**Final Exam** December 10, 2009

1. (15 pts) Suppose $X, Y$ are random variables with the following joint distribution:

$$f(x,y) = \begin{cases} xy & 0 < x < 1, 0 < y < 2 \\ 0 & otherwise \end{cases}$$

   (a) Are $X$ and $Y$ independent? {**answer:** *Yes. Follows from:* $f(x,y) = f_X(x)f_Y(y)$.}

   (b) What is the covariance between $X$ and $Y$?{**answer:** *Zero. It is always zero for independent RV's.*}

2. (15 pts) Let $X_1, X_2$ and $X_3$ be three IID Poisson random variables with parameter $\lambda$. What is

$$\text{cov}(X_1 + X_2, X_2 + X_3)?$$

{**answer:** $cov(X_1 + X_2, X_2 + X_3) = cov(X_1, X_2) + cov(X_1, X_3) + cov(X_2, X_2) + cov(X_2, X_3) = cov(X_2, X_2) = var(X_2) = \lambda$.}

3. (15 pts) Consider an insurance company who has $n = 10,000$ customers. They are worried primarilly about big law suits. They come up with the following simple model. Each customer has a probability $p_i$ of having a million dollar claim against the company over the course of one year. If they don't make a million dollar claim, they will not have to be paid anything. Let $T$ be the total amount paid over the course of one year.

   (a) It is argued that an important parameter is $\pi = \sum_{i=1}^{n} p_i$. Why is $\pi$ imporant? {**answer:** $E(T) = 1000000\pi$.}

   (b) Assume $E(T) = 2$ million dollars. But, last year, a total of 30 million was claimed. This seems to have been a rare event. Use Markov's inequality to estimate the probability of an event this extreme. {**answer:** $P(T > 30) \leq 2/30$.}

   (c) Assume that $E(T) = 3$ million dollars. But, last year, actually nothing was claimed. Use a Poisson approximation to estimate the chance of this happening.{**answer:** *As long as the* $\sum p_i^2$*'s are small, a poisson should be a good approximation to the number of claims. Since the "typical"* $p_i$ *is about* $1/3000$*, we can estimate that the* $\sum p_i^2$ *should be about 1/1000. So using a Poisson with parameter* $\pi = 3$ *should be a good answer.* $P(T = 0) = e^{-3} \approx 1/20$.}

4. (10 pts) Let $X_1, X_2, \ldots, X_n$ be a sequence if IID random variables with CDF of $F(\cdot)$.

   (a) If each of them has a continuous distribution, what is the probability that $X_1 < X_2 < X_3 < \cdots < X_n$? {**answer:** $1/n!$}

   (b) Give an example of $F(\cdot)$ for which $P(X_1 < X_2 < X_3 < \cdots < X_n)$ is zero. {**answer:** *Let* $F(x) = I_{x \geq 0}$*. Then* $P(X_i = 0) = 1$*, so they are all the same.*}

5. (25 pts) Let $W = \prod_{i=1}^{n} R_i$, where $R_i$ is a sequence of non-negative IID random variables. Suppose $E(R_i) = \mu$ and $E(\log(R)) = \gamma$.

   (a) What is $E(W)$? {**answer:** $E(W) = E(\prod R_i) = \prod E(R_i) = \mu^n$}

   (b) What is $e^{E(\log W)}$? {**answer:** $E(\log W) = E(\sum logR_i) = n\mu$*. So answer is* $e^{n\mu}$.}

   (c) Which of the above two calculations is going to be a better approximation to the actual value of $W$? {**answer:** *The second since the WLLN's tells us that* $\log W$ *will be close to its mean, but nothing tells us the* $W$ *will be close to its mean (and in fact it will often be very very far away.)*}

6. (10 pts) Let $X_1, X_2, \ldots, X_n$ be integer Cauchy, namely, $P(X_i = x) = (6/\pi^2)x^{-2}$ for $x = 1, 2, 3, \ldots$. Let $\overline{X}_n = \sum_{i=1}^{n} X_i/n$. What does the weak law of large numbers tell us about $\overline{X}_n$?

{**answer:** *$E(X_i) = \infty$, or if you perfer, $E(X_i)$ doesn't exist. Hence WLLN tells us nothing about $\overline{X}_n$.*
*But we can do better. If we consider $Y \equiv \min(X, M)$ then $E(Y)$ exists, but is arbitarilly large as $M$ goes to infinity. Clearly $\overline{X}_n \geq \overline{Y}_n$ and the WLLN tells us that $\overline{Y}_n \to E(Y)$. Hence we see that $\overline{X}_n \to \infty$.*}

7. (15 pts) Bayesians often use random variables where typical probabilitist use parameters. So when a Bayesian talks about a normal, the mean $\mu$ is often random itself. We will be discussing with is called the Gamma-Poisson.

Let $X$ be a Poisson random variable with parameter $Y$. Let $Y$ be a exponential with parameter $\lambda$, namely

$$f_Y(y) = \lambda e^{-\lambda y} I_{y \geq 0}.$$

(a) What is $E(X|Y)$? {**answer:** *$E(X|Y) = Y$.*}

(b) What is $E(X)$? {**answer:** *$E(X) = E(E(X|Y)) = E(Y) = 1/\lambda$*}

(c) Why is it resonable to call this a Gamma-Poisson?

{**answer:** *An exponential is a special case of a Gamma. Hence the distribution of $Y, X$ is that of a Gamma,Poisson.*}

(bonus) What is $E(Y|X)$?

{**answer:**

$$\begin{aligned}
E(Y|X = x) &= \frac{\int_0^\infty y f(x,y)dy}{\int_0^\infty f(x,y)dy} \\
&= \frac{\int_0^\infty y e^{-y} y^x/x! e^{-\lambda y} dy}{\int_0^\infty e^{-y} y^x/x! e^{-\lambda y} dy} \\
&= \frac{\int_0^\infty e^{-(1+\lambda)y} y^{x+1} dy}{\int_0^\infty e^{-(y+\lambda)} y^x dy} \\
&= \frac{\int_0^\infty e^{-u}(u/(1+\lambda)^{x+1} dy}{\int_0^\infty e^{-u}(u/(1+\lambda)^x dy} \\
&= \frac{1}{1+\lambda} \frac{\int_0^\infty e^{-u} u^{x+1} dy}{\int_0^\infty e^{-u} u^x dy} \\
&= \frac{1}{1+\lambda} \frac{\Gamma(x+2)}{\Gamma(x+1)} \\
&= \frac{1}{1+\lambda} \frac{(x+1)!}{x!} \\
&= \frac{x+1}{1+\lambda}
\end{aligned}$$

*So, as random variables $E(Y|X) = \frac{X+1}{1+\lambda}$. As a check we can compute $EY = E(E(Y|X)) = \frac{EX+1}{1+\lambda} = \frac{1/\lambda+1}{1+\lambda} = 1/\lambda$ which is about right.*}