Probability Midterm

You may use one sheet of paper of notes. No calculators, cell phones, PDA's, laptops, or HAL 9000's. Show your reasoning. Don't just give the answer.

1. (20 points) You can convert an 10 card deck from $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ to the sequence $8, 2, 3, 5, 1, 7, 10, 9, 4, 6$ via either shuffles or unshuffles.

   (a) What is the smallest α such that an α-shuffle can do the conversion?

   (b) What is the minimum number of riffle shuffles that it would take to do this conversion?

   (c) What is the smallest α such that an α-unshuffle can do the conversion?

   (d) Compute the inverse of this permutation and show that the number of rising sequences equals the number of falls.

   {**answer:** *2-shuffle: 6 shuffles, or 5 unshuffles. The inverse is* $5, 2, 3, 9, 4, 10, 6, 1, 8, 7$ *which has 5 rising sequences and 6 falls.*}

2. (20 points) Suppose there are three horses in a race, called A, B and C. Only one can win. One person believe the probabilities are $P(A) = .5$, $P(B) = .3$ and $P(C) = .2$. The other person believes the probabilities are $Q(A) = .4$, $Q(B) = .3$ and $Q(C) = .3$. What is the L1 distance between these two distributions?

   {**answer:** *There are three ways of computing this:*

   - $|.5 - .4| + |.3 - .3| + |.2 - .3| = .2 = 2L1$ *is the easiest way of getting* $L1 = .1$.
   - *More difficult is to try all possible subsets A and look at* $P(A) - Q(A)$. *Doing this gets a list of difference which are* $0, .1, .0, -.1, .1, 0, -.1, 0$ *for which the maximum is* $.1$.
   - *Finally you can come up with an arbitrage of betting A to win vs A to lose. The payoffs to P would be .5 and -.5 and to B would be -.4 and .6. This generates a guarantee of .1.*

   }

3. (10 points) What is the probability of getting four of a kind in poker? (For those who don't gamble this means a hand of 5 cards that looks something like,

   - 5 Spades, 5 Hearts, 5 Diamonds, 5 Clubs, 8 Spades

   I.e. 4 of one type plus 1 of another. Recall there are 52 cards in a standard deck.)

4. (10 points) Suppose you choose a real number X from the interval $[-1, 1]$ with a density function of the form

   $$f(x) = Cx^2$$

   where C is a constant. Find C.

   {**answer:** $1 = \int_{-1}^{1} Cx^2 = x^3/3|_{-1}^{1} = 1^3/3 - -1^3/3 = 2/3$. *So* $C = 3/2$.}

5. (15 points) Consider the probability triple $(\Omega, \mathcal{F}, P)$. Suppose $\Omega = \{a, b, c, d, e\}$. We are also told that $P(\{a, b\}) = 1/2$, and $P(\{b, c, d\}) = 1/3$.

(a) Why isn't $\{a\} \in \Omega$? {**answer:** *Because* $\{a\}$ *is a set.*}

(b) Prove $\{a\} \in \mathcal{F}$. {**answer:** $\{b, c, d\}$ *is in* $\mathcal{F}$ *so* $\{a, e\} \in \mathcal{F}$. *Now intersect this with* $\{a, b\}$ *to see that* $\{a\} \in \mathcal{F}$.}

(c) Clearly $0 \leq P(\{a\}) \leq 1$. But in fact, it has to be more tightly bounded. What are the best possible bounds for $P(\{a\})$? {**answer:** $1/6 \leq P(\{a\}) \leq 1/2$.}

6. (20 points) And now for something educational.

In quantum mechanics there is something called the Bell inequalities. We will derive them in this problem. Suppose that the spin of an electron is measured in one of three directions. Each time the measurement is taken, either a spin-up results or a spin-down results. Let A be the event that a spin-up results from the first direction. Let B be the event that a spin-up results from the second direction. Let C be the event that a spin-up results from the third direction. We know that:

$$P(A) = P(B) = P(C) = .5$$

Further from doing experiments we can determine that

$$r = P(A \cap B) = P(A \cap C) = P(B \cap C)$$

Let

$$s = P(A \cap B \cap C)$$

(a) What is $P(A \cup B \cup C)$ in terms of r and s? {**answer:** *By inclusion/exclusion, or Venn diagrams, or whatever technology you prefer:* $P(A \cup B \cup C) = 1.5 - 3r + s$.}

(b) From the fact that $P(A \cup B \cup C) \leq 1$ find an inequality for r and s. {**answer:** $1.5 - 3r + s \leq 1$ *or* $-3r + s \leq -.5$ *or* $3r - s \geq .5$.}

(c) From the fact that $s \geq 0$ find an inequality for r. {**answer:** *From above:* $3r \geq s + .5$, *so* $3r \geq .5$ *so* $r \geq 1/6$. *This contradicts the actual measurement which is around .1. Quantum mechanics is strange!*}

(Measurements of r have recently yielded a value of .1. This property of quantum mechanics bothered Einstein–and so he thought QM was incorrect. Some call it Einstein's biggest mistake.)

7. (5 points) We have worked out the birthday problem many times in class. We have come up with two answers: a crude guess of $1 - \binom{n}{r}/n$, and an exact answer of n_r/n^r (where n is typically 365 and r is the number of students in the class). In this question you will show that these two are approximately equal.

(a) (5 points) Using Stirling's formula compute an approximation to $\log(n_r/n^r)$. {**answer:** *Recall Stirling's formula,* $\log(n!) \approx n\log(n/e) + 1/2\log(\sqrt{n2\pi})$. *We get the following algebra:*

$$\begin{aligned}
\log(n_r/n^r) &= \log(n!/(n-r)!) - r\log(n) \\
&\approx n\log(n/e) - (n-r)\log((n-r)/e) + 1/2\log(n/(n-r-1)) - r\log(n) \\
&\approx (n-r)\log(n/(n-r)) + r\log(e) + 1/2\log(n/(n-r-1)) \\
&\approx (n-r)\log(1 + r/(n-r)) + r\log(e) + 1/2\log(n/(n-r-1))
\end{aligned}$$

}

(b) (bonus) Using a two term Taylor series for the log show that for r much less than n, our approximation is close to the correct answer.