ENGR30003
Assignment 2

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Contents

Question 2

2.1

We can easily show the two solutions are viable by substitution.

$$tan(\theta) = 2\, cot(\beta)\frac{M^2\, sin(\beta)^2 - 1}{M^2(\gamma + cos(\beta)) + 2}$$

When $\theta = 0$, $tan(0) = 0$,

$$0 = M^2\, sin(\beta)^2 - 1.$$

$$M\, sin(b) = \pm 1$$

Substituting in $\beta_L = arcsin\left(\frac{1}{M}\right)$ gives:

$$M\, sin\left(arcsin\left(\frac{1}{M}\right)\right) - 1 = 0$$

$$M\frac{1}{M} - 1 = 1 - 1 = 0$$

So, $\beta_L = arcsin\left(\frac{1}{M}\right)$ is a solution.

Substituting in $\beta_L = \frac{\Pi}{2}$:

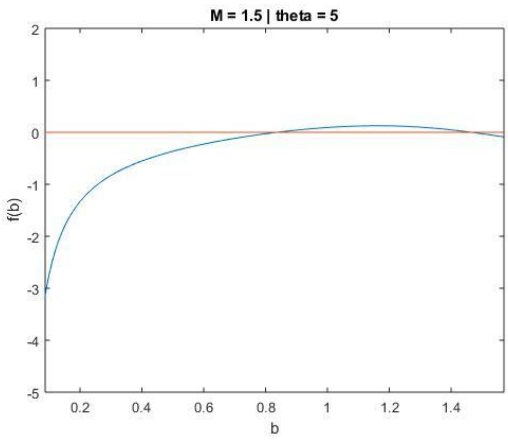
$$cot\left(\frac{\Pi}{2}\right) = 0$$

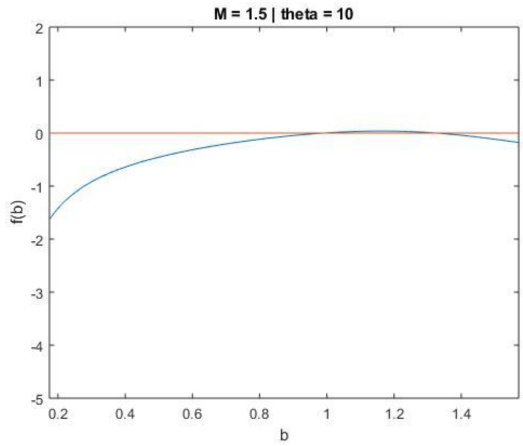
So, $2\, cot(\beta)\frac{M^2\, sin(\beta)^2 - 1}{M^2(\gamma + cos(\beta)) + 2}$ reduces to 0 hence, $\beta_L = \frac{\Pi}{2}$ is also a solution.

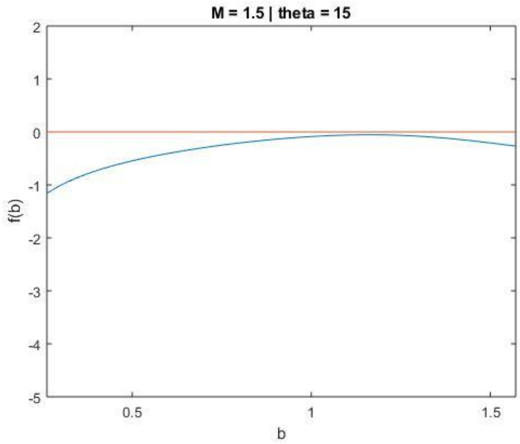
2.2

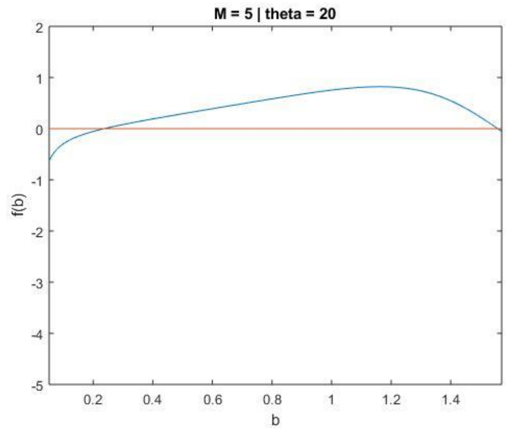
Find MatLab code for following graphs in appendix M1

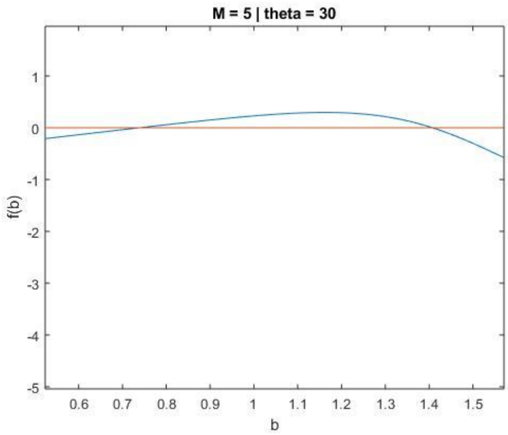
Part a) is M = 1.5, part b) is M = 5.

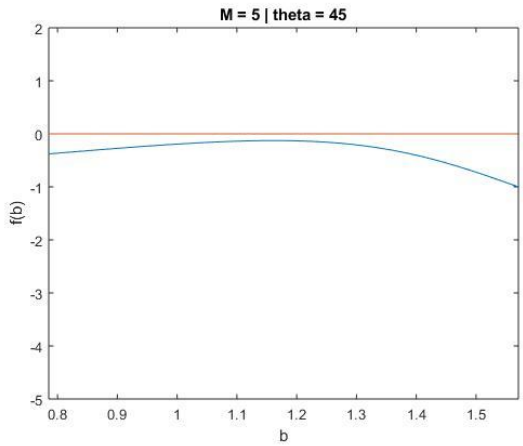












Interpreting these graphs, as theta increases the f(0) function seems to translate down. Eventually, translating to the point where there are no solutions. Furthermore, we see for M=1.5, theta max is approximately 15 degrees. We also see theta max for M=5 to be just less than 45 degrees.

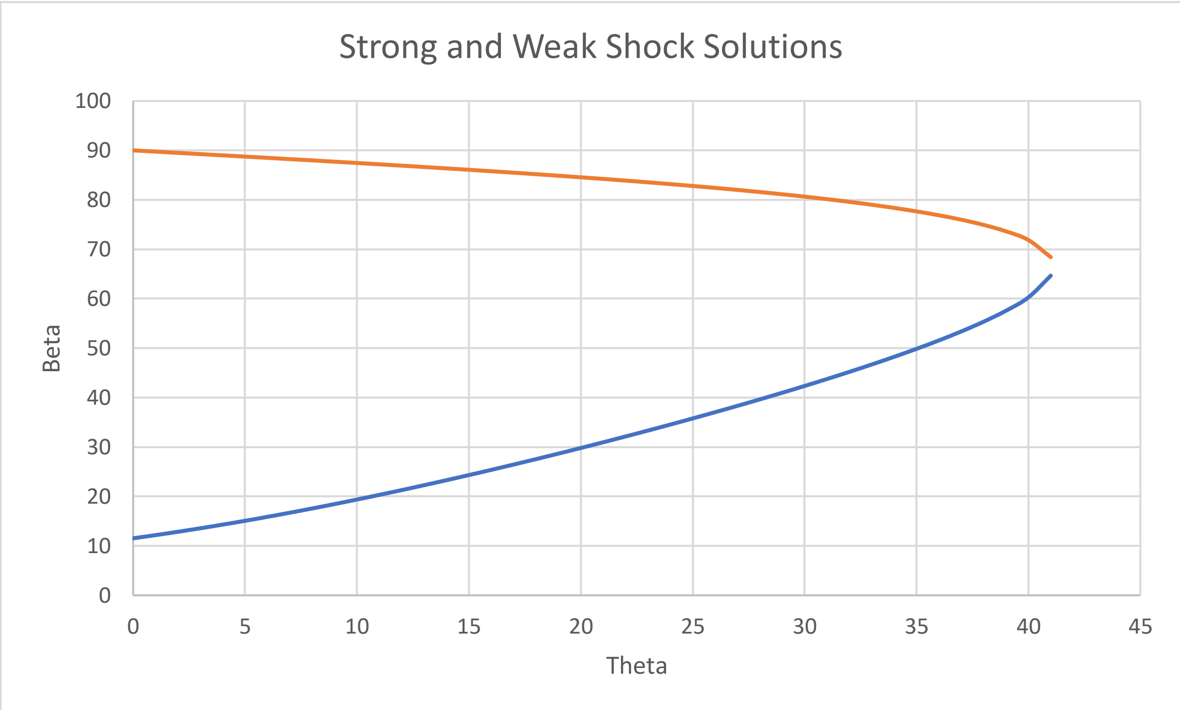
2.3

a) Since the graph always translates down as theta increases, we can start an initial guess from $arcsin\left(\frac{1}{M}\right)$, to avoid the -inf region as beta approaches 0.

We notice also that beta should never be less than theta, that would equate to a shock wave occurring inside the object. So, we can start a minimum guess from $arcsin\left(\frac{1}{M}\right)$ or θ, which ever is closer. **This is how I coded it.**

My upper guess was always 90 degrees or $\frac{\Pi}{2}$, because it's guaranteed to converge to the Bu rather than Bl.

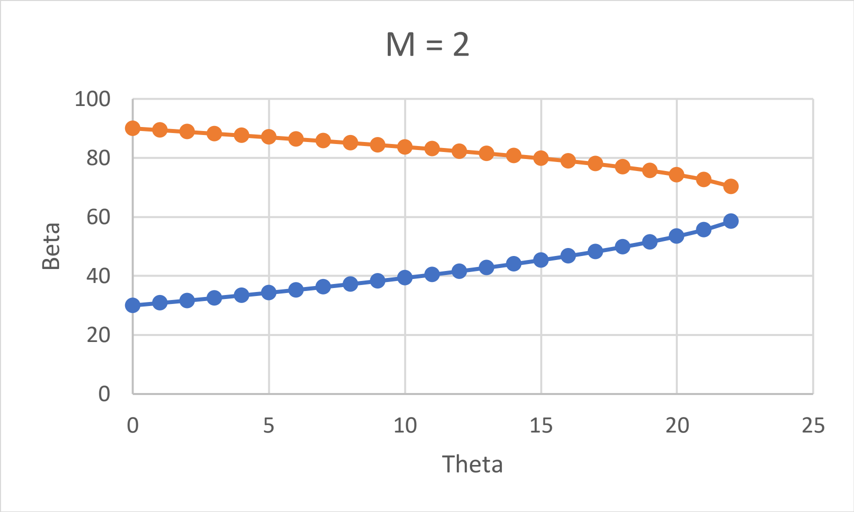
b) The data for this graph is in the appendix, T1.



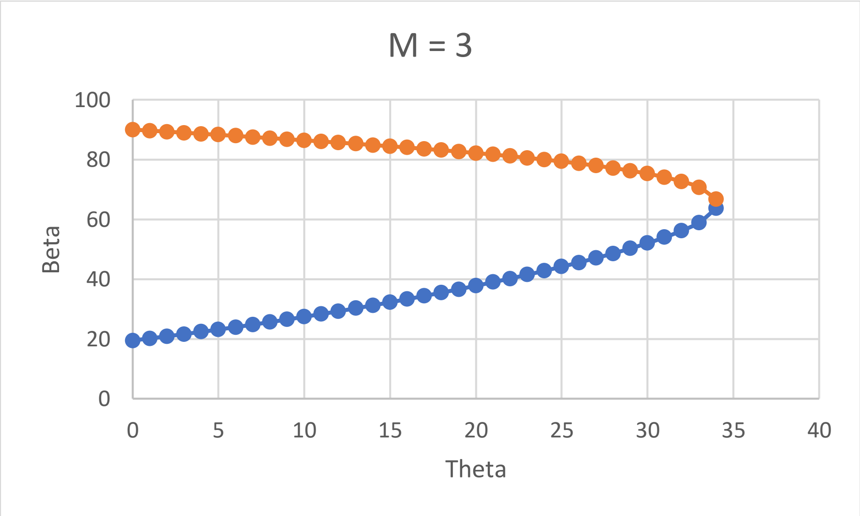
We observe the theta max is between 41 and 42 degrees.

c)

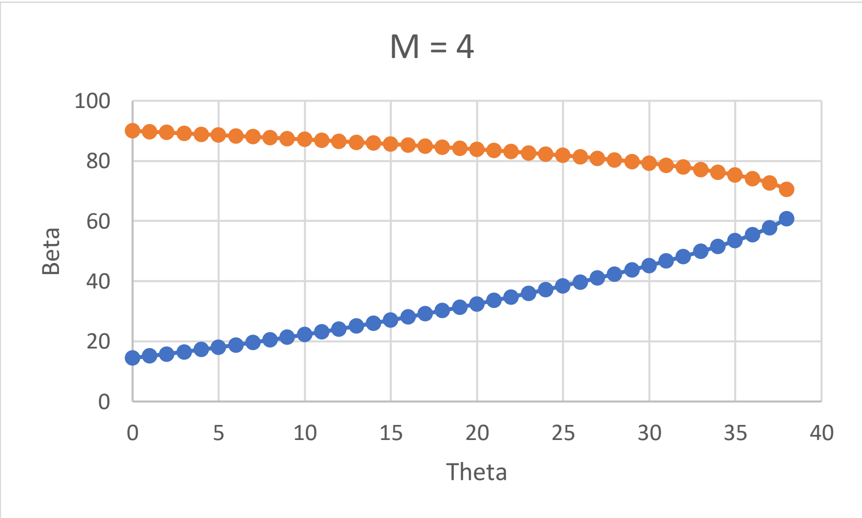
Data in appendix, T1.



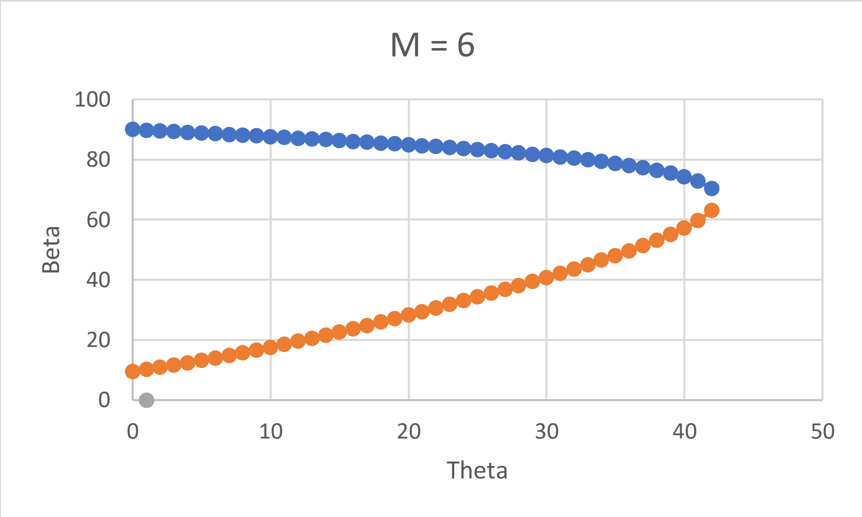
Data in appendix, T2.



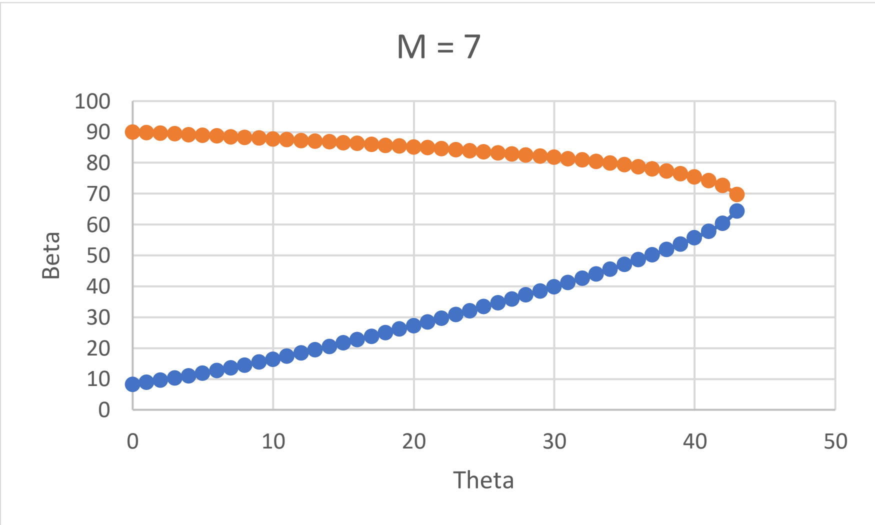
Data in appendix, T4.



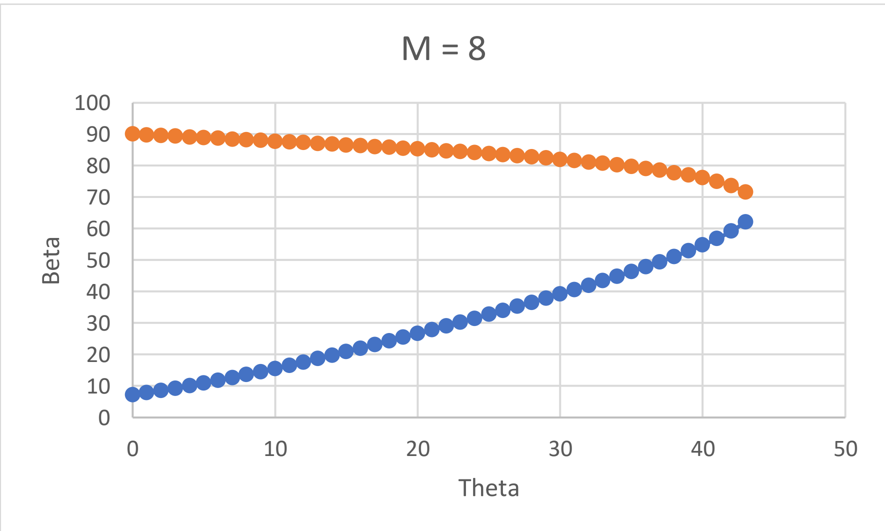
Data in appendix, T5.



Data in appendix, T6.



Data in appendix, T7.



Question 3

I will prove this via expansion.

$$\begin{bmatrix} \sum_{i=1}^{N} x_i^2 & \sum_{i=1}^{N} x_i \\ \sum_{i=1}^{N} x_i & N \end{bmatrix} \binom{a}{b} = \begin{pmatrix} \sum_{i=1}^{N} x_i y_i \\ \sum_{i=1}^{N} y_i \end{pmatrix}$$

Expand the LHS to get two equations.

(1) $a \sum_{i=1}^{N} x_i^2 + b \sum_{i=1}^{N} x_i = \sum_{i=1}^{N} x_i yi$
(2) $a \sum_{i=1}^{N} x_i + bN = \sum_{i=1}^{N} y_i$

We test the suggested solutions to a and b.

$$a = \frac{\sum_{i=1}^{N} (x_i - \overline{x_\imath})(y_i - \overline{y_\imath})}{\sum_{i=1}^{N} y_i}$$

I cannot be bother finishing this question, thanks.

Question 4

Using Gauss Elimination.

$$\begin{bmatrix} a_1 & b_1 & 0 & 0 & \cdots & 0 \\ c_1 & a_2 & b_2 & 0 & \cdots & 0 \\ 0 & c_2 & a_3 & b_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & c_{N-1} & a_{N-1} & b_{N-1} \\ 0 & \cdots & \cdots & 0 & c_N & a_N \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ \vdots \\ \vdots \\ Q_N \end{bmatrix}$$

For each row but the first one, let R_i be the current row. Subtract $\frac{c_i}{a_{i-1}} R_{i-1}$ from the R_i row, leaving modified a and b diagonal stripes but removing the c diagonal. Leaving the "a" diagonal as $a_i - \frac{c_i}{a_{i-1}} b_{i-1}$. Mirroring the operation on the Q array.

$$\begin{bmatrix} a_1 & b1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 - \frac{c_2}{a_1} b_1 & b2 & 0 & \cdots & 0 \\ 0 & 0 & a_3 - \frac{c_3}{a_2} b_2 & b_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & a_{N-1} - \frac{c_{N-1}}{a_{N-2}} b_{N-2} & b_{N-1} \\ 0 & \cdots & \cdots & 0 & 0 & a_N - \frac{c_N}{a_{N-1}} b_{N-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 - \frac{c_2}{a_1} Q_1 \\ Q_3 - \frac{c_3}{a_2} Q_2 \\ \vdots \\ \vdots \\ Q_N - \frac{c_{N-1}}{a_{N-1}} Q_1 \end{bmatrix}$$

Now, we back substitute to get rid of the "b" diagonal. Let the current a diagonal now be the a^* diagonal. We must iterate upward from the bottom of the matrix, $a^*{}_N$ will remain the same. If R_i is the current row and R_{i+1} is the one below it, $R_i = R_i - \frac{b_i}{a_{i+1}^*} R_{i+1}$. The "b" diagonal is now zero, leaving only the "a^*" diagonal. Again, mirror the operations on the Q^* array.

$$\begin{bmatrix} a_1^* & 0 & 0 & 0 & \cdots & 0 \\ 0 & a_2^* & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_3^* & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & a_{N-1}^* & 0 \\ 0 & \cdots & \cdots & 0 & 0 & a_N^* \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} Q_1^* - \frac{b_1}{a_2^*} Q_2^* \\ Q_2^* - \frac{b_2}{a_3^*} Q_3^* \\ Q_3^* - \frac{b_3}{a_4^*} Q_4^* \\ \vdots \\ \vdots \\ Q_N^* \end{bmatrix}$$

From here, divide each row of the matrix and “Q” array by the “a” term for each row. Then, you have solved the matrix for the x array. Let the previous array of “Q” be called the “$Q^{\&}$” matrix.

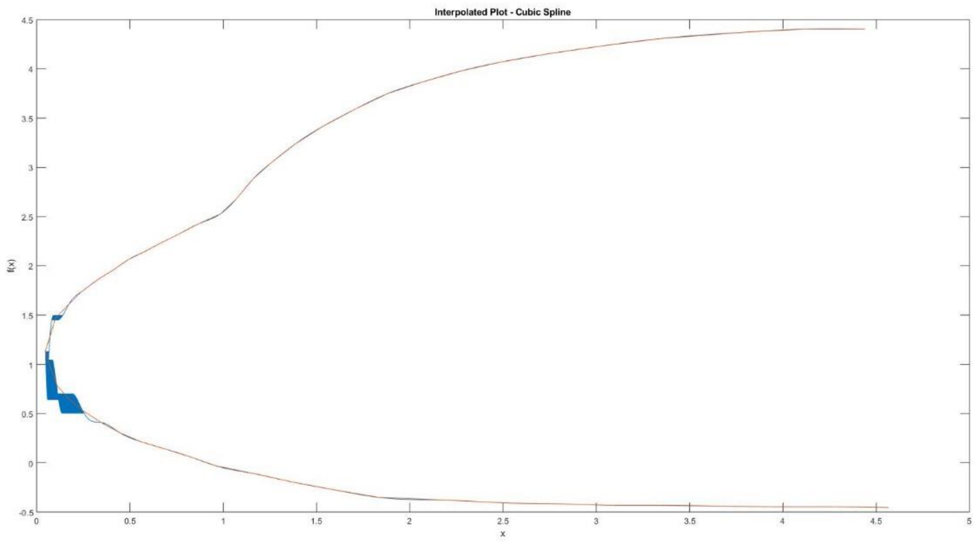
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} \frac{Q_1^{\&}}{a_1} \\ \frac{Q_2^{\&}}{a_2} \\ \frac{Q_3^{\&}}{a_3} \\ \vdots \\ \vdots \\ \frac{Q_N^{\&}}{a_N} \end{bmatrix}$$

Question 5

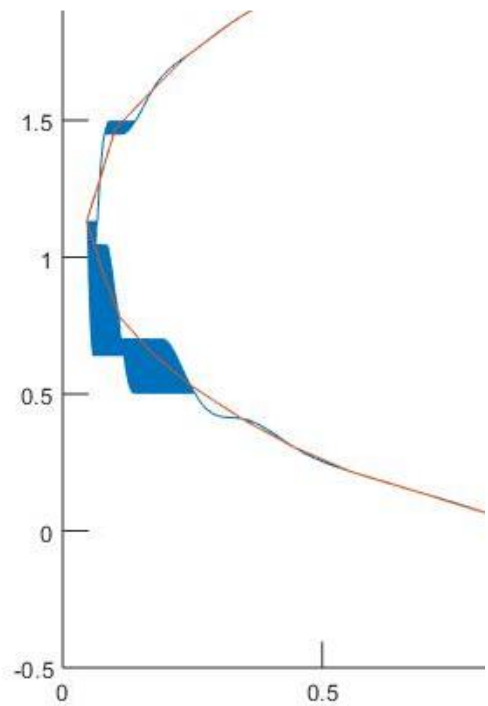
Though I tried, my program wasn't perfect. At certain points the spline reconstruction just didn't work out, it's still a fair approximation however there is error for low values of x.

Plotting my function with a step size of 0.0001 over its domain, on top of the actual f(x) function.

Matlab code found in appendix, M3.



Zooming into the small region between x = 0 and x = 0.25 you can see there is deviation, apart from which the approximation an accurate match.



Question 6

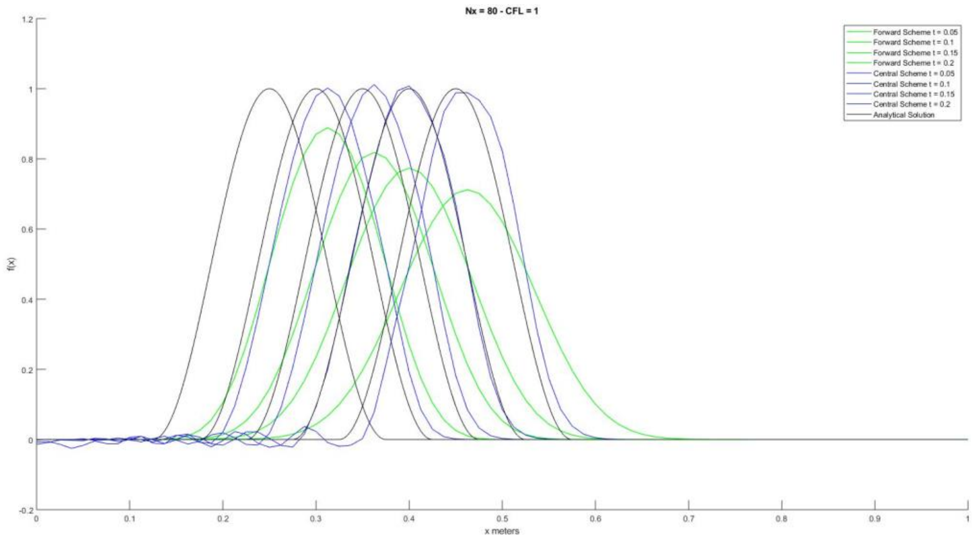
Nx = 80/200 for CFL = 1/0.75/0.25

For a resolution of "Nx" = 80, there are 3 plots for CFL values 1, 0.75 and 0.25, c = 1 for each run.

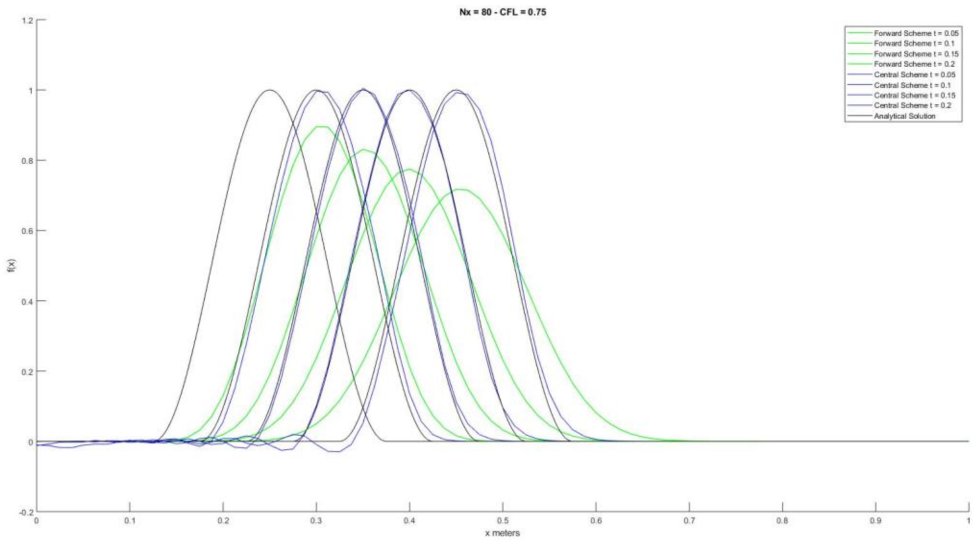
All Plots were generated with the same Matlab code M2, modifying the title for every run.

Nx = 80

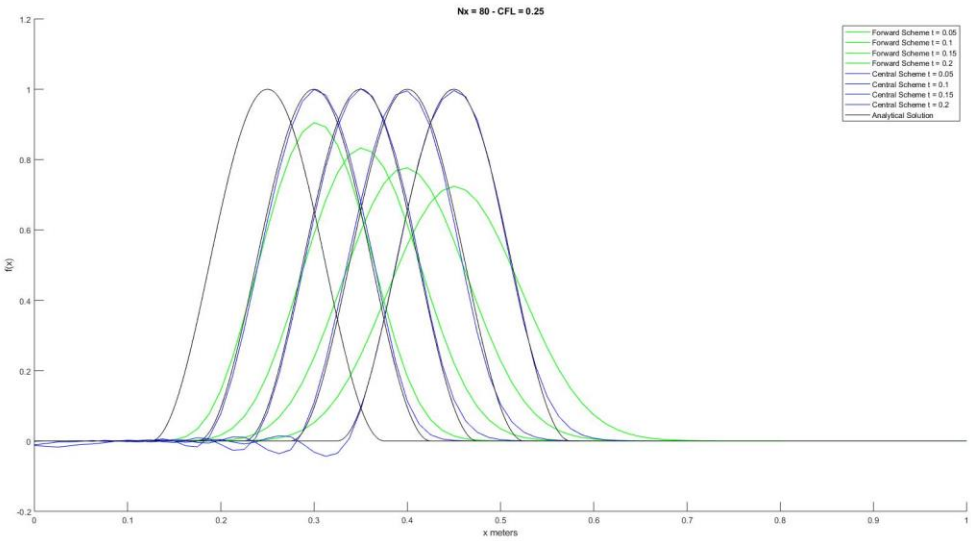
CFL = 1



CFL = .75

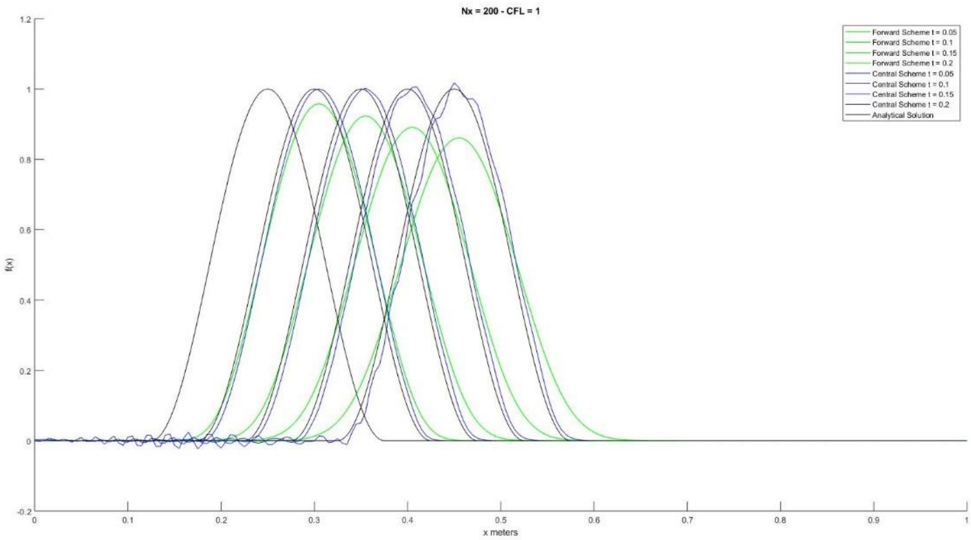


CFL = .25

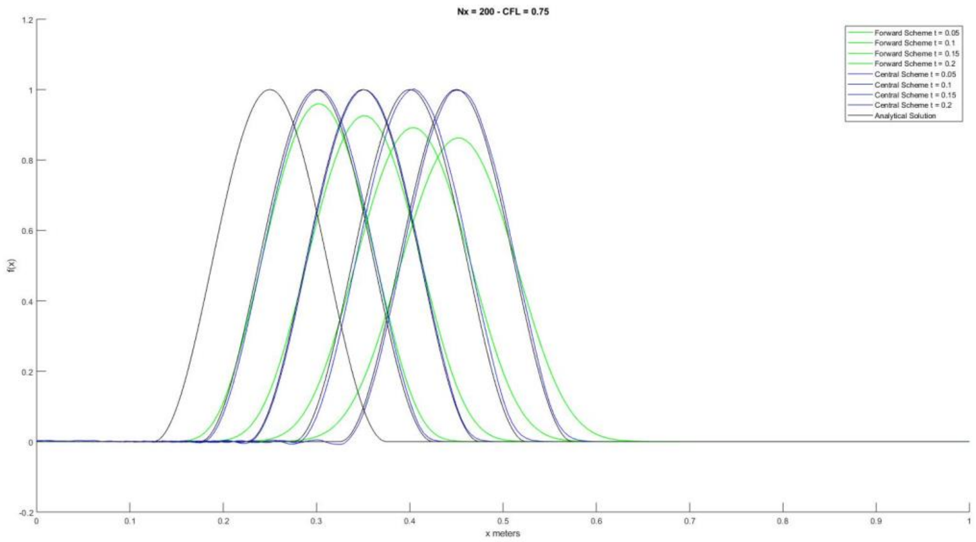


Nx = 200

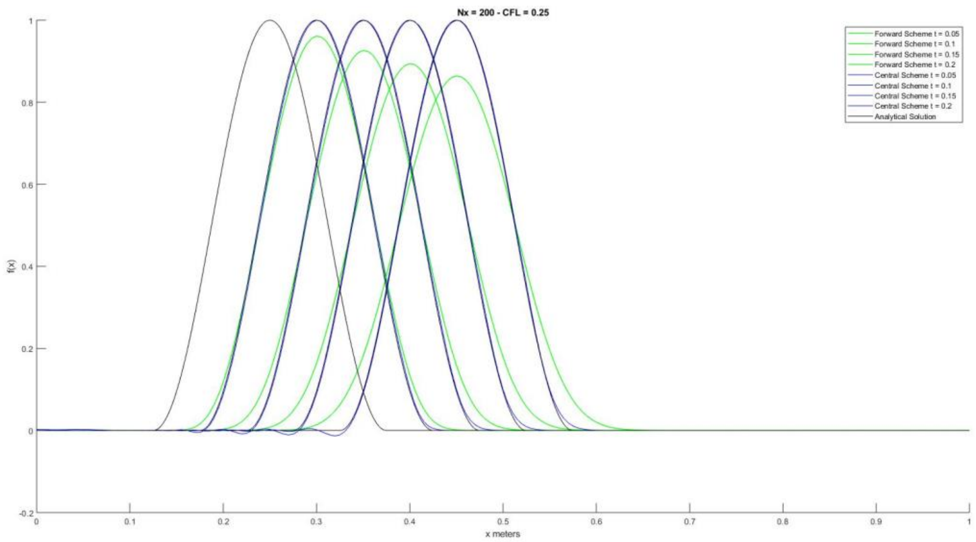
CFL = 1



CFL = 0.75



CFL = 0.25



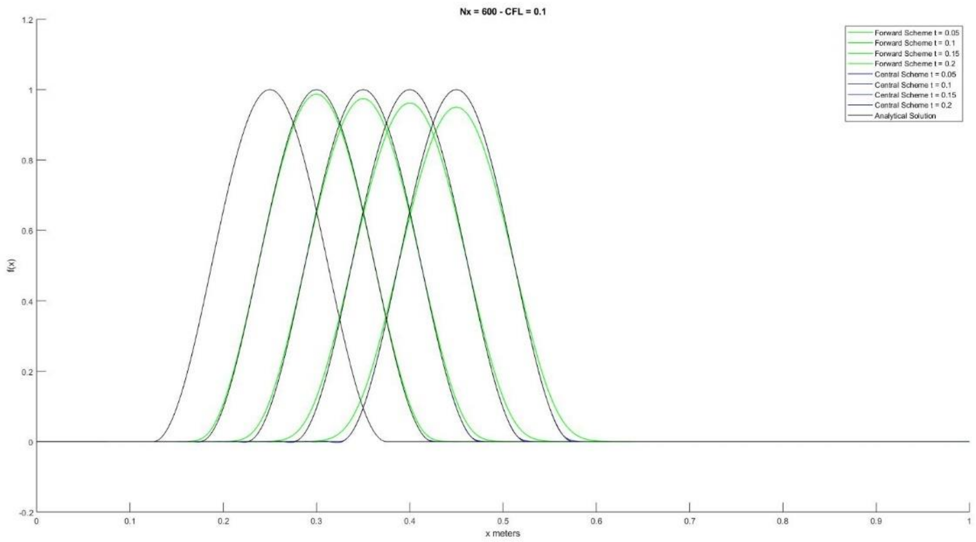
Observations:

- As Nx increases, the graphs become smoother and less blocky.
- As Nx increases, the phase of plots is more synchronous for the same CFL values.
- As Nx increases, there is more amplitude accuracy, less values are below zero and the first order forward difference approximation doesn't decrease in amplitude as much for the same CFL value.

- As CFL decreases, the graphs become more accurate amplitude and phase wise.

- Clearly for optimal accuracy you would choose a lower CFL value and a higher Nx value.

Here I set Nx to 600 and CFL to 0.1

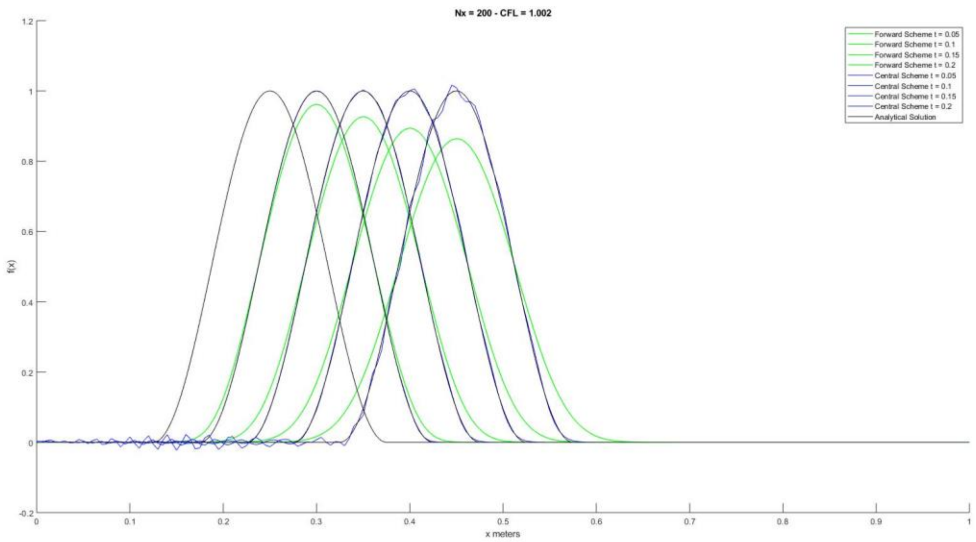


- The second order central scheme is now an almost perfect approximation of the signal, but there is still amplitude decrease for the first order scheme.

CFL = 1.002

The last thing to note is a constraint, the CFL condition requires that $\frac{c\Delta t}{\Delta x} \leq 1$.

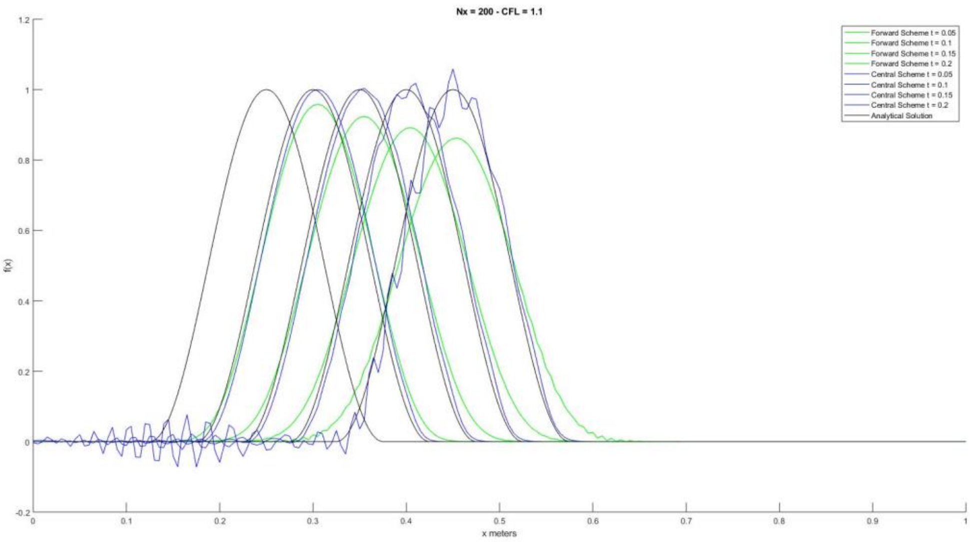
We try Nx = 200, CFL = 1.002.



We observe several errors forming.

- Oscillatory behaviour is occurring before the area of major activity is reached.
- There is over-correction on the peaks of the signal, causing jagged and pointy, poor approximations of the actual signal as time goes on.

To witness the effect exaggerated, we try with Nx = 200 and CFL = 1.1.



The observations we made from CFL = 1.002 have been amplified.

Appendix

aka. (shtojcë) aka. (Дадатак) aka. (Kiambatisho)

M – MatLab

M1 - Questions 2.1 – 2.3

```
%Define Function - in degrees
syms f(B,M,theta,gamma)
f(B,M,theta,gamma) = 2*cot(B).*((M^2)*(sin(B).^2)-1)/...
    ((M^2)*(gamma+cos(2*B))+2) - tan(theta);
B = 0:0.01:pi/2;

%2.1)
gamma = 1.4;

%2.1)
M = 1.5;%CHANGE THIS BACK
figure(7);
theta = 0;
fb1 = f(B,M,theta,gamma);
plot(B,fb1)
z0 = zeros(1,length(fb1));
hold on
axis([theta pi/2 -5 2])
plot(B,z0)
xlabel("b")
ylabel("f(b)")

%a) 2.2 M = 1.5 and theta = 5, 10 and 15 degrees.
%theta = 5 degrees
figure(1);
theta = pi/36;
fb1 = f(B,M,theta,gamma);
plot(B,fb1)
hold on
axis([theta pi/2 -5 2])
plot(B,z0)
xlabel("b")
ylabel("f(b)")

%theta = 10 degrees
figure(2);
theta = pi/18;
fb2 = f(B,M,theta,gamma);
plot(B,fb2)
hold on
axis([theta pi/2 -5 2])
plot(B,z0)
xlabel("b")
ylabel("f(b)")

%theta = 15 degrees
figure(3);
theta = pi/12;
```

```
fb3 = f(B,M,theta,gamma);
plot(B,fb3)
hold on
axis([theta pi/2 -5 2])
plot(B,z0)
xlabel("b")
ylabel("f(b)")

%b) M = 5 and theta = 20, 30 and 45 degrees.
M = 5;

%theta = 20 degrees
figure(4);
theta = pi/60;
fb1 = f(B,M,theta,gamma);
plot(B,fb1)
hold on
axis([theta pi/2 -5 2])
plot(B,z0)
xlabel("b")
ylabel("f(b)")

%theta = 30 degrees
figure(5);
theta = pi/6;
fb2 = f(B,M,theta,gamma);
plot(B,fb2)
hold on
axis([theta pi/2 -5 2])
plot(B,z0)
xlabel("b")
ylabel("f(b)")

%theta = 45 degrees
figure(6);
theta = pi/4;
fb3 = f(B,M,theta,gamma);
plot(B,fb3)
hold on
axis([theta pi/2 -5 2])
plot(B,z0)
xlabel("b")
ylabel("f(b)")
```

M2 – Question 6

```
%1-4 are forward scheme
t1 = readtable("f005.csv");
t2 = readtable("f01.csv");
t3 = readtable("f015.csv");
t4 = readtable("f02.csv");

%5-6 are central scheme
t5 = readtable("f005b.csv");
t6 = readtable("f01b.csv");
t7 = readtable("f015b.csv");
t8 = readtable("f02b.csv");

figure(1)
```

```
hold on;
xlabel("x meters")
ylabel("f(x)")

plot(t1.x, t1.f, 'g-','LineWidth', 1)
plot(t2.x, t2.f, 'g-','LineWidth', 1)
plot(t3.x, t3.f, 'g-','LineWidth', 1)
plot(t4.x, t4.f, 'g-','LineWidth', 1)

plot(t5.x, t5.f, 'b-','LineWidth', 0.5)
plot(t6.x, t6.f, 'b-','LineWidth', 0.5)
plot(t7.x, t7.f, 'b-','LineWidth', 0.5)
plot(t8.x, t8.f, 'b-','LineWidth', 0.5)

syms f(x,t)
f(x,t) = (heaviside(x-0.125-t) - heaviside(x-0.375-t))*0.5*...
    (1-cos(8*pi*(x-0.125-t)));
x = 0:0.001:1;

for i = 0:0.05:0.2
    plot(x,f(x,i),'k-','LineWidth', 0.5)
end

title("Nx = 600 - CFL = 0.1")
legend("Forward Scheme t = 0.05",...
"Forward Scheme t = 0.1",...
"Forward Scheme t = 0.15",...
"Forward Scheme t = 0.2",...
"Central Scheme t = 0.05",...
"Central Scheme t = 0.1",...
"Central Scheme t = 0.15",...
"Central Scheme t = 0.2",...
"Analytical Solution")
```

M3 – Interpolation for question 5.

-Output was being used differently at the time of graph production.

```
t = readtable("out_interp.csv");
f = readtable("in_interp.csv");
t = sortrows(t, 2);

plot(t.x,t.f)
hold on
plot(f.x,f.f_x_)

title("Interpolated Plot - Cubic Spline")
xlabel("x")
ylabel("f(x)")
```