Probability Midterm

You may use one sheet of paper of notes. No calculators, cell phones, PDA's, laptops, or HAL 9000's. Show your reasoning. Don't just give the answer.

1. (10 points) Suppose we know that $P(A) = .5$, $P(B) = .5$ and $P(C) = .5$. Likewise, suppose $P(A \cap B) = 1/6$, $P(B \cap C) = 1/6$, $P(A \cap C) = 1/6$.
   (a) If the probability of $A \cap B \cap C$ is zero, then what is the probability of $A \cup B \cup C$?
   (b) Why can't the probability $A \cap B \cap C$ be .1?

2. (10 points) Suppose you choose a real number X from the interval $[1, 3]$ with a density function of the form
$$f(x) = Cx^2$$
   where C is a constant. Find C.

3. (15 points) Assume that the probability that a radio isotope decays in t years has an exponential density
$$f(t) = \lambda e^{-\lambda t} \; .$$
   where t is chosen from $[0, \infty)$. In this context, λ is often called the *decay rate*. The half life is the point where $P(T < x) = .5$. If the half life is one million years, what is λ?

4. (15 points) If a patient presents with flu like symptoms sometime this month, the CDC has told us that there is only a 5% chance that they have swine flu. You give a patient a test for swine flu. If she has it, the test will be positive with probability 100%. If she doesn't have it, the test will be positive with probability 10%. What is the probability that a sick patient has swine flu given the test came back positive?

5. (10 points) Let X be choosen from the unit interval (i.e. $[0, 1]$). Consider the events $\mathcal{A} \equiv \{X < .5\}$ and $\mathcal{B} \equiv \{X > 1/3\}$. Are $\mathcal{A}$ and $\mathcal{B}$ are independent? Give an argument for your answer.

6. (10 points) You can convert an 8 card deck from $1, 2, 3, 4, 5, 6, 7, 8$ to the sequence $1, 4, 5, 2, 6, 7, 8, 3$ via either shuffles or unshuffles.
   (a) What is the smallest α such that an α-shuffle can do the conversion?
   (b) What is the smallest α such that an α-unshuffle can do the conversion?

7. (points 15) Consider the probability triple $(\Omega, \mathcal{F}, P)$. Suppose $\Omega = \{a, b, c, d, e\}$. We are also told that $P(\{a, b\}) = 1/2$, and $P(\{b, c, d\}) = 1/3$.
   (a) Why isn't $a \in \mathcal{F}$?
   (b) Prove $\{e\} \in \mathcal{F}$.
   (c) Clearly $0 \leq P(\{e\}) \leq 1$. But in fact, it has to be more tightly bounded. What are the best possible bounds for $P(\{e\})$?

8. (15 points) And now for something educational. A trick that Erdös championed is to determine "yes" or "no" questions by probability. The idea is if you can prove the probability of an event actually is zero, then it can't occur. We will try it with a version of the birthday problem.

   We will consider peoples initials as their "birthday." So mine, are "DF" for Dean Foster. We will only use two letter initials.

   (a) If every pair of initials is equally likely, in a room of n people, what is the probability that two people have the same initials?

   (b) Why is the assumption of equally likely untenable?

   (c) (Bonus) Use your formula to compute the probability of two people having the same initials if there are 700 people. Use Erdös trick to make a logical claim.