Theory and Practice of Finite Element Methods

Transformations of local Finite Element Spaces

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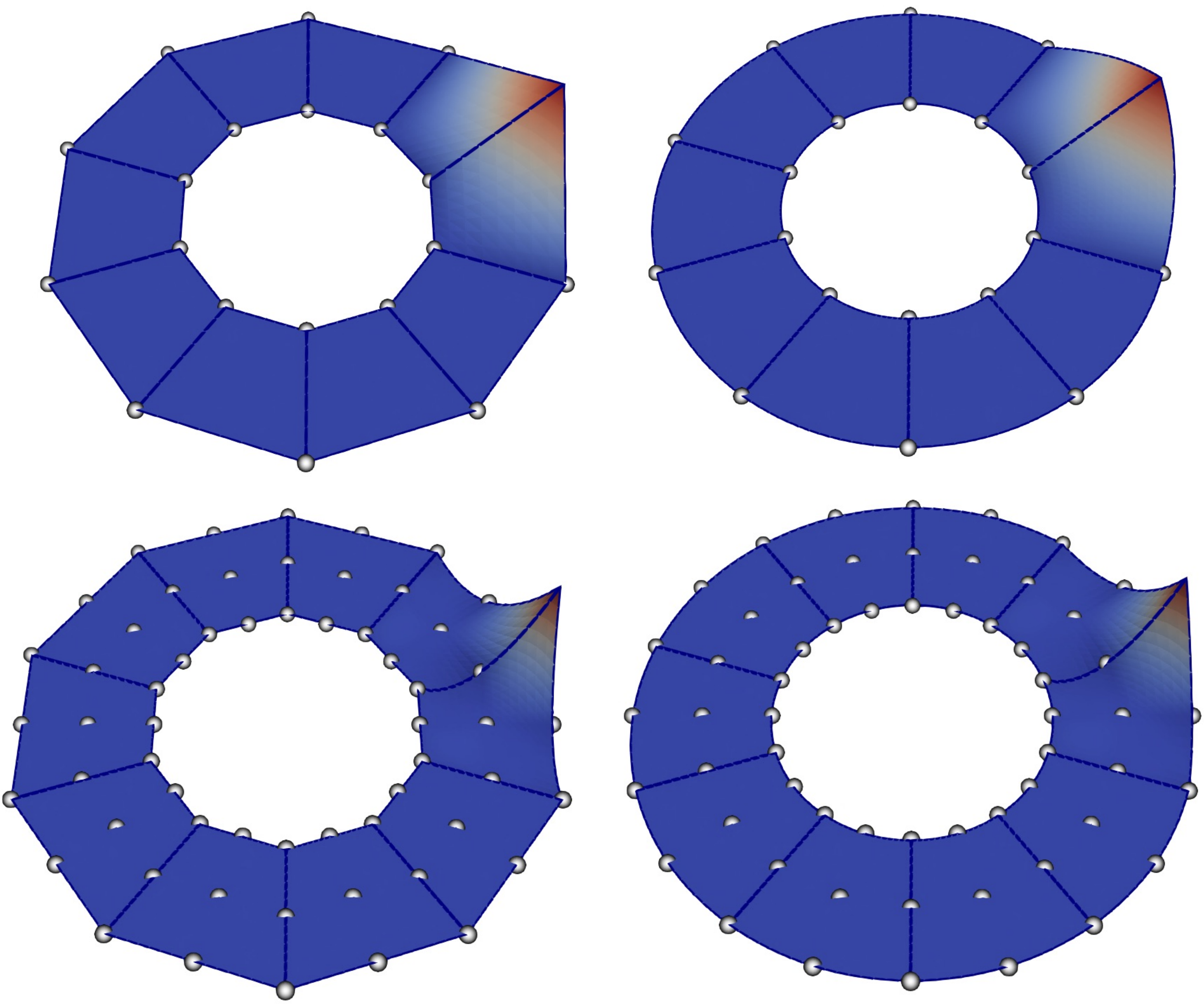


Short recap

Local finite element space: triple $\{T_m, P_{T_m}, \Sigma_{T_m}\}$ $\quad \Omega = \overset{\circ}{\overline{\left(\bigcup_{m=1}^{M} \overline{T}_m\right)}}$

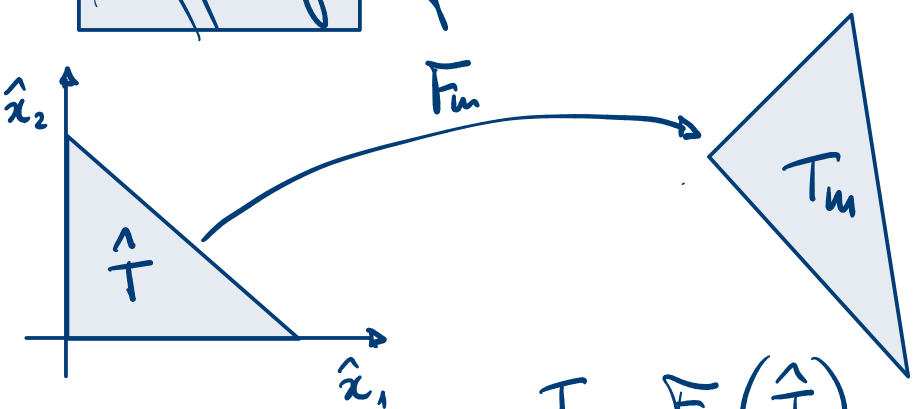
- T_m: simple geometrical domain (cell)
- P_{T_m}: finite dimensional space on T_m ($\dim(P_{T_m}) = N_{T_m}$)
- Σ_{T_m}: basis for $(P_{T_m})'$ (nodal basis $\Sigma_T := \{v_m^i\}_{i=1}^{N_{T_m}}$)

$$P_{T_m} = \operatorname{span}\{v_{mi}\}_{i=1}^{N_T} \quad \text{where} \quad v_{mi} \in P_{T_m}, \quad v^j(v_i) = \delta^j_i$$

1) Define triple $\{\hat{T}, P_{\hat{T}} \equiv \mathbb{P}^k(\hat{T}), \Sigma_{\hat{T}}\}$ $\boxed{\text{Finite Element}}$ *

2) Define, for each $T_m \in \mathcal{T}_h$ $\quad \left(\mathcal{T}_h = \overset{\circ}{\overline{\left(\bigcup_{m=1}^{M} \overline{T}_m\right)}}\right)$ $\quad \mathcal{T}_h \equiv \Omega$

a INVERTIBLE $\boxed{\text{Mapping}}$ * from $\hat{T}$ to T_m



$$T_m = F_m(\hat{T})$$

Let's consider Affine Mapping for the theory

Define P_m as $\#_{F_m} P_{\hat{T}} \equiv \mathbb{P}^k(\hat{T})$, $\Sigma_m := \#_{F_m} \Sigma_{\hat{T}}$

$$P_m := \{v_{mi}\}_{i=1}^{N_{T_m}} = \{\hat{v}_i \circ T_m^{-1}\}$$

3) Make sure that each local dof maps to a unique global dof s.t. the resulting $V_h = \text{span}\{v_i\}_{i=1}^{N}$ is conforming: $V_h \subset V$

Exploit this in the assembly of linear systems

3) $\rightarrow$ $\forall\, T_m$, $\exists$ local to global numbering.

s.t. $\hat{v}_i \circ T_m^{-1} = V_{I_{mi}}$

$\boxed{I_{mi} = l2g_{mi}}$

$I_{mi} \in (0, N)$ and

$\{V_I\}_{I=0}^{N-1}$ is a basis for V_h

$\boxed{\text{DoFHandler}}$ *

is responsible for the numbering

How to use all of this?

$$a(v_J, v_I) := A_{IJ} \qquad a(u^J v_J, v_I) = \langle f, v_I \rangle$$

$$\langle f, v_I \rangle := F_I$$

$$\Rightarrow \quad \underline{Au = F} \quad \Leftrightarrow \quad \underline{a(u^J v_J, v_I) = \langle f, v_I \rangle}$$

For $-\Delta u = f$ $\Longrightarrow$ $a(v_J, v_I) := \int_\Omega \nabla v_J \cdot \nabla v_I$

$$F_I = \underline{\int_\Omega f\, v_I}$$

How to compute F_I ?

$\Rightarrow$ split $\int_\Omega \leadsto \sum_{m=1}^{M} \int_{T_m} dT_m$

$$\boxed{\hat{J} = \det(DF_m)}$$

$$\int_{T_m} f\, v_{I_{mi}}\, dT_m = \int_{\hat{T}} f \circ F_m\, \hat{v}_i\, \hat{J}\, d\hat{T}$$

$$\simeq \boxed{\sum_{q=1}^{N_q} \underbrace{(f \circ F_m)(\hat{x}_q)}_{f(\underbrace{F_m(\hat{x}_q)}_{x_{mq}})}\, \hat{v}_i(\hat{x}_q)\, \underbrace{\hat{J}(\hat{x}_q)\, W_q}_{JxW_{mq}}}$$

$$f_{mi} = \sum_{i=1}^{N_m} \sum_{q=1}^{N_q} f(x_{mq})\, \hat{v}_i(\hat{x}_q)\, JxW_{mq}$$

$F = 0 \longrightarrow F(I_{mi}) \mathrel{+}= f_{mi}$

$$F_i := \int_\Omega v_I\, f = \int_{\mathrm{supp}(v_I)} v_I\, f$$

FOR EACH CELL T_m

- COMPUTE LOCAL INTEGRALS ON T_m
- STORE RESULTS ON LOCAL RHS, and MATRIX
  - $\boxed{f_{m_i}} \in \mathbb{R}^{N_{T_m}}$
  - $\boxed{a_{m_{ij}}} \in \mathbb{R}^{N_{T_m} \times N_{T_m}}$
- Distribute local integrals to global matrices:

FOR $i = [0, \dots, N_{T_m})$

FOR $j = [0, \dots, N_{T_m})$

$$A_{I_{m_i} I_{m_j}} = A_{I_{m_i} I_{m_j}} + a_{m_{ij}}$$

$$F_{I_{m_i}} = F_{I_{m_i}} + f_{m_i}$$

How about $a_{m\,ij}$?

$$a_{m\,ij} := \int_{T_m} \nabla(\hat{v}_j \circ F_m^{-1}) \cdot \nabla(\hat{v}_j \circ F_m^{-1})\, dT_m$$

$$= \int_{\hat{T}} \underbrace{\hat{D}F_m^{-T} \hat{\nabla} \hat{v}_j}_{\nabla v_j \circ F_m^{-1}} \cdot \underbrace{\hat{D}T_m^{-T} \hat{\nabla} \hat{v}_i}_{\nabla v_i \circ F_m^{-1}} \hat{J}\, d\hat{T}$$

$$(\nabla v)_i := \frac{\partial}{\partial x_i}\left(\hat{v} \circ F_m^{-1}\right) = \left[\left(\frac{\partial}{\partial \hat{x}_j} \hat{v}\right) \circ F_m^{-1}\right] \frac{\partial F_{m_i}^{-1}}{\partial \hat{x}_j}$$

$$\nabla v = \underbrace{DF_m^{-T}}_{\text{depends on } m \ (F_m : \hat{T} \longrightarrow T_m)} \hat{\nabla} \hat{v} \longrightarrow \text{does not depend on } m$$

only need evaluation on quadrature points

Given : 1) $\boxed{\text{Quadrature}}^{*}$ formula

2) $\boxed{\text{Finite Element}}^{*}$ $\Rightarrow \hat{v}_i \,,\ \hat{\nabla}\hat{v}_i \quad i = 1, \dots, N_{\hat{T}}$

3) $\boxed{\text{Mapping}}^{*}$: $F_m : \hat{T} \longrightarrow T$

We compute $\boxed{\text{FE Values}}^{*}$ on quadrature points.

1) Values $\qquad \hat{v}_i(\hat{x}_q)$

2) quadrature points $\qquad x_{mq} = F_m(\hat{x}_q)$

2) JxW $\qquad JxW_{mq} := \det(DF_m) \cdot w_q$

3) gradients $\qquad (\nabla v)(F_m(\hat{x}_q)) =$

$$DF_m^{-T} \hat{\nabla} \hat{v}(\hat{x}_q)$$

With these three, we can assemble

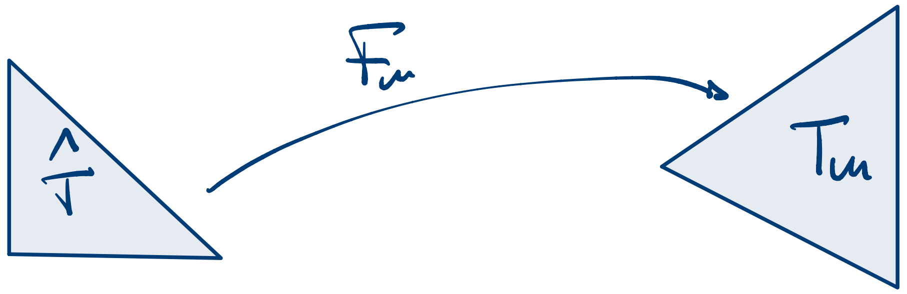
$$f_{mi} := \sum_q f\big(F_m(\hat{x}_q)\big)\, J_x W_{mq}$$

$$a_{miJ} := \boxed{\sum_q \underbrace{DF_m^{-T}(\hat{x}_q)(\hat{\nabla}\hat{v}_J)(\hat{x}_q)}_{\nabla v_J|_{F_m(\hat{x}_q)}} \cdot \underbrace{DF_m^{-T}(\hat{x}_q)\,\hat{\nabla}\hat{v}_J(\hat{x}_q)}_{\nabla v_i|_{F_m(x_q)}}\, J_x W_{mq}}$$

$$a_{miJ} \simeq \int_{T_m} \nabla v_{I_{mi}}\, \nabla v_{I_{mJ}}\, dT_m$$

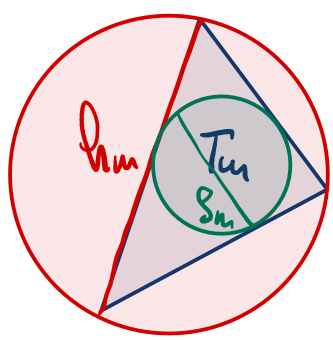
$\Longrightarrow$ Distribute to $\mathbb{A}_{I_{mi} I_{mJ}}$

AFFINE MAPPINGS



$$F_m = B\hat{x} + b \qquad B \in \mathbb{R}^{d\times d},\ b \in \mathbb{R}^d$$

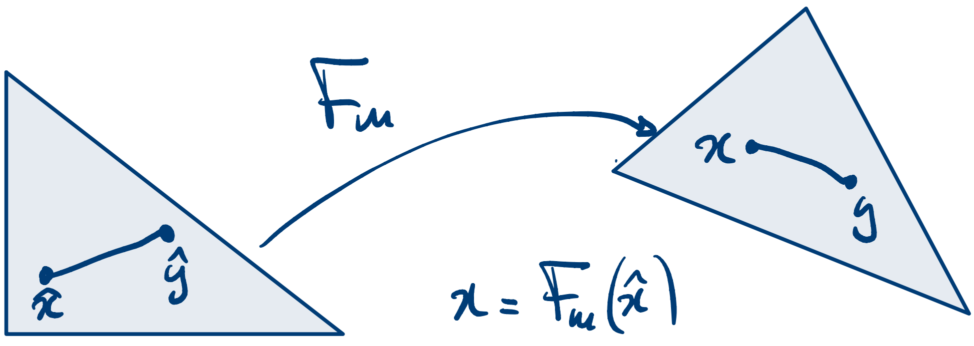
$$h_m := \max_{x,y\in T_m} |x-y|$$



$$\rho_m := \sup_{B_r \subseteq T_m} \operatorname{diam}(B_r)$$

$$\|B\| := \sup_{|\xi| = \hat{\rho}} \frac{|B\xi|}{|\xi|} \leq \frac{h_m}{\hat{\rho}}$$

$$\xi := \hat{x} - \hat{y} \qquad F_m(\hat{x}) - F_m(\hat{y})$$



$$x = F_m(\hat{x})$$
$$y = F_m(\hat{y})$$

$$F_m(\hat{x}) - F_m(\hat{y}) = B(\hat{x} - \hat{y}) \leq h_m$$

In the opposite direction:

$$\|B^{-1}\| := \sup_{|x-y| = \rho_m} \frac{|B^{-1}(x-y)|}{|x-y|} \leq \frac{\hat{h}}{\rho_m} = \hat{h}\, \rho_m^{-1}$$

$$\Rightarrow \|B\| \leq c_1 h_m \qquad \|B^{-1}\| \leq c_2 \rho_m^{-1}$$

Lets see how F_m transforms Sobolev norms:

$$|v|^p_{k,p,T} := \int_T (D^k v)^p \, d\Gamma$$

k is a multiindex in $\mathbb{N}_0^d$

$$\hat{D}^k (v \circ F_m) = \left[(D^k v) \circ F_m \right] (\underbrace{DF_m}_{B})^{|k|}$$

$$|v|^p_{k,p,T} := \int_T \left((D^k v) B^{|k|} B^{-|k|} \right)^p J J^{-1} dT$$

$$\leq \|B^{-1}\|^{|k|p} J \int_{\hat{T}} (\hat{D}^k \hat{v})^p d\hat{T}$$

$$|v|_{k,p,T} \leq C \underbrace{\|B^{-1}\|}_{\rho_m^{-k}}{}^{|k|} J^{\frac{1}{p}} |v \circ F_m|_{k,p,\hat{T}}$$

$$|v \circ F_m|_{k,p,\hat{T}} \leq C \underbrace{\|B\|}_{h_m^k}{}^{|k|} J^{-\frac{1}{p}} |v|_{k,p,T}$$

For $v \in \mathbb{P}^e(T)$ for $0 \leq k, s \leq e$

$\exists\, c_1, c_2$ s.t.

$$c_1 |v \circ F_m|_{k,p,\hat{T}} \leq |v \circ F_m|_{s,p,\hat{T}} \leq c_2 |v \circ F_m|_{k,p,\hat{T}}$$

$$\boxed{\|v\|_{k,p,T_m} \leq C \rho_m^{-k} h_m^s \|v\|_{s,p,T_m}}$$