An Operational Measure of Riskiness[1]

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Abstract

We propose a measure of *riskiness* of "gambles" (risky assets) that is objective: it depends only on the gamble. The measure is based on identifying for every gamble the *critical wealth* level below which it becomes "risky" to accept the gamble.

1 Introduction

You are offered a gamble (a "risky asset") g where it is equally likely that you gain \$120 or lose \$100. What is the risk in accepting g? Is there an *objective* way to measure the *riskiness* of g? "Objective" means that the measure should depend on the gamble itself and not on the decision-maker; that is, only the outcomes and the probabilities (the "distribution") of the gamble should matter.

Such objective measures exist for the "return" of the gamble—its *expectation*, here $\mathbf{E}[g] = \$10$—and for the "spread" of the gamble—its *standard deviation*, here $\boldsymbol{\sigma}[g] = \$110$. While the standard deviation is at times used also as a measure of riskiness, it is well known that it is not a very good measure in general. One important drawback is that it is *not monotonic*: a better gamble h, i.e., a gamble with higher gains and lower losses, may well have a higher standard deviation and thus be wrongly viewed as having a higher riskiness.[1]

We propose here a *measure of riskiness* for gambles that is, like the expectation and the standard deviation, objective and measured in the same units as the outcomes; moreover, it is monotonic and has a simple "operational" interpretation.

Let us return to our gamble g. The risk in accepting g clearly depends on how much wealth you have. If all you have is \$100 or less, then it is extremely risky to accept g: you risk going bankrupt (assume there is no "Chapter 11," etc.). But if your wealth is, say, \$1,000,000, then accepting g is not risky at all (and recall that the expectation of g is positive). While one might expect a smooth transition between these two situations, we will show that there is in fact a well-defined *critical wealth level* that separates between two very different "regimes": one where it is "risky" to accept the gamble, the other where it isn't.[2]

[1] Take g above; increasing the gain from \$120 to \$150 and decreasing the loss from \$100 to \$90 makes the standard deviation increase from \$110 to \$120.

[2] We distinguish between the terms "risky" and "riskiness": the former is a property of decisions, the latter of gambles. Thus, accepting a gamble in a certain situation may be a *risky decision* (or not), whereas the *riskiness of a gamble* is a measure that, as we shall see, determines when the decision to accept the gamble is risky.

What does "risky" mean, and what is this critical level? For this purpose we consider a very simple model, in which a decision-maker faces an unknown sequence of gambles. Each gamble is offered in turn, and may be either accepted or rejected; or, in a slightly more general setup, any proportion of the gamble may be accepted.

We show that for every gamble g there exists a unique critical wealth level $\mathbf{R}(g)$ such that accepting gambles g when the current wealth is below the corresponding $\mathbf{R}(g)$ leads to "bad" outcomes, such as decreasing wealth and even bankruptcy in the long run; in contrast, not accepting gambles g when the current wealth is below $\mathbf{R}(g)$ yields "good" outcomes: no-bankruptcy is guaranteed, and wealth can only increase in the long run.[3] In fact, almost any reasonable criterion—such as no-loss, or an assured gain, or no-bankruptcy—will be shown to lead to exactly the same critical point $\mathbf{R}(g)$. We will call $\mathbf{R}(g)$ the *riskiness* of the gamble g, as it provides a sharp distinction between the "risky" and the "non-risky" decisions. The risky decisions are precisely those of accepting gambles g whose riskiness $\mathbf{R}(g)$ is too high, specifically, higher than the current wealth W (i.e., $\mathbf{R}(g) > W$): they lead to bad outcomes and possibly bankruptcy.

Moreover, the riskiness measure $\mathbf{R}$ that we obtain satisfies all our initial desiderata: it is objective (it depends only on the gamble), it is scale-invariant and thus measured in the same unit as the outcomes,[4] it is monotonic (increasing gains and/or decreasing losses lowers the riskiness), it has a simple operational interpretation, and, finally, is given by a simple formula. We emphasize that our purpose is not to analyze the most general investment and bankruptcy models, but rather to use such simple operational setups as a sort of "thought experiment" in order to determine the riskiness of gambles.

In summary, what we show is that there is a clear and robust way to identify exactly when it becomes risky to accept a gamble, and then

the *riskiness* of a gamble g is defined as

[3] All these occur with probability one (i.e., almost surely); see Sections 3 and 4 for the precise statements.

[4] I.e., the unit ("currency") in which the outcomes are measured does not matter: rescaling all outcomes by a constant factor $\lambda > 0$ rescales the riskiness by the same λ.

the *critical wealth* below which accepting g becomes risky.

The paper is organized as follows. The basic model of no-bankruptcy is presented in Section 2, followed in Section 3 by the result that yields the measure of riskiness. Section 4 extends the setup and shows the robustness of the riskiness measure; an illustrating example is provided at the end of the section. The properties of the riskiness measure are studied in Section 5. We conclude with a discussion of the literature and other pertinent issues in Section 6; this includes, in particular, the recent "economic index of riskiness" of Aumann and Serrano (2007)—which was the starting point of our research—and the "calibration" of Rabin (2000). The proofs are relegated to the Appendix.

2 The Basic Model

This section and the next deal with the simple basic model; it is generalized in Section 4.

2.1 Gambles

A *gamble* g is a real-valued random variable[5] having some negative values—losses are possible—and positive expectation, i.e., $\mathbf{P}[g < 0] > 0$ and $\mathbf{E}[g] > 0$. For simplicity[6] we assume that each gamble g has finitely many values, say $x_1, x_2, ..., x_m$, with respective probabilities $p_1, p_2, ..., p_m$ (where $p_i > 0$ and $\sum_{i=1}^m p_i = 1$). Let $\mathcal{G}$ denote the collection of all such gambles.

Some useful notations: $L(g) := -\min_i x_i > 0$ is the *maximal loss* of g; $M(g) := \max_i x_i > 0$ is the *maximal gain* of g; and $\|g\| := \max_i |x_i| = \max\{M(g), L(g)\}$ is the (ℓ_∞-)*norm* of g. One way to view g is that one buys a "ticket" to g at a cost of $L(g) > 0$; this ticket yields various prizes $L(g) + x_i$ with probability p_i each (and so there is a positive probability of getting no prize—when $x_i = -L(g)$).

[5] We take g to be a random variable for convenience; only the distribution of g will matter. $\mathbf{P}$ denotes "probability."

[6] A significant assumption here is that of "limited liability"; see Section 6.5 (g).

2.2 Gambles and Wealth

Let the initial wealth be W_1. At every period $t = 1, 2, ...$, the decision-maker, whose current wealth we denote W_t, is offered a gamble $g_t \in \mathcal{G}$ that he may either accept or reject. If he accepts g_t then his wealth next period will be $W_{t+1} = W_t + g_t$; and, if he rejects g_t, then $W_{t+1} = W_t$. Exactly which gamble g_t is offered may well depend on the period t and the past history (of gambles, wealth levels, and decisions); thus, there are no restrictions on the stochastic dependence between the random variables g_t. Let G denote the *process* $(g_t)_{t=1,2,...}$. We emphasize that there is no underlying probability distribution on the space of processes from which G is drawn; the setup is *non-Bayesian*, and the analysis is "worst-case." Thus, at time t the decision-maker knows nothing about which future gambles he will face nor how his decisions will influence them.

To avoid technical issues, it is convenient to consider only *finitely generated* processes; such a process G is generated by a finite set of gambles $\mathcal{G}_0 = \{g^{(1)}, g^{(2)}, ..., g^{(m)}\} \subset \mathcal{G}$ such that each g_t is a nonnegative multiple of some gamble in $\mathcal{G}_0$; more precisely, the gamble g_t that is offered following any history belongs to the finitely generated[7] cone $\mathcal{G}_0 = \{\lambda g : \lambda \geq 0 \text{ and } g \in \mathcal{G}_0\}$.

2.3 Critical Wealth and Simple Strategies

As discussed in the Introduction, we are looking for simple rules that distinguish between situations that are deemed "risky" and those that are not: the offered gamble is rejected in the former and accepted in the latter. Such rules—think of them as candidate riskiness measures—are given by a "threshold" that depends only on the distribution of the gamble and is scale-invariant. That is, there is a *critical-wealth function* Q that associates to each gamble $g \in \mathcal{G}$ a number $Q(g)$ in $[0, \infty]$, with $Q(\lambda g) = \lambda Q(g)$ for every $\lambda > 0$, and is used as follows: a gamble g is rejected at wealth W if $W < Q(g)$, and is accepted if $W \geq Q(g)$. We will refer to the behavior induced by such a function Q as a *simple strategy*, and denote it s_Q. Thus

[7] λg means that the values of g are rescaled by the factor λ, whereas the probabilities do not change (this is not to be confused with the "dilution" of Section 5).

s_Q accepts g at wealth $Q(g)$ and at any higher wealth, and rejects g at all lower wealths: $Q(g)$ is the minimal wealth at which g is accepted. In the two extreme cases, $Q(g) = 0$ means that g is always accepted (i.e., at every wealth $W > 0$), whereas $Q(g) = \infty$ means that g is always rejected.[8]

2.4 No-Bankruptcy

Since risk has to do with losing money, and in the extreme, bankruptcy, we start by studying the simple objective of avoiding bankruptcy.

Assume that the initial wealth is positive (i.e., $W_1 > 0$), and that borrowing is not allowed (so $W_t \geq 0$ for all t).[9] *Bankruptcy* occurs when the wealth becomes zero, or, more generally, when it converges to zero, i.e, $\lim_{t\to\infty} W_t = 0$. The strategy s yields *no-bankruptcy* for the process G and the initial wealth W_1 if the probability of bankruptcy is zero, i.e.,[10] $\mathbf{P}\left[\lim_{t\to\infty} W_t = 0\right] = 0$. Finally, the strategy s *guarantees no-bankruptcy* if it yields no-bankruptcy for every process G and every initial wealth W_1. Thus, no matter what the initial wealth is and what the sequence of gambles will be, the strategy s guarantees that the wealth will not go to zero (with probability one).

3 The Measure of Riskiness

The result in the basic setup is:

Theorem 1 *For every gamble $g \in \mathcal{G}$ there exists a unique real number $\mathbf{R}(g) > 0$ such that a simple strategy $s \equiv s_Q$ with critical-wealth function Q guarantees no-bankruptcy if and only if $Q(g) \geq \mathbf{R}(g)$ for every gamble $g \in \mathcal{G}$. Moreover, $\mathbf{R}(g)$ is uniquely determined by the equation*

$$\mathbf{E}\left[\log\left(1 + \frac{1}{\mathbf{R}(g)} g\right)\right] = 0. \tag{1}$$

[8] See Section 6 (f) and Appendix A.7 for more general strategies.

[9] If borrowing is allowed up to some maximal credit limit C, then shift everything by C (see also Section 6.3).

[10] $\mathbf{P} \equiv \mathbf{P}_{W_1,G,s}$ is the probability distribution induced by the initial wealth W_1, the process G, and the strategy s.

The condition $Q(g) \geq \mathbf{R}(g)$ says that the minimal wealth level $Q(g)$ where g is accepted must be $\mathbf{R}(g)$ or higher, and so g is for sure rejected at all wealth levels below $\mathbf{R}(g)$, i.e., at all $W < \mathbf{R}(g)$. Therefore we get

Corollary 2 *A simple strategy s guarantees no-bankruptcy if and only if for every gamble $g \in \mathcal{G}$*

$$s \text{ rejects } g \text{ at all } W < \mathbf{R}(g). \tag{2}$$

Thus $\mathbf{R}(g)$ is the minimal wealth level where g may be accepted; as discussed in the Introduction, it is the *measure of riskiness of* g.

Simple strategies s satisfying (2) differ in terms of which gambles are accepted. The "minimal" strategy, with $Q(g) = \infty$ for all g, never accepts any gamble; the "maximal" one, with $Q(g) = \mathbf{R}(g)$ for all g, accepts g as soon as the wealth is at least as large as the riskiness of g; these two strategies, as well as any strategy in between, guarantee no-bankruptcy (see also Proposition 12 in Appendix A.1). We emphasize that condition (2) does *not* say when to accept gambles, but merely when a simple strategy *must reject* them, to avoid bankruptcy. Therefore $\mathbf{R}(g)$ may also be viewed as a sort of minimal "reserve" needed for g.

Consider for example gambles g where it is equally likely to gain a or lose b (with $0 < b < a$ so that $g \in \mathcal{G}$); then $\mathbf{E}[\log(1 + g/R)] = 0$ if and only if $(1 + a/R)(1 - b/R) = 1$, and so $\mathbf{R}(g) = ab/(a - b)$ by formula (1). In particular, for $a = 120$ and $b = 100$ we get $\mathbf{R}(g) = 600$, and for $a = 105$ and $b = 100$ we get $\mathbf{R}(g) = 2100$.

The proof of Theorem 1 is relegated to Appendix A.1; an illustrating example is provided in Section 4.

4 Extension: The Shares Setup

We will now show that the distinction made in the previous section between the two "regimes" is robust, and does not hinge on the extreme case of long-run bankruptcy. To do so we slightly extend our setup by allowing the decision-maker to take *any proportion* of the offered gamble. This results in a sharper distinction—in the limit, bankruptcy on one side and wealth growing to infinity on the other—which moreover becomes evident already after finitely many periods. The intuitive reason is that it is now possible to overcome short-term losses by taking appropriately small proportions of the offered gambles (which is not the case in the basic model of Section 3, where all future gambles may turn out to be too risky relative to the wealth).

Formally, in this setup—which we call the *shares setup*—the decision-maker can accept any nonnegative multiple of the offered gamble g_t (i.e., $\alpha_t g_t$ for some $\alpha_t \geq 0$) rather than just accept or reject g_t (which corresponds to $\alpha_t \in \{0, 1\}$). Think for instance of investments that can be made in arbitrary amounts (shares of equities). Let $Q : \mathcal{G} \to (0, \infty)$ be a critical-wealth function (we no longer allow $Q(g) = 0$ and $Q(g) = \infty$) with $Q(\lambda g) = \lambda Q(g)$ for all $\lambda > 0$. The corresponding *simple shares strategy* $s \equiv s_Q$ is as follows: at wealth $Q(g)$ one accepts g (i.e., $\alpha = 1$), and at any wealth W one accepts the proportion $\alpha = W/Q(g)$ of g; that is, the gamble αg that is taken is exactly the one for which $Q(\alpha g) = W$. The result is:

Theorem 3 *Let $s \equiv s_Q$ be a simple shares strategy with critical-wealth function Q. Then:*

(i) $\lim_{t\to\infty} W_t = \infty$ *(a.s.) for every process G if and only if $Q(g) > \mathbf{R}(g)$ for every gamble $g \in \mathcal{G}$.*

(ii) $\lim_{t\to\infty} W_t = 0$ *(a.s.) for some process G if and only if $Q(g) < \mathbf{R}(g)$ for some gamble $g \in \mathcal{G}$.*

Theorem 3 is proved in Appendix A.2 (Proposition 14 there provides a more precise result). Thus, our measure of riskiness $\mathbf{R}$ provides the threshold between two very different "regimes": bankruptcy (i.e., $W_t \to 0$ almost surely, when the riskiness of the accepted gambles is higher than the wealth), and infinite wealth (i.e., $W_t \to \infty$ almost surely, when the riskiness of the accepted gambles is lower than the wealth). As a consequence, one may replace the "no-bankruptcy" criterion with various other criteria, such as:

- **no-loss**: $\liminf_{t\to\infty} W_t \geq W_1$ (a.s.);

- **bounded loss**: $\liminf_{t\to\infty} W_t \geq W_1 - C$ (a.s.) for some $C < W_1$, or $\liminf_{t\to\infty} W_t \geq cW_1$ (a.s.) for some $c > 0$;

- **assured gain**: $\liminf_{t\to\infty} W_t \geq W_1 + C$ (a.s.) for some $C > 0$, or $\liminf_{t\to\infty} W_t \geq (1+c)W_1$ (a.s.) for some $c > 0$;

- **infinite gain**: $\lim_{t\to\infty} W_t = \infty$ (a.s.).

Moreover, in the no-bankruptcy as well as any of the above conditions, one may replace "almost surely" (a.s.) by "with positive probability." For each one of these criteria, Theorem 3 implies that the threshold is the same: it is given by the riskiness function $\mathbf{R}$. For example:

Corollary 4 *A simple shares strategy s_Q guarantees no-loss* if $Q(g) > \mathbf{R}(g)$ *for all g, and* only if $Q(g) \geq \mathbf{R}(g)$ *for all g.*

By way of illustration, take the gamble g of the Introduction where it is equally likely to gain \$120 or lose \$100, and consider the situation where one faces a sequence of gambles g_t that are independent draws from g. Let $q := Q(g)$ be the critical wealth that is used for g; then in each period one takes the proportion $\alpha_t = W_t/q$ of g_t. Therefore $W_{t+1} = W_t + \alpha_t g_t = W_t + (W_t/q)g_t = W_t(1 + (1/q)g_t)$, and so $W_{t+1} = W_1 \prod_{i=1}^{t}(1 + (1/q)g_t)$. Assume first that $Q(g) = \$200$; then $1 + (1/q)g_t$ equals either $1 + 120/200 = 160\%$ or $1 - 100/200 = 50\%$ with equal probabilities (these are the *relative gross returns* of g when the wealth is \$200; in *net* terms, a gain of 60% or a loss of 50%). In the long run, by the Law of Large Numbers, about half the time the wealth will be multiplied by a factor of 160%, and about half the time by a factor of 50%. So, on average, the wealth will be multiplied by a factor of $\gamma = \sqrt{1.6 \cdot 0.5} < 1$ per period, which implies that it will almost surely converge to zero[11]: bankruptcy! Now assume that we use $Q(g) = \$1000$ instead; the relative gross returns become $1+120/1000 = 112\%$ or $1-100/1000 = 90\%$, which yield a factor of $\gamma = \sqrt{1.12 \cdot 0.9} > 1$ per period,

[11] W_{t+1} will be close to $W_1(1.6)^{t/2}(0.5)^{t/2} = W_1\gamma^t \to_{t\to\infty} 0$. Intuitively, to offset a loss of 50% it needs to be followed by a gain of 100% (since the basis has changed); a 60% gain does not suffice.

and so the wealth will almost surely go to infinity rather than to zero. The critical point is at $Q(g) = \$600$, where the per-period factor becomes $\gamma = 1$; the *riskiness* of g is precisely $\mathbf{R}(g) = \$600$.

Indeed, accepting g when the wealth is less than \$600 yields "risky" returns—returns of the kind that if repeated lead in the long run to bankruptcy; in contrast, accepting g only when the wealth is more than \$600 yields returns of the kind that guarantee no-bankruptcy, and lead to increasing wealth in the long run. We point out that these conclusions do not depend on the independent identically distributed sequence that we have used in the illustration above; any sequence of returns of the first kind leads to bankruptcy, and of the second kind, to infinite growth.

The criteria up to now were all formulated in terms of the limit as t goes to infinity. However, the distinction between the two situations can be seen already after relatively few periods: the distribution of wealth will be very different. In the example above, the probability that there is no loss after t periods (i.e., $\mathbf{P}[W_{t+1} \geq W_1]$) is, for $t = 100$, about 2.7% when one uses $Q(g) = \$200$, and about 64% when $Q(g) = \$1000$; these probabilities become $10^{-7}\%$ and 87%, respectively, for $t = 1000$. In terms of the median wealth, after $t = 100$ periods it is only 0.0014% of the original wealth when $Q(g) = \$200$ vs. 148% of it when $Q(g) = \$1000$ (for $t = 1000$ these numbers are $10^{-46}\%$ and $5,373\%$, respectively).[12]

5 Properties of the Measure of Riskiness

The riskiness measure enjoys many useful properties; they all follow from definition (1). A number of basic properties are collected in Proposition 5 below, following which we discuss two issues of particular interest: stochastic dominance and continuity.

Some notations: Given $0 < \lambda < 1$ and the gamble g that takes the values $x_1, x_2, ..., x_m$ with respective probabilities $p_1, p_2, ..., p_m$, the λ-*dilution of* g, denoted $\lambda * g$, is the gamble that takes the same values $x_1, x_2, ..., x_m$ but now

[12]Taking $Q(g) = \$500$ and $Q(g) = \$700$ (closer to $\mathbf{R}(g) = \$600$) yields after $t = 1000$ a median wealth that is 1.8% and 766%, respectively, of the original wealth.

with probabilities $\lambda p_1, \lambda p_2, ..., \lambda p_m$, and takes the value 0 with probability $1-\lambda$; that is, with probability λ the gamble g is performed, and with probability $1-\lambda$ there is no gamble.

Proposition 5 *For all gambles*[13] $g, h \in \mathcal{G}$*:*

(i) Distribution*: If* g *and* h *have the same distribution then* $\mathbf{R}(g) = \mathbf{R}(h)$.

(ii) Homogeneity*:* $\mathbf{R}(\lambda g) = \lambda \mathbf{R}(g)$ *for every* $\lambda > 0$.

(iii) Maximal loss*:* $\mathbf{R}(g) > L(g)$.

(iv) Subadditivity*:* $\mathbf{R}(g+h) \leq \mathbf{R}(g) + \mathbf{R}(h)$.

(v) Convexity*:* $\mathbf{R}(\lambda g+(1-\lambda)h) \leq \lambda \mathbf{R}(g)+(1-\lambda)\mathbf{R}(h)$ *for every* $0 < \lambda < 1$.

(vi) Dilution*:* $\mathbf{R}(\lambda * g) = \mathbf{R}(g)$ *for every* $0 < \lambda \leq 1$.

(vii) Independent gambles*: If* g *and* h *are independent random variables then* $\min\{\mathbf{R}(g), \mathbf{R}(h)\} < \mathbf{R}(g+h) < \mathbf{R}(g) + \mathbf{R}(h)$.

Moreover, there is equality in (iv) and (v) if and only if g *and* h *are proportional (i.e.,* $h = \lambda g$ *for some* $\lambda > 0$*).*

Thus, only the distribution of a gamble determines its riskiness; the riskiness is always larger than the maximal loss (which may be viewed as an "immediate one-shot risk"); the riskiness measure is positively homogeneous of degree one and subadditive, and thus convex; diluting a gamble does not affect the riskiness[14]; and the riskiness of the sum of independent gambles lies between the minimum of the two riskinesses and their sum.

Proofs can be found in Appendix A.3; see Proposition 15 there for sequences of independent and identically distributed gambles.

[13] In (iv), (v), and (vii) g and h are random variables defined on the same probability space.

[14] In our setup of sequences of gambles, dilution by a factor λ translates into "rescaling time" by a factor of $1/\lambda$ (for example $\lambda = 1/2$ corresponds to being offered a gamble on average once every two periods). Such a rescaling does not affect the long-run outcome, and therefore explains why the riskiness does not change.

5.1 Stochastic Dominance

There are certain situations where one gamble g is clearly "less risky" than another gamble h. One such case is when in every instance the value that g takes is larger than the value that h takes. Another is when some values of h are replaced in g by their expectation (this operation of going from g to h is called a "mean-preserving spread"). These two cases correspond to "first-order stochastic dominance" and "second-order stochastic dominance," respectively; see Hadar and Russell (1969), Hanoch and Levy (1969), Rothschild and Stiglitz (1970, 1971).

Formally, a gamble g *first-order stochastically dominates* a gamble h, which we write $g \,\mathrm{SD}_1\, h$, if there exist gambles g' and h' defined on the same probability space such that: g and g' have the same distribution; h and h' have the same distribution; $g' \geq h'$; and $g' \neq h'$. Similarly, g *second-order stochastically dominates* h, which we write $g \,\mathrm{SD}_2\, h$, if there exist g' and h' as above, but now the condition "$g' \geq h'$" is replaced by "h' is obtained from g' by a finite sequence of mean-preserving spreads, or as the limit of such a sequence."

The importance of stochastic dominance lies in the fact that, for expected-utility decision-makers (who have a utility function u on outcomes and evaluate each gamble g by[15] $\mathbf{E}\left[u(g)\right]$), we have the following: $g \,\mathrm{SD}_1\, h$ if and only if g is strictly preferred to h whenever the utility function is strictly increasing; and $g \,\mathrm{SD}_2\, h$ if and only if g is strictly preferred to h whenever the utility function is also strictly concave.

Our riskiness measure is monotonic with respect to stochastic dominance: a gamble that dominates another has a lower riskiness. In contrast, this desirable property is not satisfied by most existing measures of riskiness (see Section 6.4).

Proposition 6 *If g first-order stochastically dominates h, or if g second-order stochastically dominates h, then $\mathbf{R}(g) < \mathbf{R}(h)$.*

Proposition 6 is proved in Appendix A.4.

[15] Or, if the wealth W is taken into account, by $\mathbf{E}\left[u(W+g)\right]$.

5.2 Continuity

The natural notion of convergence for gambles is convergence in distribution (after all, only the distribution of the gamble determines the riskiness; see Proposition 5 (i)). Roughly speaking, gambles are close in distribution if they take similar values with similar probabilities. Formally, a sequence of gambles $(g_n)_{n=1,2,\ldots} \subset \mathcal{G}$ converges *in distribution* to a gamble $g \in \mathcal{G}$, denoted $g_n \overset{\mathcal{D}}{\to} g$, if $\mathbf{E}[\phi(g_n)] \to \mathbf{E}[\phi(g)]$ for every bounded and uniformly continuous real function ϕ (see Billingsley 1968). We get the following result:

Proposition 7 *Let $(g_n)_{n=1,2,\ldots} \subset \mathcal{G}$ be a sequence of gambles with uniformly bounded values; i.e., there exists a finite K such that $|g_n| \leq K$ for all n. If $g_n \overset{\mathcal{D}}{\to} g \in \mathcal{G}$ and $L(g_n) \to L(g)$ as $n \to \infty$, then $\mathbf{R}(g_n) \to \mathbf{R}(g)$ as $n \to \infty$.*

Proposition 7 is proved in Appendix A.5, as a corollary of a slightly more general continuity result (Proposition 16).

To see that the condition $L(g_n) \to L(g)$ is indispensable, let g_n take the values $2, -1$, and -3 with probabilities $(1/2)(1 - 1/n)$, $(1/2)(1 - 1/n)$, and $1/n$, respectively, and let g take the values 2 and -1 with probabilities $1/2$, $1/2$. Then $g_n \overset{\mathcal{D}}{\to} g$ but $L(g_n) = 3 \neq 1 = L(g)$, and $\mathbf{R}(g_n) \to 3 \neq 2 = \mathbf{R}(g)$.

Though at first sight the discontinuity in the above example may seem disconcerting, it is nevertheless natural, and our setup helps to clarify it.[16] Even if the maximal loss $L(g_n)$ has an arbitrarily small—but *positive*—probability, it still affects the riskiness. After all, this maximal loss *will eventually occur*, and to avoid bankruptcy the wealth must be sufficiently large to overcome it. The fact that the probability is small only implies that it may take a long time to occur. But occur it will!

Interestingly, a similar point has been recently made by Taleb (2005): highly improbable events that carry a significant impact (called "black swans") should *not* be ignored. One may make money for a very long time, but if one ignores the very low probability possibilities, then one will eventually lose everything.

[16]Other measures of riskiness, such as that of Aumann and Serrano (2007), are continuous even when $L(g_n)$ does *not* converge to $L(g)$.

6 Discussion and Literature

This section is devoted to several pertinent issues and connections to the existing literature. We start with the recently proposed "index of riskiness" of Aumann and Serrano (2007), continue with matters concerning utility, risk aversion, wealth, and the "calibration" of Rabin (2000), discuss other measures of riskiness, and conclude with a number of general comments.

6.1 Aumann and Serrano's Index of Riskiness

Aumann and Serrano (2007) (henceforth "**A&S**") have recently proposed an *economic index of riskiness*, which associates to every gamble $g \in \mathcal{G}$ a unique number $R^{AS}(g)$ as follows. Consider the decision-maker with constant (Arrow–Pratt) absolute risk aversion coefficient α (his utility function is thus[17] $u(x) = -\exp(-\alpha x)$) who is indifferent between accepting and rejecting g; put $R^{AS}(g) = 1/\alpha$. The following equation thus defines $R^{AS}(g)$ uniquely:

$$\mathbf{E}\left[\exp\left(-\frac{1}{R^{AS}(g)}g\right)\right] = 1. \tag{3}$$

The approach of A&S is based on comparing the riskiness of gambles. They say that a gamble g is *less risky* than a gamble h if whenever a "more risk-averse" decision-maker accepts h, a "less risk-averse" decision-maker accepts g. Here, "more" and "less" risk aversion must hold at any two wealth levels (which leads one to consider decision-makers that have constant absolute risk aversion, i.e., CARA utilities). The result of A&S is essentially as follows. An *index* Q defined on all gambles respects the riskiness order (i.e., $Q(g) < Q(h)$ when g is less risky than h) and is homogeneous of degree one if and only if Q is proportional to R^{AS} (as defined by (3)).

Comparing this to our approach, we note the following distinctions:

(i) R^{AS} is an *index of riskiness*, based on *comparing* the gambles in terms of their riskiness. Our $\mathbf{R}$ is a *measure* of riskiness, defined for each gamble separately (see Section 6.5 (b)).

[17] This is the class of *CARA* utility functions; $\exp(x)$ stands for e^x.

(ii) R^{AS} is based on risk-averse expected-utility decision-makers. Our approach completely dispenses with utility functions and risk aversion, and just compares two situations: bankruptcy vs. no-bankruptcy, or, even better (Section 4), bankruptcy vs. infinite growth.

(iii) R^{AS} is based on the *critical* level of risk aversion, whereas our $\mathbf{R}$ is based on the *critical* level of wealth. Moreover, the comparison between decision-makers in A&S—being "more" or "less" risk averse—must hold at *all wealth levels*. We thus have an interesting "duality": R^{AS} looks for the *critical* risk-aversion coefficient *regardless* of wealth, whereas $\mathbf{R}$ looks for the *critical* wealth *regardless* of risk aversion.

(iv) Our approach yields a measure $\mathbf{R}$ whose unit and normalization are well defined, while A&S are free to choose any positive multiple of R^{AS}. Moreover, the number $\mathbf{R}(g)$ has a clear operational interpretation, which, at this point, has not yet been obtained for $R^{AS}(g)$. In fact, our work originally started as an attempt to provide such an interpretation for $R^{AS}(g)$, but it led to a different measure of riskiness.

The two approaches thus appear quite different in many respects, both conceptually and practically. Nevertheless, they share many properties (compare Section 5 above with Section 5 in A&S).[18] Moreover, they turn out to yield similar values in many examples. To see why, rewrite (3) as $\mathbf{E}\left[1 - \exp\left(-g/R^{AS}(g)\right)\right] = 0$, and compare it to our equation (1), $\mathbf{E}\left[\log(1 + g/\mathbf{R}(g))\right] = 0$. Now the two relevant functions, $\log(1 + x)$ and $1 - \exp(-x)$, are close for small x: their Taylor series around $x = 0$ are

$$\begin{aligned} \log(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \quad \text{and} \\ 1 - \exp(-x) &= x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 + \cdots . \end{aligned}$$

The two series differ only from their third-order terms on; this suggests that when $g/R(g)$ is small—i.e., when the riskiness is large relative to the gamble—the two approaches should yield similar answers.

To see this formally, it is convenient to keep the gambles bounded, from

[18] The only differences concern continuity and independent gambles.

above and from below, and let their riskiness go to infinity (recall that both $\mathbf{R}$ and R^{AS} are homogeneous of degree one); as we will see below, this is equivalent to letting their expectation go to zero. The notation $a_n \sim b_n$ means that $a_n/b_n \to 1$ as $n \to \infty$.

Proposition 8 *Let $(g_n)_{n=1,2,\ldots} \subset \mathcal{G}$ be a sequence of gambles such that there exist $K < \infty$ and $\kappa > 0$ with $|g_n| \leq K$ and $\mathbf{E}[|g_n|] \geq \kappa$ for all n. Then the following three conditions are equivalent:*

(i) $\mathbf{E}[g_n] \to 0$ *as* $n \to \infty$.

(ii) $\mathbf{R}(g_n) \to \infty$ *as* $n \to \infty$.

(iii) $R^{AS}(g_n) \to \infty$ *as* $n \to \infty$.

Moreover, in this case $\mathbf{R}(g_n) \sim R^{AS}(g_n)$ as $n \to \infty$.

Thus, when the expectation goes to zero, both measures go to infinity; and if one of them goes to infinity, then the other does so too—and moreover, they become approximately equal. Proposition 8 is proved in Appendix A.6. We note here another general relation that has been obtained in A&S[19]: for every $g \in \mathcal{G}$,

$$-L(g) < R^{AS}(g) - \mathbf{R}(g) < M(g). \tag{4}$$

6.2 Utility and Risk Aversion

Consider an expected-utility decision-maker with utility function u, where $u(x)$ is the utility of wealth x. The utility function u generates a strategy $s \equiv s^u$ as follows: accept the gamble g when the wealth is W if and only if by doing so the expected utility will not go down, i.e.,

$$\text{accept } g \text{ at } W \text{ if and only if } \mathbf{E}[u(W+g)] \geq u(W); \tag{5}$$

[19] A&S show that a decision-maker with log utility (see Section 6.2) accepts g at all $W > R^{AS}(g) + L(g)$ and rejects g at all $W < R^{AS}(g) - M(g)$, and so $\mathbf{R}(g)$ must lie between these two bounds.

equivalently, the expected utility from accepting g at W is no less than the utility from rejecting g at W.

A special case is the *logarithmic utility* $u(x) = \log x$ (also known as the "Bernoulli utility"). The riskiness measure $\mathbf{R}$ turns out to be characterized by the following property. For every gamble g, the logarithmic utility decision-maker is indifferent between accepting and rejecting g when his wealth W equals exactly $\mathbf{R}(g)$, and he strictly prefers to reject g at all $W < \mathbf{R}(g)$, and to accept g at all $W > \mathbf{R}(g)$; this follows from (1) and Lemma 9 in Appendix A.1, since $\mathbf{E}[\log(1 + g/\mathbf{R}(g))] = \mathbf{E}[\log(\mathbf{R}(g) + g)] - \log(\mathbf{R}(g))$. Therefore the condition (2) of rejecting a gamble when its riskiness is higher than the current wealth, i.e., when $W < \mathbf{R}(g)$, can be restated as follows: *reject any gamble that the logarithmic utility rejects.*

The logarithmic utility is characterized by a *constant relative risk aversion* coefficient of 1 (i.e., $\gamma_u(x) := -xu''(x)/u'(x) = 1$ for all $x > 0$). More generally, consider the class *CRRA* of utility functions that have a constant relative risk aversion coefficient, i.e., $\gamma_u(x) = \gamma > 0$ for all $x > 0$ (the corresponding utility functions are $u_\gamma(x) = x^{1-\gamma}/(1-\gamma)$ for $\gamma \neq 1$, and $u_1(x) = \log x$ for $\gamma = 1$). It can be checked that these are exactly the utility functions for which the resulting strategy s^u turns out to be a simple strategy (i.e., with a critical-wealth function that is homogenenous; use for instance Corollary 3 and Lemma 4 in Section 8.1 of Aumann and Serrano 2007). Since a higher risk aversion means that more gambles are rejected, our main result (see Corollary 2) yields the following: *no-bankruptcy is guaranteed for a CRRA utility u_γ if and only if the relative risk aversion coefficient γ satisfies*[20] $\gamma \geq 1$.

Given a general utility function u (which is not necessarily CRRA, and therefore the resulting strategy s^u is not necessarily homogeneous), assume for simplicity that the relative risk aversion coefficient at 0 is well defined; i.e., the limit $\gamma_u(0) := \lim_{x \to 0^+} \gamma_u(x)$ exists. Then Proposition 17 yields the following result: $\gamma_u(0) > 1$ guarantees no-bankruptcy, and guaranteed no-

[20] It is interesting how *absolute* risk aversion and CARA utilities have come out of the Aumann and Serrano (2007) approach, and *relative* risk aversion and CRRA utilities out of ours—in each case, as a *result* and not an assumption.

bankruptcy implies that[21] $\gamma_u(0) \geq 1$. It is interesting how the conclusion of a *relative risk aversion coefficient of at least* 1 has been obtained from the simple and basic requirement of no-bankruptcy, or any of the alternative criteria in Section 4.[22]

6.3 Wealth and Calibration

We come now to the issue of what is meant by "wealth."

Our basic setup assumes that the decision-maker wants to avoid bankruptcy (i.e., $W_t \to 0$). This can be easily modified to accommodate any other minimal level of wealth $\underline{W}$ which must be guaranteed: just add $\underline{W}$ throughout. Thus, rejecting g at W when $W < \underline{W} + \mathbf{R}(g)$ guarantees that $W_t \geq \underline{W}$ for all t and $\mathbf{P}\left[\lim_{t\to\infty} W_t = \underline{W}\right] = 0$ (this follows from Proposition 12 in Appendix A.1).

If, say, $\underline{W}$ is the wealth that is needed and earmarked for purposes like living expenses, housing, consumption, and so on, then $\mathbf{R}(g)$ should be viewed as the least "reserve wealth" that is required to cover the possible losses without going bankrupt, or, more generally, without going below the minimal wealth level $\underline{W}$. That is, $\mathbf{R}(g)$ is not the total wealth needed, but only the additional amount above $\underline{W}$. Therefore, if that part of the wealth that is designated for no purpose other than taking gambles—call it "gambling wealth" or "risky investment wealth"—is below $\mathbf{R}(g)$, then g must be rejected.

This brings us to the "calibration" of Rabin (2000). Take a risk-averse expected-utility decision-maker and consider for example the following two gambles: the gamble g where he gains \$105 or loses \$100 with equal probabilities, and the gamble h where he gains \$5,500,000 (five and a half million dollars) or loses \$10,000 with equal probabilities. Rabin proves that: *if* (i) g is rejected at *all wealth levels* $W <$ \$300,000, *then* (ii) h must be rejected at wealth $W =$ \$290,000.

[21] The knife-edge case of $\gamma_u(0) = 1$ can go either way: consider $u^1(x) = \log x$, and $u^2(x) = \exp(-\sqrt{-\log x})$ for small x.

[22] So, perhaps, the reason that many empirical studies indicate relative risk aversion coefficients larger than 1 (see, e.g., Palacios-Huerta and Serrano 2006) is that agents with coefficient less than 1 may already be bankrupt (and thus not part of the studies).

If one were to interpret the wealth W as gambling wealth, then our result suggests that the premise (i) that one rejects g at all $W < \$300,000$ is not plausible, since $\mathbf{R}(g)$ is only $\$2,100$. If, on the other hand, wealth were to be interpreted as total wealth, then, as we saw above, (i) is consistent with wanting to preserve a minimal wealth level $\underline{W}$ of at least $\$297,900 = \$300,000 - \$2,100$. If that is the case then a wealth of $\$290,000$ is below the minimal level $\underline{W}$, and so it makes sense to reject h there.

Thus, if wealth in the Rabin setup is gambling wealth, then the assumption (i) is not reasonable and so it does not matter whether the conclusion (ii) is reasonable or not.[23] And if it is total wealth, then both (i) and (ii) are reasonable, because such behavior is consistent with wanting to keep a certain minimal wealth level $\underline{W} \geq \$297,000$. In either case our setup suggests that there is nothing "implausible" here, as Rabin argues there is (and which leads him to cast doubts on the usefulness and appropriateness of expected utility theory).[24,25]

6.4 Other Measures of Riskiness

Risk is a central issue, and various measures of riskiness have been proposed. We have already discussed in Section 6.1 the recent index of Aumann and Serrano (2007), which is the closest to ours.

Almost all the riskiness measures in the literature (and in practice) turn out to be non-monotonic with respect to first-order stochastic dominance, which, as has been repeatedly pointed out by various authors, is a very reasonable—if not necessary—requirement. Indeed, if in every instance the gains increase and the losses decrease, how can the riskiness increase? Nevertheless, riskiness measures, particularly those based on the variance or other

[23]Palacios-Huerta and Serrano (2006) argue that (i) is unreasonable from an *empirical* point of view (their paper led to the theoretical work of Aumann and Serrano 2007).

[24]Safra and Segal (2006) show that similar issues arise in many non-expected utility models as well. Rubinstein (2001) makes the point that expected utility need not be applied to final wealth, and there may be inconsistencies between the preferences at different wealth levels.

[25]Of course, this applies provided that there is no "friction," such as hidden costs (e.g., in collecting the prizes) or "cheating" in the realization of the gambles.

measures of "dispersion" of the gamble, do not satisfy this monotonicity condition (see the survey of Machina and Rothschild 2007, and Section 7 in Aumann and Serrano 2007).

Artzner et al. (1999) have proposed the notion of a "coherent measure of risk," which is characterized by four axioms: "translation invariance" (T), "subadditivity" (S), "positive homogeneity" (PH), and "monotonicity" (M). Our measure $\mathbf{R}$ satisfies the last three axioms: (PH) and (S) are the same as (iii) and (iv) in Proposition 5, and (M), which is essentially monotonicity with respect to first-order stochastic dominance, is in Proposition 6. However, $\mathbf{R}$ does *not* satisfy (T), which requires that $R(g + c) = R(g) - c$ for every constant c (assuming no discounting; see Section 6.5 (g)); that is, adding the same number c to all outcomes of a gamble decreases the riskiness by exactly c. To see why this requirement is not appropriate in our setup, take for example the gamble g of the Introduction where one gains 120 or loses 100 with equal probabilities; its riskiness is $\mathbf{R}(g) = 600$. Now add $c = 100$ to all payoffs; the new gamble $g + 100$ has no losses, and so its riskiness should be 0, not[26] $500 = 600 - 100$.

See also Section 6.5 (a) below.

6.5 General Comments

(a) Universal and objective measure. Our approach looks for a "universal" and "objective" measure of riskiness. First, it abstracts away from the goals and the preference order of specific decision-makers (and so, *a fortiori*, from utility functions, risk aversion, and so on). The only property that is assumed is that no-bankruptcy is preferred to bankruptcy; or, in the shares setup, that infinite growth is preferred to bankruptcy.[27] Second, we make no assumptions on the sequence of gambles the decision-maker will face. And third, our measure does not depend on any ad hoc parameters that need

[26] Formally, $g + 100$ is not a gamble; so take instead, say, $g + 99.99$, where one gains 219.99 or loses 0.01—its riskiness can hardly be 500.01. The index of Aumann and Serrano (2007) likewise satisfies (S), (PH), and (M), but not (T).

[27] In particular, the fact that gambles with positive expectation are sometimes rejected—i.e., "risk-aversion"—is a *consequence* of our model, not an assumption.

to be specified (as is the case, for example, with the measure "Variance-at-Risk"—"VaR" for short—which depends on a "confidence level" $\alpha \in (0, 1)$).

Of course, if one is in a situation where additional specifications are available, then a different measure may result. The measure that we propose here may then be viewed as an ideal benchmark.

(b) Single gamble. While our model allows arbitrary sequences of gambles, the analysis can be carried out separately for any single gamble g (and its multiples); see the example in the shares setup of Section 4 and Proposition 13 in Appendix A.1. The riskiness $\mathbf{R}(g)$ of a gamble g is thus determined by considering g *only*; no comparisons with other gambles are needed.

(c) Returns. One may restate our model in terms of returns: accepting a gamble g at wealth W yields *relative gross returns* $X = (W + g)/W = 1 + g/W$. We will say that X has *B-returns* if $\mathbf{E}[\log X] < 0$, and that it has *G-returns* if $\mathbf{E}[\log X] > 0$ ("B" stands for "Bad" or "Bankruptcy," and "G" for "Good" or "Growth")[28]: a sequence of i.i.d. B-returns leads to bankruptcy, and of G-returns to infinite wealth (a.s.). Now, accepting g at W yields B-returns if and only if $W < \mathbf{R}(g)$, and G-returns if and only if $W > \mathbf{R}(g)$ (see Lemma 9 in Appendix A.1), and so $\mathbf{R}(g)$ is the *critical wealth level below which the returns become B-returns.*

(d) Acceptance. As pointed out in Section 3, our approach tells us when we must reject gambles—namely, when their riskiness exceeds the available wealth—but it does not say when to accept gambles. Any strategy satisfying condition (2) guarantees no-bankruptcy (see Proposition 12 in Appendix A.1). Therefore additional criteria are needed to decide when to accept a gamble (such as in (e) below).

(e) Maximal growth rate. Given a utility function, it is natural to assume that decisions are made according to condition (5) in Section 6.2. However, that is not the only way to use a utility function. For instance, in the shares setup, one may choose that proportion of the gamble that *maximizes the expected growth rate* (rather than just guarantees that it is at least 1, as the riskiness measure does). This yields a number $K \equiv K(g)$ where

[28]The returns in the knife-edge case $\mathbf{E}[\log X] = 0$ may be called *C-returns* ("C" for "Critical" or "Constant").

$\mathbf{E}[\log(1+g/K)]$ is maximal over $K > 0$; equivalently (taking the derivative), $K(g)$ is the unique positive solution of the equation $\mathbf{E}[g/(1+g/K(g))] = 0$; for example, when g takes the values 105 and -100 with equal probabilities, $K(g) = 4200$ and[29] $\mathbf{R}(g) = 2100$. There is an extensive literature on the maximal growth rate; see, e.g., Kelly (1958), Samuelson (1979), Chapter 6 in Cover and Thomas (1991), and Algoet (1992). While the log function appears there too, our approach is different. We do not ask who will win and get more than everyone else (see, e.g., Blume and Easley 1992), but rather who will not go bankrupt and will get good returns. It is like the difference between "optimality" and "satisficing."

(f) Non-homogeneous strategies. A simple strategy is based on a riskiness-like function, and thus homogeneous of degree 1. This raises the question of what happens in the case of general non-homogenenous strategies, where the critical-wealth function $Q : \mathcal{G} \to [0, \infty]$ may be arbitrary. In the basic no-bankruptcy setup of Section 3, for instance, condition (2) that $Q(g) \geq \mathbf{R}(g)$ for all g is sufficient to guarantee no-bankruptcy, whether Q is homogeneous or not (see Proposition 12 in Appendix A.1). However, this condition is no longer necessary: a non-homogeneous Q allows one to behave differently depending on whether the wealth is large or small. It turns out that no-bankruptcy is guaranteed if and only if, roughly speaking, condition (2) holds when the wealth is small—provided that immediate ruin is always avoided and so the wealth remains always positive (i.e., when $Q(g) > L(g)$ for all g). See Appendix A.7.

(g) Limited liability. Our approach yields infinite riskiness when the losses are unbounded (since $\mathbf{R}(g) > L(g)$; see also the discussion in Section 5.2). This may explain the need to bound the losses, i.e., have *limited liability.* It is interesting that, historically, the introduction of limited-liability contracts did in fact induce many people to invest who would otherwise have been hesitant to do so.

(h) Risk-free asset and discounting. We have assumed no discounting

[29]For $1/2 - 1/2$ gambles g it is easy to prove that $K(g) = 2\mathbf{R}(g)$; of course, $K(g) > \mathbf{R}(g)$ holds for every gamble $g \in \mathcal{G}$ (see Lemma 9 and Figure 1 (b) in Appendix A.1: $K(g)$ is the point where ψ is maximal).

over time and a risk-free rate of return $r_f = 1$. Allowing for discounting and an r_f different from 1 can, however, be easily accommodated—either directly, or by interpreting future outcomes as being expressed in present-value terms.

(i) Riskiness instead of standard deviation and VaR. As pointed out in Sections 6.4 and 5.1, commonly used measures of risk—such as the standard deviation $\boldsymbol{\sigma}$ and the "Value at Risk" (VaR)—may be problematic. *We propose the use of* $\mathbf{R}$ *instead.*

Indeed, $\mathbf{R}$ shares many good properties with $\boldsymbol{\sigma}$ (see Proposition 5); but it has the added advantage of being monotonic with respect to stochastic dominance (see Proposition 6). For instance, one could use $\mathbf{R}$ to determine "efficient portfolios" (Markowitz 1952, 1959; Sharpe 1964): rather than maximize the expected return for a fixed standard deviation, *maximize the expected return for a fixed riskiness.* Moreover, one may try to use $\mathbf{E}[g]/\mathbf{R}(g)$ in place of the Sharpe (1966) ratio $\mathbf{E}[g]/\boldsymbol{\sigma}[g]$.

The measures VaR are used for determining bank reserves. Since our measure $\mathbf{R}$ may be viewed as the minimum "reserve" needed to guarantee no-bankruptcy, it is a natural candidate to apply in this setup.

All this of course requires additional study.

A Appendix: Proofs

The proofs are collected in this Appendix, together with a number of additional results.

A.1 Proof of Theorem 1

We prove here the main result, Theorem 1, together with a number of auxiliary results (in particular Lemma 9) and extensions (Propositions 12 and 13). We start by showing that $\mathbf{R}(g)$ is well defined by equation (1).

Lemma 9 *For every $g \in \mathcal{G}$ there exists a unique number $R > 0$ such that $\mathbf{E}[\log(1 + g/R)] = 0$. Moreover, $R > L \equiv L(g)$ (the maximal loss of g); $\mathbf{E}[\log(1 + g/r)] < 0$ if and only if $L < r < R$; and $\mathbf{E}[\log(1 + g/r)] > 0$ if and only if $r > R$.*

Proof. Let $\phi(\lambda) := \mathbf{E}[\log(1+\lambda g)] = \sum_{i=1}^{m} p_i \log(1+\lambda x_i)$ for $0 \le \lambda < 1/L$. It is straightforward to verify that: $\phi(0) = 0$; $\lim_{\lambda\to(1/L)^-} \phi(\lambda) = -\infty$; $\phi'(\lambda) = \sum_i p_i x_i/(1+\lambda x_i)$; $\phi'(0) = \sum_i p_i x_i = \mathbf{E}[g] > 0$; and $\phi''(\lambda) = -\sum_i p_i x_i^2/(1+\lambda x_i)^2 < 0$ for every $\lambda \in [0, 1/L)$. Therefore the function ϕ is a strictly concave function that starts at $\phi(0) = 0$ with a positive slope ($\phi'(0) > 0$), and goes to $-\infty$ as λ increases to $1/L$. Hence (see Figure 1 (a)) there exists a unique $0 < \lambda^* < 1/L$ such that $\phi(\lambda^*) = 0$, and moreover $\phi(\lambda) > 0$ for $0 < \lambda < \lambda^*$ and $\phi(\lambda) < 0$ for $\lambda^* < \lambda < 1/L$. Now let $R = 1/\lambda^*$. $\square$

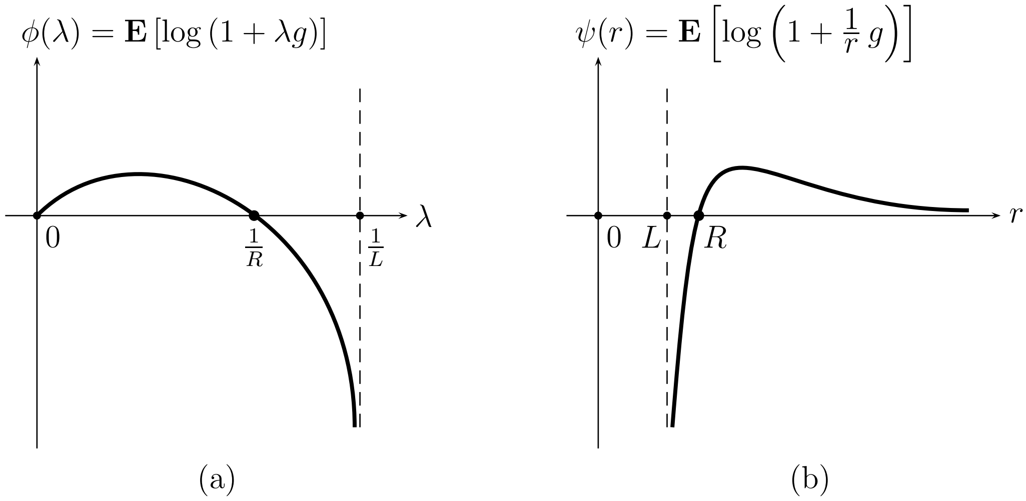

Figure 1: The functions ϕ and ψ (see Lemma 9)

Note that the function $\psi(r) := \mathbf{E}[\log(1+g/r)]$ is *not* monotonic in r, since g has negative values (see Figure 1 (b)).

From Lemma 9 it follows that $\mathbf{R}$ is positively homogeneous of degree one:

Lemma 10 $\mathbf{R}(\lambda g) = \lambda \mathbf{R}(g)$ *for every* $g \in \mathcal{G}$ *and* $\lambda > 0$.

Proof. $0 = \mathbf{E}[\log(1+g/\mathbf{R}(g))] = \mathbf{E}[\log(1+(\lambda g)/(\lambda \mathbf{R}(g))]$, and so $\lambda \mathbf{R}(g) = \mathbf{R}(\lambda g)$ since equation (1) determines $\mathbf{R}$ uniquely. $\square$

We recall a result on martingales:

Proposition 11 *Let* $(X_t)_{t=1,2,\ldots}$ *be a martingale defined on a probability space* $(\Omega, \mathcal{F}, \mathbf{P})$ *and adapted to the increasing sequence of* σ*-fields* $(\mathcal{F}_t)_{t=1,2,\ldots}$. *Assume that* (X_t) *has bounded increments; i.e., there exists a finite* K *such that* $|X_{t+1} - X_t| \leq K$ *for all* $t \geq 1$. *Then for almost every* $\omega \in \Omega$ *either:* ***(i)*** $\lim_{t\to\infty} X_t(\omega)$ *exists and is finite; or* ***(ii)*** $\liminf_{t\to\infty} X_t(\omega) = -\infty$ *and* $\limsup_{t\to\infty} X_t(\omega) = +\infty$. *Moreover, define the random variable* $A_\infty := \sum_{t=1}^{\infty} \mathbf{E}\left[(X_{t+1} - X_t)^2 \,|\mathcal{F}_t\right] \in [0, \infty]$; *then (i) holds for almost every* $\omega \in \Omega$ *with* $A_\infty(\omega) < \infty$, *and (ii) holds for almost every* $\omega \in \Omega$ *with* $A_\infty(\omega) = \infty$.

Proof. Follows from Proposition VII-3-9 in Neveu (1975). □

Thus, almost surely either the sequence of values of the martingale converges, or it oscillates infinitely often between arbitrarily large and arbitrarily small values. A_∞ may be interpreted as the "total one-step conditional variance."

Theorem 1 will follow from the next two propositions, which provide slightly stronger results.

Proposition 12 *If a strategy* s *satisfies condition (2) then* s *guarantees no-bankruptcy.*

We emphasize that this applies to *any* strategy, not only to simple strategies; the function Q may be non-homogeneous, or there may not be a critical-wealth function at all.

Proof of Proposition 12. Consider a process G generated by a finite set $\mathcal{G}_0 \subset \mathcal{G}$. When g_t is accepted at W_t we have $W_t \geq \mathbf{R}(g_t) > L(g_t)$, and so $W_{t+1} \geq W_t - L(g_t) > 0$; by induction, it follows that $W_t > 0$ for every t. Put

$$Y_t := \log W_{t+1} - \log W_t, \tag{6}$$

and let d_t be the decision at time t; the history before d_t is $f_{t-1} := (W_1, g_1, d_1; \ldots; W_{t-1}, g_{t-1}, d_{t-1}; W_t, g_t)$. We have $\mathbf{E}\left[Y_t \mid f_{t-1}\right] \geq 0$; indeed, $Y_t = 0$ when g_t is rejected, and $Y_t = \log(W_t + g_t) - \log W_t = \log(1 + g_t/W_t)$ when it is accepted, and then $\mathbf{E}\left[Y_t \mid f_{t-1}\right] = \mathbf{E}\left[\log(1 + g_t/W_t) \middle| f_{t-1}\right] \geq 0$ by (2) and Lemma 9.

If g_t is accepted then $W_t \geq \mathbf{R}(g_t)$, which implies that $1 + g_t/W_t \leq 1 + M(g_t)/\mathbf{R}(g_t) < \infty$ and $1 + g_t/W_t \geq 1 - L(g_t)/\mathbf{R}(g_t) > 0$. Therefore $Y_t = \log(1 + g_t/W_t) \leq \sup_{g \in \operatorname{cone} \mathcal{G}_0} \log(1 + M(g)/\mathbf{R}(g)) = \max_{g \in \mathcal{G}_0} \log(1 + M(g)/\mathbf{R}(g)) < \infty$ and, similarly, $Y_t \geq \min_{g \in \mathcal{G}_0} \log(1 - L(g)/\mathbf{R}(g)) > -\infty$ (since $\mathcal{G}_0$ is finite and the functions M, L, and $\mathbf{R}$ are homogeneous of degree one); the random variables Y_t are thus uniformly bounded.

Put

$$X_T := \sum_{t=1}^{T} \left(Y_t - \mathbf{E}\left[Y_t \mid f_{t-1}\right]\right); \tag{7}$$

then $(X_T)_{T=1,2,\ldots}$ is a martingale with bounded increments. Recalling that $\mathbf{E}\left[Y_t \mid f_{t-1}\right] \geq 0$ we have

$$X_T \leq \sum_{t=1}^{T} Y_t = \sum_{t=1}^{T} (\log W_{t+1} - \log W_t) = \log W_{T+1} - \log W_1.$$

Now bankruptcy means $\log W_T \to -\infty$, and so $X_T \to -\infty$; but the event $\{X_T \to -\infty\}$ has probability zero by Proposition 11, and so bankruptcy occurs with probability zero. $\square$

Proposition 13 *Let s_Q be a simple strategy with $Q(\tilde{g}) < \mathbf{R}(\tilde{g})$ for some $\tilde{g} \in \mathcal{G}$. Then there exists a process $G = (g_t)$ such that $\lim_{t\to\infty} W_t = 0$ (a.s.); moreover, all the g_t are multiples of $\tilde{g}$.*

Thus there is bankruptcy with probability one, not just with positive probability.

Proof of Proposition 13. Let $q := Q(\tilde{g})$; we have $q > L(g)$ (otherwise there is immediate bankruptcy starting with $W_1 = q$ and accepting $g_1 = \tilde{g}$; indeed, once the wealth becomes 0 it remains so forever by the no-borrowing condition $W_t \geq 0$, since no gambles may be accepted). Therefore $L(\tilde{g}) < q < \mathbf{R}(\tilde{g})$, and so $\mu := \mathbf{E}\left[\log(1 + \tilde{g}/q)\right] < 0$ by Lemma 9. Let $(\tilde{g}_t)_{t=1,2,\ldots}$ be a sequence of independent and identically distributed (i.i.d.) gambles with each one having the same distribution as $\tilde{g}$, and take $g_t = \lambda_t \tilde{g}_t$ with $\lambda_t = W_t/q$. Now $Q(g_t) = (W_t/q)Q(\tilde{g}) = W_t$, and so g_t is accepted at W_t.

Therefore $Y_t = \log(1 + g_t/W_t) = \log(1 + \tilde{g}_t/Q)$ is an i.i.d. sequence and so, as $T \to \infty$,

$$\frac{1}{T}(\log W_{T+1} - \log W_1) = \frac{1}{T}\sum_{t=1}^{T} Y_t \to \mathbf{E}\left[\log\left(1 + \frac{1}{q}\tilde{g}\right)\right] = \mu < 0$$

(a.s.), by the Strong Law of Large Numbers. Therefore $\log W_T \to -\infty$, or $W_T \to 0$ (a.s.). $\square$

A.2 Proof of Theorem 3

The result in the shares setup will follow from the following proposition that gives a more precise result.

Proposition 14 *Let s_Q be a simple shares strategy, and let G be a process generated by a finite $\mathcal{G}_0$. Then:*

($>$) *If $Q(g) > \mathbf{R}(g)$ for every $g \in \mathcal{G}_0$ then $\lim_{t\to\infty} W_t = \infty$ a.s.*

($\geq$) *If $Q(g) \geq \mathbf{R}(g)$ for every $g \in \mathcal{G}_0$ then $\limsup_{t\to\infty} W_t = \infty$ a.s.*

($=$) *If $Q(g) = \mathbf{R}(g)$ for every $g \in \mathcal{G}_0$ then $\limsup_{t\to\infty} W_t = \infty$ and $\liminf_{t\to\infty} W_t = 0$ a.s.*

($\leq$) *If $Q(g) \leq \mathbf{R}(g)$ for every $g \in \mathcal{G}_0$ then $\liminf_{t\to\infty} W_t = 0$ a.s.*

($<$) *If $Q(g) < \mathbf{R}(g)$ for every $g \in \mathcal{G}_0$ then $\lim_{t\to\infty} W_t = 0$ a.s.*

Proof. Define Y_t and X_T as above, by (6) and (7), respectively. Since the gamble taken at time t is $\alpha_t g_t$, where $\alpha_t = W_t/Q(g_t)$, we have

$$Y_t = \log\left(1 + \frac{\alpha_t}{W_t} g_t\right) = \log\left(1 + \frac{1}{Q(g_t)} g_t\right).$$

Next,

$$\lim \frac{1}{T} X_T = 0, \tag{8}$$

$$\limsup X_T = \infty \quad \text{and} \quad \liminf X_T = -\infty \tag{9}$$

a.s. as $T \to \infty$. Indeed, the random variables Y_t are uniformly bounded (since each g has finitely many values and $\mathcal{G}_0$ is finite), and so (8) follows from the Strong Law of Large Numbers for Dependent Random Variables (see Loève 1978, Theorem 32.1.E). As for (9), it follows from Proposition 11 applied to the martingale X_T, since for every history f_{t-1}

$$\mathbf{E}\left[(X_t - X_{t-1})^2 \mid f_{t-1}\right] \geq \min_{g \in \mathcal{G}_0} \mathbf{Var}\left[\log\left(1 + \frac{1}{Q(g)}\, g\right)\right] =: \delta > 0$$

(we have used the homogeneity of Q and the finiteness of $\mathcal{G}_0$; **Var** denotes variance), and so $A_\infty = \infty$.

We can now complete the proof in the five cases.

($>$) The assumption that $Q(g) > \mathbf{R}(g)$ for every $g \in \mathcal{G}_0$ implies by Lemma 9 that

$$\mathbf{E}\left[Y_t \mid f_{t-1}\right] \geq \min_{g \in \mathcal{G}_0} \mathbf{E}\left[\log\left(1 + \frac{1}{Q(g)}\, g\right)\right] =: \delta' > 0,$$

and so, as $T \to \infty$ (a.s.),

$$\liminf \frac{1}{T}\left(\log W_{T+1} - \log W_1\right) = \liminf \frac{1}{T} \sum_{t=1}^{T} Y_t \geq \lim \frac{1}{T} X_T + \delta' = \delta' > 0$$

(recall (8)); therefore $\lim W_T = \infty$.

($<$) Similar to the proof of ($>$), using

$$\mathbf{E}\left[Y_t \mid f_{t-1}\right] \leq \max_{g \in \mathcal{G}_0} \mathbf{E}\left[\log(1 + g/Q(g)\right] =: \delta'' < 0.$$

($\geq$) Here we have $\mathbf{E}\left[Y_t \mid f_{t-1}\right] \geq 0$ and so $\limsup(\log W_{T+1} - \log W_1) = \limsup \sum_{t=1}^{T} Y_t \geq \limsup X_T = \infty$ by (9).

($\leq$) Similar to the proof of ($\geq$), using now $\mathbf{E}\left[Y_t \mid f_{t-1}\right] \leq 0$.

($=$) Combine ($\geq$) and ($\leq$). □

Proof of Theorem 3. Follows from Proposition 14: ($>$) and ($\leq$) yield (i) (if $Q(\tilde{g}) \leq \mathbf{R}(\tilde{g})$ then take G to be an i.i.d. sequence $(\tilde{g}_t)$ with all $\tilde{g}_t$ having the same distribution as $\tilde{g}$), and similarly ($\geq$) and ($<$) yield (ii) . □

A.3 Proof of Proposition 5

We prove here the basic properties of the riskiness measure, followed by an additional result on sequences of independent and identically distributed gambles.

Proof of Proposition 5. **(i)**, **(ii)**, and **(iii)** are immediate from (1) and Lemmata 9 and 10.

(iv) Let $r := \mathbf{R}(g)$ and $r' := \mathbf{R}(h)$, and put $\lambda := r/(r+r') \in (0,1)$. Since $(g+h)/(r+r') = \lambda(g/r) + (1-\lambda)(h/r')$, the concavity of the log function gives

$$\mathbf{E}\left[\log\left(1+\frac{g+h}{r+r'}\right)\right] \geq \lambda\mathbf{E}\left[\log\left(1+\frac{g}{r}\right)\right] + (1-\lambda)\mathbf{E}\left[\log\left(1+\frac{h}{r'}\right)\right] = 0,$$

and so $r + r' \leq \mathbf{R}(g+h)$ by Lemma 9.

(v) follows from (ii) and (iv).

(vi) Put $h := \lambda * g$; then $\mathbf{E}\left[\log\left(1 + h/\mathbf{R}(g)\right)\right] = \lambda\mathbf{E}\left[\log\left(1 + g/\mathbf{R}(g)\right)\right] + (1-\lambda)\log(1+0) = 0$ and so $\mathbf{R}(h) = \mathbf{R}(g)$.

(vii) The second inequality is (iv) (it is strict since only constant random variables can be both independent and equal [or proportional], and gambles in $\mathcal{G}$ are never constant). To prove the first inequality, recall the concave function $\phi(\lambda) := \mathbf{E}\left[\log\left(1+\lambda g\right)\right]$ of the proof of Lemma 9 (see Figure 1 (a)): it increases at $\lambda = 1/\mathbf{R}(g)$, and so $\phi'(\lambda) = \mathbf{E}\left[g/(1+\lambda g)\right] < 0$ for $\lambda = 1/\mathbf{R}(g)$, and thus for $\lambda \leq 1/\mathbf{R}(g)$.

Without loss of generality assume that $\mathbf{R}(g) \leq \mathbf{R}(h)$. Put $\rho := 1/\mathbf{R}(g)$; then $\mathbf{E}\left[\log(1+\rho g)\right] = 0 \geq \mathbf{E}\left[\log(1+\rho h)\right]$ and, as we have seen above,

$$\mathbf{E}\left[\frac{g}{1+\rho g}\right] < 0 \quad \text{and} \quad \mathbf{E}\left[\frac{h}{1+\rho h}\right] < 0 \tag{10}$$

(since $\rho = 1/\mathbf{R}(g)$ and $\rho \leq 1/\mathbf{R}(h)$). Now

$$\begin{aligned}\mathbf{E}\left[\log\left(1+\rho(g+h)\right)\right] &= \mathbf{E}\left[\log(1+\rho g)\right] + \mathbf{E}\left[\log\left(1+\rho h\right)\right] \\ &\quad + \mathbf{E}\left[\log\left(1 - \frac{\rho^2 gh}{(1+\rho g)(1+\rho h)}\right)\right].\end{aligned}$$

The first term vanishes, the second is ≤ 0, and for the third we get

$$\begin{aligned}
\mathbf{E}\left[\log\left(1-\frac{\rho^2 gh}{(1+\rho g)(1+\rho h)}\right)\right] &\leq \mathbf{E}\left[-\frac{\rho^2 gh}{(1+\rho g)(1+\rho h)}\right] \\
&= -\rho^2\,\mathbf{E}\left[\frac{g}{1+\rho g}\right]\mathbf{E}\left[\frac{h}{1+\rho h}\right] < 0
\end{aligned}$$

(we have used $\log(1-x) \leq -x$, the independence of g and h, and (10)). Altogether $\mathbf{E}\left[\log\left(1+\rho(g+h)\right)\right] < 0$, and so $1/\rho < \mathbf{R}(g+h)$ (by Lemma 9), proving our claim (recall that $1/\rho = \mathbf{R}(g) = \min\{\mathbf{R}(g), \mathbf{R}(h)\}$). $\square$

Let $g_1, g_2, ..., g_n, ...,$ be a sequence of independent and identically distributed (i.i.d.) gambles; then (vii) implies that $\mathbf{R}(g_1) < \mathbf{R}(g_1 + g_2 + ... + g_n) < n\mathbf{R}(g_1)$. In fact, we can get a better estimate.

Proposition 15 *Let $(g_n)_{n=1}^{\infty} \subset \mathcal{G}$ be a sequence of i.i.d. gambles. Then*

$$\max\{\mathbf{R}(g_1), nL(g_1)\} < \mathbf{R}(g_1 + g_2 + ... + g_n) < \mathbf{R}(g_1) + nL(g_1) + M(g_1).$$

Moreover, $\lim_{t\to\infty} \mathbf{R}(\bar{g}_n) = L(\bar{g}_n) = L(g_1)$ *where* $\bar{g}_n := (g_1 + g_2 + ... + g_n)/n$.

Proof. Let $h_n := g_1 + g_2 + ... + g_n$. The left-hand side inequality follows from Proposition 5 (vii) and (ii); for the right-hand side inequality, use (4), $R^{AS}(h_n) = R^{AS}(g_1)$ (see Section 5.7 in Aumann and Serrano 2007), and again (4): $\mathbf{R}(h_n) < R^{AS}(h_n) + L(h_n) = R^{AS}(g_1) + nL(g_1) < \mathbf{R}(g_1) + M(g_1) + nL(g_1)$. The "moreover" statement follows from the homogeneity of $\mathbf{R}$. $\square$

For small n, if $\mathbf{R}(g_1)$ is large relative to g_1, then $\mathbf{R}(g_1 + g_2 + ... + g_n)$ is close to $\mathbf{R}(g_1)$ (compare Section 5.7 in Aumann and Serrano 2007). For large n, the average gamble $\bar{g}_n$ converges to the positive constant $\mathbf{E}[g_1]$ by the Law of Large Numbers, and so its riskiness decreases; however, as the maximal loss stays constant ($L(\bar{g}_n) = L(g_1)$), the riskiness of $\bar{g}_n$ converges to it (compare Section 5.2).

A.4 Proof of Proposition 6

We prove here that $\mathbf{R}$ is monotonic with respect to stochastic dominance.

Proof of Proposition 6. Let $u(x) := \log(1 + x/\mathbf{R}(g))$. If $g\,\mathrm{SD}_1\,h$ then $\mathbf{E}[u(g)] > \mathbf{E}[u(h)]$ since u is strictly monotonic; if $g\,\mathrm{SD}_2\,h$ then $\mathbf{E}[u(g)] > \mathbf{E}[u(h)]$ since u is also strictly concave. But $\mathbf{E}[u(g)] = 0$ by (1), and so $\mathbf{E}[\log(1 + h/\mathbf{R}(g))] < 0$, which implies that $\mathbf{R}(g) < \mathbf{R}(h)$ by Lemma 9. $\square$

A.5 Proof of Proposition 7

We will prove a slightly more precise continuity result that implies Proposition 7.

Proposition 16 *Let* $(g_n)_{n=1,2,\ldots} \subset \mathcal{G}$ *be a sequence of gambles satisfying* $\sup_{n\geq 1} M(g_n) < \infty$. *If* $g_n \xrightarrow{\mathcal{D}} g \in \mathcal{G}$ *and* $L(g_n) \to L$ *as* $n \to \infty$, *then* $\mathbf{R}(g_n) \to \max\{\mathbf{R}(g), L\}$ *as* $n \to \infty$.

Thus $\mathbf{R}(g_n) \to \mathbf{R}(g)$ except when the limit L of the maximal losses $L(g_n)$ exceeds $\mathbf{R}(g)$, in which case $\mathbf{R}(g_n) \to L$.

Proof. Denote $R := \max\{\mathbf{R}(g), L\}$. First, note that $L(g_n) \to L$ implies that $L(g) \leq \liminf_n L(g_n)$ (for every open neighborhood O of $L(g)$ we have $\liminf_n \mathbf{P}[g_n \in O] \geq \mathbf{P}[g \in O] > 0$; see Billingsley 1968, Theorem 2.1 (iv)), and thus $L(g) \leq L$. Let r be a limit point of the sequence $\mathbf{R}(g_n)$, possibly $\pm\infty$; without loss of generality, assume that $\mathbf{R}(g_n) \to r$. Since $R(g_n) \geq L(g_n) \to L$ it follows that: **[1]** $r \geq L$ (and so either r is finite or $r = \infty$).

We will now show that $r \geq \mathbf{R}(g)$. Indeed, when r is finite (if $r = \infty$ there is nothing to prove here), let $0 < \varepsilon < 1$ and $q := (1+\varepsilon)^2 r$; then for all large enough n we have: **[2]** $\mathbf{R}(g_n) < q$, and **[3]** $L(g_n) < (1+\varepsilon)L \leq (1+\varepsilon)r = q/(1+\varepsilon)$ (the second inequality by [1] above). Hence $\mathbf{E}[\log(1 + g_n/q)] > 0$ (by Lemma 9 and [2]), and $\log(1 + g_n/q)$ is uniformly bounded: from above by $\log(1 + \sup_n M(g_n)/q)$, and from below by $\log(\varepsilon/(1+\varepsilon))$ (since $g_n/q \geq -L(g_n)/q > -1/(1+\varepsilon)$ by [3]). Therefore $\mathbf{E}[\log(1 + g/q)] = \lim_n \mathbf{E}[\log(1 + g_n/q)] \geq 0$ (since $g_n \xrightarrow{\mathcal{D}} g$), which implies that $q = (1+\varepsilon)^2 r \geq \mathbf{R}(g)$ (again by Lemma 9). Now $\varepsilon > 0$ was arbitrary, and so we got: **[4]** $r \geq \mathbf{R}(g)$.

[1] and [4] imply that $r \geq R$. If $r > R$, then take $0 < \varepsilon < 1$ small enough so that $q := (1+\varepsilon)^2 R < r$. For all large enough n we then have: **[5]** $q < \mathbf{R}(g_n)$,

and **[6]** $L(g_n) < (1+\varepsilon)L \leq (1+\varepsilon)R = q/(1+\varepsilon)$. Hence $\mathbf{E}[\log(1+g_n/q)] < 0$ (by Lemma 9 and [5]), and $\log(1+g_n/q)$ is again uniformly bounded (the lower bound by [6]). Therefore $\mathbf{E}[\log(1+g/q)] = \lim_n \mathbf{E}[\log(1+g_n/q)] \leq 0$ (since $g_n \xrightarrow{\mathcal{D}} g$), contradicting $q = (1+\varepsilon)^2 R \geq (1+\varepsilon)^2\mathbf{R}(g) > \mathbf{R}(g)$ (by Lemma 9 and $\mathbf{R}(g) > L(g) > 0$).

Therefore $r = R$ for every limit point of $\mathbf{R}(g_n)$, or $\mathbf{R}(g_n) \to R$. $\square$

Proof of Proposition 7. If $L = L(g)$ then $\max\{\mathbf{R}(g), L\} = \mathbf{R}(g)$; apply Proposition 16. $\square$

To see why the values need to be uniformly bounded from above (i.e., $\sup_{n\geq 1} M(g_n) < \infty$), let g_n take the values $-3/4$, 3, and $2^{n-1}-1$ with probabilities $(3/4)(1-1/n)$, $(1/4)(1-1/n)$, and $1/n$, respectively, and let g take the values $-3/4$ and 3 with probabilities 3/4, 1/4. Then $g_n \xrightarrow{\mathcal{D}} g$ and $L(g_n) = L(g) = -3/4$, but $M(g_n) \to \infty$ and $\mathbf{R}(g_n) = 1 \neq 5.72... = \mathbf{R}(g)$.

A.6 Proof of Proposition 8

We prove the result connecting the measures of riskiness.

Proof of Proposition 8. (4) yields the equivalence of (ii) and (iii) and the "moreover" statement.

(ii) implies (i): Let $r_n := \mathbf{R}(g_n) \to \infty$. Using $\log(1+x) = x - x^2/2 + o(x^2)$ as $x \to 0$ for each value of g_n/r_n (all these values are uniformly bounded) and then taking expectation yields $0 = \mathbf{E}[\log(1+g_n/r_n)] = \mathbf{E}[g_n]/r_n - \mathbf{E}[g_n^2]/(2r_n^2) + o(1/r_n^2)$. Multiplying by r_n^2 gives $r_n\mathbf{E}[g_n] - \mathbf{E}[g_n^2]/2 \to 0$, and thus $\mathbf{E}[g_n] \to 0$ (since $r_n \to \infty$, and the $\mathbf{E}[g_n^2]$ are bounded), i.e., (i).

(i) implies (ii): Assume that $\mathbf{E}[g_n] \to 0$. For every $0 < \delta < 1$, let $q_n = (1-\delta)\mathbf{E}[g_n^2]/(2\mathbf{E}[g_n])$; then $q_n \to \infty$ and

$$
\begin{aligned}
q_n\mathbf{E}\left[\log\left(1+\frac{1}{q_n}g_n\right)\right] &= \mathbf{E}[g_n] - \frac{\mathbf{E}[g_n^2]}{2q_n} + o\left(\frac{1}{q_n}\right) \\
&= -\frac{\delta}{1-\delta}\mathbf{E}[g_n] + o\left(\frac{1}{q_n}\right).
\end{aligned}
$$

Therefore for all large enough n we have $\mathbf{E}[\log(1+g_n/q_n)] < 0$, and thus $\mathbf{R}(g_n) > q_n \to \infty$. $\square$

Remark. One may define another measure on gambles: $R^0(g) = \mathbf{E}[g^2]/(2\,\mathbf{E}[g])$ for every $g \in \mathcal{G}$. It is easy to see that $R^0(g) \to \infty$ if and only if (i)–(iii), and then $R^0(g) \sim \mathbf{R}(g_n) \sim R^{AS}(g_n)$ as $n \to \infty$. However, R^0 does not satisfy monotonicity.

A.7 Non-Homogeneous Strategies

As discussed in Section 6.5 (f), we take the basic setup of Section 3 and consider strategies s_Q with arbitrary critical-wealth functions $Q : \mathcal{G} \to [0, \infty]$. To avoid inessential technical issues we make a mild *regularity* assumption: the limit $Q_1(g) := \lim_{\lambda \to 0^+} Q(\lambda g)/\lambda$ exists for every[30] $g \in \mathcal{G}$ (see Remark 2 below for general strategies). The result is:

Proposition 17 *Let $s \equiv s_Q$ be a regular strategy with $Q(g) > L(g)$ for all $g \in \mathcal{G}$. Then s guarantees no-bankruptcy* if *$Q_1(g) > \mathbf{R}(g)$ for every $g \in \mathcal{G}$, and* only if *$Q_1(g) \geq \mathbf{R}(g)$ for every $g \in \mathcal{G}$.*

Thus, for non-homogeneous strategies, one needs to consider only "small" gambles (i.e., λg with $\lambda \to 0$); but, again, $\mathbf{R}(g)$ provides the critical threshold.

Proof of Proposition 17. We start by showing that $Q_1(g) > \mathbf{R}(g)$ for every $g \in \mathcal{G}$ implies that for every finite set of gambles $\mathcal{G}_0 \subset \mathcal{G}$ there exists $\varepsilon > 0$ such that $Q(g) \geq \min\{\mathbf{R}(g), \varepsilon\}$ for every $g \in \operatorname{cone} \mathcal{G}_0$. Indeed, otherwise we have sequences $\varepsilon_n \to 0^+$ and $g_n \in \operatorname{cone} \mathcal{G}_0$ with $Q(g_n) < \min\{\mathbf{R}(g_n), \varepsilon_n\}$ for every n. Since $\mathcal{G}_0$ is finite, without loss of generality we can take all g_n to be multiples of the same $g_0 \in \mathcal{G}_0$, say $g_n = \lambda_n g_0$. Now $\lambda_n \to 0$ since $\varepsilon_n > Q(g_n) > L(g_n) = \lambda_n L(g_0) > 0$ (the second inequality since $Q(g) > L(g)$ for every g); also $Q(\lambda_n g_0)/\lambda_n < \mathbf{R}(\lambda_n g_0)/\lambda_n = \mathbf{R}(g_0)$ (since $\mathbf{R}$ is homogeneous of degree 1 by Lemma 10), and so $Q_1(g_0) \leq \mathbf{R}(g_0)$, contradicting our assumption.

Assume $Q_1(g) > \mathbf{R}(g)$ for all g. Given a process G generated by a finite set $\mathcal{G}_0 \subset \mathcal{G}$, fix $\varepsilon > 0$ that satisfies (13). Let $Z_t := W_{t+1}/W_t$ and define

[30] Note that Q_1 is by definition positively homogeneous of degree 1.

$Z'_t := Z_t$ if $W_t < \varepsilon$ and $Z'_t := 1$ otherwise. Now $W_T \to 0$ implies that $W'_T := \prod_{t=1}^T Z'_t \to 0$ (indeed, let T_0 be such that $W_T < \varepsilon$ for all $T \geq T_0$; then $W'_T = \left(W'_{T_0}/W_{T_0}\right) W_T$ for all $T \geq T_0$ and so $W'_T \to 0$ too). We proceed as in the proof of the first part of Theorem 1, but with $Y_t := \log Z'_t$, to obtain $\mathbf{P}[W'_T \to 0] = 0$, and thus $\mathbf{P}[W_T \to 0] = 0$.

Conversely, assume that there is $\tilde{g} \in \mathcal{G}$ with $Q_1(\tilde{g}) < \mathbf{R}(\tilde{g})$. Let q be such that $Q_1(\tilde{g}) < q < \mathbf{R}(\tilde{g})$; then there exists $\delta > 0$ such that for all $\lambda < \delta$ we have $Q(\lambda\tilde{g}) < \lambda q$, and thus $\lambda\tilde{g}$ is accepted at λq. Equivalently, $(W/q)\tilde{g}$ is accepted at W for all $W < \delta q$. We now proceed as in the proof of Proposition 13. Let $\tilde{g}_t$ be an i.i.d. sequence with $\tilde{g}_t$ having the same distribution as $\tilde{g}$ for every t; let $G = (g_t)$ be the process with $g_t = (W_t/q)\tilde{g}_t$ for every t; put $U_T := \sum_{t=1}^T Y_t = \sum_{t=1}^T \log(1 + \tilde{g}_t/q)$. Then $U_T/T \to \mu := \mathbf{E}[\log(1 + \tilde{g}/q)] < 0$, and so

$$U_T \to -\infty \tag{11}$$

a.s. as $T \to \infty$.

This does not yield bankruptcy, however, since the wealth W_T may go above δq, where we have no control over the decisions, and then $\log(W_{T+1}/W_T)$ need no longer equal Y_T. What we will thus show is that the probability of that happening is strictly less than 1, and so bankruptcy indeed occurs with positive probability.

First, we claim that there exists $K > 0$ large enough such that

$$\mathbf{P}[U_T \leq K \text{ for all } T] > 0. \tag{12}$$

Indeed, the Y_t are i.i.d., with $\mathbf{E}[Y_t] = \mu < 0$ and $a \leq Y_t \leq b$ (for $a = \log(1 - L(\tilde{g})/q)$ and $b = \log(1 + M(\tilde{g})/q))$); applying the "large deviations" inequality of Hoeffding (1963, Theorem 2) yields

$$\begin{aligned}\mathbf{P}[U_T > K] &= \mathbf{P}[U_T - \mu T > K + |\mu| T] \\ &\leq \exp\left(-\frac{2(K + |\mu|\, T)^2}{T(b-a)^2}\right) < \exp(-cT - dK)\end{aligned}$$

for appropriate constants $c, d > 0$ (specifically, $c = 2\mu^2/(b-a)^2$ and $d = 4|\mu|/(b-a)^2$). Therefore

$$\mathbf{P}\left[U_T > K \text{ for some } T\right] < \sum_{T=1}^{\infty} \exp(-cT - dK) = \frac{\exp(-c - dK)}{1 - \exp(-c)},$$

which can be made < 1 for an appropriately large K; this proves (12).

Start with $W_1 < \delta q \exp(-K)$. We claim that if $U_T \leq K$ for all T then g_T is accepted for all T. Indeed, assume by induction that $g_1, g_2, ..., g_{T-1}$ have been accepted; then $W_T = W_1 \exp(U_{T-1}) \leq W_1 \exp(K) < \delta q$, and so $g_T = (W_T/q)\tilde{g}_T$ is also accepted (at W_T). But if g_T is accepted for all T, then $W_T = W_1 \exp(U_{T-1})$ for all T; since $U_T \to -\infty$ a.s. (see (11)), it follows that $W_T \to 0$ a.s. on the event $\{U_T \leq K \text{ for all } T\}$. Therefore $\mathbf{P}\left[W_T \to 0\right] \geq \mathbf{P}\left[U_T \leq K \text{ for all } T\right] > 0$ (see (12)), and so the process G leads to bankruptcy with positive probability. □

Remark 1. In the proof we have shown that $Q_1(g) > \mathbf{R}(g)$ for all g implies that for every finite set of gambles $\mathcal{G}_0 \subset \mathcal{G}$ there exists $\varepsilon > 0$ such that $Q(g) \geq \min\{\mathbf{R}(g), \varepsilon\}$ for every $g \in \operatorname{cone} \mathcal{G}_0$, or[31]

$$s \text{ rejects } g \text{ at all } W < \mathbf{R}(g) \text{ with } W < \varepsilon. \tag{13}$$

Compare (2): the addition here is "$W < \varepsilon$". Condition (13) means that the policy of rejecting gambles whose riskiness exceeds the wealth (i.e., $W < \mathbf{R}(g)$) applies only at small wealths (i.e., $W < \varepsilon$); see Section 6.5 (f).

Remark 2. Slight modifications of the above proof show that for a general strategy s that need not be regular nor have a critical-wealth function (but does reject g when $W \leq L(g)$), a sufficient condition for guaranteeing no-bankruptcy is that for every $g \in \mathcal{G}$, if $W < \mathbf{R}(g)$ then s rejects λg at λW for *all* small enough λ (i.e., there is $\delta > 0$ such that this holds for all $\lambda < \delta$); a necessary condition is that for every $g \in \mathcal{G}$, if $W < \mathbf{R}(g)$ then s rejects λg at λW for *arbitrarily* small λ (i.e., for every $\delta > 0$ there is $\lambda < \delta$ where this holds). If we let $Q_s(g) := \inf\{W > 0 : s \text{ accepts } g \text{ at } W\}$,

[31]Moreover, it is easy to see that this condition implies that $Q_1(g) \geq \mathbf{R}(g)$ for all g.

then $\liminf_{\lambda\to 0^+} Q_s(\lambda g)/\lambda > \mathbf{R}(g)$ for all g is a sufficient condition, and, when s is a threshold strategy (i.e., s accepts g at all $W > Q_s(g)$), then $\limsup_{\lambda\to 0^+} Q_s(\lambda g)/\lambda \geq \mathbf{R}(g)$ for all g is a necessary condition.

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