Testing Mean Reversion in Stock Market Volatility*

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ABSTRACT

This paper presents a comprehensive study of continuous time GARCH modeling with the thin-tailed normal and the fat-tailed Student-*t* and generalized error distributions. The paper measures the degree of mean reversion in stock return volatility based on the relationship between discrete time GARCH and continuous time diffusion models. The convergence results based on the aforementioned distribution functions are shown to have similar implications for testing mean reversion in stochastic volatility. Alternative models are compared in terms of their ability to capture mean-reverting behavior of stock return volatility. The empirical evidence obtained from several stock market indices indicates that the conditional variance, log-variance, and standard deviation of stock market returns are pulled back to some long-run average level over time.

I. Introduction

This paper deals with continuous time GARCH modeling with fat-tailed distributions, and provides implications for testing mean reversion in stock market volatility based on the relationship between discrete time GARCH and continuous time diffusion models.

Modeling and estimating the volatility of economic time series has been high on the agenda of financial economists over the last two decades. Engle (1982) put forward the Autoregressive Conditional Heteroskedastic (ARCH) class of models for conditional variances which proved to be extremely useful for analyzing financial return series. Since then an extensive literature has been developed for modeling the conditional distribution of stock prices, interest rates, and exchange rates.[1] Following the introduction of ARCH models by Engle (1982) and their generalization by Bollerslev (1986), there have been numerous refinements of this approach to estimating conditional volatility. Most of the refinements have been driven by empirical regularities in financial data.

First, asset prices are generally nonstationary, often have a unit root whereas returns are usually stationary. Second, return series usually show no or little autocorrelation.[2] Serial independence between the squared values of the series however is often rejected pointing towards the existence of nonlinear relationships between subsequent observations. Volatility of the return series appears to be clustered, i.e., heavy fluctuations occur for longer periods whereas small values for returns tend to be followed by small values. These phenomena point towards time-varying conditional variances. Third, normality is rejected frequently in favor of some thick-tailed distribution. The presence of unconditional excess kurtosis in the series could be related to the time-variation in the conditional variance. This leptokurtosis is reduced when returns are normalized by time-varying variances of GARCH models, but it is by no means eliminated.[3] Fourth, some series exhibit the so-called leverage effects [e.g., Christie (1982)], i.e, changes in stock prices tend to be negatively correlated with changes in volatility. Some series have skewed unconditional empirical distributions pointing towards the inappropriateness of the normal distribution.[4]

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[1] This vast literature on the theory and empirical evidence for ARCH modeling has been surveyed in Bollerslev, Chou, and Kroner (1992), Bollerslev, Engle, and Nelson (1994), Andersen (1994), Hentschel (1995), Pagan (1996), Duan (1997), and Bali (2000) among others. A detailed treatment of ARCH models at a textbook level is also given by Gourieroux (1997).

[2] Some researchers find that stock index returns are positively autocorrelated at high frequencies. The autocorrelation in index returns has been attributed to nonsynchronous trading [e.g., Fisher (1966), Scholes and Williams (1977), Lo and MacKinlay (1990)] and to time varying short term expected returns, or risk premia [e.g., Fama and French (1988), Conrad and Kaul (1988)].

[3] To deal with this problem, many researchers have used more general distributions [e.g., Mandelbrot (1963), Fama (1965), Bollerslev (1987), Nelson (1991), Hansen (1994), Jondeau and Rockinger (2003), and Bali (2003, 2005)].

[4] These stylized facts, such as fat-tails and volatility clustering, of financial return series can be explained by models introduced by Brock and Hommes (1998), Gaunersdorfer (2000), and Chiarella and He (2002, 2003).

GARCH models have been developed to account for these empirical regularities in financial data. In contrast to the stochastic differential equations frequently used in the theoretical finance literature to model time-varying volatility, GARCH processes are discrete time stochastic difference equations. The discrete time approach has been favored by empiricists because observations are usually recorded at discrete points in time, and the likelihood functions implied by the GARCH models are usually easy to compute and maximize. By contrast, the likelihood of a nonlinear diffusion process observed at discrete intervals can be very difficult to derive, especially when there are unobservable state variables in the system.

Relatively little work has been done so far on the relation between continuous time diffusion and discrete time GARCH models. Indeed, the two literatures have developed quite independently, with little attempt to reconcile the discrete and continuous time models. Nelson (1990) was the first to study the continuous time limits of GARCH models. Nelson and Foster (1994) derive the diffusion limits of the standard GARCH, exponential GARCH, and absolute GARCH processes. Following Nelson (1990) and Nelson and Foster (1994), weak converge results have been developed by Drost and Werker (1996), Fornari and Mele (1996), Duan (1997), and Corradi (2000) for some of the GARCH processes, but clear results do not exist for others. In this paper, we present the diffusion limits of many symmetric and asymmetric GARCH models, and then use the drift of these limiting models to test mean reversion in stock return volatility. The earlier research on continuous time GARCH modeling assumes that the error process is drawn from the normal density despite the mounting evidence of fat-tailed errors. One of our contributions is to develop conditions under which many GARCH models *with a fat-tailed density* converge weakly to Ito processes as the length of the discrete time intervals goes to zero.

The former studies test for mean reversion in stock prices, but mean reversion in stock return volatility has not been tested yet. Despite a bewildering array of GARCH models, relatively little is known about how these models can be used to detect the presence of mean reversion in stochastic volatility. The present paper, to our knowledge, represents the first study of testing and measuring the degree of mean reversion in volatility. The theoretical and empirical specifications of the stock price process often assume a normal distribution for modeling unexpected information shocks. However, most studies find that the normality assumption is violated, i.e., the distribution of equity returns has much thicker tails compared to the normal distribution. To take this into account, we use the fat-tailed standardized Student-*t* and generalized error distribution as a more flexible density function with degrees of freedom estimated endogenously. The convergence results based on the aforementioned distribution functions are shown to have similar implications for testing mean reversion in stochastic volatility.

The geometric Brownian motion assumption utilized by standard options pricing models imply that log-stock price changes are identically and independently distributed (iid) normal variables, thus exhibit no moment dependencies, such as asymmetric and conditional volatility with skewed and/or fat-tailed distributions. However, our empirical results from alternative specifications of the stock price process indicate the presence of significant volatility clustering and leptokurtosis in stock returns. To accommodate tail-thickness of the return distribution as well as time-variation in volatility, recent studies derive an option pricing model with fat-tailed distributions and stochastic volatility. The underlying assumption in these models is that stochastic volatility follows a mean-reverting process. Without this assumption, it is extremely difficult to come up with a closed form solution for option prices. Hence, testing the presence of mean-reversion in stock return volatility has important implications for newly proposed option pricing models.

This paper is organized as follows. Section II provides continuous time limits of the GARCH models with fat-tailed distributions. Section III describes the data and estimation methodology. Sections IV and V present the empirical results. Section VI concludes the paper.

II. Continuous-Time GARCH Modeling with Fat-Tailed Distributions

In continuous time diffusion models, stock price movements are described by the following stochastic differential equation,

$$dP_t = \mu P_t dt + \sigma P_t dW_t \, , \tag{1}$$

where P_t is a stock price at time t, W_t is a standard Wiener process with zero mean and variance of dt, μ and σ are the constant drift and diffusion parameters of the geometric Brownian motion.

Applying Ito's Lemma to equation (1), we can derive the process followed by $\ln P_t$:

$$d \ln P_t = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dW_t \tag{2}$$

where $\mu - \sigma^2/2$ is the constant drift and σ^2 is the constant variance of log-stock price changes. The one-factor continuous time model in equation (2) can be extended by incorporating a stochastic volatility factor into the diffusion function:

$$d \ln P_t = \left(\mu_t - \frac{\sigma_t^2}{2} \right) dt + \sigma_t dW_{1,t} \tag{3}$$

$$df(\sigma_t) = f_\mu(\sigma_t) dt + f_\sigma(\sigma_t) dW_{2,t} \tag{4}$$

where $\mu_t - \sigma_t^2 / 2$ and σ_t are the (instantaneous) time-varying mean and volatility of stock returns, $W_{1,t}$ and $W_{2,t}$ are standard Brownian motion processes so that $dW_{1,t}$ and $dW_{2,t}$ are normally distributed with zero mean and variance of dt. In equation (4), the stochastic volatility factor is specified with the instantaneous *variance* [i.e., $f(\sigma_t) = \sigma_t^2$], the *log-variance* [i.e., $f(\sigma_t) = \ln \sigma_t^2$], or the *standard deviation* of stock returns [i.e., $f(\sigma_t) = \sigma_t$]. $f_\mu(\sigma_t)$ and $f_\sigma(\sigma_t)$ are the drift and diffusion functions of the volatility process, respectively.

Different parameterization of $f_\mu(\sigma_t)$ and $f_\sigma(\sigma_t)$ yields different GARCH processes in discrete time. Our discussion on continuous time GARCH modeling and stock return volatility is focused on ten parametric versions of $f_\mu(\sigma_t)$ and $f_\sigma(\sigma_t)$. To have a more precise discussion, the discrete time GARCH models are formally defined below.

$$\ln P_{t+\Delta} - \ln P_t = \left(\mu_t - \frac{\sigma_t^2}{2} \right) \Delta + \sigma_t z_t \sqrt{\Delta} \tag{5}$$

AGARCH: Asymmetric GARCH model of Engle (1990)

$$\sigma_t^2 = \beta_0 + \beta_1 (\gamma + \sigma_{t-\Delta} z_{t-\Delta})^2 + \beta_2 \sigma_{t-\Delta}^2 \tag{6}$$

EGARCH: Exponential GARCH model of Nelson (1991)

$$\ln \sigma_t^2 = \beta_0 + \beta_1 [| z_{t-\Delta} | - E(| z_{t-\Delta} |)] + \beta_2 \ln \sigma_{t-\Delta}^2 + \gamma z_{t-\Delta} \tag{7}$$

GARCH: Linear symmetric GARCH model of Bollerslev (1986)

$$\sigma_t^2 = \beta_0 + \beta_1 \sigma_{t-\Delta}^2 z_{t-\Delta}^2 + \beta_2 \sigma_{t-\Delta}^2 \tag{8}$$

GJR-GARCH: Threshold GARCH model of Glosten, Jagannathan, and Runkle (1993)

$$\sigma_t^2 = \beta_0 + \beta_1 \sigma_{t-\Delta}^2 z_{t-\Delta}^2 + \beta_2 \sigma_{t-\Delta}^2 + \gamma S_{t-\Delta}^- \sigma_{t-\Delta}^2 z_{t-\Delta}^2 \tag{9}$$

$$S_{t-\Delta}^- = 1 \text{ if } \sigma_{t-\Delta} z_{t-\Delta} < 0, \text{ and } S_{t-\Delta}^- = 0 \text{ otherwise}$$

NGARCH: Nonlinear asymmetric GARCH model of Engle and Ng (1993)

$$\sigma_t^2 = \beta_0 + \beta_1 \sigma_{t-\Delta}^2 (\gamma + z_{t-\Delta})^2 + \beta_2 \sigma_{t-\Delta}^2 \tag{10}$$

QGARCH: Quadratic GARCH model of Sentana (1995)

$$\sigma_t^2 = \beta_0 + \beta_1 \sigma_{t-\Delta}^2 z_{t-\Delta}^2 + \beta_2 \sigma_{t-\Delta}^2 + \gamma \sigma_{t-\Delta} z_{t-\Delta} \tag{11}$$

SQR-GARCH: Square-Root GARCH model of Heston and Nandi (1998)[5]

$$\sigma_t^2 = \beta_0 + \beta_1 (\gamma \sigma_{t-\Delta} + z_{t-\Delta})^2 + \beta_2 \sigma_{t-\Delta}^2 \tag{12}$$

[5] As will be shown in Appendices A-C, the continuous time limit of equation (12) converges to a stochastic variance process which is generated by the "square-root" diffusion popularized as a model of the short term interest rate by Cox, Ingersoll, and Ross (1985).

TGARCH: Threshold GARCH model of Zakoian (1994)

$$\sigma_t = \beta_0 + \beta_1 \sigma_{t-\Delta} |z_{t-\Delta}| + \beta_2 \sigma_{t-\Delta} + \gamma S^{-}_{t-\Delta} \sigma_{t-\Delta} z_{t-\Delta} \tag{13}$$

$$S^{-}_{t-\Delta} = 1 \text{ if } \sigma_{t-\Delta} z_{t-\Delta} < 0, \text{ and } S^{-}_{t-\Delta} = 0 \text{ otherwise}$$

TS-GARCH: The specification proposed by Taylor (1986) and Schwert (1989)

$$\sigma_t = \beta_0 + \beta_1 \sigma_{t-\Delta} |z_{t-\Delta}| + \beta_2 \sigma_{t-\Delta} \tag{14}$$

VGARCH: A version proposed in Engle and Ng (1993)

$$\sigma_t^2 = \beta_0 + \beta_1 (\gamma + z_{t-\Delta})^2 + \beta_2 \sigma^2_{t-\Delta} \tag{15}$$

where Δ is the length of the time interval, z_t is a random variable drawn from the standardized normal distribution [$z_t \sim N(0,1)$], $\beta_0 > 0$, $0 \le \beta_1 < 1$, $0 \le \beta_2 < 1$, and $\gamma < 0$. The diffusion parameter γ allows for asymmetric volatility response to past positive and negative information shocks.

Following Nelson (1990) and Nelson and Foster (1994), it can be shown that the symmetric and asymmetric GARCH models in (6)-(15) converge in distribution to continuous time stochastic volatility models as Δ goes to zero. Now we consider the properties of the stochastic difference equation system (5)-(6) as we partition time more and more finely. We allow the parameters of the system β_0, β_1, and β_2 to depend on the discrete time interval Δ, and make both the drift term in (5) and the variance of z_t proportional to Δ. We apply the Nelson's (1990) approximation scheme to equations (5)-(6):

$$\ln P_{k\Delta} - \ln P_{(k-1)\Delta} = (\mu_{(k-1)\Delta} - \sigma^2_{(k-1)\Delta} / 2)\, \Delta + \sigma_{k\Delta} z_{k\Delta} \tag{16}$$

$$\sigma^2_{k\Delta} = \beta_{0\Delta} + \beta_{1\Delta} (\gamma + \Delta^{-1/2} \sigma_{(k-1)\Delta} z_{(k-1)\Delta})^2 + \beta_{2\Delta} \sigma^2_{(k-1)\Delta} \tag{17}$$

where $z_{k\Delta} \overset{iid}{\sim} N(0, \Delta)$ and $(k-1)\Delta \le t \le k\Delta$.

We allow β_0, β_1, and β_2 to depend on Δ because our objective is to discover which sequences $\{\beta_{0\Delta}, \beta_{1\Delta}, \beta_{2\Delta}\}$ make the $\{\sigma_t^2 - \sigma_{t-1}^2, \ln P_t - \ln P_{t-1}\}$ process converge in distribution to the Ito process given in equations (3)-(4) as Δ goes to zero. We will compute the first two conditional moments, and then, after few mild technical conditions, appeal to the theorems for weak convergence of Markov chains to diffusion processes by Strook and Varadhan (1979, Chapter 11) or by Ethier and Kurtz (1986, Chapter 8). The first and second moments per unit of time are given by:[6]

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[6] The continuous time limit of eqs. (18) and (19) yield the Ito process in (3) because the drift of the log-stock price changes equals $\mu_t - \sigma_t^2 / 2$ limit as Δ goes to zero, and the instantaneous variance of stock returns equals σ_t^2 since

$$\Delta^{-1} \text{Var}\, [(\ln P_{k\Delta} - \ln P_{(k-1)\Delta}) | \Omega_{(k-1)\Delta}] = \Delta^{-1} \{E[(\ln P_{k\Delta} - \ln P_{(k-1)\Delta})^2 | \Omega_{(k-1)\Delta}] - (E[(\ln P_{k\Delta} - \ln P_{(k-1)\Delta}) | \Omega_{(k-1)\Delta}])^2\} = \sigma_t^2$$

$$\Delta^{-1}\,\mathrm{E}[(\ln \mathrm{P}_{k\Delta} - \ln \mathrm{P}_{(k-1)\Delta})|\Omega_{(k-1)\Delta}] = (\mu_{(k-1)\Delta} - \sigma^2_{(k-1)\Delta}/2) \tag{18}$$

$$\Delta^{-1}\,\mathrm{E}[(\ln \mathrm{P}_{k\Delta} - \ln \mathrm{P}_{(k-1)\Delta})^2|\Omega_{(k-1)\Delta}] = \Delta(\mu_{(k-1)\Delta} - \sigma^2_{(k-1)\Delta}/2)^2 + \sigma^2_{k\Delta} \tag{19}$$

$$\Delta^{-1}\,\mathrm{E}[(\sigma^2_{k\Delta} - \sigma^2_{(k-1)\Delta})|\Omega_{(k-1)\Delta}] = \Delta^{-1}(\beta_{0\Delta} + \beta_{1\Delta}\gamma^2) + \Delta^{-1}[\beta_{1\Delta} + \beta_{2\Delta} - 1] \tag{20}$$

$$\begin{aligned}\Delta^{-1}\,\mathrm{E}[(\sigma^2_{k\Delta} - \sigma^2_{(k-1)\Delta})^2|\Omega_{(k-1)\Delta}] &= \Delta^{-1}(\beta_{0\Delta} + \beta_{1\Delta}\gamma^2)^2 \\ &\quad + \Delta^{-1}[4\beta^2_{1\Delta}\gamma^2 + 2(\beta_{0\Delta} + \beta_{1\Delta}\gamma^2)(\beta_{1\Delta} + \beta_{2\Delta} - 1)]\,\sigma^2_{(k-1)\Delta} \\ &\quad + \Delta^{-1}[2\beta^2_{1\Delta} + (\beta_{1\Delta} + \beta_{2\Delta} - 1)^2]\,\sigma^4_{(k-1)\Delta}\end{aligned} \tag{21}$$

where $\Omega_{(k-1)\Delta}$ is the information set at time $(k-1)\Delta$. To obtain the stochastic variance process in (4), we will consider the following parameterization for $\beta_{0\Delta}$, $\beta_{1\Delta}$, and $\beta_{2\Delta}$ as a function of Δ:

$$\lim_{\Delta\to 0} \Delta^{-1}(\beta_{0\Delta} + \beta_{1\Delta}\gamma^2) = \rho_0, \tag{22}$$

$$\lim_{\Delta\to 0} \Delta^{-1}[\beta_{1\Delta} + \beta_{2\Delta} - 1] = \rho_1 < 0, \tag{23}$$

$$\lim_{\Delta\to 0} \Delta^{-1} 2\beta^2_{1\Delta} = \lambda^2 > 0. \tag{24}$$

Equations (22) and (23) give the drift, $\rho_0 + \rho_1\sigma^2_t$, of the stochastic volatility model. Equation (24) yields the standard deviation, $\lambda\sigma_t\sqrt{2\gamma^2 + \sigma^2_t}$, of the variance process in (4) because

$$\begin{aligned}\Delta^{-1}\,\mathrm{Var}[(\sigma^2_{k\Delta} - \sigma^2_{(k-1)\Delta})|\Omega_{(k-1)\Delta}] &= \Delta^{-1}\{\mathrm{E}[(\sigma^2_{k\Delta} - \sigma^2_{(k-1)\Delta})^2|\Omega_{(k-1)\Delta}] - (\mathrm{E}[(\sigma^2_{k\Delta} - \sigma^2_{(k-1)\Delta})|\Omega_{(k-1)\Delta}])^2\} \\ &= \Delta^{-1}\, 2\beta^2_{1\Delta}\sigma^2_{(k-1)\Delta}(2\gamma^2 + \sigma^2_{(k-1)\Delta}).\end{aligned}$$

One can easily show that equations (22), (23), and (24) are all satisfied if

$$\beta_{0\Delta} = \rho_0\Delta - \gamma^2\sqrt{\frac{\lambda^2}{2}}\Delta^{1/2} \tag{25}$$

$$\beta_{1\Delta} = \sqrt{\frac{\lambda^2}{2}}\,\Delta^{1/2} \tag{26}$$

$$\beta_{2\Delta} = 1 + \rho_1\Delta - \sqrt{\frac{\lambda^2}{2}}\,\Delta^{1/2}. \tag{27}$$

We find that the AGARCH model of Engle (1990) in eq. (6) converges weakly to the stochastic volatility model in eq. (4) with $f_\mu(\sigma_t) = \rho_0 + \rho_1\sigma^2_t$ and $f_\sigma(\sigma_t) = \lambda\sigma_t\sqrt{2\gamma^2 + \sigma^2_t}$. We should note that there is no specific reason for us to present the diffusion limit of AGARCH model in this section, except for the illustration purpose. The diffusion limits of alternative GARCH processes are presented in Appendix A.

In the existing literature, diffusion limits of the GARCH models are obtained under the assumption that innovation process is normally distributed. However, Drost and Werker (1996) show that the normality assumption of an underlying continuous time GARCH(1,1) model leads to kurtosis parameters of the corresponding discrete time processes which are greater than three, implying heavy tails. Their results provide an explanation why fat-tailed conditional distributions are obtained, without exception, in empirical work. To capture this phenomenon, we use a fat-tailed innovation distribution in continuous time GARCH modeling.

We let the distribution of $z_t \sim t[0, v/(v–2)]$ be Student-*t* with degrees of freedom v:

$$f(z_t) = \Gamma\left(\frac{v+1}{2}\right)\Gamma\left(\frac{v}{2}\right)^{-1}(v\pi)^{-1/2}\left[1+\frac{z_t^2}{v}\right]^{-(v+1)/2} \tag{28}$$

where $\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}dx$ is the gamma function. The *t*-distribution is symmetric around zero, and the first, second, and fourth moments of z_t and $|\, z_t \,|$ are equal to:

$$E(z_t) = 0, \quad E(|\, z_t \,|) = 2\Gamma\left(\frac{v+1}{2}\right)\Gamma\left(\frac{v}{2}\right)^{-1}\sqrt{\frac{v}{(v-1)^2\pi}}, \quad E(z_t^2) = \frac{v}{v-2}, \quad E(z_t^4) = \frac{3v^2}{(v-2)(v-4)}$$

Applying the Nelson's (1990) approximation scheme to the heavy-tailed innovation process with the statistical properties of z_t and $|\, z_t \,|$, we show in Appendix B that the diffusion limits of discrete time GARCH processes with Student-*t* density are given by equations (B.1)-(B.10). Appendix B also displays the parameter restrictions that form the relation between continuous time diffusion and discrete time GARCH models with Student *t*-distribution.

Several other conditional distributions have been employed in the literature to fully capture the degree of tail fatness in speculative prices. One of these heavy-tailed distributions known as the generalized error distribution has been widely used by financial economists.[7] In addition to the Student-*t* distribution, we utilize the density function for the generalized error distribution (GED):

$$f(z_t) = \frac{v\ \exp[(-1/2)\,|\, z_t / \Pi \,|^v\,]}{\Pi\ 2^{[(v+1)/v]}\ \Gamma(1/v)} \tag{29}$$

[7] The generalized error distribution (GED) is initially introduced by Subbotin (1923), and then used by Box and Tiao (1962) to model prior densities in Bayesian estimation and by Nelson (1991) to model the distribution of stock market returns.

where $\Pi = \left[\frac{2^{(-2/v)}\Gamma(1/v)}{\Gamma(3/v)} \right]^{1/2}$. For the tail thickness parameter v = 2, the GED density equals the standard normal density. For v < 2 the distribution has thicker tails than the normal, while v > 2 results in a distribution with thinner tails than the normal.

Applying the same approximation scheme to the fat-tailed innovation process along with the statistical properties of z_t and $|z_t|$ for the GED density,

$$E(z_t) = 0, \quad E(|z_t|) = \frac{\Gamma\left(\frac{2}{v}\right)}{\Gamma\left(\frac{3}{v}\right)^{1/2}\Gamma\left(\frac{1}{v}\right)^{1/2}}, \quad E(z_t^2) = 1, \quad E(z_t^4) = \frac{\Gamma\left(\frac{1}{v}\right)\Gamma\left(\frac{5}{v}\right)}{\Gamma\left(\frac{3}{v}\right)^2},$$

we present in Appendix C that the continuous time limits of GARCH models are given by equations (C.1)-(C.10). Appendix C also shows the parameter restrictions that form the relation between diffusion and GARCH processes with GED density.

The continuous time stochastic volatility models presented in Appendices A-C have similar implications for testing mean reversion in stock return volatility. Table 1 shows the mean reversion rates based on the relation between discrete time GARCH and continuous time diffusion models. The degree of mean reversion in stochastic volatility is measured by the negative values of ρ_1 for the normal distribution, κ_1 for the Student-*t* distribution, and θ_1 for the generalized error distribution. As will be discussed in the paper, the degree (or speed) of mean reversion in stock return volatility is found to be robust across different distribution functions.

III. Data and Estimation

The data consist of daily prices for the Dow Jones Industrial Average (DJIA). The time period of investigation for the Dow 30 equity index extends from May 26, 1896 through December 29, 2000, giving a total of 28,758 daily observations.[8] To compute stock market returns (r_t) we use the formula, $r_t = \ln P_t - \ln P_{t-1}$, where P_t is the value of the stock market index for period t. Table 2 provides

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[8] At an earlier stage of the study, we also use the daily S&P 500, NASDAQ Composite, and New York Stock Exchange (NYSE) indices. The time period of investigation for the S&P 500 composite index extends from January 4, 1965 through December 29, 2000, giving a total of 9,059 daily observations. The data for the NASDAQ Composite index cover the period from February 5, 1971 to December 29, 2000, yielding 7,554 daily observations. The data for the NYSE index consist of 8,803 daily returns for the period January 3, 1966 - December 29, 2000. To save space, we do not present the empirical findings based on the S&P 500, NASDAQ, and NYSE indices. They are available upon request. However, we should note that the qualitative results regarding mean reversion in stock return volatility are found to be robust across different data sets.

descriptive statistics for the daily percentage returns on DJIA. The unconditional mean of the daily returns is about 0.019% with a standard deviation of 1.09%. The maximum and minimum values are about 14.27% and –27.96%, respectively.

We also report the skewness, excess kurtosis, first-order autocorrelation, and the Ljung-Box statistics for testing the null hypotheses of independent and identically distributed normal variates. The skewness statistic for daily returns is negative and statistically significant at the 1% level. The excess kurtosis statistic is considerably high and significant at the 1% level, implying that the distribution of equity returns has much ticker tails than the normal distribution. The fat-tail property is more dominant than skewness in the sample. The first-order autocorrelation coefficient is found to be positive and large enough to reject the first-order zero correlation null hypothesis at the 1% level. The Ljung-Box $Q_1(12)$ statistic for the cumulative effect of up to twelfth-order autocorrelation in the standardized residual exceeds 26.22 (1 percentile critical value from a χ^2 distribution with 12 d.f.). This provides evidence of temporal dependencies in the first moment of the distribution of returns. The Ljung-Box $Q_2(12)$ statistic on the squared standardized residuals provides us with a test of intertemporal dependence in the variance. The $Q_2(12)$ statistic rejects the zero correlation null hypothesis, indicating that the distribution of the next squared return depends not only on the current return but on several previous returns.

To estimate the conditional distribution of stock market returns, we use the partial adjustment model of Amihud and Mendelson (1987). The model distinguishes the intrinsic value of a security at time t, V_t, and its observed price, P_t:[9]

$$P_t - P_{t-1} = (1-\alpha)(V_t - P_{t-1}) \tag{30}$$

where both V_t and P_t are expressed in natural logarithm, and the adjustment coefficient $(1-\alpha)$ measures the speed of adjustment of transaction prices towards the security's value, and satisfies $0 < (1-\alpha) < 2$.[10] An economic justification of equation (30) can be based on the assumption that the market price, P_t, is arrived at through minimization of the following cost function:

$$C = g_e(P_t - V_t)^2 + g_a(P_t - P_{t-1})^2 \tag{31}$$

[9] As indicated by Black (1986), the difference between the intrinsic value of a security and its price is attributable to noise. As discussed in Amihud and Mendelson (1987), this noise, which pushes the observed price of the security away from its value, comes from two main sources. First, it is the result of noise trading caused by errors in the analysis and interpretation of information and by transitory liquidity needs of investors. Second, it reflects the impact of the trading mechanism by which prices are set in the market.

[10] In particular, $\alpha = 1$ represents the extreme case of no price reaction to changes in value, and $0 < \alpha < 1$ represents partial price adjustment. A unit adjustment coefficient $(\alpha = 0)$ represents full price adjustment, and when $\alpha < 0$, we have overshooting or over-reaction of traders to new information.

where $g_e(P_t - V_t)^2$ measures the cost of being out of equilibrium, and $g_a(P_t - P_{t-1})^2$ measures the cost of making the adjustment [see, for example, Hwang (1985)]. Minimization of equation (31) implies that:

$$P_t - P_{t-1} = \left(\frac{g_e}{g_e + g_a}\right)(V_t - P_{t-1}) \tag{32}$$

which is essentially equation (30) with speed of adjustment given by $g_e/(g_e + g_a)$. Following Amihud and Mendelson (1987), Damodaran (1993), and others, we assume that the logarithms of security values, V_t, follow a random walk process with drift:

$$V_t = m + V_{t-1} + u_t \tag{33}$$

where m represents the expected daily value return and u_t is a zero mean innovation process. Equations (30) and (33) imply that returns follow an autoregressive process of order one, i.e.,

$$r_t = \omega + \alpha r_{t-1} + \varepsilon_t \tag{34}$$

where $\omega = m(1 - \alpha)$ and $\varepsilon_t = (1 - \alpha)u_t$.

To complete the estimation framework, we let the error process $\{\varepsilon_t = z_t\sigma_t\}$ be conditionally heteroskedastic with time-varying variance, for example, given by the GARCH (1,1) process:

$$\sigma_t^2 = \beta_0 + \beta_1 z_{t-1}^2 \sigma_{t-1}^2 + \beta_2 \sigma_{t-1}^2 \tag{35}$$

where z_t is an independent normal N(0,1) variate, and can be viewed as an unexpected shock to the stock market. Since z_t is drawn from the standard normal distribution, the density function for r_t is

$$f(\Theta; r_t) = \frac{1}{\sqrt{2\pi\sigma_{t|t-1}^2}} \exp\left\{-\frac{1}{2}\left(\frac{r_t - \mu_{t|t-1}}{\sigma_{t|t-1}}\right)^2\right\}. \tag{36}$$

where $\Theta \equiv (\omega, \alpha, \beta_0, \beta_1, \beta_2)$ is the parameter vector, $\mu_{t|t-1} = \omega + \alpha\, r_{t-1}$ is the conditional mean, and $\sigma_{t|t-1}^2$ is the conditional variance of equity returns. Given the initial values of ε_t and σ_t^2, the parameters can be estimated by maximizing the log-likelihood function over the sample period. The normal density in equation (36) yields the following log-likelihood function:

$$\text{LogL}_{\text{normal}} = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\sigma_{t|t-1}^2 - \frac{1}{2}\sum_{t=1}^{n}\left(\frac{r_t - \omega - \alpha r_{t-1}}{\sigma_{t|t-1}}\right)^2. \tag{37}$$

In most empirical studies the normal density is used even though the standardized residuals obtained from ARCH-type models, which assume normality, remain leptokurtic. As shown in Table 2, the excess kurtosis statistic for daily returns on DJIA is extremely high and statistically significant, implying that the tails of the actual distribution are much thicker than the tails of the normal distribution.

In light of the empirical evidence of fat-tailed errors, Bollerslev (1987) and Nelson (1991) use leptokurtic distributions such as the Student-*t* distribution and the generalized error distribution, respectively. In this paper, in addition to the thin-tailed normal distribution, we use the heavy-tailed standardized Student-*t* and GED distributions.

We let the conditional distribution of r_t be standardized *t* with mean $\mu_{t|t-1} = \omega + \alpha\ r_{t-1}$, variance $\sigma^2_{t|t-1}$, and degrees of freedom $v > 2$, i.e.,

$$r_t = \mu_{t|t-1} + \varepsilon_t, \quad \varepsilon_t|\Omega_{t-1} \sim f_v(\varepsilon_t|\Omega_{t-1}) \tag{38}$$

$$f_v(\varepsilon_t \mid \Omega_{t-1}) = \Gamma\left(\frac{v+1}{2}\right)\Gamma\left(\frac{v}{2}\right)^{-1}[(v-2)\sigma^2_{t|t-1}]^{-1/2}\left[1+\frac{\varepsilon_t^2}{(v-2)\sigma^2_{t|t-1}}\right]^{-(v+1)/2} \tag{39}$$

where Ω_{t-1} denotes the sigma-field generated by all the information up through time t-1, and $f_v(\varepsilon_t|\Omega_{t-1})$ the conditional density function for the error term $\varepsilon_t = z_t\sigma_t$. It is well known that for $1/v \to 0$ the *t*-distribution approaches a normal distribution with variance $\sigma^2_{t|t-1}$, but for $1/v > 0$ the *t*-distribution has fatter tails than the corresponding normal distribution. The standardized-*t* density in (39) gives the following log-likelihood function that can be used to estimate the model's parameters:

$$\text{LogL}_{\text{Student-}t} = n\ln\Gamma\left(\frac{v+1}{2}\right) - n\ln\Gamma\left(\frac{v}{2}\right) - \frac{n}{2}\ln[(v-2)\sigma^2_{t|t-1}] - \left(\frac{v+1}{2}\right)\sum_{t=1}^{n}\ln\left[1+\frac{\varepsilon_t^2}{(v-2)\sigma^2_{t|t-1}}\right]. \tag{40}$$

In addition to the heavy-tailed Student-*t* distribution, we employ the GED density to model the conditional variance and the observed leptokurtosis in daily stock return data:

$$f_v(z_t \mid \Omega_{t-1}) = \frac{v\ \exp[(-1/2)\,|\,\mathrm{z_t}/\Pi\,|^{v}]}{\Pi\ 2^{[(v+1)/v]}\ \Gamma(1/v)} \tag{41}$$

where $z_t = \dfrac{r_t - \mu_{t|t-1}}{\sigma_{t|t-1}}$ and $\Pi = \left[\dfrac{2^{(-2/v)}\Gamma(1/v)}{\Gamma(3/v)}\right]^{1/2}$. The simplified version of the log-likelihood function for the GED density is

$$\begin{aligned}\text{LogL}_{\text{GED}} &= \ln(v/2) + 0.5\ln\Gamma(3/v) - 1.5\ln\Gamma(1/v) - 0.5\sum_{t=1}^{n}\ln\sigma^2_{t|t-1} \\ &\quad - \exp\left[(v/2)(\ln\Gamma(3/v) - \ln\Gamma(1/v))\right]\times\sum_{t=1}^{n}\left|\frac{r_t-\mu_{t|t-1}}{\sigma_{t|t-1}}\right|^{v}\end{aligned} \tag{42}$$

The log-likelihood functions implied by the conditional Student-*t* and GED density yield parameter estimates, which are not excessively influenced by extreme observations that occur with low probability (e.g., stock market booms and crashes). In addition, the standard errors of the estimated parameters are robust, allowing for more reliable statistical inference. Another advantage is that one can formally test the empirical validity of standard models that assume normality.

IV. Estimation Results

Table 3 presents the maximum likelihood estimates of the symmetric and asymmetric GARCH models for the normal, Student-*t* and generalized error distributions. For all density functions, the parameters in the conditional mean equation (ω, α) are usually statistically significant at the 1 or 5 percent level, and those in the conditional variance (β_0, β_1, β_2, γ) are significant at the 1 percent level without any exception. The speed of adjustment coefficient, $(1-\alpha)$, is estimated to be between zero and one, implying partial adjustment of transaction prices towards the security's value. The maximum likelihood estimates of α are found to be in the range of 0.07 to 0.10, which implies neither the overshooting or over-reaction of traders to new information nor the extreme case of no reaction to changes in the security's value.

A notable point in Table 3 is that allowing for fat-tailed disturbances consistently improves the ability of capturing the stock price dynamics. In addition, the speed of adjustment coefficient is slightly overestimated when the innovation process is drawn from the normal distribution. The estimation results confirm the presence of rather extreme conditionally heteroskedastic volatility effects in the stock price process. For example, the symmetric GARCH parameters, β_1 and β_2, are found to be highly significant, and the sum $\beta_1 + \beta_2$ is close to one for all density functions considered in the paper. This implies the existence of substantial volatility persistence in stock market returns.

Another notable point in Table 3 is that the asymmetry parameter, γ, in the conditional variance, log-variance, and standard deviation models turns out to be highly significant. The parameter γ allows for an asymmetric volatility response in the diffusion function to past positive and negative information shocks. It is estimated to be negative for all volatility models, implying that negative shocks (or unexpected decrease in stock prices) have a larger impact than positive shocks (or unexpected rise in stock prices) of the same size on conditional volatility. This is also consistent with the leverage effect that changes in stock prices are negatively related with changes in volatility. The presence of asymmetry in return volatility is formally determined on the basis of the likelihood ratio (LR) test by comparing the maximized log-likelihood values of the asymmetric and symmetric GARCH models.[11] Although not

[11] The likelihood ratio test (LR) statistic is calculated as LR = -2[log-L* - Log-L], where Log-L* is the value of the log-likelihood under the null hypothesis ($\gamma = 0$), and Log-L is the log-likelihood under the alternative. The statistic is distributed as the Chi-square, χ^2, with one degree of freedom.

presented in the paper, the estimated LR statistics are well above the critical value, implying that negative returns are followed by greater increases in volatility than equally large positive returns.

Normality requires that the estimated degrees of freedom parameter of the Student-*t* density be close to infinity, $\nu \to \infty$, or $1/\nu$ be equal to zero. As presented in Panel B of Table 3, the estimated ν is found to be in the range of 5.44 to 6.38. These estimates are highly significant for all GARCH processes considered in the paper. Bollerslev (1987) discusses that when testing against the null hypothesis of conditionally normal errors, i.e., $1/\nu = 0$, $1/\nu$ is on the boundary of the admissible parameter space, and the usual test statistics will likely be more concentrated towards the origin than a χ_1^2 distribution. Bollerslev finds that for moderately sized samples the correct 5 % critical value for the LR test statistic for null hypothesis $1/\nu = 0$ is approximately 2.70. To save space we choose not to present the LR statistics, which are substantially greater than the critical value for all volatility specifications, indicating that the distribution of daily returns is much more leptokurtic than the corresponding normal distribution.

As discussed earlier, for the tail thickness parameter $\nu = 2$, the GED density equals the standard normal density. However, the estimates of ν turn out to be highly significant and less than two for each model specification. As shown in Panel C of Table 3, the degrees of freedom parameter, ν, is estimated to be in the range of 1.26 to 1.35. Although not presented in the paper, comparing the maximized log-likelihood functions of the models estimated with the GED and normal distributions indicates that ν is statistically different from two, implying strong rejection of the models that utilize normally distributed innovation process. The results also suggest that the double exponential or Laplace with $\nu = 1$ is a more appropriate density function. An intuitive understanding can be gained by realizing that for the normal distribution ($\nu = 2$) the degree of kurtosis is equal to 3, while for the double exponential or Laplace distribution ($\nu = 1$) the degree of kurtosis is equal to 6.

V. Testing Mean Reversion in Stock Return Volatility

Before testing and measuring the degree of mean reversion in stock return volatility within discrete time GARCH models, we run a simple experiment. Our objective is to determine whether a simple measure of the variance and standard deviation is pulled back to some long-run average level over time. A 5-day moving average variance of equity returns is taken as a simple measure of the variance, and similarly the 5-day moving average standard deviation of stock returns is considered as a measure of the volatility. First, we generate the time-varying volatility of equity returns for DJIA. Then the last period's variance, σ_{t-1}^2, is plotted against the change in variance, $\sigma_t^2 - \sigma_{t-1}^2$, to test mean reversion in stock return volatility. When the level of the variance is high, mean reversion tends to cause it to have a negative drift, and when it is low, mean reversion tends to cause it to have a positive drift.

In other words, mean reversion in moving average volatility implies a negative slope in the conditional mean of the volatility process because in this simple framework the change in variance is a function of the last period's variance as well as its lagged values. When $\sigma_t^2 - \sigma_{t-1}^2 = f(\sigma_{t-1}^2)$ with the first derivative of this function being negative, $df / d\sigma_{t-1}^2 < 0$, stock return volatility follows a mean reverting process. To measure the degree of this relationship, we calculate the correlation coefficient between σ_{t-1}^2 and $\sigma_t^2 - \sigma_{t-1}^2$, and it is found to be –0.2471, and statistically significant at the 1% level. However, the degree of mean reversion in standard deviation turns out to be slightly lower than that in variance. More specifically, the correlation coefficient between σ_{t-1} and $\sigma_t - \sigma_{t-1}$ is found to be –0.2162 and significant at the 1% level. Overall, this simple experiment indicates mean reversion in return volatility.[12]

The first-order Euler approximation of the stochastic volatility models in Appendices A-C implies that the change in variance is a function of the last period's variance. For example, the continuous time limit of the GARCH model in equation (A.3) in Appendix A is approximated as

$$\sigma_t^2 - \sigma_{t-\Delta}^2 = (\rho_0 + \rho_1 \sigma_{t-\Delta}^2)\,\Delta + \lambda\,\sigma_{t-\Delta}^2\,\Delta W_{2,t} \tag{43}$$

where Δ is the length of the time interval, $\Delta W_{2,t} = z_{2,t}\sqrt{\Delta}$ can be viewed as a normally distributed random variable with zero mean and variance of Δ since $z_{2,t} \sim N(0,1)$, $\rho_0 + \rho_1 \sigma_{t-\Delta}^2$ is the drift (or the conditional mean) of the volatility process, and $\lambda\sigma_{t-\Delta}^2$ is the conditional standard deviation of the volatility process. As shown in Table 1, the speed of mean reversion parameters (ρ_1, κ_1, θ_1) include the GARCH parameters (β_1, β_2, γ), the length of time interval Δ, and the estimated degrees of freedom parameter (v) for the Student-*t* and GED distributions.

As indicated by equation (43), mean reversion in variance (or standard deviation) of equity returns can formally be tested for the discrete time GARCH models. Stock return volatility follows a

[12] The method used to obtain a simple measure of the variance and standard deviation describes a simple structure on how volatility evolves through time and places constant weights on past observations. However, if stock return volatility clusters, then more recent rates should be given more weight. Equity returns in recent periods provide more information about the current volatility than the ones in the distant past. To take this into account, at an earlier stage of the study, we also utilized an exponentially weighted moving average (EWMA) volatility that places more weight on more recent observations. This weighting is done by using a smoothing constant (or a damping parameter) which is chosen to be 0.90 in this study. We should note that the variance and standard deviation estimates of the EWMA model turn out to be very similar to those obtained from the constantly weighted moving average volatility. To save space we do not to show the EWMA model estimates. They are available upon request.

mean-reverting process if the volatility drift parameter ρ_1 in equations (A.1)-(A.10) is negative, so a test for mean reversion is a test of whether $\rho_1 = 0$ against the alternative that $\rho_1 < 0$. Rewriting the drift function, $\rho_0 + \rho_1 \sigma_t^2$, as $\rho_1(\sigma_t^2 - \varphi)$ reveals that ρ_1 can be viewed as a measure of the speed of mean reversion in stock return volatility. The more negative ρ_1 is, the faster the return volatility σ_t^2 responds to deviations from its long-run average level, $\varphi = -\rho_0/\rho_1$.

Table 4 displays the estimated mean reversion rates based on the relationship between discrete time GARCH and continuous time diffusion models. All of the GARCH processes with the thin-tailed normal, the fat-tailed Student-*t* and GED density indicate a mean-reverting behavior of stochastic volatility. A notable point in Table 4 is that the speed of mean reversion in the conditional variance, log-variance, and standard deviation is slightly overestimated by the normal density compared to the estimates from the Student-*t* and GED distributions.[13] For the aforementioned distribution functions, the degree of mean reversion in the conditional *variance* is found to be slightly greater than that in the *log-variance* and *standard deviation*.[14] Specifically, the speed of mean reversion in the variance of stock returns is estimated to be in the range of –0.0124 to –0.0796 for the normal, –0.0082 to –0.0624 for the Student-*t*, and –0.0105 to –0.0711 for the GED density. The speed of mean reversion in the conditional log-variance is estimated to be –0.0221 for the normal, –0.0186 for the Student-*t*, and –0.0199 for the GED density. The mean reversion rates for the standard deviation are in the range of –0.0070 and –0.0093 for the normal, –0.0021 to –0.0022 for the Student-*t*, and –0.0067 to –0.0086 for the GED density.

Among the conditional variance models, the SQR-GARCH model of Heston and Nandi (1998) and the VGARCH model of Engle (1990) yield the highest mean reversion rates in stock return volatility. The conditional standard deviation models, the TGARCH model of Zakoian (1994) and the TS-GARCH model of Taylor (1986) and Schwert (1989), indicate similar mean-reverting behavior of the volatility process. The mean reversion rates based on the parameter estimates of the TGARCH, TS-GARCH, and EGARCH models indicate that the conditional log-variance of stock returns is pulled back to a long-run average level faster than the conditional standard deviation.

The convergence results presented in Table 4 are obtained from the parameter restrictions that form the relation between discrete time GARCH and continuous time diffusion models. In other words,

[13] We should note that the AGARCH, GARCH, GJR-GARCH, NGARCH, QGARCH, SQR-GARCH, and VGARCH models are used to test mean reversion in the conditional variance of stock returns. Mean reversion in the conditional log-variance is tested based on the estimates of the EGARCH model. Mean reversion in the conditional standard deviation is tested based on the TGARCH and TS-GARCH models' estimates.

[14] In other words, the conditional variance of stock returns responds faster than the standard deviation and log-variance to deviations from the long-run average level.

the Nelson's (1990) approximation scheme is applied to develop conditions under which discrete time GARCH models converge weakly to Ito processes as the length of time interval goes to zero. However, the existing literature does not provide clear evidence on the magnitude of the approximation error especially for the heavy-tailed innovation process, and it is beyond the scope of this paper. Therefore, we decide to find the degree of mean reversion in stock return volatility in an alternative framework, and estimate a series of OLS regressions of the form:

$$\sigma_t^2 - \sigma_{t-1}^2 = \phi_0 + \phi_1 \sigma_{t-1}^2 + u_t \tag{44}$$

$$\mathrm{E}(u_t|\Omega_{t-1}) = 0,\ \mathrm{E}(u_t^2|\Omega_{t-1}) \equiv \Psi^2 \tag{45}$$

where σ_t^2 is the conditional variance of stock returns estimated by the moving average, symmetric and asymmetric GARCH models, and u_t is the error term, assumed to be homoscedastic and serially uncorrelated. Mean reversion in the conditional log-variance and standard deviation is tested using similar OLS regressions: $\ln\sigma_t^2 - \ln\sigma_{t-1}^2 = \phi_0 + \phi_1 \ln\sigma_{t-1}^2 + u_t$ and $\sigma_t - \sigma_{t-1} = \phi_0 + \phi_1\sigma_{t-1} + u_t$. Stock return volatility follows a mean-reverting process if the volatility drift parameter ϕ_1 in eq. (44) is negative, so a test for mean reversion in volatility is a test of whether $\phi_1 = 0$ against the alternative that $\phi_1 < 0$.

To take into account time-variation and serial correlation (or persistence) in the volatility of conditional variance or standard deviation, we also consider the following econometric specifications when testing mean reversion in volatility:

$$\sigma_t^2 - \sigma_{t-1}^2 = \phi_0 + \phi_1 \sigma_{t-1}^2 + u_t \tag{46}$$

$$\mathrm{E}(u_t|\Omega_{t-1}) = 0,\ \mathrm{E}(u_t^2|\Omega_{t-1}) \equiv \Psi_t^2 = \zeta_0 + \zeta_1 u_{t-1}^2 + \zeta_2 \Psi_{t-1}^2 \tag{47}$$

$$\mathrm{E}(u_t^2|\Omega_{t-1}) \equiv \Psi_t^2 = \zeta\sigma_{t-1}^2 \tag{48}$$

where the current conditional variance of stock return volatility (denoted by Ψ_t^2) is parameterized in eq. (47) as a function of the last period's volatility shocks, u_{t-1}^2, and the last period's variance Ψ_{t-1}^2. Equation (48) defines Ψ_t^2 as a function of the conditional variance of stock returns σ_{t-1}^2. The parameters in equations (46)-(48) are estimated using the maximum likelihood (ML) methodology. Here, our main objective is to test whether $\phi_1 = 0$ against the alternative that $\phi_1 < 0$. The ML estimates indicate that the magnitude and statistical significance of ϕ_1 is not affected much by modeling the variance of return volatility, Ψ_t^2, as a function of past volatility shocks $(u_{t-1}^2, \Psi_{t-1}^2)$ or the last period's return volatility,

σ_{t-1}^2.[15] In fact, the volatility of volatility is found to be very small relative to the conditional mean of the variance (or standard deviation) process.

Table 5 presents the estimated mean reversion rates (ϕ_1) for the normal, Student-*t*, and GED distributions. The degree of mean reversion in stock return volatility is measured by the negative values of ϕ_1. Although not presented in Table 5, the moving average volatility estimates indicate that both the variance and standard deviation of stock returns are pulled back to their unconditional means over time, but the variance responds faster than the standard deviation to deviations from the long-run average level. Specifically, ϕ_1 is estimated to be –0.1221 (–43.289) for the 5-day moving average variance and –0.0935 (–37.594) for the 5-day moving average standard deviation. Based on the *t*-statistics given in parentheses, the mean reversion rates are statistically significant at the 1% level.

Table 5 provides supporting evidence that all of the GARCH processes with the normal, Student-*t*, and GED distributions indicate a mean-reverting behavior of stochastic volatility. Similar to our findings from the Nelson's (1990) approximation scheme, the speed of mean reversion in the conditional variance, log-variance, and standard deviation is slightly overestimated by the normal density compared to the estimates from the Student-*t* and GED distributions. For the density functions considered here, the degree of mean reversion in the conditional variance turns out to be slightly greater than that in the log-variance and standard deviation. Specifically, the speed of mean reversion in the variance of stock returns is estimated to be in the range of –0.0442 to –0.0586 for the normal, –0.0345 to –0.0497 for the Student-*t*, and –0.0387 to –0.0533 for the GED density. The speed of mean reversion in the conditional log-variance is estimated to be –0.0294 for the normal, –0.0203 for the Student-*t*, and –0.0228 for the GED density. The mean reversion rates for the standard deviation are in the range of –0.0215 and –0.0264 for the normal, –0.0105 to –0.0148 for the Student-*t*, and –0.0196 to –0.0201 for the GED density. It is important to note that the mean reversion rates presented in Table 5 are statistically significant at the 1% level without any exception. The results also imply that the moving average volatilities are pulled back to a long-run average level faster than the GARCH volatilities.

VI. Conclusion

This paper presents a comprehensive study of continuous time GARCH modeling with the thin-tailed normal and the fat-tailed Student-*t* and generalized error distributions. The paper represents the first study of measuring the degree of mean reversion in stock market volatility based on the relationship

[15] To preserve space, Table 5 only presents the mean reversion rates based on equations (44)-(45). The results obtained from equations (46)-(48) are very similar and available upon request.

between discrete time GARCH and continuous time diffusion models. The convergence results based on the aforementioned distribution functions are shown to have similar implications for testing mean reversion in stochastic volatility. Alternative models are compared in terms of their ability to capture mean-reverting behavior of stock return volatility. The empirical findings indicate that the conditional variance, log-variance, and standard deviation of stock market returns are pulled back to some long-run average level over time. The results are found to be robust across different periods, data sets, and distributional assumptions for the innovation process.

Appendix A
Continuous Time Limits of the GARCH models with Normal Distribution

	Discrete Time GARCH Models	*Continuous Time Limits*	
AGARCH:	$\sigma_t^2 = \beta_0 + \beta_1(\gamma + \sigma_{t-\Delta} z_{t-\Delta})^2 + \beta_2 \sigma_{t-\Delta}^2$	$d\sigma_t^2 = (\rho_0 + \rho_1 \sigma_t^2)\, dt + \lambda \sigma_t \sqrt{\sigma_t^2 + 2\gamma^2}\, dW_{2,t}$	(A.1)
EGARCH:	$\ln \sigma_t^2 = \beta_0 + \beta_1[\mid z_{t-\Delta} \mid - E(\mid z_{t-\Delta} \mid)] + \beta_2 \ln \sigma_{t-\Delta}^2 + \gamma z_{t-\Delta}$	$d \ln \sigma_t^2 = (\rho_0 + \rho_1 \ln \sigma_t^2)\, dt + \sqrt{\lambda^2 + \gamma^2}\, dW_{2,t}$	(A.2)
GARCH:	$\sigma_t^2 = \beta_0 + \beta_1 \sigma_{t-\Delta}^2 z_{t-\Delta}^2 + \beta_2 \sigma_{t-\Delta}^2$	$d\sigma_t^2 = (\rho_0 + \rho_1 \sigma_t^2)\, dt + \lambda \sigma_t^2\, dW_{2,t}$	(A.3)
GJR-GARCH:	$\sigma_t^2 = \beta_0 + \beta_1 \sigma_{t-\Delta}^2 z_{t-\Delta}^2 + \beta_2 \sigma_{t-\Delta}^2 + \gamma S_{t-\Delta}^- \sigma_{t-\Delta}^2 z_{t-\Delta}^2$	$d\sigma_t^2 = (\rho_0 + \rho_1 \sigma_t^2)\, dt + \sqrt{\lambda^2 + 0.75\gamma^2}\, \sigma_t^2\, dW_{2,t}$	(A.4)
NGARCH:	$\sigma_t^2 = \beta_0 + \beta_1 \sigma_{t-\Delta}^2 (\gamma + z_{t-\Delta})^2 + \beta_2 \sigma_{t-\Delta}^2$	$d\sigma_t^2 = (\rho_0 + \rho_1 \sigma_t^2)\, dt + \lambda \sqrt{1 + 2\gamma^2}\, \sigma_t^2\, dW_{2,t}$	(A.5)
QGARCH:	$\sigma_t^2 = \beta_0 + \beta_1 \sigma_{t-\Delta}^2 z_{t-\Delta}^2 + \beta_2 \sigma_{t-\Delta}^2 + \gamma \sigma_{t-\Delta} z_{t-\Delta}$	$d\sigma_t^2 = (\rho_0 + \rho_1 \sigma_t^2)\, dt + \sqrt{\lambda^2 \sigma_t^2 + \gamma^2}\, \sigma_t\, dW_{2,t}$	(A.6)
SQR-GARCH:	$\sigma_t^2 = \beta_0 + \beta_1(\gamma \sigma_{t-\Delta} + z_{t-\Delta})^2 + \beta_2 \sigma_{t-\Delta}^2$	$d\sigma_t^2 = (\rho_0 + \rho_1 \sigma_t^2)\, dt + \lambda \sqrt{1 + 2\gamma^2 \sigma_t^2}\, dW_{2,t}$	(A.7)
TGARCH:	$\sigma_t = \beta_0 + \beta_1 \sigma_{t-\Delta} \mid z_{t-\Delta} \mid + \beta_2 \sigma_{t-\Delta} + \gamma S_{t-\Delta}^- \sigma_{t-\Delta} z_{t-\Delta}$	$d\sigma_t = (\rho_0 + \rho_1 \sigma_t)\, dt + \sqrt{\lambda^2 + 0.5\gamma^2}\, \sigma_t\, dW_{2,t}$	(A.8)
TS-GARCH:	$\sigma_t = \beta_0 + \beta_1 \sigma_{t-\Delta} \mid z_{t-\Delta} \mid + \beta_2 \sigma_{t-\Delta}$	$d\sigma_t = (\rho_0 + \rho_1 \sigma_t)\, dt + \lambda \sigma_t\, dW_{2,t}$	(A.9)
VGARCH:	$\sigma_t^2 = \beta_0 + \beta_1(\gamma + z_{t-\Delta})^2 + \beta_2 \sigma_{t-\Delta}^2$	$d\sigma_t^2 = (\rho_0 + \rho_1 \sigma_t^2)\, dt + \lambda \sqrt{1 + 2\gamma^2}\, dW_{2,t}$	(A.10)

Parameter Restrictions for the GARCH Models with Normal Distribution

AGARCH: $\lim_{\Delta\to 0} \Delta^{-1}(\beta_{0\Delta} + \beta_{1\Delta}\gamma^2) = \rho_0, \quad \lim_{\Delta\to 0} \Delta^{-1}[\beta_{1\Delta} + \beta_{2\Delta} - 1] = \rho_1, \quad \lim_{\Delta\to 0} \Delta^{-1}\beta_{1\Delta}^2 = \lambda^2$

EGARCH: $\lim_{\Delta\to 0} \Delta^{-1}\beta_{0\Delta} = \rho_0, \quad \lim_{\Delta\to 0} \Delta^{-1}(\beta_{2\Delta} - 1) = \rho_1, \quad \lim_{\Delta\to 0} \Delta^{-1}\beta_{1\Delta}^2\left(1 - \frac{2}{\pi}\right) = \lambda^2$

GARCH: $\lim_{\Delta\to 0} \Delta^{-1}\beta_{0\Delta} = \rho_0, \quad \lim_{\Delta\to 0} \Delta^{-1}[\beta_{1\Delta} + \beta_{2\Delta} - 1] = \rho_1, \quad \lim_{\Delta\to 0} \Delta^{-1}\, 2\beta_{1\Delta}^2 = \lambda^2$

GJR-GARCH: $\lim_{\Delta\to 0} \Delta^{-1}\beta_{0\Delta} = \rho_0, \quad \lim_{\Delta\to 0} \Delta^{-1}[\beta_{1\Delta} + \beta_{2\Delta} + 0.5\gamma - 1] = \rho_1, \quad \lim_{\Delta\to 0} \Delta^{-1}\beta_{1\Delta}^2 = \lambda^2$

NGARCH: $\lim_{\Delta\to 0} \Delta^{-1}\beta_{0\Delta} = \rho_0, \quad \lim_{\Delta\to 0} \Delta^{-1}[\beta_{1\Delta}(1+\gamma^2) + \beta_{2\Delta} - 1] = \rho_1, \quad \lim_{\Delta\to 0} \Delta^{-1}\beta_{1\Delta}^2 = \lambda^2$

QGARCH: $\lim_{\Delta\to 0} \Delta^{-1}\beta_{0\Delta} = \rho_0, \quad \lim_{\Delta\to 0} \Delta^{-1}[\beta_{1\Delta} + \beta_{2\Delta} - 1] = \rho_1, \quad \lim_{\Delta\to 0} \Delta^{-1}\beta_{1\Delta}^2 = \lambda^2$

SQR-GARCH: $\lim_{\Delta\to 0} \Delta^{-1}(\beta_{0\Delta} + \beta_{1\Delta}) = \rho_0, \quad \lim_{\Delta\to 0} \Delta^{-1}[\beta_{1\Delta}\gamma^2 + \beta_{2\Delta} - 1] = \rho_1, \quad \lim_{\Delta\to 0} \Delta^{-1}\beta_{1\Delta}^2 = \lambda^2$

TGARCH: $\lim_{\Delta\to 0} \Delta^{-1}\beta_{0\Delta} = \rho_0, \quad \lim_{\Delta\to 0} \Delta^{-1}\left[\sqrt{\frac{2}{\pi}}\beta_{1\Delta} + \beta_{2\Delta} - 1\right] = \rho_1, \quad \lim_{\Delta\to 0} \Delta^{-1}\beta_{1\Delta}^2\left(1 - \frac{2}{\pi}\right) = \lambda^2$

TS-GARCH: $\lim_{\Delta\to 0} \Delta^{-1}\beta_{0\Delta} = \rho_0, \quad \lim_{\Delta\to 0} \Delta^{-1}\left[\sqrt{\frac{2}{\pi}}\beta_{1\Delta} + \beta_{2\Delta} - 1\right] = \rho_1, \quad \lim_{\Delta\to 0} \Delta^{-1}\beta_{1\Delta}^2\left(1 - \frac{2}{\pi}\right) = \lambda^2$

VGARCH: $\lim_{\Delta\to 0} \Delta^{-1}[\beta_{0\Delta} + \beta_{1\Delta}(1+\gamma^2)] = \rho_0, \quad \lim_{\Delta\to 0} \Delta^{-1}(\beta_{2\Delta} - 1) = \rho_1, \quad \lim_{\Delta\to 0} \Delta^{-1}\beta_{1\Delta}^2 = \lambda^2$

Appendix B
Continuous Time Limits of the GARCH models with Student-*t* Distribution

	Discrete Time GARCH Models	*Continuous Time Limits*	
AGARCH:	$\sigma_t^2 = \beta_0 + \beta_1(\gamma + \sigma_{t-\Delta} z_{t-\Delta})^2 + \beta_2\sigma_{t-\Delta}^2$	$d\sigma_t^2 = (\kappa_0 + \kappa_1\sigma_t^2)\,dt + \eta\sigma_t\sqrt{4\Phi_v\gamma^2 + \Psi_v\sigma_t^2}\,dW_{2,t}$	(B.1)
EGARCH:	$\ln\sigma_t^2 = \beta_0 + \beta_1[\lvert z_{t-\Delta}\rvert - E(\lvert z_{t-\Delta}\rvert)] + \beta_2\ln\sigma_{t-\Delta}^2 + \gamma z_{t-\Delta}$	$d\ln\sigma_t^2 = (\kappa_0 + \kappa_1\ln\sigma_t^2)\,dt + \sqrt{\Psi_{\lvert v\rvert}\eta^2 + \Phi_v\gamma^2}\,dW_{2,t}$	(B.2)
GARCH:	$\sigma_t^2 = \beta_0 + \beta_1\sigma_{t-\Delta}^2 z_{t-\Delta}^2 + \beta_2\sigma_{t-\Delta}^2$	$d\sigma_t^2 = (\kappa_0 + \kappa_1\sigma_t^2)\,dt + \eta\sigma_t^2\,dW_{2,t}$	(B.3)
GJR-GARCH:	$\sigma_t^2 = \beta_0 + \beta_1\sigma_{t-\Delta}^2 z_{t-\Delta}^2 + \beta_2\sigma_{t-\Delta}^2 + \gamma S_{t-\Delta}^{-}\sigma_{t-\Delta}^2 z_{t-\Delta}^2$	$d\sigma_t^2 = (\kappa_0 + \kappa_1\sigma_t^2)\,dt + \sqrt{\Psi_v(\eta+\gamma)\eta + \phi_v\gamma^2}\;\sigma_t^2\,dW_{2,t}$	(B.4)
NGARCH:	$\sigma_t^2 = \beta_0 + \beta_1\sigma_{t-\Delta}^2(\gamma + z_{t-\Delta})^2 + \beta_2\sigma_{t-\Delta}^2$	$d\sigma_t^2 = (\kappa_0 + \kappa_1\sigma_t^2)\,dt + \eta\sigma_t^2\sqrt{4\Phi_v\gamma^2 + \Psi_v}\,dW_{2,t}$	(B.5)
QGARCH:	$\sigma_t^2 = \beta_0 + \beta_1\sigma_{t-\Delta}^2 z_{t-\Delta}^2 + \beta_2\sigma_{t-\Delta}^2 + \gamma\,\sigma_{t-\Delta} z_{t-\Delta}$	$d\sigma_t^2 = (\kappa_0 + \kappa_1\sigma_t^2)\,dt + \sigma_t\sqrt{\Phi_v\gamma^2 + \Psi_v\eta^2\sigma_t^2}\,dW_{2,t}$	(B.6)
SQR-GARCH:	$\sigma_t^2 = \beta_0 + \beta_1(\gamma\,\sigma_{t-\Delta} + z_{t-\Delta})^2 + \beta_2\sigma_{t-\Delta}^2$	$d\sigma_t^2 = (\kappa_0 + \kappa_1\sigma_t^2)\,dt + \eta\sqrt{\Psi_v + 4\Phi_v\gamma^2\sigma_t^2}\,dW_{2,t}$	(B.7)
TGARCH:	$\sigma_t = \beta_0 + \beta_1\sigma_{t-\Delta}\lvert z_{t-\Delta}\rvert + \beta_2\sigma_{t-\Delta} + \gamma S_{t-\Delta}^{-}\sigma_{t-\Delta} z_{t-\Delta}$	$d\sigma_t = (\kappa_0 + \kappa_1\sigma_t)\,dt + \sqrt{\eta^2\Psi_{\lvert v\rvert} + 0.5\Phi_v\gamma^2}\;\sigma_t\,dW_{2,t}$	(B.8)
TS-GARCH:	$\sigma_t = \beta_0 + \beta_1\sigma_{t-\Delta}\lvert z_{t-\Delta}\rvert + \beta_2\sigma_{t-\Delta}$	$d\sigma_t = (\kappa_0 + \kappa_1\sigma_t)\,dt + \eta\sigma_t\,dW_{2,t}$	(B.9)
VGARCH:	$\sigma_t^2 = \beta_0 + \beta_1(\gamma + z_{t-\Delta})^2 + \beta_2\sigma_{t-\Delta}^2$	$d\sigma_t^2 = (\kappa_0 + \kappa_1\sigma_t^2)\,dt + \eta\sqrt{\Psi_v + 4\Phi_v\gamma^2}\,dW_{2,t}$	(B.10)

Φ_v, Ψ_v, $\Phi_{|v|}$ and $\Psi_{|v|}$ are the first moment and variance of z_t^2 and $|z_t|$, respectively: $\Phi_v = E(z_t^2)$, $\Psi_v = Var(z_t^2)$, $\Phi_{|v|} = E(|z_t|)$, and $\Psi_{|v|} = Var(|z_t|)$:

$$\Phi_v = \frac{v}{v-2},\quad \Psi_v = \frac{2v^2(v-1)}{(v-2)^2(v-4)},\quad \Phi_{|v|} = 2\Gamma\left(\frac{v+1}{2}\right)\Gamma\left(\frac{v}{2}\right)^{-1}\sqrt{\frac{v}{(v-1)^2\pi}},\quad \Psi_{|v|} = \frac{v}{v-2} - 4\left[\Gamma\left(\frac{v+1}{2}\right)\right]^2\left[\Gamma\left(\frac{v}{2}\right)\right]^{-2}\left(\frac{v}{(v-1)^2\pi}\right),\quad \phi_v = \frac{5v^2+2v+8}{4(v-2)(v-4)}$$

Parameter Restrictions for the GARCH Models with Student-*t* Distribution

AGARCH: $\lim_{\Delta\to 0} \Delta^{-1}(\beta_{0\Delta} + \beta_{1\Delta}\gamma^2) = \kappa_0, \quad \lim_{\Delta\to 0} \Delta^{-1}[\Phi_v\beta_{1\Delta} + \beta_{2\Delta} - 1] = \kappa_1, \quad \lim_{\Delta\to 0} \Delta^{-1}\beta_{1\Delta}^2 = \eta^2$

EGARCH: $\lim_{\Delta\to 0} \Delta^{-1}\beta_{0\Delta} = \kappa_0, \quad \lim_{\Delta\to 0} \Delta^{-1}(\beta_{2\Delta} - 1) = \kappa_1, \quad \lim_{\Delta\to 0} \Delta^{-1}\beta_{1\Delta}^2 = \eta^2$

GARCH: $\lim_{\Delta\to 0} \Delta^{-1}\beta_{0\Delta} = \kappa_0, \quad \lim_{\Delta\to 0} \Delta^{-1}[\Phi_v\beta_{1\Delta} + \beta_{2\Delta} - 1] = \kappa_1, \quad \lim_{\Delta\to 0} \Delta^{-1}\Psi_v\beta_{1\Delta}^2 = \eta^2$

GJR-GARCH: $\lim_{\Delta\to 0} \Delta^{-1}\beta_{0\Delta} = \kappa_0, \quad \lim_{\Delta\to 0} \Delta^{-1}[\Phi_v(\beta_{1\Delta} + 0.5\gamma) + \beta_{2\Delta} - 1] = \kappa_1, \quad \lim_{\Delta\to 0} \Delta^{-1}\beta_{1\Delta}^2 = \eta^2$

NGARCH: $\lim_{\Delta\to 0} \Delta^{-1}\beta_{0\Delta} = \kappa_0, \quad \lim_{\Delta\to 0} \Delta^{-1}[\beta_{1\Delta}(\Phi_v + \gamma^2) + \beta_{2\Delta} - 1] = \kappa_1, \quad \lim_{\Delta\to 0} \Delta^{-1}\beta_{1\Delta}^2 = \eta^2$

QGARCH: $\lim_{\Delta\to 0} \Delta^{-1}\beta_{0\Delta} = \kappa_0, \quad \lim_{\Delta\to 0} \Delta^{-1}[\Phi_v\beta_{1\Delta} + \beta_{2\Delta} - 1] = \kappa_1, \quad \lim_{\Delta\to 0} \Delta^{-1}\beta_{1\Delta}^2 = \eta^2$

SQR-GARCH: $\lim_{\Delta\to 0} \Delta^{-1}(\beta_{0\Delta} + \beta_{1\Delta}\Phi_v) = \kappa_0, \quad \lim_{\Delta\to 0} \Delta^{-1}[\beta_{1\Delta}\gamma^2 + \beta_{2\Delta} - 1] = \kappa_1, \quad \lim_{\Delta\to 0} \Delta^{-1}\beta_{1\Delta}^2 = \eta^2$

TGARCH: $\lim_{\Delta\to 0} \Delta^{-1}\beta_{0\Delta} = \kappa_0, \quad \lim_{\Delta\to 0} \Delta^{-1}[\Phi_{|v|}\beta_{1\Delta} + \beta_{2\Delta} - 1] = \kappa_1, \quad \lim_{\Delta\to 0} \Delta^{-1}\beta_{1\Delta}^2 = \eta^2$

TS-GARCH: $\lim_{\Delta\to 0} \Delta^{-1}\beta_{0\Delta} = \kappa_0, \quad \lim_{\Delta\to 0} \Delta^{-1}[\Phi_{|v|}\beta_{1\Delta} + \beta_{2\Delta} - 1] = \kappa_1, \quad \lim_{\Delta\to 0} \Delta^{-1}\Psi_{|v|}\beta_{1\Delta}^2 = \eta^2$

VGARCH: $\lim_{\Delta\to 0} \Delta^{-1}[\beta_{0\Delta} + \beta_{1\Delta}(\Phi_v + \gamma^2)] = \kappa_0, \quad \lim_{\Delta\to 0} \Delta^{-1}(\beta_{2\Delta} - 1) = \kappa_1, \quad \lim_{\Delta\to 0} \Delta^{-1}\beta_{1\Delta}^2 = \eta^2$

Appendix C
Continuous Time Limits of the GARCH models with Generalized Error Distribution

	Discrete Time GARCH Models	*Continuous Time Limits*	
AGARCH:	$\sigma_t^2 = \beta_0 + \beta_1(\gamma + \sigma_{t-\Delta} z_{t-\Delta})^2 + \beta_2 \sigma_{t-\Delta}^2$	$d\sigma_t^2 = (\theta_0 + \theta_1 \sigma_t^2)\,dt + \vartheta \sigma_t \sqrt{4\gamma^2 + \Psi_v \sigma_t^2}\; dW_{2,t}$	(C.1)
EGARCH:	$\ln \sigma_t^2 = \beta_0 + \beta_1[\lvert z_{t-\Delta}\rvert - E(\lvert z_{t-\Delta}\rvert)] + \beta_2 \ln \sigma_{t-\Delta}^2 + \gamma z_{t-\Delta}$	$d\ln \sigma_t^2 = (\theta_0 + \theta_1 \ln \sigma_t^2)\,dt + \sqrt{\Psi_{\lvert v\rvert} \vartheta^2 + \gamma^2}\; dW_{2,t}$	(C.2)
GARCH:	$\sigma_t^2 = \beta_0 + \beta_1 \sigma_{t-\Delta}^2 z_{t-\Delta}^2 + \beta_2 \sigma_{t-\Delta}^2$	$d\sigma_t^2 = (\theta_0 + \theta_1 \sigma_t^2)\,dt + \vartheta \sigma_t^2\, dW_{2,t}$	(C.3)
GJR-GARCH:	$\sigma_t^2 = \beta_0 + \beta_1 \sigma_{t-\Delta}^2 z_{t-\Delta}^2 + \beta_2 \sigma_{t-\Delta}^2 + \gamma S_{t-\Delta}^- \sigma_{t-\Delta}^2 z_{t-\Delta}^2$	$d\sigma_t^2 = (\theta_0 + \theta_1 \sigma_t^2)dt + \sqrt{\Psi_v(\vartheta^2 + 0.5\gamma^2 + \vartheta\gamma) + 0.25\gamma^2}\,\sigma_t^2\, dW_{2,t}$	(C.4)
NGARCH:	$\sigma_t^2 = \beta_0 + \beta_1 \sigma_{t-\Delta}^2 (\gamma + z_{t-\Delta})^2 + \beta_2 \sigma_{t-\Delta}^2$	$d\sigma_t^2 = (\theta_0 + \theta_1 \sigma_t^2)\,dt + \vartheta \sqrt{\Psi_v + 4\gamma^2}\; \sigma_t^2\, dW_{2,t}$	(C.5)
QGARCH:	$\sigma_t^2 = \beta_0 + \beta_1 \sigma_{t-\Delta}^2 z_{t-\Delta}^2 + \beta_2 \sigma_{t-\Delta}^2 + \gamma\, \sigma_{t-\Delta} z_{t-\Delta}$	$d\sigma_t^2 = (\theta_0 + \theta_1 \sigma_t^2)\,dt + \sqrt{\Psi_v \vartheta^2 \sigma_t^2 + \gamma^2}\; \sigma_t\, dW_{2,t}$	(C.6)
SQR-GARCH:	$\sigma_t^2 = \beta_0 + \beta_1(\gamma\, \sigma_{t-\Delta} + z_{t-\Delta})^2 + \beta_2 \sigma_{t-\Delta}^2$	$d\sigma_t^2 = (\theta_0 + \theta_1 \sigma_t^2)\,dt + \vartheta \sqrt{\Psi_v + 4\gamma^2 \sigma_t^2}\; dW_{2,t}$	(C.7)
TGARCH:	$\sigma_t = \beta_0 + \beta_1 \sigma_{t-\Delta} \lvert z_{t-\Delta}\rvert + \beta_2 \sigma_{t-\Delta} + \gamma S_{t-\Delta}^- \sigma_{t-\Delta} z_{t-\Delta}$	$d\sigma_t = (\theta_0 + \theta_1 \sigma_t)\,dt + \sqrt{\Psi_{\lvert v\rvert} \vartheta^2 + 0.5\gamma^2}\; \sigma_t\, dW_{2,t}$	(C.8)
TS-GARCH:	$\sigma_t = \beta_0 + \beta_1 \sigma_{t-\Delta} \lvert z_{t-\Delta}\rvert + \beta_2 \sigma_{t-\Delta}$	$d\sigma_t = (\theta_0 + \theta_1 \sigma_t)\,dt + \vartheta \sigma_t\, dW_{2,t}$	(C.9)
VGARCH:	$\sigma_t^2 = \beta_0 + \beta_1(\gamma + z_{t-\Delta})^2 + \beta_2 \sigma_{t-\Delta}^2$	$d\sigma_t^2 = (\theta_0 + \theta_1 \sigma_t^2)\,dt + \vartheta \sqrt{\Psi_v + 4\gamma^2}\; dW_{2,t}$	(C.10)

$$\Psi_v = \frac{\Gamma(1/v)\,\Gamma(5/v)}{[\Gamma(3/v)]^2} - 1,\quad \Phi_{\lvert v\rvert} = \frac{\Gamma(2/v)}{[\Gamma(3/v)]^{1/2}[\Gamma(1/v)]^{1/2}},\quad \Psi_{\lvert v\rvert} = 1 - \left(\frac{[\Gamma(2/v)]^2}{\Gamma(3/v)\Gamma(1/v)}\right)$$

Parameter Restrictions for the GARCH Models with Generalized Error Distribution

AGARCH: $\lim_{\Delta\to 0} \Delta^{-1}(\beta_{0\Delta} + \beta_{1\Delta}\gamma^2) = \theta_0$, $\lim_{\Delta\to 0} \Delta^{-1}[\beta_{1\Delta} + \beta_{2\Delta} - 1] = \theta_1$, $\lim_{\Delta\to 0} \Delta^{-1}\beta_{1\Delta}^2 = \vartheta^2$

EGARCH: $\lim_{\Delta\to 0} \Delta^{-1}\beta_{0\Delta} = \theta_0$, $\lim_{\Delta\to 0} \Delta^{-1}(\beta_{2\Delta} - 1) = \theta_1$, $\lim_{\Delta\to 0} \Delta^{-1}\beta_{1\Delta}^2 = \vartheta^2$

GARCH: $\lim_{\Delta\to 0} \Delta^{-1}\beta_{0\Delta} = \theta_0$, $\lim_{\Delta\to 0} \Delta^{-1}[\beta_{1\Delta} + \beta_{2\Delta} - 1] = \theta_1$, $\lim_{\Delta\to 0} \Delta^{-1}\Psi_v\beta_{1\Delta}^2 = \vartheta^2$

GJR-GARCH: $\lim_{\Delta\to 0} \Delta^{-1}\beta_{0\Delta} = \theta_0$, $\lim_{\Delta\to 0} \Delta^{-1}[\beta_{1\Delta} + \beta_{2\Delta} + 0.5\gamma - 1] = \theta_1$, $\lim_{\Delta\to 0} \Delta^{-1}\beta_{1\Delta}^2 = \vartheta^2$

NGARCH: $\lim_{\Delta\to 0} \Delta^{-1}\beta_{0\Delta} = \theta_0$, $\lim_{\Delta\to 0} \Delta^{-1}[\beta_{1\Delta}(1+\gamma^2) + \beta_{2\Delta} - 1] = \theta_1$, $\lim_{\Delta\to 0} \Delta^{-1}\beta_{1\Delta}^2 = \vartheta^2$

QGARCH: $\lim_{\Delta\to 0} \Delta^{-1}\beta_{0\Delta} = \theta_0$, $\lim_{\Delta\to 0} \Delta^{-1}[\beta_{1\Delta} + \beta_{2\Delta} - 1] = \theta_1$, $\lim_{\Delta\to 0} \Delta^{-1}\beta_{1\Delta}^2 = \vartheta^2$

SQR-GARCH: $\lim_{\Delta\to 0} \Delta^{-1}(\beta_{0\Delta} + \beta_{1\Delta}) = \theta_0$, $\lim_{\Delta\to 0} \Delta^{-1}[\beta_{1\Delta}\gamma^2 + \beta_{2\Delta} - 1] = \theta_1$, $\lim_{\Delta\to 0} \Delta^{-1}\beta_{1\Delta}^2 = \vartheta^2$

TGARCH: $\lim_{\Delta\to 0} \Delta^{-1}\beta_{0\Delta} = \theta_0$, $\lim_{\Delta\to 0} \Delta^{-1}[\Phi_{|v|}\beta_{1\Delta} + \beta_{2\Delta} - 1] = \theta_1$, $\lim_{\Delta\to 0} \Delta^{-1}\beta_{1\Delta}^2 = \vartheta^2$

TS-GARCH: $\lim_{\Delta\to 0} \Delta^{-1}\beta_{0\Delta} = \theta_0$, $\lim_{\Delta\to 0} \Delta^{-1}[\Phi_{|v|}\beta_{1\Delta} + \beta_{2\Delta} - 1] = \theta_1$, $\lim_{\Delta\to 0} \Delta^{-1}\Psi_{|v|}\beta_{1\Delta}^2 = \vartheta^2$

VGARCH: $\lim_{\Delta\to 0} \Delta^{-1}[\beta_{0\Delta} + \beta_{1\Delta}(1+\gamma^2)] = \theta_0$, $\lim_{\Delta\to 0} \Delta^{-1}(\beta_{2\Delta} - 1) = \theta_1$, $\lim_{\Delta\to 0} \Delta^{-1}\beta_{1\Delta}^2 = \vartheta^2$

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Table 1
Speed of Mean Reversion in Stock Return Volatility

This table displays the drift parameters of the stochastic volatility models that determine the presence of mean reversion in stock return volatility. The speed of mean reversion in volatility is measured by ρ_1 for the normal distribution, κ_1 for the Student-*t* distribution, and θ_1 for the GED distribution.

Models	Normal Distribution	Student-*t* Distribution♣	GED Distribution♥
AGARCH	$\rho_1 = \lim_{\Delta\to 0} \Delta^{-1} [\beta_{1\Delta} + \beta_{2\Delta} - 1]$	$\kappa_1 = \lim_{\Delta\to 0} \Delta^{-1} [\Phi_v \beta_{1\Delta} + \beta_{2\Delta} - 1]$	$\theta_1 = \lim_{\Delta\to 0} \Delta^{-1} [\beta_{1\Delta} + \beta_{2\Delta} - 1]$
EGARCH	$\rho_1 = \lim_{\Delta\to 0} \Delta^{-1} (\beta_{2\Delta} - 1)$	$\kappa_1 = \lim_{\Delta\to 0} \Delta^{-1} (\beta_{2\Delta} - 1)$	$\theta_1 = \lim_{\Delta\to 0} \Delta^{-1} (\beta_{2\Delta} - 1)$
GARCH	$\rho_1 = \lim_{\Delta\to 0} \Delta^{-1} [\beta_{1\Delta} + \beta_{2\Delta} - 1]$	$\kappa_1 = \lim_{\Delta\to 0} \Delta^{-1} [\Phi_v \beta_{1\Delta} + \beta_{2\Delta} - 1]$	$\theta_1 = \lim_{\Delta\to 0} \Delta^{-1} [\beta_{1\Delta} + \beta_{2\Delta} - 1]$
GJR-GARCH	$\rho_1 = \lim_{\Delta\to 0} \Delta^{-1} [\beta_{1\Delta} + \beta_{2\Delta} + 0.5\gamma - 1]$	$\kappa_1 = \lim_{\Delta\to 0} \Delta^{-1} [\Phi_v (\beta_{1\Delta} + 0.5\gamma) + \beta_{2\Delta} - 1]$	$\theta_1 = \lim_{\Delta\to 0} \Delta^{-1} [\beta_{1\Delta} + \beta_{2\Delta} + 0.5\gamma - 1]$
NGARCH	$\rho_1 = \lim_{\Delta\to 0} \Delta^{-1} [\beta_{1\Delta}(1+\gamma^2) + \beta_{2\Delta} - 1]$	$\kappa_1 = \lim_{\Delta\to 0} \Delta^{-1} [\beta_{1\Delta}(\Phi_v + \gamma^2) + \beta_{2\Delta} - 1]$	$\theta_1 = \lim_{\Delta\to 0} \Delta^{-1} [\beta_{1\Delta}(1+\gamma^2) + \beta_{2\Delta} - 1]$
QGARCH	$\rho_1 = \lim_{\Delta\to 0} \Delta^{-1} [\beta_{1\Delta} + \beta_{2\Delta} - 1]$	$\kappa_1 = \lim_{\Delta\to 0} \Delta^{-1} [\Phi_v \beta_{1\Delta} + \beta_{2\Delta} - 1]$	$\theta_1 = \lim_{\Delta\to 0} \Delta^{-1} [\beta_{1\Delta} + \beta_{2\Delta} - 1]$
SQR-GARCH	$\rho_1 = \lim_{\Delta\to 0} \Delta^{-1} [\beta_{1\Delta}\gamma^2 + \beta_{2\Delta} - 1]$	$\kappa_1 = \lim_{\Delta\to 0} \Delta^{-1} [\beta_{1\Delta}\gamma^2 + \beta_{2\Delta} - 1]$	$\theta_1 = \lim_{\Delta\to 0} \Delta^{-1} [\beta_{1\Delta}\gamma^2 + \beta_{2\Delta} - 1]$
TGARCH	$\rho_1 = \lim_{\Delta\to 0} \Delta^{-1} \left[\sqrt{\frac{2}{\pi}}\beta_{1\Delta} + \beta_{2\Delta} - 1\right]$	$\kappa_1 = \lim_{\Delta\to 0} \Delta^{-1} [\Phi_{\lvert v\rvert} \beta_{1\Delta} + \beta_{2\Delta} - 1]$	$\theta_1 = \lim_{\Delta\to 0} \Delta^{-1} [\Phi_{\lvert v\rvert} \beta_{1\Delta} + \beta_{2\Delta} - 1]$
TS-GARCH	$\rho_1 = \lim_{\Delta\to 0} \Delta^{-1} \left[\sqrt{\frac{2}{\pi}}\beta_{1\Delta} + \beta_{2\Delta} - 1\right]$	$\kappa_1 = \lim_{\Delta\to 0} \Delta^{-1} [\Phi_{\lvert v\rvert} \beta_{1\Delta} + \beta_{2\Delta} - 1]$	$\theta_1 = \lim_{\Delta\to 0} \Delta^{-1} [\Phi_{\lvert v\rvert} \beta_{1\Delta} + \beta_{2\Delta} - 1]$
VGARCH	$\rho_1 = \lim_{\Delta\to 0} \Delta^{-1} (\beta_{2\Delta} - 1)$	$\kappa_1 = \lim_{\Delta\to 0} \Delta^{-1} (\beta_{2\Delta} - 1)$	$\theta_1 = \lim_{\Delta\to 0} \Delta^{-1} (\beta_{2\Delta} - 1)$

♣: $\Phi_v = \frac{v}{v-2}$ and $\Phi_{\lvert v\rvert} = 2\Gamma\left(\frac{v+1}{2}\right)\Gamma\left(\frac{v}{2}\right)^{-1}\sqrt{\frac{v}{(v-1)^2\pi}}$ for the Student-*t* distribution.

♥: $\Phi_{\lvert v\rvert} = \frac{\Gamma(2/v)}{[\Gamma(3/v)]^{1/2}[\Gamma(1/v)]^{1/2}}$ for the generalized error distribution.

Table 2
Summary Statistics

This table shows the daily percentage returns on the Dow Jones Industrial Average (DJIA). The sample period, number of observations, maximum and minimum values, mean, standard deviation, skewness, excess kurtosis, first-order autocorrelation coefficient AC(1), and the Ljung-Box statistics for the standardized residuals $Q_1(12)$ and the squared standardized residuals $Q_2(12)$ are reported for daily returns. * denotes the 1% level of significance.

	DJIA (5/26/1896 - 12/29/00)
# of obs.	28,758
Maximum	14.27
Minimum	–27.96
Mean	0.019
Std. Dev.	1.09
Skewness	–1.1735*
Kurtosis	39.391*
AC(1)	0.0421*
$Q_1(12)$	181.77*
$Q_2(12)$	2565.10*

Table 3
Maximum Likelihood Estimates of the GARCH Models

This table displays the maximum likelihood estimates of the GARCH models for Dow Jones Industrial Average. Panels A, B, and C present the estimated parameters of the thin-tailed normal, and the fat-tailed Student-*t* and GED distributions. The parameter estimates with asymptotic *t*-statistics are shown in parentheses for each model. The maximized log-likelihood values are used to test the presence of asymmetry in volatility and the empirical validity of normality assumption.

$$r_t - r_{t-1} = (\omega + \alpha r_{t-1}) + \sigma_t z_t\,,\ \ \varepsilon_t = \sigma_t z_t$$

$$f(\sigma_t) = \hbar(\sigma_{t-1}, z_{t-1}, |z_{t-1}|; \beta_0, \beta_1, \gamma) + \beta_2 f(\sigma_{t-1})$$

$$\text{where}\quad f(\sigma_t) = \sigma_t,\ \sigma_t^2,\ \text{or}\ \ln\sigma_t^2$$

Panel A: Normal Distribution

Models	ω	α	β_0	β_1	β_2	γ	*Log-L*
AGARCH	0.00018 (3.8994)	0.0979 (16.479)	0.0000011 (17.596)	0.0981 (66.573)	0.8819 (538.10)	-0.00347 (-26.283)	95713.33
EGARCH	0.00016 (3.4451)	0.0988 (17.223)	-0.3482 (-49.283)	0.1850 (67.666)	0.9779 (741.24)	-0.0781 (-40.794)	95761.62
GARCH	0.00039 (8.5979)	0.0938 (16.273)	0.0000016 (35.216)	0.1019 (82.449)	0.8857 (650.49)	0.0	95527.61
GJRGARCH	0.00019 (4.1918)	0.1026 (17.518)	0.0000019 (39.014)	0.0431 (17.198)	0.8874 (487.27)	-0.1046 (-34.455)	95758.13
NGARCH	0.00015 (3.2526)	0.1017 (17.177)	0.0000021 (42.086)	0.0976 (60.589)	0.8566 (425.53)	-0.5476 (-28.949)	95798.36
QGARCH	0.00017 (3.8991)	0.0987 (16.469)	0.0000012 (17.596)	0.0980 (66.571)	0.8819 (538.21)	-0.00346 (-26.279)	95713.33
SQRGARCH	0.00010 (2.9865)	0.1076 (19.957)	0.00000061 (41.156)	0.0000068 (58.340)	0.8649 (548.51)	-97.582 (-57.061)	95036.78
TGARCH	0.00014 (3.8129)	0.1002 (18.645)	0.000299 (62.111)	0.1514 (36.536)	0.8722 (564.20)	-0.1122 (-28.450)	95751.23
TSGARCH	0.00016 (3.7806)	0.0747 (13.536)	0.000292 (63.035)	0.1403 (35.005)	0.8787 (582.11)	0.0	95640.12
VGARCH	0.00011 (2.8200)	0.0985 (18.187)	0.00000051 (37.420)	0.0000073 (63.229)	0.9204 (676.61)	-0.5799 (-69.367)	94895.24

Panel B: Standardized Student-*t* Distribution

Models	ω	α	β_0	β_1	β_2	γ	v	*Log-L*
AGARCH	0.00035 (8.2373)	0.0834 (14.020)	0.0000006 (3.8979)	0.0864 (23.725)	0.8991 (245.78)	-0.00338 (-13.815)	6.2785 (30.846)	113141.54
EGARCH	0.00032 (10.451)	0.0802 (13.223)	-0.2246 (-21.101)	0.1756 (30.543)	0.9814 (741.24)	-0.0678 (-40.794)	6.1134 (31.777)	113199.12
GARCH	0.00048 (11.455)	0.0794 (13.442)	0.0000011 (11.748)	0.0865 (24.462)	0.9053 (261.07)	0.0	5.9398 (31.374)	113020.78
GJRGARCH	0.00037 (8.8086)	0.0857 (14.367)	0.0000014 (13.527)	0.0448 (10.971)	0.8992 (248.56)	-0.0857 (-14.984)	6.3001 (30.288)	113137.36
NGARCH	0.00033 (7.8988)	0.0854 (14.325)	0.0000015 (14.242)	0.0864 (23.776)	0.8765 (213.92)	-0.5366 (-16.228)	6.3833 (30.600)	113190.56
QGARCH	0.00034 (8.2369)	0.0836 (14.103)	0.0000007 (3.8980)	0.0865 (23.726)	0.8992 (245.81)	-0.00339 (-13.816)	6.2786 (30.842)	113141.54
SQRGARCH	0.00015 (3.4526)	0.0803 (14.358)	0.00000013 (4.0154)	0.0000054 (19.279)	0.8935 (199.86)	-93.827 (-18.312)	5.5757 (33.552)	112749.46
TGARCH	0.00034 (8.0986)	0.0813 (14.117)	0.000159 (14.843)	0.1041 (31.767)	0.9147 (302.25)	-0.0787 (-18.803)	6.1119 (34.154)	113163.24
TSGARCH	0.00049 (11.817)	0.0758 (13.389)	0.000122 (12.170)	0.0928 (28.829)	0.9178 (315.58)	0.0	5.6597 (35.504)	112991.96
VGARCH	0.00019 (4.1798)	0.0773 (13.849)	0.00000011 (3.9687)	0.0000057 (18.999)	0.9376 (340.17)	-0.5917 (-16.637)	5.4436 (34.138)	112695.12

Panel C: Generalized Error Distribution

Models	ω	α	β_0	β_1	β_2	γ	v	*Log-L*
AGARCH	0.00034 (8.3736)	0.0784 (13.590)	0.0000007 (5.7353)	0.0907 (28.430)	0.8926 (269.32)	-0.00330 (-14.215)	1.3381 (136.20)	96504.49
EGARCH	0.00035 (9.2006)	0.0811 (14.123)	-0.2135 (-19.908)	0.1834 (31.537)	0.9801 (423.12)	-0.0711 (-27.334)	1.3467 (134.88)	96573.42
GARCH	0.00047 (11.495)	0.0745 (13.027)	0.0000013 (15.427)	0.0922 (31.219)	0.8973 (303.89)	0.0	1.3166 (132.54)	96387.47
GJRGARCH	0.00036 (8.7330)	0.0810 (14.040)	0.0000015 (16.825)	0.0452 (11.072)	0.8943 (271.51)	-0.0901 (-17.063)	1.3438 (132.84)	96512.73
NGARCH	0.00033 (7.9697)	0.0805 (13.931)	0.0000016 (17.386)	0.0905 (28.147)	0.8702 (239.51)	-0.5225 (-16.486)	1.3472 (136.07)	96553.44
QGARCH	0.00035 (8.3734)	0.0785 (13.592)	0.0000007 (5.7354)	0.0908 (28.432)	0.8927 (269.34)	-0.00331 (-14.216)	1.3381 (136.20)	96504.49
SQRGARCH	0.00016 (3.7368)	0.0756 (13.950)	0.00000032 (18.169)	0.0000059 (21.702)	0.8839 (222.92)	-91.577 (-19.916)	1.2720 (137.43)	96075.91
TGARCH	0.00029 (7.2239)	0.0812 (14.517)	0.000311 (48.157)	0.1262 (42.970)	0.8988 (318.67)	-0.1022 (-22.765)	1.3014 (131.14)	96425.24
TSGARCH	0.00048 (11.827)	0.0747 (13.536)	0.000289 (46.965)	0.1249 (39.665)	0.8984 (341.08)	0.0	1.2625 (129.65)	96231.13
VGARCH	0.00019 (4.5586)	0.0722 (13.351)	0.00000010 (10.194)	0.0000062 (21.546)	0.9289 (322.56)	-0.5790 (-19.887)	1.2593 (139.21)	96015.97

Table 4
Speed of Mean Reversion in Stock Return Volatility
Based on the Relation Between GARCH and Diffusion Models

This table displays the speed of mean reversion in the conditional variance, log-variance, and standard deviation of daily returns based on the relation between discrete time GARCH and continuous time diffusion models. The speed of mean reversion in stock return volatility is measured by the negative values of ρ_1 for the normal distribution, κ_1 for the Student-*t* distribution, and θ_1 for the generalized error distribution.

Models	**Normal Distribution (ρ_1)**	**Student-*t* Distribution (κ_1)**	**GED Distribution (θ_1)**
AGARCH	–0.0201	–0.0145	–0.0167
EGARCH	–0.0221	–0.0186	–0.0199
GARCH	–0.0124	–0.0082	–0.0105
GJR-GARCH	–0.0172	–0.0132	–0.0155
NGARCH	–0.0165	–0.0122	–0.0146
QGARCH	–0.0201	–0.0145	–0.0167
SQR-GARCH	–0.0703	–0.0590	–0.0666
TGARCH	–0.0070	–0.0022	–0.0067
TS-GARCH	–0.0093	–0.0021	–0.0086
VGARCH	–0.0796	–0.0624	–0.0711

Table 5
Measuring the Degree of Mean Reversion in Stock Return Volatility

This table displays the speed of mean reversion in stock return volatility for the normal, Student-*t* and GED distributions. The degree of mean reversion in volatility is measured by the negative values of ϕ_1 from a series of OLS regressions of the form:

$$f(\sigma_t) - f(\sigma_{t-1}) = \phi_0 + \phi_1 f(\sigma_{t-1}) + u_t$$

where $f(\sigma_t) = \sigma_t^2,\ \ln\sigma_t^2,\ \sigma_t$ for the conditional variance, log-variance, and standard deviation of daily returns, respectively. The estimated mean reversion rates (ϕ_1) are presented with their *t*-statistics in parentheses. * denotes the 1% level of significance.

Models	**Normal Distribution**	**Student-*t* Distribution**	**GED Distribution**
AGARCH	–0.0463 (–26.124)*	–0.0376 (–23.510)*	–0.0412 (–24.622)*
EGARCH	–0.0294 (–18.597)*	–0.0203 (–15.944)*	–0.0228 (–17.012)*
GARCH	–0.0442 (–25.525)*	–0.0345 (–22.474)*	–0.0387 (–23.857)*
GJR-GARCH	–0.0586 (–29.497)*	–0.0497 (–27.091)*	–0.0533 (–28.086)*
NGARCH	–0.0508 (–27.393)*	–0.0411 (–24.595)*	–0.0450 (–25.737)*
QGARCH	–0.0463 (–26.124)*	–0.0376 (–23.510)*	–0.0412 (–24.622)*
SQR-GARCH	–0.0511 (–27.739)*	–0.0456 (–33.987)*	–0.0466 (–27.574)*
TARCH	–0.0264 (–19.642)*	–0.0148 (–14.655)*	–0.0201 (–17.453)*
TS-GARCH	–0.0215 (–17.685)*	–0.0105 (–12.343)*	–0.0196 (–14.617)*
VGARCH	–0.0516 (–27.958)*	–0.0467 (–35.942)*	–0.0471 (–27.840)*