

# Static Hedging of Standard Options\*

PETER CARR<sup>†</sup>

*Courant Institute, New York University*

LIUREN WU<sup>‡</sup>

*Graduate School of Business, Fordham University*

First draft: July 26, 2002

This version: November 26, 2002

Filename: `static hedge14.tex`

---

\*We thank David Hait and OptionMetrics for providing the option data and Alex Mayus for clarifying trading practices. We also thank Steven Posner and the participants of finance workshops at Vanderbilt University, Princeton University and Oak Hill Asset Management for comments and suggestions. We assume full responsibility for any errors.

<sup>†</sup>251 Mercer Street, New York, NY 10012; tel: (212) 260-3765; [pcarr@nyc.rr.com](mailto:pcarr@nyc.rr.com); [www.math.nyu.edu/research/carrp/papers](http://www.math.nyu.edu/research/carrp/papers).

<sup>‡</sup>113 West 60th Street, Suite 616, New York, NY 10023; tel: (212) 636-6117; fax: (212) 765-5573; [wu@fordham.edu](mailto:wu@fordham.edu); [www.bnet.fordham.edu/lwu](http://www.bnet.fordham.edu/lwu).

# Static Hedging of Standard Options

## ABSTRACT

We consider the hedging of derivative securities when the price movement of the underlying asset can exhibit jumps of random size. Working in a single factor Markovian setting, we derive a new spanning relation between a given option and a continuum of shorter term options written on the same asset. In the portfolio of shorter term options, the portfolio weights do not vary with changes in stock price or time. We then implement this static relation using a finite set of shorter term options based on a quadrature rule and use Monte Carlo simulation to determine the hedging error thereby introduced. We compare this hedging error to that of a delta hedging strategy based on daily rebalancing in the underlying futures. The simulation results indicate that the two types of hedging strategies exhibit comparable performance in the classic Black-Scholes environment, but that our static hedge strongly outperforms delta-hedging when the underlying asset price is governed by Merton (1976)'s jump-diffusion model. Further simulation exercises indicate that these results are robust to model misspecification, so long as one performs *ad hoc* adjustments based on the observed implied volatility.

We also compare the hedging effectiveness of the two types of strategies using more than six years of data on S&P 500 index options. We find that in all cases considered, a static hedge using just five calls outperforms daily delta hedging with the underlying futures. The consistency of this result with our jump model simulations lends empirical support for the existence of jumps of random size in the movement of the S&P 500 index. We also find that the performance of our static hedge deteriorates moderately as we increase the time between the maturity date of the target call and the common maturity of the calls in the hedge. We interpret this result as evidence of the existence of additional random factors such as stochastic volatility.

JEL CLASSIFICATION CODES: G12, G13, C52.

KEY WORDS: Jumps; option pricing; static hedging; Monte Carlo; S&P 500 index options; stochastic volatility.

## Static Hedging of Standard Options

Over the past two decades, the derivatives market has been expanding dramatically. Accompanying this expansion is an increased urgency in understanding and effectively managing the risks associated with derivative securities. In an ideal setting where the price of the underlying security moves continuously (such as in a diffusion) or with fixed size steps (such as in a binomial tree), derivatives pricing theory provides a framework in which the risks inherent in a derivatives position can be eliminated via frequent trading in only a small number of securities.

In reality, however, large and random price movements happen much more often than typically assumed in the above ideal settings. During the last two decades, we have repeatedly witnessed turmoil in the financial markets such as the 1987 stock market crash, the 1997 Asian crisis, the 1998 Russian default and the ensuing hedge fund crisis, and the tragic events of September 11, 2001. Juxtaposed between these large crises are many more mini-crises, during which prices move sufficiently fast so as to trigger circuit breakers and trading halts. When these crises occur, a dynamic hedging strategy based on small or fixed size movements often breaks down. Worse yet, strategies which involve dynamic hedging in the underlying asset tend to fail precisely when liquidity dries up or when the market makes large moves in either direction. Unfortunately, it is during financial crises such as liquidity gaps or market crashes that effective hedging is most dearly needed. Indeed, several prominent critics have gone further and blamed the emergence of some financial crises on the pursuit of dynamic hedging strategies.

This article presents an alternative approach to hedging derivative securities. We work in a fairly general one-factor Markovian setting, where the market price of a security is allowed not only to move diffusively, but also to jump randomly to any non-negative value. In this setting, we derive a simple spanning relation between the value of a long term European option and the value of a continuum of shorter term European options. The required position in each of the shorter term options is proportional to the gamma (second price derivative) that the target option will have when the options in the hedge portfolio expire. As the target option's

future gamma does not change with the passage of time or changes in the underlying price, the weights in the replicating portfolio of shorter term options are static over their life. Given this static spanning result, no arbitrage implies that the target option and the replicating portfolio have the same value for all times until the shorter term options expire. As a result, a long term option can be theoretically hedged, even though large random jumps are allowed in the security price movement. Furthermore, given the static nature of the strategy, one does not need to rebalance the hedging portfolio until the shorter term options mature and hence one does not need to worry about market shutdowns and liquidity gaps in the intervening period. Therefore, the strategy remains viable and can become even more useful when the market is in stress.

As transactions costs and illiquidity render the formation of a portfolio with a continuum of options physically impossible, we develop an approximation for the static hedging strategy using only a finite number of options. This discretization of the ideal trading strategy is analogous to the discretization that must be performed when a continuous time dynamic trading strategy (eg., delta-hedging) is implemented in practice. To discretize our static hedge, we choose the strike levels and the associated portfolio weights based on a Gauss-Hermite quadrature method. We use Monte Carlo simulation to gauge the magnitude and distributional characteristics of the hedging error introduced by the quadrature approximation. We compare this hedging error to the hedging error from a delta hedging strategy based on daily rebalancing with the underlying futures. The simulation results indicate that the two strategies have comparable hedging effectiveness in the classic Black-Scholes environment. The mean absolute hedging errors from the two strategies are comparable when the same number of transactions are involved. Nevertheless, since the bid-ask spread is typically lower for the underlying asset than it is for any option, these results favor delta-hedging.

However, this conclusion changes when the simulation is performed under the Merton (1976) jump-diffusion environment, where the underlying asset price can exhibit discontinuous price movements of random size. We find that the performance of daily delta hedging deteriorates

dramatically when jumps are added, while the performance of the static strategy hardly varies. As a result, under the Merton model, a static strategy with merely three options outperforms delta hedging with daily updating. Further simulations indicate that these results are robust to model misspecification, so long as one performs *ad hoc* adjustments based on the observed implied volatility. Finally, we also find that increasing the rebalancing frequency in the delta hedging strategy does not rescue its performance as long as the underlying price can jump by a random amount. In contrast, the performance of the static hedging strategy can be further improved with an increase in the number of strikes used and with the appropriate choice of maturities for the hedging portfolio. We conclude that the superior performance of static hedging over daily delta-hedging in the jump model simulations is not due to model misspecification, nor is it due to the approximation error introduced via discrete rebalancing. Rather, this outperformance is due to the fact that delta hedging is inherently incapable of dealing with jumps of random size in the underlying asset price movement. In contrast, the continuum static hedge does hedge these jumps and our approximation of it performs equally well with and without jumps.

The static hedging strategy is derived in a one factor Markovian setting. The simulation exercises are also performed in this type of environment. To determine whether the superiority of the static hedge in the jump model simulations is robust with respect to possible relaxation of the Markovian assumption, we also compare the performance of the two types of hedging strategies using more than six years of data on S&P 500 index options. We find that in all the cases considered, a static hedge using no more than five options outperforms daily delta hedging with the underlying futures. The consistency of this result with our jump model simulations lends empirical support for the existence of jumps of random size in the movement of the S&P 500 index.

We also find that our static strategy performs better when the maturity of the options in the hedging portfolio is brought closer to the maturity of the target option being hedged. As

the maturity gap increases, the hedging performance deteriorates moderately, indicating the likely existence of additional random factors such as stochastic volatility.

The hedging of derivatives via a static option hedge is still in its infancy both theoretically and empirically. An interesting facet of this approach is that the very existence of a such a hedge and the exact composition of the portfolio relies heavily on the nature of the contract being hedged. For example, consider the seminal results of Breeden and Litzenberger (1978), as foreshadowed in the work of Ross (1976). The elaborations of these results by Green and Jarrow (1987) and Nachman (1988) show how a continuum of options with a common maturity can be used to provide a static hedge, provided that the target payoff depends only upon the price of a single asset at this common maturity. However, if the target payoff is that of a European option as in our case, the spanning relation proposed in Breeden and Litzenberger (1978) degenerates into a tautology. By applying more constraints on the underlying process, e.g., a one-factor Markovian process, we are able to span the value of a target European option by a continuum of European options which mature at an earlier date and are thus also potentially more liquid.

As another example of the type of contract which can be statically hedged, Carr, Ellis, and Gupta (1998) show that barrier options can be hedged by semi-static option trading in the Black model and certain mild generalizations. More recent work by Hodges and Neuberger (2002) provides new model-independent bounds on barrier options by considering semi-static trading strategies in their underlying asset. Similarly, Britten-Jones and Neuberger (2000) show how a static position in European options can be combined with dynamic trading in their underlying to hedge contracts paying the realized variance of an underlying asset. While barrier options and variance swaps represent important and common forms of exotic options, there is far greater trading volume in standard options. Thus, our work focuses on the hedging of the simpler standard European options, although it is straightforward to extend our results to certain path-dependent options such as Bermudan, compound, and forward-starting options.

The remainder of this paper is organized as follows. Section I develops the theoretical results underlying our static hedging strategy. Section II uses Monte Carlo simulation to enact a wide variety of scenarios under which the market not only moves diffusively, but also jumps randomly. Under each scenario, we analyze the hedging performance of our static strategy and compare it with dynamic hedging. Section III applies both strategies to the S&P 500 index options data. Section IV concludes.

## I. Spanning Options with Options

We develop our main theoretical results in this section. In particular, working in a continuous time single factor Markovian setting, we show how the risk of a European option can be spanned by holding a continuum of shorter term European options. The weights in the portfolio are static i.e. invariant to changes in the initial stock price or time. We then illustrate how a quadrature rule can be used to approximate the static hedge using a finite number of shorter term options.

### A. Assumptions and Notation

We assume frictionless markets and no arbitrage. To fix notation, let  $S_t$  denote the spot price of an asset (say, a stock or stock index) at time  $t \in [0, \mathcal{T}]$ , where  $\mathcal{T}$  is some arbitrarily distant horizon. For realism, we assume that the owners of this asset enjoy limited liability, and hence  $S_t \geq 0$  at all times. For simplicity, we assume that the continuously compounded riskfree rate  $r$  and dividend yield  $q$  are constant. No arbitrage implies that there exists a risk-neutral probability measure  $\mathbb{Q}$  defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{Q})$  such that the instantaneous expected rate of return on every asset equals this instantaneous riskfree rate  $r$ . Also for simplicity, we restrict our analysis to the class of models in which the risk-neutral evolution of the stock price is Markov in the stock price  $S$  and time  $t$ . Our class of models includes local volatility models, e.g., Dupire (1994), and models based on Lévy processes, e.g., Merton

(1976), Madan, Carr, and Chang (1998), and Carr and Wu (2002a), but does not include stochastic volatility models such as Heston (1993).

Consider the time- $t$  price of a European call with strike  $K$  and maturity  $T$ , denoted as  $C_t(K, T)$ . Our assumption that the state is fully described by the stock price and time implies that there exists a call pricing function  $C(S, t; K, T; \Theta)$  such that:

$$C_t(K, T) = C(S_t, t; K, T; \Theta), \quad t \in [0, T], K \geq 0, T \in [t, T]. \quad (1)$$

Thus, the call pricing function relates the call price at  $t$  to the state variables  $(S_t, t)$ , the contractual parameters  $(K, T)$ , and to a vector of deterministic model parameters  $\Theta$ .

Finally, let  $q(S, t; K, T; \Theta)$  denote the probability density function of the asset price under the risk-neutral measure  $\mathbb{Q}$ , evaluated at the future price level  $K$  and the future time  $T$  and conditional on the stock price starting at level  $S$  at some earlier time  $t$ . Breeden and Litzenberger (1978) show that this risk-neutral density is related to the second strike derivative of the call pricing function by

$$q(S, t; K, T; \Theta) = e^{-r(T-t)} \frac{\partial^2 C}{\partial K^2}(S, t; K, T; \Theta). \quad (2)$$

## B. Spanning Standard European Options with Shorter Term European Options

The main theoretical result of the paper comes from the following theorem, which introduces a new spanning relation between the value of a European option at one maturity and the value of a continuum of European options at some nearer maturity. The practical implication of this theorem is that one can span the risk of a given option by taking a static position in a continuum of shorter term, usually more liquid, options.



**Theorem 1** *Under no arbitrage and the Markovian assumption in (1), the time- $t$  value of a European call option maturing at a fixed time  $T \geq t$  is related to the time- $t$  value of a continuum of European call options at a shorter maturity  $u \in [t, T]$  by,*

$$C(S, t; K, T; \Theta) = \int_0^\infty w(\mathcal{K}) C(S, t; \mathcal{K}, u; \Theta) d\mathcal{K}, \quad u \in [t, T], \quad (3)$$

*for all possible nonnegative values of  $S$  and at all times  $t \leq u$ . The weighting function  $w(\mathcal{K})$  is given by*

$$w(\mathcal{K}) = \frac{\partial^2}{\partial \mathcal{K}^2} C(\mathcal{K}, u; K, T; \Theta). \quad (4)$$

Note that the spanning relation holds for all possible values of the spot price  $S$  and at all times up to the expiry of the options in the spanning portfolio. The option weights  $w(\mathcal{K})$  are independent of  $S$  and  $t$ . This property characterizes the *static* nature of the spanning relation. Under no arbitrage, once the spanning portfolio is formed, no rebalancing is necessary until the maturity date of the options in the spanning portfolio. Also note that the weight  $w(\mathcal{K})$  on a call option at maturity  $u$  and strike  $\mathcal{K}$  is proportional to the gamma that the target call option will have at time  $u$ , should the underlying price level be at  $\mathcal{K}$  then. As the gamma of a call option is typically given by a bell shaped curve centered near the call option's strike price, the greatest weight is given to the options whose strikes are close to that of the target option. Furthermore, as we let the common maturity  $u$  of the spanning portfolio approach the target call option's maturity  $T$ , the gamma becomes more concentrated around  $K$ . In the limit when  $u = T$ , all of the weight is on the call option of strike  $K$ , and equation (3) reduces to a tautology.

**Proof.** Under the Markovian assumption in (1), the initial value of the target call option is given by discounting the expected value it will have at some future date  $u$ ,

$$C(S, t; K, T; \Theta) = e^{-r(u-t)} \int_0^\infty q(S, t; \mathcal{K}, u; \Theta) C(\mathcal{K}, u; K, T; \Theta) d\mathcal{K}$$

$$= \int_0^\infty \frac{\partial^2}{\partial \mathcal{K}^2} C(S, t; \mathcal{K}, u; \Theta) C(\mathcal{K}, u; K, T; \Theta) d\mathcal{K}. \quad (5)$$

The first line follows from the Markovian property: the call option value at any time  $u$  depends only upon the underlying security's price at that time. The second line is obtained via a substitution of equation (2). We then integrate equation (5) by parts twice and observe the following boundary conditions,

$$\frac{\partial}{\partial K} C(S, t; K, u; \Theta)|_{K=\infty} = 0, \quad C(0, u; K, T; \Theta) = 0, \quad C(S, t; \infty, u; \Theta) = 0, \quad \frac{\partial}{\partial S} C(0, u; K, T; \Theta) = 0.$$

The final result is as in equation (3). ■

Equation (3) represents a constraint imposed by no-arbitrage and the Markovian assumption on the relation between prices of options at two different maturities. Given that the Markovian assumption is correct, a violation of (3) implies an arbitrage opportunity. For example, suppose that at time  $t$ , the market price of a call option with strike  $K$  and maturity  $T$  (left hand side) exceeds the price of a gamma weighted portfolio of call options for some earlier maturity  $u$  (right hand side). Then conditional on the validity of the Markovian assumption (1), the arbitrage is to sell the call option of strike  $K$  and maturity  $T$ , and to buy the gamma weighted portfolio of all call options maturing at the earlier date  $u$ . The cash received from selling the  $T$  maturity call exceeds the cash spent buying the portfolio of nearer dated calls. At time  $u$ , the portfolio of expiring calls pays off:

$$\int_0^\infty \frac{\partial^2}{\partial \mathcal{K}^2} C(\mathcal{K}, u; K, T; \Theta) (S_u - \mathcal{K})^+ d\mathcal{K}.$$

Integrating by parts twice implies that this payoff reduces to  $C(S_u, u; K, T; \Theta)$ , which can be used to close the short call position.

To understand the implications of our theorem for risk management, suppose that at time  $t$  there are no call options of maturity  $T$  available in the listed market. However, it is known that such a call will be available in the listed market by the future date  $u \in (t, T)$ . An options

trading desk could consider writing such a call option of strike  $K$  and maturity  $T$  to a customer in return for a (hopefully sizeable) premium. Given the validity of the Markov assumption, the risk exposure arising from writing the call option can be hedged away over the time period  $[t, u]$  using a static position in available shorter term options. The maturity of the shorter term options should be equal to or longer than  $u$  and the portfolio weight is determined by equation (3). Then at date  $u$ , the assumed validity of the Markov condition (1) implies that the desk can use the proceeds from the sale of the shorter term call options to purchase the  $T$  maturity call in the listed market. Thus, this hedging strategy is semi-static in that it involves rolling over call options once. In contrast to a purely static strategy, there is a risk that the Markov condition will not hold at the rebalancing date  $u$ . We will continue to use the terser term “static” to describe this semi-static strategy when it is contrasted to a classical dynamic strategy. However, we warn the practically minded reader that our use of this term does not imply that there is no model risk.

Theorem 1 states the spanning relation in terms of call options. One can readily show that the spanning relation also holds if one replaces the call options on both sides of the equation by their corresponding put options of the same strike and maturity. The relation on put options can either be proved analogously or via the application of put-call parity to the call option spanning relation in (3).

These static spanning relations stand in sharp contrast to traditional dynamic hedging strategies, which are based on continuous rebalancing of positions in the underlying asset. In what follows, we investigate the effectiveness of the two types of strategies using both Monte Carlo simulation and an empirical study.

### C. Finite Approximation with Gaussian Quadrature Rules

In practice, one can neither rebalance a portfolio continuously, nor can one form a static portfolio involving a continuum of securities. Both strategies involve an infinite number of transactions and hence in the presence of discrete transaction costs, both would lead to financial

ruin. As a result, dynamic strategies are only rebalanced discretely in practice. The trading times are chosen to balance the costs arising from the hedging error with the cost arising from transacting in the underlying. Similarly, to implement our static hedging strategy in practice, we need to approximate it using a finite number of call options. The number of call options used in the portfolio is analogously chosen to balance the cost from the hedging error with the cost from transacting in these options.

Specifically, we approximate the spanning integral in (3) by a weighted sum of a finite number ( $N$ ) of call options at strikes  $\mathcal{K}_j, j = 1, 2, \dots, N$ ,

$$\int_0^\infty w(\mathcal{K})C(S, t; \mathcal{K}, u; \Theta)d\mathcal{K} \approx \sum_{j=1}^N \mathcal{W}_j C(S, t; \mathcal{K}_j, u; \Theta), \quad (6)$$

where we choose the strike points  $\mathcal{K}_j$  and their corresponding weights based on the Gauss-Hermite quadrature rule.

The Gauss-Hermite quadrature rule is designed to approximate an integral of the form  $\int_{-\infty}^\infty f(x) e^{-x^2} dx$ , where  $f(x)$  is an arbitrary smooth function. After some rescaling, the integral can be regarded as an expectation of  $f(x)$  where  $x$  is a normally distributed random variable with zero mean and variance of two. For a given target function  $f(x)$ , the Gauss-Hermite quadrature rule generates a set of weights  $w_i$  and nodes  $x_i, i = 1, 2, \dots, N$ , that are defined by

$$\int_{-\infty}^\infty f(x) e^{-x^2} dx = \sum_{j=1}^N w_j f(x_j) + \frac{N! \sqrt{\pi}}{2^N} \frac{f^{(2N)}(\xi)}{(2N)!} \quad (7)$$

for some  $\xi \in (-\infty, \infty)$ . Note that the approximation error vanishes if the integrand  $f(x)$  is a polynomial of degree equal or less than  $2N - 1$ . See Davis and Rabinowitz (1984) for details.

To apply the quadrature rules, we need to map the quadrature nodes and weights  $\{x_i, w_j\}_{j=1}^N$  to our choice of option strikes  $\mathcal{K}_j$  and the portfolio weights  $\mathcal{W}_j$ . One reasonable choice of a mapping function between the strikes and the quadrature nodes is given by

$$\mathcal{K}(x) = K e^{x\sigma\sqrt{2(T-u)} + (q-r-\sigma^2/2)(T-u)}, \quad (8)$$

where  $\sigma^2$  denotes the annualized variance of the log asset return. This choice is motivated by the gamma weighting function under the Black-Scholes model, which is given by

$$\mathcal{W}(\mathcal{K}) = \frac{\partial^2 C(\mathcal{K}, u; K, T; \Theta)}{\partial \mathcal{K}^2} = e^{-q(T-u)} \frac{n(d_1)}{\mathcal{K} \sigma \sqrt{T-u}}, \quad (9)$$

where  $n(\cdot)$  denotes the probability density of a standard normal and the standardized variable  $d_1$  is defined as

$$d_1 \equiv \frac{\ln(\mathcal{K}/K) + (r - q + \sigma^2/2)(T - u)}{\sigma \sqrt{T - u}}.$$

We can then obtain the mapping in (8) by replacing  $d_1$  with  $\sqrt{2}x$ , which can also be regarded as a standard normal variable.

Thus, given the Gauss-Hermite quadrature  $\{w_j, x_j\}_{j=1}^N$ , we choose the strike points as

$$\mathcal{K}_j = K e^{x_j \sigma \sqrt{2(T-u)} + (q-r-\sigma^2/2)(T-u)}. \quad (10)$$

The portfolio weights are then given by

$$\mathcal{W}_j = \frac{w(\mathcal{K}_j) \mathcal{K}'_j(x_j)}{e^{-x_j^2}} w_j = \frac{w(\mathcal{K}_j) \mathcal{K}_j \sigma \sqrt{2(T-t)}}{e^{-x_j^2}} w_j. \quad (11)$$

The next section uses simulation to determine the effectiveness of this approximation under different scenarios.

## II. Simulation Analysis Based on Popular Models

Consider the problem faced by the writer of a call option on a certain stock. For concreteness, suppose that the call matures in one year and is written at-the-money. The writer intends to hold this short position for a month, after which the option position will be closed. During this month, the writer can hedge his market risk using various exchange traded liquid assets such as the underlying stock, futures, and/or options on the same stock. In particular, we compare

the performance of the following two strategies: (1) a static hedging strategy using one-month standard options, and (2) a dynamic delta hedging strategy using the underlying stock futures. The static strategy is based on the spanning relation in (3) and is approximated by a finite number of options, with the portfolio strikes and weights determined by the quadrature method. The dynamic strategy is discretized by rebalancing the futures position daily. The choice of using futures instead of the stock itself for the delta hedge is intended to be consistent with our empirical study in the next section on S&P 500 index options. For these options, direct trading in the 500 stocks comprising the index is impractical, so almost all delta hedging is done in the very liquid index futures market. Given our assumption of constant interest rates and dividend yields, the simulated performances of the delta hedges based on the stock or its futures are almost identical. Hence, this choice does not affect our results.

We compare the performance of the above two strategies based on Monte Carlo simulation. For the simulation, we consider two data generating processes: the benchmark Black-Scholes model (BS) and the Merton (1976) jump-diffusion model (MJ). Under the objective measure,  $\mathbb{P}$ , the stock price dynamics in the two models are controlled by the following stochastic differential equations,

$$\begin{aligned} \text{BS: } dS_t/S_t &= \mu dt + \sigma dW_t, \\ \text{MJ: } dS_t/S_t &= (\mu - \lambda g) dt + \sigma dW_t + dJ(\lambda), \end{aligned} \tag{12}$$

where  $W$  denotes a standard Brownian motion in both models, and  $J(\lambda)$  in the MJ model denotes a compound Poisson jump process with constant intensity  $\lambda > 0$ . Conditional on a jump occurring, the MJ model assumes that the log price relative is normally distributed with mean  $\mu_j$  and variance  $\sigma_j^2$ , with the mean percentage price change induced by a jump given by  $g = e^{\mu_j + \frac{1}{2}\sigma_j^2} - 1$ .

While the data generating processes are specified under the objective measure  $\mathbb{P}$ , we also need to price the relevant options and compute the weights in the hedge portfolios, both of

which are determined by the dynamics of the underlying security under a risk-neutral measure  $\mathbb{Q}$ . The risk-neutral dynamics of the above two models are assumed to take the following form,

$$\begin{aligned} \text{BS: } dS_t/S_t &= (r - q)dt + \sigma dW_t^*, \\ \text{MJ: } dS_t/S_t &= (r - q - \lambda^* g^*)dt + \sigma dW_t^* + dJ^*(\lambda^*), \end{aligned} \tag{13}$$

where  $W^*$  denotes a standard Brownian motion under measure  $\mathbb{Q}$ . The compound Poisson process under measure  $\mathbb{Q}$ ,  $J^*$ , is assumed to have constant intensity  $\lambda^* > 0$ . Conditional on a jump occurring, the jump size is normally distributed with mean  $\mu_j^*$  and variance  $\sigma_j^2$ . See Bates (1991) for an equilibrium economy that supports such a measure change. For the simulation, we benchmark the parameter values of the two models to the S&P 500 index. Specifically, we set  $\mu = 0.10$ ,  $r = 0.06$ , and  $q = 0.02$  for both models. We further set  $\sigma = 0.27$  for the Black-Scholes model and  $\lambda = \lambda^* = 2.00$ ,  $\mu_j = \mu_j^* = -0.10$ ,  $\sigma_j = 0.13$ , and  $\sigma = 0.14$  for the Merton jump-diffusion model.

In each simulation, we generate a time series of daily underlying asset prices according to an Euler approximation of the respective data generating process. The starting value for the stock price is set to \$100. We consider a hedging horizon of one month and simulate paths over this period. We assume that there are 21 business days in a month. To be consistent with the empirical study on S&P 500 index options in the next section, we think of the simulation as starting on a Wednesday and ending on a Thursday four weeks later, spanning a total of 21 week days and 29 actual days. The hedging performance is recorded and, when needed, updated only on week days, but the interest earned on the money market account is computed based on actual over 360.

At each week day, we compute the relevant option prices based on the realization of the security price and the specification of the risk-neutral dynamics. For the dynamic delta hedge, we also compute the delta each day based on the risk-neutral dynamics and rebalance the portfolio accordingly. For both strategies, we monitor the hedging error (profit and loss) at each week day based on the simulated security price and the option prices. The hedging error

at each date  $t$ ,  $e_t$ , is defined as the difference between the value of the hedging portfolio and the value of the target call option being hedged,

$$\begin{aligned} e_t^D &= B_{t-\Delta t}e^{r\Delta t} + \Delta_{t-\Delta t}(F_t - F_{t-\Delta t}) - C(S_t, t; K, T); \\ e_t^S &= \mathcal{W}_j C(S_t, t; \mathcal{K}_j, u) + B_0 e^{rt} - C(S_t, t; K, T), \end{aligned} \quad (14)$$

where the superscript  $D$  and  $S$  denote the dynamic and static strategies, respectively,  $\Delta_t$  denotes the delta of the target call option with respect to the futures price at time  $t$ ,  $\Delta t$  denotes the amount of time between stock trades, and  $B_t$  denotes the time- $t$  balance in the money market account. It includes the receipts from selling the one-year call option, less the cost of initiating and possibly changing the hedging portfolio. In case of the static hedging strategy, under no arbitrage, the value of the portfolio of the shorter term options should be equal to the value of the long term target option, and hence  $B_0$  should be zero. However, since we use a finite number of call options in the static hedge to approximate the spanning relation, the money market account captures the value difference due to the approximation error, which is normally very small. Note that there is no rebalancing in the static strategy.

Under each model, the delta is given by the partial derivative  $\partial C(S, t; K, T; \Theta)/\partial F$ , with  $F = Se^{(r-q)(T-t)}$  denoting the forward/futures price. If an investor could trade continuously, this delta hedge removes all of the risk in the BS model. It does not remove all risk in the MJ model, but has nonetheless emerged as the market standard for implementing delta-hedges in jump models. The hedging portfolio for the static strategy is formed based on the weighting function  $w(\mathcal{K})$  in equation (4) implied by each model, the Gauss-Hermite quadrature nodes and weights  $\{x_i, w_i\}$ , and the mapping from the quadrature nodes and weights to the option strikes and weights, as given in equations (10) and (11). In computing the strike points, the annualized variance is  $v = \sigma^2$  for the Black-Scholes model and  $v = \sqrt{\sigma^2 + \lambda(\mu_j^2 + \sigma_j^2)}$  for the Merton jump-diffusion model. Given the chosen model parameters,  $v \doteq 0.27^2$  for both models. Appendix A details the option pricing formula, the formula for delta  $\partial C(S, t; K, T; \Theta)/\partial F$ , and the weighting function  $w(\mathcal{K})$  for both models.



## A. Hedging Comparison under the Diffusive Black-Scholes World

Table I reports the summary statistics of the simulated hedging errors, based upon 1,000 simulations. The results based on the Black-Scholes model are summarized in Panel A of Table I. Entries are the summary statistics of the hedging errors at the last step (at the end of the 21 business days) based on both strategies. For the dynamic strategy (the last column), we perform daily updating. For the static strategy, we consider hedging portfolios with  $N = 3, 5, 10, 15, 21$  one-month options. If the transaction cost for a single option is comparable to the transaction cost for revising a position in the underlying security, we would expect that the transaction cost induced by buying 21 options at one time is close to the cost rebalancing a position in the underlying stock 21 times. Hence, it would be of interest to compare the performance of daily delta hedging with the performance of the static hedge with 21 options.

Our simulations of the underlying geometric Brownian motion indicate that the daily updating strategy beats the static strategy with 21 options in terms of the standard error, the root mean squared error (RMSE), the mean absolute error (MAE), and the mean short fall (MSF). The static strategy with 21 options does slightly better in terms of maximum profit or loss (Min and Max). Overall, the two strategies are comparable with a slight edge to the dynamic strategy. Thus, since the stock market is much more liquid than the stock options market, the dynamic delta strategy should be favored over the static strategy, if indeed stock prices move as in the Black-Scholes world.

The hedging errors from the two strategies also exhibit different distributional properties. In particular, the kurtosis of the hedging errors from the dynamic strategy is larger than that from all the static strategies. The kurtosis for the dynamic hedging errors is 4.68, while that for the static hedges are below two in all cases. Therefore, when one is particularly concerned about avoiding large losses, the static strategy may be preferred.

The last row in each panel shows the accuracy of the Gauss-Hermite quadrature approximation of the integral in pricing the target options. Under the Black-Scholes model, the theoretical value of the target call is \$12.35, given under the dynamic hedging column. The approximation error is about one cent when applying a 21-node quadrature. The approximation error gradually increases as we reduce the number of quadrature nodes in the approximation.

## B. Hedging Comparison in the Presence of Random Jumps as in the Merton World

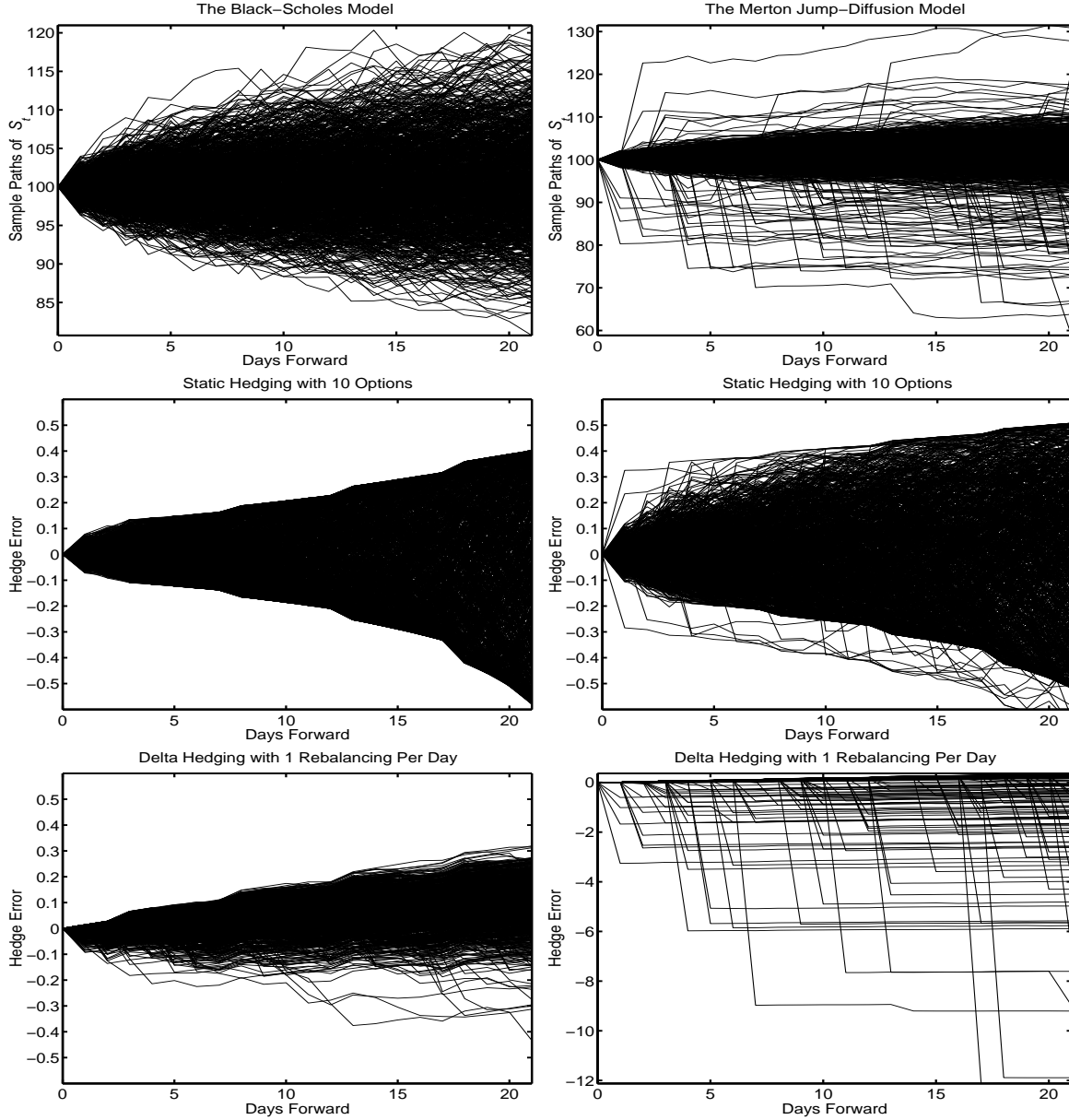
The hedging performance under the Merton jump-diffusion model is shown in Panel B of Table I. The performance of all the static strategies are comparable to their corresponding cases under the Black-Scholes world. If anything, most of the performance measures for the static strategies become slightly better under the Merton jump-diffusion case. In contrast, the performance of the dynamic strategy deteriorates dramatically as we move from the diffusion-based Black-Scholes model to the jump-diffusion process of Merton (1976). The standard error and the root mean squared error increase by a factor of ten for the dynamic strategy. The mean absolute error increases by a factor of four. As a result, the performance of the dynamic strategy is worse than the static strategy with merely three options.

Furthermore, the distributional differences between the hedging errors of the two strategies become even more obvious under the Merton model. While the kurtosis of the static hedge errors remains small, the kurtosis of the dynamic hedge errors explodes from 4.68 in the BS model to 59.79 in the MJ model. The maximum loss from the dynamically hedged portfolio is \$12.12, even larger than the initial revenue from writing the call option (\$11.99). In contrast, the maximum loss is less than two dollars for the static hedge with merely three call options.

Figure 1 depicts the simulated sample paths and the corresponding hedging errors under the two models: the BS model in the left panels and the Merton model in the right panels. In the top two panels of Figure 1, we compare the simulated sample paths of the underlying security price under the two models. The daily movements under the Black-Scholes model are

usually small, while both small and large movements are quite evident under the Merton-jump diffusion model. The middle two panels of Figure 1 compare the sample paths of the hedging errors from the static hedging strategy using ten options. We apply the same scale for ease of comparison. While the sample paths of the static hedging errors look different under the two models, the relative magnitudes of the errors are similar. That is, the performance of the static hedging strategy is relatively insensitive to the specification of the underlying process. In contrast, the bottom two panels illustrate the sample paths of the dynamic hedging error under the two models. Under the Black-Scholes model, the dynamic hedging errors are smaller than the static hedging errors (the scale of the graphs remains the same); but under the Merton jump-diffusion model, the hedging errors from the dynamic strategy explode and we are forced to use a much larger scale in plotting the error paths (right, bottom panel). In particular, the large hedging errors from the dynamic strategy seem to correspond to the large moves in the underlying security price. Another interesting feature is that, under the Merton model, most of the large errors from the dynamic strategy are negative, irrespective of the direction of the large move in the underlying security price. This is because the option price function exhibits positive convexity with the underlying futures price. Under a large movement, the value of the delta portfolio is always below the value of the option contract. Thus, most of the large hedge errors for selling an option contract are losses (negative values).

In summary, the daily delta hedging strategy performs reasonably well under the diffusion-based Black-Scholes model, but fails miserably when the underlying price jumps randomly. In contrast, the performance of the static hedging strategy with a few shorter term options is much less sensitive to the nature of the underlying price process. These simulation results parallel what theory predicts for continuously rebalanced delta hedges and for static hedges with a continuum of short term options. The continuously revised delta hedge was not designed to handle jumps of random size, while the static hedge with a continuum of short term options takes these jumps in stride. The discretizations needed to implement both strategies do not change the result that introducing jumps destroys the effectiveness of the delta strategy, but has little impact on our static hedging strategy.



**Figure 1. Hedging Performance Under Different Sample Paths**

The two panels in the first row depict the simulated sample paths for the underlying security price movement based on the Black-Scholes model (left) and the Merton jump-diffusion model (right). The two panels in the second row depict the sample paths of the hedging errors from the static hedging strategy with ten option contracts under the two models. The last row depicts the corresponding sample paths of the hedging errors from the dynamic delta strategy with the underlying futures and daily updating.

### C. Effects of Model Uncertainty and Misspecification

The above simulations are performed under the assumption that the hedger knows exactly under which model the options are priced. In practice, however, we can only use different models to approximately fit market option prices. Hence, model uncertainty is an inherent part of both pricing and hedging. To investigate the sensitivity of the hedging performance to model misspecification, we further compare the two types of hedging strategies when the hedger does not know the data generating process and hence must develop a hedging approach in the absence of this information. In particular, we assume that the actual underlying asset prices and the option prices are generated from the Merton jump diffusion model, but the hedge portfolios are formed using the Black-Scholes model, with an ad hoc adjustment using the observed option implied volatility.

Specifically, for the static strategy, we compute the weighting function  $w(\mathcal{K})$  based on the Black-Scholes model, but use the eleven month at-the-money option implied volatility as the input for annualized volatility. For the dynamic strategy, the daily delta computation is based on the Black-Scholes formula using the implied volatility of the target call option as the volatility input. The practice of updating Black-Scholes deltas based on the market observed implied volatilities is in wide use in practice as an ad hoc defense against model risk. Also, empirical studies in, for example, Engle and Rosenberg (2002), Jackwerth and Rubinstein (1996), and Bollen and Raisel (2002) have generally found that this approach works as well or better than the alternative approach of estimating a sophisticated model and delta-hedging with it. For the static hedge, we analogously assume that the hedger uses the Black-Scholes gamma to pick strike prices and portfolio weights with the initial implied volatility used as the volatility input.

The results are shown in Panel C of Table I. For the dynamic strategy, as long as we compute the delta based on the market implied volatility, the impact of model misspecification is minimal. For the static strategy, we observe some slight deterioration in performance when there are more than ten option contracts in the hedging portfolio, but the performance actually

improves slightly when fewer option contracts are used in the hedging portfolio. Overall, model misspecification is not an over-riding concern in hedging.

These results are quite remarkable. They imply that in hedging, being able to span the right space is much more important than specifying the right model. For example, even if an investor has perfect knowledge of the stochastic process governing the underlying asset price, and hence can compute the perfectly correct delta, a dynamic strategy in the underlying asset still fails miserably when the underlying asset price can jump by a random amount. In contrast, as long as an investor uses a few short term call options of different strikes in the hedging portfolio, the hedging error is about the same regardless of whether jumps can occur or not. This result holds even if the investor does not know which model to use to pick the appropriate strikes and portfolio weights.

## D. Effects of Rebalancing Frequency in Delta Hedging

In the above simulations, we approximate the sample paths of the underlying stock price process using an Euler approximation with daily time steps and only consider dynamic delta strategies with daily updating. We are interested in knowing how much of the failure of the delta hedging strategy under the Merton jump-diffusion model is due to this somewhat arbitrary choice of discretization step. Under the Black-Scholes environment, the dependence of the delta hedging error on the discretization step has been studied extensively in, for example, Black and Scholes (1972), Boyle and Emanuel (1980), Bhattacharya (1980), Galai (1983), Leland (1985), and Toft (1996). Several of these authors show that, under the Black-Scholes environment, the standard deviation of the hedging error arising from discrete rebalancing over a time step of length  $\Delta t$  declines to zero slowly like  $O(\sqrt{\Delta t})$ . Thus, doubling the trading frequency reduces the standard deviation by about thirty percent. In contrast, the discretization error in the Gaussian quadrature method is  $\frac{N! \sqrt{\pi}}{2^N} \frac{f^{(2N)}(\xi)}{(2N)!}$ . This drops by much more when the number of strikes  $N$  is doubled. Indeed, our simulations indicate that the standard deviation of the hedging error drops rapidly as the number of strikes is increased.

This subsection focuses on relating the delta hedging error to the rebalancing frequency under the Merton-jump diffusion model. Nevertheless, we also simulate the Black-Scholes model as a benchmark reference. Table II illustrates the impacts of the rebalancing frequency on the hedging performance under three different cases: (A) the Black-Scholes model, (B) the Merton jump-diffusion model, assuming that the hedger knows the underlying data generating process, and (C) ad hoc Black-Scholes delta hedging under the Merton world, where the hedger is assumed not to have knowledge of the data generating process. We consider cases where the rebalancing frequency increases from once per day, to twice per day, five times per day, and then to ten times per day. To ease comparisons, all of the hedging exercises are performed on the same simulated sample paths. To accommodate the more frequent rebalancing, we now simulate the sample paths based on the Euler approximation with a time interval of one-tenth of a business day. The slight differences between the dynamic hedging with daily updating in this table and in Table I is due to this difference in the simulation of the sample paths.

Our simulation of the Black-Scholes model is consistent with the results in previous studies. As we increase the updating frequency from one to two, five, and ten times per day, the standard error of the hedging error reduces from 0.10 to 0.07, 0.04 and to 0.03, adhering fairly closely to the  $\sqrt{\Delta t}$  rule.

This speed of improvement in hedging performance is no longer valid, however, when the underlying data generating process is based on the Merton jump diffusion model, irrespective of whether the hedger knows the model or not. In the case when the process is known (Panel B), the standard error of the hedging errors remains around 1.02–1.03 as we increase the rebalancing frequency. In the ad hoc rebalancing case (Panel C), the standard error hovers around 0.88 – 0.93 and exhibits no obvious dependence on the rebalancing frequency. Therefore, the failure of the delta hedging strategy under the Merton model is neither due to model misspecification, nor due to infrequent updating, but due to its inherent incapability in spanning risks associated with jumps of random size.

Note, however, that the Achilles heel of delta hedging in jump models is not the large size of the movement per se, but rather the randomness of the jump size. For example, Cox and Ross (1976) and Dritschel and Protter (1999) show that dynamic delta hedging can span all risks arising in their pure jump models. Under these jump models, the jump size is known just prior to any jump. This is analogous to the binomial model where only two subsequent asset prices are possible. Under both cases, delta hedging can remove all risks. Therefore, it is the a priori randomness in the jump size that creates the difficulty in dynamic delta hedging.

## **E. Effects of Target and Hedging Instrument Choice**

For concreteness, the above simulations have focused on the hedging of a one-year call option. Furthermore, although the static spanning relation permits many option maturities in the spanning portfolio, we have only used one-month options. In this subsection, we compare the hedging performance when we choose different target options being hedged and different maturities for the options in the static hedging portfolio. In theory, if we use a continuum of options at a certain maturity, the spanning is perfect regardless of the exact maturity choice for the hedging portfolio. In practice, however, the gaussian quadrature approximation error may vary with the target strike and with the maturities used. The simulation analyzes how the hedging error introduced by the quadrature approximation varies over different choices of target and hedging options. Along the same lines, we also analyze how the dynamic hedging error varies with the choice of the target option.

The results are reported in Table III. To save space, we only report static hedges with three and five options and compare their performance with that of delta hedging with daily updating. First, we investigate the impact of varying the target option maturity given the same hedging instruments. We choose target option maturities of two months, four months, and 12 months. For the static hedging strategy, as we lower the target option's maturity, the hedging errors become smaller in the Black-Scholes model but slightly larger in the Merton



jump diffusion model. We conjecture that these variations in performance are related to the different accuracies of the quadrature approximation for different integrands.

For the dynamic strategy, the hedging errors are larger for hedging shorter term options than for hedging longer term options under all simulated scenarios. This deteriorating performance with declining maturity is probably linked to the gamma of the target option. The shorter the maturity, the larger is the gamma for an at-the-money option. Since the delta strategy can be regarded as a linear approximation, the hedging error normally increases with gamma, especially in the presence of large moves.

Our static spanning relation allows the use of different maturities in forming the static hedging portfolios. Thus, holding the same one-year option as the target option, we also compare how different maturity options fare in spanning the risk of this target option. Under all three scenarios, we find that the hedging performance improves quite significantly when one increases the maturity of the hedging options. For example, under the Black-Scholes environment, the standard error of the hedging error is 0.66 when we use five one-month options to hedge the one-year option. This performance is much worse than daily delta hedging, which generates a standard error of merely 0.10. However, as we replace the one-month options in the portfolio by two month options and then by four month options, the performance of the static hedge improves quite dramatically. The standard error of the hedging error declines to 0.25 when using two month options and to a meager 0.04 when using four month options. Thus, when moderately longer term options are liquid and available in the market, we can further improve the performance of the static hedging strategy such that it outperforms daily delta hedging even under the Black-Scholes environment. Comparing this to Table II, we see that to achieve the static hedging error of 0.04, a dynamic delta strategy must be updated five to ten times per day.

The same trend follows under the Merton jump diffusion world, although the improvement in performance is not as dramatic. Under the Merton world, the standard error of the hedging error is 0.47 when hedging one-year options with five one-month options. The standard error

is reduced to 0.29 when using five two-month options and is further reduced to 0.16 when using five four-month options, which is much smaller than the standard error of the dynamic hedging error (1.05) under daily updating.

To summarize, the simulation exercises illustrate that when the underlying asset price can jump by a random amount, our static strategy with a few appropriately chosen options delivers much smaller hedging errors than the dynamic delta strategy. But probably the biggest advantage of the static strategy lies in its flexibility. For the same target option, we have the freedom to choose options at different maturities to form the hedging portfolio. Furthermore, while the Gauss-Hermite quadrature rule provides a convenient way in performing finite approximations, there is ample room left for future research in developing better approximating schemes that can potentially further improve the performance of the static strategy.

The fact that a static hedging strategy with merely three to five options can outperform a dynamic strategy with daily updating is remarkable. In addition to the above mentioned flexibility and potentially reduced transaction costs due to fewer transactions, there are other advantages in implementing the static strategy. First, since the static hedge employs neither short stock positions nor substantial borrowing,<sup>1</sup> it is not subject to either short sales restrictions or leverage constraints. In contrast, delta hedges of options always involve a short position in either the risky asset or a riskfree bond, and hence always face one of these restrictions. Furthermore, the use of a static hedge also allows one to economize on the monitoring costs (e.g., paying for traders and real time data feeds) associated with dynamic rebalancing. These costs are much larger in practice than typically assumed in theory and potentially explain the current situation where dynamic hedging is usually only performed by specialized institutions.

---

<sup>1</sup>The money market account induced by the approximation error for the static strategy is normally very small, and can be reduced to zero via a rescaling of portfolio weights without much effect on the hedging performance.

### III. Hedging S&P 500 Index Options: An Applied Example

The simulation study in the previous section compares the performance of the two different types of hedging strategies under controlled conditions. In this section, we investigate the historical performance of the two strategies in hedging the sale of S&P 500 index options. While the simulations allow us to benchmark the magnitude of the approximation error in various Markov models, only an empirical study can gauge the likely effectiveness of the two types of hedging strategies in practice. Furthermore, since the simulations indicate that the two strategies exhibit comparable performance when the stock price follows geometric Brownian motion, but the static strategy has superior performance when the stock price follows Merton’s jump diffusion process, the relative performance of the two strategies in the past can also serve as an indirect test on whether the S&P 500 index has moved purely diffusively or has also experienced jumps of random size.

#### A. Data and Estimation

The data on S&P 500 index options are obtained from OptionMetrics, a financial research and consulting firm specializing in econometric analysis of the options markets. The “Ivy DB” data set from OptionMetrics is the first widely-available, up-to-date, and comprehensive source of high-quality historical price and implied volatility data for the US equity and index options markets. Encompassing six years of data, Ivy DB contains accurate historical prices of options and their associated underlying instruments, correctly calculated implied volatilities, and option sensitivities. The index options data we have obtained from OptionMetrics are from January 1996 to August 2002. They are standard European options on the spot index and are listed at the Chicago Board of Options Exchange (CBOE). The data set includes, among other information, the closing quotes on each options contract (bid and ask) and implied volatilities based on the mid quote. Also included in the data set is a unique option contract identifier to facilitate the tracking of an option contract over time. The underlying index level at close, the interest rate curve, and the projected dividend yield for the calculation of implied volatility

are also supplied by OptionMetrics. Our hedging exercises are based on the mid option price quotes.

In parallel with the hedging exercises in the simulation studies, we perform month-long hedging exercises on the index options. The S&P 500 index options expire on the Saturday following the third Friday. Since the terminal payoff is computed based on the opening price on that Friday morning, trades and quotes on the expiring options effectively stop on the preceding Thursday. Hence, we start the hedging exercise each month 30 days prior to the expiring Friday, which is a Wednesday. The available number of one-month option contracts at each of the starting dates ranges from 48 to 142, half of them call options and half of them put options. From these starting dates, we can perform month long hedging exercises for 79 non-overlapping months, from January 1996 to July 2002. Sampling properties of the hedging errors can then be computed from the 79 hedging experiments. To be comparable with the simulations, we normalize the option prices and hedging errors as percentages of the underlying index level at the starting date of each month.

At each starting date, we classify options into four maturity groups, matching those used in the simulations: (1) one-month options, (2) two-month options, (3) options with maturities four to six months, and (4) options with maturities twelve to seventeen months. The variations in maturities in the last two maturity groups are needed to obtain a monthly series as we do not have four- and twelve-month options in all months. As in the simulations, we use the first three groups (one, two, and four month options) in forming static hedging portfolios and the last three maturity groups (two, four, and twelve month options) for the target option being hedged. For each month, the target option strike is chosen to be as close as possible to the spot index level at the starting date.

Since we do not know the “true” data generating process nor the option pricing model underlying the market prices, we adopt an ad hoc strategy using the Black-Scholes model. For the dynamic strategy, we delta hedge with the underlying futures based on the Black model, using the observed implied volatility to compute the delta. For the static strategy, the portfolio

is formed based on the Black-Scholes formula, using the at-the-money implied volatility of the appropriate maturity as the needed volatility input. Recall that our simulations indicate that these ad hoc hedging strategies perform about as well as the hedges conducted when the true process is known.

When quotes at the appropriate strikes are not available, we use the nearest available strike contract as a replacement. For the static strategy, we can pick any number of shorter term options based on the quadrature rule. However, a large order quadrature rule often requires some deep out-of-the-money or deep in-the-money option contracts that are not available on the market. Thus, we focus on analyzing the performance of the static hedge with only three to five option contracts.

We follow both strategies for 29 actual days, running from the starting date to the Thursday of the fourth following week, the last day of trading for the one-month options used in the static hedge. For the static strategy, we only need to track the price of the short term options at each date and record the difference between the price of the hedge portfolio and the price of the target call. When there is a discrepancy between the price of the target call and the cost of the quadrature-determined hedge portfolio at the starting date, we also monitor the typically small money market account balance. For the dynamic strategy, we need to compute a new delta at each date based on the newly observed underlying price level and implied volatility and perform the appropriate rebalancing. For obvious reasons, we do not rebalance during weekends, holidays, or other market closures. For ease of comparison, we align the hedging errors based on the week days of each week and then compute the sample properties of the hedging errors at each week day.

## B. Static Versus Dynamic Hedging in Practice

Table IV presents the summary statistics for the hedging errors of the various hedging exercises on S&P 500 index options. To ease comparisons, we present the results in a similar format to those from the simulations summarized in Table III. Analogously, the option prices and

hedging errors are also represented as percentages of the underlying index level at the starting date of each month.

As in the simulations, we consider the hedging of three maturity groups of target options: (1) two-month calls, (2) four-to-six month calls, and (3) twelve-month and longer calls. We find that the performance varies with the maturity group of the option being hedged. First, consider the hedging of the sale of a two-month call. Daily delta hedging with the underlying futures generates hedging errors with a standard error of 0.57, a root mean squared error of 0.59, and a mean absolute error of 0.47. The corresponding statistics for the static strategy with three one month options are 0.27, 0.27, and 0.21, respectively, less than half of the corresponding values for the dynamic strategy. Using five one-month options makes the hedging errors even smaller. Therefore, a static hedge with just three one-month options significantly outperforms daily delta hedging in reducing the risks associated with writing two-month call options.

In hedging the sale of a four-to-six month call, the dynamic hedging strategy generates a standard error of 0.65 and a mean absolute error of 0.49. In contrast, the standard error from the static strategy with three one-month calls is 0.62, slightly smaller than the dynamic strategy; but the mean absolute error from this static strategy is 0.50, slightly larger than that from the dynamic strategy. Overall, the performance from the two strategies are on par in hedging the sale of a four-to-six month call. However, when using five one-month options rather than three, the static strategy generates smaller errors than the dynamic strategy.

When hedging the sale of a call with a time to maturity of twelve months or longer, the dynamic strategy generates a standard error of 0.88 for the hedging error. The mean absolute error is smaller at 0.64. Both numbers are larger in absolute magnitudes than those for hedging a shorter term call. Furthermore, this dynamic hedging performance is better than the static strategy with three one-month calls, but on par with the static strategy with five one-month calls. For example, the standard error from the static strategy with three one-month calls is 1.06, larger than 0.88 from the dynamic strategy. Nevertheless, the standard error from the

static strategy with five one-month calls is 0.87, about the same as that from the dynamic strategy.

Consistent with the results observed in the simulations, the performance of the static strategy improves if we increase the time to maturity of the options in the hedge. In hedging the sale of a twelve month or longer call, the standard error of the hedging errors from the static strategy with three calls declines from 1.06 to 0.74 and then to 0.44, as the time to maturity of the three calls in the hedge increases from one month to two months, and then to four-six months. A similar reduction is also observed when using five calls in the static hedge.

Overall, the performance of static hedging with merely three to five calls is on par with or better than the performance of daily delta hedging. In addition, the performance of our static strategy can be further improved by choosing slightly longer maturities for the options in the hedge. Therefore, the static strategy not only works in theory and in simulations, but it also works on historical data, at least for S&P 500 index options, one of the most actively traded derivative contracts.

## C. Implications for the Index Movement

By comparing the hedging results from the Monte Carlo simulations with those from the historical data, we can draw inferences on the type of stochastic process underlying the S&P 500 index movement. The issues that we can draw inferences on include (1) whether the stock index movement displays jumps of random size and (2) whether the risk-neutral stock index process is Markovian in the spot index level and calendar time.

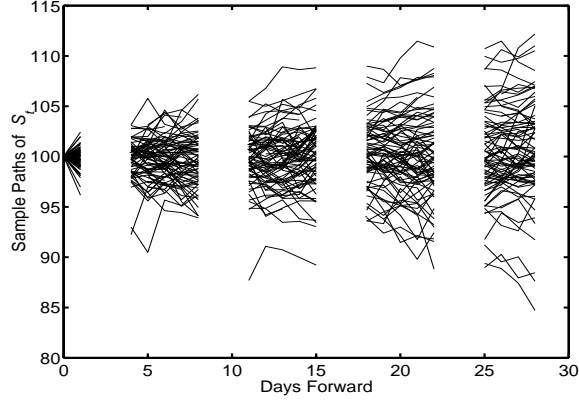
Recall that our Monte Carlo simulations indicate that the dynamic hedging strategy works very well in the Black-Scholes environment. Also recall that this dynamic strategy deteriorates dramatically when we simulate Merton's jump-diffusion model, so that a static hedge with merely three options has a hedging performance on par with or even better than daily delta hedging. For example, as shown in the last column of Table III, the standard error for hedging

twelve month options using daily delta hedging is 0.10 under the Black-Scholes model, but 1.05 under the Merton model, more than ten times larger. In contrast, the standard error from the static strategy with three one-month options is 1.00 under the Black-Scholes model, but 0.72 under the Merton model. Comparing these numbers to those for S&P 500 index options in Table IV, we find that the standard error from the static strategy with three one-month options is 1.06, while that from the dynamic strategy is 0.88, only slightly smaller. Overall, the performance difference is much closer to that under the Merton jump-diffusion case than under the purely diffusive Black-Scholes world. Hence, we infer that the S&P 500 index movement may have exhibited jumps of random magnitudes. This result is consistent with the findings from many parametric studies, and also with the results from the more generic tests such as in Aït-Sahalia (2002) and Carr and Wu (2002b).

Figure 2 depicts the normalized sample paths of the S&P 500 index level over the 79 month-long hedging experiments. The four major breaks in the sample paths reflect the four weekends of the month. There may also be other breaks due to holidays. When we compare this to the simulated sample paths under the Black-Scholes model and the Merton model, we see that the sample paths of the index exhibit both small and large movements. The jumps, however, are not as dramatic as those shown in the simulated paths of the Merton jump-diffusion model.

The hedging performance on S&P 500 index options, however, does not always match the simulated results on the Merton jump-diffusion model. In particular, under the Merton model, the simulated hedging error is smaller for hedging longer term options than for hedging shorter term options. This holds for both static and dynamic strategies. For example, as shown in Panel B of Table III, the standard error of the hedging errors from the static strategy with three one-month options is 0.84 for hedging two month options, 0.77 for hedging four month options, and 0.72 for hedging twelve month options. A similar downward trend is also observed under the dynamic strategy. However, the trend is quite the opposite when hedging S&P 500 index options. As shown in Table IV, the magnitude of the hedging errors increases when the maturity of the target option increases. This observation holds for both the static strategy and,



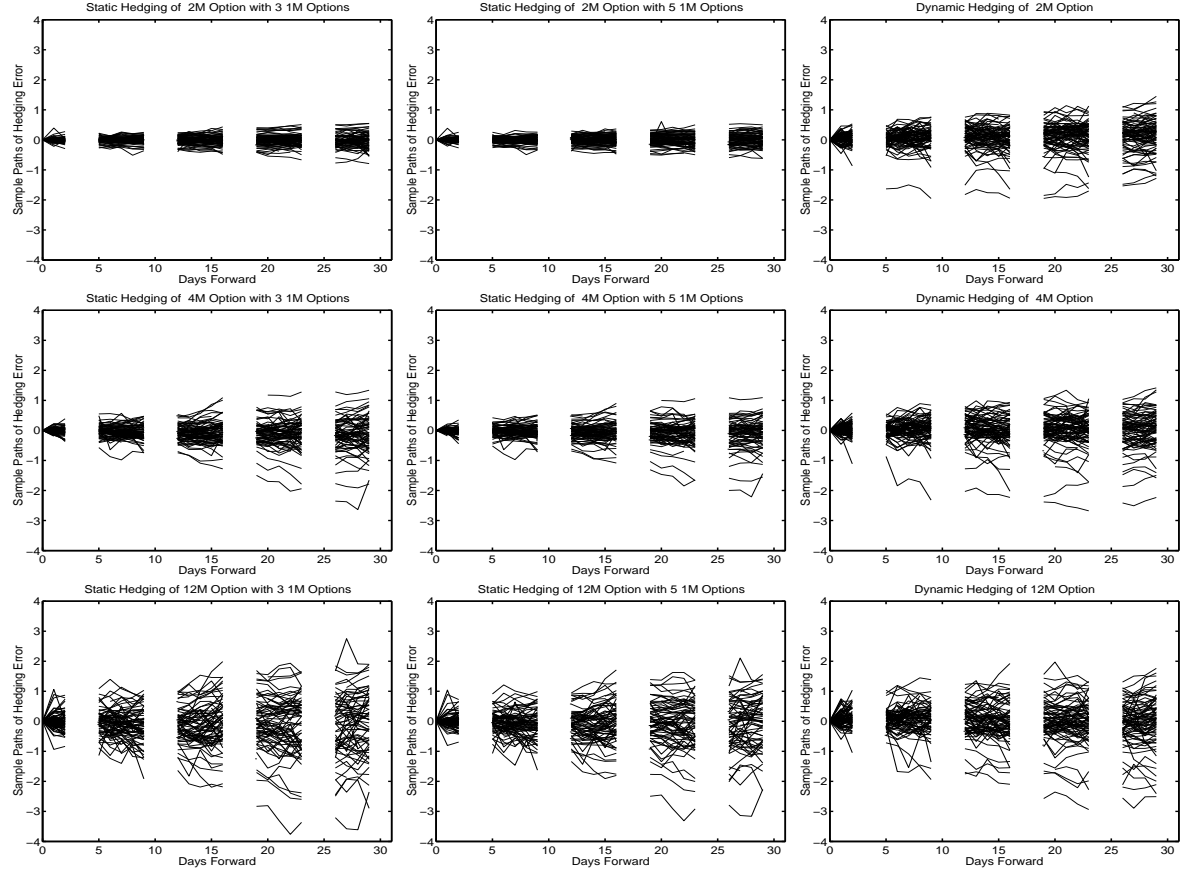


**Figure 2. Normalized Sample Paths of the S&P 500 Index**

Plots are the sample paths of the S&P 500 index level over the month-long horizon of the hedging exercises. The index level is normalized to 100 at the start of each hedging exercise. The data are aligned based on week days, starting on a Wednesday and ending on a Thursday four weeks later, spanning 29 days.

to a lesser extent, the dynamic strategy. For example, when hedging with three one-month options, the standard error of the hedging errors is 0.27 for hedging two-month index options, 0.62 for hedging four-to-six month index options, but 1.06 for hedging twelve-month or longer index options, a large increase. A similar trend is also observed for the hedging errors under the dynamic strategy.

The impact of the maturity of the target index option on the hedging performance is vividly shown in Figure 3, where we depict the 79 sample paths of the hedging errors for the hedging of near-the-money index options at maturities of (1) two months (top row), (2) four-to-six months (middle row) and (3) one year or longer (bottom row). The horizontal axis is based on the actual number of days forward. Again, the four breaks in the sample paths represent the four weekends during the month long monitoring of the hedging performance. One also observes occasional path breaks during the week days, which can be either due to holidays or missing data. The performance of the static hedge is recorded only when the market quotes for all the relevant options (the options in the hedge portfolio and the target call option) are available.



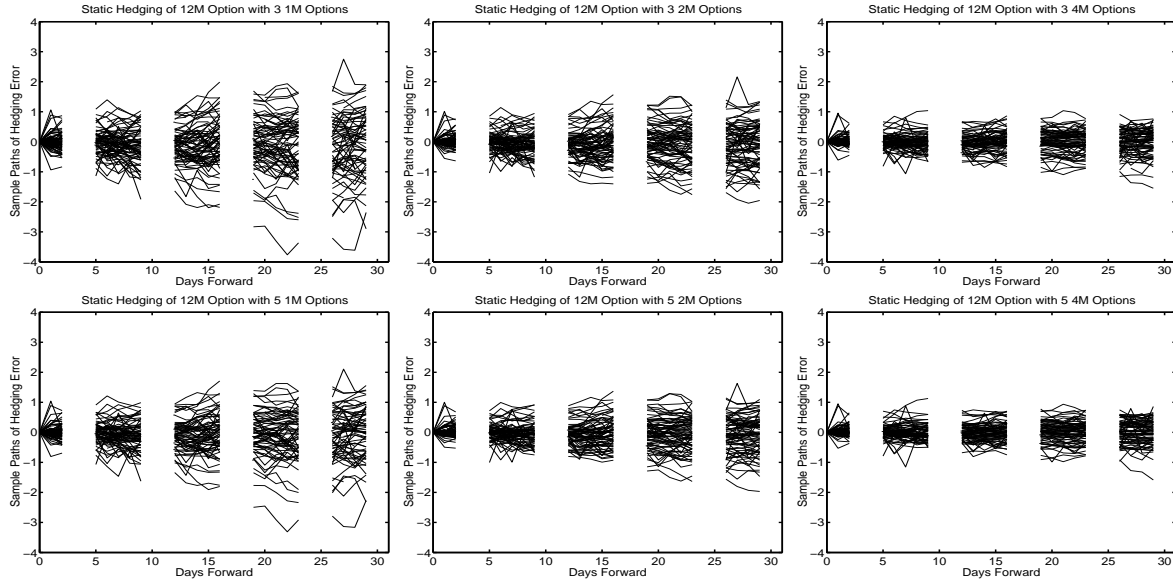
**Figure 3. Hedging Errors from Static and Dynamic Strategies**

Plots are the sample paths of the hedging errors based on (1) static strategies with a portfolio of three one-month options (left column), (2) static strategies with a portfolio of five one-month options (middle column) and (3) dynamic delta hedging strategies with the underlying futures and daily updating (right column). The options being hedged are at the money and have maturities of (1) two months (top row), (2) four-to-six months (middle row), and (3) one year and longer (bottom row).

The three panels on the left give the hedging errors based on the static hedge portfolio using three one-month options. The panels in the middle are based on a static hedge portfolio of five one-month options, while those on the right are errors based on daily delta hedging with futures. For ease of comparison, we apply the same scale on all panels in the figure. For all three hedging strategies, the magnitudes of the hedging errors increase with the maturity of the target option, more so for the two static strategies than for the dynamic strategy. These results stand in sharp contrast to what we have observed from simulating either the Merton model or the Black-Scholes model.

These differing results between the Monte Carlo simulations of single factor Markovian processes and the historical record of S&P500 index options prompts us to conjecture that the underlying index movement may not be Markovian in itself. In particular, there may be additional sources of risk, other than the underlying index level, that are also affecting the index option prices. One such risk could be stochastic volatility. When there are additional sources of risk, the hedging performance of both strategies should deteriorate as the maturity of the target call increases away from the maturity of the hedging portfolio. For the pure delta hedging strategy, the effects of neglecting to vega hedge would become more pronounced as we increase the target call option's maturity and hence its vega. For the static hedging strategy, the difference between the vega of the hedge portfolio and the vega of the target call also increases with the maturity gap between the target call and the calls in the hedge portfolio. In particular, as the calls in the static hedge approach expiry, the resulting payoffs from the maturing options are purely determined by the realized index level and do not depend upon any other state variables such as volatility. However, the value at that time of the unexpired target call will be sensitive to factors other than price, which will result in replication error. The magnitude of this replication error increases as this sensitivity of the target call increases.

Given the same target index option, we also observe that the performance of the static hedge improves as we increase the maturity of the options in the portfolio. This phenomenon is shown clearly in Figure 4, where we plot the sample paths of the hedging errors from the static



**Figure 4. Errors of Static Hedging With Options of Different Maturities**

Plots are the sample paths of the hedging errors for the static hedging strategy with a portfolio of three options. The time to maturity of the options being hedged is one year or longer. The time to maturity of options in the hedge are (1) one month (left), (2) two months (middle), and (3) four-to-six months (right). The hedging portfolio contains three options in the first row, and five options in the second row.

hedge of a target call with maturity one-year or longer. From left to right, the time to maturity of the options in the hedge increases from one month (left panel) to two months (middle panel) and then to four-to-six months (right panel). The top panels are based on portfolios with three options, while the bottom panels are based on portfolios with five options. We observe that from left to right, as the time to maturity of the options in the hedge increases, the magnitude of the hedging error declines. This result is consistent with the simulation results, confirming that longer term options are more effective in spanning the target option when a quadrature approximation is applied in setting up the portfolio. Furthermore, when there exists additional sources of risk such as stochastic volatility, the reduced maturity gap between the target option and the options in the hedge also reduces the exposure of the hedge portfolio to these additional risks.

## IV. Concluding Remarks

To hedge the risk associated with the sale of a standard European option, we develop a static hedging strategy using a portfolio of shorter-term European options. The portfolio is designed to hedge the risk associated with jumps of random size, a source of risk that cannot be effectively dealt with by delta hedging. Since a perfect static hedge requires a continuum of options in the hedging portfolio, we develop a discrete approximation of the static hedge and test its effectiveness using simulation. The simulation indicates that the static hedge approximation has about the same effectiveness as delta hedging with daily rebalancing in the Black-Scholes environment. However, when the simulated underlying price process can also experience jumps of random size, the performance of the delta hedge deteriorates dramatically. In contrast, the performance of our static option hedge is relatively insensitive to the change from a purely diffusive process to a jump diffusion. The conclusions are unchanged when we switch to ad hoc static and dynamic hedging practices necessitated by a lack of knowledge of the driving process. Further simulations indicate that the inferior performance of the delta hedge in the presence of jumps cannot be improved by increasing the rebalancing frequency, but the superior performance of the static hedging strategy can be further enhanced by using more strikes or by optimizing on the common maturity in the hedge portfolio. As a result, superior risk reduction can be achieved by the static hedge in this setting with as few as three options in the portfolio. Thus, also accompanying the superior performance are the potentially lower transaction and monitoring costs. Furthermore, since delta hedging also requires short positions in either the risky asset or the money market account, complications can arise from short sales restrictions and leverage constraints. These complications, however, are completely circumvented with our static hedging strategy.

To investigate how our static strategy performs in a realistic setting, we investigate its effectiveness in hedging S&P 500 index options and compare its performance with daily delta hedging with the index futures. We find that the superior performance of our static hedge found in the simulations of the Merton model also extends to the index options data. This finding

lends indirect support to the existence of jumps of random size as part of the S&P 500 index dynamics. We also find that the hedging errors from both the static and the dynamic strategies become larger when the maturity of the target call increases, indicating the potential existence of additional risk factors affecting option prices. One such risk can come from stochastic volatility. Hence, based on the availability and liquidity of the relevant option contracts, future research should be directed towards developing and testing static hedging strategies which account for a second risk factor. Such strategies would in general involve simultaneous positions in multiple strikes and maturities. In the interests of brevity, this extension is best left for future research.

## Appendix A. Option Pricing and Hedging under BS and MJ models

### Appendix A.1. The Black-Scholes Model

Under the Black-Scholes model, the spot price follows geometric Brownian motion under measure  $\mathbb{Q}$ ,

$$dS_t/S_t = (r - q)dt + \sigma dW_t^*.$$

The time- $t$  value of a European call with strike  $K$  and maturity  $T$  is given by

$$C(S, t; K, T; \Theta) = S e^{-q(T-t)} N(d_1(S, t; K, T; r, q, \sigma)) - K e^{-r(T-t)} N(d_1(S, t; K, T; r, q, \sigma) - \sigma\sqrt{T-t}),$$

where  $N(\cdot)$  denotes the standard normal distribution function and

$$d_1(S, t; K, T; r, q, \sigma) = \frac{\ln(S/K) + (r - q + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}. \quad (\text{A1})$$

The delta and strike weighting functions are given by

$$\begin{aligned} \Delta &\equiv \frac{\partial C(S, t; K, T; \Theta)}{\partial F} = e^{-r(T-t)} N(d_1(S, t; K, T; r, q, \sigma)), \\ w(\mathcal{K}) &= \frac{\partial^2 C(\mathcal{K}, u; K, T; \Theta)}{\partial \mathcal{K}^2} = e^{-q(T-u)} \frac{n(d_1(\mathcal{K}, u; K, T; r, q, \sigma))}{\mathcal{K}\sigma\sqrt{T-u}}, \end{aligned}$$

where  $n(\cdot)$  denotes the probability density function of a standard normal,

$$n(d_1) = \frac{1}{\sqrt{2\pi}} e^{-d_1^2/2}.$$

Given the Gauss-Hermite quadrature  $\{x_j, w_j\}_{j=1}^N$ , the static portfolio strikes and weights are given by,

$$\begin{aligned} \mathcal{K}_j &= K e^{x_j \sigma \sqrt{2(T-u)} + (q-r-\sigma^2/2)(T-u)}, \\ \mathcal{W}_j &= \frac{w(\mathcal{K}_j) \mathcal{K}_j \sigma \sqrt{2(T-u)}}{e^{-x_j^2}} w_j = \frac{e^{-q(T-u)}}{\sqrt{\pi}} w_j. \end{aligned}$$

Note that given the definition of  $\mathcal{K}_j$ , we have  $d_1(\mathcal{K}_j, u; K, T) = \sqrt{2}x_j$  and hence

$$n(d_1(\mathcal{K}_j, u; K, T; r, q, \sigma)) = \frac{1}{\sqrt{2\pi}} e^{-x_j^2}.$$

The cancellation then follows for the portfolio weight  $\mathcal{W}_j$ .

## Appendix A.2. The Merton Jump-Diffusion Model

The Merton (1976) jump-diffusion model assumes the following risk-neutral dynamics for the underlying security price movement,

$$dS_t/S_t = (r - q - \lambda^* g^*) dt + \sigma dW_t^* + dJ^*(\lambda^*),$$

where  $dJ^*$  denotes a compound Poisson jump with intensity  $\lambda^*$ . Conditional on a jump occurring, the log price relative is normally distributed with mean  $\mu_j^*$  and variance  $\sigma_j^2$ . Conditional on a jump occurring, the mean percentage price change is given by  $g^* = (e^{\mu_j^* + \sigma_j^2/2} - 1)$ .

The price of a European call can be written as a weighted average of the Black-Scholes call pricing functions, with the weights given by the Poisson distribution,

$$C(S, t; K, T; \Theta) = e^{-r(T-t)} \sum_{n=0}^{\infty} \Pr(n) \left[ S e^{(r_n - q)(T-t)} N(d_{1n}(S, t; K, T)) - K N(d_{1n}(S, t; K, T) - \sigma_n \sqrt{T-t}) \right],$$

where  $\Pr(n)$  is the Poisson probability mass function, which gives the probability of having  $n$  jumps between time  $t$  and  $T$ ,

$$\Pr(n) = e^{-\lambda^*(T-t)} \frac{(\lambda^*(T-t))^n}{n!},$$

$d_{1n}(S, t; K, T)$  is defined as

$$d_{1n}(S, t; K, T) = \frac{\ln(S/K) + (r_n - q - \sigma_n^2/2)(T-t)}{\sigma_n \sqrt{T-t}},$$

with

$$r_n = r - \lambda^* g^* + n(\mu_j^* + \sigma_j^2/2)/(T-t),$$



$$\sigma_n^2 = \sigma^2 + n\sigma_j^2/(T-t).$$

In Merton's jump diffusion model, the delta and strike weighting function are given by

$$\begin{aligned}\Delta &= e^{-2r(T-t)} \sum_{n=0}^{\infty} \Pr(n) e^{r_n(T-t)} N(d_{1n}(S, t; K, T)), \\ w(\mathcal{K}) &= e^{-r(T-u)} \sum_{n=0}^{\infty} \Pr(n) e^{(r_n-q)(T-u)} \frac{n(d_{1n}(\mathcal{K}, u; K, T))}{\mathcal{K}\sigma_n\sqrt{T-u}},\end{aligned}$$

We define the strike price points based on the Gauss-Hermite quadrature  $\{x_j, w_j\}_{j=1}^N$  as follows,

$$\mathcal{K}_j = K e^{x_j \sqrt{2v(T-u)} + (q-r-v/2)(T-u)},$$

where

$$v = \sigma^2 + \lambda^* ((\mu_j^*)^2 + \sigma_j^2),$$

is the annualized variance of the asset return under measure  $\mathbb{Q}$ . The portfolio weights are then given by

$$\mathcal{W}_j = \frac{w(\mathcal{K}_j) \mathcal{K}_j \sqrt{2v(T-u)}}{e^{-x_j^2}} w_j.$$

Note that in this case we no longer have the equality  $d_{1n} = \sqrt{2}x_j$  as  $v \neq \sigma_n^2$ . One can regard  $v$  as a weighted average of  $\sigma_n^2$  for all  $n$ 's.

## References

- Aït-Sahalia, Yacine, 2002, Telling from discrete data whether the underlying continuous-time model is a diffusion, *Journal of Finance* 57, 2075–2112.
- Bates, David, 1991, The crash of 87: Was it expected? The evidence from option markets, *Journal of Finance* 46, 1009–1044.
- Bhattacharya, Mihir, 1980, Empirical properties of the Black-Scholes formula under ideal conditions, *Journal of Financial and Quantitative Analysis* 15, 1081–1105.
- Black, Fisher, and Myron Scholes, 1972, The valuation of option contracts and a test of market efficiency, *Journal of Finance* 27, 399–417.
- Bollen, Nick, and Emma Raisel, 2002, The performance of alternative valuation models in the OTC currency option market, *Journal of International Money and Finance* forthcoming.
- Boyle, Phelim P., and David Emanuel, 1980, Discretely adjusted option hedges, *Journal of Financial Economics* 8, 259–282.
- Breeden, Douglas T., and Robert H. Litzenberger, 1978, Prices of state-contingent claims implicit in option prices, *Journal of Business* 51, 621–651.
- Britten-Jones, Mark, and Anthony Neuberger, 2000, Option prices, implied price processes, and stochastic volatility, *Journal of Finance* 55, 839–866.
- Carr, Peter, Katrina Ellis, and Vishal Gupta, 1998, Static hedging of exotic options, *Journal of Finance* 53, 1165–1190.
- Carr, Peter, and Liuren Wu, 2002a, Finite moment log stable process and option pricing, *Journal of Finance* forthcoming.
- Carr, Peter, and Liuren Wu, 2002b, What type of process underlies options? A simple robust test, *Journal of Finance* forthcoming.
- Cox, John C., and Stephen A. Ross, 1976, The valuation of options for alternative stochastic processes, *Journal of Financial Economics* 3, 145–166.
- Davis, Philip J., and Philip Rabinowitz, 1984, *Methods of Numerical Integration*. (Academic Press New York).

- Dritschel, M., and Philip Protter, 1999, Complete markets with discontinuous security price, *Finance and Stochastics* 3, 203–214.
- Dupire, Bruno, 1994, Pricing with a smile, *Risk* 7, 18–20.
- Engle, Robert F., and Joshua V. Rosenberg, 2002, Empirical pricing kernels, *Journal of Financial Economics* 64, 341–372.
- Galai, Dan, 1983, The components of the return from hedging options against stocks, *Journal of Business* 56, 45–54.
- Green, Richard, and Robert A. Jarrow, 1987, Spanning and completeness in markets with contingent claims, *Journal of Economic Theory* 41, 202–210.
- Heston, Stephen, 1993, Closed-form solution for options with stochastic volatility, with application to bond and currency options, *Review of Financial Studies* 6, 327–343.
- Hodges, Stewart, and Anthony Neuberger, 2002, Rational bounds and the robust risk management of derivatives, manuscript, London Business School.
- Jackwerth, Jens Carsten, and Mark Rubinstein, 1996, Recovering probability distributions from contemporary security prices, *Journal of Finance* 51, 347–369.
- Leland, Hayne E., 1985, Option pricing and relation with transaction costs, *Journal of Finance* 40, 1283–1301.
- Madan, Dilip B., Peter P. Carr, and Eric C. Chang, 1998, The variance gamma process and option pricing, *European Finance Review* 2, 79–105.
- Merton, Robert C., 1976, Option pricing when underlying stock returns are discontinuous, *Journal of Financial Economics* 3, 125–144.
- Nachman, David, 1988, Spanning and completeness with options, *Review of Financial Studies* 3, 311–328.
- Ross, Stephen A., 1976, Options and efficiency, *Quarterly Journal of Economics* 90, 75–89.
- Toft, Klaus Bjerre, 1996, On the mean-variance tradeoff in option replication with transactions costs, *Journal of Financial and Quantitative Analysis* 31.

**Table I**  
**Simulated Hedge Performance Comparisons of Static and Dynamic Strategies**

Entries report the summary statistics (mean, standard error, root mean squared error (RMSE), mean absolute error (MAE), mean short fall (MSF), minimum, maximum, skewness, and kurtosis) of the hedging error of a one-year call option based on both static and dynamic strategies. The hedging error is defined as the difference between the value of the hedging portfolio and the value of the target call option being hedged at the closing of the month-long hedging exercise. The static hedging portfolios consist of several one-month call options with strikes and weights determined by the static relation and a Gauss-Hermite quadrature approximation. The portfolios are then hold for one month without rebalancing. The dynamic hedging portfolio is a simple delta hedging with the underlying futures, but with daily rebalancing. The statistics are computed based on 1,000 simulated paths of the Black-Scholes model (Panel A) and the Merton jump-diffusion model (Panel B), assuming that the hedger knows the exact model in forming the portfolios. In Panel C, the sample paths and option prices are simulated based on the Merton model, but we assume that the hedger does not know this information and is formed to form the hedging portfolios based on the Black-Scholes formula, with ad hoc adjustments to accommodate the observed implied volatility. The last row of each panel reports the value of the target call option approximated by the quadrature method, with the theoretical value given under the dynamic hedging column.

Hedge Error	Static with Options					Dynamic with
No. of Assets	3	5	10	15	21	Underlying
<i>A. The Black-Scholes Model</i>						
Mean	-0.00	0.01	-0.02	0.02	0.01	0.10
Std Err	1.00	0.66	0.33	0.20	0.14	0.10
RMSE	1.00	0.66	0.33	0.20	0.14	0.14
MAE	0.86	0.57	0.29	0.18	0.13	0.12
MSF	-0.43	-0.28	-0.15	-0.08	-0.06	-0.01
Min	-1.62	-1.13	-0.58	-0.38	-0.25	-0.43
Max	1.86	0.93	0.40	0.24	0.17	0.32
Skewness	0.01	-0.26	-0.32	-0.61	-0.48	-0.84
Kurtosis	1.87	1.79	1.70	1.96	1.78	4.68
Call Value	11.72	12.20	12.34	12.37	12.36	12.35
<i>B. The Merton Jump-Diffusion Model</i>						
Mean	-0.01	0.00	-0.01	0.02	0.02	0.07
Std Err	0.72	0.47	0.31	0.16	0.12	1.05
RMSE	0.72	0.47	0.31	0.16	0.12	1.05
MAE	0.53	0.35	0.27	0.13	0.10	0.48
MSF	-0.27	-0.18	-0.14	-0.05	-0.04	-0.20
Min	-1.73	-1.28	-0.67	-0.58	-0.41	-12.12
Max	2.84	1.48	0.51	0.20	0.14	0.37
Skewness	0.56	-0.16	-0.10	-1.77	-1.65	-6.82
Kurtosis	5.23	4.07	1.83	5.74	5.15	59.79
Call Value	9.52	11.14	11.93	12.09	12.06	11.99
<i>C. Ad Hoc Black-Scholes Hedge under the Merton World</i>						
Mean	-0.00	0.01	-0.02	0.01	0.01	0.03
Std Err	0.61	0.38	0.46	0.19	0.21	1.04
RMSE	0.61	0.38	0.46	0.19	0.21	1.04
MAE	0.48	0.29	0.39	0.12	0.15	0.45
MSF	-0.24	-0.14	-0.21	-0.05	-0.07	-0.21
Min	-1.57	-1.30	-0.72	-0.92	-0.82	-12.48
Max	1.63	1.10	0.85	0.68	0.65	2.08
Skewness	-0.26	-0.43	0.12	-0.43	-0.61	-6.45
Kurtosis	3.13	4.07	1.87	9.23	5.61	56.05
Call Value	11.53	11.96	11.84	12.07	12.03	11.99

**Table II**  
**Effect of Different Rebalancing Frequencies on the Dynamic Delta Hedge**

Entries report the summary statistics (mean, standard error, root mean squared error (RMSE), mean absolute error (MAE), mean short fall (MSF), minimum, maximum, skewness, and kurtosis) of the hedging error of a one-year call option based on a dynamic delta hedge with different rebalancing frequencies. The hedging error is defined as the difference between the value of the hedging portfolio and the value of the target call option at the closing time of the month-long exercise. The statistics are computed based on 1,000 simulated paths of the Black-Scholes model (Panel A) and the Merton jump-diffusion model (Panel B) assuming that the hedger knows the exact model in forming the portfolios. In Panel C, the sample paths and option prices are simulated based on the Merton model, but we assume that the hedger does not know this information and is formed to form the hedging portfolios based on the Black-Scholes formula, with ad hoc adjustments to accommodate the observed implied volatility.

Statistics	Rebalancing Frequency Per Day			
	1	2	5	10
<i>A. The Black-Scholes Model</i>				
Mean	0.11	0.11	0.11	0.11
Std Err	0.10	0.07	0.04	0.03
RMSE	0.15	0.13	0.12	0.12
MAE	0.13	0.12	0.11	0.11
MSF	-0.01	-0.00	-0.00	0.00
Min	-0.36	-0.15	-0.03	0.02
Max	0.32	0.28	0.22	0.19
Skewness	-0.77	-0.41	-0.34	-0.16
Kurtosis	4.21	3.23	3.12	2.86
<i>B. The Merton Jump-Diffusion Model</i>				
Mean	0.09	0.09	0.09	0.09
Std Err	1.02	1.03	1.03	1.02
RMSE	1.02	1.03	1.03	1.03
MAE	0.49	0.49	0.49	0.49
MSF	-0.20	-0.20	-0.20	-0.20
Min	-11.78	-11.84	-11.72	-11.72
Max	0.38	0.37	0.35	0.35
Skewness	-6.27	-6.30	-6.24	-6.20
Kurtosis	50.97	51.40	50.34	49.79
<i>C. Ad Hoc Black-Scholes Hedge under the Merton World</i>				
Mean	0.08	0.09	0.09	0.07
Std Err	0.88	0.92	0.93	0.88
RMSE	0.88	0.93	0.93	0.88
MAE	0.41	0.41	0.42	0.41
MSF	-0.17	-0.16	-0.16	-0.17
Min	-10.13	-10.19	-10.08	-10.07
Max	1.21	8.97	9.43	0.76
Skewness	-6.31	-4.74	-4.43	-6.25
Kurtosis	52.66	54.10	53.41	51.59

**Table III**  
**Effect of Different Target and Hedging Instrument Choice**

Entries report the summary statistics (mean, standard error, root mean squared error (RMSE), mean absolute error (MAE), mean short fall (MSF), minimum, maximum, skewness, and kurtosis) of the hedging errors when hedging different target options and using different hedging instruments. The first row denotes the maturity of the target at-the-money option being hedged. The second row denotes the strategy, and the third row denotes the maturity of the options in the case of the static hedging strategy. The statistics are computed based on 1,000 simulated paths of the Black-Scholes model (Panel A) and the Merton jump-diffusion model (Panel B) assuming that the hedger knows the exact model in forming the portfolios. In Panel C, the sample paths and option prices are simulated based on the Merton model, but we assume that the hedger does not know this information and is forced to form the hedging portfolios based on the Black-Scholes formula, with ad hoc adjustments to accommodate the observed implied volatility. The last row of each panel reports the value of the target call option approximated by the quadrature method, with the theoretical value given under the dynamic hedging column.

Target Mat	2	4	12	12	12	2	4	12	12	12	2	4	12
Strategy	Static with Three Options					Static with Five Options					Daily Delta		
Instruments	1	1	1	2	4	1	1	1	2	4	Underlying Futures		
<i>A. The Black-Scholes Model</i>													
Mean	-0.02	-0.07	-0.00	-0.02	-0.01	-0.01	-0.03	0.01	-0.00	-0.00	0.30	0.21	0.11
Std Err	0.28	0.56	1.00	0.50	0.15	0.14	0.33	0.66	0.25	0.04	0.26	0.18	0.10
RMSE	0.28	0.57	1.00	0.50	0.15	0.14	0.33	0.66	0.25	0.04	0.40	0.27	0.15
MAE	0.24	0.50	0.86	0.43	0.12	0.12	0.29	0.57	0.22	0.03	0.34	0.23	0.13
MSF	-0.13	-0.28	-0.43	-0.22	-0.07	-0.06	-0.16	-0.28	-0.11	-0.02	-0.02	-0.01	-0.01
Min	-0.50	-0.90	-1.62	-0.59	-0.16	-0.26	-0.56	-1.13	-0.31	-0.04	-1.11	-0.77	-0.42
Max	0.32	0.79	1.86	1.22	0.43	0.18	0.41	0.93	0.44	0.07	0.97	0.63	0.33
Skewness	-0.42	-0.01	0.01	0.70	0.93	-0.31	-0.24	-0.26	0.36	0.54	-0.62	-0.76	-0.83
Kurtosis	1.73	1.60	1.87	2.37	2.90	1.81	1.61	1.79	1.75	1.99	4.14	4.45	4.64
Target Call	4.68	6.60	11.72	11.94	12.20	4.70	6.77	12.20	12.28	12.34	4.71	6.81	12.35
<i>B. The Merton Jump-Diffusion Model</i>													
Mean	-0.10	-0.09	-0.01	-0.02	-0.02	-0.06	-0.05	0.00	-0.00	-0.01	0.23	0.16	0.09
Std Err	0.84	0.77	0.72	0.50	0.27	0.40	0.46	0.47	0.29	0.16	2.55	1.92	1.05
RMSE	0.85	0.77	0.72	0.50	0.27	0.40	0.46	0.47	0.29	0.16	2.56	1.93	1.05
MAE	0.68	0.64	0.53	0.35	0.14	0.34	0.37	0.35	0.21	0.06	1.33	0.95	0.49
MSF	-0.39	-0.36	-0.27	-0.19	-0.08	-0.20	-0.21	-0.18	-0.11	-0.03	-0.55	-0.40	-0.20
Min	-1.20	-1.20	-1.73	-0.81	-0.33	-0.69	-0.74	-1.28	-0.72	-0.64	-25.22	-19.83	-12.11
Max	6.08	2.76	2.84	2.49	1.95	1.87	1.26	1.48	1.25	0.98	1.14	0.76	0.38
Skewness	1.29	0.51	0.56	1.76	4.76	0.11	0.66	-0.16	1.13	4.44	-5.51	-5.85	-6.81
Kurtosis	7.89	2.65	5.23	9.43	29.43	2.67	3.38	4.07	7.10	26.23	39.24	43.87	59.66
Target Call	3.76	4.80	9.52	9.76	10.15	3.72	5.33	11.14	11.19	11.18	4.11	6.34	11.99
<i>C. Ad Hoc Black-Scholes Hedge under the Merton World</i>													
Mean	-0.10	-0.07	-0.01	-0.02	-0.02	-0.06	-0.04	0.00	-0.00	-0.00	0.18	0.11	0.05
Std Err	0.84	0.55	0.63	0.46	0.27	0.40	0.31	0.41	0.26	0.14	2.26	1.73	1.04
RMSE	0.85	0.55	0.63	0.46	0.27	0.40	0.31	0.41	0.26	0.14	2.26	1.73	1.05
MAE	0.68	0.43	0.45	0.31	0.11	0.34	0.25	0.32	0.19	0.06	1.18	0.85	0.47
MSF	-0.39	-0.25	-0.23	-0.17	-0.06	-0.20	-0.14	-0.16	-0.10	-0.03	-0.50	-0.37	-0.21
Min	-1.20	-0.81	-1.42	-0.70	-0.65	-0.69	-0.53	-1.10	-0.67	-0.59	-22.01	-17.43	-12.46
Max	6.08	1.59	2.53	2.38	1.87	1.87	0.89	1.12	1.07	0.83	2.36	2.16	2.10
Skewness	1.29	1.06	0.81	2.16	5.12	0.11	0.47	-0.42	0.92	4.13	-5.38	-5.68	-6.45
Kurtosis	7.89	4.26	5.93	11.28	31.09	2.67	3.25	3.53	6.27	24.24	38.74	42.88	55.98
Target Call	3.76	4.94	10.03	9.94	10.24	3.72	5.80	11.55	11.38	11.34	4.11	6.34	11.99



**Table IV**  
**Static and Dynamic Hedging of S&P 500 Index Options**

Entries report the summary statistics (mean, standard error, root mean squared error (RMSE), mean absolute error (MAE), mean short fall (MSF), minimum, maximum, skewness, and kurtosis) of the hedging errors for the hedging exercises on S&P 500 index options. The maturities (in months) of target options being hedged are given in the first row. They are near-the-money options. The hedging strategy is either static with a portfolio of three options, five options, or dynamic with the underlying futures and daily updating. The maturity of the options in the static hedging portfolio (in months) are given in the third row. The statistics are computed based on the 79 non-overlapping month long hedging exercises over a sample period of six years (from January 1996 to August 2002). The hedging errors are computed in percentages of the spot index level at the starting date of each exercise. The hedging strategies are designed based on the Black model with ad hoc adjustments to the observed implied volatilities. The last row reports the sample average of the value of the target call option approximated by the quadrature-based hedging portfolio. Numbers under the dynamic hedging columns are the sample average of the observed target call option price, all in percentages of the underlying spot index level at the starting date of each month.

Target Mat	2	4	12	12	12	2	4	12	12	12	2	4	12
Strategy	Static with Three Options					Static with Five Options					Daily Delta		
Instruments	1	1	1	2	4	1	1	1	2	4	Underlying Futures		
Mean	-0.03	-0.13	-0.03	-0.09	0.00	-0.00	-0.10	-0.01	-0.07	0.01	0.16	0.03	-0.01
Std Err	0.27	0.62	1.06	0.74	0.44	0.23	0.50	0.87	0.64	0.42	0.57	0.65	0.88
RMSE	0.27	0.63	1.05	0.74	0.44	0.23	0.51	0.86	0.64	0.42	0.59	0.65	0.87
MAE	0.21	0.50	0.85	0.61	0.35	0.18	0.38	0.69	0.52	0.33	0.47	0.49	0.64
MSF	-0.12	-0.31	-0.44	-0.35	-0.17	-0.09	-0.24	-0.35	-0.29	-0.16	-0.15	-0.23	-0.32
Min	-0.79	-1.78	-2.90	-1.95	-1.55	-0.62	-1.55	-2.32	-1.97	-1.58	-1.28	-2.23	-2.51
Max	0.55	1.34	1.90	1.34	0.76	0.50	1.09	1.56	1.10	0.86	1.45	1.42	1.76
Skewness	-0.19	-0.21	-0.35	-0.22	-0.87	-0.28	-0.51	-0.46	-0.45	-0.74	-0.49	-0.78	-0.76
Kurtosis	3.00	3.20	2.81	2.51	4.16	3.12	3.70	2.91	2.89	4.51	3.34	4.47	3.83
Target Call	3.63	5.92	10.49	10.41	11.47	3.68	6.16	10.94	10.76	11.52	3.65	6.14	11.58