Diversification of Equity with VIX Futures:
Personal Views and Skewness Preference

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ABSTRACT

A comprehensive description of the trading and statistical characteristics of VIX futures and their exchange-traded notes motivates our study of their benefits to equity investors seeking to diversify their exposure. We analyze when diversification into VIX futures is ex-ante optimal for standard mean-variance investors, then extend this to include (a) skewness preference, and (b) a moderation of personal forecasts by equilibrium returns, as in the Black-Litterman framework. An empirical study shows that skewness preference increases the frequency of diversification, but out-of-sample the optimally-diversified portfolios rarely out-perform equity alone, even according to a generalized Sharpe ratio that incorporates skewness preference, except during an extreme crisis period or when the investor has personal access to accurate forecasts of VIX futures returns.

1 Introduction

The average holding time of a VIX futures contract has fallen, but still remains between one week and a month, so some investors clearly believe they are useful instruments for diversification. Trading volume on VIX futures tracker exchange-traded notes (ETNs) is now so great that they are starting to lead the VIX futures market. The potential recursive influence has led to considerable concern by the U.S. Securities and Exchange Commission that the risks of trading these products are not fully understood.[1]

The first volatility exchange-traded notes (ETN) were launched at the height of the banking crisis, as the trading of VIX futures intensified.[2] Extolling the virtues of volatility as a new and effective equity diversifier, between January 2009 and December 2011 Barclays issued a total principal of \$27.5bn on the VXX, a short-term VIX futures tracker product. At the time of writing there are about 30 volatility ETNs and most were issued by Barclays, Credit Suisse and UBS.[3] These products allow individual and institutional investors that are not allowed to trade derivatives, such as pension funds and mutual funds, to take direct and inverse positions related to VIX futures.

Between January 2009 and December 2011 the VXX lost over 90% of its value, and a 4:1 reverse split occurred in November 2010. Nevertheless, secondary market trading on VXX and other volatility ETNs has now become intense. For instance, in February 2012, when Credit Suisse announced it was suspending issue of further shares in the 2 × leveraged TVIX, over 27 million shares traded in one session alone and its price peaked at a 13% premium to its indicative value. At the onset of the Eurozone crisis in the first two weeks of August 2011, about \$2.87bn was traded on VIX futures, on average, per day. For comparison, over the same period the average value of shares traded each day on the VXX alone was about \$2.52bn.

During 2011 the average holding period of a VXX share fell from about three weeks to about one week, but the ETNs that track longer-term VIX futures are typically held for several months. Similarly, average turnover times are currently a little more than one week for short-term VIX futures, down from about a month in 2008. Longer-dated contracts are typically held for almost a month, down from about three months in 2008.[4] These

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[1]Source: "Out of control? Volatility ETN triggers risk concerns". Reuters, February 23, 2012.

[2]The VIX index represents the 30-day implied volatility of the Standard and Poor's (S&P) 500 index. It is calculated and disseminated on a real-time basis by the Chicago Board Options Exchange (CBOE). The ETNs linked to VIX futures are issued with a fixed tenor, of 10 – 40 years, and their redemption value is their indicative value at redemption. Indicative returns on each note are typically direct or inverse returns to an S&P constant maturity VIX futures index, minus an annual fee of between 0.85% and 5.35%, and a small early redemption fee. Some are leveraged products and most have callable and automatic termination features. There is an active secondary market for these notes, especially on the NYSE exchange, and the CBOE lists options on some of the more long-standing ETNs. Unlike VIX futures, ETNs have a credit risk resulting from possible issuer default.

[3]Further details are on the web at www.ipathetn.com, www.velocityshares.com and www.ibb.ubs.com.

[4]Details of the data used and calculation methods under-pinning the facts and figures quoted in the

turnover times indicate that VIX futures and ETNs are not only traded for speculation, they are also used as diversification instruments. Indeed the CBOE, S&P and some issuers of ETNs are actively promoting the use of VIX futures and ETNs for diversification.[5]

VIX futures and their ETNs are certainly amongst the most risky of all exchange-traded products. For instance, at the onset of the Eurozone crisis in 2011 the volatility of the prompt VIX futures topped 200%. Hence, that of the $2 \times$ leveraged ETNs that are linked to its returns exceeded 400%. This motivates the question whether such highly speculative instruments should even be regarded as suitable diversification tools for pension funds, mutual funds, public companies and commercial banks.

The search for effective diversifiers has intensified since the banking crisis. International equities have become more highly correlated, as have international bonds,[6] and the equity–bond correlation has also risen. Moreover, Daskalaki and Skiadopoulos (2011) provides a convincing analysis showing that even commodities now offer little or no diversification gains.[7] In this environment equity volatility arises as a natural diversification choice because its negative correlation with equity increases exactly when diversification is needed most, a fact that has been well documented in the literature since Bekaert and Wu (2000). For example, if on April 1, 2010 an S&P 500 investor had put 30% of his capital in the risk-free asset and taken an equivalent long position in the June 2010 VIX futures contract, closing the position a week before expiry, he would have achieved an annualized Sharpe ratio (SR) of 3.61. Holding the SPY alone gave a negative mean excess return over the same period.

However, the empirical evidence in favor of VIX futures and ETNs as effective instruments for diversification is scarce. Most of the academic papers that advocate this studied variance swaps, or data on the spot VIX index, which have very different statistical characteristics to VIX futures.[8] There are just four previous studies on equity diversification using VIX futures, and three employ only an ex-post analysis: Szado (2009) demon-

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introduction are provided in Section 2.

[5]See the CBOE micro/vxx and S&P spvixviews websites, CBOE (2009), the propsectuses of ETNs and Jhunjhunwala (2010).

[6]Indeed, many years ago Errunza, Hogan, and Hung (1999) warned U.S. investors that the gains from international diversification can be adequately captured by home-made diversification and any further gains are statistically and economically insignificant.

[7]Simple calculations support this: e.g. the sample correlation between the daily returns on the S&P 500 Stock index and those on the S&P Goldman Sachs Commodity index was only 0.02 (January 2005 to December 2007) but then rose to 0.45 (January 2008 to December 2011).

[8]Copeland and Copeland (1999) used spot VIX as an indicator to alter equity allocation between market regimes; Dash and Moran (2005) added spot VIX to a hedge fund portfolio, proposing static and dynamic allocations that have greater mean–variance efficiency than the hedge fund portfolio alone, over the period 1995–2004; and Daigler and Rossi (2006) showed that a combination of S&P 500 and VIX has greater mean–variance efficiency than an equity-only exposure over the period 1992–2002. Variance swaps have been shown to provide good instruments for volatility diversification by Hafner and Wallmeier (2008) and Egloff, Leippold, and Wu (2010).

strates positive diversification benefits based on an arbitrarily-chosen allocation between VIX futures and other assets; Warren (2012) and Chen, Chung, and Ho (2010) examine the in-sample diversification benefits of adding long or short VIX futures to different portfolios. Warren (2012) uses risk-return plots to show that VIX futures could enhance the characteristics of typical portfolios held by pension funds, and Chen, Chung, and Ho (2010) uses a mean–variance approach to show that VIX futures could enhance the in-sample performance of equity returns only if shorted.

This poses an important question. Is a short volatility position really offering any diversification benefits in the outset? Adding an asset with very high positive correlation, exceptionally high volatility and a very large negative skewness would be sub-optimal for long equity investors because they offer a highly positively correlated exposure with risk–return characteristics far inferior to those of the equity index.

For results to be put into practice they must be based on an ex-ante analysis. Only one such study has previously been published: Brière, Burgues, and Signori (2010) find diversification benefits for long equity investors based on minimum-variance optimization. But the sample ends in August 2008, before the onset of the banking crisis, prices prior to 2004 were based on the unrealistic assumption of a stable relationship between the futures and spot VIX, and the strong positive skewness in VIX futures was ignored. In sum, there is scant evidence that the desirable diversification qualities of spot VIX and variance swaps could carry over in practice to exchange-traded products such as VIX futures and their ETNs. Our paper aims to fill a gap in this literature by providing a comprehensive study of the potential for using VIX futures and their ETNs for equity diversification, using ex-ante optimal allocations and out-of-sample performance analysis that also adjusts the standard mean-variance criterion for skewness and personal views.

The first section motivates our study by analyzing the trading and statistical characteristics of VIX futures and ETNs, providing new data by tracking the indicative value of two popular ETNs back to the inception of VIX futures trading in March 2004. Thereafter we assess the potential diversification benefits of VIX futures and ETNs under the classical optimal allocation approach of Markowitz (1952), extending this first by allowing the investor to have a preference for the positive skewness of VIX futures, and then supposing personal forecasts are moderated by market equilibrium returns in the Black and Litterman (1992) optimization framework.[9] We also derive useful analytic formulas for the optimal diversification threshold, which is the expected return on VIX futures that would justify diversification for a long equity investor, under the Markowitz (1952) and

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[9]This should be distinguished from a hedging exercise, where minimizing equity risk is the main purpose. Although Stanton (2011) describes long equity investors as being implicitly short volatility and thus considers long volatility as a hedge rather than a diversifier, hedging an equity with its own futures must be more effective and less costly than buying volatility futures.

the Black and Litterman (1992) portfolio optimization frameworks.

We use daily data from March 2004 (the inception of VIX futures) to December 2011. The total sample is divided into three sub-samples of almost equal size that respectively capture: stable-trending markets leading up to the credit crisis; the excessively volatile period encompassing both credit and banking crises; and the nervous, range-bounded markets during the Eurozone crisis. Our empirical study begins with the ex-post justification for diversification with VIX futures, and we find that volatility diversification was only optimal during the sub-sample that covers the credit and banking crises, even after accounting for skewness preference. We continue with an ex-ante analysis, and we find that equity investors would frequently choose to diversify, in each sub-sample, even when they employ the standard mean-variance criterion with an expected return on VIX futures that is negative and their forecasts are simple static forecasts based on recent historical performance. Skewness preference increases diversification frequency, but on tracking the performance of optimally-diversified portfolios out-of-sample we find that it is only when investors have very accurate personal forecasts for VIX futures returns – e.g. via access to privileged information – that they would always benefit from volatility diversification. Otherwise, the only period when the ex-ante optimally diversified portfolio outperformed equity was during the middle sub-sample, i.e. the credit and banking crises period.

In the following: Section 2 describes the data and investigates the trading and statistical characteristics of VIX futures and their ETNs; Section 3 discusses our ex-post results, followed by an extensive out-of-sample analysis; Section 4 presents our methodology and results for mean-variance investors; Section 5 analyzes how optimal behaviour and portfolio performance changes for skewness-aware investors; Section 6 considers the diversification decision problem from the perspective of an investor using the Black-Litterman model; and Section 7 summarizes and concludes.

2   Data, Trading Characteristics, and Sample Statistics

Equity exposure is characterized by a long position in the spider (SPY), the S&P 500 exchange-traded fund. Volatility exposure is a long position on a VIX futures contract or the purchase of shares in a volatility ETN. We have nearly 8 years of Bloomberg daily data on the closing prices of SPY and the close, bid and ask prices, as well as the daily spread, open interest and volume levels for the full set of VIX futures contracts that traded between March 26, 2004, the inception of the VIX futures contract, and December 31, 2011. Data on the ETNs is based on the Bloomberg US ticker, covering all U.S. exchanges including the primary exchange, NYSE Arca. But even the 1-month and 5-month constant maturity VIX futures trackers VXX and VXZ have data starting only on January 30, 2009. So we replicate the indicative values exactly using the S&P constant

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maturity VIX futures indices, which are available from December 2005. Prior to this we constructed the S&P indices ourselves using the Standard & Poor's (2011) methodology. The 1-month U.S. Treasury bill is used for the risk-free rate, downloaded from the U.S. Federal Reserve website. It is convenient to separate the sample into three sub-samples of roughly equal size that capture different market circumstances: from April 1, 2004–September 30, 2006 (a tranquil period), October 1, 2006–March 31, 2009 (the excessively volatile period of the banking crisis) and April 1, 2009–December 31, 2011 (with volatile periods during the Greek crisis of 2010 and the wider Eurozone crisis of 2011).

2.1   Prices, investable indices and roll costs

For a continuous series of VIX futures prices we consider three possible rollover frequencies: the monthly rollover commences with the prompt futures contract on the first day of the sample and on each rollover date moves to the next available contract. The quarterly rollover series moves to the next contract on the March quarterly cycle and the long-term rollover series rolls into the contract with the longest maturity available, provided that is actively traded (we require a minimum 10-day average volume of at least 50 contracts).
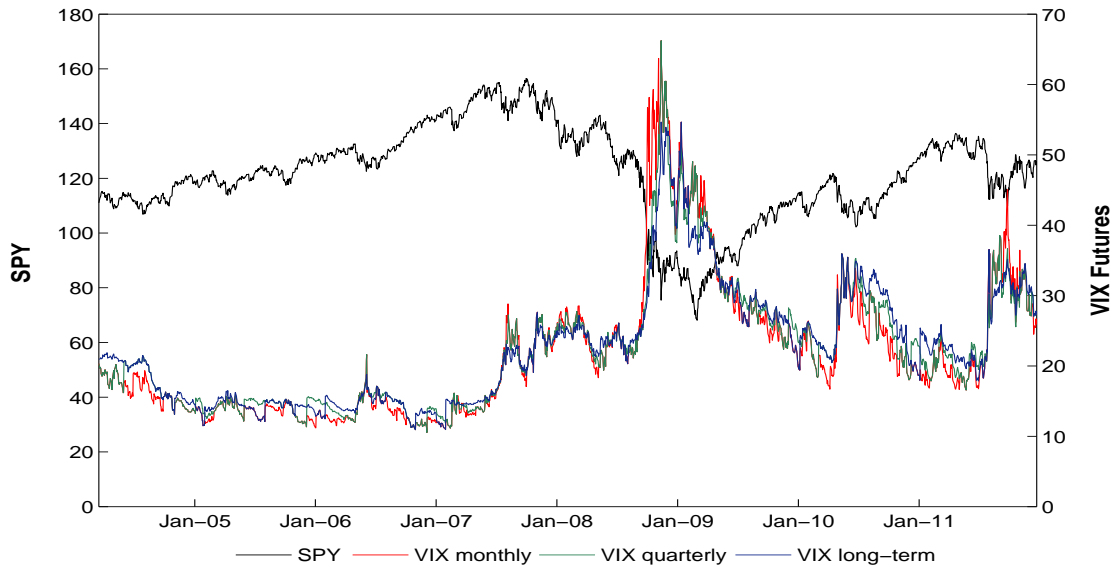

Figure 1 SPY and VIX Futures
Price series evolution of the SPY and the three VIX futures series, from March 2004 to December 2011.

Figure 1 depicts the daily evolution of the prices of the SPY and the three VIX futures series, rolling over 5 days before expiry. Prices of VIX futures have been higher and more variable since the 2007 credit crisis,[10] and the strong negative correlation between SPY

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[10] At the time of writing the press suggests that such behavior is the "new normal," as various forces combine to make stock trading permanently more erratic. See, for example, "The New Standard: Volatility", front page, New York Times, September 18, 2011.

and VIX futures is particularly evident during the 2008 banking crisis. The prices for VIX futures shown in Figure 1 do not correspond to realizable returns, because at each rollover date there is a cost for rolling from the shorter to the longer term contract, which depends on the slope of the term structure. The roll cost is positive (roll yield is negative) if and only if the term structure is in contango. Roll costs are usually small and positive because the VIX futures market is typically in contango. However, brief periods of steep backwardation tend to occur at the onset of a period of unusually high volatility, and at such times the roll costs can be high and negative, leading to greatly enhanced realised returns on VIX futures.
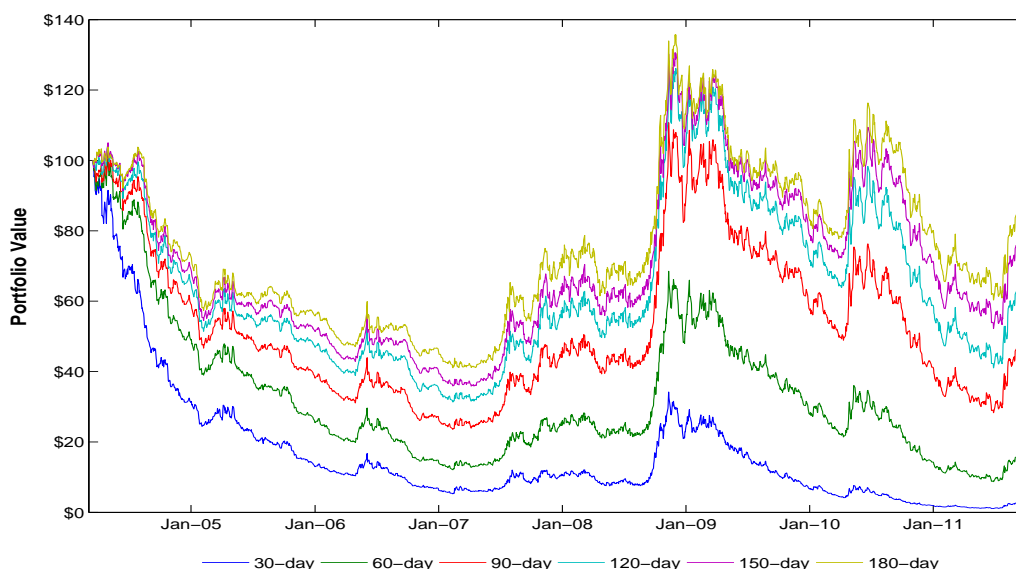

Figure 2 S&P VIX Futures Indices
Evolution of the S&P 1-month and 5-month VIX futures indices, December 2005 to December 2011.

The indicative values for VIX futures direct tracker ETNs are based on the returns to the S&P constant-maturity VIX futures indices, respectively, depicted in Figure 2. The S&P quote an investable index by interpolating linearly between returns, not prices, of the two VIX futures that straddle the target constant maturity. The indices maintain a constant maturity by rolling over a small amount daily from the shorter to the longer maturity contract. The total roll on the S&P indices is similar to the once-off roll cost paid periodically at every rebalancing on the VIX futures series that we analyze, so returns on our VIX futures series have very similar features to those on the S&P indices.

Because the slope of the term structure is greater at the short end, the roll-yield effect is larger for the VXX than it is for the 5-month tracker, the VXZ. For instance, between January 1, 2008 and December 31, 2011 the 1-month index lost almost 80% of its value, whereas the 5-month index gained more than 11%. However, when the term structure swings into strong backwardation, as it did at the start of the banking crisis, the roll yield

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becomes high and positive and the ETNs make remarkable gains in price. Indeed, the 1-month index jumped from about 55,000 on September 2, 2008 to more than 170,000 on October 20, 2008 – more than 200% return in less than 2 months!

2.2 Trading characteristics of VIX futures and ETNs

A prevalent characteristic of the volatility market is the negative carry, i.e. a negative return on a buy-and-hold position until the rolling over to the next contract. Figure 3 compares the carry on the prompt futures when the contract is rolled over upon maturity ($t = 0$) with a rollover date 5 business days before expiry ($t = 5$). These discrete returns include transactions costs and are translated into a monthly equivalent (22 trading days) for uniformity over time. Most contracts had negative carry, until the banking crisis when VIX futures produced very large positive returns. On the few occasions that the carry was positive it was most often followed by a negative carry, indicating that positive returns are not easy to forecast based on historical evidence. Only during the credit crisis of 2007 and the banking crisis of 2008 was a positive carry followed by another positive carry. The familiar maturity effect is exacerbated by the settlement process for VIX futures,[11] so most contracts have a lower and less variable carry when rolled over 5 days prior to maturity. This finding is consistent with other evidence on optimal rollover in futures markets, see e.g. Ma, Mercer, and Walker (1992). From henceforth we shall rollover 5 days prior to expiry.

Figure 4 depicts the average daily trading volume for the prompt VIX futures contract, indicated on the right-hand scale, with the average taken over the previous 10 trading days. During the Eurozone crisis of August 2011 trading volume on the prompt VIX futures contract averaged more than 40,000 contracts per day, and the value traded averaged $1,234,704,930 per day. On the left-hand scale we show the average basis point (bp) spread over the same 10-day period, measured at close of trading each day as the ratio of the absolute bid–ask spread to the mid price. Although spreads are still very much higher than those on S&P 500 futures, they have clearly diminished as trading volume has soared during the last few years. At the onset of the Eurozone crisis in the first two weeks of August 2011, about $2.87bn was traded on VIX futures, on average, per day. For comparison, over the same period the average value of shares traded each day on the VXX alone was about $2.52bn.

Spread and volume data for the VXX are depicted in Figure 5. Data on the spread

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[11]The underlying VIX index is based on the average of bid and ask prices of options entering the calculation formula but VIX futures are settled on the special opening quotation (SOQ) price. The SOQ is extracted using *actual* traded prices of SPX options during the market open at settlement day. Consequently, the difference between the VIX futures settlement price and VIX open deviates from zero. This convergence problem leads to increased arbitrage trading activity over the last few days prior to expiry, causing increased volatility in VIX futures prices as they approach expiry.
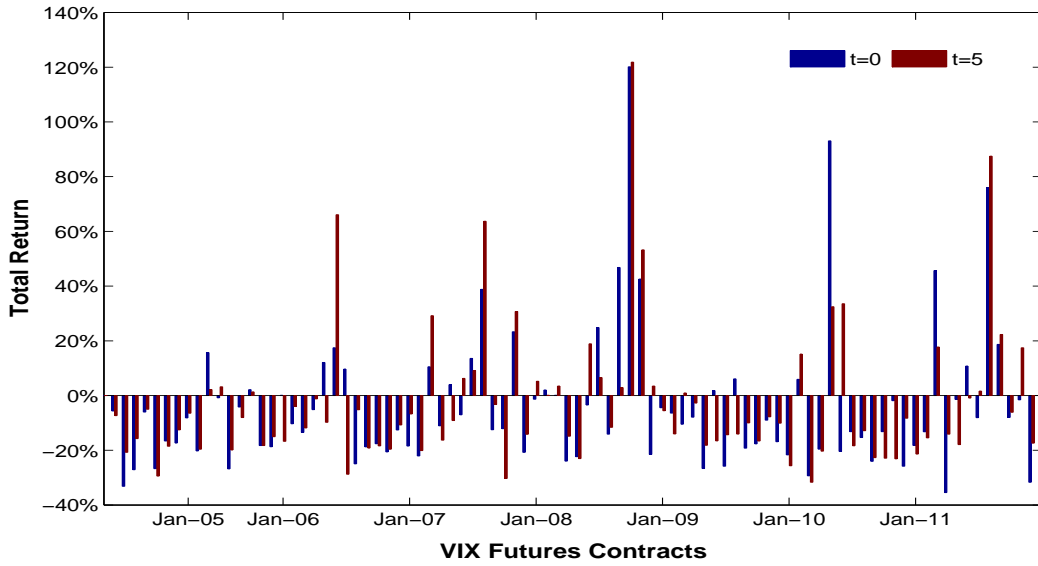
Figure 3 Carry and Rollover Times
A comparison of the carry on the prompt VIX futures when rolled over at maturity ($t = 0$) or 5 trading days prior to expiry ($t = 5$). Each carry is quoted as a monthly equivalent return.
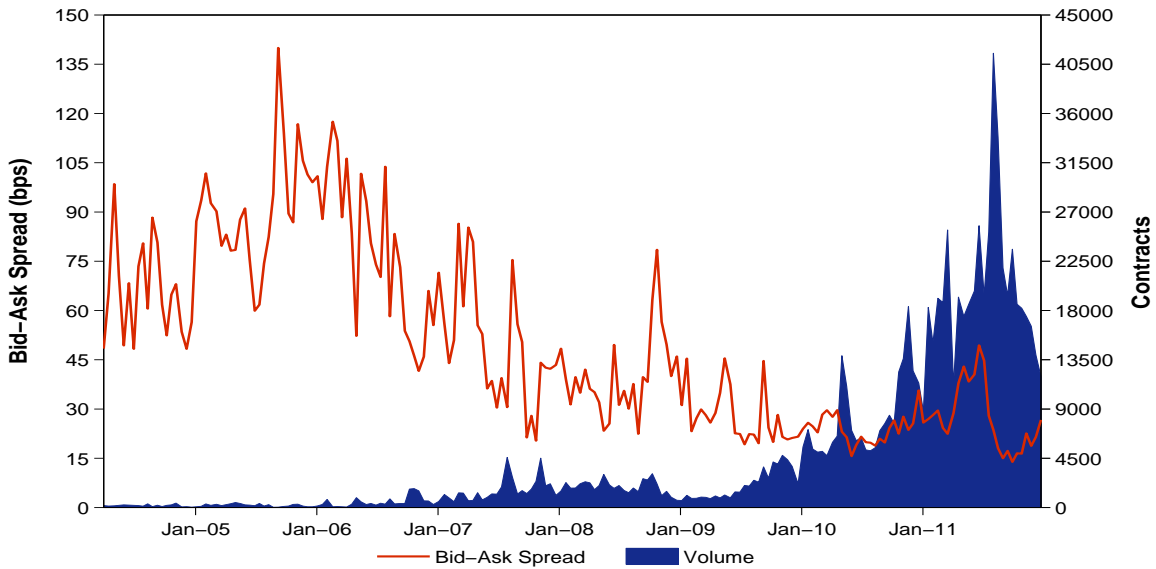
Figure 4 Spread and Trading Volume of Prompt VIX Futures
Bid–ask spread and volume on VIX (monthly rollover). The spread is defined as the difference between the ask and bid prices as a percentage of the mid price (left scale) while trading volume is the number of contracts traded per day (right scale). Series are smoothed by averaging over the last 10 trading days.

are only available since January 2010 and before July 2010 the spread was high with low trading volume. Since then it has remained in the region of 10 – 20bps, except during the Eurozone crisis in 2011, when the average daily trading volume peaked at almost 80 million contracts per day. On August 11, 2011 there were nearly 84 million transactions

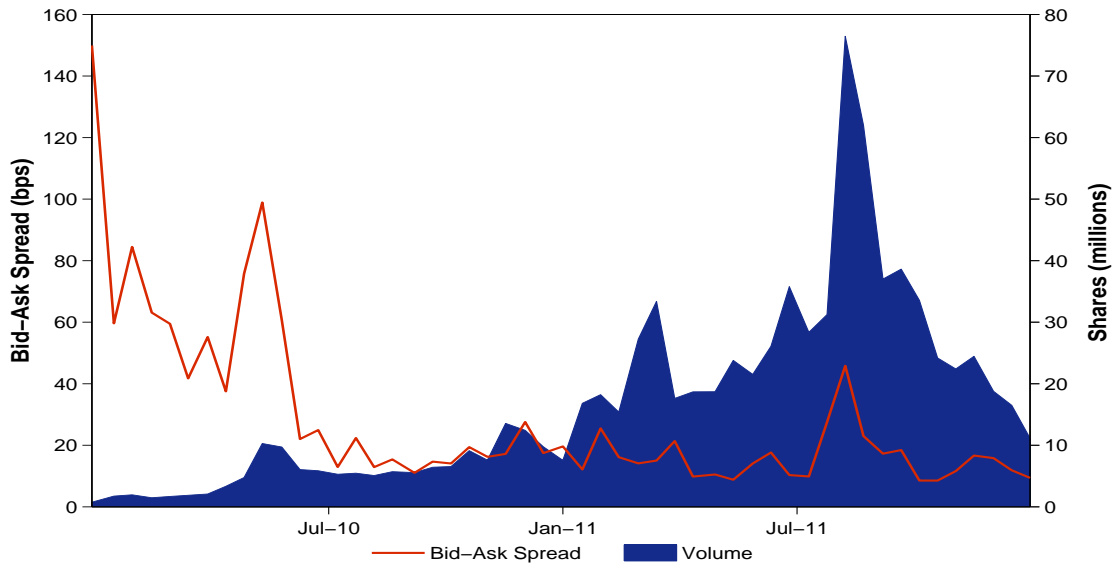**Figure 5 Spread and Trading Volume of VXX**
Bid–ask spread and volume on VXX. The spread is defined as the difference between the ask and bid prices as a percentage of the mid price (left scale) while trading volume is the number of shares traded per day (right scale, in millions). Series are smoothed by averaging over the last 10 trading days.

and, with the price closing at \$33.8, around \$2.8bn was traded on VXX on a single day!

Table 1 : Spread, Volume, and Total Return on VIX Futures and ETNs.

	Average Spread	Spread Stdev	Average Volume	Total Return
Sub-sample 1: April 1, 2004 - September 30, 2006				
VIX monthly	0.82%	0.41%	241	-90.36%
VIX quarterly	0.75%	0.38%	266	-75.29%
VIX long-term	0.74%	0.37%	281	-74.51%
Sub-sample 2: October 1, 2006 - March 31, 2009				
VIX monthly	0.44%	0.32%	1,587	161.63%
VIX quarterly	0.58%	0.60%	911	191.90%
VIX long-term	0.71%	0.59%	631	96.55%
Sub-sample 3: April 1, 2009 - December 30, 2011				
VIX monthly	0.26%	0.15%	10,906	-92.61%
VIX quarterly	0.31%	0.20%	6,661	-81.52%
VIX long-term	0.33%	0.20%	3,484	-54.47%
VXX	0.36%	0.45%	12,265,049	-92.18%
VXZ	1.29%	1.23%	395,004	-46.39%

Columns 1 and 2 are the mean and standard deviation of the daily bid–ask spread; for the relatively illiquid VXZ, only days with a spread of less than 5% are used in the calculations. Column 3 is the mean of the daily number of contracts traded, or shares traded for the ETNs; the VXX data are adjusted for the 1:4 reverse split in November 2010. Column 4 is the total return on a buy-and-hold strategy, rebalanced according to our three VIX futures series. The VXX and VXZ data are only available for sub-sample 3.

Table 1 reports the daily trading volume and basis point spread when averaged over the three sub-samples, and the total return on the VIX futures series over each sub-sample.

We also exhibit the same data for the VXX and VXZ but these are only available for sub-sample 3. The total return is the percentage growth of an investment in the futures initiated at the start of the sub-sample and held until it ends. It is large and negative, except in sub-sample 2, and even in sub-sample 2 the total return on the prompt futures rollovers is less than it is when rolling to the next quarterly or long-term futures contract. The reason is that roll costs are greater with monthly rollovers; even though they hold more liquid contracts with lower spreads, the rollovers are more frequent. In sub-sample 3 spreads were higher and more variable on the VXX than on VIX futures and trading volume and spreads on the VXZ were much less than on the VXX.

Since trading volumes record both buy and sell as separate transactions the average number of trading days that a futures contract is held can be approximated using:

$$\text{Average turnover time in days} = 2 \times \frac{\text{Average daily open interest}}{\text{Average daily trading volume}}. \tag{1}$$

Table 2 reports the results of applying (1) to annual averages. Average turnover times have generally been decreasing since 2005 so that the prompt and next contracts were held, on average a little more than one week during 2011. The longer-term contracts are typically held for longer: during 2011 the 3rd and 4th to mature contracts were held for about 2 weeks and the 5th and 6th to mature contracts were held for $3-4$ weeks. This shows that VIX futures are not only transacted by day traders and other speculators. They are promoted as effective diversifiers, so it is reasonable to assume that they are being held for such a purpose.

Table 2 : Average Turnover Times for VIX Futures Contracts.

			Contract			
Year	1	2	3	4	5	6
2004	26.0	37.1	40.0	25.8	N/A	N/A
2005	37.9	54.2	56.4	47.0	N/A	N/A
2006	26.1	46.2	72.0	47.9	N/A	N/A
2007	18.8	35.6	54.1	30.6	33.7	31.4
2008	16.4	21.2	34.9	41.4	58.5	40.3
2009	11.3	11.6	19.3	23.7	28.3	22.7
2010	8.6	10.6	20.0	20.5	20.5	15.7
2011	5.9	5.8	10.1	12.4	16.6	17.9

Figures in each cell report the average number of days a VIX futures contract is held, computed using (1) based on annual averages. Columns headed $1-6$ represent the 1st, 2nd, ..., 6th to mature contracts.

We estimate the average number of trading days that an ETN share is held using (1) with the number of shares outstanding in place of open interest, neglecting the small adjustment for trading on the primary market since volume typically far exceeds the absolute change in shares outstanding. Figure 6 depicts the calculation (1) computed using rolling averages over 22 days. Average holding times have remained high and
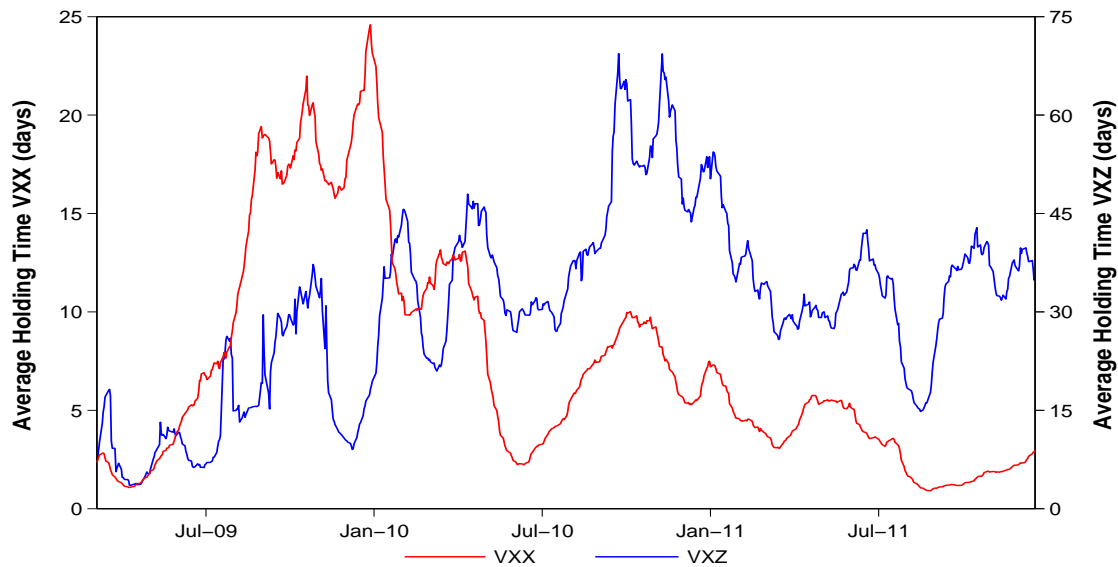
Figure 6 Average number of trading days an ETN is held
Average number of days an ETN is held, computed using (1) for the VXX (left scale) and VXZ (right scale). Sample period is January 2009 – December 2011.

variable on the VXZ and during 2011 shares were held for about 6 weeks, on average. By contrast, the VXX has been held for progressively shorter periods, so that the average holding period has been less than one week most of 2011. Barclays now state that VXX shares are suitable for investments of up to one week, whereas VXZ investors may have an horizon of up to one month. Naturally, some investors will be holding shares for much longer than average, especially in the VXX market where the exceptionally high trading volumes indicate a considerable number of very short-term speculative trades.

2.3   Sample statistics

Table 3 reports descriptive statistics for returns on the SPY, the spot VIX, VXX and VXZ and the three VIX futures series. The percentage change in a futures price already represents the excess return above the current risk-free rate earned on a futures position, but for the SPY and ETNs we adjust the percentage price change for the risk-free rate. For the VXX and VXZ prior to sub-sample 3 we use the indicative value derived from the S&P constant-maturity indices, or from VIX futures prices directly following the S&P methodology.

The different statistical characteristics of spot VIX and its futures are immediately apparent, with spot VIX having a much higher mean return and standard deviation and a positive Sharpe ratio (SR) which is typically greater than the SR of VIX futures. Due to volatility mean-reversion the contracts in the long-term series represent longer-term expectations of spot volatility, on average, and are therefore less variable than the shorter rollover series. The characteristics of VXX and the monthly rollover series are similar, the basic difference being only that the roll cost is taken all at once in the latter. In

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Table 3 : Summary Statistics: SPY, VIX, VIX Futures, VXX, and VXZ.

	Annualized Mean	Volatility	Sharpe Ratio	Skewness	Kurtosis
Sub-sample 1: April 1, 2004 - September 30, 2006					
SPY	5.82%	10.64%	0.547	-0.121	3.23
VIX	20.50%	87.06%	0.235	0.732	6.66
VIX monthly	-85.08%	37.76%	-2.253	0.235	8.13
VIX quarterly	-50.12%	31.83%	-1.575	0.814	10.04
VIX long-term	-50.30%	26.81%	-1.876	-0.539	8.32
VXX	-86.44%	35.23%	-2.453	-0.188	8.91
VXZ	-27.36%	20.30%	-1.347	-0.661	8.85
Sub-sample 2: October 1, 2006 - March 31, 2009					
SPY	-17.84%	30.76%	-0.580	0.499	12.96
VIX	124.81%	125.73%	0.993	1.381	10.81
VIX monthly	63.46%	72.12%	0.880	1.156	6.85
VIX quarterly	59.57%	59.07%	1.009	1.359	9.94
VIX long-term	36.82%	44.86%	0.821	0.798	7.20
VXX	53.88%	62.91%	0.856	0.881	5.91
VXZ	42.06%	35.15%	1.196	0.738	5.76
Sub-sample 3: April 1, 2009 - December 30, 2011					
SPY	18.39%	20.39%	0.902	-0.306	5.25
VIX	43.01%	118.39%	0.363	1.531	9.11
VIX monthly	-68.23%	72.44%	-0.942	1.474	8.95
VIX quarterly	-44.87%	56.83%	-0.790	1.132	8.37
VIX long-term	-16.71%	49.03%	-0.341	2.134	19.39
VXX	-72.97%	61.64%	-1.184	1.215	6.98
VXZ	-17.42%	31.75%	-0.549	0.787	6.89

Summary statistics for SPY, VIX, VXX and VXZ daily excess returns and daily returns on the three VIX futures series with rollover 5 days prior to expiry. Means, standard deviations and SR are annualized.

sub-sample 2 SPY had a negative average excess return whilst average returns on VIX futures were large and positive; but in sub-samples 1 and 3 this situation was reversed. The volatility of all VIX futures and ETNs was higher in sub-samples 2 and 3 than in the first, particularly so for the VXX, and since October 2006 the average volatility of the monthly rollover series has been about 70%. Although expected returns on VIX futures are becoming increasing difficult to predict their skew and kurtosis tend to be large and positive so extreme positive returns are more frequent than extreme negative returns.[12] During sub-sample 2 (the credit and banking crises) VIX futures yielded an average SR of around 1, whereas in sub-samples 1 and 3 they had very large negative mean returns. The VXZ clearly outperforms VXX in each sample, even in sub-sample 2 when the roll yield is positive. Thus, with the term structure being in contango most of the time, the larger roll costs on VXX are depressing the returns on this product considerably.

Clearly, large negative returns are built into trades on VIX futures and their ETNs. The reason for this is that the roll yield is almost always negative. The roll-yield effect

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[12]The high kurtosis on the long-term VIX futures in sub-sample 3 is due to frenetic trading on the contract rolled on 10 August 2011, at the onset of the Eurozone crisis. If we had rolled the contract on 3 August instead the kurtosis would drop to 7.75.

is greatest for short-term futures and their tracker ETNs, because the term structure is steepest at the short end. This negative roll yield implies a zero value for long-term investments in VIX futures and ETNs – for instance, the VXX lost 90% of its value between January 2009 and December 2011. Nevertheless, it may still be optimal for equity investors to diversify into these products, provided it is for no more than a few months, because there is a very large negative correlation between SPY returns and VIX returns. This correlation is pronounced in all three sub-samples: they range between -0.71 for the VIX long-term series in sub-sample 1, to -0.85 for the prompt VIX futures (monthly rollovers) in sub-sample 3. Correlations with SPY returns are generally significantly greater in magnitude when rolling into the prompt futures contract. This may explain why trading volumes have focused on the short-end of the term structure, even though the negative roll-yield effect depresses their expected return far more than it does medium-to long-maturity VIX futures and their tracker ETNs.

3 Ex–post analysis

We start with an analysis of the market conditions under which it would be ex-post optimal for long equity investors to diversify using VIX futures, first within the standard Markowitz (1952) framework and then allowing for skewness preference. Later on we present a comprehensive ex-ante analysis covering mean-variance optimality, skewness preference and the effect of personal views and equilibrium returns.

The allocation of funds between two risky assets, in this case SPY and VIX futures, and a risk-free asset (discount bond) may be considered in two stages: (i) find all MV efficient combinations of SPY and VIX futures; and (ii) find the optimal mix of one of these portfolios with the risk-free asset. The portfolio chosen from the stage (i) analysis is the tangency portfolio that when connected with the risk-free asset yields a linear efficient frontier with slope equal to the maximized SR; the optimal choice along this frontier in stage (ii) is the portfolio that maximizes investor's expected utility. In stage (i) we need not consider specific risk preferences because we are only concerned with the convex frontier in {expected return, standard deviation} space. In particular, we seek to identify the times and conditions in the past under which a MV optimal allocation that was positive to SPY would also be positive to VIX futures. Thus, our problem may be stated mathematically as:

$$\max_{w} SR = \frac{w'\hat{q}}{\sqrt{w'\hat{\Sigma}w}}, \quad w_s + w_v = 1, \quad w_s > 0, \, w_v \geq 0, \tag{2}$$

where $w = (w_s, w_v)'$, $\hat{q}$ is the vector of mean excess returns and $\hat{\Sigma}$ is their covariance matrix, both estimated ex post using historical data.

As will be discussed in detail later, the SR criterion is based on mean-variance optimization which assumes that portfolio returns are normal and the investor has an exponential utility with some level of risk aversion, γ. We extend our analysis to accommodate non-normal returns but retain the exponential utility assumption. In this case Hodges (1998) showed that the optimal allocation is that which maximizes the generalized Sharpe ratio (GSR) defined as $(-2\log(-\mathbb{E}(U)^*))^{1/2}$, where $\mathbb{E}(U)^*$ denotes the maximum expected utility. We keep the same constraints as in (2), and again use numerical optimization to search for the combination of w_s and w_v that give the maximum expected utility. Performance measurement is provided through (5), presented in detail in Section 5.

The solution to the ex-post optimization problem (2) is achieved with $w_v = 0$ in periods 1 (April 1, 2004 - September 30, 2006) and 3 (April 1, 2009 - December 31, 2011), with the exception of the long-term series in period 3 where the optimal allocation on VIX would have been 20.91%. However, in period 2 (October 1, 2006 - March 31, 2009) the SR of the portfolio is substantially greater when the long VIX futures positions summarized in the upper part of Table 4 are taken. In that period the ex-post optimal VIX exposure was 65.83% using the quarterly rollover series, 66.97% using the monthly rollover series and a complete switch to volatility ($w_v = 1$) for the long-term rollover series. The highest SR of 1.0348 was achieved by a VIX investment based on the quarterly rollover strategy.

For the skewness-aware investor we maximize GSR, rather than SR and obtain similar results for sub-samples 1 and 3 in that the investor would never find it ex-post optimal to diversify into VIX futures, with one exception: if his risk aversion is relatively high ($\gamma = 4$) then a small position (less than 8%) in long-term VIX futures slightly improves on the ex-post SPY-only GSR during sub-sample 3. In sub-sample 2 the skewness-aware investor would have been better off allocating all funds to VIX futures if his risk tolerance is high ($\gamma = 1$), due to the combination of high positive skewness and high negative correlation between SPY and VIX futures. On the other hand, the relatively risk averse ($\gamma = 4$) investor is still attracted by positive skewness but more deterred by the high volatility of VIX futures, and would have found it best, ex post, to take the positions shown in the lower part of Table 4.

Figure 7 depicts the evolution of an investment on October 1, 2006 of $100 invested in: (a) the mean-variance optimal combination of SPY and the quarterly rollover VIX futures, with the allocation to the futures in the risk-free asset, an equivalent exposure being taken as a long position on VIX futures; (b) the GSR optimal combination of SPY and the same VIX futures, for the skewness-aware investor; and (c) $100 in SPY alone. The performance of (a) and (b) are similar, and they diverge from (c) especially at the onset of the banking crisis in September 2008. Except during a crisis the diversified portfolios tend to lose value. Overall, the SPY–VIX futures mean-variance optimal portfolio produced an annualized

Table 4 : Ex-post Analysis for Optimal SPY–VIX Portfolios.

VIX futures	w_s	w_v	SR/GSR with VIX	SR/GSR SPY only
Panel A: Mean-Variance Investor				
Monthly rollover	33.03%	66.97%	0.8914	
Quarterly rollover	34.17%	65.83%	1.0348	−0.5799
Long-term rollover	0.00%	100.00%	0.8208	
Panel B: Skewness-Aware Investor, $\gamma = 4$				
Monthly rollover	49.35%	50.65%	0.8820	
Quarterly rollover	39.27%	60.73%	1.0505	−0.5779
Long-term rollover	32.44%	67.56%	0.7902	

Optimal weights on SPY and VIX futures in the tangency portfolio for a long equity investor, calculated ex post between October 1, 2006 - March 31, 2009; Sharpe ratio (SR) or generalized Sharpe ratio (GSR) for the ex-post optimal diversified portfolios; and SR or GSR for the portfolio containing SPY alone.
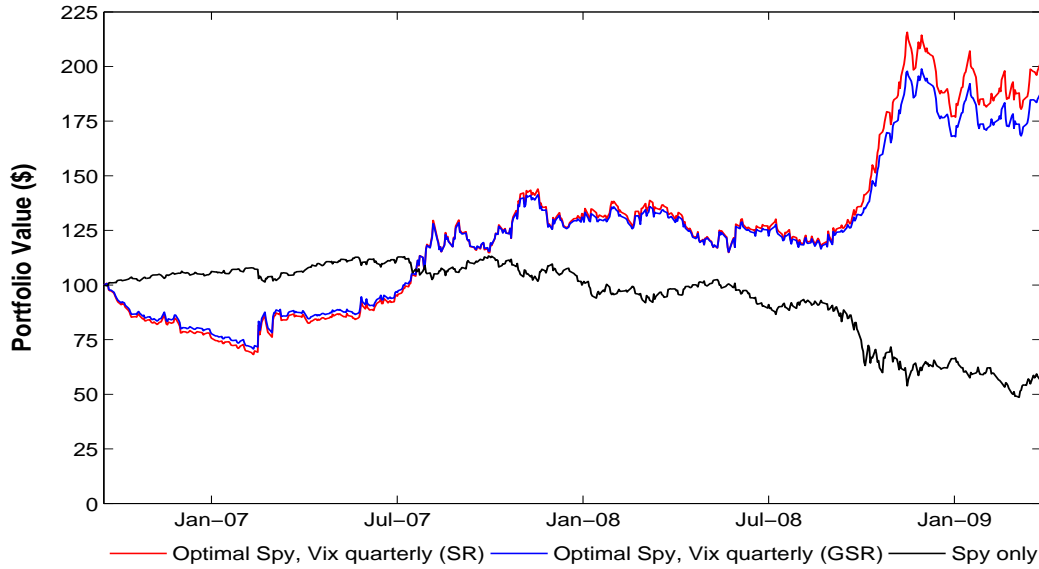
Figure 7 Comparison of Equity-Only and Diversified Portfolio, Ex Post
Portfolio growth of a theoretical $100 investment in SPY alone and in the ex-post optimal diversified portfolio. The diversified portfolio gains during the credit crisis in 2007 and again during the onset of the banking crisis in September 2008.

excess return of over 33% with 32% volatility, compared to 29% with 28% volatility for the skewness-aware optimal portfolio and an annualized negative excess return of −18% with 31% volatility for the SPY alone. However, one should be very skeptical about such results. They do not extend to either of the sub-samples before and after this period, and the results are highly sensitive to the historical mean inputs for VIX futures returns, which have a very large sampling error because VIX futures returns are excessively volatile.

4 Mean-Variance Optimality

When an investor allocates between SPY, VIX futures, and a discount bond according to the mean-variance (MV) criterion the unconstrained solution is $w = \gamma^{-1}\Sigma^{-1}q$, where:

$w = (w_s, w_v)'$ describes the allocation to the risky assets (w_s and w_v are not constrained to sum to 1, the allocation being completed with the residual invested in the risk-free asset); γ denotes the investor's coefficient of risk aversion; $q = (q_s, q_v)'$ is the vector of expected SPY excess returns and VIX futures returns; and Σ denotes their covariance matrix with elements σ_s^2, σ_v^2 and σ_{sv}. The solution may be rewritten:

$$w_s = \gamma^{-1} |\Sigma|^{-1}(\sigma_v^2 q_s - \sigma_{sv} q_v) \quad , \quad w_v = \gamma^{-1} |\Sigma|^{-1}(\sigma_s^2 q_v - \sigma_{sv} q_s), \tag{3}$$

where $|\Sigma| = \sigma_s^2 \sigma_v^2 - \sigma_{sv}^2$ is the determinant of Σ. Since $\gamma > 0$, $|\Sigma| > 0$ and $\sigma_{sv} < 0$, requiring both $w_s > 0$ and $w_v > 0$ simultaneously results in the condition:

$$q_v > \max\left(\frac{q_s \sigma_v^2}{\sigma_{sv}}, \frac{q_s \sigma_{sv}}{\sigma_s^2}\right). \tag{4}$$

The right-hand side of (4) is the mean–variance optimal diversification threshold, i.e. the expected return on VIX futures that would justify an equity investor to take a long position in VIX futures, for the purpose of diversification. Since $\sigma_{sv} < 0$ this threshold is positive iff $q_s < 0$ and it does not depend on the investor's risk aversion, γ.

Now consider an investor who uses the previous month of historical data on SPY and prompt VIX futures to make forecasts for q and Σ from one rebalancing point to the next. Rebalancing is either every month, 5 business days before expiry of the futures, or every week. At each rebalancing point he compares his forecast q_v of returns on prompt VIX futures with the optimal diversification threshold (4), and if q_v is greater than this threshold he holds the ex-ante MV optimal portfolio of the risk-free asset with long positions on both SPY and VIX futures; otherwise he holds SPY alone, or the optimal combination of SPY and the risk-free asset; until the next rebalancing point where the optimization is repeated. We find that the threshold is exceeded at 16% of the rebalancing point during sub-sample 1, at 37% of the point in sub-sample 2 and at 27% of the points in sub-sample 3. Thus, the investor would perceive that it is optimal to diversify into prompt VIX futures quite frequently, even during stable trending and range-bounded market conditions.

Figure 8 compares the out-of-sample performance of the optimal portfolio with two possible alternatives, i.e. holding SPY alone for the entire period, or holding a MV optimal combination of SPY and the risk-free asset from one rebalancing point to the next. Using the last 250 observations, we compute the SR of (a) the optimally diversified portfolios, and (b) the SPY only and (c) the (SPY, risk-free asset) MV optimal portfolios. The lines in the figure depict the difference [(a) − (b)] as solid red and blue lines and the difference [(a) − (c)] as dashed red and blue lines, with red indicating weekly rebalancing and blue indicating monthly rebalancing. Three observations can be made: (i) The ex-ante
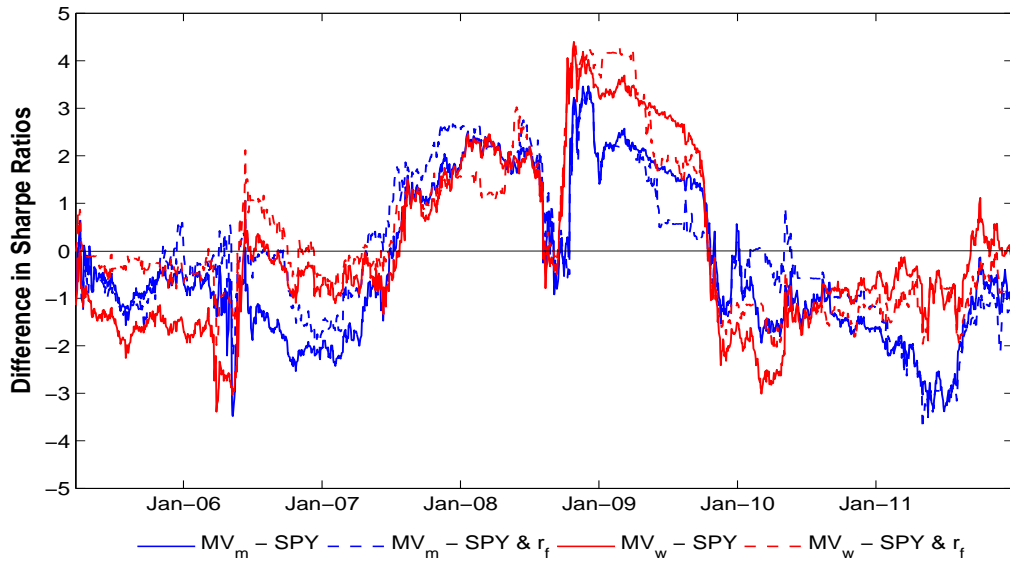
Figure 8 Difference between SR on optimally diversified portfolio using prompt VIX futures and SR on SPY only (solid line) or SR on (SPY, risk-free) MV optimal portfolios (dashed lines).
Annualized Sharpe ratios computed using the last 250 observations based on MV optimization with historical forecasts for covariance matrix and expected returns, using one-month look-back. Portfolios are re-optimized weekly (w) or monthly (m). Transactions costs not included.

optimal diversified portfolios only outperformed equity during the credit crisis of 2007 and the banking crisis of 2008–9. At all other times it was better to keep the SPY holding, or to allocate some capital to the risk-free asset, but never to diversify into prompt VIX futures; (ii) Comparing the solid and dashed lines, the qualitative conclusions are the same whether performance is measured relative to the (SPY, risk-free) MV optimal portfolio, or whether it is relative to holding the SPY alone; and (iii) More frequent rebalancing does not automatically lead to a better performance. Since these results do not include transactions costs which are currently about 25bp per round trip, but were previously much higher, it is better to rebalance monthly than weekly.

In Figure 8 all forecasts were historical, based on the previous month of daily data, and diversification was only considered using prompt VIX futures, because these have greater negative correlation with SPY returns than longer-maturity futures. However, similar figures (for other look-back periods and using different VIX futures) lead to similar conclusions. That is, the only periods since the inception of VIX futures when it was beneficial to the MV equity investor to diversify into VIX futures was during the credit crisis and the banking crisis. Also, given the transactions costs, it is better to rebalance the portfolio monthly rather than weekly. Finally, it makes little difference whether we compare performance relative a SPY-only holding, or the optimal holding of SPY and the riskless asset. Hence, for brevity, we present all further results on this basis.

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5 Skewness Preference

Mean-variance optimization is based on two main assumptions: that portfolio returns r are normally distributed and that the investor has an exponential utility $U(r) = -\exp(-\gamma\, r)$ with risk aversion γ. The optimal portfolio weights are the same whether we maximize the expected utility $\mathbb{E}[U(r)]$ or its certainty equivalent, $\mathrm{CE} = -\gamma^{-1}\log(\mathbb{E}[U(r)])$. Indeed, the CE is the standard mean-variance criterion when returns are normal.

But Table 3 provides strong empirical evidence that returns on VIX futures are not normal, in particular because they have high positive skewness. So now we extend our analysis to accommodate non-normal returns, still using the exponential utility assumption. In this case the CE has no simple analytic form, although several approximations exist, so we compute the ex-ante allocations for optimal portfolios of SPY and VIX futures numerically. We shall present results based only on a one-month in-sample period for brevity and for consistency with previous results, but the qualitative conclusions are similar when we use a 3-month look-back period. We take the empirical joint distribution of returns to derive portfolio returns for a given weights vector w and optimize these weights by maximizing the expected (exponential) utility of portfolio returns. As before, we rebalance the portfolio whenever the VIX futures is 5 business days from expiry, and compare the results when investing in either prompt, mid-term or long-term VIX futures.

For the out-of-sample performance assessment we need a measure that converges to the ordinary SR as the skewness and excess kurtosis tend to zero. Just as the optimal allocation for a MV investor can be obtained by maximizing the SR, Hodges (1998) showed that the optimal allocation for a general investor is that which maximizes the generalized Sharpe ratio (GSR) defined as $(-2\log(-\mathbb{E}^*[U(r)]))^{1/2}$, where $\mathbb{E}^*[U(r)]$ denotes the maximum expected utility obtained over all possible portfolio weights. The GSR is a simple transformation of the certainty equivalent which indeed converges to the SR as returns become more normal. So that we can compare the GSR with the SR, which can be negative as we have seen in Figure 8, we measure performance for the skewness-aware investor using the standard approximation (see Alexander (2008), for instance)

$$\mathrm{GSR} \approx \mathrm{SR}\{1 + \frac{\tau}{3}\mathrm{SR} - \frac{\kappa}{6}\mathrm{SR}^2\}^{1/2}. \tag{5}$$

Table 5 records the frequency of rebalancing points when it is optimal to diversify into VIX futures, and measures the out-of-sample performance of each optimally-diversified portfolio using the GSR and, for reference, the ordinary SR in annualized terms, for both MV and skewness-aware investors. The results show that the skewness-aware investor diversifies more frequently than the MV investor, but the out-of-sample performance of their portfolios is very similar in aggregate. Outperformance of the SPY-only portfolio is

again only evident during sub-sample 2, and during that period more-risk averse skewness-aware investors (with $\gamma = 4$) do better than less risk-averse skewness-aware investors (with $\gamma = 1$) and both do better than MV variance investors. The conclusions are the same, whether we use the SR or the GSR approximation (5) to measure performance.

Table 5 : Ex-ante Diversification for Mean-Variance and Skewness-Aware Investor.

| | Mean-Variance | | Skewness-Aware | | | | | |
| | | | $\gamma = 1$ | | | $\gamma = 4$ | | |
Subsample	Frequency	SR	Frequency	SR	GSR	Frequency	SR	GSR
Panel A: VIX monthly								
1	16%	-0.515	16%	-0.751	-0.746	24%	-0.641	-0.638
2	37%	1.399*	60%	1.535*	1.520*	60%	1.633*	1.587*
3	27%	-0.610	30%	-0.826	-0.827	36%	-0.489	-0.489
Panel B: VIX quarterly								
1	20%	-0.587	20%	-0.381	-0.378	20%	-0.330	-0.328
2	36%	0.460*	64%	0.522*	0.518*	43%	0.653*	0.647*
3	18%	-0.545	27%	-0.576	-0.577	27%	-0.518	-0.519
Panel C: VIX long-term								
1	0%	0.547	0%	0.546	0.547	0%	0.546	0.547
2	20%	-0.616	40%	-0.191*	-0.199*	40%	-0.530*	-0.531*
3	40%	0.906*	60%	0.531	0.531	60%	0.674	0.675

The frequency of rebalancing points at which it is ex-ante optimal to diversify into VIX futures, and the annualized GSR and SR for a SPY investor that diversifies with VIX futures at each rebalancing point if this maximizes expected utility, or otherwise remains fully invested in SPY. * denotes exceedance of the SPY-only GSR or SPY-only SR, respectively. SR (GSR) for SPY is 0.547 (0.546), -0.580 (-0.578) and 0.821 (0.818) for the three sub-samples under consideration.

Although this is not evident from Table 5, the GSR criterion can diverge significantly from the SR criterion. Figure 9 compares the difference in rolling SRs against SPY for the MV investor in prompt VIX futures (in blue, and as in figure 8 this is based on the last 250 business days) with that for the skewness-aware investor (in red) and the difference in rolling GSRs against SPY for the skewness-aware investor is depicted by the dotted red line. Even though the skewness-aware investor typically chooses to diversify more frequently than the MV investor, the out-of-sample performance of the optimally-diversified portfolios is very similar when measured by the SR. However, there are a few very large positive or negative VIX futures returns during sub-sample 2, which impact the GSR much more than the SR, so that the two measures actually deviate significantly, when computed on a rolling basis. Nevertheless, their overall averages are similar: 1.633 for the SR and 1.587 for the GSR, as shown in the last two columns of table 5.

6   Personal Views and Equilibrium Returns

Conclusions based on mean–variance optimization should be regarded with caution, because errors in the model parameters can lead to extremely unstable allocations. This
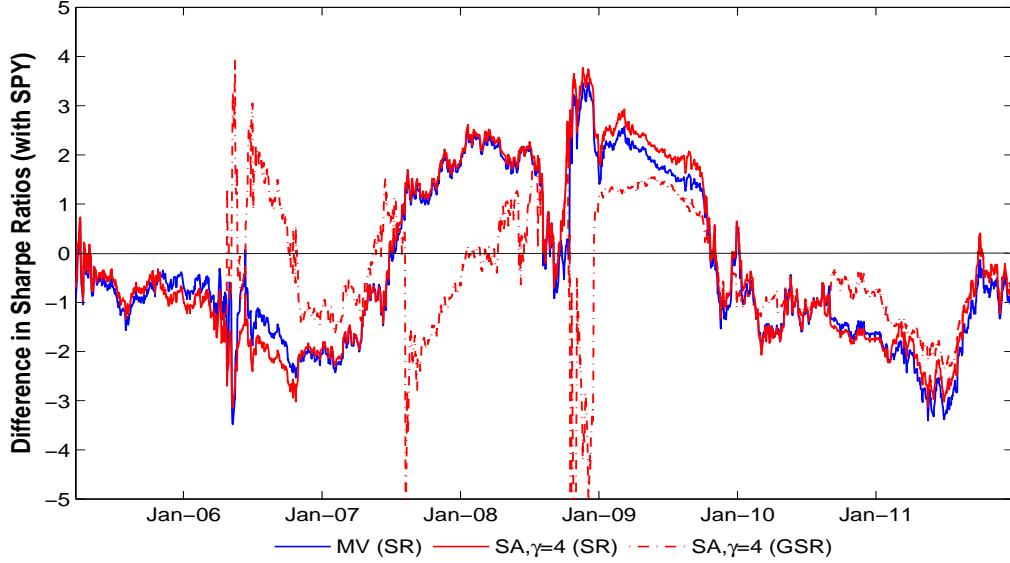
Figure 9 Difference between SR (or GSR) on optimally diversified portfolio using prompt VIX futures and SR (or GSR) on SPY only.
Annualized Sharpe ratios computed using the last 250 observations with historical forecasts for covariance matrix and expected returns, using one-month look-back. Difference between the two SRs [SPY-VIX futures minus SPY only] for MV investor (blue line) and skewness-aware investor (red line). Difference between the two GSRs for skewness-aware investor shown by dotted red line. Transactions costs not included.

has been thoroughly discussed in the literature – see Chopra and Ziemba (1993) among others – and errors in the expected returns have much greater effect on results than errors in the covariance matrix forecast.

Black and Litterman (1992) argue that investors should not base their decisions entirely on historical data, or more generally on their own personal views about expected returns. Any long-term investment should also take account of equilibrium expected returns, and if their personal views are highly uncertain then the resulting allocations will be more stable than MV allocations, because they will not deviate too far from the equilibrium returns.

In this section we consider the VIX futures diversification problem from the perspective of an equity investor that uses the classical BL model, as interpreted and implemented by He and Litterman (1999). This assumes that asset returns follow a normal distribution with unknown expected returns vector μ and covariance matrix Σ. An expression for the posterior distribution of returns is obtained by conjugating two normal distributions, one for the investor's personal views and the other for the equilibrium returns.

In a capital asset pricing model (CAPM) market equilibrium, all investors hold the market portfolio w^{eq} and share the same beliefs about expected returns, encapsulated by a normal prior distribution with mean Π and covariance matrix $\tau\Sigma$, where Σ is the historical

covariance matrix. The parameter τ is a positive constant representing the uncertainty in the prior distribution for expected returns. In addition to prior beliefs, which are based on equilibrium returns, an individual investor holds his own, subjective views about the distribution of expected returns. These views might be about the distribution of expected returns on individual assets, and/or about certain portfolios of these assets. The views are represented using a matrix P, such that $P\mu$ follows a normal distribution with mean vector Q and diagonal covariance matrix Ω, which defines the investor's confidence in each view. Now, blending equilibrium with subjective views yields a posterior normal distribution for expected returns with mean:

$$\mu^{BL} = \left[(\tau\Sigma)^{-1} + P'\Omega^{-1}P\right]^{-1}\left[(\tau\Sigma)^{-1}\Pi + P'\Omega^{-1}Q\right] \tag{6}$$

and covariance matrix $M = [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}$. Then the assets' actual returns follow a normal distribution with the same mean, μ^{BL} but covariance matrix $\bar{\Sigma} = \Sigma + M$. Finally, He and Litterman (1999) apply the MV optimizer to the posterior distribution for actual returns to obtain the solution for the unconstrained optimal portfolio weights:

$$w^{BL} = (1+\tau)^{-1}(w^{eq} + P'\Lambda), \tag{7}$$

where $w^{eq} = \gamma^{-1}\Sigma^{-1}\Pi$ are the equilibrium portfolio weights with $\gamma > 0$ being the coefficient of absolute risk aversion, and where the vector Λ is given by:

$$\Lambda = \gamma^{-1}(1+\tau)B^{-1}Q - B^{-1}P\Sigma w^{eq}, \tag{8}$$

with $B = P\Sigma P' + \tau^{-1}(1+\tau)\Omega$.

All futures contracts are in zero net supply, i.e. for every buy order there must be a corresponding sell order, so they have zero weight in the equilibrium portfolio. Hence, in our case $w^{eq} = (1,0)'$ corresponds to an equity-only reference portfolio, against which we assess the BL alternative of a diversified portfolio.[13] Let Σ denote the covariance matrix of SPY excess returns and VIX futures returns with elements σ_s^2, σ_v^2, and σ_{sv}. In the BL model the equilibrium returns are obtained via reverse MV optimization, so that $\Pi = \gamma\Sigma w^{eq}$ and, knowing w^{eq}, γ and Σ we have:

$$\Pi := (\mu_s^{eq}, \mu_v^{eq})' = \gamma(\sigma_s^2, \sigma_{sv})'. \tag{9}$$

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[13]Although our analysis does not consider bonds or any other risky assets in the equilibrium portfolio, such as real estate or commodities, we continue to refer to Π as the "equilibrium" expected returns vector. But some might argue that, when the BL model is applied to a context that excludes one or more of the risky assets in the economy, a more appropriate term for the prior would be a "reference" rather than an "equilibrium" portfolio.

For illustration we depict the equilibrium returns for SPY and VIX futures with monthly rollover calculated according to (9) and based on an historical covariance matrix with a 1-month in-sample period. These are displayed in Figure 10. We report the returns between each rebalancing point in monthly terms, only for $\gamma = 1$, since equilibrium returns for other values of γ can easily be deduced from these, given (9). Equilibrium expected returns are positive for equity, negative for volatility, and they have a very large negative correlation. The same general features are displayed by equilibrium returns for other VIX futures, and based on different sample sizes for the historical covariance matrix; results are available on request.
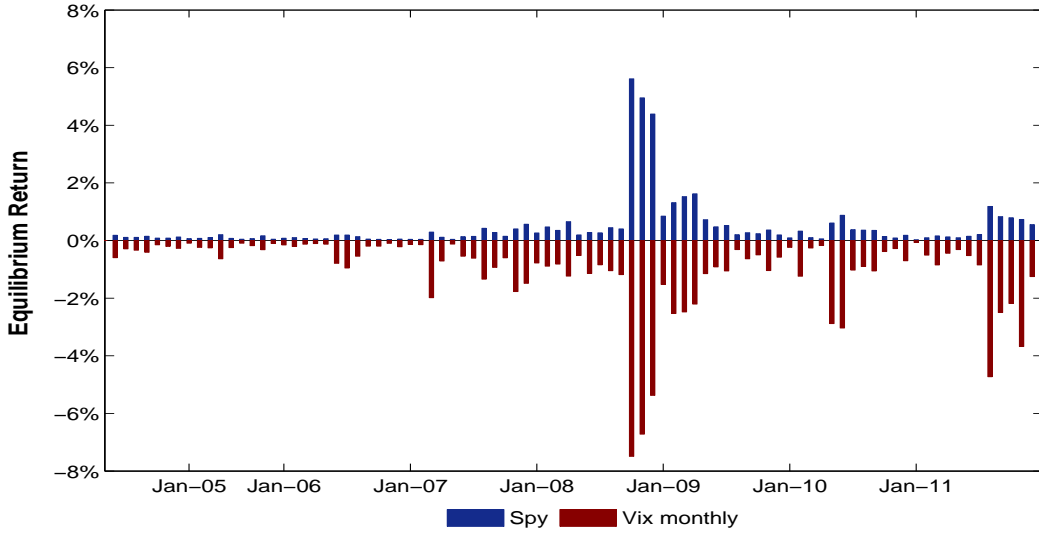

Figure 10 Equilibrium Returns
The equilibrium expected returns on SPY and VIX futures between monthly rebalancing points. We assume the investor has risk aversion $\gamma = 1$, employs the VIX monthly rollover series and, uses 1 month as the in-sample period.

Now we derive the optimal diversification threshold in the Black-Litterman framework. Black and Litterman (1992) and He and Litterman (1999) assume that Ω is a diagonal matrix. Meucci (2005) relaxed this assumption, suggesting that Ω be directly proportional to $P\Sigma P'$. We meld the two assumptions by setting

$$\Omega = \eta \operatorname{diag}(P\Sigma P'). \tag{10}$$

Thus the uncertainty in each personal view is proportional to the historical variance, with the same proportionality constant η.[14] Consider two possibilities for the current views

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[14]Note that a more restricting assumption, where η is set equal to τ, has been used in the implementations of He and Litterman (1999) and Da Silva, Lee, and Pornrojnangkool (2009). We prefer to include η as a free parameter so that we may investigate how the BL solution behaves as the investor becomes relatively more or less confident in his own views, i.e. as η decreases or increases, respectively, but τ

of the investor: (a) only one view on VIX futures and (b) views on both SPY and VIX futures. In each case we compute the minimum expected return on VIX futures that will justify a long position for them in the BL portfolio.

6.1 One view on VIX

Under this scenario the matrix of views becomes $P = (0,1)$ and the vectors Λ and Q each have only one element, i.e. $\Lambda = \lambda_v$ and $Q = q_v$. We also have

$$P\Sigma P' = \sigma_v^2, \quad P\Pi = \mu_v^{eq}, \quad \Omega = \eta\sigma_v^2, \quad B = \left(1 + \eta + \eta\tau^{-1}\right)\sigma_v^2 > 0. \tag{11}$$

Since $w^{eq} = (1,0)'$ the BL allocation from (7) becomes:

$$w_s^{BL} = (1 + \tau)^{-1} \quad , \quad w_v^{BL} = (1 + \tau)^{-1}\lambda_v. \tag{12}$$

Clearly, $w_s^{BL} > 0$ always, and requiring $w_v^{BL} > 0$ yields the condition: $\lambda_v > 0$, or equivalently $\gamma\lambda_v > 0$. Substituting (11) in (8) yields

$$\gamma\lambda_v = [(1 + \tau)q_v - \mu_v^{eq}]\left(1 + \eta + \eta\tau^{-1}\right)^{-1}\sigma_v^{-2},$$

so $w_v^{BL} > 0$ if, and only if $q_v > (1 + \tau)^{-1}\mu_v^{eq}$. Thus a fund manager having no personal views on SPY will allocate positively to VIX futures whenever his expected return is greater than the equilibrium return, scaled for the uncertainty about the prior that is captured by the parameter τ. Using (9), the condition becomes:

$$q_v > (1 + \tau)^{-1}\gamma\sigma_{sv}. \tag{13}$$

The optimal diversification threshold on the right-hand side of (13) is negative and it decreases as γ increases. Thus, for a given expected return q_v, diversification is more likely to occur by a more risk-averse investor.[15]

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remains fixed.

[15]However, it is independent of η, and would also be independent of τ under the modification of the BL model suggested by Pézier (2007), which argues that, since $M = \tau\Sigma$ in the absence of any personal views, we should set $\Pi = \gamma(1 + \tau)\Sigma w^{eq}$ rather than $\Pi = \gamma\Sigma w^{eq}$, so that in (6) Π should be replaced by $(1 + \tau)\Pi$ and (7) becomes simply $w^{BL} = w^{eq} + P'\Lambda$. See Pézier (2007) for further details. The factor $(1 + \tau)^{-1}$ would not appear in (7) and consequently nor in (13), so that diversification would be optimal simply when the expected return on volatility exceeds its equilibrium return.

6.2 Views on both SPY and VIX

We now consider the case where current views on both SPY and VIX returns are expressed directly, i.e. P equals the identity matrix. Since $P\Sigma P' = \Sigma$ the BL allocation (7) becomes:

$$w_s^{BL} = (1+\tau)^{-1} \left[\gamma^{-1} |\Sigma|^{-1} (\sigma_v^2 \mu_s^{eq} - \sigma_{sv} \mu_v^{eq}) + \lambda_s \right],$$

$$w_v^{BL} = (1+\tau)^{-1} \left[\gamma^{-1} |\Sigma|^{-1} (\sigma_s^2 \mu_v^{eq} - \sigma_{sv} \mu_s^{eq}) + \lambda_v \right],$$

where $|\Sigma| = \sigma_s^2 \sigma_v^2 - \sigma_{sv}^2 > 0$ is the determinant of Σ. Hence, the conditions for a positive allocation to both SPY and VIX futures ($w_s^{BL} > 0$ and $w_v^{BL} > 0$) become:

$$\sigma_{sv} \mu_v^{eq} - \sigma_v^2 \mu_s^{eq} < \gamma |\Sigma| \lambda_s, \quad \text{and} \quad \sigma_{sv} \mu_s^{eq} - \sigma_s^2 \mu_v^{eq} < \gamma |\Sigma| \lambda_v. \tag{14}$$

After some algebra, given in the Appendix with some remarks about the result, we have:

Proposition 1. *When an investor has asset-specific views on SPY and on VIX futures with means $Q = (q_s, q_v)'$ then positive allocations $w_s^{BL} > 0$ and $w_v^{BL} > 0$ are guaranteed if, and only if,*

$$q_v > \max\left(a_3/a_1, a_4 \right) \tag{15}$$

where

$$\begin{bmatrix} a_3 \\ a_4 \end{bmatrix} = q_s \begin{bmatrix} \sigma_{sv}/\sigma_s^2 \\ a_1 \sigma_v^2/\sigma_{sv} \end{bmatrix} + a_2 \begin{bmatrix} (a_1+1)\sigma_{sv}/\sigma_s^2 & -(a_1+\beta^2) \\ (a_1+\beta^2)\sigma_v^2/\sigma_{sv} & -(a_1+1) \end{bmatrix} \begin{bmatrix} \mu_s^{eq} \\ \mu_v^{eq} \end{bmatrix}$$

with $a_1 = \tau^{-1}\eta + \eta + 1$, $a_2 = \tau^{-1}\eta(1-\beta^2)^{-1}$ and $\beta = \sigma_{sv}/(\sigma_s \sigma_v)$.

6.3 Ex-ante Optimal Diversification Frequency

For the BL diversification thresholds (13) and (15) we set Σ equal to an historical covariance matrix based on a sample of size n. Here we report results only for the 1-month in-sample period. Those for a 3-month period are qualitatively very similar and so are not reported for brevity. Black and Litterman (1992) propose that τ should be set close to zero, as the investor is more certain about the distribution of expected returns than for returns themselves. Since, under the i.i.d. assumption, the variance of a sample mean is inversely proportional to the sample size, we follow He and Litterman (1999) and Blamont and Firoozy (2003) and set $\tau = n^{-1} = 0.05$ assuming $n = 20$.[16] We consider two possible values for η, viz. $\eta = \tau$ and $\eta = 1$. When $\eta = 1$ the investor's personal views are held

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[16]Note that $\tau = 0.015$ for a 3-month in-sample period.

with far greater uncertainty than the equilibrium returns, so the optimal allocations will deviate from the equilibrium portfolio less than they do when $\eta = \tau$.
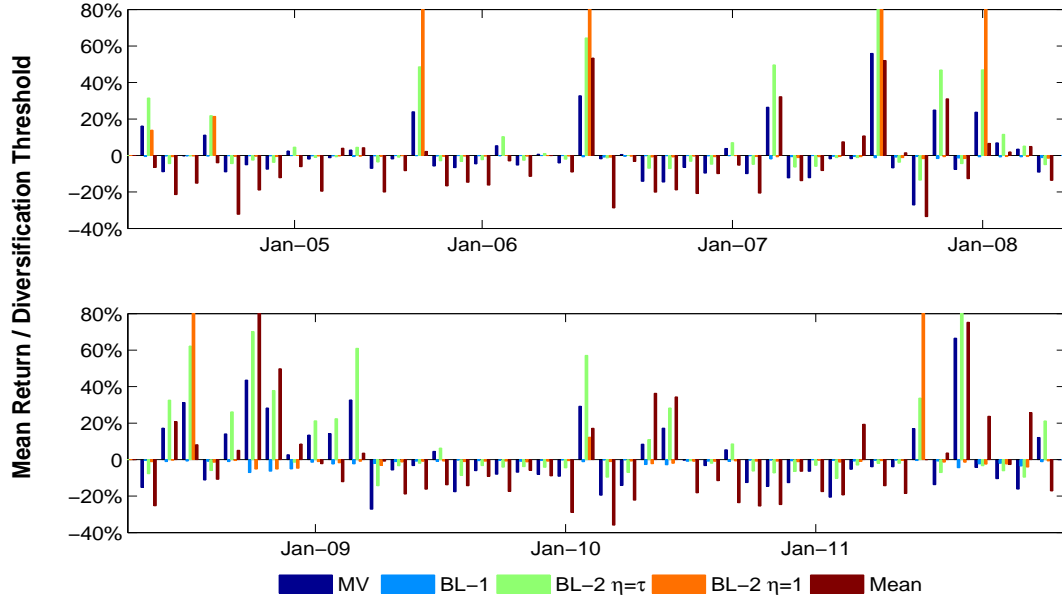

Figure 11 Optimal Diversification Thresholds
The four optimal diversification thresholds based on (4), (13), and on (15) with $\eta = \tau$ and with $\eta = 1$, and the mean daily return since the last rebalancing. The investor has $\gamma = 1$, employs a 1-month in-sample period and rebalances his portfolio on the rollover days of VIX futures (i.e. monthly). The thresholds for (15) are sometimes cut off the graph, being greatly in excess of the top of the vertical scale.

Figure 11 depicts the optimal diversification thresholds under the BL approach with one view on VIX futures only, and with two views on VIX futures and SPY, and the MV optimal diversification thresholds, for comparison. These thresholds are typically negative but occasionaly very large and positive, especially during the credit and banking crises. These results are for an investor with $\gamma = 1$ who employs a 1-month in-sample period and rebalances his allocation to SPY and VIX futures simultaneously on the same days that rollovers VIX futures (in this case, monthly). The smallest and most stable diversification thresholds are derived from the BL model with a view on volatility alone. From Remark 1 in the appendix, the BL threshold based on both equity and volatility views will be similar to the MV diversification threshold when η is small – e.g. when $\eta = \tau$ the BL thresholds are similar but the BL are slightly above the MV thresholds. Much of the time the BL model with two individual views on SPY and VIX futures returns provides thresholds that are similar to the one-view case, but in a handful of cases the thresholds are extremely large and positive. The figure also depicts (in brown) the in-sample mean daily return on VIX futures, computed since the last rebalancing. Even though these are very often negative, they exceed the diversification thresholds in many cases. So if the investor has a "naïve" approach where the VIX futures return forecast is the in-sample

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mean return, volatility diversification could frequently be considered optimal.

The success of any ex-ante diversification strategy depends on the quality of the forecasts for the returns and covariance matrix. We aim to identify which variables must be predicted accurately so that volatility diversification will be beneficial to the investor. We disregard the case that the SPY realized return can be forecast accurately (as this facility would preclude the need to diversify) so the expected return on SPY is set equal to its in-sample mean return. For the other forecasts we consider two hypothetical extremes: the naïve forecasts, which are the historical mean return and the historical covariance matrix, and the exact forecasts where the investor forecasts the realized return on VIX futures and SPY–VIX futures covariance matrix exactly, until the next rebalancing point. By considering combinations of naïve (H) and exact (R) forecasts for the VIX futures return q_v and the covariance matrix Σ we identify which of these quantities we must forecast accurately so that the optimally diversified portfolios will out-perform equity in an out-of-sample analysis.

Table 6 : Frequency of Optimal Volatility Diversification.

| (q_v, Σ) | $\gamma = 1$ | | | | $\gamma = 4$ | | | |
	(H,H)	(H,R)	(R,H)	(R,R)	(H,H)	(H,R)	(R,H)	(R,R)
Sub-sample 1: April 1, 2004 - September 30, 2006								
BL–1	16%	16%	20%	16%	28%	16%	24%	20%
BL–2 $\eta = \tau$	8%	8%	20%	20%	12%	12%	20%	20%
BL–2 $\eta = 1$	8%	8%	20%	20%	20%	16%	24%	20%
Sub-sample 2: October 1, 2006 - March 31, 2009								
BL–1	57%	57%	60%	57%	60%	60%	60%	60%
BL–2 $\eta = \tau$	23%	33%	30%	30%	27%	37%	47%	40%
BL–2 $\eta = 1$	47%	47%	53%	50%	60%	60%	60%	60%
Sub-sample 3: April 1, 2009 - December 30, 2011								
BL–1	24%	24%	27%	27%	24%	27%	30%	30%
BL–2 $\eta = \tau$	18%	21%	27%	33%	21%	21%	27%	33%
BL–2 $\eta = 1$	24%	21%	24%	27%	24%	27%	30%	30%

The proportion of rebalancing periods when diversification is perceived as optimal, i.e. q_v exceeds the lower bound given by (4), (11) or (15), respectively, for the MV, BL (1 view) and BL (2 views) optimization models. For (q_v, Σ) the investor uses either the historical estimate (H) or forecasts the exact value realized until the next rebalancing point (R). Results are based on monthly rebalancing with a 1-month in-sample period for historical forecasts. Figures for MV are the same for both $\gamma = 1$ and for $\gamma = 4$ because (4) is independent of γ.

Table 6 reports the proportion of rebalancing times when diversification is perceived as optimal, i.e. the expected return q_v exceeds the diversification threshold given by the model. Results are disaggregated according to the forecast quality of both q_v and Σ, according to the naïve or exact forecasts, and for each optimization model the proportion corresponding to a threshold derived from $\gamma = 1$ is followed by the proportion corresponding to $\gamma = 4$. Regarding the full sample results, diversification becomes more frequent

when using the higher value of γ in the BL model. In the case that the investor has views on both SPY and VIX futures and $\eta = \tau$, he tends to diversify less frequently than when using other models. For all models the diversification frequency increases when the VIX futures return can be forecast precisely. By contrast, precise forecasting of the covariance matrix may even decrease the diversification frequency.

Although (4) is independent of γ, the diversification frequency could increase with γ in the BL models. In fact, we often find little or no increase in diversification frequencies as we increase γ from 1 to 4 in the table. For each model–forecast pair the least (most) diversification occurred during sub-sample 1 (sub-sample 2). The maximum, where holding VIX futures is perceived as optimal at 60% of the rebalancing points, occurs when using the BL model with one view on VIX futures alone, or with relatively uncertain views on both VIX futures and SPY, during sub-sample 2. During sub-sample 3 volatility diversification would occur in 18% – 36% of the rebalancing periods, depending on the investor's risk tolerance and the forecasts used.

6.4   Out-of-Sample Performance

Table 7 reports the SR for the optimally diversified portfolios based on their out-of-sample performance. The investor's forecasts for q_v and Σ are specified in the second row of the table: again we suppose he employs either naïve (historical, H) or exact (realized, R) forecasts, so the proportion of rebalancing periods that diversification is perceived as optimal is specified in Table 6. We use two possible values for the coefficient of risk aversion ($\gamma = 1$ and 4). We attach an asterisk (*) to indicate that the SR of the optimally diversified portfolio exceeds the SR for the SPY-only portfolio, which is 0.55 in sub-sample 1, -0.58 in sub-sample 2, and 0.82 in sub-sample 3.

The results provide no evidence that performance is improved by having a precise forecast for the SPY–VIX futures covariance matrix Σ – indeed this may even depress the SR in an out-of-sample setting. By contrast, every model yields SR exceeding SPY-only SR when q_v equals the realized return, but when q_v equals the historical return it is only during sub-sample 2, covering the credit and banking crises, that we see a consistent superior performance for the diversified portfolios, with the best results obtained using the prompt VIX futures as the diversification instrument. Outside of this period the performance is typically worse than holding SPY alone. Even when the covariance matrix forecast is exact there is only one instance when the diversified portfolio would out-perform a holding in SPY alone, and that is using the BL model with highly uncertain views on expected returns ($\eta = 1$): a SR of 0.92 is obtained when an investor who is only mildy risk-averse ($\gamma = 1$) diversifies into prompt VIX futures during sub-sample 3. In all other cases the SR of the diversified portfolio is less the SR of holding SPY alone.

Table 7 : Sharpe Ratios of Diversified Portfolios.

Sub-sample	(q_v, Σ)	$\gamma = 1$				$\gamma = 4$			
		(H,H)	(H,R)	(R,H)	(R,R)	(H,H)	(H,R)	(R,H)	(R,R)
Panel A: VIX monthly									
	BL–1 $\eta = \tau$	-0.84	-1.02	1.44*	1.62*	-0.69	-0.82	1.47*	1.85*
1	BL–2 $\eta = \tau$	0.14	-0.08	1.42*	0.97*	0.42	-0.38	1.45*	1.21*
	BL–2 $\eta = 1$	0.58*	0.45	1.50*	1.16*	0.41	0.27	1.67*	0.83*
	BL–1 $\eta = \tau$	1.12*	0.30*	1.97*	2.68*	0.90*	0.28*	2.04*	2.84*
2	BL–2 $\eta = \tau$	1.14*	1.04*	1.91*	1.96*	0.65*	1.03*	2.07*	2.14*
	BL–2 $\eta = 1$	0.17*	-0.03*	1.80*	0.68*	-0.48*	-0.56*	0.31*	-0.29*
	BL–1 $\eta = \tau$	0.59	-0.29	1.03*	1.95*	1.16*	0.25	1.18*	2.67*
3	BL–2 $\eta = \tau$	-0.05	0.12	1.17*	1.49*	0.64	0.69	1.37*	1.93*
	BL–2 $\eta = 1$	0.56	0.92*	1.65*	1.45*	0.87*	0.79	1.60*	1.14*
Panel B: VIX quarterly									
	BL–1 $\eta = \tau$	-0.90	-0.86	1.43*	1.52*	-0.55	-0.83	1.45*	1.67*
1	BL–2 $\eta = \tau$	-0.17	-0.80	1.45*	1.47*	0.30	-0.79	1.46*	1.59*
	BL–2 $\eta = 1$	0.46	0.38	1.50*	1.37*	0.32	-0.27	1.64*	0.83*
	BL–1 $\eta = \tau$	0.30*	-1.22	1.94*	2.07*	-0.17*	-1.23	1.93*	2.00*
2	BL–2 $\eta = \tau$	0.35*	-1.31	1.86*	1.89*	-0.20*	-1.27	1.81*	1.52*
	BL–2 $\eta = 1$	-0.34*	-0.47*	0.40*	-0.01*	-0.62	-0.97	-0.36*	-0.44*
	BL–1 $\eta = \tau$	-0.42	-1.01	1.14*	1.56*	0.28	-0.57	1.49*	2.05*
3	BL–2 $\eta = \tau$	-0.10	-0.75	0.83*	1.18*	0.48	0.02	1.07*	1.81*
	BL–2 $\eta = 1$	0.62	0.02	0.86*	1.19*	0.83*	0.77	1.24*	1.06*
Panel C: VIX long-term									
	BL–1 $\eta = \tau$	0.55	0.55	0.55	0.55	0.55	0.55	0.62*	0.61*
1	BL–2 $\eta = \tau$	0.55	0.55	0.55	0.55	0.55	0.55	0.55	0.55
	BL–2 $\eta = 1$	0.55	0.55	0.55	0.55	0.55	0.55	0.59*	0.59*
	BL–1 $\eta = \tau$	-0.48*	-0.11*	1.41*	1.55*	-0.60	-0.34*	1.00*	1.03*
2	BL–2 $\eta = \tau$	-0.58	-0.21*	0.40*	1.57*	-0.58	-0.41*	0.30*	1.01*
	BL–2 $\eta = 1$	-0.51*	-0.46*	-0.10*	-0.23*	-0.56*	-0.58	-0.41*	-0.44*
	BL–1 $\eta = \tau$	-0.33	-0.46	0.91*	1.46*	0.41	-0.39	1.29*	1.91*
3	BL–2 $\eta = \tau$	0.90*	0.16	0.17	0.92*	0.48	0.33	0.32	1.15*
	BL–2 $\eta = 1$	0.52	-0.39	0.16	0.94*	0.79	0.56	0.61	0.99*

The annualized SR for a SPY investor that diversifies with VIX futures at each rebalancing point when q_v exceeds the model's diversification threshold, or otherwise remains fully invested in SPY. For (q_v, Σ) the investor uses either the historical estimate (H) or forecasts the exact value realized until the next rebalancing point (R). The in-sample period for historical inputs is 1 month and * denotes exceedance of the SPY-only SR.

7 Conclusions

The rapid proliferation of exchange-traded volatility products since the banking crisis of 2008 proves that volatility trading is becoming increasingly popular, and small investors or those that are not allowed to trade futures can now gain access to VIX futures returns via almost 30 related ETNs. During 2011 the average holding time for a VIX futures contract ranged from just over a week for short-term contracts to 3 – 4 weeks on longer-term contracts. Their bid–ask spreads have steadily decreased to 25 – 35 bps as trading has reached very high volumes. Activity is now even more intense on the ETNs that

are linked to VIX futures, and those that track mid-term futures contracts are held, on average, for about 6 weeks. This shows that VIX futures and some ETNs are currently being held for diversification purposes.

Volatility appears to be an attractive diversifier, now that traditional diversification channels are less effective. Even though VIX futures/ETNs lose money due to the negative roll yield, except during a crisis, their strong negative correlation with equity and positive skewness could justify their addition to many portfolios. Yet, previous academic support for the benefits of VIX futures diversification in an ex-ante framework is meager. Virtually all evidence in favor employs variance swaps or spot VIX, which have very different characteristics to VIX futures, or else the empirical study is based entirely on an ex-post analysis. Only an ex-ante analysis can be used in practice, and so the comprehensive ex-ante analysis and empirical study that we have presented is an essential contribution to the literature on this very important topic.

We have approached the problem of volatility diversification from the perspective of three different types of equity investors: the standard investors that use the MV criterion; the skewness-aware investors that use a generalized criterion, still based on an exponential utility function but now allowing returns distributions to be different from normal; and the BL investors that balance personal views on equity and/or volatility returns with equilibrium returns that are the same for all investors. We have found that skewness preference increases the frequency of diversification but does not lead to an improved out-of-sample performance for optimally diversify portfolios. Indeed, there is only one way for diversification to be profitable outside of a period of excessive volatility, and that is for the investor to have personal views that are based on precise forecasts of VIX futures returns. The difficulty of acquiring such forecasts, without access to privileged information, sheds rather a dubious light on the scope for volatility diversification by equity investors. Indeed, since the successful performance of volatility-diversified portfolios rests entirely on the accuracy of predicting returns on VIX futures, it becomes an exercise in short-term speculation rather than long-term diversification.

Appendix

To prove Proposition 1, we write:

$$B = (1+\tau)\frac{\eta}{\tau}\begin{bmatrix} \sigma_s^2 & 0 \\ 0 & \sigma_v^2 \end{bmatrix} + \Sigma = \begin{bmatrix} x\sigma_s^2 & \sigma_{sv} \\ \sigma_{sv} & x\sigma_v^2 \end{bmatrix},$$

with $x = \eta\tau^{-1} + \eta + 1 > 0$, and

$$\gamma\Lambda = B^{-1}\left\{(1+\tau)\begin{bmatrix} q_s \\ q_v \end{bmatrix} + \begin{bmatrix} \mu_s^{eq} \\ \mu_v^{eq} \end{bmatrix}\right\} \quad \text{so}$$

$$\gamma\,|\Sigma|\,\Lambda = |\Sigma|\,|B|^{-1}\begin{bmatrix} x\sigma_v^2 & -\sigma_{sv} \\ -\sigma_{sv} & x\sigma_s^2 \end{bmatrix}\begin{bmatrix} (1+\tau)q_s - \mu_s^{eq} \\ (1+\tau)q_v - \mu_v^{eq} \end{bmatrix},$$

where $|B| = x^2\sigma_s^2\sigma_v^2 - \sigma_{sv}^2 > 0$ is the determinant of B. Thus,

$$\gamma\,|\Sigma|\,\lambda_s = |\Sigma|\,|B|^{-1}\left\{x\sigma_v^2\left[(1+\tau)q_s - \mu_s^{eq}\right] - \sigma_{sv}\left[(1+\tau)q_v - \mu_v^{eq}\right]\right\}, \text{ and}$$

$$\gamma\,|\Sigma|\,\lambda_v = |\Sigma|\,|B|^{-1}\left\{x\sigma_s^2\left[(1+\tau)q_v - \mu_v^{eq}\right] - \sigma_{sv}\left[(1+\tau)q_s - \mu_s^{eq}\right]\right\}.$$

It follows that the conditions (14) may be written:

$$\sigma_{sv}\mu_v^{eq} - \sigma_v^2\mu_s^{eq} < |\Sigma|\,|B|^{-1}\left\{x\sigma_v^2\left[(1+\tau)q_s - \mu_s^{eq}\right] - \sigma_{sv}\left[(1+\tau)q_v - \mu_v^{eq}\right]\right\}, \text{ and}$$

$$\sigma_{sv}\mu_s^{eq} - \sigma_s^2\mu_v^{eq} < |\Sigma|\,|B|^{-1}\left\{x\sigma_s^2\left[(1+\tau)q_v - \mu_v^{eq}\right] - \sigma_{sv}\left[(1+\tau)q_s - \mu_s^{eq}\right]\right\};$$

i.e.

$$|\Sigma|^{-1}\,|B|\left(\sigma_{sv}\mu_v^{eq} - \sigma_v^2\mu_s^{eq}\right) + \left(x\sigma_v^2\mu_s^{eq} - \sigma_{sv}\mu_v^{eq}\right) < (1+\tau)\left(xq_s\sigma_v^2 - q_v\sigma_{sv}\right), \text{ and}$$

$$|\Sigma|^{-1}\,|B|\left(\sigma_{sv}\mu_s^{eq} - \sigma_s^2\mu_v^{eq}\right) + \left(x\sigma_s^2\mu_v^{eq} - \sigma_{sv}\mu_s^{eq}\right) < (1+\tau)\left(xq_v\sigma_s^2 - q_s\sigma_{sv}\right).$$

Substituting in

$$|\Sigma|^{-1}\,|B| = (x^2 - \rho^2)(1 - \rho^2)^{-1},$$

where $\rho = \sigma_{sv}/(\sigma_s\sigma_v)$, the conditions may be written:

$$(1+\tau)^{-1}\left(1 - \rho^2\right)^{-1}\left[\left(x^2 - 1\right)\sigma_{sv}\mu_v^{eq} - \left(x + \rho^2\right)(x - 1)\sigma_v^2\mu_s^{eq}\right] - xq_s\sigma_v^2 < -q_v\sigma_{sv}, \text{ and}$$

$$(1+\tau)^{-1}\left(1 - \rho^2\right)^{-1}\left[\left(x^2 - 1\right)\sigma_{sv}\mu_s^{eq} - \left(x + \rho^2\right)(x - 1)\sigma_s^2\mu_v^{eq}\right] + q_s\sigma_{sv} < xq_v\sigma_s^2.$$

Now, after some tedious algebra, the conditions for a positive allocation to both SPY and VIX futures may be written $q_v > \max(a_3/a_1, a_4)$ with a_1, a_2, a_3, a_4 as defined in Proposition 1.

Remark 1. As $\eta \to 0$, the condition (15) converges to the MV condition (4). Indeed, as already noted by Black and Litterman (1992), for $\eta \to 0$ the BL portfolio converges to the views portfolio when $P = I$. As $\eta \to \infty$ the BL (posterior) portfolio converges to the

equilibrium (prior) portfolio as the views become less and less informative.

Remark 2. The diversification threshold in (15) decreases as γ increases, so for a given q_v diversification is more likely to occur by more risk-averse investors.

Remark 3. It is commonly assumed in the literature that $\eta = \tau$. In that case $a_1 = 2 + \tau$ and $a_2 = (1 - \rho^2)^{-1}$, so

$$a_3 = \left(\frac{\sigma_{sv}}{\sigma_s^2} \right) q_s + \left(\frac{3 + \tau}{1 - \rho^2} \right) \left(\frac{\sigma_{sv}}{\sigma_s^2} \right) \mu_s^{eq} - \left(\frac{2 + \tau + \rho^2}{1 - \rho^2} \right) \mu_v^{eq}$$

and

$$a_4 = (2 + \tau) \left(\frac{\sigma_v^2}{\sigma_{sv}} \right) q_s + \left(\frac{2 + \tau + \rho^2}{1 - \rho^2} \right) \left(\frac{\sigma_v^2}{\sigma_{sv}} \right) \mu_s^{eq} - \left(\frac{3 + \tau}{1 - \rho^2} \right) \mu_v^{eq}.$$

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