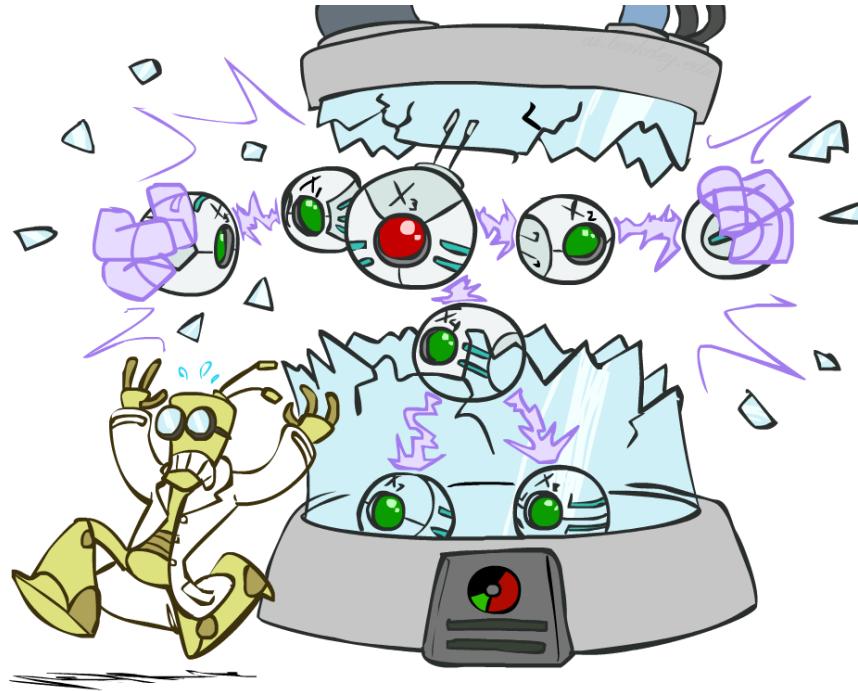


# CMPT 310: Artificial Intelligence

## Bayes' Nets: Independence



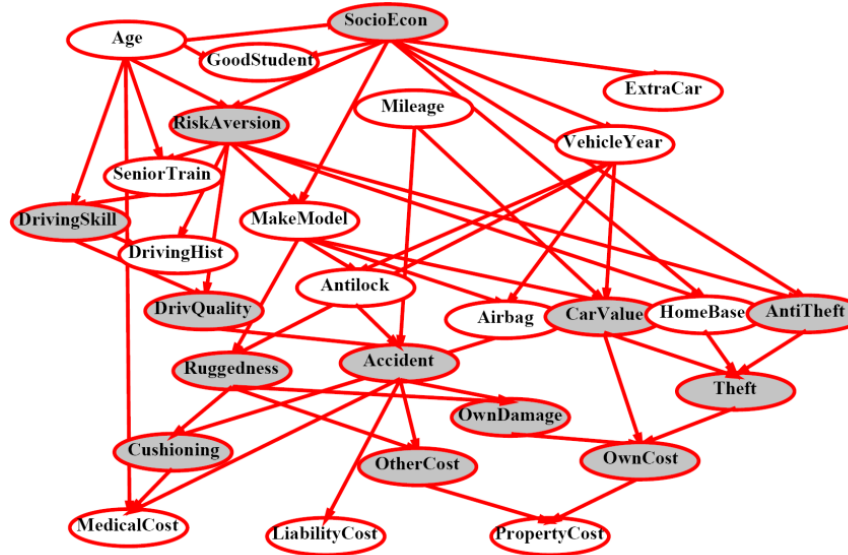
Instructors: Steven Bergner --- Simon Fraser University

# Probability Recap

- Conditional probability  $P(x|y) = \frac{P(x, y)}{P(y)}$
- Product rule  $P(x, y) = P(x|y)P(y)$
- Chain rule 
$$\begin{aligned} P(X_1, X_2, \dots, X_n) &= P(X_1)P(X_2|X_1)P(X_3|X_1, X_2) \dots \\ &= \prod_{i=1}^n P(X_i|X_1, \dots, X_{i-1}) \end{aligned}$$
- X, Y independent if and only if:  $\forall x, y : P(x, y) = P(x)P(y)$
- X and Y are conditionally independent given Z if and only if:  $X \perp\!\!\!\perp Y | Z$   
$$\forall x, y, z : P(x, y|z) = P(x|z)P(y|z)$$

# Bayes' Nets

- A Bayes' net is an efficient encoding of a probabilistic model of a domain



- Questions we can ask:
  - Inference: given a fixed BN, what is  $P(X \mid e)$ ?
  - Representation: given a BN graph, what kinds of distributions can it encode?
  - Modeling: what BN is most appropriate for a given domain?

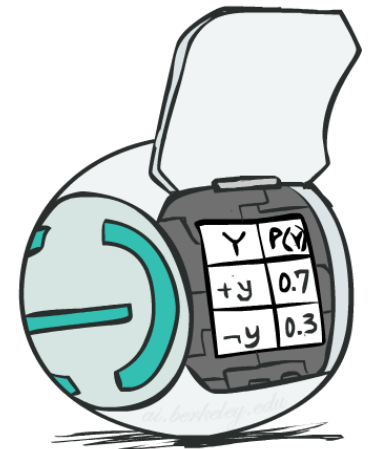
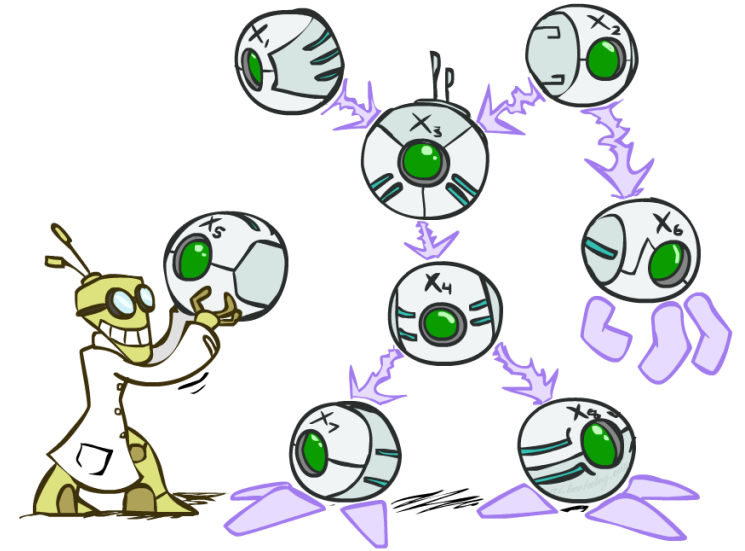
# Bayes' Net Semantics

- A directed, acyclic graph, one node per random variable
- A conditional probability table (CPT) for each node
  - A collection of distributions over  $X$ , one for each combination of parents' values

$$P(X|a_1 \dots a_n)$$

- Bayes' nets implicitly encode joint distributions
  - As a product of local conditional distributions
  - To see what probability a BN gives to a full assignment, multiply all the relevant conditionals together:

$$P(x_1, x_2, \dots, x_n) = \prod_{i=1}^n P(x_i | \text{parents}(X_i))$$



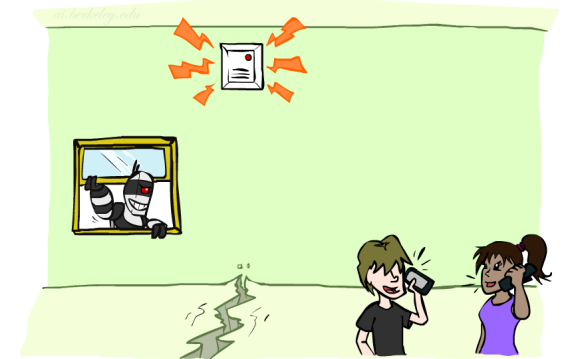
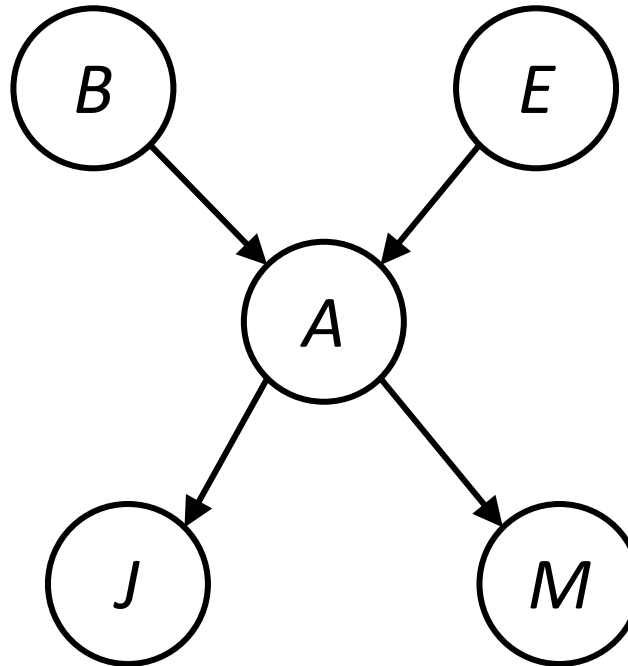
# Example: Alarm Network

| B  | P(B)  |
|----|-------|
| +b | 0.001 |
| -b | 0.999 |

| E  | P(E)  |
|----|-------|
| +e | 0.002 |
| -e | 0.998 |

| A  | J  | P(J A) |
|----|----|--------|
| +a | +j | 0.9    |
| +a | -j | 0.1    |
| -a | +j | 0.05   |
| -a | -j | 0.95   |

| A  | M  | P(M A) |
|----|----|--------|
| +a | +m | 0.7    |
| +a | -m | 0.3    |
| -a | +m | 0.01   |
| -a | -m | 0.99   |

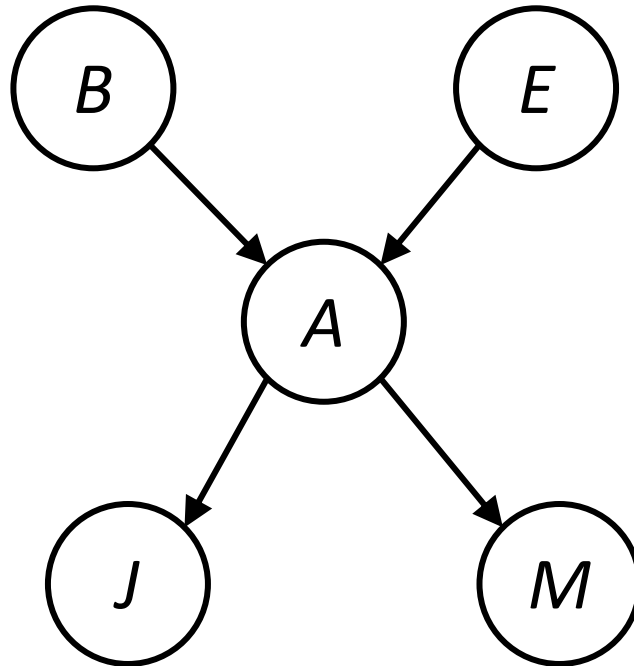


$$P(+b, -e, +a, -j, +m) =$$

| B  | E  | A  | P(A B,E) |
|----|----|----|----------|
| +b | +e | +a | 0.95     |
| +b | +e | -a | 0.05     |
| +b | -e | +a | 0.94     |
| +b | -e | -a | 0.06     |
| -b | +e | +a | 0.29     |
| -b | +e | -a | 0.71     |
| -b | -e | +a | 0.001    |
| -b | -e | -a | 0.999    |

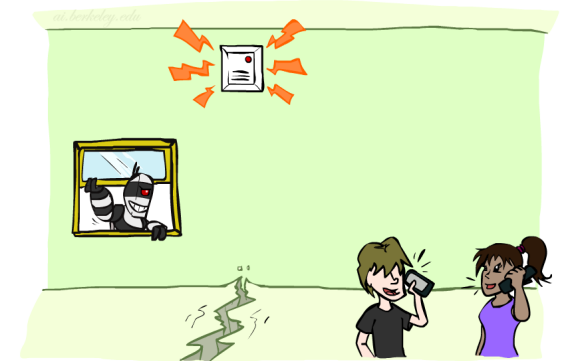
# Example: Alarm Network

| B  | P(B)  |
|----|-------|
| +b | 0.001 |
| -b | 0.999 |



| E  | P(E)  |
|----|-------|
| +e | 0.002 |
| -e | 0.998 |

| A  | M  | P(M A) |
|----|----|--------|
| +a | +m | 0.7    |
| +a | -m | 0.3    |
| -a | +m | 0.01   |
| -a | -m | 0.99   |



| A  | J  | P(J A) |
|----|----|--------|
| +a | +j | 0.9    |
| +a | -j | 0.1    |
| -a | +j | 0.05   |
| -a | -j | 0.95   |

| B  | E  | A  | P(A B,E) |
|----|----|----|----------|
| +b | +e | +a | 0.95     |
| +b | +e | -a | 0.05     |
| +b | -e | +a | 0.94     |
| +b | -e | -a | 0.06     |
| -b | +e | +a | 0.29     |
| -b | +e | -a | 0.71     |
| -b | -e | +a | 0.001    |
| -b | -e | -a | 0.999    |

$$\begin{aligned}
 P(+b, -e, +a, -j, +m) &= \\
 P(+b)P(-e)P(+a|+b, -e)P(-j|+a)P(+m|+a) &= \\
 0.001 \times 0.998 \times 0.94 \times 0.1 \times 0.7 &
 \end{aligned}$$

# Size of a Bayes' Net

- How big is a joint distribution over  $N$  Boolean variables?

$$2^N$$

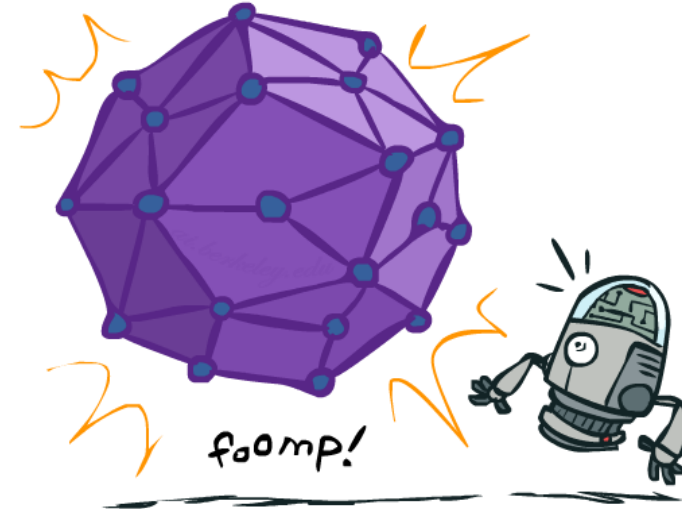
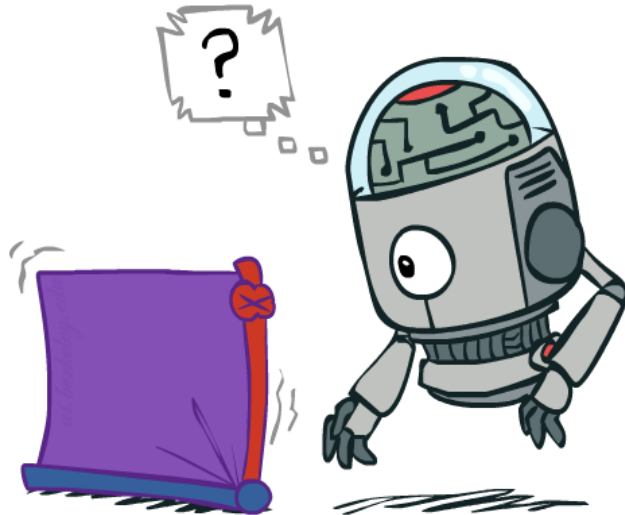
- How big is an  $N$ -node net if nodes have up to  $k$  parents?

$$O(N * 2^{k+1})$$

- Both give you the power to calculate

$$P(X_1, X_2, \dots, X_n)$$

- BNs: Huge space savings!
- Also easier to elicit local CPTs
- Also faster to answer queries (coming)



# Bayes' Nets

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## Representation

- Conditional Independences
- Probabilistic Inference
- Learning Bayes' Nets from Data



# Conditional Independence

- X and Y are independent if

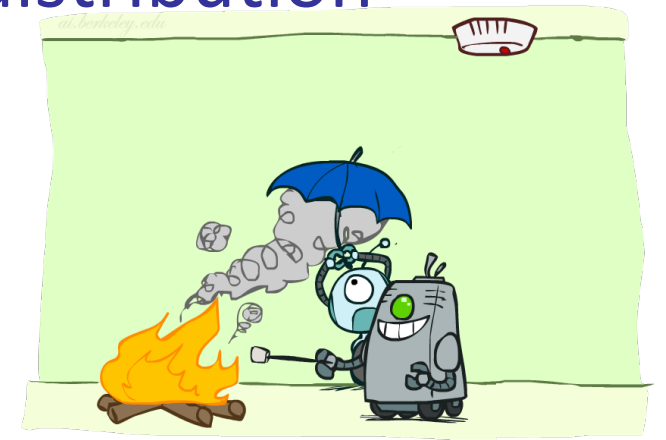
$$\forall x, y \quad P(x, y) = P(x)P(y) \quad \text{---} \rightarrow \quad X \perp Y$$

- X and Y are conditionally independent given Z

$$\forall x, y, z \quad P(x, y|z) = P(x|z)P(y|z) \quad \text{---} \rightarrow \quad X \perp Y|Z$$

- (Conditional) independence is a property of a distribution

- Example:  $Alarm \perp Fire|Smoke$

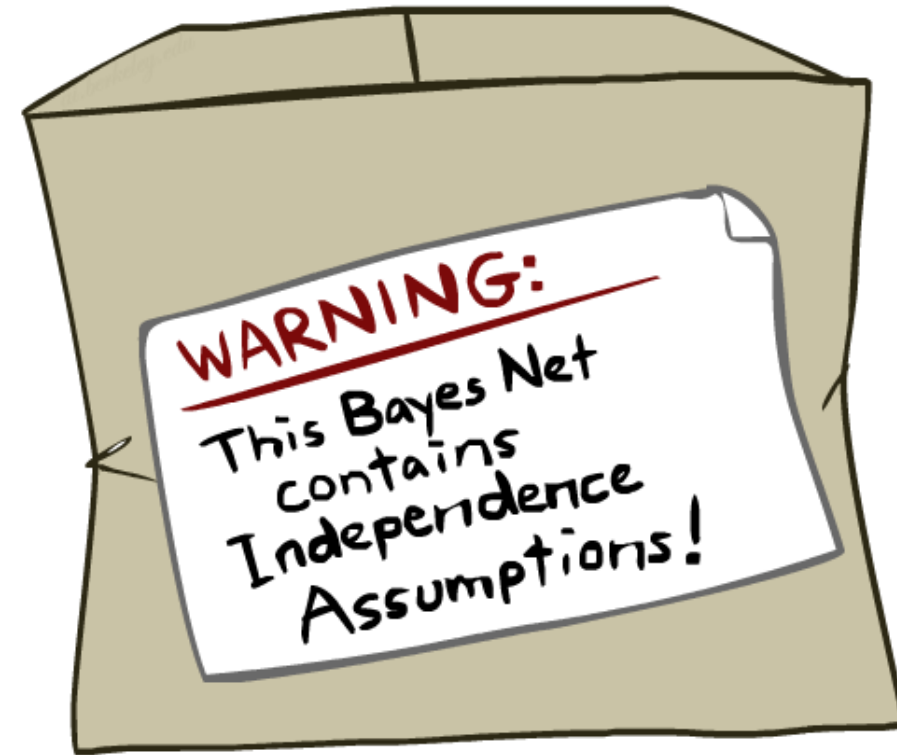


# Bayes Nets: Assumptions

- Assumptions we are required to make to define the Bayes net when given the graph:

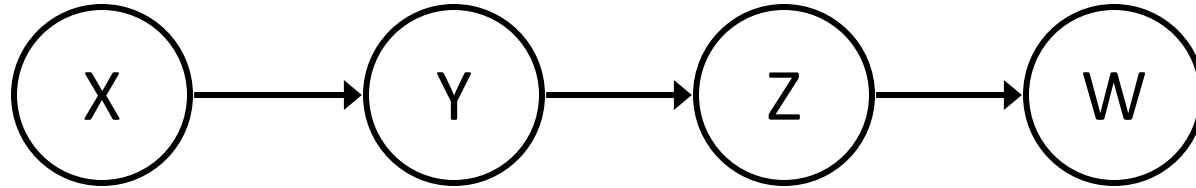
$$P(x_i | x_1 \cdots x_{i-1}) = P(x_i | \text{parents}(X_i))$$

- Beyond above “chain rule → Bayes net” conditional independence assumptions
  - Often additional conditional independences
  - They can be read off the graph
- Important for modeling: understand assumptions made when choosing a Bayes net graph



# Example

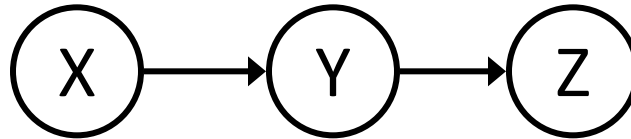
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- Conditional independence assumptions directly from simplifications in chain rule:
- Additional implied conditional independence assumptions?

# Independence in a BN

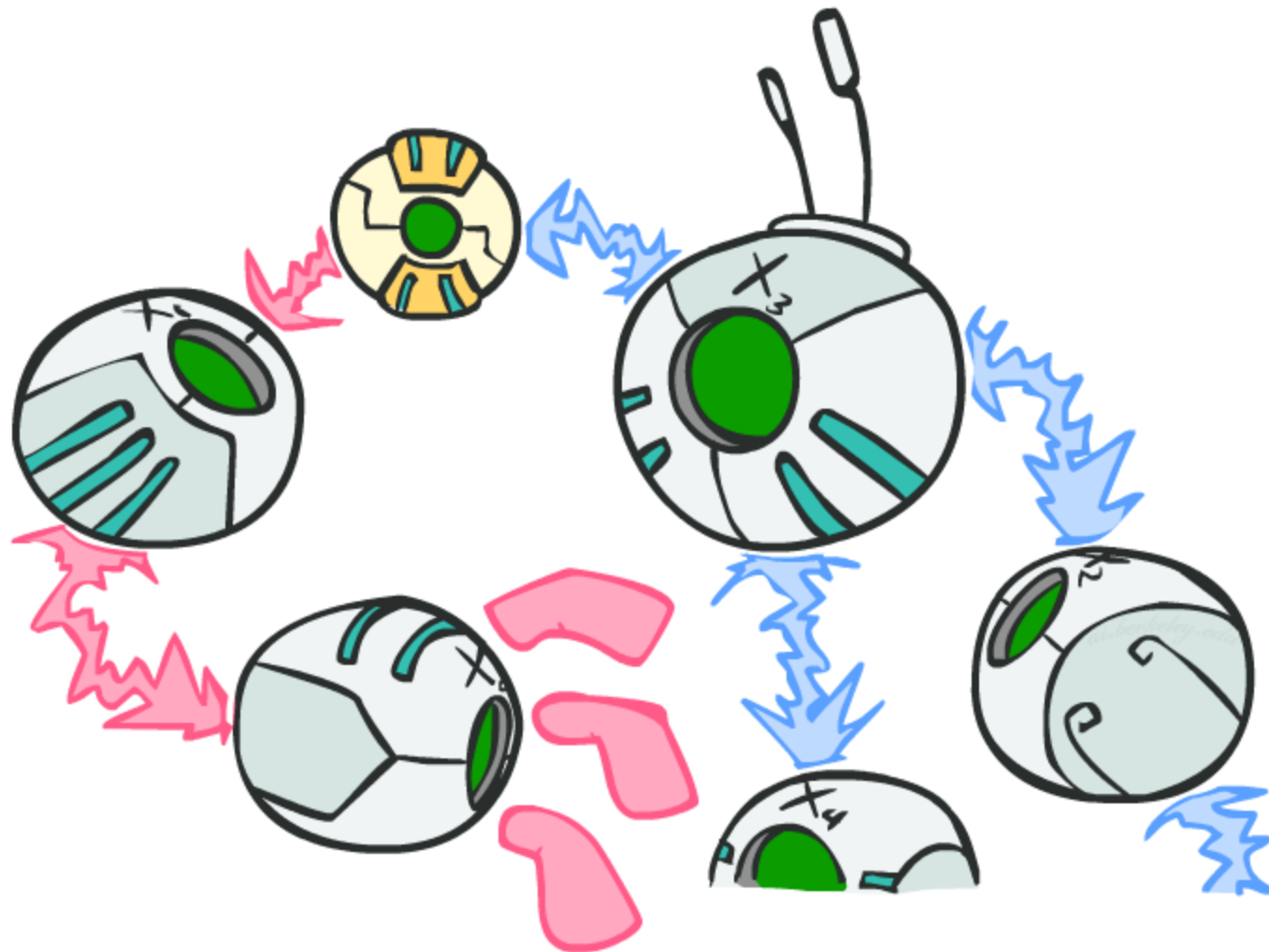
- Important question about a BN:
  - Are two nodes independent given certain evidence?
  - If yes, can prove using algebra (tedious in general)
  - If no, can prove with a counter example
  - Example:



- Question: are X and Z necessarily independent?
  - Answer: no. Example: low pressure causes rain, which causes traffic.
  - X can influence Z, Z can influence X (via Y)
  - Addendum: they *could* be independent: how?

# D-separation: Outline

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# D-separation: Outline

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- Study independence properties for triples
- Analyze complex cases in terms of member triples
- D-separation: a condition / algorithm for answering such queries

# Causal Chains

- This configuration is a “causal chain”



X: Low pressure

Y: Rain

Z: Traffic

$$P(x, y, z) = P(x)P(y|x)P(z|y)$$

- Guaranteed X independent of Z ? **No!**

- One example set of CPTs for which X is not independent of Z is sufficient to show this independence is not guaranteed.

- Example:

- Low pressure causes rain causes traffic, high pressure causes no rain causes no traffic

- In numbers:

$$P(+y \mid +x) = 1, P(-y \mid -x) = 1, \\ P(+z \mid +y) = 1, P(-z \mid -y) = 1$$

# Causal Chains

- This configuration is a “causal chain”

- Guaranteed X independent of Z given Y?



X: Low pressure

Y: Rain

Z: Traffic

$$P(x, y, z) = P(x)P(y|x)P(z|y)$$

$$\begin{aligned} P(z|x, y) &= \frac{P(x, y, z)}{P(x, y)} \\ &= \frac{P(x)P(y|x)P(z|y)}{P(x)P(y|x)} \\ &= P(z|y) \end{aligned}$$

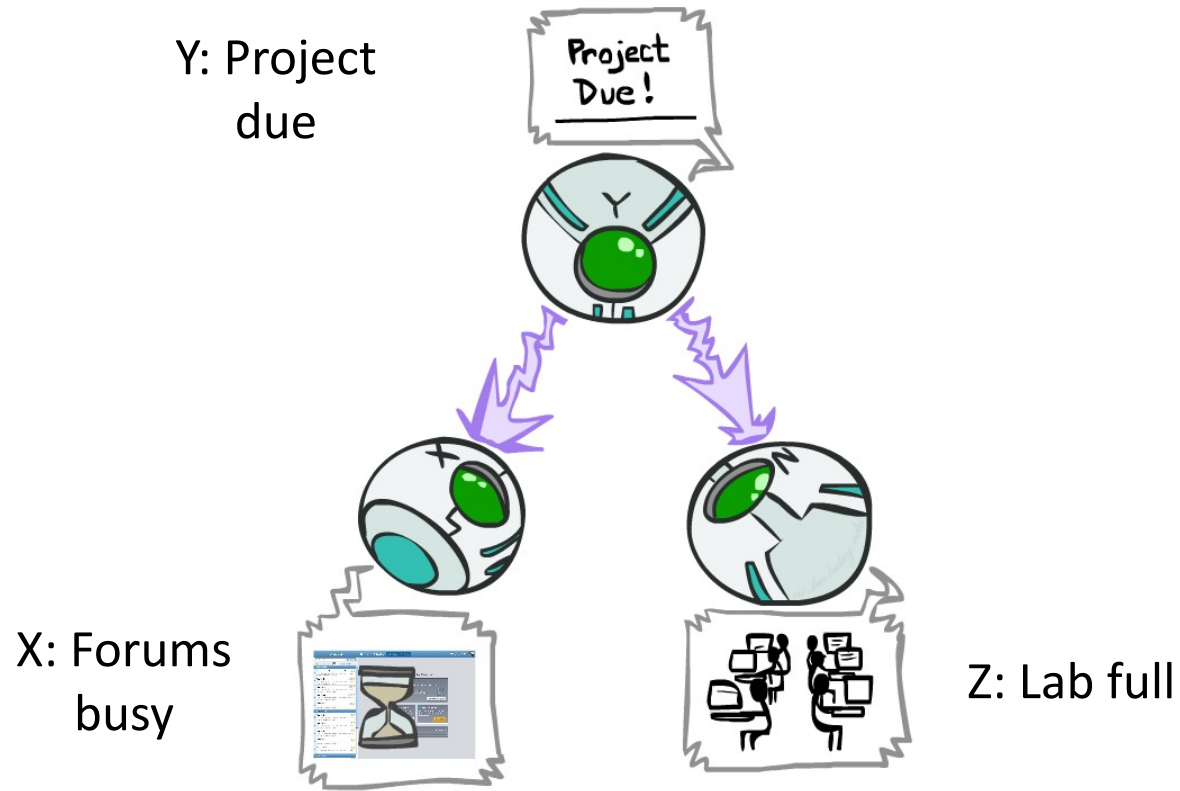
*Yes!*

- Evidence along the chain “blocks” the influence



# Common Cause

- This configuration is a “common cause”



$$P(x, y, z) = P(y)P(x|y)P(z|y)$$

- Guaranteed X independent of Z ? **No!**

- One example set of CPTs for which X is not independent of Z is sufficient to show this independence is not guaranteed.

- Example:

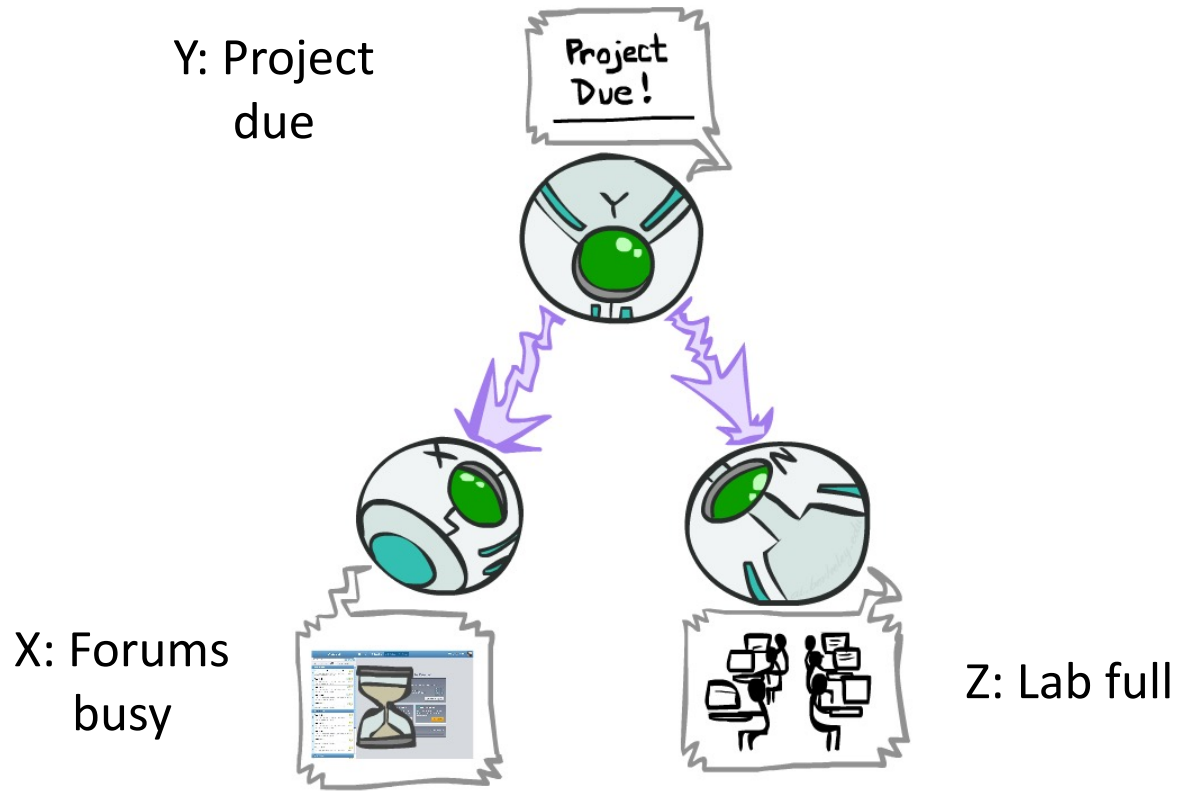
- Project due causes both forums busy and lab full

- In numbers:

$$P(+x \mid +y) = 1, P(-x \mid -y) = 1, \\ P(+z \mid +y) = 1, P(-z \mid -y) = 1$$

# Common Cause

- This configuration is a “common cause”
- Guaranteed X and Z independent given Y?



$$P(x, y, z) = P(y)P(x|y)P(z|y)$$

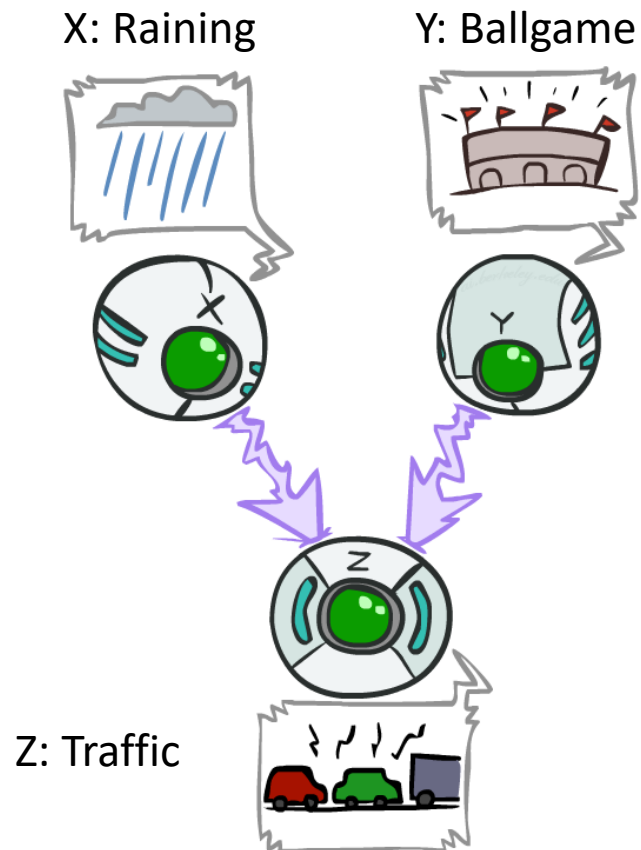
$$\begin{aligned} P(z|x, y) &= \frac{P(x, y, z)}{P(x, y)} \\ &= \frac{P(y)P(x|y)P(z|y)}{P(y)P(x|y)} \\ &= P(z|y) \end{aligned}$$

**Yes!**

- Observing the cause blocks influence between effects.

# Common Effect

- Last configuration: two causes of one effect (v-structures)



- Are X and Y independent?
  - **Yes**: the ballgame and the rain cause traffic, but they are not correlated
  - Still need to prove they must be (try it!)
- Are X and Y independent given Z?
  - **No**: seeing traffic puts the rain and the ballgame in competition as explanation.
- **This is backwards from the other cases**
  - Observing an effect **activates** influence between possible causes.

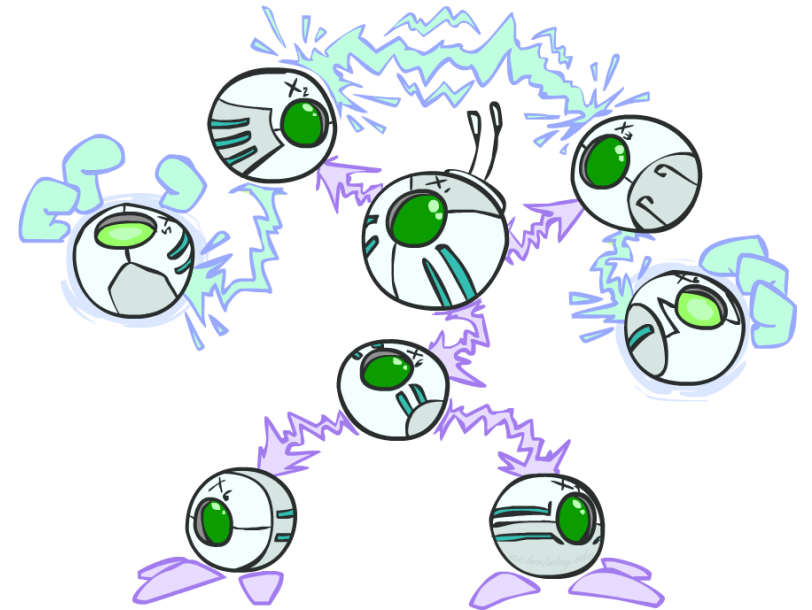
# The General Case

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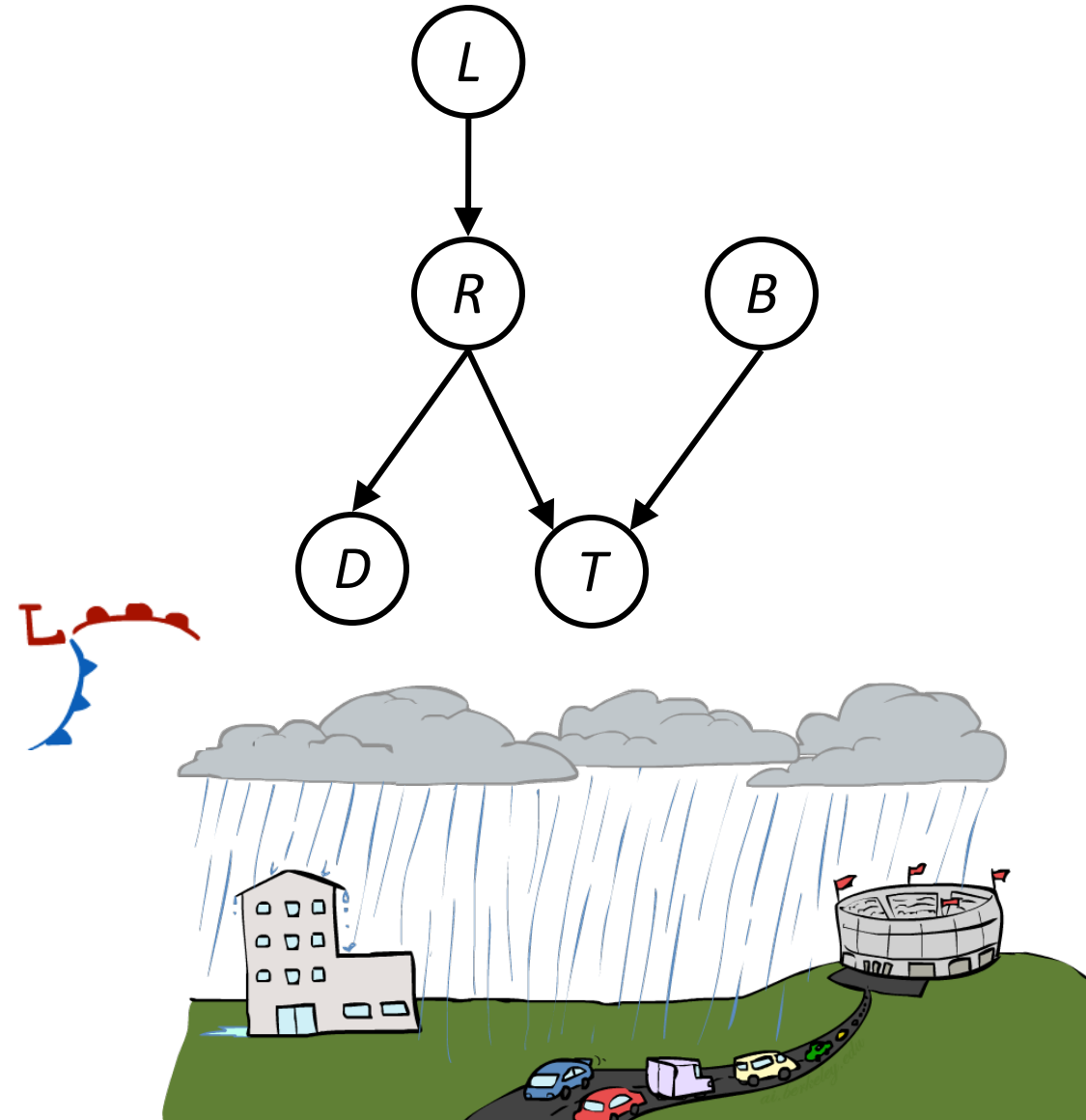
# The General Case

- General question: in a given BN, are two variables independent (given evidence)?
- Solution: analyze the graph
- Any complex example can be broken into repetitions of the three canonical cases



# Reachability

- Recipe: shade evidence nodes, look for paths in the resulting graph
- Attempt 1: if two nodes are connected by an undirected path not blocked by a shaded node, they are conditionally independent
- Almost works, but not quite
  - Where does it break?
  - Answer: the v-structure at T doesn't count as a link in a path unless "active"

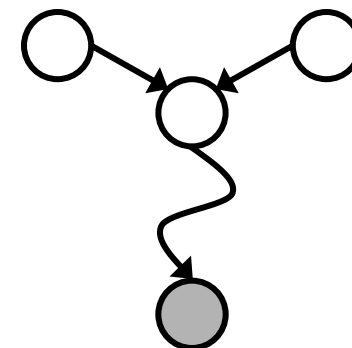
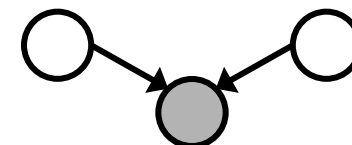
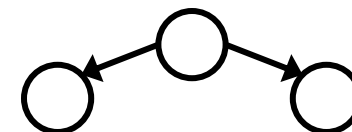


# Active / Inactive Paths

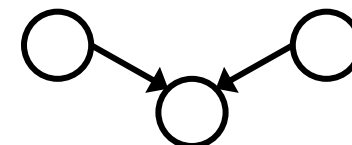
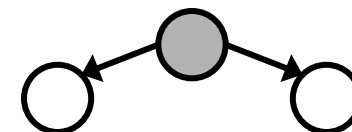
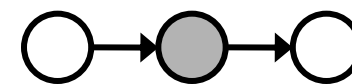
● Variable observed  
○ Variable not observed

- Question: Are X and Y conditionally independent given evidence variables {Z}?
  - Yes, if X and Y “d-separated” by Z
  - Consider all (undirected) paths from X to Y
  - No active paths = independence!
- A path is active if each triple is active:
  - Causal chain  $A \rightarrow B \rightarrow C$  where B is unobserved (either direction)
  - Common cause  $A \leftarrow B \rightarrow C$  where B is unobserved
  - Common effect (aka v-structure)  
 $A \rightarrow B \leftarrow C$  where B or one of its descendants is observed
- All it takes to block a path is a single inactive segment

Active Triples



Inactive Triples



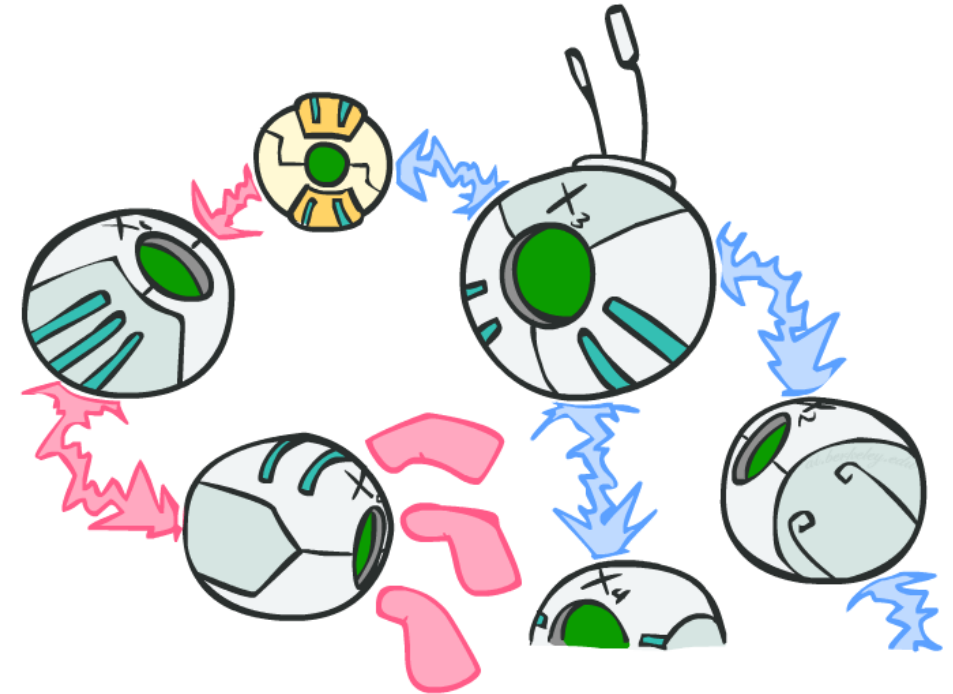
# D-Separation

- Query:  $X_i \perp\!\!\!\perp X_j \mid \{X_{k_1}, \dots, X_{k_n}\} ?$
- Check all (undirected!) paths between  $X_i$  and  $X_j$ 
  - If one or more active, then independence not guaranteed

$$X_i \not\perp\!\!\!\perp X_j \mid \{X_{k_1}, \dots, X_{k_n}\}$$

- Otherwise (i.e. if all paths are inactive), then independence is guaranteed

$$X_i \perp\!\!\!\perp X_j \mid \{X_{k_1}, \dots, X_{k_n}\}$$





# Example

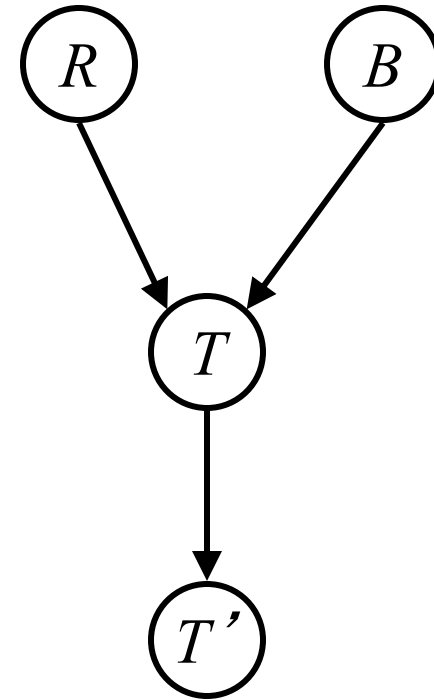
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$$R \perp\!\!\!\perp B$$

*Yes*

$$R \perp\!\!\!\perp B | T$$

$$R \perp\!\!\!\perp B | T'$$



# Example

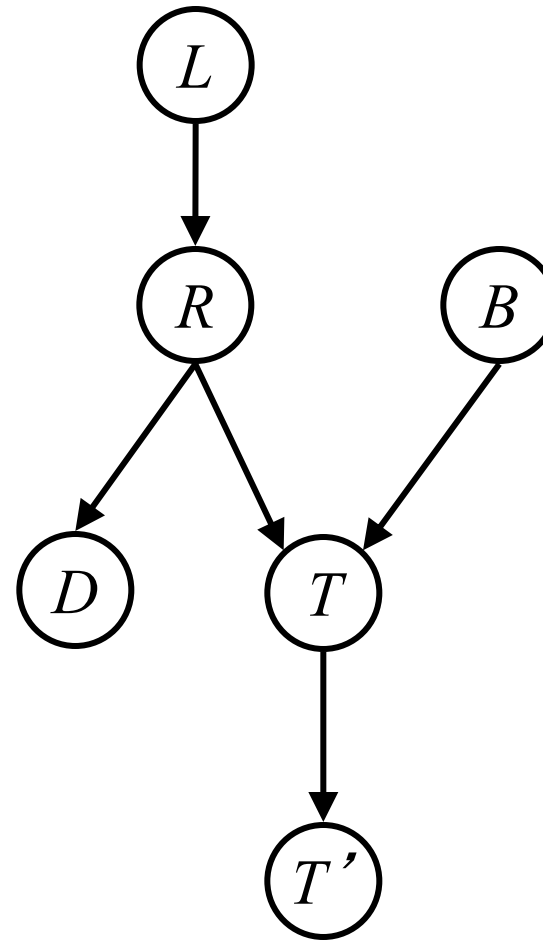
$L \perp\!\!\!\perp T' | T$       *Yes*

$L \perp\!\!\!\perp B$       *Yes*

$L \perp\!\!\!\perp B | T$

$L \perp\!\!\!\perp B | T'$

$L \perp\!\!\!\perp B | T, R$       *Yes*



# Example

- Variables:

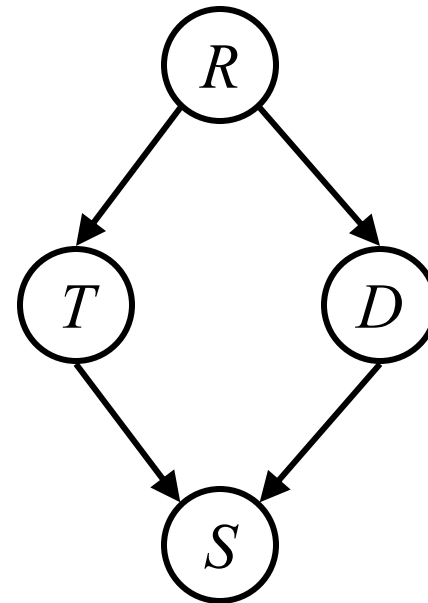
- R: Raining
- T: Traffic
- D: Roof drips
- S: I'm sad

- Questions:

$$T \perp\!\!\!\perp D$$

$$T \perp\!\!\!\perp D | R \quad \text{Yes}$$

$$T \perp\!\!\!\perp D | R, S$$

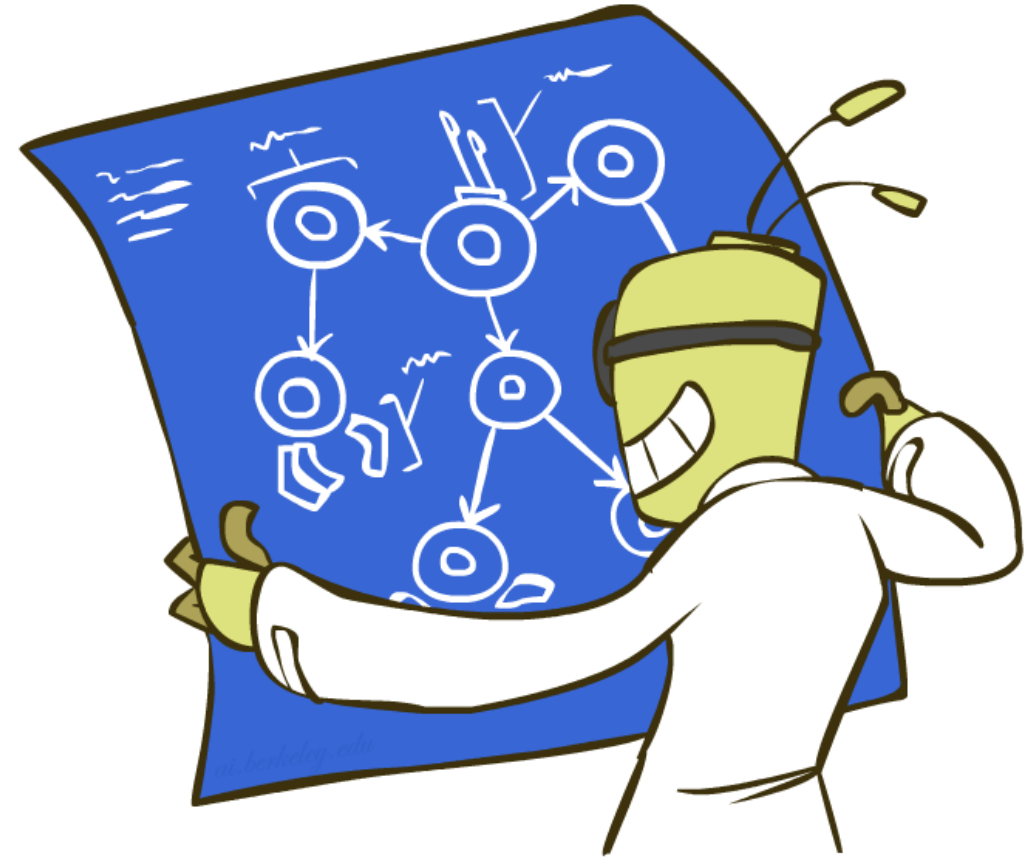


# Structure Implications

- Given a Bayes net structure, can run d-separation algorithm to build a complete list of conditional independences that are necessarily true of the form

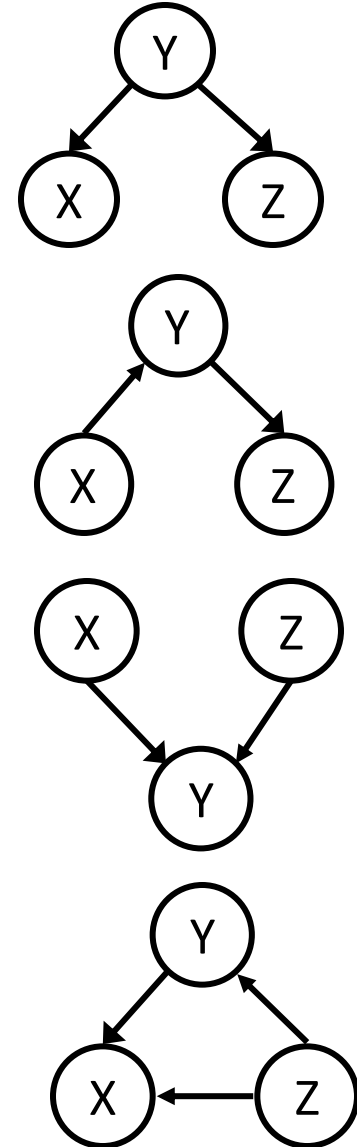
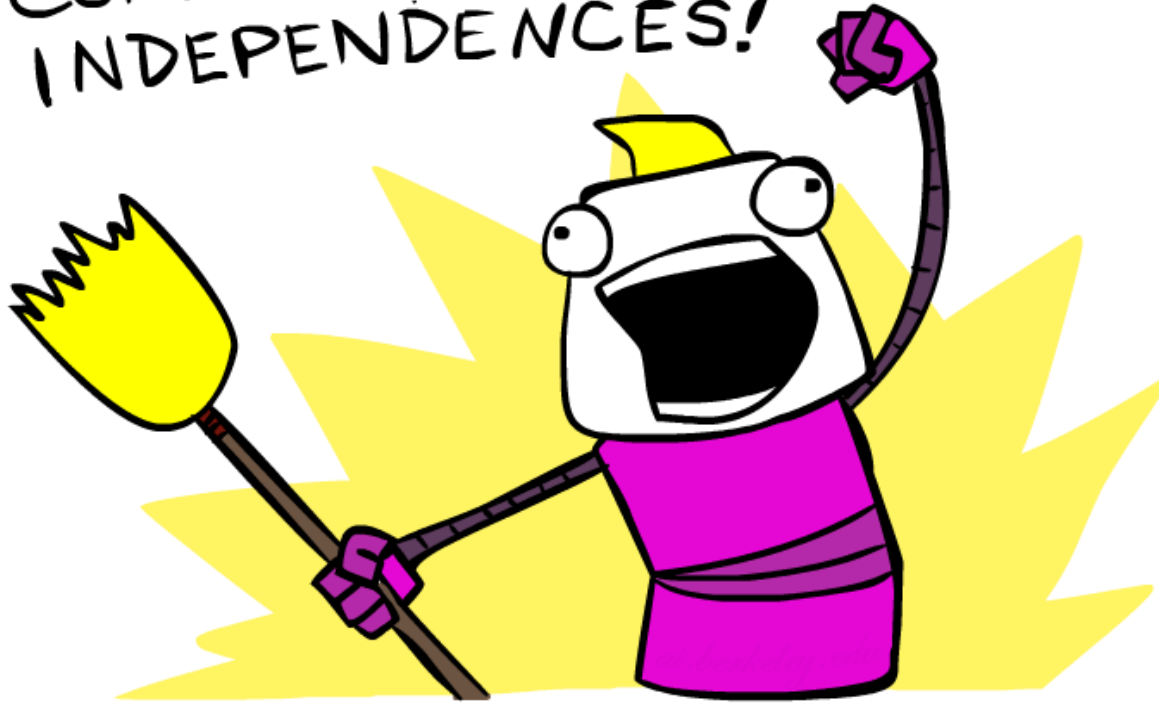
$$X_i \perp\!\!\!\perp X_j | \{X_{k_1}, \dots, X_{k_n}\}$$

- This list determines the set of probability distributions that can be represented



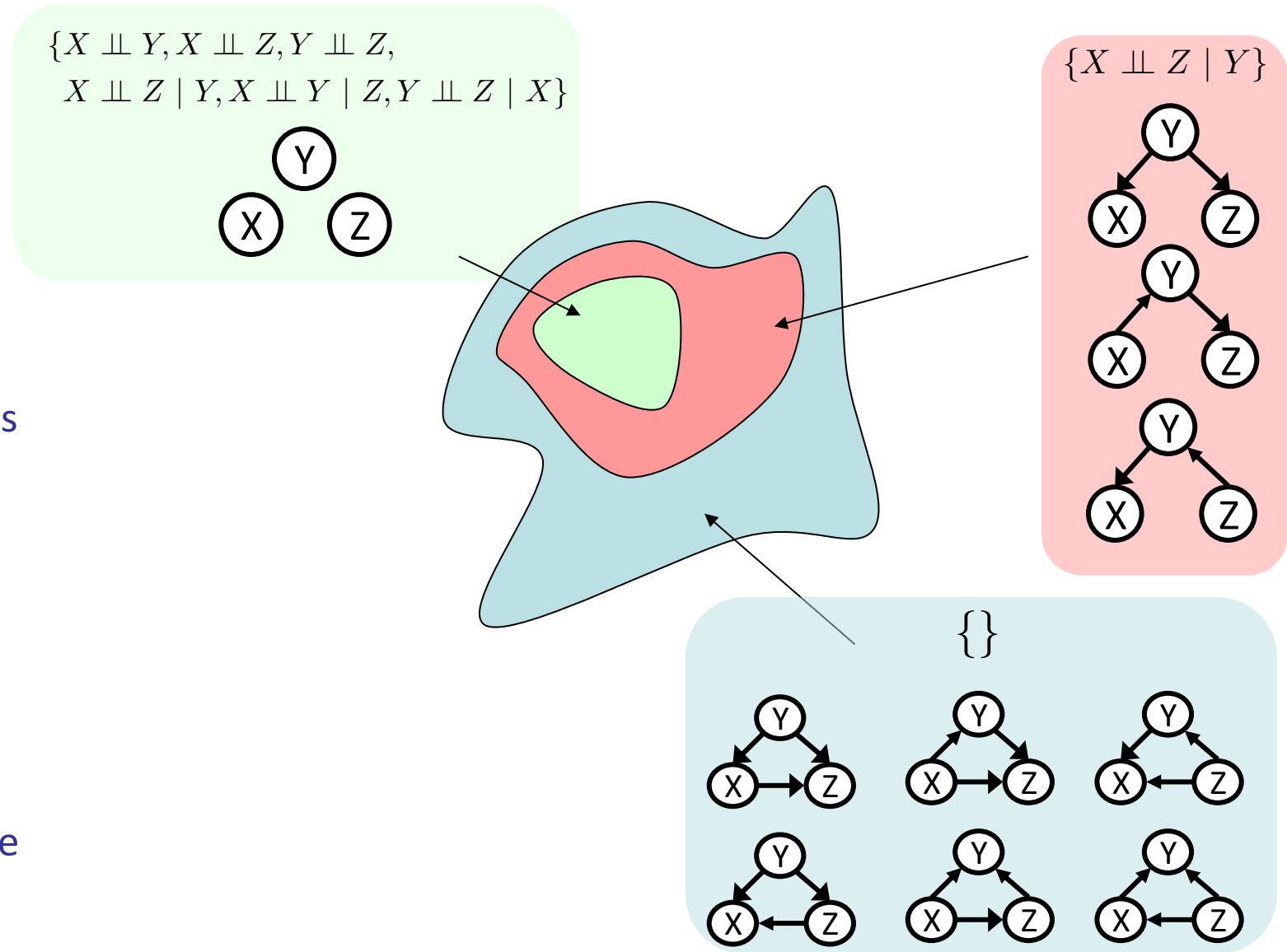
# Computing All Independences

COMPUTE ALL THE  
INDEPENDENCES!



# Topology Limits Distributions

- Given some graph topology  $G$ , only certain joint distributions can be encoded
- The graph structure guarantees certain (conditional) independences
- (There might be more independence)
- Adding arcs increases the set of distributions, but has several costs
- Full conditioning can encode any distribution



# Bayes Nets Representation Summary

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- Bayes nets compactly encode joint distributions
- Guaranteed independencies of distributions can be deduced from BN graph structure
- D-separation gives precise conditional independence guarantees from graph alone
- A Bayes' net's joint distribution may have further (conditional) independence that is not detectable until you inspect its specific distribution

# Bayes' Nets

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✓ Representation

✓ Conditional Independences

- Probabilistic Inference

- Enumeration (exact, exponential complexity)
- Variable elimination (exact, worst-case exponential complexity, often better)
- Probabilistic inference is NP-complete
- Sampling (approximate)

- Learning Bayes' Nets from Data