__________________________ **Midterm Exam** October 8, 2009

(name)

Write your name both on the exam and on the blue book. Circle the one best answer for the multiple choice below. Tear off the multiple choice and turn them in with your blue book. (5pts each)

1. The set $(A \cup C)(B \cup C)$ is best represented as

   (a) C.

   (b) $C \cup AB$.

   (c) $AB \cup BC \cup CB \cup CC$.

   (d) $(A^c B \cup B^c A)^c$

   (e) none of the above.

   {**answer:** *1b, draw the picture. 1c is too complex to qualify as best!*}

2. The sum $\displaystyle\sum_{i=1}^{n} \binom{n}{i}$ is

   (a) 2^n

   (b) $2^n + 1$

   (c) 1^n

   (d) $\sum_{i=0}^{n-1} 2^i$

   (e) more than one of the above

   {**answer:** *2d both are* $2^n - 1$}

3. The identity $(1-1)^n = 0$ can be used to evaluate which of the following

   (a) $\sum_{j=1}^{n} \binom{n}{j}\binom{j}{i}$

   (b) $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} \cdots \binom{n}{n}$

   (c) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} \cdots \binom{n}{n}$

   (d) $\sum_{i=0}^{n} 2^i$

   (e) none of the above

   {**answer:** *3b*}

4. Suppose a person chooses one letter from MISSISSIPPI and one letter from TENNESSEE, what is the probability that they spell IT:

(a) 4/99

(b) 0

(c) 4/11

(d) $4/11 + 1/9$

(e) none of the above

{**answer:** *4a*}

5. If $A \subset B$ then

   (a) $P(A \cup B) > P(B)$

   (b) $P(B|A) < P(B)$

   (c) $P(A) < 1$

   (d) $P(A|B) > P(A)$

   (e) $P(B|A) = 1$

   (f) More than one of the above

   {**answer:** *5e, $P(B|A) = P(AB)/P(A) = P(A)/P(A) = 1$. In 5a $A = B = S$ shows it wrong. In 5b B=A=S agan shows it wrong. Likewise in the other two, A=B=S show them wrong.*}

6. Which one of the following is always true

   (a) $P(E|E \cup F) \leq P(E|F)$

   (b) $P(E|F) \geq P(E|E)$

   (c) $P(E|E \cup F) \geq P(E|F)$

   (d) $P(E|EF) \leq P(E|F)$

   {**answer:** *6c $P(E|E \cup F) = P(E(E \cup F))/P(E \cup F) = P(E)/P(E \cup F) = (P(EF) + P(EF^c)/(P(F) + P(EF^c)) > P(EF)/P(F)$. Alternatvely the book suggests conditioning on F to do the computation.*}

7. In a clinical trial we are looking for patients which satisfy our rules for entry. Suppose that the probabily of a patient doing so is 2/3's. What is the probability that we get our 2nd patient when the 4th candidate walks in the door:

   (a) $\binom{4}{2}(1/3)^2(2/3)^2$

   (b) 4/27

   (c) 4/81

(d) $ppqp + pqpp + qppp$

(e) none of the above

{**answer:** *7b:* $\binom{3}{2}(2/3)^2(1/3)^2 = 3 * 4/81 = 4/27$}

8. For the probability space $(\Omega, \mathcal{A}, P)$ the function $P()$

(a) maps Ω to the real line

(b) maps subset of Ω to the real line

(c) maps elements of Ω to the real line

(d) maps subsets of $\mathcal{A}$ to the real line

(e) maps elements of $\mathcal{A}$ to the real line

{**answer:** *8e*}

Answer the rest in your blue book (20 pts each)

9. We will say that a sequence of events $E_1, E_2, E_3, \ldots$ converges quickly to zero if $P(E_i) \leq p^i$ for some $p < 1$.

(A) Prove that there exists some m such that $P(\cup_{i=m}^{\infty} E_i) < .05$.

{**answer:** $P(\cup_{i=m}^{\infty} E_i) \leq \sum_{i=m}^{\infty} P(E_i) \leq \sum_{i=m}^{\infty} p^i \leq p^{m-1}/(1-p) < \infty$}

(B) Let $X = \sum_i I_{E_i}$. Prove that $E(X) < \infty$.

{**answer:** *By linearity of expectation* $E(X) = \sum_i E(I_{E_i}) = \sum_i P(E_i) \leq \sum_i p^i \leq 1/(1-p) < \infty$.}

10. Let X be a random variable with density function $f(x) = .5^x$ for $x = 1, 2, 3, \ldots$. Let Y be a random variable defined by the equation $Y = a^X$.

(A) What is $E(Y)$?

{**answer:** $E(Y) = \sum a^x f(x) = \sum (a/2)^x = 1/(2/a - 1)$}

(B) For what values of a is $E(Y) < \infty$?

{**answer:** *If* $|a| < 2$ *this* $E(Y)$ *is well defined and finite.*}

(C) If $a = 1 + \epsilon$, for $\epsilon \approx 0$, show $X \approx \frac{Y-1}{\epsilon}$.

{**answer:** $\frac{Y-1}{\epsilon} = \frac{(1+\epsilon)^X - 1}{\epsilon} \approx \frac{e^{\epsilon X} - 1}{\epsilon} \approx \frac{1 + \epsilon X - 1}{\epsilon} = X$}

(D) From this, compute $E(X)$. {**answer:** $E(X) \approx E(\frac{Y-1}{\epsilon}) = \frac{1/(2/(1+\epsilon)-1)}{\epsilon} = 2 + O(\epsilon) \approx 2$}

11. Let $\mathcal{A} \equiv$ {the event A works}, and likewise for $\mathcal{B}, \mathcal{C}$ and $\mathcal{D}$. Assume that each part fails independently of the other pieces.

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                  +----------  B  --------+
                  |                       |
                  |                       |
s ------  A  -----+                       +----------  S
                  |                       |
                  |                       |
                  +-----  C  ----- D  ----+
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Let $\mathcal{T} \equiv \{$ the event a path from s to S works$\}$.

(A) Write $\mathcal{T}$ in terms of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$.

$\{$**answer:** $\mathcal{T} = \mathcal{A}(\mathcal{B} \cup \mathcal{C}\mathcal{D})\}$

(B) If $P(\mathcal{A}) = P(\mathcal{B}) = P(\mathcal{C}) = P(\mathcal{D}) = 3/5$, what is the $P(\mathcal{T})$?

$\{$**answer:** $3/5(1 - 2/5(1 - (3/5)^2))\}$

(Bonus) If you could pay a dollar to increase one of the probabilities by a small amount say ϵ, which of $P(\mathcal{A})$, $P(\mathcal{B})$, $P(\mathcal{C})$ or $P(\mathcal{D})$ would you choose? Why?

$\{$**answer:** *The best to increase is "clearly" $P(\mathcal{A})$. Unfortunately, this is wrong! In the extreme case where all the probabilites are close to 1 except one of the four, (say $P(\mathcal{C})$) then increasing this smallest one will get more bounce than increasing $P(\mathcal{A})$. The best way to see this is to compute all 4 partial derivatives and show this has the largest partial derivative. Using $\alpha, \beta, \gamma, \delta$ for the four probabilities, we have $\alpha(1 - (1 - \beta)(1 - \gamma\delta)$ is our function. The partial with respect to α is $(1 - (1 - \beta)(1 - \gamma\delta)$. And the the partial with respects to β which is $\alpha(1 - \gamma\delta)$. There are some situations where the first of these is large and some where the second is large. Likewise for the other 2 possibilities.*$\}$