Homework 6 Solutions

6.2

4. (a) 10015, (b) 310, (c) -100, (d) 15, (e) $\sqrt{15}$.

8. $E(S_{2400}) = 960, \quad V(S_{2400}) = 576, \quad \sigma = D(S_{2400}) = 24.$

12. $E\Big(X - \mu/\sigma\Big) = (1/\sigma)(E(X) - \mu) = 0,$

$V\Big(X - \mu/\sigma\Big)^2 = (1/\sigma^2)E(X-\mu)^2 = \sigma^2/\sigma^2 = 1.$

15. (a) $P_{X_i} = \begin{pmatrix} 0 & 1 \\ \frac{n-1}{n} & \frac{1}{n} \end{pmatrix}$. Therefore, $E(X_i)^2 = 1/n$ for $i \neq j$.

(b) $P_{X_iX_j} = \begin{pmatrix} 0 & 1 \\ 1 - \frac{1}{n(n-1)} & \frac{1}{n(n-1)} \end{pmatrix}$ for $i \neq j$.

Therefore, $E(X_iX_j) = \dfrac{1}{n(n-1)}$.

(c)

$$\begin{aligned} E(S_n)^2 &= \sum_i E(X_i)^2 + \sum_i \sum_{j \neq i} E(X_iX_j) \\ &= n \cdot \frac{1}{n} + n(n-1) \cdot \frac{1}{n(n-1)} = 2 \, . \end{aligned}$$

(d)

$$\begin{aligned} V(S_n) &= E(S_n)^2 - E(S_n)^2 \\ &= 2 - (n \cdot (1/n))^2 = 1 \, . \end{aligned}$$

6.3

1. (a) $\mu = 0, \ \sigma^2 = 1/3$

(b) $\mu = 0, \ \sigma^2 = 1/2$

(c) $\mu = 0, \ \sigma^2 = 3/5$

(d) $\mu = 0, \ \sigma^2 = 3/5$

7. $f(a) = E(X-a)^2 = \int (x-a)^2 f(x)dx$. Thus

$$\begin{aligned} f'(a) &= -\int 2(x-a)f(x)dx \\ &= -2\int xf(x)dx + 2a\int f(x)dx \end{aligned}$$

$$= -2\mu(X) + 2a \ .$$

Since $f''(a) = 2$, $f(a)$ achieves its minimum when $a = \mu(X)$.

14. Break up the integral into three parts: from (-infinity, -1), (-1,+1) and (+1,+infinity). Use the hint on the first and third integrals. Then recombine to get the integral with x^2 as the integrand.

17. (a)

$$\begin{aligned} \mathrm{Cov}(X,Y) &= E(XY) - \mu(X)E(Y) - E(X)\mu(Y) + \mu(X)\mu(Y) \\ &= E(XY) - \mu(X)\mu(Y) = E(XY) - E(X)E(Y) \ . \end{aligned}$$

(b) If X and Y are independent, then $E(XY) = E(X)E(Y)$, and so $\mathrm{Cov}(X,Y) = 0$.

(c)

$$\begin{aligned} V(X+Y) &= E(X+Y)^2 - (E(X+Y))^2 \\ &= E(X^2) + 2E(XY) + E(Y^2) \\ &\quad -E(X)^2 - 2E(X)E(Y) - E(Y)^2 \\ &= V(X) + V(Y) + 2\mathrm{Cov}(X,Y) \ . \end{aligned}$$

7.1

1. (a) .625

(b) .5

5. (a) $\begin{pmatrix} 3 & 4 & 5 & 6 \\ \frac{1}{12} & \frac{4}{12} & \frac{4}{12} & \frac{3}{12} \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ \frac{1}{12} & \frac{4}{12} & \frac{4}{12} & \frac{3}{12} \end{pmatrix}$

7. (a) $P(Y_3 \le j) = P(X_1 \le j, X_2 \le j, X_3 \le j) = P(X_1 \le j)^3$.
Thus

$$p_{Y_3} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \frac{1}{216} & \frac{7}{216} & \frac{19}{216} & \frac{37}{216} & \frac{61}{216} & \frac{91}{216} \end{pmatrix} .$$

This distribution is not bell-shaped.

(b) In general,

$$P(Y_n \le j) = P(X_1 \le j)^3 = \left(\frac{j}{6}\right)^n .$$

Therefore,

$$P(Y_n = j) = \left(\frac{j}{6}\right)^n - \left(\frac{j-1}{6}\right)^n .$$

This distribution is not bell-shaped for large n.

7.2

2. (a) $f_Z(x) = \dfrac{x+2}{4}$ on $[-2, 0]$ and $\dfrac{2-x}{4}$ on $[0, 2]$.

6. Z is normally distributed with mean $\mu = \mu_1 + \mu_2$ and variance $\sigma^2 = \sigma_1^2 + \sigma_2^2$.

10. $P(\min(X_1, \ldots, X_n > x) = (P(X_1 > x))^n = (e^{-x/\mu})^n = e^{-(n/\mu)x}$. Thus

$$f_{\min(X_1,\ldots,X_n)} = \frac{n}{\mu} e^{-(n/\mu)x} .$$

This is the exponential density with mean μ/n.

14. $X_3 = -X_2$ has density

$$f_{-X_2}(x) = \begin{cases} e^{\lambda x}, & -\infty < x \le 0; \\ 0, & \text{otherwise}. \end{cases}$$

Thus $Z = X_1 + X_3$ has density

$$\begin{aligned} f_Z(x) &= \int_0^\infty e^{\lambda(x-2y)} dy = \frac{1}{2\lambda} e^{\lambda x}, & x < 0; \\ &= \int_x^\infty e^{\lambda(x-2y)} dy = \frac{1}{2\lambda} e^{\lambda x} \left(e^{-2\lambda x}\right) = \frac{1}{2\lambda} e^{-\lambda x}, & x \ge 0. \end{aligned}$$

8.1

4. You will lose on the average 1.41 percent of the money that you bet. Thus if you play a long time, you will lose a lot. The law of large numbers tells you that the probability that you will be ahead in the long run tends to 0.

6. $V\left(\frac{S_n}{n}-p\right) = V\left(\frac{S_n}{n}\right) = \frac{p(1-p)}{n}$. Thus $P\left(\left|\frac{S_n}{n}-p\right| \ge \epsilon\right) \le \frac{p(1-p)}{n\epsilon^2}$.

8. No.

8.2

1. (a) 1

(b) 1

(c) 100/243

(d) 1/12

4. (a) $E(X) = 1/\lambda = 10$, $V(X) = (1/\lambda)^2 = 100$.

(b) For the first three probabilities Chebyshev's estimate is greater than 1, and so the best estimate is 1. For the last one Chebyshev's estimate gives $P(|X-10| \ge 20) \le .25$.

(c) Comparing these Chebyshev's estimates with the exact values, we have:

$$(1, .852),\ (1, .617),\ (1, .245),\ (.25, .0498).$$