A Hybridizable Discontinuous Galerkin formulation for Fluid-Structure Interaction

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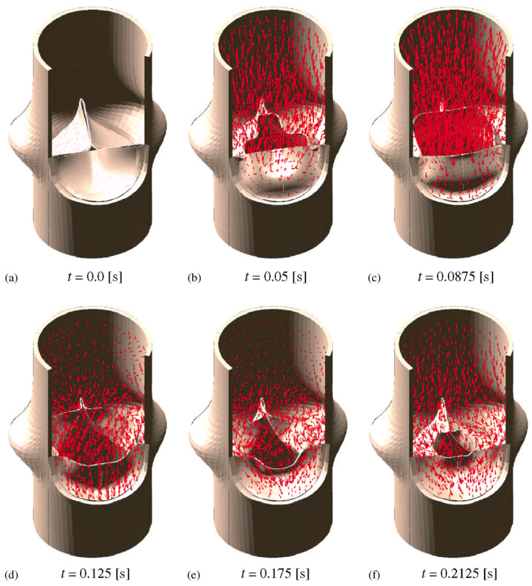
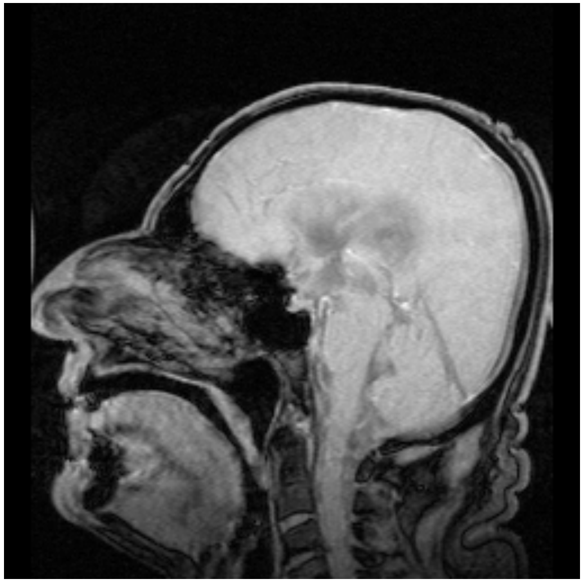
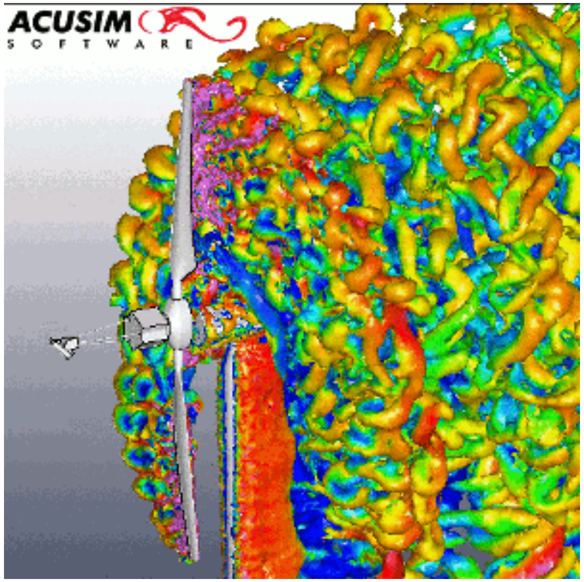
Motivation

Fluid-Structure Interaction (FSI)

The interaction of a deformable structure with a surrounding or internal fluid







<u>Source</u>

- http://www.sciencedirect.com/science/article/pii/S0021929002002440
- http://en.wikipedia.org/wiki/Cerebrospinal_fluid
- http://www.acusim.com/html/apps/windTurbFSI.html

Problem statement

Problem statement

High-order accurate FSI calculations are often too computationally expensive for real-world problems

Solution requirements

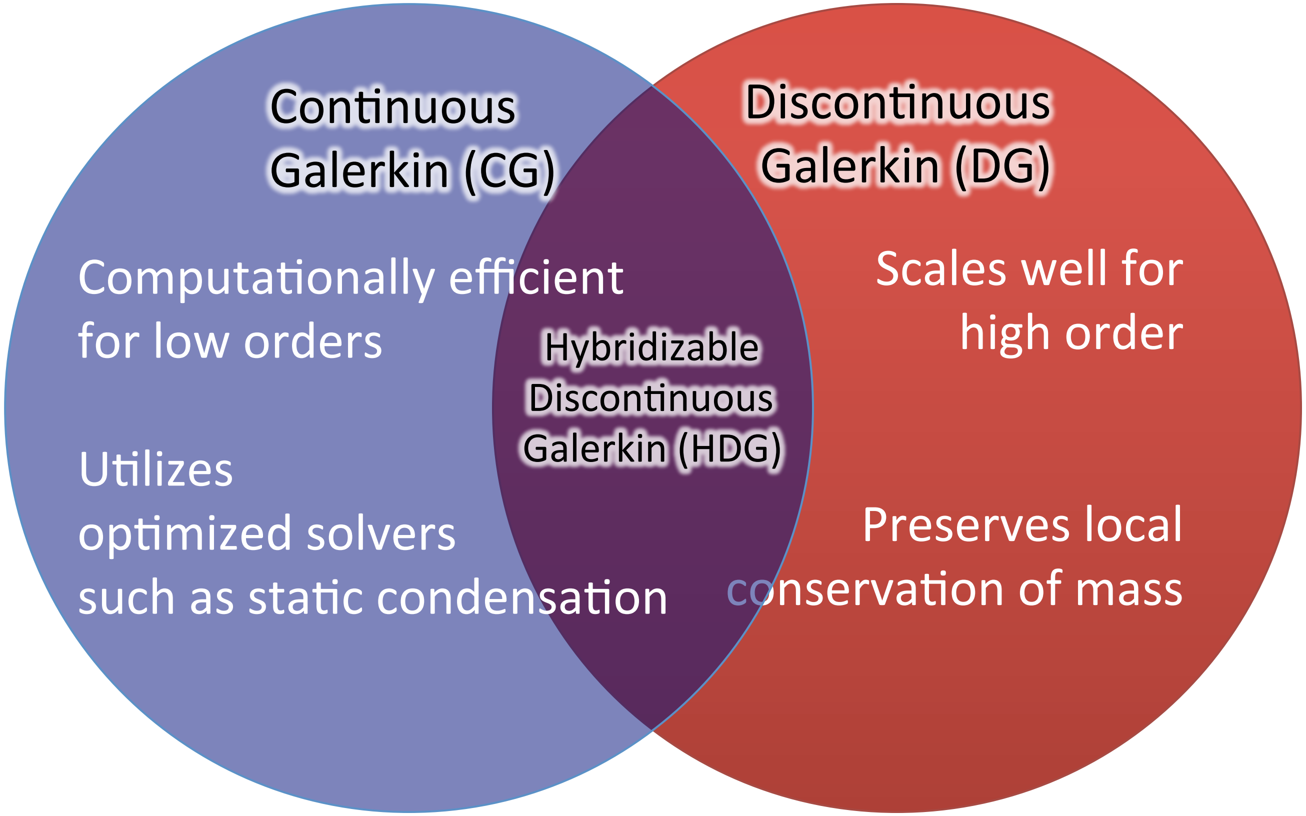
High-order accurate computations

Reasonable computational expenses

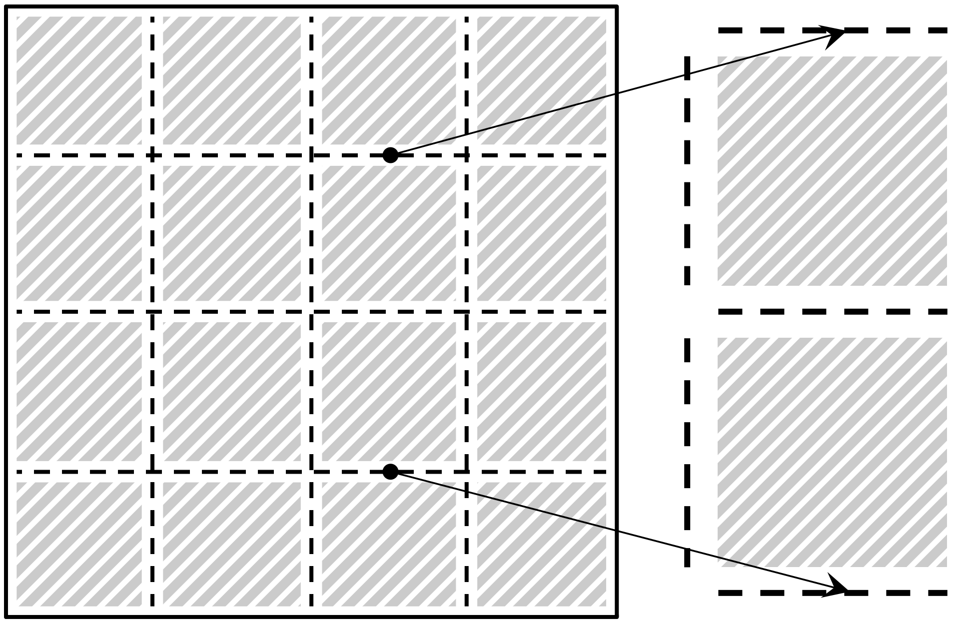
Proposed solution

Develop the first FSI model that uses the state of the art hybridizable discontinuous Galerkin (HDG) method

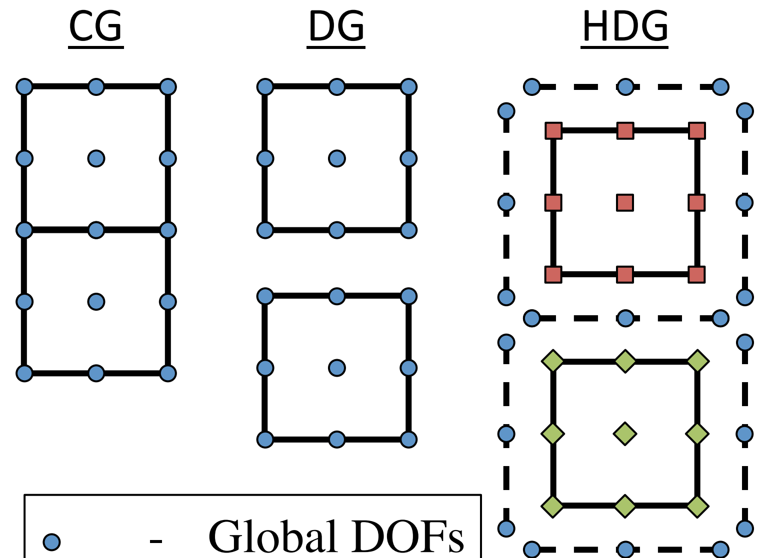
HDG – best of both worlds



HDG discretized domain



- Local solution
- Global solution trace
- Global boundary



- Global DOFs
- Local DOFs

Example: 2D 1x2 quadratic elements

Variables: $\mathbf{u}$, $\mathbf{v}$, $\mathbf{e}$ - 8 global DOFS per (∘)
CG: 120 global DOFs
DG: 144 global DOFs

Variables: $\mathbf{u}$, $\mathbf{v}$, $\mathbf{e}$ - 8 local DOFS per (■, ◆)
$\boldsymbol{\mu}$ - 2 global DOFS per (∘)
HDG: 42 global DOFs
+ 72 local DOFs per cell

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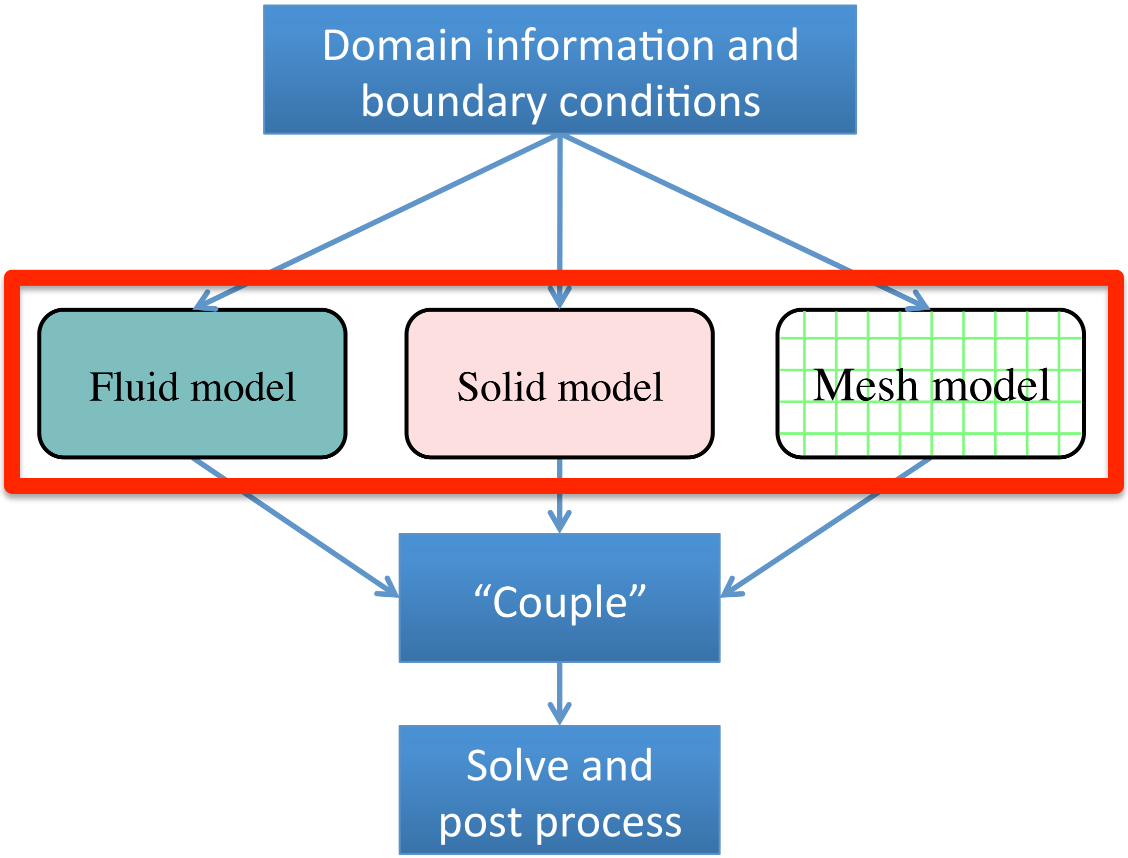
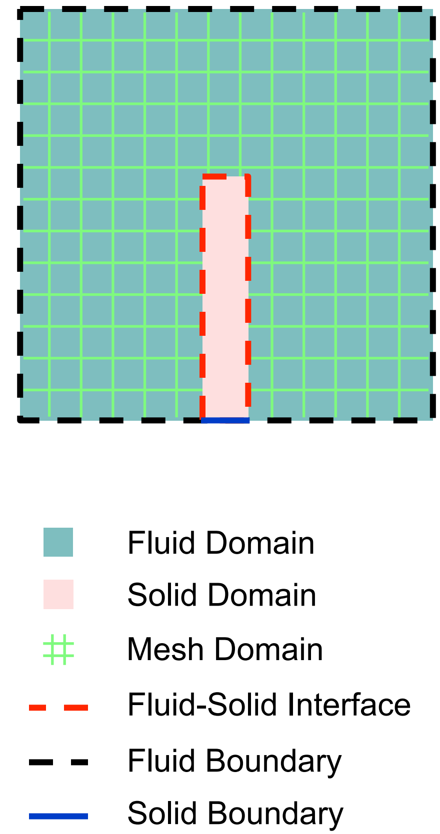
Fluid-structure interaction modeling





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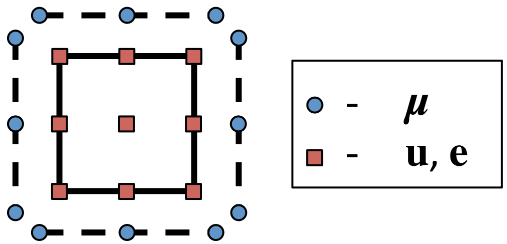
(Mesh Motion)

Linear elastostatics: HDG discretization

Strong form

$$-\mathrm{Div}\;(\mathbb{C}\,[\mathbf{e}]) = \mathbf{b}$$

$$\mathbf{e} - \mathrm{Sym}(\mathrm{Grad}\,\mathbf{u}) = \mathbf{0}$$



1. Weight by weighting functions $(\tilde{\mathbf{u}}, \tilde{\mathbf{e}})$
2. Integrate over cell (K)
3. Integrate by parts > boundary terms (Γ)

Weak form

$$\int_K \mathrm{Grad}\,\tilde{\mathbf{u}} : \mathbb{C}\,(\mathbf{e})\,\mathrm{d}\Omega - \int_\Gamma \tilde{\mathbf{u}} \cdot \widehat{\mathbb{C}\,(\mathbf{e})}\mathbf{n}\,\mathrm{d}\Gamma = \int_K \tilde{\mathbf{u}} \cdot \mathbf{b}\,\mathrm{d}\Gamma$$

$$\int_K \tilde{\mathbf{e}} : \mathbf{e}\,\mathrm{d}\Omega - \int_K \mathrm{Sym}(\tilde{\mathbf{e}}) : \mathrm{Grad}\,\mathbf{u}\,\mathrm{d}\Omega + \int_\Gamma \mathrm{Sym}(\tilde{\mathbf{e}})\mathbf{n} \cdot (\mathbf{u} - \boldsymbol{\mu})\;\mathrm{d}\Gamma = 0$$

Flux continuity

$$\int_\Gamma \tilde{\boldsymbol{\mu}} \cdot \widehat{\mathbb{C}\,(\mathbf{e})}\mathbf{n}\,\mathrm{d}\Gamma = 0$$

Stabilization

$$\widehat{\mathbb{C}\,(\mathbf{e})} := \mathbb{C}\,(\mathbf{e}) - \mathbf{S}\,(\mathbf{u} - \boldsymbol{\mu}) \otimes \mathbf{n}$$

Linear elastostatics: HDG discretization

Weak form

$$\int_K \operatorname{Grad}\widetilde{\mathbf{u}} : \mathbb{C}(\mathbf{e})\,\mathrm{d}\Omega - \int_\Gamma \widetilde{\mathbf{u}} \cdot \widehat{\mathbb{C}(\mathbf{e})}\mathbf{n}\,\mathrm{d}\Gamma = \int_K \widetilde{\mathbf{u}} \cdot \mathbf{b}\,\mathrm{d}\Gamma$$

$$\int_K \widetilde{\mathbf{e}} : \mathbf{e}\,\mathrm{d}\Omega - \int_K \operatorname{Sym}(\widetilde{\mathbf{e}}) : \operatorname{Grad}\mathbf{u}\,\mathrm{d}\Omega + \int_\Gamma \operatorname{Sym}(\widetilde{\mathbf{e}})\mathbf{n} \cdot (\mathbf{u} - \boxed{\boldsymbol{\mu}})\,\mathrm{d}\Gamma = 0$$

$$\widehat{\mathbb{C}(\mathbf{e})} := \mathbb{C}(\mathbf{e}) - \mathbf{S}(\mathbf{u} - \boxed{\boldsymbol{\mu}}) \otimes \mathbf{n} \qquad \int_\Gamma \widetilde{\boldsymbol{\mu}} \cdot \widehat{\mathbb{C}(\mathbf{e})}\mathbf{n}\,\mathrm{d}\Gamma = 0$$

$$\left(\begin{array}{c|c} \text{local} \times \text{local} & \text{local} \times \text{global} \\ \hline \text{global} \times \text{local} & \text{global} \times \text{global} \end{array}\right) \left\{\begin{array}{c} \text{sol}_{\text{local}} \\ \hline \boxed{\text{sol}_{\text{global}}} \end{array}\right\} = \left\{\begin{array}{c} \text{RHS}_{\text{local}} \\ \hline \text{RHS}_{\text{global}} \end{array}\right\}$$

Using the Schur complement, any equation of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} U \\ \boxed{\Lambda} \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix},$$

can be rearranged into the following system of equations:

$$(D - CA^{-1}B)\boxed{\Lambda} = G - CA^{-1}F,$$
$$AU = F - B\Lambda.$$

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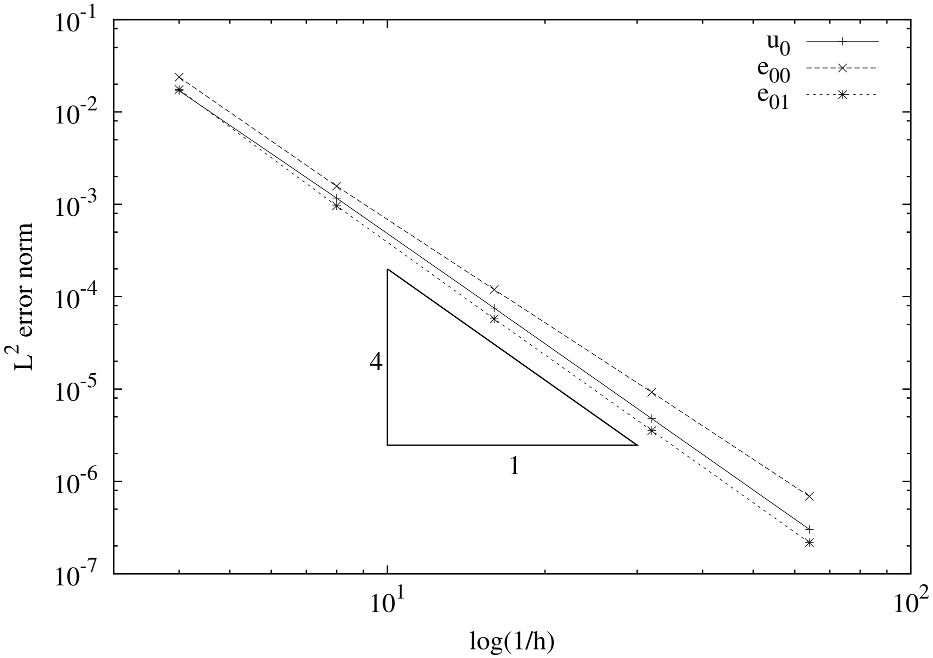
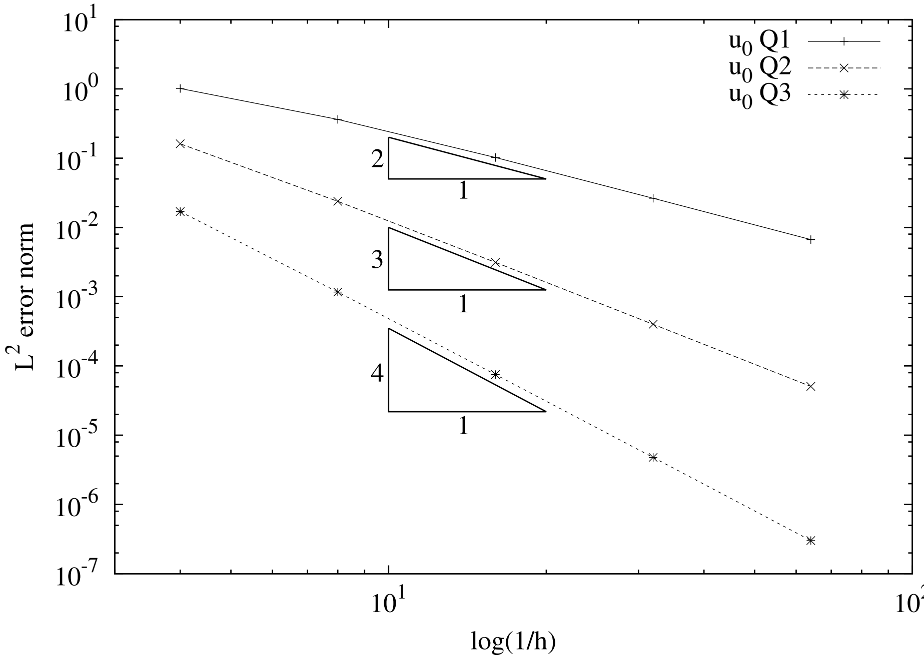
Linear elastostatics: verification

h-refinement (Q3)



p-refinement



All errors converge at the expected rate of h^{p+1}

Manufactured solution function of sines and cosines

Pure Dirichlet boundary conditions

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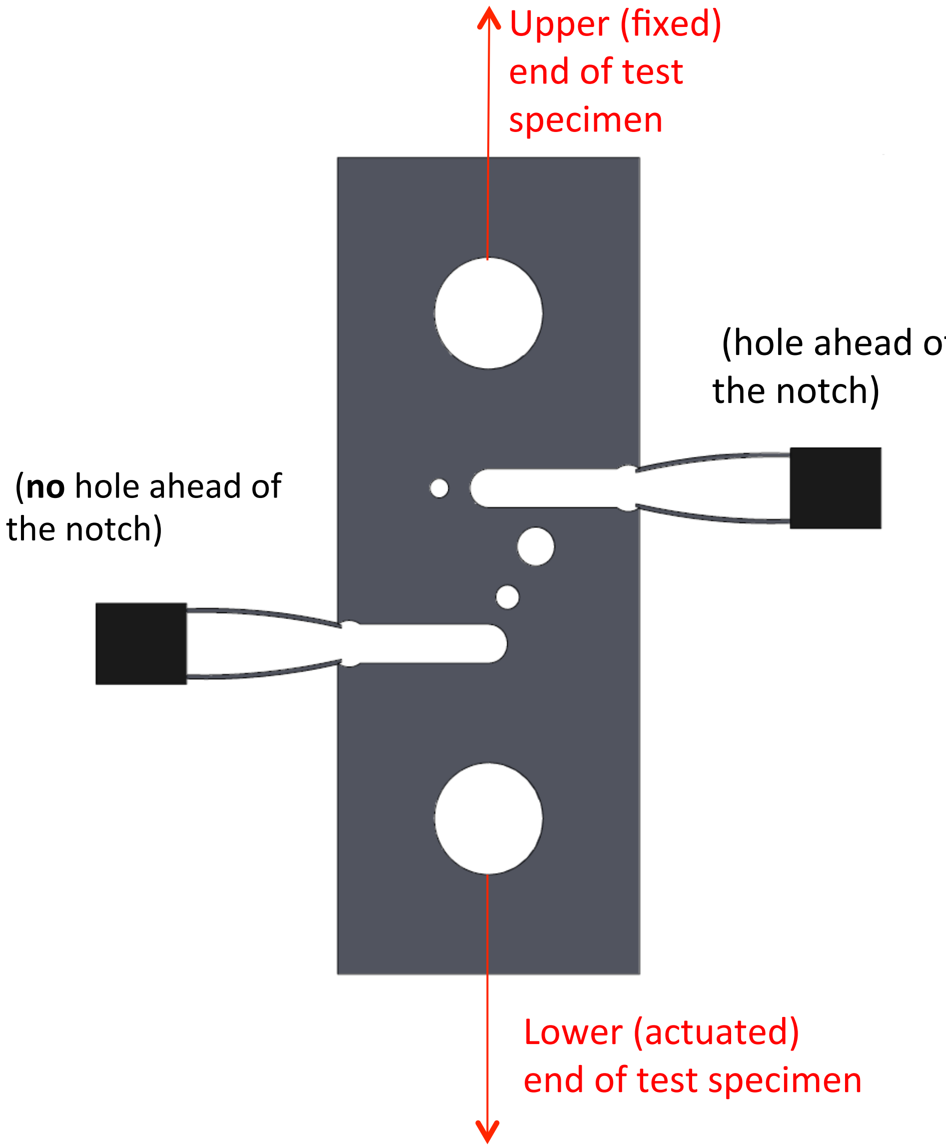
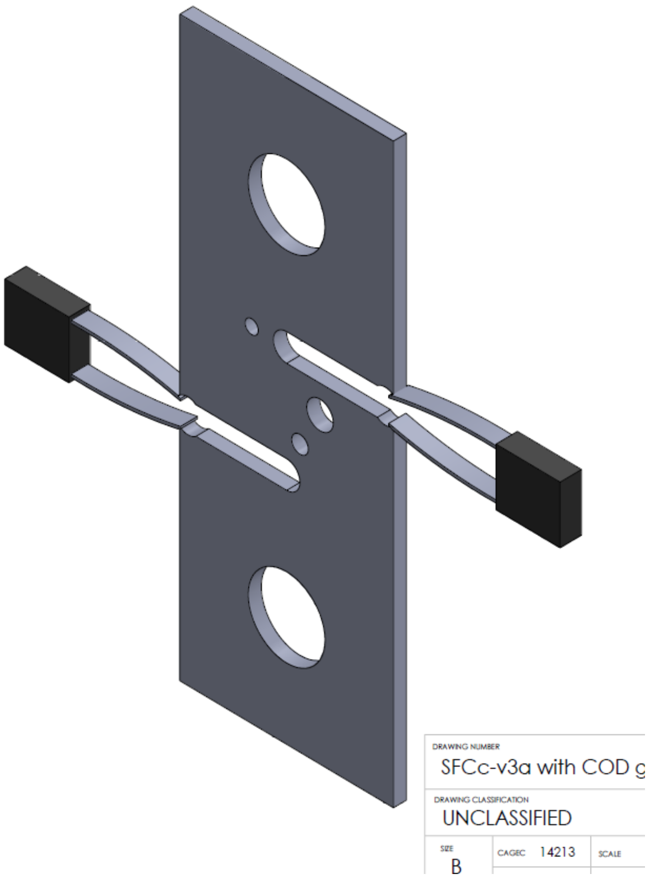
Preliminary results: Sandia geometry



Source
- The 2014 Sandia Fracture Challenge (SFC2)



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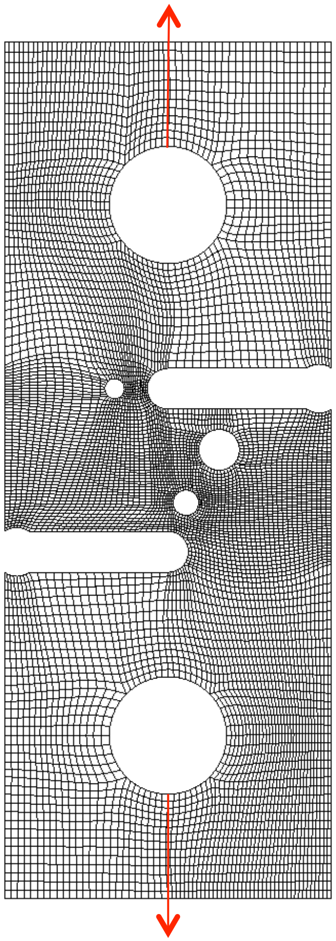
Preliminary results: Sandia geometry

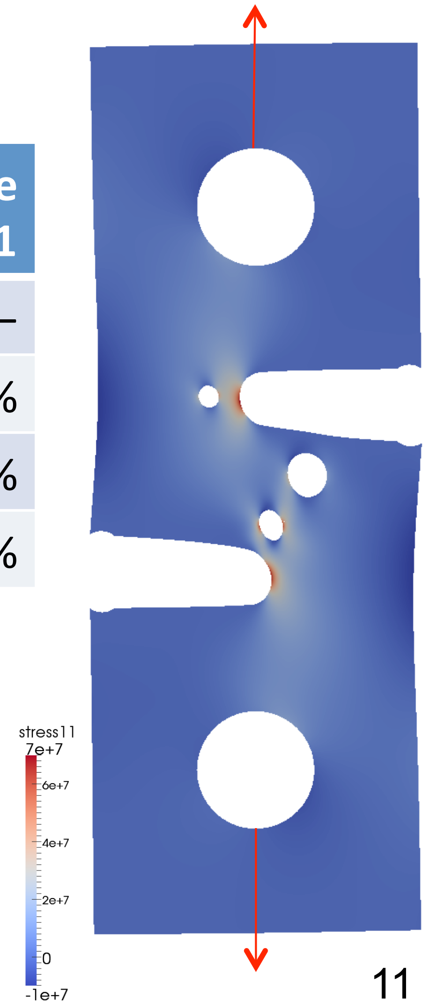
Undeformed mesh
7777 cells



Element Order	Local # of DOFs	Increase from Q1	Global # of DOFs	Increase from Q1
Q1	186648	–	63720	–
Q3	746592	300%	127440	100%
Q5	1679832	800%	191160	200%
Q7	2986368	1500%	254880	300%

Size of global system increases minimally, even for high orders!

Deformed solution
Showing stress



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Nonlinear elastodynamics: governing equations

Restricted to hyperelastic materials (Saint Venant-Kirchhoff)

Variables: $\mathbf{u}, \mathbf{v}, \mathbf{E}$ - local

$\boldsymbol{\mu}$ - global

Weak form

$$\int_K \widetilde{\mathbf{u}}\cdot\rho\frac{\partial \mathbf{v}}{\partial t}\,\mathrm{d}K + \int_K \mathrm{Grad}\,\widetilde{\mathbf{u}}:\mathbf{F}\,\mathbb{C}\left(\mathbf{E}\right)\,\mathrm{d}K - \int_\Gamma \widetilde{\mathbf{u}}\cdot\widehat{\mathbf{h}}\,\mathrm{d}\Gamma = \int_K \widetilde{\mathbf{u}}\cdot\mathbf{b}\,\mathrm{d}K$$

$$\int_K \widetilde{\mathbf{E}}:\mathbf{E}\,\mathrm{d}K - \int_K \mathrm{Sym}(\widetilde{\mathbf{E}}):\mathrm{Grad}\,\mathbf{u}\,\mathrm{d}K - \int_K \tfrac{1}{2}\widetilde{\mathbf{E}}:(\mathrm{Grad}\,\mathbf{u})^{\mathsf{T}}\mathrm{Grad}\,\mathbf{u}\,\mathrm{d}K + \int_\Gamma \mathrm{Sym}(\widetilde{\mathbf{E}})\mathbf{n}\cdot\left(\mathbf{u}-\boldsymbol{\mu}\right)\,\mathrm{d}\Gamma = 0$$

$$\int_K \widetilde{\mathbf{v}}\cdot\frac{\partial \mathbf{u}}{\partial t}\,\mathrm{d}K - \int_K \widetilde{\mathbf{v}}\cdot\mathbf{v}\,\mathrm{d}K = 0$$

$$\int_\Gamma \widetilde{\boldsymbol{\mu}}\cdot\widehat{\mathbf{h}}\,\mathrm{d}\Gamma = 0$$

where $\widehat{\mathbf{h}} := \left[\mathbf{F}\,\mathbb{C}\left(\mathbf{E}\right)\right]\mathbf{n} - \mathbf{S}\left(\mathbf{u}-\boldsymbol{\mu}\right))$

$$\mathbf{F} = \mathbf{I} + \mathrm{Grad}\,\mathbf{u}$$

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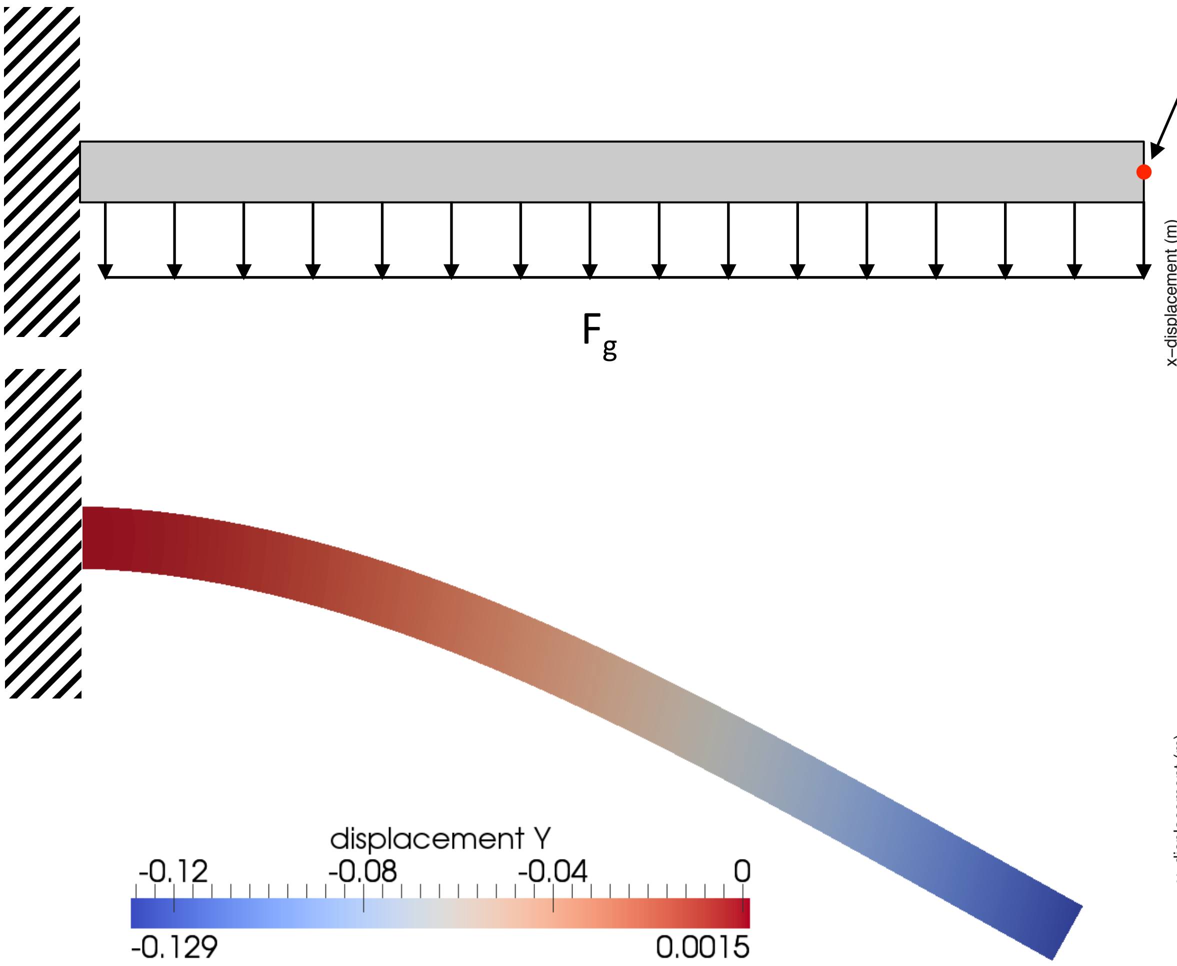
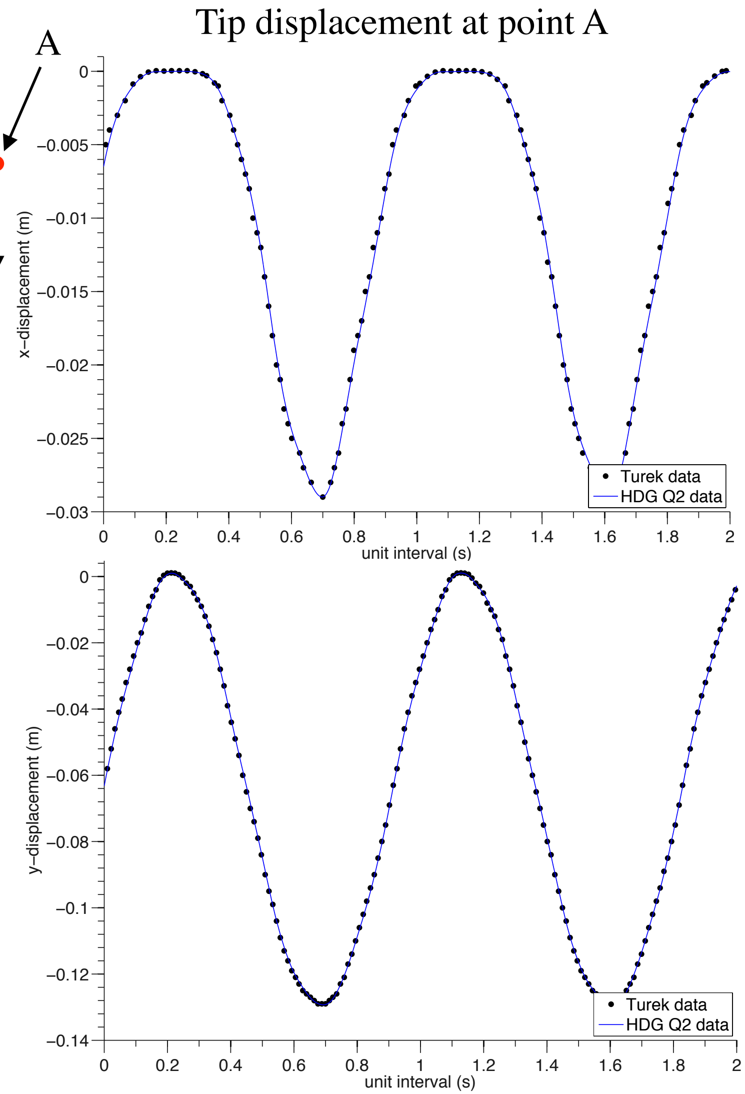
Preliminary results: Turek and Hron CSM3





Source

- Proposal for numerical benchmarking of fluid-structure interaction between an elastic object and laminar incompressible flow, Turek 2006

Source: HDG Navier-Stokes formulation from Martin Kronbichler's unpublished work

Navier-Stokes: governing equations

Variables: $\mathbf{v}, p, \mathbf{L}$ - local
$\boldsymbol{v}, \psi$ - global

Weak form

$$\int_K \widetilde{\mathbf{v}} \cdot \frac{\partial \mathbf{v}}{\partial t}\,\mathrm{d}K - \int_K \operatorname{grad}\widetilde{\mathbf{v}} : \mathbf{v}\otimes\mathbf{v} - \nu\mathbf{L}\,\mathrm{d}K + \int_K \widetilde{\mathbf{v}} \cdot \operatorname{grad} p_k\,\mathrm{d}K + \int_\Gamma \widetilde{\mathbf{v}} \cdot \widehat{\mathbf{h}}^*\,\mathrm{d}\Gamma = \int_K \widetilde{\mathbf{v}} \cdot \mathbf{f}\,\mathrm{d}K$$

$$\int_K \widetilde{\mathbf{L}} : \mathbf{L}\,\mathrm{d}K - \int_K \widetilde{\mathbf{L}} : \operatorname{grad}\mathbf{v}\,\mathrm{d}K + \int_\Gamma \widetilde{\mathbf{L}}\mathbf{n} \cdot (\mathbf{v} - \boldsymbol{v})\,\mathrm{d}\Gamma = 0$$

$$-\int_K \operatorname{grad}\widetilde{p_k} \cdot \mathbf{v}\,\mathrm{d}K + \int_\Gamma \widetilde{p_k}\boldsymbol{v} \cdot \mathbf{n}\,\mathrm{d}\Gamma = 0$$

$$\int_\Gamma \widetilde{\boldsymbol{v}} \cdot \widehat{\mathbf{h}}\,\mathrm{d}\Gamma = 0$$

$$\int_\Gamma \widetilde{\psi}\boldsymbol{v} \cdot \mathbf{n}\,\mathrm{d}\Gamma = 0$$

Flux stabilization

$$\widehat{\mathbf{h}} := [\boldsymbol{v}\otimes\boldsymbol{v} - \nu\mathbf{L} + p\mathbf{I}]\,\mathbf{n} + \mathbf{S}\,(\mathbf{v} - \boldsymbol{v})$$

$$\widehat{\mathbf{h}}^* := [\boldsymbol{v}\otimes\boldsymbol{v} - \nu\mathbf{L}]\,\mathbf{n} + \mathbf{S}\,(\mathbf{v} - \boldsymbol{v})$$

Introduce ψ to close the system

$$\psi := \frac{1}{|K|}\int_K p\,\mathrm{d}K$$

Remove the kernel of the gradient

$$\int_\Omega p\,\mathrm{d}\Omega = 0$$

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Arbitrary Lagrangian-Eulerian (ALE) transformation

HDG ALE transform

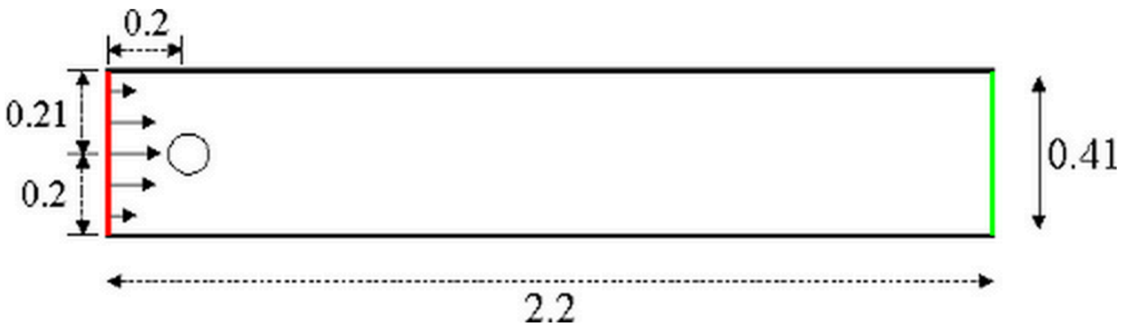
$$\mathbf{L} = \mathrm{grad}\,\mathbf{v} = \mathrm{Grad}\,\mathbf{v}_{\mathrm{f}}\mathbf{F}_{\mathrm{m}}^{-1} = \mathbf{L}_{\mathrm{f}},$$
$$\frac{\partial \mathbf{v}}{\partial t} = \frac{\partial \mathbf{v}_{\mathrm{f}}}{\partial t} - \mathbf{L}_{\mathrm{f}}\left[\mathbf{v}_{\mathrm{m}}\right],$$

Auxiliary HDG variables simplify ALE transformation

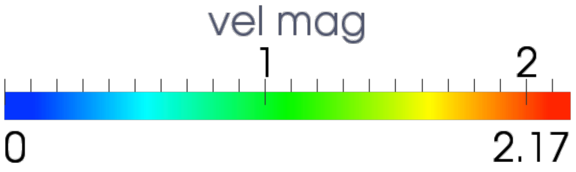
Integration domain ALE transform

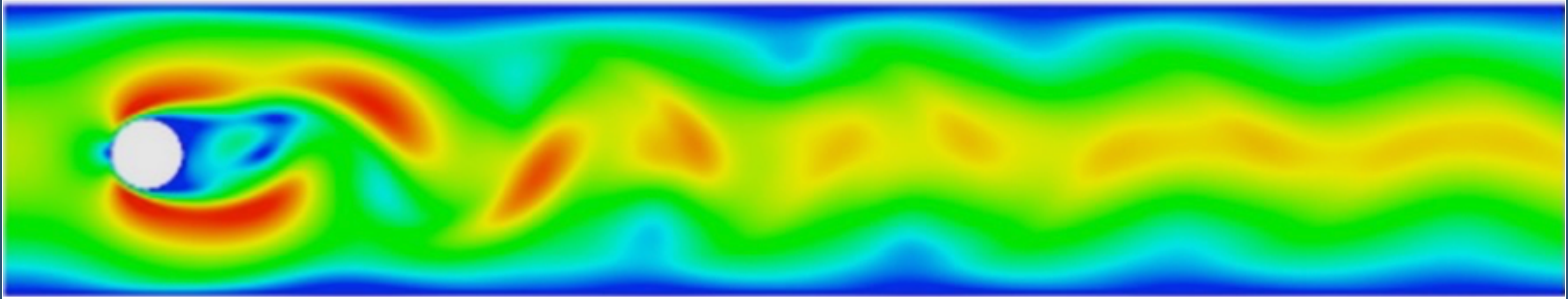
$$\int_{\Omega_{\mathrm{F}}(t)} f\left(\mathbf{x}\right)\mathrm{d}\mathbf{x} = \int_{\Omega_{\mathrm{F}}(0)} f_{\mathrm{f}}\left(\mathbf{X}\right) J_{\mathrm{m}}\mathrm{d}\mathbf{X},$$

German research association (DFG) Benchmark



2D channel flow around a cylinder
Reynolds number Re=100





Source

- http://www.featflow.de/en/benchmarks/cfdbenchmarking/flow/dfg_benchmark2_re100.html

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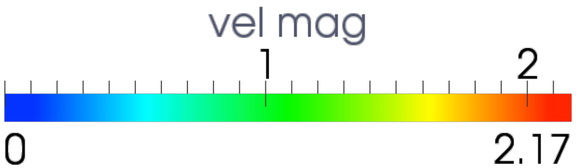
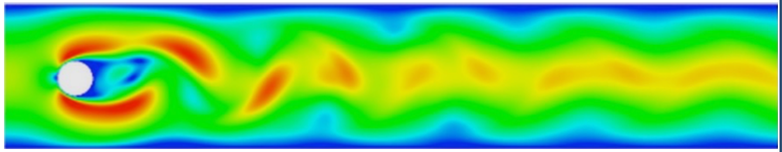
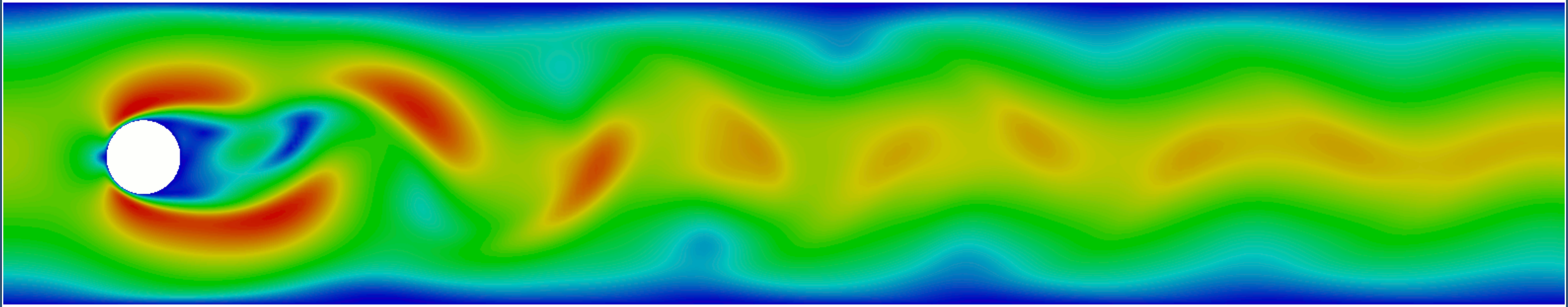
Preliminary results: flow over cylinder



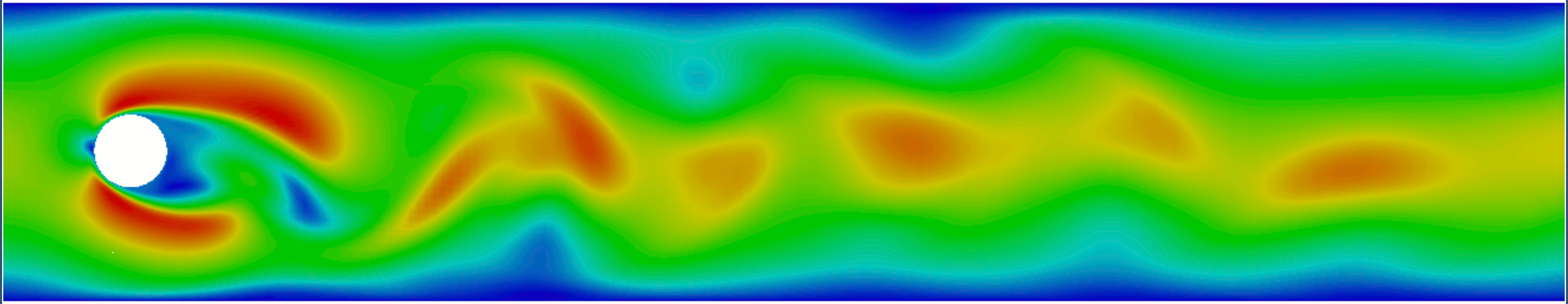
DFG benchmark ->



Matching HDG Navier-Stokes with DFG benchmark



ALE HDG Navier-Stokes proof of concept: oscillating cylinder



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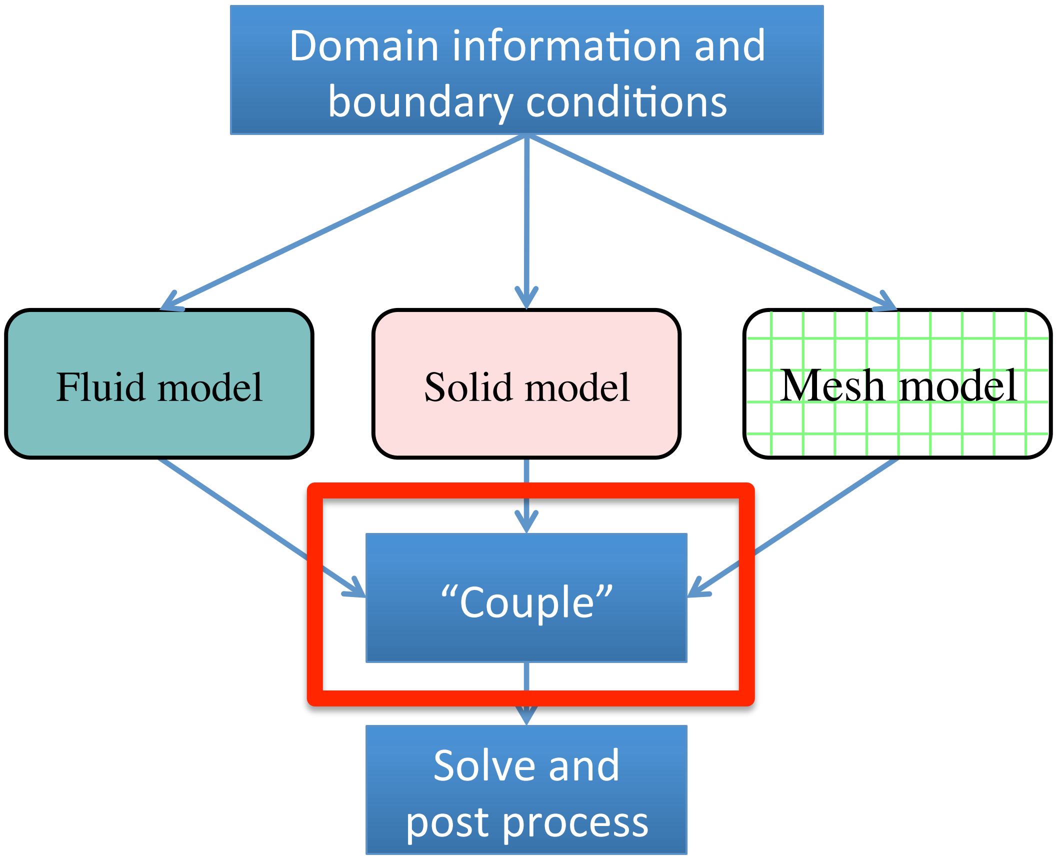
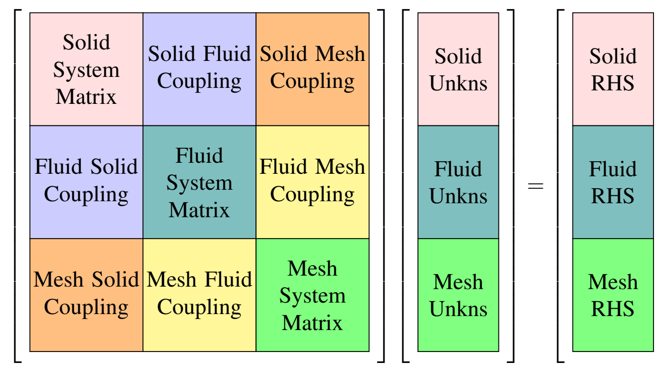
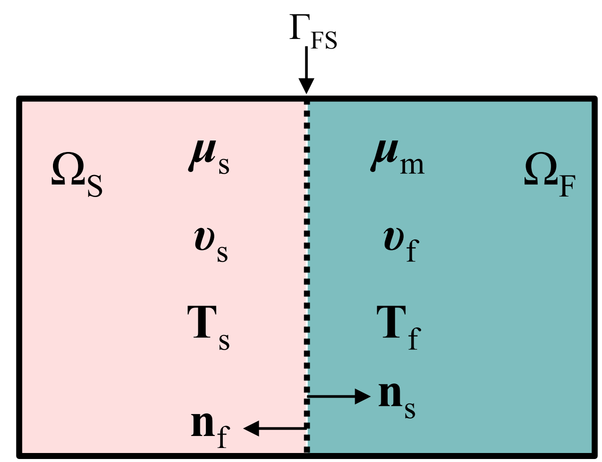
Fluid-Structure interaction modeling



Monolithic Coupling





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Proposed HDG FSI formulation

Problem 1 (Fluid sub-problem)
Find $\{\mathbf{L}_\mathrm{f}^\mathrm{h}, \mathbf{v}_\mathrm{f}^\mathrm{h}, p_\mathrm{f}^\mathrm{h}, \boldsymbol{\upsilon}_\mathrm{f}^\mathrm{h}, \psi_\mathrm{f}^\mathrm{h}\} \in \boldsymbol{\mathcal{L}}_\mathrm{f}^\mathrm{h} \times \boldsymbol{V}_\mathrm{f}^\mathrm{h} \times Q_\mathrm{f}^\mathrm{h} \times \boldsymbol{\mathcal{V}}_\mathrm{f}^\mathrm{h} \times \Psi_\mathrm{f}^\mathrm{h}$ *such that*

$$
\begin{aligned}
\left(\widetilde{\mathbf{v}}_\mathrm{f}, \rho_\mathrm{f} J_\mathrm{m} \frac{\partial \mathbf{v}_\mathrm{f}^\mathrm{h}}{\partial t}\right)_{K_\mathrm{f}} + \left(\widetilde{\mathbf{v}}_\mathrm{f}, \rho_\mathrm{f} J_\mathrm{m} \mathbf{L}_\mathrm{f}^\mathrm{h}\left[\mathbf{v}_\mathrm{f}^\mathrm{h} - \mathbf{v}_\mathrm{m}^\mathrm{h}\right]\right)_{K_\mathrm{f}} + \left(\widetilde{\mathbf{v}}_\mathrm{f}, J_\mathrm{m} \mathbf{F}_\mathrm{m}^{-\intercal}\left[\mathrm{Grad}\, p_\mathrm{f}^{k}\right]\right)_{K_\mathrm{f}} & \\
+ \left(\mathrm{Grad}\,\widetilde{\mathbf{v}}_\mathrm{f}, \mu_\mathrm{f} J_\mathrm{m} \mathbf{L}_\mathrm{f}^\mathrm{h} \mathbf{F}_\mathrm{m}^{-\intercal}\right)_{K_\mathrm{f}} + \left\langle \widetilde{\mathbf{v}}_\mathrm{f}, \widehat{\mathbf{T}}\left[\mathbf{n}\right]_\mathrm{f}^{*}\right\rangle_{\partial K_\mathrm{f}} &= \left(\widetilde{\mathbf{v}}_\mathrm{f}, J_\mathrm{m} \mathbf{f}_\mathrm{f}\right)_{K_\mathrm{f}} , \\
\left(\widetilde{\mathbf{L}}_\mathrm{f}, J_\mathrm{m} \mathbf{L}_\mathrm{f}^\mathrm{h}\right)_{K_\mathrm{f}} - \left(\widetilde{\mathbf{L}}_\mathrm{f}, J_\mathrm{m} \mathrm{Grad}\,\mathbf{v}_\mathrm{f}^\mathrm{h} \mathbf{F}_\mathrm{m}^{-1}\right)_{K_\mathrm{f}} + \left\langle \widetilde{\mathbf{L}}_\mathrm{f} \mathbf{F}_\mathrm{m}^{-\intercal}\left[\mathbf{n}_\mathrm{f}\right], J_\mathrm{m}\left(\mathbf{v}_\mathrm{f}^\mathrm{h} - \boldsymbol{\upsilon}_\mathrm{f}^\mathrm{h}\right)\right\rangle_{\partial K_\mathrm{f}} &= 0 , \\
- \left(\mathbf{F}_\mathrm{m}^{-\intercal}\left[\mathrm{Grad}\,\widetilde{p}_\mathrm{f}^{k}\right], J_\mathrm{m} \mathbf{v}_\mathrm{f}^\mathrm{h}\right)_{K_\mathrm{f}} + \left\langle \mathbf{F}_\mathrm{m}^{-\intercal}\left[\mathbf{n}_\mathrm{f}\right] \widetilde{p}_\mathrm{f}^{k}, J_\mathrm{m} \boldsymbol{\upsilon}_\mathrm{f}^\mathrm{h}\right\rangle_{\partial K_\mathrm{f}} &= 0 , \\
\left\langle \widetilde{\boldsymbol{\upsilon}}_\mathrm{f}, \widehat{\mathbf{T}}\left[\mathbf{n}\right]_\mathrm{f}\right\rangle_{\partial \mathcal{T}_\mathrm{f}^\mathrm{h} \setminus \Gamma_\mathrm{FSI}} \boxed{+ \left\langle \widetilde{\boldsymbol{\upsilon}}_\mathrm{f}, \widehat{\mathbf{T}}\left[\mathbf{n}\right]_\mathrm{f} + \widehat{\mathbf{T}}\left[\mathbf{n}\right]_\mathrm{s}\right\rangle_{\Gamma_\mathrm{FSI}} + \left\langle \widetilde{\boldsymbol{\upsilon}}_\mathrm{f}, \mathbf{S}_\mathrm{f}\left(\boldsymbol{\upsilon}_\mathrm{f}^\mathrm{h} - \boldsymbol{\upsilon}_\mathrm{s}^\mathrm{h}\right)\right\rangle_{\Gamma_\mathrm{FSI}}} &= \left\langle \widetilde{\boldsymbol{\upsilon}}_\mathrm{f}, J_\mathrm{m} \mathbf{g}_{\mathrm{N}_\mathrm{f}}\right\rangle_{\Gamma_\mathrm{N}} , \\
\left\langle \widetilde{\psi}_\mathrm{f}, J_\mathrm{m} \mathbf{F}_\mathrm{m}^{-\intercal}\left[\mathbf{n}_\mathrm{f}\right] \cdot \boldsymbol{\upsilon}_\mathrm{f}^\mathrm{h}\right\rangle_{\partial \mathcal{T}_\mathrm{f}^\mathrm{h}} &= 0 , \\
\left(p_\mathrm{f}^{0}, J_\mathrm{m} p_\mathrm{f}^{0}\right)_{\mathcal{T}_\mathrm{f}^\mathrm{h}} &= 0 ,
\end{aligned}
$$

$\forall\, \{\widetilde{\mathbf{L}}_\mathrm{f}, \widetilde{\mathbf{v}}_\mathrm{f}, \widetilde{p}_\mathrm{f}, \widetilde{\boldsymbol{\upsilon}}_\mathrm{f}, \widetilde{\psi}_\mathrm{f}\} \in \boldsymbol{\mathcal{L}}_\mathrm{f}^\mathrm{h} \times \boldsymbol{V}_\mathrm{f}^\mathrm{h} \times Q_\mathrm{f}^\mathrm{h} \times \widetilde{\boldsymbol{\mathcal{V}}}_\mathrm{f} \times \Psi_\mathrm{f}^\mathrm{h}$, *where*

$$
\begin{aligned}
p_\mathrm{f}^{0} + p_\mathrm{f}^{k} &= p_\mathrm{f}^\mathrm{h} \\
\widehat{\mathbf{T}}\left[\mathbf{n}\right]_\mathrm{f}^{*} &:= -\mu_\mathrm{f} J_\mathrm{m} \mathbf{L}_\mathrm{f}^\mathrm{h} \mathbf{F}_\mathrm{m}^{-\intercal}\left[\mathbf{n}_\mathrm{f}\right] + \mathbf{S}_\mathrm{f}\left(\mathbf{v}_\mathrm{f}^\mathrm{h} - \boldsymbol{\upsilon}_\mathrm{f}^\mathrm{h}\right) , \\
\widehat{\mathbf{T}}\left[\mathbf{n}\right]_\mathrm{f} &:= J_\mathrm{m}\left[-\mu_\mathrm{f} \mathbf{L}_\mathrm{f}^\mathrm{h} + \left(p_\mathrm{f}^{k} + \psi_\mathrm{f}^\mathrm{h}\right)\mathbf{I}\right] \mathbf{F}_\mathrm{m}^{-\intercal}\left[\mathbf{n}_\mathrm{f}\right] + \mathbf{S}_\mathrm{f}\left(\mathbf{v}_\mathrm{f}^\mathrm{h} - \boldsymbol{\upsilon}_\mathrm{f}^\mathrm{h}\right) , \\
\mathbf{S}_\mathrm{f} &:= \left(\frac{\mu_\mathrm{f}}{l_\mathrm{f}} + \rho_\mathrm{f}\left|\mathbf{v}_\mathrm{f}^\mathrm{h}\right|_{\mathrm{L2}}\right)\mathbf{I} ,
\end{aligned}
$$

Problem 2 (Solid sub-problem)
Find $\{\mathbf{u}_\mathrm{s}^\mathrm{h}, \mathbf{v}_\mathrm{s}^\mathrm{h}, \mathbf{E}_\mathrm{s}^\mathrm{h}, \boldsymbol{\mu}_\mathrm{s}^\mathrm{h}, \boldsymbol{\upsilon}_\mathrm{s}^\mathrm{h}\} \in \boldsymbol{U}_\mathrm{s}^\mathrm{h} \times \boldsymbol{V}_\mathrm{s}^\mathrm{h} \times \boldsymbol{\mathcal{E}}_\mathrm{s}^\mathrm{h} \times \boldsymbol{M}_\mathrm{s}^\mathrm{h} \times \boldsymbol{\mathcal{V}}_\mathrm{s}^\mathrm{h}$, *such that*

$$
\begin{aligned}
\left(\widetilde{\mathbf{v}}_\mathrm{s}, \rho \frac{\partial \mathbf{v}_\mathrm{s}^\mathrm{h}}{\partial t}\right)_{K_\mathrm{s}} + \left(\mathrm{Grad}\,\widetilde{\mathbf{v}}_\mathrm{s}, \mathbf{F}_\mathrm{s}\, \mathbb{C}_\mathrm{s}\left(\mathbf{E}_\mathrm{s}^\mathrm{h}\right)\right)_{K_\mathrm{s}} + \left\langle \widetilde{\mathbf{v}}_\mathrm{s}, \widehat{\mathbf{T}}\left[\mathbf{n}\right]_\mathrm{s}\right\rangle_{\partial K_\mathrm{s}} &= \left(\widetilde{\mathbf{v}}_\mathrm{s}, \mathbf{b}_\mathrm{s}\right)_{K_\mathrm{s}} , \\
\left(\widetilde{\mathbf{E}}_\mathrm{s}, \mathbf{E}_\mathrm{s}^\mathrm{h}\right)_{K_\mathrm{s}} - \left(\mathrm{Sym}(\widetilde{\mathbf{E}})_\mathrm{s}, \mathrm{Grad}\,\mathbf{u}_\mathrm{s}^\mathrm{h}\right)_{K_\mathrm{s}} - \left(\tfrac{1}{2}\widetilde{\mathbf{E}}_\mathrm{s}, \left(\mathrm{Grad}\,\mathbf{u}_\mathrm{s}^\mathrm{h}\right)^{\intercal} \mathrm{Grad}\,\mathbf{u}_\mathrm{s}^\mathrm{h}\right)_{K_\mathrm{s}} & \\
+ \left\langle \mathrm{Sym}(\widetilde{\mathbf{E}})_\mathrm{s} \mathbf{n}, \left(\mathbf{u}_\mathrm{s}^\mathrm{h} - \boldsymbol{\mu}_\mathrm{s}^\mathrm{h}\right)\right\rangle_{\partial K_\mathrm{s}} &= 0 , \\
\left(\widetilde{\mathbf{u}}_\mathrm{s}, \frac{\partial \mathbf{u}_\mathrm{s}^\mathrm{h}}{\partial t}\right)_{K_\mathrm{s}} - \left(\widetilde{\mathbf{u}}_\mathrm{s}, \mathbf{v}_\mathrm{s}^\mathrm{h}\right)_{K_\mathrm{s}} &= 0 , \\
\left\langle \widetilde{\boldsymbol{\mu}}_\mathrm{s}, \mathbf{u}_\mathrm{s}^\mathrm{h}\right\rangle_{\partial K_\mathrm{s}} - \left\langle \widetilde{\boldsymbol{\mu}}_\mathrm{s}, \boldsymbol{\mu}_\mathrm{s}^\mathrm{h}\right\rangle_{\partial K_\mathrm{s}} &= 0 , \\
\left\langle \widetilde{\boldsymbol{\upsilon}}_\mathrm{s}, \widehat{\mathbf{T}}\left[\mathbf{n}\right]_\mathrm{s}\right\rangle_{\partial \mathcal{T}_\mathrm{s}^\mathrm{h} \setminus \Gamma_\mathrm{FSI}} \boxed{+ \left\langle \widetilde{\boldsymbol{\upsilon}}_\mathrm{s}, \widehat{\mathbf{T}}\left[\mathbf{n}\right]_\mathrm{s} + \widehat{\mathbf{T}}\left[\mathbf{n}\right]_\mathrm{f}\right\rangle_{\Gamma_\mathrm{FSI}} + \left\langle \widetilde{\boldsymbol{\upsilon}}_\mathrm{s}, \mathbf{S}_\mathrm{s}\left(\boldsymbol{\upsilon}_\mathrm{s}^\mathrm{h} - \boldsymbol{\upsilon}_\mathrm{f}^\mathrm{h}\right)\right\rangle_{\Gamma_\mathrm{FSI}}} &= \left\langle \widetilde{\boldsymbol{\upsilon}}_\mathrm{s}, \mathbf{g}_{\mathrm{N}_\mathrm{s}}\right\rangle_{\Gamma_{\mathrm{N}_\mathrm{s}}} ,
\end{aligned}
$$

$\forall\, \{\widetilde{\mathbf{u}}_\mathrm{s}, \widetilde{\mathbf{v}}_\mathrm{s}, \widetilde{\mathbf{E}}_\mathrm{s}, \widetilde{\boldsymbol{\mu}}_\mathrm{s}, \widetilde{\boldsymbol{\upsilon}}_\mathrm{s}\} \in \boldsymbol{U}_\mathrm{h} \times \boldsymbol{V}_\mathrm{s}^\mathrm{h} \times \boldsymbol{\mathcal{E}}_\mathrm{s}^\mathrm{h} \times \widetilde{\boldsymbol{M}}_\mathrm{s} \times \boldsymbol{\mathcal{V}}_\mathrm{s}^\mathrm{h}$, *where*

$$
\begin{aligned}
\mathbf{F}_\mathrm{s} &= \mathbf{I} + \mathrm{Grad}\,\mathbf{u}_\mathrm{s}^\mathrm{h} \\
\widehat{\mathbf{T}}\left[\mathbf{n}\right]_\mathrm{s} &:= -\left[\mathbf{F}_\mathrm{s}\, \mathbb{C}_\mathrm{s}\left(\mathbf{E}_\mathrm{s}^\mathrm{h}\right)\right] \mathbf{n}_\mathrm{s} + \mathbf{S}_\mathrm{s}\left(\mathbf{v}_\mathrm{s}^\mathrm{h} - \boldsymbol{\upsilon}_\mathrm{s}^\mathrm{h}\right) , \\
\mathbf{S}_\mathrm{s} &:= \frac{\mu_\mathrm{s}}{l_\mathrm{s}} \mathbf{I} .
\end{aligned}
$$

Problem 3 (Mesh sub-problem)
Find $\{\mathbf{u}_\mathrm{m}^\mathrm{h}, \mathbf{F}_\mathrm{m}^\mathrm{h}, \boldsymbol{\mu}_\mathrm{m}^\mathrm{h}\} \in \boldsymbol{U}_\mathrm{m}^\mathrm{h} \times \boldsymbol{\mathcal{F}}_\mathrm{m}^\mathrm{h} \times \boldsymbol{M}_\mathrm{m}^\mathrm{h}$ *such that*

$$
\begin{aligned}
&\left(\mathrm{Grad}\,\widetilde{\mathbf{u}}_\mathrm{m}, \mathbb{C}_\mathrm{m}\left(\mathbf{F}_\mathrm{m}^\mathrm{h}\right)\right)_K - \left\langle \widetilde{\mathbf{u}}_\mathrm{m}, \widehat{\mathbf{T}}\left[\mathbf{n}\right]_\mathrm{m}\right\rangle_{\partial K} = \left(\mathrm{Grad}\,\widetilde{\mathbf{u}}_\mathrm{m}, \mathbb{C}_\mathrm{m}\left(\mathbf{I}\right)\right)_K - \left\langle \widetilde{\mathbf{u}}_\mathrm{m}, \mathbb{C}_\mathrm{m}\left(\mathbf{I}\right)\mathbf{n}_\mathrm{m}\right\rangle_{\partial K} + \left(\widetilde{\mathbf{u}}_\mathrm{m}, \mathbf{b}\right)_K , \\
&\left(\widetilde{\mathbf{F}}_\mathrm{m}, \mathbf{F}_\mathrm{m}^\mathrm{h}\right)_K - \left(\widetilde{\mathbf{F}}_\mathrm{m}, \mathrm{Grad}\,\mathbf{u}_\mathrm{m}^\mathrm{h}\right)_K + \left\langle \widetilde{\mathbf{F}}_\mathrm{m} \mathbf{n}_\mathrm{m}, \left(\mathbf{u}_\mathrm{m}^\mathrm{h} - \boldsymbol{\mu}_\mathrm{m}^\mathrm{h}\right)\right\rangle_{\partial K} = \left(\widetilde{\mathbf{F}}_\mathrm{m}, \mathbf{I}\right)_K , \\
&\left\langle \widetilde{\boldsymbol{\mu}}_\mathrm{m}, \widehat{\mathbf{T}}\left[\mathbf{n}\right]_\mathrm{m}\right\rangle_{\partial \mathcal{T}_\mathrm{m}^\mathrm{h} \setminus \Gamma_\mathrm{FSI}} \boxed{+ \left\langle \widetilde{\boldsymbol{\mu}}_\mathrm{m}, \boldsymbol{\mu}_\mathrm{m}^\mathrm{h} - \boldsymbol{\mu}_\mathrm{s}^\mathrm{h}\right\rangle_{\Gamma_\mathrm{FSI}}} = \left\langle \widetilde{\boldsymbol{\mu}}_\mathrm{m}, \mathbb{C}_\mathrm{m}\left(\mathbf{I}\right)\mathbf{n}_\mathrm{m}\right\rangle_{\partial \mathcal{T}_\mathrm{m}^\mathrm{h} \setminus \Gamma_\mathrm{FSI}} + \left\langle \widetilde{\boldsymbol{\mu}}_\mathrm{m}, \mathbf{g}_\mathrm{N}\right\rangle_{\partial \Omega_\mathrm{N}} ,
\end{aligned}
$$

$\forall\, \{\widetilde{\mathbf{u}}_\mathrm{m}, \widetilde{\mathbf{F}}, \widetilde{\boldsymbol{\mu}}\} \in \boldsymbol{U}_\mathrm{m}^\mathrm{h} \times \boldsymbol{\mathcal{F}}_\mathrm{m}^\mathrm{h} \times \widetilde{\boldsymbol{M}}_\mathrm{m}$, *where*

$$
\begin{aligned}
\widehat{\mathbf{T}}\left[\mathbf{n}\right]_\mathrm{m} &:= \mathbb{C}_\mathrm{m}\left(\mathbf{F}_\mathrm{m}^\mathrm{h}\right)\mathbf{n}_\mathrm{m} - \mathbf{S}_\mathrm{m}\left(\mathbf{u}_\mathrm{m}^\mathrm{h} - \boldsymbol{\mu}_\mathrm{m}^\mathrm{h}\right) , \\
\mathbf{S}_\mathrm{m} &:= \frac{\mu}{l} \mathbf{I} .
\end{aligned}
$$

Monolithic Coupling

$$
\begin{bmatrix}
\text{Solid System Matrix} & \text{Solid Fluid Coupling} & \text{Solid Mesh Coupling} \\
\text{Fluid Solid Coupling} & \text{Fluid System Matrix} & \text{Fluid Mesh Coupling} \\
\text{Mesh Solid Coupling} & \text{Mesh Fluid Coupling} & \text{Mesh System Matrix}
\end{bmatrix}
\begin{bmatrix}
\text{Solid Unkns} \\
\text{Fluid Unkns} \\
\text{Mesh Unkns}
\end{bmatrix}
=
\begin{bmatrix}
\text{Solid RHS} \\
\text{Fluid RHS} \\
\text{Mesh RHS}
\end{bmatrix}
$$

Actually implementing this in deal

- **Multiple Triangulations (for each material)**
  - and sets of dofhandlers, fe-sytems, etc
- **Using blockmatrix system**
  - Each diagonal block gets a different physics model
- **Fluid-solid interface (off-diagonal blocks)**
  - We have two sets of global (trace) degrees of freedom unlike everywhere else in the domain
    - Face/point tracking between abutting trias
    - Use weak form of coupling conditions with single point quadrature from the adjacent tria
  - Need global-global coupling!
    - Don't want to couple the local and global solutions

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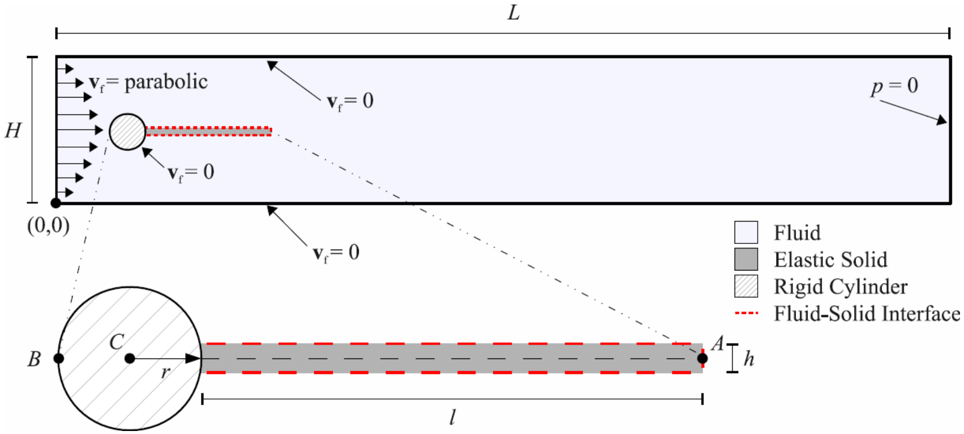
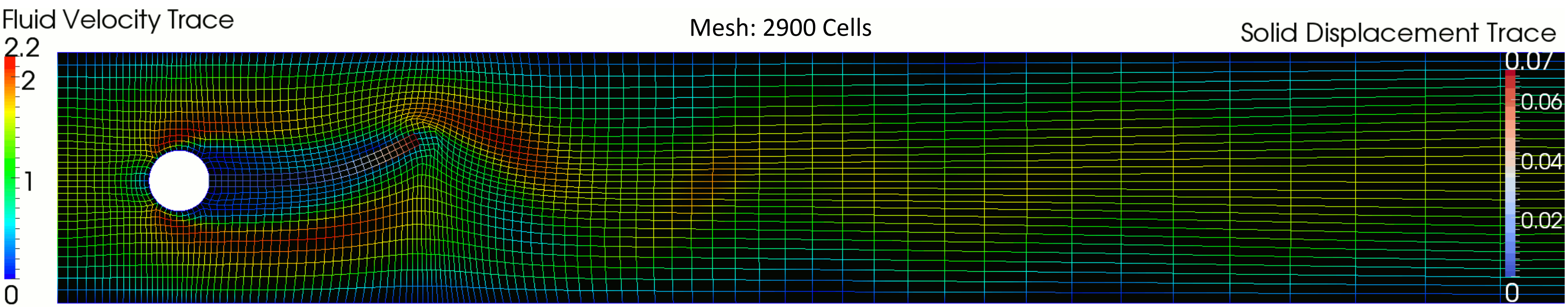
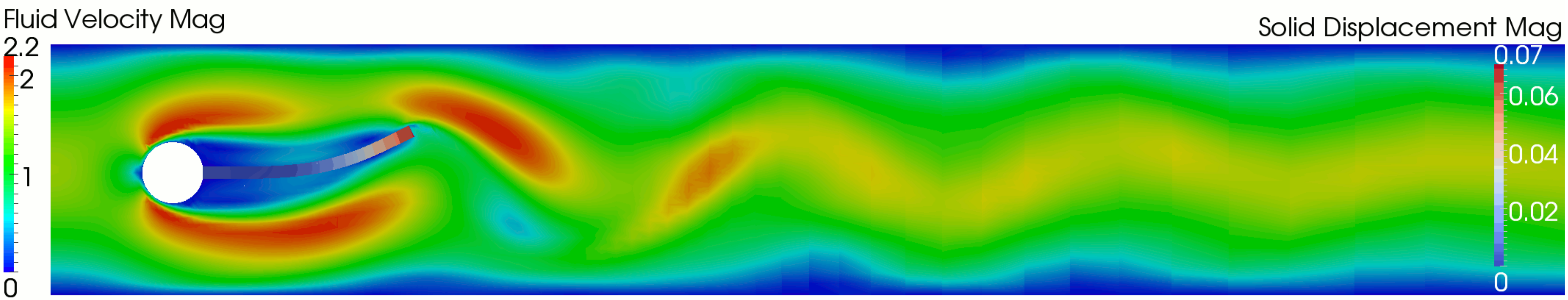
Turek-Hron FSI benchmark



parameter	symbol	value [m]	property	value
Channel length	L	2.5	$\rho_s\ [\frac{kg}{m^3}]$	10000
Channel height	H	0.41	ν_s	0.4
Cylinder center	C	(0.2, 0.2)	$E_s\ [\mathrm{E6}\ \frac{kg}{ms^2}]$	5.6
Cylinder radius	r	0.05	$\rho_f\ [\frac{kg}{m^3}]$	1000
Flag length	l	0.35	$\mu_f\ [\frac{kg}{ms}]$	1
Flag height	h	0.02	$\bar{U}\ [\frac{m}{s}]$	1
Reference point	A	(0.6, 0.2)		
Reference point	B	(0.15, 0.2)	$\mathbf{v}_f\vert_{in}\ [\frac{m}{s}]$	$1.5\,\bar{U}\,\frac{y(H-y)}{(H/2)^2}$





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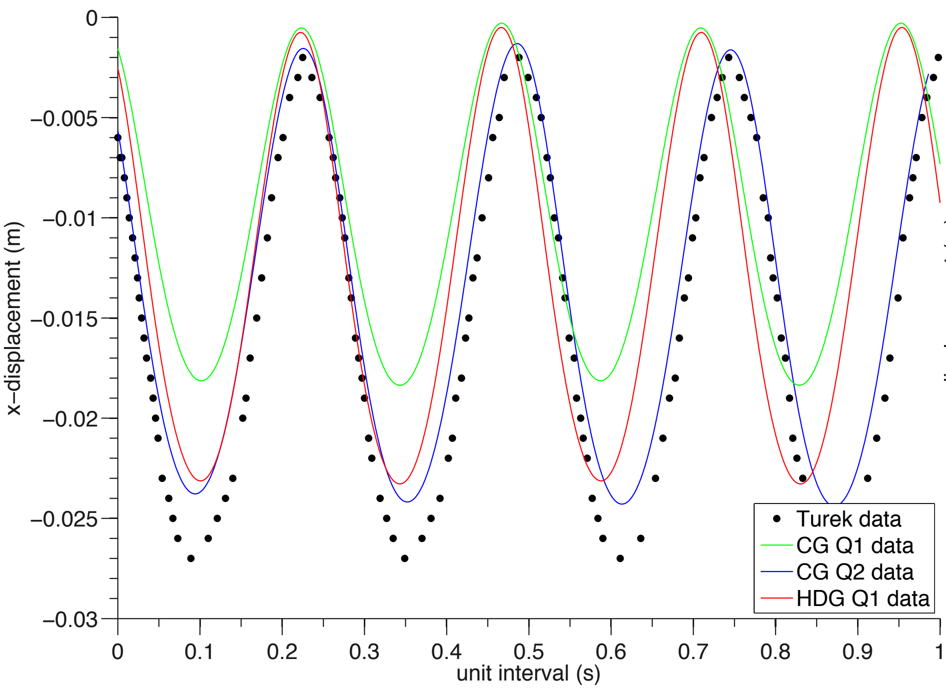
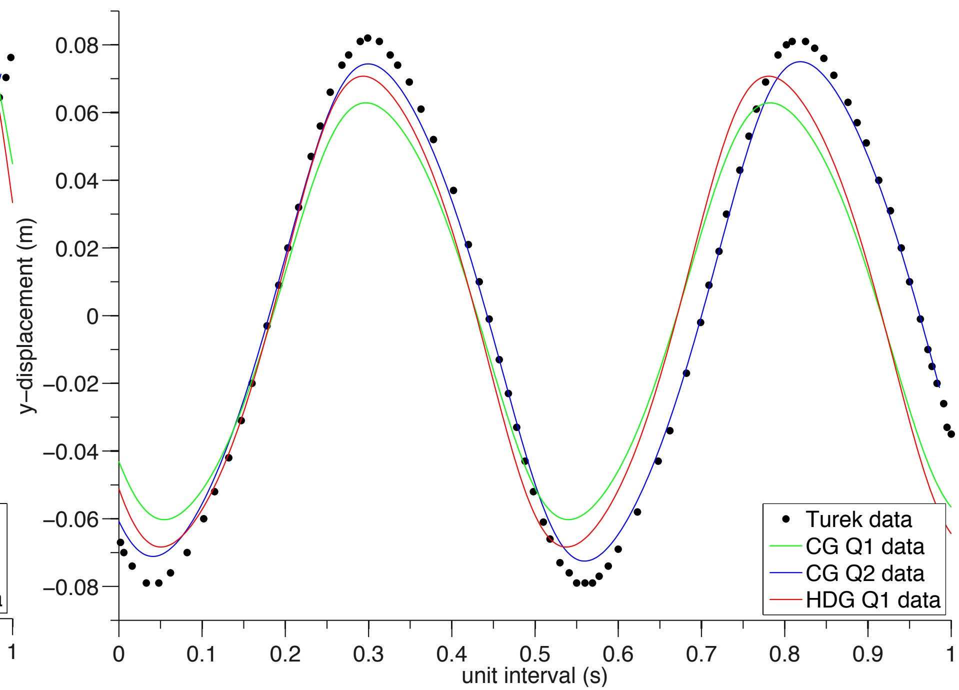
Tip displacement plots (at A)

Turek-Hron FSI benchmark tip displacement

Plotted against CG and HDG results, both using backward Euler time-stepping on same mesh





- Q1 HDG results match oscillation period for Q1 CG results
- Q1 HDG results are closer in amplitude to Q2 CG results than Q1

Summary

- **Goal:** Develop new computational tool, a HDG FSI model
  - HDG allows for arbitrarily high order polynomials and their inherently parallel nature leads to significant increase in computational efficiency
- **Progress:** Models implemented, tested, and coupled
  - Implemented and verified HDG elasticity and Navier-Stokes models
  - Considered preliminary test cases
  - Implemented HDG FSI formulation
- **HDG FSI:** Benchmarked against Turek and Hron (backward Euler in time)
  - Q1 HDG results match oscillation period for Q1 CG results
  - Q1 HDG results are closer in amplitude to Q2 CG results than Q1

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Future research tasks / questions

- Implement higher order time-stepping, such as Diagonally implicit Runge-Kutta
- Perform verification with the method of manufactured solutions for the fully-coupled FSI problem.
- Implement superconvergent postprocess capabilities for HDG FSI as done for individual components.
- Are the various HDG formulations for the different physics models algebraically equivalent? Are there optimal formulations in terms of accuracy, computational efficiency, and/or stability?
- Can we quantify the effects on convergence and dissipation of varying the stability parameter **S**?
- Plotting error vs. wall time, at what order element does using the HDG method for FSI become more computationally efficient than using other methods, such as our CG FSI model?

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Questions

Thank you

Questions?