Final Exam December 10, 2009

1. (15 pts) Suppose X, Y are random variables with the following joint distribution:

$$f(x,y) = \begin{cases} xy & 0 < x < 1, 0 < y < 2 \\ 0 & otherwise \end{cases}$$

   (a) Are X and Y independent?

   (b) What is the covariance between X and Y?

2. (15 pts) Let X_1, X_2 and X_3 be three IID Poisson random variables with parameter λ. What is

$$\mathrm{cov}(X_1 + X_2, X_2 + X_3)?$$

3. (15 pts) Consider an insurance company who has $n = 10,000$ customers. They are worried primarilly about big law suits. They come up with the following simple model. Each customer has a probability p_i of having a million dollar claim against the company over the course of one year. If they don't make a million dollar claim, they will not have to be paid anything. Let T be the total amount paid over the course of one year.

   (a) It is argued that an important parameter is $\pi = \sum_{i=1}^{n} p_i$. Why is this?

   (b) Assume $E(T) = 2$ million dollars. But, last year, a total of 30 million was claimed. This seems to have been a rare event. Use Markov's inequality to estimate the probability of an event this extreme.

   (c) Assume that $E(T) = 3$ million dollars. But, last year, actually nothing was claimed. Use a Poisson approximation to estimate the chance of this happening.

4. (10 pts) Let X_1, X_2, ..., X_n be a sequence if IID random variables with CDF of $F(\cdot)$.

   (a) If each of them has a continuous distribution, what is the probability that $X_1 < X_2 < X_3 < \cdots < X_n$?

   (b) Give an example of $F(\cdot)$ for which $P(X_1 < X_2 < X_3 < \cdots < X_n)$ is zero.

5. (25 pts) Let $W = \prod_{i=1}^{n} R_i$, where R_i is a sequence of non-negative IID random variables. Suppose $E(R) = \mu$ and $E(\log(R)) = \gamma$.

   (a) What is $E(W)$?

   (b) What is $e^{E(\log W)}$?

   (c) Which of the above two calculations is going to be a better approximation to the actual value of W?

6. (10 pts) Let $X_1, X_2, \ldots, X_n$ be integer Cauchy, namely, $P(X = x) = (\pi^2/6)x^{-2}$ for $i = 1, 2, 3, \ldots$. Let $\overline{X}_n = \sum_{i=1}^{n} X_i$. What does the weak law of large numbers tell us about $\overline{X}_n$?

7. (15 pts) Bayesians often use random variables where typical probabilitist use parameters. So when a Bayesian talks about a normal, the mean μ is often random itself. We will be discussing with is called the Gamma-Poisson.

   Let X be a Poisson random variable with parameter Y. Let Y be a exponential with parameter λ, namely

   $$f_Y(y) = \lambda e^{-\lambda y} I_{y \geq 0}.$$

   (a) What is $E(X|Y)$?

   (b) What is $E(X)$?

   (c) Why is it resonable to call this a Gamma-Poisson?

(bonus) What is $E(Y|X)$?