

# CMPT 310: Artificial Intelligence

## Hidden Markov Models



Instructor: Steven Bergner --- Simon Fraser University

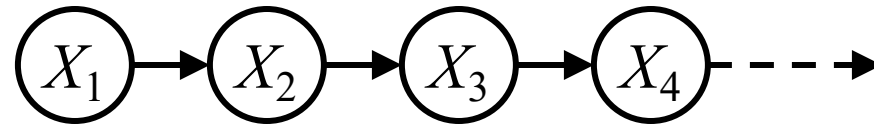
# Reasoning over Time or Space

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- Often, we want to reason about a sequence of observations
  - Speech recognition
  - Robot localization
  - User attention
  - Medical monitoring
- Need to introduce time (or space) into our models

# Markov Models

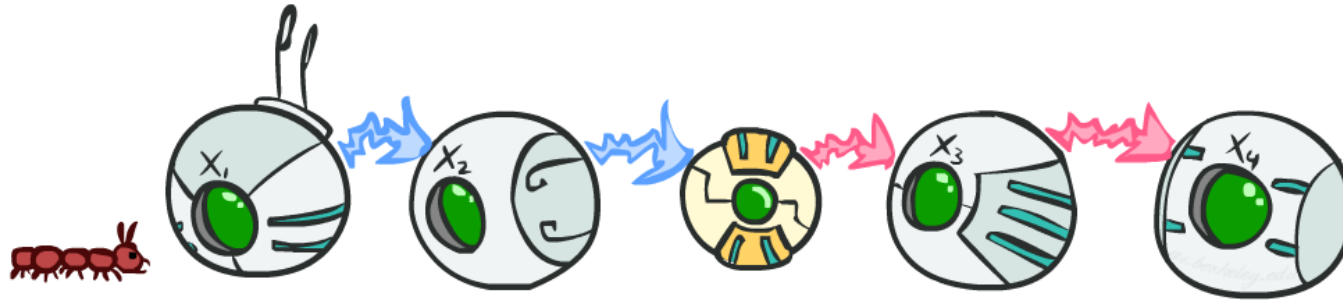
- Value of  $X$  at a given time is called the **state**



$$P(X_1) \quad P(X_t|X_{t-1})$$

- Parameters: called **transition probabilities** or dynamics, specify how the state evolves over time (also, initial state probabilities)
- Stationarity assumption: transition probabilities the same at all times
- Same as MDP transition model, but no choice of action

# Conditional Independence

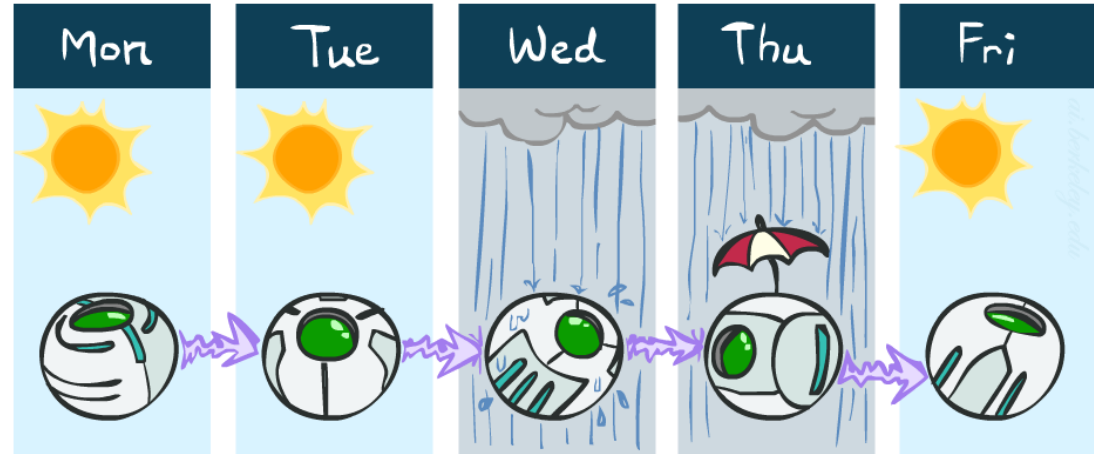


- Basic conditional independence:
  - Past and future independent of the present
  - Each time step only depends on the previous
  - This is called the (first order) Markov property
- Note that the chain is just a (growable) BN
  - We can always use generic BN reasoning on it if we truncate the chain at a fixed length

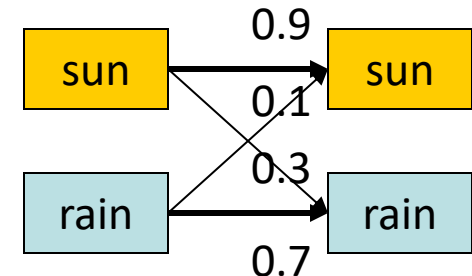
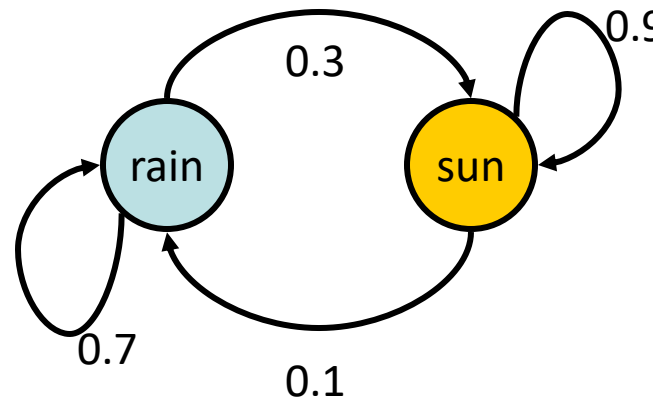
# Example Markov Chain: Weather

- States:  $X = \{\text{rain}, \text{sun}\}$
- Initial distribution: 1.0 sun
- CPT  $P(X_t \mid X_{t-1})$ :

$X_{t-1}$	$X_t$	$P(X_t \mid X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

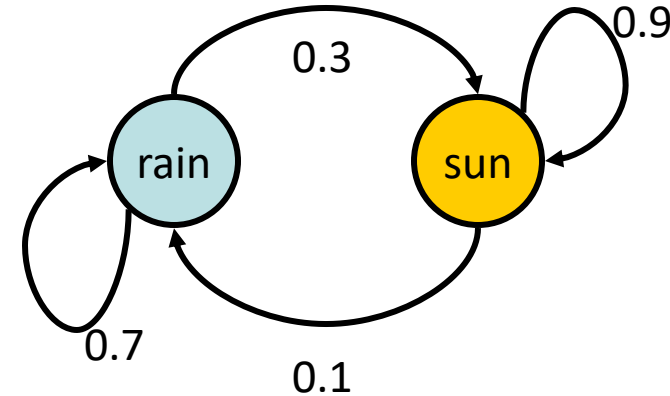


Two new ways of representing the same CPT



# Example Markov Chain: Weather

- Initial distribution: 1.0 sun



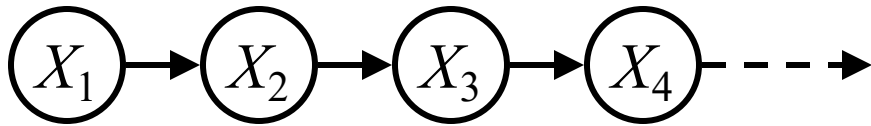
- What is the probability distribution after one step?

$$P(X_2 = \text{sun}) = P(X_2 = \text{sun} | X_1 = \text{sun})P(X_1 = \text{sun}) + P(X_2 = \text{sun} | X_1 = \text{rain})P(X_1 = \text{rain})$$

$$0.9 \cdot 1.0 + 0.3 \cdot 0.0 = 0.9$$

# Mini-Forward Algorithm

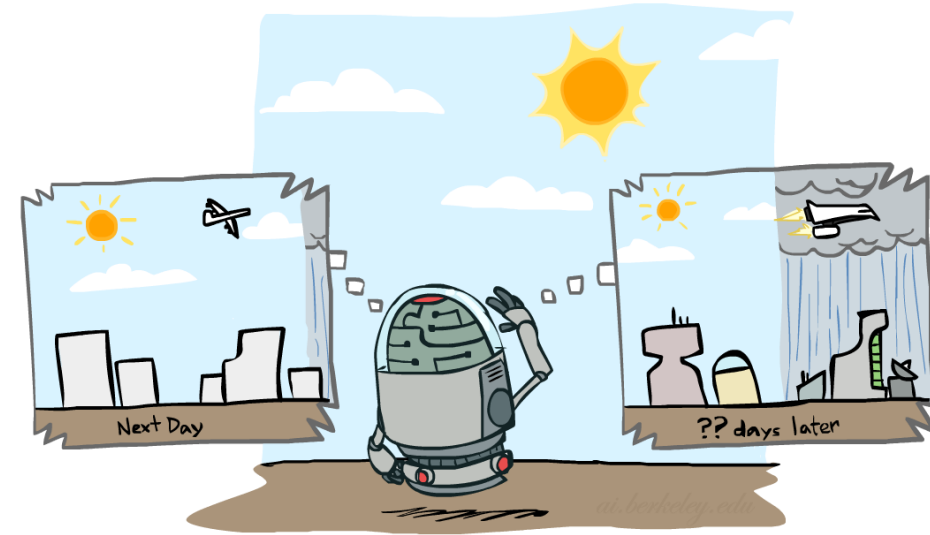
- Question: What's  $P(X)$  on some day  $t$ ?



$P(x_1)$  = known

$$\begin{aligned} P(x_t) &= \sum_{x_{t-1}} P(x_{t-1}, x_t) \\ &= \sum_{x_{t-1}} P(x_t \mid x_{t-1}) P(x_{t-1}) \end{aligned}$$

*Forward simulation*



# Example Run of Mini-Forward Algorithm

- From initial observation of sun

$$\begin{array}{ccccc}
 \left\langle \begin{array}{c} 1.0 \\ 0.0 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.9 \\ 0.1 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.84 \\ 0.16 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.804 \\ 0.196 \end{array} \right\rangle & \longrightarrow \left\langle \begin{array}{c} 0.75 \\ 0.25 \end{array} \right\rangle \\
 P(X_1) & P(X_2) & P(X_3) & P(X_4) & P(X_\infty)
 \end{array}$$

- From initial observation of rain

$$\begin{array}{ccccc}
 \left\langle \begin{array}{c} 0.0 \\ 1.0 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.3 \\ 0.7 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.48 \\ 0.52 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.588 \\ 0.412 \end{array} \right\rangle & \longrightarrow \left\langle \begin{array}{c} 0.75 \\ 0.25 \end{array} \right\rangle \\
 P(X_1) & P(X_2) & P(X_3) & P(X_4) & P(X_\infty)
 \end{array}$$

- From yet another initial distribution  $P(X_1)$ :

$$\begin{array}{ccc}
 \left\langle \begin{array}{c} p \\ 1 - p \end{array} \right\rangle & \dots & \longrightarrow \left\langle \begin{array}{c} 0.75 \\ 0.25 \end{array} \right\rangle \\
 P(X_1) & & P(X_\infty)
 \end{array}$$



# Video of Demo Ghostbusters Basic Dynamics

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# Video of Demo Ghostbusters Circular Dynamics

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# Video of Demo Ghostbusters Whirlpool Dynamics

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# Stationary Distributions

- For most chains:

- Influence of the initial distribution gets less and less over time.
- The distribution we end up in is independent of the initial distribution

- Stationary distribution:

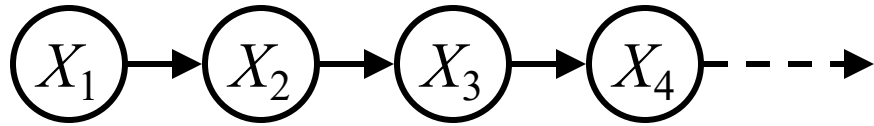
- The distribution we end up with is called the **stationary distribution**  $P_\infty$  of the chain
- It satisfies

$$P_\infty(X) = P_{\infty+1}(X) = \sum_x P(X|x)P_\infty(x)$$



# Example: Stationary Distributions

- Question: What's  $P(X)$  at time  $t = \text{infinity}$ ?



$$P_{\infty}(\text{sun}) = P(\text{sun}|\text{sun})P_{\infty}(\text{sun}) + P(\text{sun}|\text{rain})P_{\infty}(\text{rain})$$

$$P_{\infty}(\text{rain}) = P(\text{rain}|\text{sun})P_{\infty}(\text{sun}) + P(\text{rain}|\text{rain})P_{\infty}(\text{rain})$$

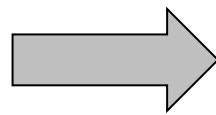
$$P_{\infty}(\text{sun}) = 0.9P_{\infty}(\text{sun}) + 0.3P_{\infty}(\text{rain})$$

$$P_{\infty}(\text{rain}) = 0.1P_{\infty}(\text{sun}) + 0.7P_{\infty}(\text{rain})$$

$$P_{\infty}(\text{sun}) = 3P_{\infty}(\text{rain})$$

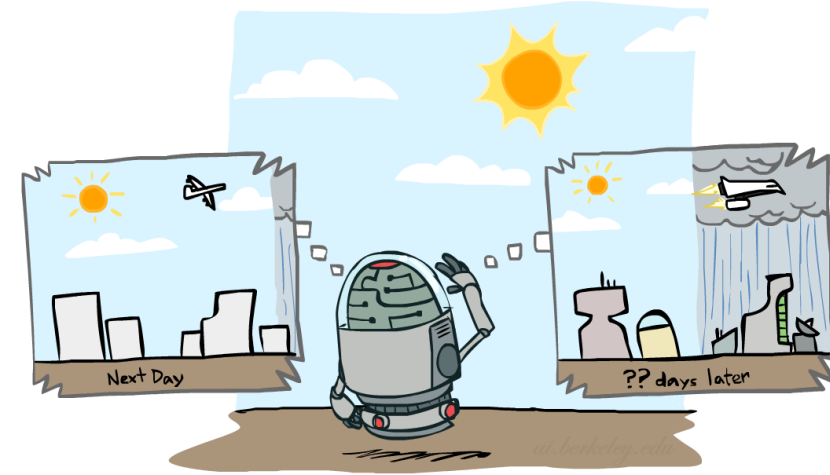
$$P_{\infty}(\text{rain}) = 1/3P_{\infty}(\text{sun})$$

Also:  $P_{\infty}(\text{sun}) + P_{\infty}(\text{rain}) = 1$



$$P_{\infty}(\text{sun}) = 3/4$$

$$P_{\infty}(\text{rain}) = 1/4$$

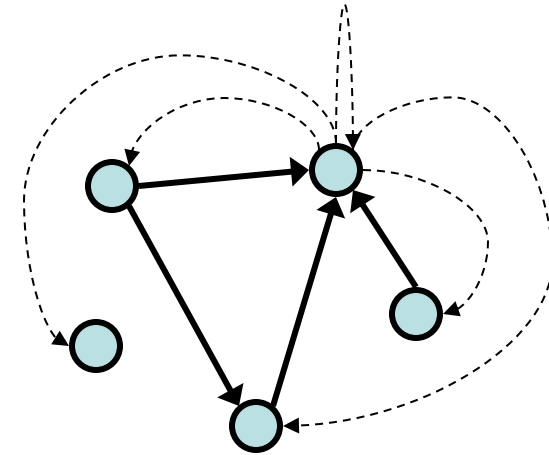


$X_{t-1}$	$X_t$	$P(X_t X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

# Application of Stationary Distribution: Web Link Analysis

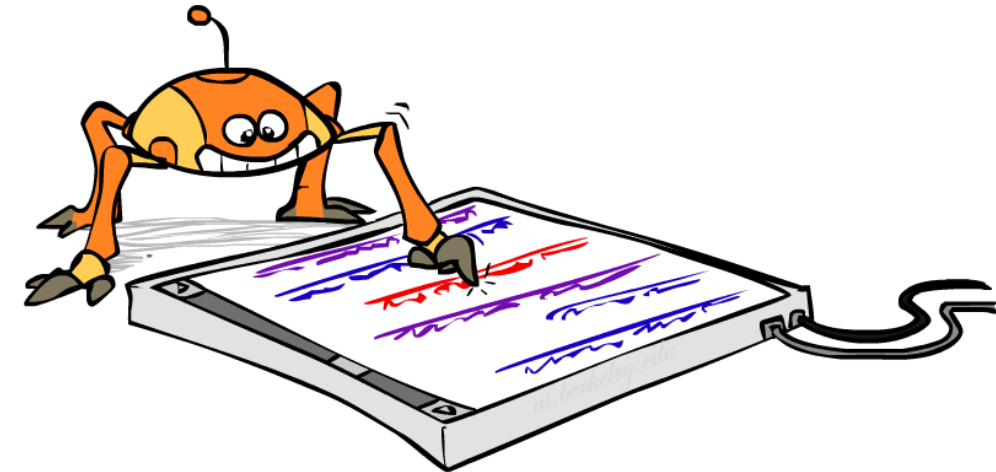
- PageRank over a web graph

- Each web page is a state
- Initial distribution: uniform over pages
- Transitions:
  - With prob.  $c$ , uniform jump to a random page (dotted lines, not all shown)
  - With prob.  $1-c$ , follow a random outlink (solid lines)

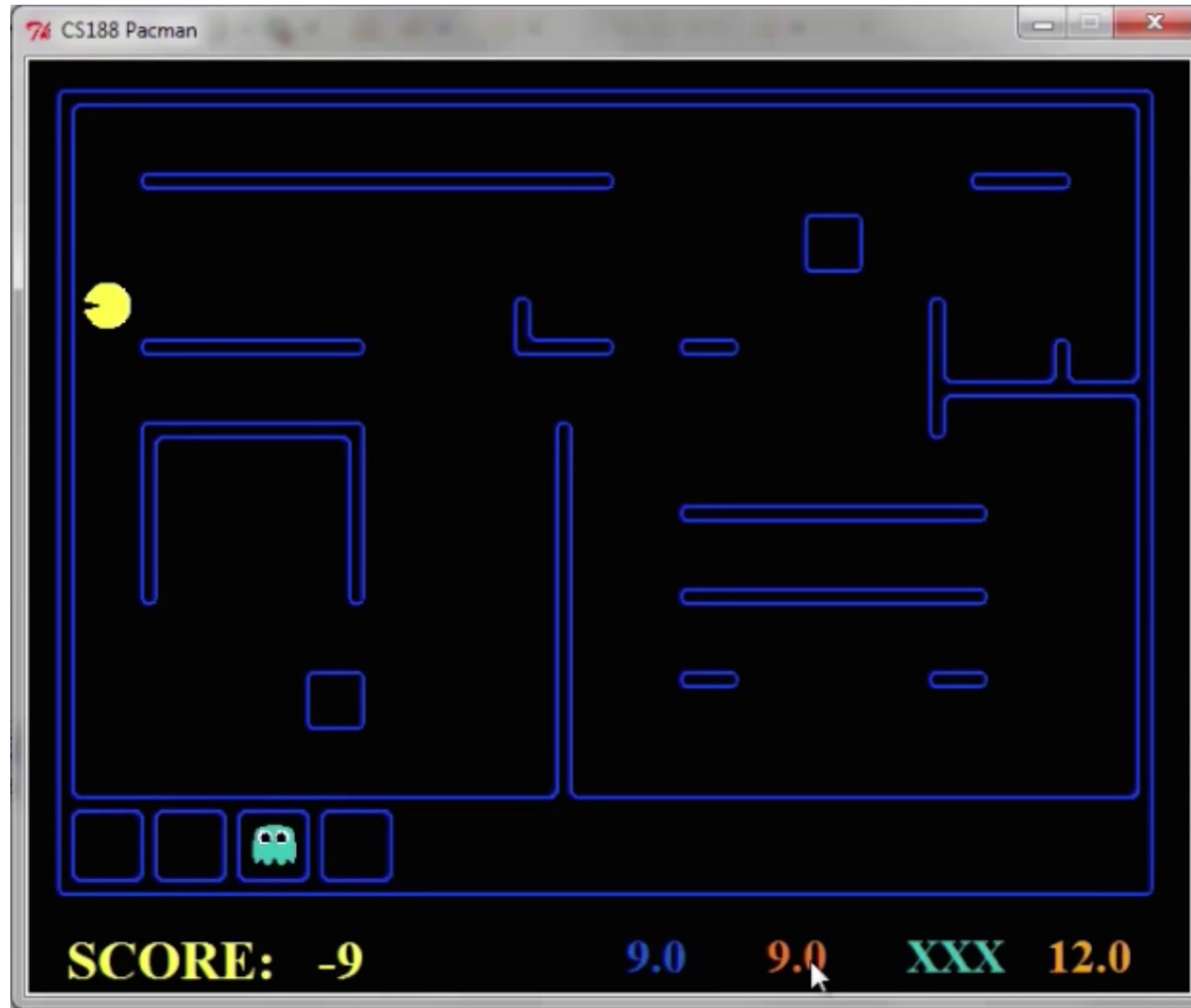


- Stationary distribution

- Will spend more time on highly reachable pages
- E.g. many ways to get to the Acrobat Reader download page
- Somewhat robust to link spam
- Google 1.0 returned the set of pages containing all your keywords in decreasing rank, now all search engines use link analysis along with many other factors (rank actually getting less important over time)



# Pacman – Sonar (P4)



# Video of Demo Pacman – Sonar (no beliefs)

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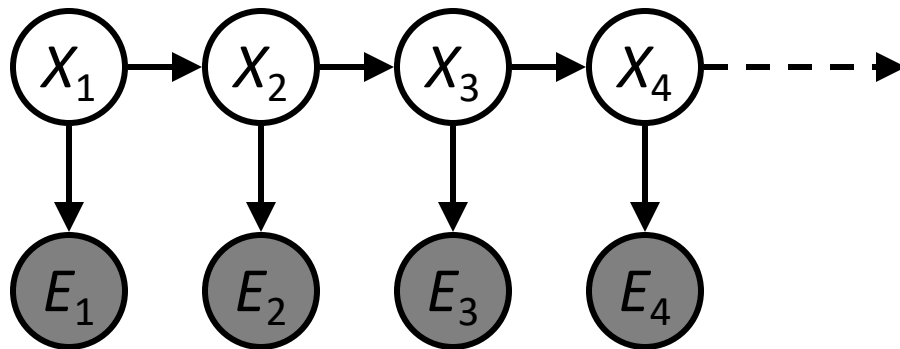
# Hidden Markov Models

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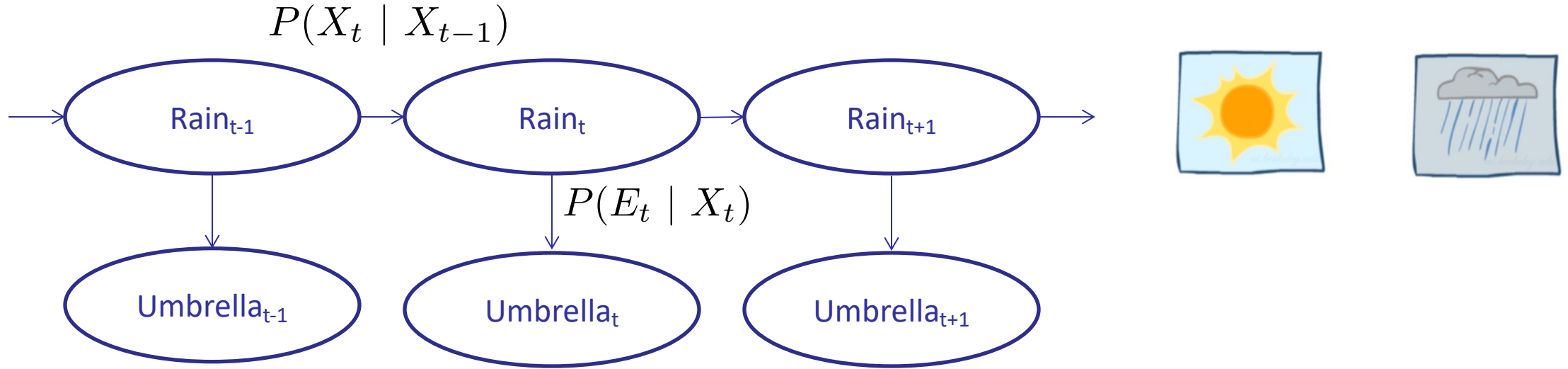


# Hidden Markov Models

- Markov chains not so useful for most agents
  - Need observations to update your beliefs
- Hidden Markov models (HMMs)
  - Underlying Markov chain over states  $X$
  - You observe outputs (effects) at each time step



# Example: Weather HMM



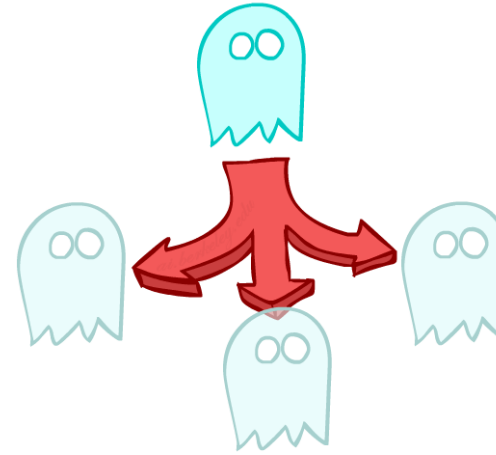
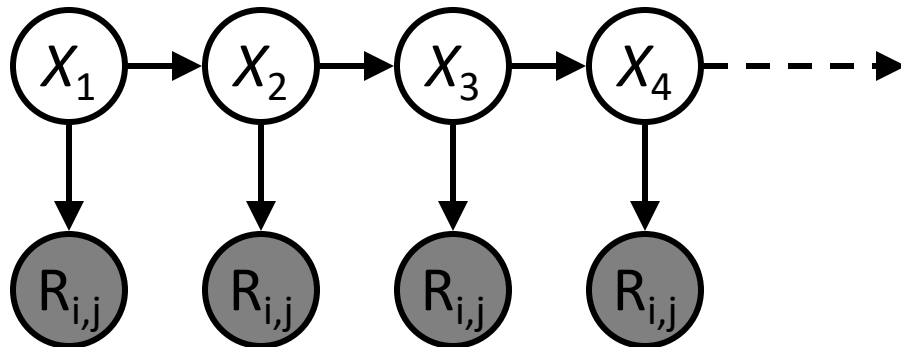
- An HMM is defined by:
  - Initial distribution:  $P(X_1)$
  - Transitions:  $P(X_t | X_{t-1})$
  - Emissions:  $P(E_t | X_t)$

$R_t$	$R_{t+1}$	$P(R_{t+1}   R_t)$
+r	+r	0.7
+r	-r	0.3
-r	+r	0.3
-r	-r	0.7

$R_t$	$U_t$	$P(U_t   R_t)$
+r	+u	0.9
+r	-u	0.1
-r	+u	0.2
-r	-u	0.8

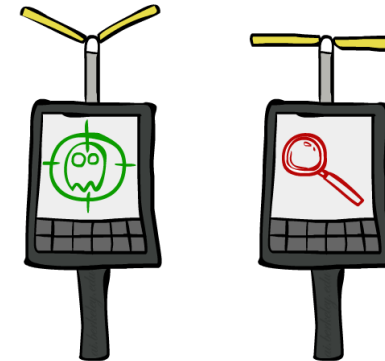
# Example: Ghostbusters HMM

- $P(X_1)$  = uniform
- $P(X|X')$  = usually move clockwise, but sometimes move in a random direction or stay in place
- $P(R_{ij}|X)$  = same sensor model as before: red means close, green means far away.



1/9	1/9	1/9
1/9	1/9	1/9
1/9	1/9	1/9

$P(X_1)$



1/6	1/6	1/2
0	1/6	0
0	0	0

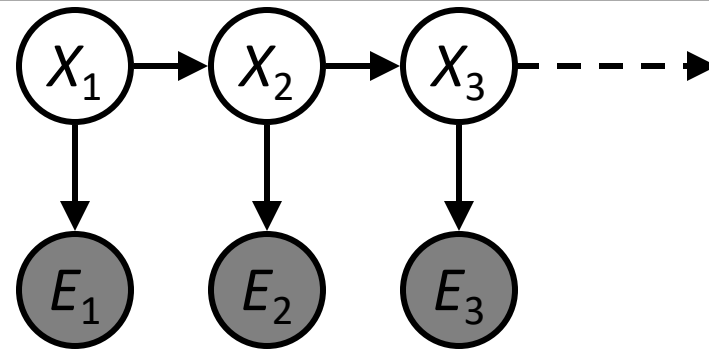
$P(X|X' = \langle 1, 2 \rangle)$

# Video of Demo Ghostbusters – Circular Dynamics -- HMM

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# Joint Distribution of an HMM



- Joint distribution:

$$P(X_1, E_1, X_2, E_2, X_3, E_3) = P(X_1)P(E_1|X_1)P(X_2|X_1)P(E_2|X_2)P(X_3|X_2)P(E_3|X_3)$$

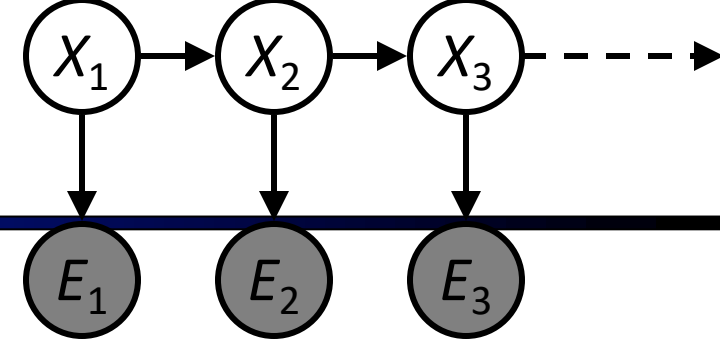
- More generally:

$$P(X_1, E_1, \dots, X_T, E_T) = P(X_1)P(E_1|X_1) \prod_{t=2}^T P(X_t|X_{t-1})P(E_t|X_t)$$

- Questions to be resolved:

- Does this indeed define a joint distribution?
- Can every joint distribution be factored this way, or are we making some assumptions about the joint distribution by using this factorization?

# Chain Rule and HMMs



- From the chain rule, *every* joint distribution over  $X_1, E_1, X_2, E_2, X_3, E_3$  can be written as:

$$P(X_1, E_1, X_2, E_2, X_3, E_3) = P(X_1)P(E_1|X_1)P(X_2|X_1, E_1)P(E_2|X_1, E_1, X_2) \\ P(X_3|X_1, E_1, X_2, E_2)P(E_3|X_1, E_1, X_2, E_2, X_3)$$

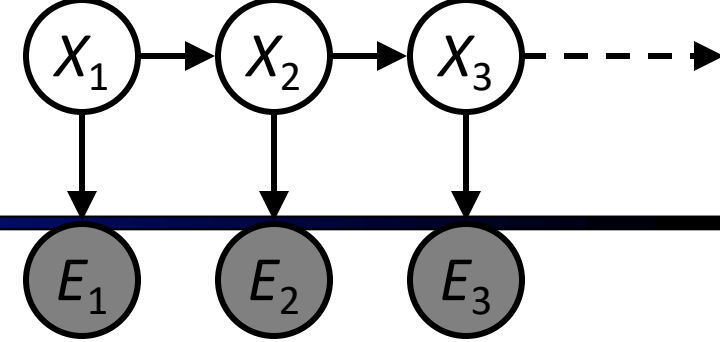
- Assuming that*

$$X_2 \perp\!\!\!\perp E_1 \mid X_1, \quad E_2 \perp\!\!\!\perp X_1, E_1 \mid X_2, \quad X_3 \perp\!\!\!\perp X_1, E_1, E_2 \mid X_2, \quad E_3 \perp\!\!\!\perp X_1, E_1, X_2, E_2 \mid X_3$$

*gives us the expression posited on the previous slide:*

$$P(X_1, E_1, X_2, E_2, X_3, E_3) = P(X_1)P(E_1|X_1)P(X_2|X_1)P(E_2|X_2)P(X_3|X_2)P(E_3|X_3)$$

# Chain Rule and HMMs



- From the chain rule, *every* joint distribution over  $X_1, E_1, \dots, X_T, E_T$  can be written as:

$$P(X_1, E_1, \dots, X_T, E_T) = P(X_1)P(E_1|X_1) \prod_{t=2}^T P(X_t|X_1, E_1, \dots, X_{t-1}, E_{t-1})P(E_t|X_1, E_1, \dots, X_{t-1}, E_{t-1}, X_t)$$

- Assuming* that for all  $t$ :

- State independent of all past states and all past evidence given the previous state, i.e.:

$$X_t \perp\!\!\!\perp X_1, E_1, \dots, X_{t-2}, E_{t-2}, E_{t-1} \mid X_{t-1}$$

- Evidence is independent of all past states and all past evidence given the current state, i.e.:

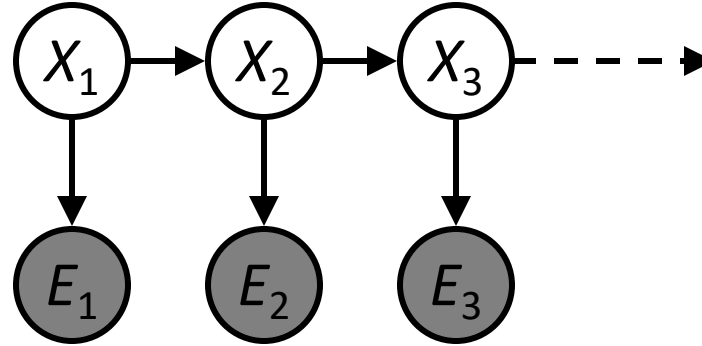
$$E_t \perp\!\!\!\perp X_1, E_1, \dots, X_{t-2}, E_{t-2}, X_{t-1}, E_{t-1} \mid X_t$$

gives us the expression posited on the earlier slide:

$$P(X_1, E_1, \dots, X_T, E_T) = P(X_1)P(E_1|X_1) \prod_{t=2}^T P(X_t|X_{t-1})P(E_t|X_t)$$



# Implied Conditional Independencies



- Many implied conditional independencies, e.g.,

$$E_1 \perp\!\!\!\perp X_2, E_2, X_3, E_3 \mid X_1$$

- To prove them

- Approach 1: follow similar (algebraic) approach to what we did in the Markov models lecture
- Approach 2: directly from the graph structure (3 lectures from now)
  - Intuition: If path between U and V goes through W, then  $U \perp\!\!\!\perp V \mid W$  [Some fineprint later]

# Real HMM Examples

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- **Speech recognition HMMs:**
  - Observations are acoustic signals (continuous valued)
  - States are specific positions in specific words (so, tens of thousands)
- **Machine translation HMMs:**
  - Observations are words (tens of thousands)
  - States are translation options
- **Robot tracking:**
  - Observations are range readings (continuous)
  - States are positions on a map (continuous)

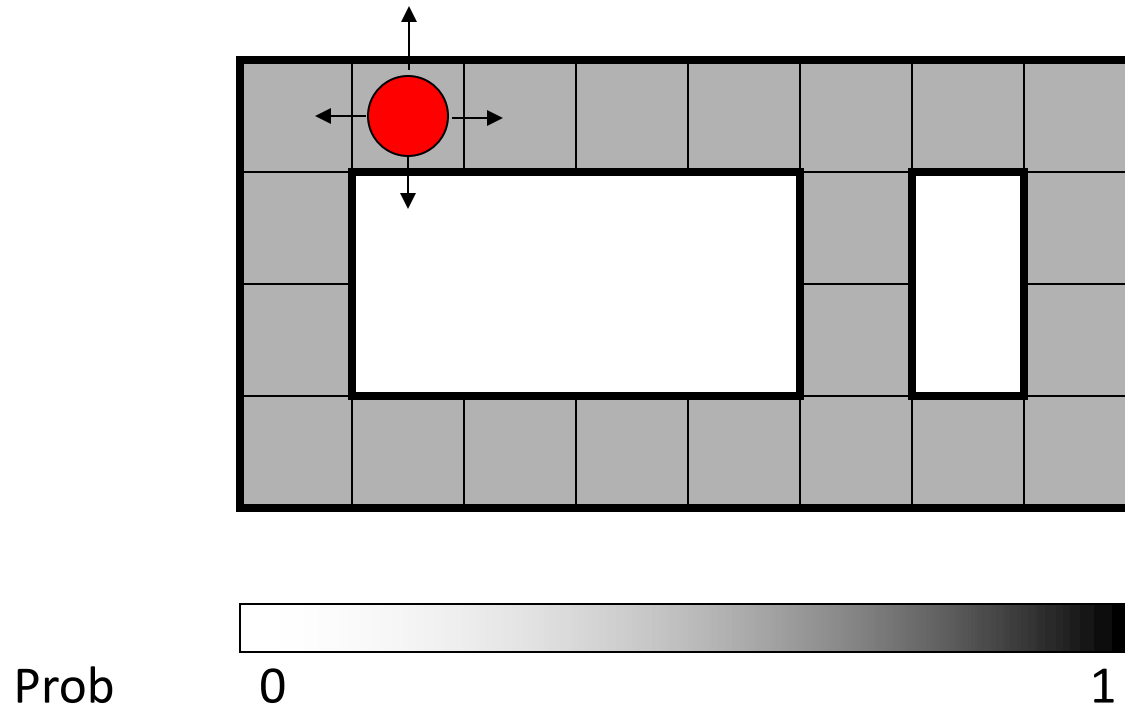
# Filtering / Monitoring

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- Filtering, or monitoring, is the task of tracking the distribution  $B_t(X) = P_t(X_t \mid e_1, \dots, e_t)$  (the belief state) over time
- We start with  $B_1(X)$  in an initial setting, usually uniform
- As time passes, or we get observations, we update  $B(X)$
- The Kalman filter was invented in the 60's and first implemented as a method of trajectory estimation for the Apollo program

# Example: Robot Localization

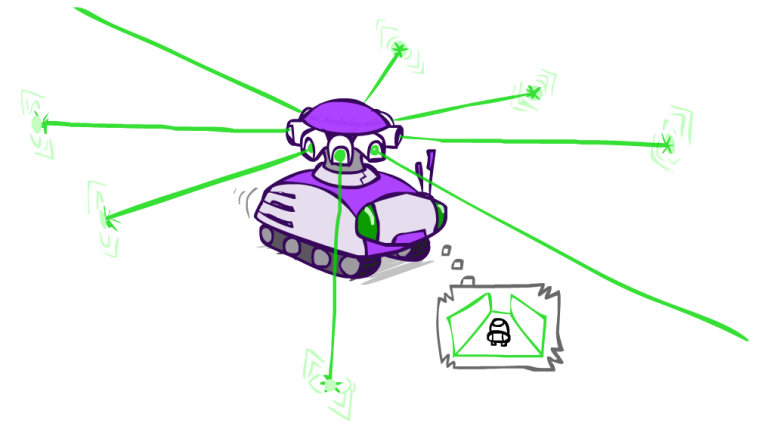
*Example from  
Michael Pfeiffer*



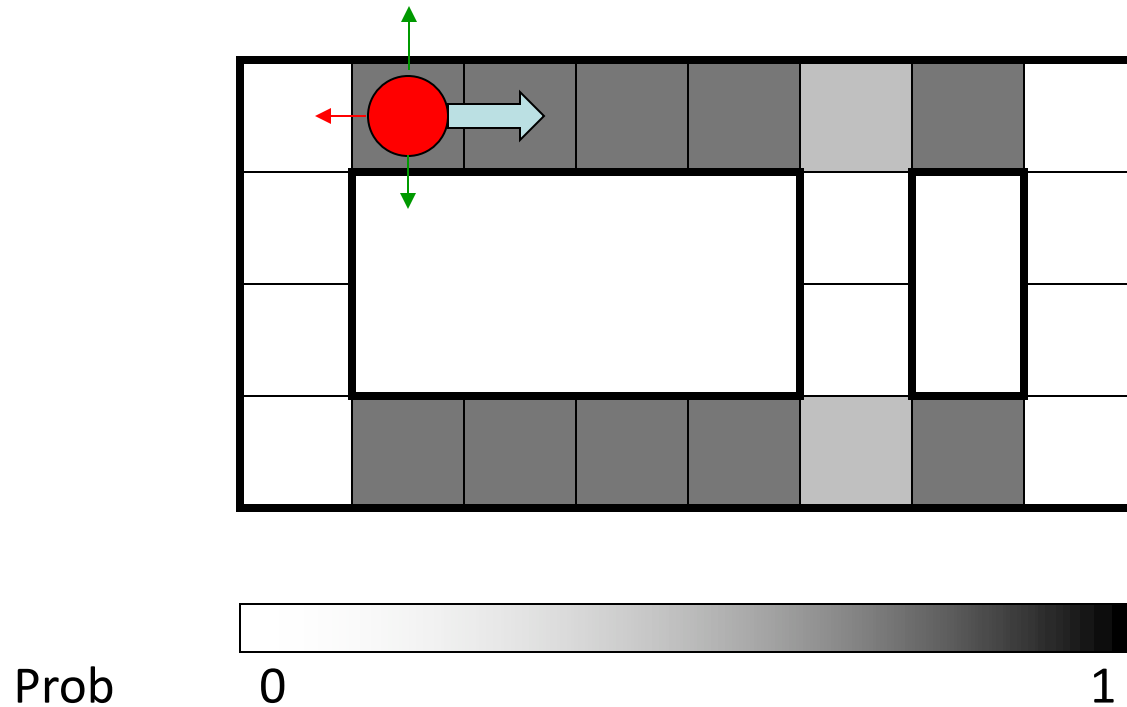
$t=0$

Sensor model: can read in which directions there is a wall,  
never more than 1 mistake

Motion model: may not execute action with small prob.

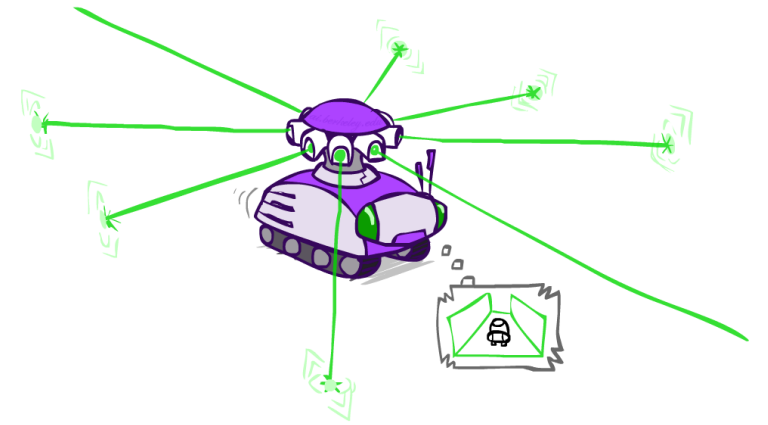


# Example: Robot Localization

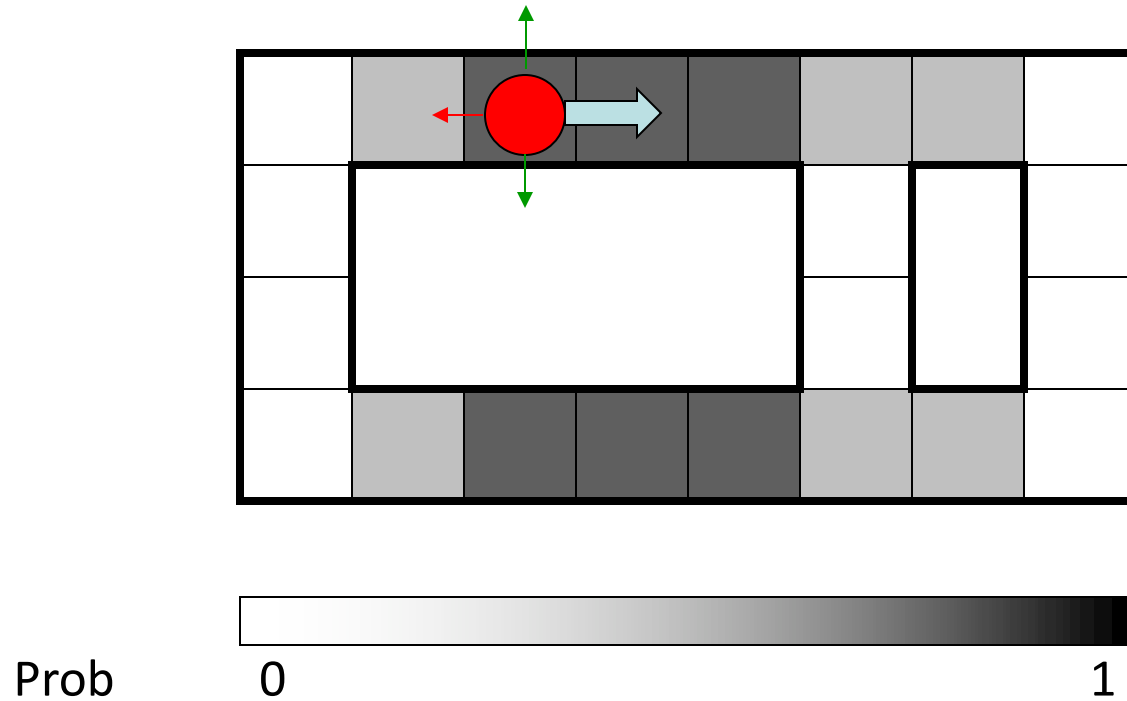


$t=1$

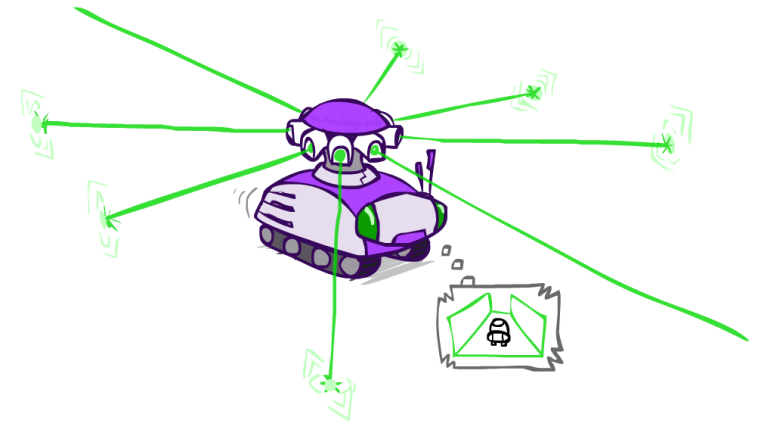
Lighter grey: was possible to get the reading, but less likely b/c required 1 mistake



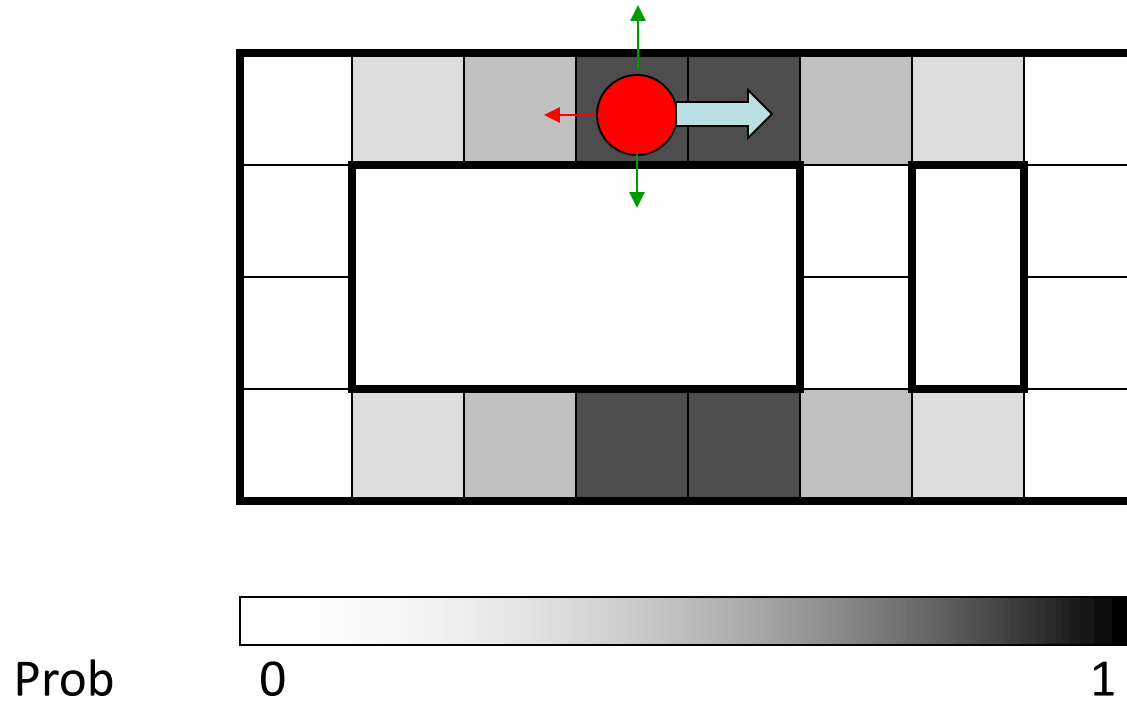
# Example: Robot Localization



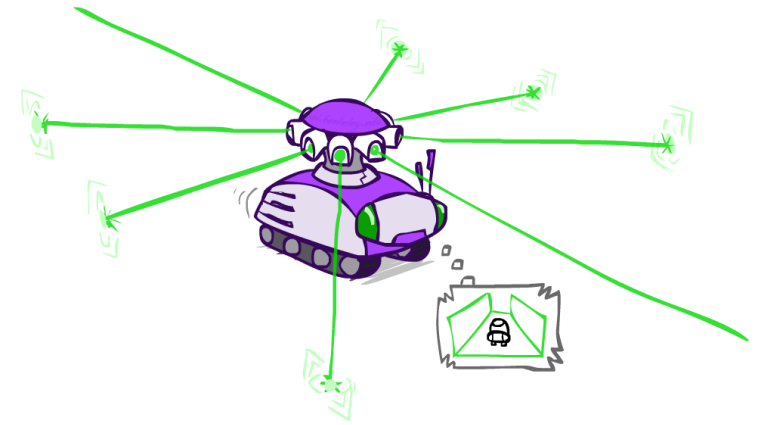
$t=2$



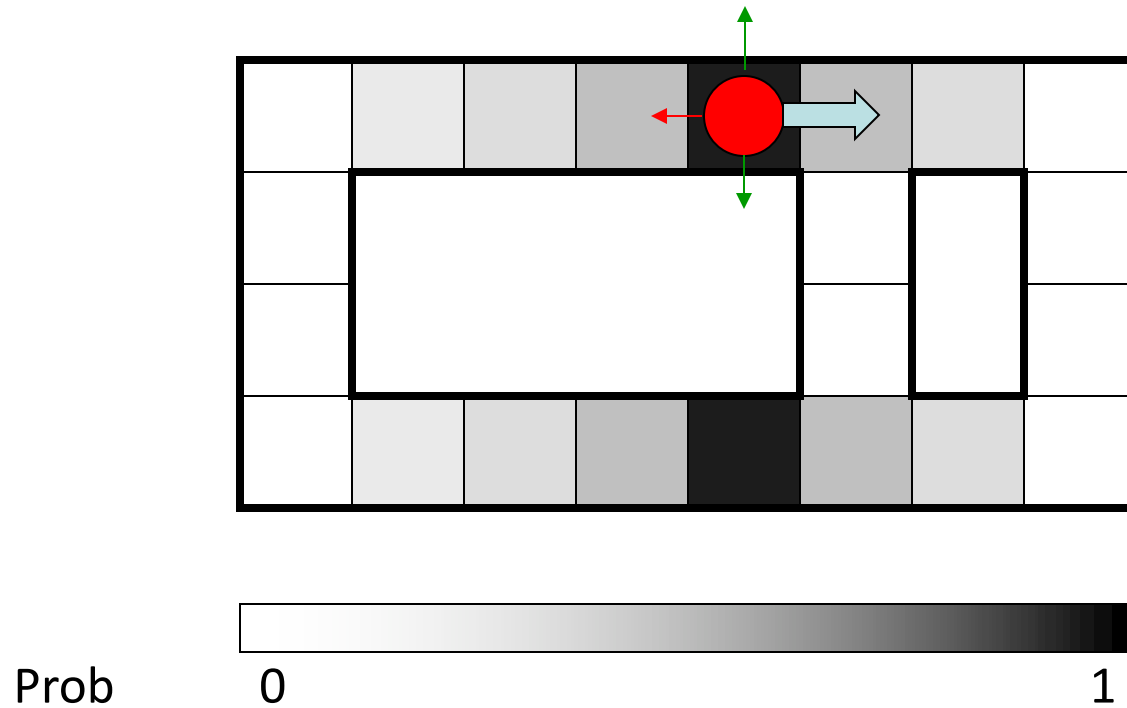
# Example: Robot Localization



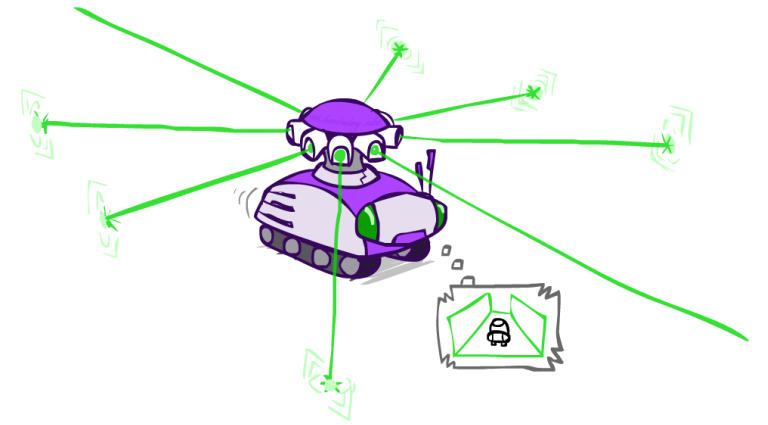
$t=3$



# Example: Robot Localization

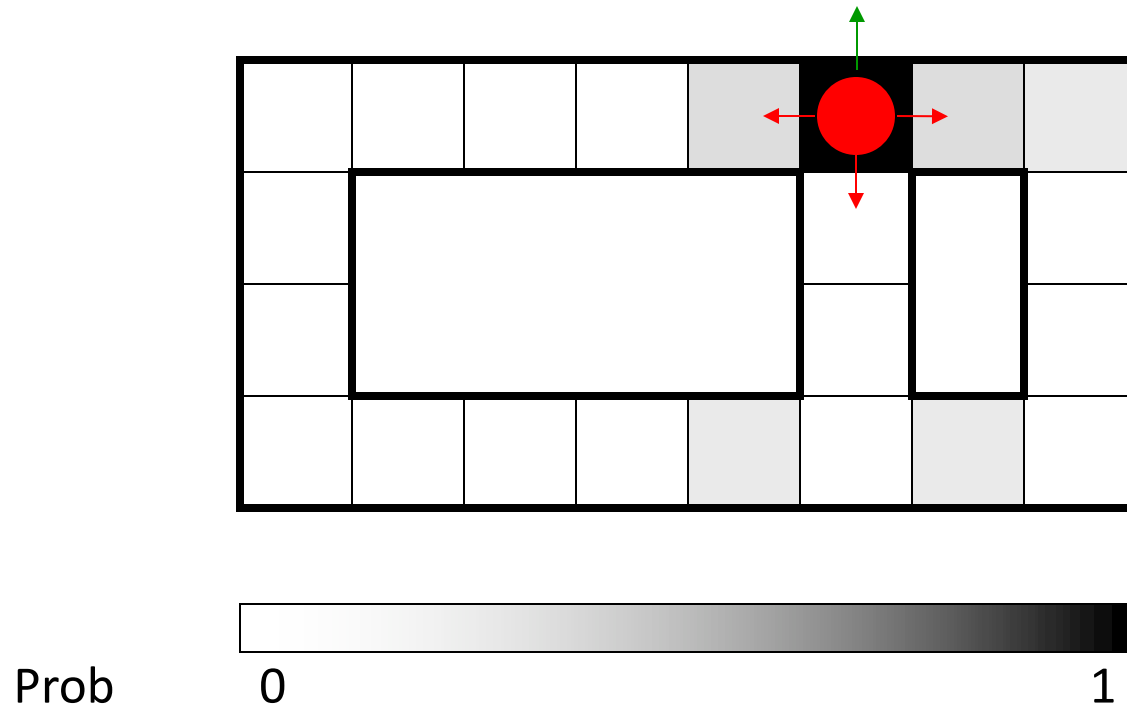


$t=4$

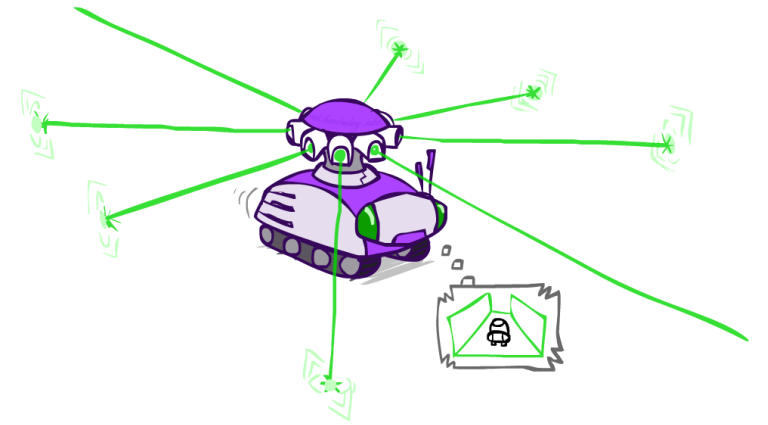




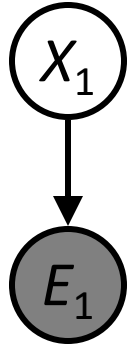
# Example: Robot Localization



t=5

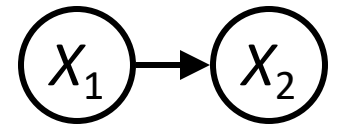
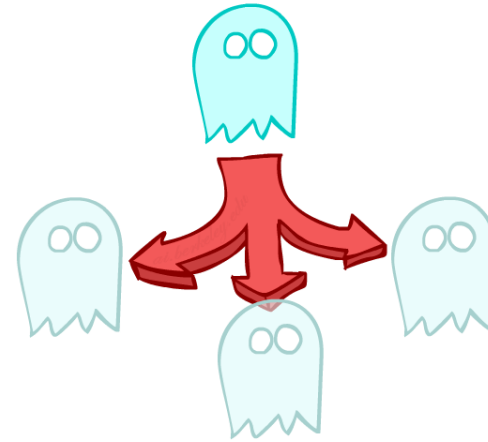


# Inference: Base Cases



$$P(X_1|e_1)$$

$$\begin{aligned} P(x_1|e_1) &= P(x_1, e_1)/P(e_1) \\ &\propto_{X_1} P(x_1, e_1) \\ &= P(x_1)P(e_1|x_1) \end{aligned}$$



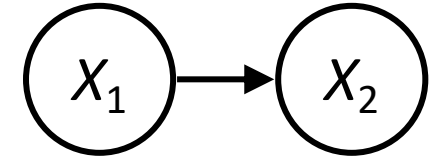
$$P(X_2)$$

$$\begin{aligned} P(x_2) &= \sum_{x_1} P(x_1, x_2) \\ &= \sum_{x_1} P(x_1)P(x_2|x_1) \end{aligned}$$

# Passage of Time

- Assume we have current belief  $P(X \mid \text{evidence to date})$

$$B(X_t) = P(X_t | e_{1:t})$$



- Then, after one time step passes:

$$\begin{aligned} P(X_{t+1} | e_{1:t}) &= \sum_{x_t} P(X_{t+1}, x_t | e_{1:t}) \\ &= \sum_{x_t} P(X_{t+1} | x_t, e_{1:t}) P(x_t | e_{1:t}) \\ &= \sum_{x_t} P(X_{t+1} | x_t) P(x_t | e_{1:t}) \end{aligned}$$

- Or compactly:

$$B'(X_{t+1}) = \sum_{x_t} P(X' | x_t) B(x_t)$$

- Basic idea: beliefs get “pushed” through the transitions
  - With the “B” notation, we have to be careful about what time step  $t$  the belief is about, and what evidence it includes

# Example: Passage of Time

- As time passes, uncertainty “accumulates”

(Transition model: ghosts usually go clockwise)

<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
<0.01	<0.01	1.00	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01

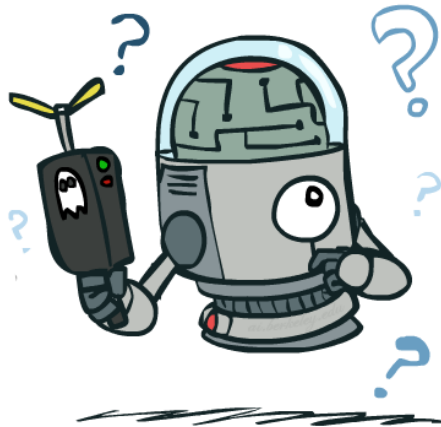
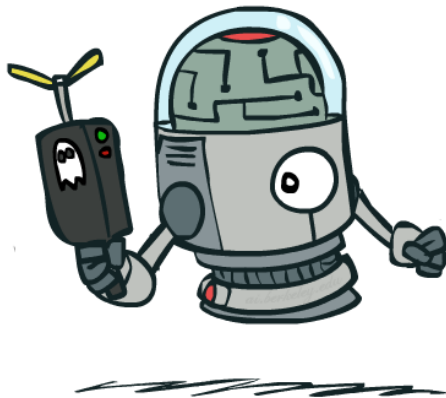
T = 1

<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
<0.01	<0.01	0.06	<0.01	<0.01	<0.01
<0.01	0.76	0.06	0.06	<0.01	<0.01
<0.01	<0.01	0.06	<0.01	<0.01	<0.01

T = 2

0.05	0.01	0.05	<0.01	<0.01	<0.01
0.02	0.14	0.11	0.35	<0.01	<0.01
0.07	0.03	0.05	<0.01	0.03	<0.01
0.03	0.03	<0.01	<0.01	<0.01	<0.01

T = 5



# Observation

- Assume we have current belief  $P(X \mid \text{previous evidence})$ :

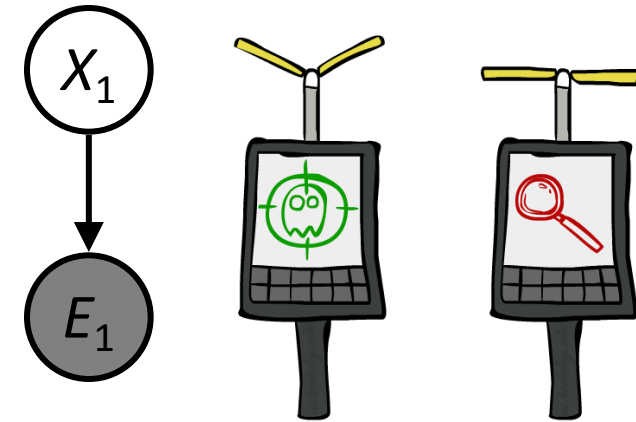
$$B'(X_{t+1}) = P(X_{t+1} | e_{1:t})$$

- Then, after evidence comes in:

$$\begin{aligned} P(X_{t+1} | e_{1:t+1}) &= P(X_{t+1}, e_{t+1} | e_{1:t}) / P(e_{t+1} | e_{1:t}) \\ &\propto_{X_{t+1}} P(X_{t+1}, e_{t+1} | e_{1:t}) \\ &= P(e_{t+1} | e_{1:t}, X_{t+1}) P(X_{t+1} | e_{1:t}) \\ &= P(e_{t+1} | X_{t+1}) P(X_{t+1} | e_{1:t}) \end{aligned}$$

- Or, compactly:

$$B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1} | X_{t+1}) B'(X_{t+1})$$



- Basic idea: beliefs “reweighted” by likelihood of evidence
- Unlike passage of time, we have to renormalize

# Example: Observation

- As we get observations, beliefs get reweighted, uncertainty “decreases”

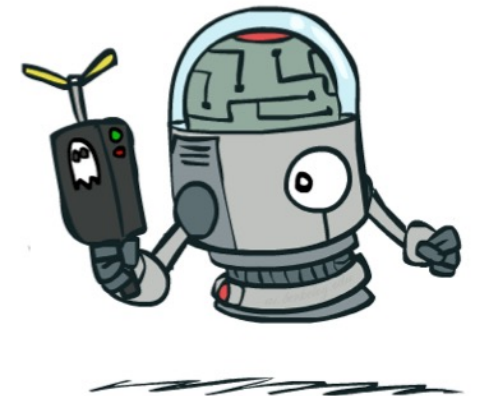
0.05	0.01	0.05	<0.01	<0.01	<0.01
0.02	0.14	0.11	0.35	<0.01	<0.01
0.07	0.03	0.05	<0.01	0.03	<0.01
0.03	0.03	<0.01	<0.01	<0.01	<0.01

Before observation

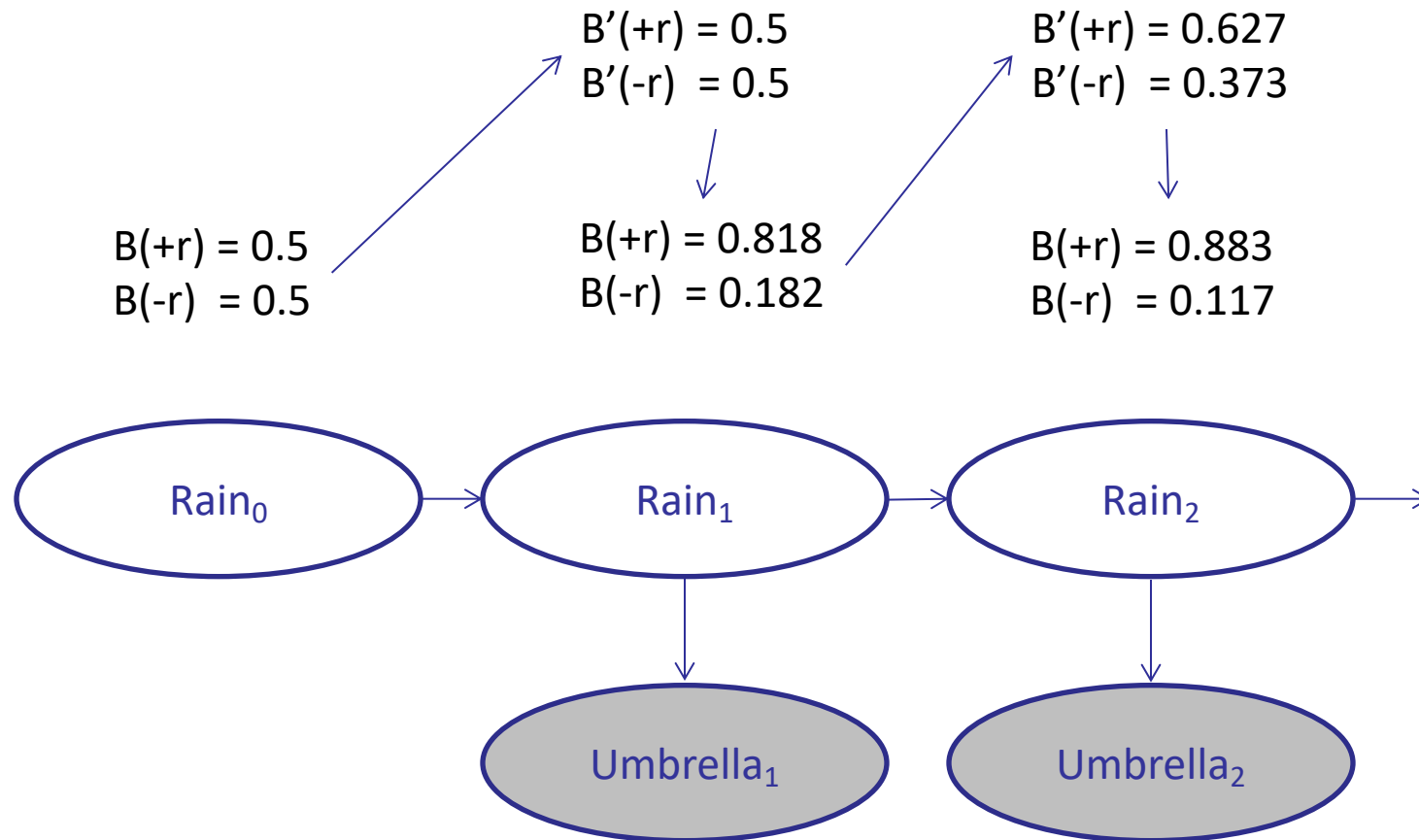
<0.01	<0.01	<0.01	<0.01	0.02	<0.01
<0.01	<0.01	<0.01	0.83	0.02	<0.01
<0.01	<0.01	0.11	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01

After observation

$$B(X) \propto P(e|X)B'(X)$$



# Example: Weather HMM



$R_t$	$R_{t+1}$	$P(R_{t+1}   R_t)$
+r	+r	0.7
+r	-r	0.3
-r	+r	0.3
-r	-r	0.7

$R_t$	$U_t$	$P(U_t   R_t)$
+r	+u	0.9
+r	-u	0.1
-r	+u	0.2
-r	-u	0.8

# The Forward Algorithm

- We are given evidence at each time and want to know

$$B_t(X) = P(X_t|e_{1:t})$$

- We can derive the following updates

$$\begin{aligned} P(x_t|e_{1:t}) &\propto_X P(x_t, e_{1:t}) \\ &= \sum_{x_{t-1}} P(x_{t-1}, x_t, e_{1:t}) \\ &= \sum_{x_{t-1}} P(x_{t-1}, e_{1:t-1}) P(x_t|x_{t-1}) P(e_t|x_t) \\ &= P(e_t|x_t) \sum_{x_{t-1}} P(x_t|x_{t-1}) P(x_{t-1}, e_{1:t-1}) \end{aligned}$$

We can normalize as we go if we want to have  $P(x|e)$  at each time step, or just once at the end...



# Online Belief Updates

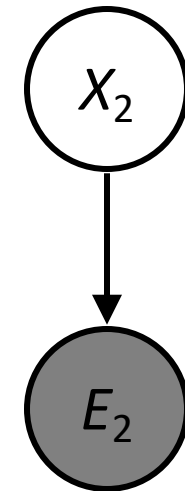
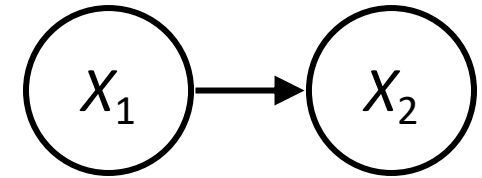
- Every time step, we start with current  $P(X \mid \text{evidence})$
- We update for time:

$$P(x_t | e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1} | e_{1:t-1}) \cdot P(x_t | x_{t-1})$$

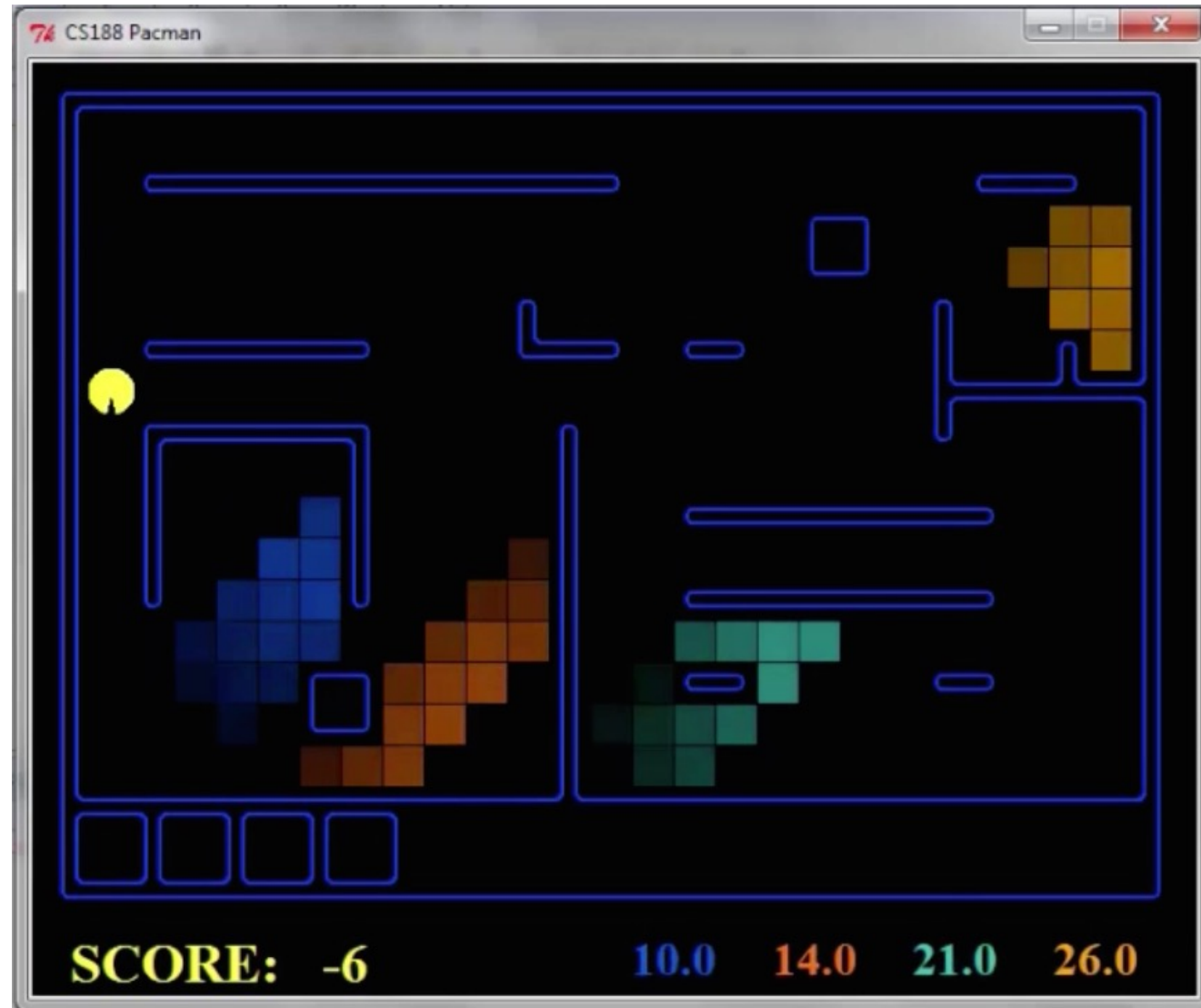
- We update for evidence:

$$P(x_t | e_{1:t}) \propto_X P(x_t | e_{1:t-1}) \cdot P(e_t | x_t)$$

- The forward algorithm does both at once (and doesn't normalize)



# Pacman – Sonar (P4)



[Demo: Pacman – Sonar – No Beliefs(L14D1)]

# Video of Demo Pacman – Sonar (with beliefs)

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# Next Time: Particle Filtering and Applications of HMMs

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