$

$$= \frac{1}{\alpha}\sup_{v\in V}\frac{|\langle A(u-u_h),v-v_h\rangle|}{\|v\|_{1,\Omega}} \qquad \forall v_h\in V_h$$

$$\langle A(u-u_h),v-v_h\rangle = \int_\Omega \nabla(u-u_h)\nabla(v-v_h) = \int_\Omega \underbrace{-\Delta u}_{=f}(v-v_h) - \int_\Omega \nabla u_h \nabla(v-v_h)$$

$$\lesssim \sum_{T\subset\mathcal{T}_h}\left(\int_T (f+\Delta u_h)(v-v_h) - \int_{\partial T} n\nabla u_h (v-v_h)\right)$$

$$\lesssim \sum_{T\subset\mathcal{T}_h}\left(\|f+\Delta u_h\|_{0,T}\|v-v_h\|_{0,T} + \sum_{F\subset\partial T\setminus\partial\Omega}\frac{1}{2}\|[\![\nabla u]\!]\|_{0,F}\|v-v_h\|_{0,F}\right)$$

* SZ properties.

$$\|v-\mathcal{SZ}v\|_{0,T} \lesssim h_T|v|_{1,\Delta T}$$

Trace inequality

$$\|v-\mathcal{SZ}v\|_{0,F} \lesssim h_F^{\frac{1}{2}}|v|_{\frac{1}{2},\Delta F} \lesssim h_F^{\frac{1}{2}}|v|_{1,\Delta T}$$

$$|\langle A(u-u_h),v-\mathcal{SZ}v\rangle| \lesssim \sum_{T\subset\mathcal{T}_h}\left[\left(\|f+\Delta u_h\|_{0,T}h_T + \sum_F\frac{1}{2}h_F^{\frac{1}{2}}\|[\![\nabla u]\!]\|_{0,F}\right)\|v\|_{1,\Delta T}\right]$$

$\tilde{N}$:= max number of neighbours of T

$$\sum_{T \subset \mathcal{T}_h} \|v\|^2_{1,\Delta T} \lesssim \tilde{N} \, \|v\|^2_{1,\Omega}$$

$$\left(\sum (a_i \cdot b_i) \right)^2 \lesssim \left(\sum a_i \right)^2 \left(\sum b_i \right)^2$$

$$\Downarrow$$

$$\| A(u-u_h)\|_{-1,\Omega} = \| f + \Delta u_h \|_{-1,\Omega} \lesssim \tilde{N} \left(\sum_{T \in \mathcal{T}_h} e_T^2 \right)^{\frac{1}{2}}$$

$$e_T := \| f + \Delta u_h \|_{0,T} \, h_T + \frac{1}{2} \sum_{F \subset \partial T \setminus \partial\Omega} h_F^{\frac{1}{2}} \| [\![\nabla u_h]\!] \|_{0,F}$$

$$\| u - u_h \|_{1,\Omega} \lesssim \frac{1}{\alpha} \left(\sum_T e_T^2 \right)^{\frac{1}{2}}$$