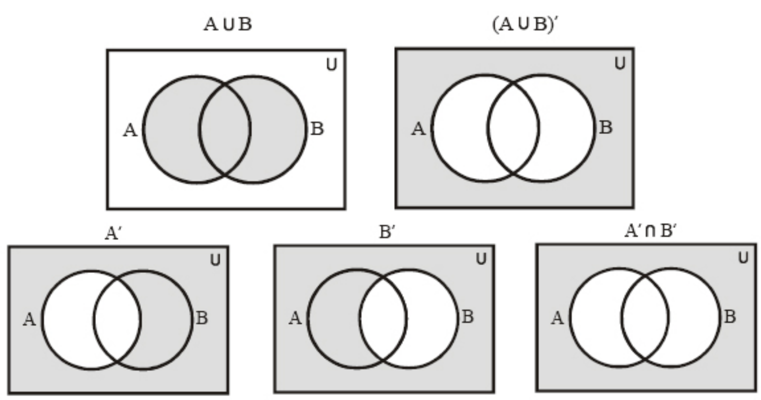
Dean Foster **Stat 430H/510: First Midterm Exam** November 2, 2011

1. (10 pts) This problem has you show deMorgan's law:

   - Draw a Venn diagram of $(A \cup B)^c$.
   - Draw a Venn diagram of $A^c \cap B^c$.

   Are they the same sets?

   Answer: Yes, they're the same set.



2. (10 points) Take $\Omega = \{1, 2, 3, 4\}$, and suppose we know that $\{1, 3\}$ and $\{3, 4\}$ are both in the algebra $\mathcal{A}$. Show that $\{1, 4\}$ is in $\mathcal{A}$.

   Answer:

$$
\begin{aligned}
\{1,3\} \in \mathcal{A} \Rightarrow &\{2,4\} \in \mathcal{A} \\
\{3,4\} \in \mathcal{A} \Rightarrow &\{1,2\} \in \mathcal{A} \\
\{1,3\} \cap \{1,2\} = &\{1\} \in \mathcal{A} \\
\{2,4\} \cap \{3,4\} = &\{4\} \in \mathcal{A} \\
\{1\} \cup \{4\} = &\{1,4\} \in \mathcal{A}
\end{aligned}
$$

   Note: unlike what many of you wrote, $A \in \mathcal{A}$ and $B \subset A$ does not imply $B \in \mathcal{A}$

3. (10 points) Suppose $\Omega = \{1, 2, 3\}$, Give an example of an algebra of sets $\mathcal{A}$ over Ω such that $\{1\} \notin \mathcal{A}$.

   Answer: Examples of such algebras are

   - $\{\phi, \Omega\}$
   - $\{\phi, \{1, 2\}, \{3\}, \Omega\}$
   - $\{\phi, \{1, 3\}, \{2\}, \Omega\}$

4. (10 points) Suppose $\Omega = \{1, 2\}$, then clearly it is impossible to get the set $\{3\}$. But, argue why it doesn't make sense to say $P(\{3\}) = 0$.

   Answer: The event $\{3\}$ is not defined as it cannot be part of any algebra derived from Ω. For an event to have zero probability, it still has to be defined. Hence, $P(\{3\})$ is not defined.

5. (20 points) Suppose Tom either takes a taxi or the bus. To save money, he only takes the taxi 1/10 of the time. When he takes a taxi, he is late 20% of the time, but when he takes the bus he is late 80% of the time. One morning, Tom, Dick and Harry are scheduled to have a meeting. When, Tom arrives on time Dick says, "I see you sprung for a cab this morning!" But before Tom can reply, Harry jumps in with "Nah, he's too cheap! I bet he was just lucky!"

   (a) What is the probability that Tom took a taxi?

   (b) What amounts should Dick and Harry bet to make this fair?

   (c) Who is more likely to be right?

   Answer:

   We have

   - $P(Taxi) = 0.1$, $P(Bus) = 0.9$
   - $P(Late \mid Taxi) = 0.2$, $P(Not\ Late \mid Taxi) = 0.8$

- $P(Late \mid Bus) = 0.8$, $P(Not\ Late \mid Taxi) = 0.2$

(a) First,

$$\begin{aligned}
P(Not\ Late) =& P(Not\ Late \mid Taxi)P(Taxi) + P(Not\ Late \mid Bus)P(Bus) \\
=& 0.8 \times 0.1 + 0.2 * 0.9 \\
=& 0.26
\end{aligned}$$

$$\begin{aligned}
P(Taxi \mid Not\ Late) =& \frac{P(Not\ Late \mid Taxi)\ P(Taxi)}{P(Not\ Late)} \\
=& \frac{0.8 \times 0.1}{0.26} \\
=& \frac{4}{13}
\end{aligned}$$

(b)

$$P(Bus \mid Not\ Late) = 1 - P(Taxi \mid Not\ Late) = \frac{9}{13}$$

Dick chooses the taxi while Harry chooses the bus. The odds ratio for Dick being right to Harry being right is 4:9. Hence for a fair bet, the ratio of the bet amounts of Dick to Harry should be 4:9, the logic being that the person taking the riskier position (Dick) should win more.

(c) Harry, since the probability of him being right is $9/13$. The probability of Dick being right is $4/13$.

6. (20 points) Suppose the generating function for the non-negative integer valued random variable X is $E(s^X) = .5 + .2s + .3s^2$.

   (a) What is $P(X = 1)$?

   (b) What is $E(X)$?

   (c) Suppose $Y = \sum_{i=1}^{n} X_i$ where each X_i is IID with the same generating function. What is the generating function for Y, and from this find what $P(Y = 0)$ is?

Answer:

(a) $P(X = 1)$ is the coefficient of s in the polynomial $= 0.2$

(b) $E[X]$ is obtained by plugging in $s = 1$ in the derivative of the generating function. So, $E[X] = 0.2 + 0.6 \times 1 = 0.8$

(c) Since the X_i are i.i.d, we have

$$E\left[s^Y\right] = E\left[s^{X_i}\right]^n = \left(0.5 + 0.2s + 0.3s^2\right)^n$$

and $P(Y = 0)$ is the constant in this polynomial $= 0.5^n$

7. (15 points) Suppose we have an asset that returns a random amount each year. We will define R_t to be the ratio of the price at the end of year t to the price at the beginning of the year. So after T years, the total value when you start with a single dollar is:

$$W_T = \prod_{t=1}^{T} R_t = R_1 \times R_2 \times R_3 \times \cdots \times R_T$$

Assume that the returns are independent and identically distributed. If $E(R_t) = 1.7$, what is the formula for $E(W_T)$? Derive it using basic calculus. What will $E(W_{20})$ equal?

Answer:

$$\begin{aligned}
E[W_T] =& E[X_1 X_2 \dots X_T] \\
=& \sum_{x_1, x_2, \dots x_T} (x_1 x_2 \dots x_T)\, f(x_1, x_2, \dots x_T) \\
=& \sum_{x_1, x_2, \dots x_T} (x_1 x_2 \dots x_T)\, f(x_1) f(x_2) \dots f(x_T) \\
=& \left(\sum_{x_1} x_1\, f(x_1)\right) \left(\sum_{x_2, \dots x_T} (x_2 \dots x_n)\, f(x_2) \dots f(x_T)\right) \\
=& \left(\sum_{x_1} x_1\, f(x_1)\right) \left(\sum_{x_2} x_2\, f(x_2)\right) \dots \left(\sum_{x_T} x_T\, f(x_T)\right) \\
=& E[X_i]^T = 1.7^T
\end{aligned}$$

and $E[W_{20}] = 1.7^{20} = 40642.31$

8. (5 points) Let X_i be a Bernulli trial, namely, $P(X_i = 1) = p = 1 - P(X_i = 0)$, where $\{X_i\}_{i=1,n}$ are an IID sequence. Let $Y = \sum_{i=1}^n X_i$. Then Y is a Binomial with parameters p and n, and hopefully you have written on your cheat sheet that $Var(Y) = np(1-p)$. Let's check that you wrote this down correctly:

(a) Warmup: What is the $var(X_i)$?

(b) Trivia: What is the $cov(X_i, X_j)$ if $i \neq j$?

(c) Real question: What is $var(\sum_{i=1}^n X_i)$?

(d) Bonus: Starting from $P(Y = y) = \binom{n}{y} p^y (1-p)^{n-y}$ compute the variance of Y.

Answer:

(a) $var(X_i) = p(1-p)$

(b) $cov(X_i, X_j) = 0$ since the X_i's are independent

(c) For independent X_i, $var\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n var(X_i) = np(1-p)$

(d)

$$
\begin{aligned}
E[Y] =& \sum_{y=0}^{n} y \, \frac{n!}{(n-y)! \, y!} p^y (1-p)^{n-y} \\
=& \sum_{y=1}^{n} y \, \frac{n!}{(n-y)! \, y!} p^y (1-p)^{n-y} \quad (y\text{=0 doesn't matter}) \\
=& np \sum_{y=1}^{n} \frac{(n-1)!}{((n-1)-(y-1))! \, (y-1)!} p^{y-1} (1-p)^{(n-1)-(y-1)} \\
& \text{(Let u = y-1)} \\
=& np \sum_{u=0}^{n-1} \frac{(n-1)!}{((n-1)-u)! \, u!} p^{u} (1-p)^{(n-1)-u} \\
=& np \text{ (The summation term is the sum of Bin}(n-1, p))
\end{aligned}
$$

$$\begin{aligned}
E[Y^2] =& \sum_{y=1}^{n} \; y^2 \; \frac{n!}{(n-y)! \; y!} p^y (1-p)^{n-y} \\
=& np \; \sum_{y=1}^{n} \; y \; \frac{(n-1)!}{((n-1)-(y-1))! \; (y-1)!} p^{y-1} (1-p)^{(n-1)-(y-1)} \\
& \quad \text{(Let u = y-1)} \\
=& np \; \sum_{u=0}^{n-1} \; (u+1) \; \frac{(n-1)!}{((n-1)-u)! \; u!} p^u (1-p)^{(n-1)-u} \\
=& np \; \left[E[u] + 1 \right] \\
=& np \; \left[(n-1)p + 1 \right] \\
=& n^2 p^2 - np^2 + np
\end{aligned}$$

$$\begin{aligned}
Var(Y) =& E[Y^2] - E[Y]^2 \\
=& n^2 p^2 - np^2 + np - (np)^2 \\
=& np(1-p)
\end{aligned}$$