Path-dependence of Leveraged ETF returns

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Abstract

It is well-known that leveraged exchange-traded funds (LETFs) don't reproduce the corresponding multiple of index returns over extended (quarterly or annual) investment horizons. In 2008, most leveraged ETFs underperformed the corresponding static strategies. In this paper, we study this phenomenon in detail. We give an exact formula linking the return of a leveraged fund with the corresponding multiple of the return of the unleveraged fund and its realized variance. This formula is tested empirically over quarterly horizons for 56 leveraged funds (44 double-leveraged, 12 triple-leveraged) using daily prices since January 2008 or since inception, according to the fund considered. The results indicate excellent agreement between the formula and the empirical data. The study also shows that leveraged funds can be used to replicate the returns of the underlying index, provided we use a dynamic rebalancing strategy. Empirically, we find that rebalancing frequencies required to achieve this goal are moderate, on the order of one week between rebalancings. Nevertheless, this need for dynamic rebalancing leads to the conclusion that leveraged ETFs as currently designed may be unsuitable for buy-and-hold investors.

1 Introduction

Leveraged ETFs (LETFs) are relative newcomers to the world of exchange-traded funds[1]. A leveraged ETF tracks the value of an index, a basket of stocks, or another ETF with the additional feature that it uses leverage. For instance, the ProShares Ultra Financial ETF (UYG) offers double exposure to

[1] To our knowledge, the first issuer of leveraged ETFs was Rydex, in late 2006.

the Dow Jones U.S. Financials index. To achieve this, the manager invests two dollars in a basket of stocks tracking the index per each dollar of UYG's net asset value, borrowing an additional dollar. This is an example of a long LETF. Short LETFs, such as the ProShares UltraShort Financial ETF (SKF), offer a negative multiple of the return of the underlying ETF. In this case, the manager sells short a basket of stocks tracking the Dow Jones U.S. Financials index (or equivalent securities) to achieve a short exposure in the index of two dollars per each dollar of NAV ($\beta = -2$). In both cases, the fund's holdings are rebalanced daily.[2]

It has been empirically established that if we consider investments over extended periods of time (*e.g* three months, one year, or more), there are significant discrepancies between LETF returns and the returns of the corresponding leveraged buy-and-hold portfolios composed of index ETFs and cash (see, Lu, Wang and Zhang, 2009). Since early 2008, the quarterly performance (over any period of 60 business days, say) of LETFs has been inferior to the performance of the corresponding static leveraged portfolios for many leveraged/unleveraged pairs tracking the same index. There are a few periods where LETFs outperform, so this is not just a one-sided effect.

For example, a portfolio consisting of two dollars invested in I-Shares Dow Jones U.S. Financial Sector ETF (IYF) and short one dollar can be compared with an investment of one dollar in UYG. Figures 1 and 2 compare the returns of UYG and a twice leveraged buy-and-hold strategy with IYF, considering all 60-day periods (overlapping) since February 2, 2008. For convenience, we present returns in arithmetic and logarithmic scales. Figures 3 and 4, display the same data for SKF and IYF.

Observing Figures 1 and 3, we see that the returns of the LETFs have predominantly underperformed the static leveraged strategy. This is particularly the case in periods when returns are moderate and volatility is high. LETFs outperform the static leveraged strategy only when return are large and volatility is small. Another observation is that the historical underperformance is more pronounced for the short LETF (SKF).

These charts can be explained by the mismatch between the quarterly investment horizon and the daily rebalancing frequency; yet there are several points that deserve attention.

First, we notice that, due to the daily rebalancing of LETFs, the geometric (continuously compounded) relation

$$\log \text{ret.(LETF)} \approx \beta \ \log \text{ret.(ETF)},$$

[2]The description of the hedging mechanism given here is not intended to be exact, but rather to illustrate the general approach used by ETF managers to achieve the targeted leveraged long and short exposures. For instance, managers can trade the stocks that compose the ETFs or indices, or enter into total-return swaps to replicate synthetically the returns of the index that they track. The fact that the returns are adjusted daily is important for our discussion. Recently Direxion Funds, a leveraged ETF manager, has announced the launch of products with monthly rebalancing.

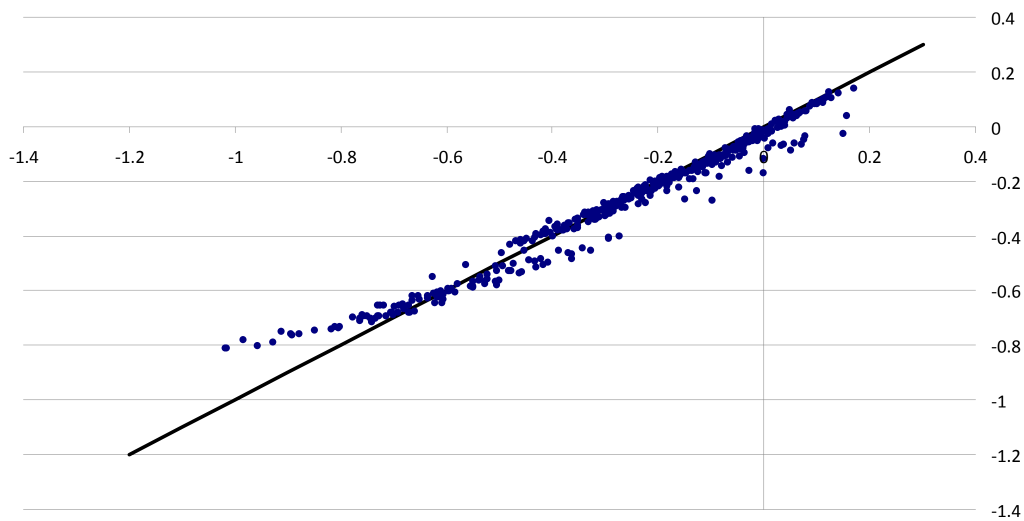

Figure 1: 60-day returns for UYG versus leveraged 60-day return of IYF. ($X = 2Ret.(IYF); Y = Ret.(UYG)$). We considered all 60-day periods (overlapping) between February 2, 2008 and March 3, 2008. The concentrated cloud of points near the 45-degree line correspond to 60-day returns prior to to September 2008, when volatility was relatively low. The remaining points correspond to periods when IYF was much more volatile.

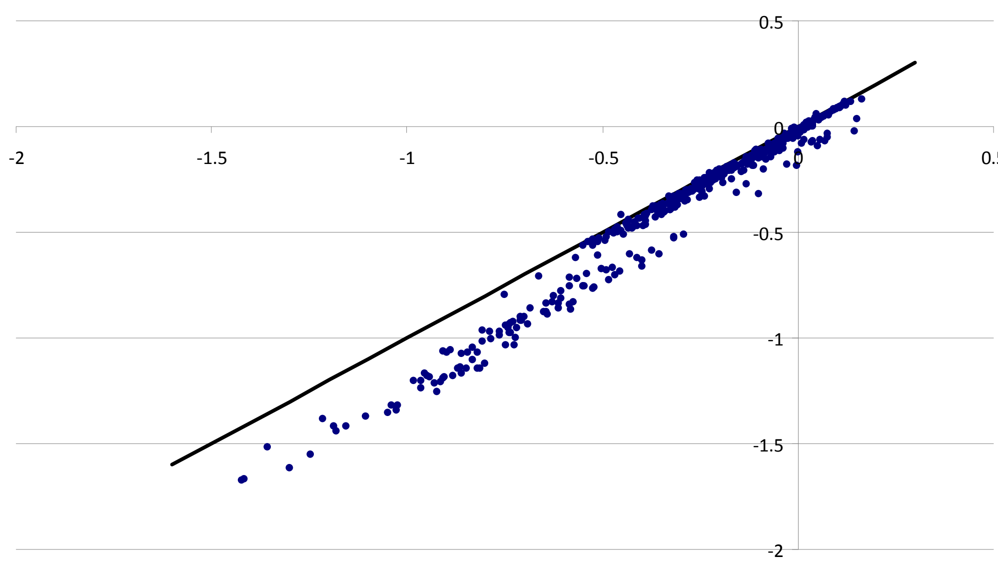

Figure 2: Same as in Figure 1, but returns are logarithmic,*i.e.* $X = 2\,ln(IYF_t/IYF_{t-60}); Y = ln(UYG_t/UYG_{t-60})$.

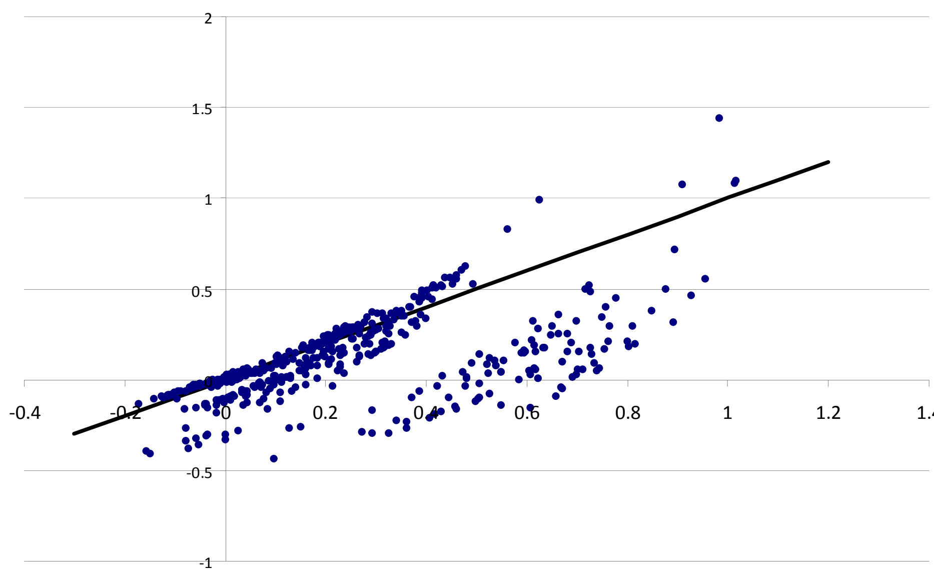

Figure 3: Overlapping 60-day returns of SKF compared with the leveraged returns of the underlying ETF, overlapping, between February 2, 2008 and March 3, 2008. ($X = -2Ret.(IYF); Y = Ret.(SKF)$).

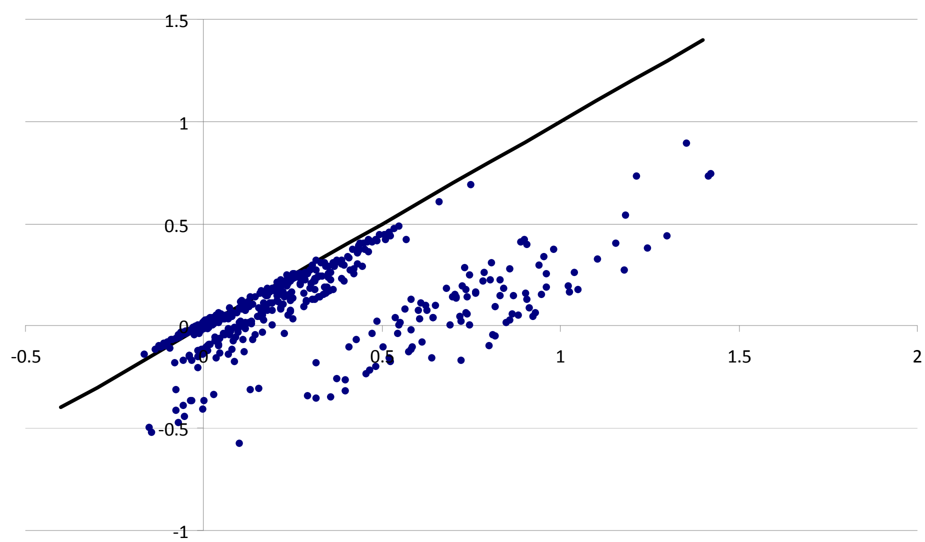

Figure 4: Comparison of logarithmic returns of SKF with the corresponding log-returns of IYF. $X = -2ln(IYF_t/IYF_{t-60}); Y = ln(SKF_t/SKF_{t-60})$

is more appropriate than the arithmetic (simply compounded) relation

$$\text{ret.(LETF)} \approx \ \beta \ \text{ret.(ETF)}, \quad \beta = \pm 2.$$

This explains the apparent alignment of the datapoints once we pass to logarithmic returns in Figures 2 and 4.

Second, we notice that the datapoints do not fall on the 45-degree line; they lie for the most part below it. This effect is due to volatility. It can be explained by the fact that the LETF manager must necessarily "buy high and sell low" in order to enforce the target leverage requirement. Therefore, frequent rebalancing will lead to under performance for the LETF relative to a static leveraged portfolio. The underperformance will be larger in periods when volatility is high, because daily rebalancing in a more volatile environment leads to more round-trip transactions, all other things being equal.

This effect is quantified using a simple model in Section 2. We derive an exact formula for the return of an LETF as a function of its expense ratio, the applicable rate of interest, and the return and realized variance of an unleveraged ETF tracking the same index (the "underlying ETF", for short). In particular, we show that the holder of an LETF has a negative exposure to the realized variance of the underlying ETF. Since the expense ratio and the funding costs can be determined in advance with reasonable accuracy, the main factor that affects LETF returns is the realized variance. In section 3, we validate empirically the formula on a set of 56 LETFs with double and triple leverage, using all the data since their inception. The empirical study suggests that the proposed formula is very accurate.

In the last section, we show that it is possible to use leveraged ETFs to replicate a pre-defined multiple of the underlying ETF returns, provided that we use dynamic hedging strategies. Specifically: in order to achieve a specified multiple of the return of the underlying index or ETF using LETFs, we must adjust the portfolio holdings in the LETF dynamically, according to the amount of variance realized up to the hedging time by the index, as well as the level of the index. We derive a formula for the dynamic hedge-ratio, which is closely related to the model for LETFs, and we validate it empirically on the historical data for 44 double-leverage LETFs. This last point – dynamic hedging – provides an interesting connection between LETFs and options.

After completing this paper, we found out that similar results were obtained independently by Cheng and Madhavan in a note issued by Barclays Global Investors (Cheng and Madhavan, 2009), which contains a formula similar to (10) without including finance and expense ratios. The application to dynamic hedging using LETFs proposed here is new, to our knowledge, as well as the empirical testing of the formula and the application to dynamic hedging over a broad universe of LETFs.

2 Modeling Leveraged ETF returns

We denote the spot price of the underlying ETF by S_t, the price of the leveraged ETF by L_t and leverage ratio by β. For instance, a double-leverage long ETF will correspond to $\beta = 2$, whereas a double-leverage short ETF corresponds to $\beta = -2$.

2.1 Discrete-time model

Assume a model where there are N trading days, and denote by R_i^S and $R_i^L, i = 1, 2, ..., N$ the one-day returns for the underlying ETF and the LETF, respectively. The leveraged ETF provides a pro-forma daily exposure of β dollars of the underlying security per dollar under management.[3] Accordingly, there is a simple link between R_i^S and R_i^L. If the leveraged ETF is bullish ($\beta > 1$):

$$R_i^L = \beta R_i^S - \beta r \Delta t - f \Delta t + r \Delta t = \beta R_i^S - \left((\beta - 1) r + f \right) \Delta t, \qquad (1)$$

where r is the reference interest rate (for instance, 3-months LIBOR), f is the expense ratio of the LETF and $\Delta t = 1/252$ represents one trading day.

If the leveraged ETF is bearish ($\beta \leq -1$), the same equation holds with a small modification, namely

$$R_i^L = \beta R_i^S - \beta(r - \lambda_t)\Delta t - f\Delta t + r\Delta t = \beta R_i^S - \left((\beta - 1) r + f - \beta \lambda_i \right) \Delta t, \qquad (2)$$

where $\lambda_i \Delta t$ represents the cost of borrowing the components of the underlying index or the underlying ETF on day i. By definition, this cost is the difference between the reference interest rate and the "short rate" applied to cash proceeds from short-sales of the underlying ETF. If the ETF, or the stocks that it holds are widely available for lending, the short rate will be approximately equal to the reference rate and the borrowing costs are negligible.[4]

Let t be a period of time (in years) covering several days ($t = N\Delta$). Compounding the returns of the LETF, we have

$$L_t = L_0 \prod_{i=1}^{N} \left(1 + R_i^L \right). \qquad (3)$$

Substituting the value of R_t^L in equation (1) or (2) (according to the sign of β), we obtain a relation between the prices of the leveraged ETF and the underlying asset. In fact, we show in the Appendix that, under mild assumptions, we have:

$$\frac{L_t}{L_0} \approx \left(\frac{S_t}{S_0} \right)^{\beta} exp \left\{ \frac{\beta - \beta^2}{2} V_t + \beta H_t + \left((1 - \beta) r - f \right) t \right\}, \qquad (4)$$

[3] In the sense that this does not account for the costs of financing positions and management fees.

[4] We emphasize the cost of borrowing, since we are interested in LETFs which track financial indices. The latter have been often hard-to-borrow since July 2008. Moreover, broad market ETFs such as SPY have also been sporadically hard-to-borrow in the last quarter of 2008; see Avellaneda and Lipkin (2009).

where

$$V_t = \sum_{i=1}^{N} \left(R_i^S - \overline{R^S} \right)^2 \text{ with } \overline{R^S} = \frac{1}{N} \sum_{i=1}^{N} R_i^S, \tag{5}$$

i.e. V_t is the realized variance of the price over the time-interval of interest, and where

$$H_t = \sum_{i=1}^{N} \lambda_i \Delta t \tag{6}$$

represents the accumulated cost of borrowing the underlying stocks or ETF. This cost is obtained by subtracting the average applicable short rate from the reference interest rate each day and accumulating this difference over the period of interest. In addition to these two factors, formula (4) also shows the dependence on the funding rate and the expense ratio of the underlying ETF. The symbol " $\approx$ " in (4) means that the difference is small in relation to the daily volatility of the ETF or LETF. In the following paragraph, we exhibit an exact relation, assuming that the price of the underlying ETF follows an Ito process.

2.2 Continuous-time model

To clarify the sense in which (4) holds, it is convenient to derive a similar formula assuming that the underlying ETF price follows an Ito process. To wit, we assume that S_t, satisfies the stochastic differential equation

$$\frac{dS_t}{S_t} = \sigma_t dW_t + \mu_t dt \tag{7}$$

where W_t is a standard Wiener process and σ_t, μ_t are respectively the instantaneous price volatility and drift. The latter processes are assumed to be random and non-anticipative with respect to W_t.[5]

Mimicking (1) and (2), we observe that if is bullish, the model for the return of the leveraged fund is now

$$\frac{dL_t}{L_t} = \beta \frac{dS_t}{S_t} - ((\beta - 1)r + f)\, dt. \tag{8}$$

If the LETF is bearish, the corresponding equation is

$$\frac{dL_t}{L_t} = \beta \frac{dS_t}{S_t} - ((\beta - 1)r - \beta\lambda_t + f)dt, \tag{9}$$

where λ_t represents the cost of borrowing the underlying ETF or the stocks that make up the ETF. In the Appendix, we show that the following formula holds:

$$\frac{L_t}{L_0} = \left(\frac{S_t}{S_0} \right)^{\beta} exp \left\{ ((1-\beta)r - f)\, t + \beta \int_0^t \lambda_s ds + \frac{(\beta - \beta^2)}{2} \int_0^t \sigma_s^2 ds \right\}, \tag{10}$$

[5] They are *not* assumed to be deterministic functions or constants.

where we assume implicitly that $\lambda_t = 0$ if $\beta > 0$.

Formulas (4) and (10) convey essentially the same information if we define

$$V_t = \int_0^t \sigma_s^2 ds, \text{ and } \quad H_t = \int_0^t \lambda_s ds.$$

The only difference is that (4) is an approximation which is valid for $\Delta t \ll 1$ whereas (10) is exact if the ETF price follows an Ito process. These equations show that the relation between the values of an LETF and its underlying ETF depends on

- the *funding rate*
- the *expense ratio* for the leveraged ETF
- the *cost of borrowing shares* in the case of short LETFs
- the *convexity* (power law) associated with the leverage ratio β
- the *realized variance* of the underlying ETF.

The first two items require no explanation. The third follows from the fact that the manager of a bearish LETF may incur additional financing costs to maintain short positions if components of the underlying ETF or the ETF itself are hard-to-borrow. The last two items are more interesting: (i) due to daily rebalancing of the beta of the LETF, we find that the prices of a leveraged and non-leveraged ETF are related by a power law with power β and (ii) the realized variance of the underlying ETF plays a significant role in determining the LETF returns.

The dependence on the realized variance might seem surprising at first. It turns out that the holder of an LETF has negative exposure to the realized variance of the underlying asset. This holds irrespective of the sign of β. For instance, if the investor holds a double-long LETF, the term corresponding to to the realized variance in formula (10) is

$$-\frac{(2^2-2)}{2}\int_0^t \sigma_s^2 = -\int_0^t \sigma_s^2.$$

In the case of a doubly bearish fund, the corresponding term is

$$-\frac{((-2)^2-(-2))}{2}\int_0^t \sigma_s^2 = -3\int_0^t \sigma_s^2.$$

We note, in particular, that the dependence on the realized variance is stronger in the case of the double-short LETF.

3 Empirical validation

To validate the formula in (10), we consider 56 LETFs which currently trade in the US markets. Of these, we consider 44 LETFs issued by ProShares, consisting 22 Ultra Long and 22 UltraShort ETFs.[6]. Table 1 gives a list of the Proshares LETFs, their tickers, together with the corresponding sectors and their ETFs. We consider the evolution of the 44 LETFs from January 2, 2008 to March 20, 2009, a period of 308 business days.

We also consider 12 triple-leveraged ETFs issued by Direxion Funds[7]. Direxion's LETFs were issued later than the ProShares funds, in November 2008; they provide a shorter historical record to test our formula. Nevertheless, we include the 3X Direxion funds for completeness' sake and also because they have triple leverage.

Double-Leveraged ETFs considered in the study

Underlying ETF	Proshares Ultra ($\beta = 2$)	Proshares Ultra Short ($\beta = -2$)	Index/Sector
QQQQ	QLD	QID	Nasdaq 100
DIA	DDM	DXD	Dow 30
SPY	SSO	SDS	S&P500 Index
IJH	MVV	MZZ	S&P MidCap 400
IJR	SAA	SDD	S&P Small Cap 600
IWM	UWM	TWM	Russell 2000
IWD	UVG	SJF	Russell 1000
IWF	UKF	SFK	Russell 1000 Growth
IWS	UVU	SJL	Russell MidCap Value
IWP	UKW	SDK	Russell MidCap Growth
IWN	UVT	SJH	Russell 2000 Value
IWO	UKK	SKK	Russell 2000 Growth
IYM	UYM	SMN	Basic Materials
IYK	UGE	SZK	Consumer Goods
IYC	UCC	SCC	Consumer Services
IYF	UYG	SKF	Financials
IYH	RXL	RXD	Health Care
IYJ	UXI	SIJ	Industrials
IYE	DIG	DUG	Oil & Gas
IYR	URE	SRS	Real Estate
IYW	ROM	REW	Technology
IDU	UPW	SDP	Utilities

Table 1: ETFs and the corresponding ProShares Ultra Long and UltraShort LETFs.

[6]For information about ProShares, see http://www.proshares.com
[7]See http://www.direxionfunds.com

Triple-Leverage ETFs considered in the study

Underlying ETF or Index	Direxion 3X Bull ($\beta = 3$)	Direxion 3X Bear ($\beta = -3$)	Index/Sector
IWB	BGU	BGZ	Russell 1000
IWM	TNA	TZA	Russell 2000
RIFIN.X	FAS	FAZ	Russell 1000 Financial Serv.
RIENG.X	ERX	ERY	Russell 1000 Energy
EFA	DZK	DPK	MSCI EAFE Index
EEM	EDC	EDZ	MSCI Emerging Markets

Table 2: ETFs and corresponding Direxion 3X LETFs.

We define the tracking error

$$\epsilon(t) = \frac{L_t}{L_0} - \left(\frac{S_t}{S_0}\right)^{\beta} exp\left\{\frac{\beta - \beta^2}{2} V_t + \beta H_t + ((1-\beta)r - f)\, t\right\}, \qquad (11)$$

where V_t is the accumulated variance, H_t is the accumulated borrowing costs (in excess of the reference interest rate), r is the interest rate and f is the management fee for the underlying ETF. The instantaneous volatility is modeled as the standard deviation of the returns of the underlying ETF sampled over a period of 5 days preceding each trading date:

$$\hat{\sigma}_s^2 = \frac{1}{5}\sum_{i=1}^{5}(R^{(S)}_{(s/\Delta t)-i})^2 - \left(\frac{1}{5}\sum_{i=1}^{5} R^{(S)}_{(t/\Delta t)-i}\right)^2, \quad 0 \le s \le t. \qquad (12)$$

For the interest rates and expense ratio, we use 3-month LIBOR rate published by the Federal Reserve Bank (H.15 Report), and the expense ratio for the Proshares LETFs published in the prospectus. In all cases, we set $\lambda_t = 0$, i.e. we do not take into account stock-borrowing costs explicitly.

The empirical results for ProShares are summarized in Tables 2 and 3 hereafter. Graphical comparisons of the tracking error for some of the major indices are also displayed in Figures 1 - 8.

In the case of long LETFs, we find that the average of the tracking error $\epsilon(t)$ over the simulation period is typically less than 100 basis points. The standard deviation is also on the order of 100 basis points, with a few slightly higher observations. This suggests that the formula (10), using the model for stochastic volatility in (12), gives a reliable model for the relation between the leveraged and the underlying ETFs across time.

In the case of short ETFs, we also assume that $\lambda_t = 0$ but expect a slightly higher tracking error, particularly between July and November of 2008, when restrictions for short-selling in U.S. stocks were put in place. We observe higher levels for the mean and the standard deviation of the tracking error and some significant departures from the exact formula during the period of October and

November 2008, especially in Financials, which we attribute to short-selling constraints. The tracking errors for the Direxion triple-leveraged ETFs have higher standard deviations, which is not surprising given that they have higher leverage. We note, in particular, that the errors for FAS and FAZ are the largest, which is consistent with the fact that they

The conclusion of the empirical analysis is that the formula (10) explains well the behavior of the price of leveraged ETFs and the deviations between LETF returns and the returns of the underlying ETFs.

Double-Leverage Ultra Long ETFs

Underlying ETF	Tracking Error (average, %)	Standard Deviation (%)	Leveraged ETF
QQQQ	0.04	0.47	QLD
DIA	0.00	0.78	DDM
SPY	-0.06	0.40	SSO
IJH	-0.06	0.38	MVV
IJR	1.26	0.71	SAA
IWM	1.26	0.88	UWM
IWD	1.00	0.98	UVG
IWF	0.50	0.59	UKF
IWS	-0.33	1.20	UVU
IWP	-0.02	0.61	UKW
IWN	2.15	1.29	UVT
IWO	0.50	0.74	UKK
IYM	1.44	1.21	UYM
IYK	1.20	0.75	UGE
IYC	1.56	1.04	UCC
IYF	-0.22	0.74	UYG
IYH	0.40	0.42	RXL
IYJ	1.05	0.74	UXI
IYE	-0.73	1.71	DIG
IYR	1.64	1.86	URE
IYW	0.51	0.55	ROM
IDU	0.25	0.55	UPW

Table 3: Average tracking error (11) and standard deviation obtained by applying formula (10) to the Proshares long LETFs from January 2, 2008 to March 20 2009. Notice that that the average tracking error is for the most part below 100bps and the standard deviation is comparable. In particular the standard deviation is inferior to the daily volatility of these assets, which often exceeds 100 basis points as well. This suggests that formula (10) gives the correct relation between the NAV of the LETFs and their underlying ETFs.

Double-Leveraged Ultra Short ETFs

Underlying ETF	Tracking Error (average, %)	Standard Deviation (%)	Leveraged ETF
QQQQ	0.22	0.80	QID
DIA	-2.01	3.24	DXD
SPY	-1.40	2.66	SDS
IJH	0.69	0.64	MZZ
IJR	-0.55	0.86	SDD
IWM	0.94	0.91	TWM
IWD	0.32	1.40	SJF
IWF	-0.30	1.34	SFK
IWS	-2.06	3.03	SJL
IWP	0.93	0.92	SDK
IWN	-2.21	1.80	SJH
IWO	-0.19	0.79	SKK
IYM	1.82	0.99	SMN
IYK	-0.76	1.98	SZK
IYC	0.79	0.92	SCC
IYF	3.30	3.03	SKF
IYH	1.04	0.91	RXD
IYJ	0.32	0.74	SIJ
IYE	0.43	3.09	DUG
IYR	2.00	2.07	SRS
IYW	0.01	0.80	REW
IDU	1.75	1.06	SDP

Table 4: Same as in Table 3, for double-short LETFs. Notice that the tracking error is relatively small, but there are a few funds where the tracking error is superior to 200 basis points. We attribute these errors to the fact that may ETFs, particularly in the Financial and Energy sectors, and the stocks in their holdings were hard-to-borrow from July to November 2008.

Triple-Leveraged Bullish ETFs

Underlying ETF/Index	Tracking Error (average, %)	Standard Deviation (%)	3X bullish LETF
IWB	0.44	0.55	BGU
IWM	0.81	0.75	TNA
RIFIN.X	3.67	2.08	FAS
RIENG.X	2.57	0.70	ERX
EFA	1.26	2.32	DZK
EEM	1.41	1.21	EDC

Table 5: Average tracking errors and standard deviations for triple-leveraged long ETFs analyzed here, since their inception in November 2008.

Triple-Leveraged Bearish ETFs

Underlying ETF/Index	Tracking Error average, %)	Standard Deviation (%)	3X bearish LETF
IWB	-0.08	0.64	BGZ
IWM	0.65	0.76	TZA
RIFIN.X	-1.63	4.04	FAZ
RIENG.X	-1.41	1.01	ERY
EFA	-1.54	1.86	DPK
EEM	0.49	1.43	EDZ

Table 6: Average tracking errors and standard deviations for triple-leveraged short ETFs analyzed here, since their inception in November 2008. Notice again that the errors for financials and energy are slightly higher than the rest.

4 Consequences for buy-and-hold investors

4.1 Comparison with buy-and-hold: break-even levels

Formula (10) suggests that an investor who is long a leveraged ETF has a "time-decay" associated with the realized variance of the underlying ETF. In other words, if the price of the underlying ETF does not change significantly over the investment horizon, but the realized volatility is large, the investor in the leveraged ETF will underperform the corresponding leveraged return on the underlying ETF. On the contrary, if the underlying ETF makes a sufficiently large move in either direction, the investor will out-perform the underlying ETF.

Consider an investor who buys one dollar of leveraged ETF and simultaneously shorts β dollars of the underlying ETF (where shorting a negative amount means buying). For simplicity, we assume that the interest rate, fees and borrowing costs are zero.

If we use equation (10), the equity in the investor's account will be equal to

$$E(t) = \left(\frac{S_t}{S_0}\right)^{\beta} e^{-\frac{(\beta^2-\beta)}{2}V_t} - \beta\frac{S_t}{S_0} - (1-\beta), \tag{13}$$

including the cash credit or debit from the initial transaction. To be concrete, we consider the case of a double long and a double short separately. Setting $Y = E(t)$ and $X = \frac{S_t}{S_0}$, we obtain

$$\begin{aligned} Y &= e^{-V_t}X^2 - 2X + 1, \quad \beta = 2, \\ Y &= e^{-3V_t}\frac{1}{X^2} + 2X - 3, \quad \beta = -2. \end{aligned} \tag{14}$$

In the case of the double-long ETFs, the equity behaves like a parabola in $X = S_t/S_0$ with a curvature tending to zero exponentially as a function of the realized variance. The investor is therefore long convexity (Gamma, in options parlance) and short variance, hence he incurrs time-decay (Theta). In the case of double-shorts, the profile is also a convex curve, which has convexity concentrated mostly for $X \ll 1$, and a much faster time-decay. From equation (14), find that the break-even levels of X, V_t needed for achieving a positive return by time t are

- Double-long LETF

$$X > X_+ = e^{V_t}\left(1 + \sqrt{1 - e^{-V_t}}\right)$$
$$X < X_- = e^{V_t}\left(1 - \sqrt{1 - e^{-V_t}}\right)$$

- Double-short LETF

$$\begin{gathered} X_+, X_- \text{ are the positive roots of the cubic equation} \\ 2X^3 - 3X^2 + e^{-3V_t} = 0 \end{gathered} \tag{15}$$

A similar analysis can be made for triple leveraged ETFs. The main observation is that, regardless of whether the LETFs are long or short, they underperform the static leveraged strategy unless the returns of the underlying ETFs overcome the above volatility-dependent break-even levels. These levels are further away from the initial level as the realized variance increases.

4.2 Targeting an investment return using dynamic replication with LETFs

Let us assume that an investor wants to replicate the return of a an ETF or an index over a given investment horizon using the corresponding leveraged ETF. We know that merely holding the LETF will not guarantee the desired return due to the convexity and volatility effects. We seek to achieve this objective by dynamically adjusting the holdings in the LETF. Denote by T be the investment horizon and by m the notional amount invested. From (10), it follows that the target investment return satisfies

$$m\left\{\frac{S_T}{S_0} - 1\right\} = m\left\{\left(\frac{L_T}{L_0}\right)^{1/\beta} e^{A(0,T)} - 1\right\} \tag{16}$$

where $A(t,T)$ is defined by the equation

$$A(t,T) = \frac{1}{\beta}\int_t^T \left(\frac{\beta^2 - \beta}{2}\sigma_s^2 - \beta\lambda_s + (\beta - 1)r + f\right) ds. \tag{17}$$

Thus, a hypothetical contract that delivers the return of the ETF over an investment horizon $(0,T)$ can be viewed as a "derivative security" contingent on the LETF with a payoff corresponding to the right-hand side of equation (16). The "fair value" of this derivative, at any intermediate time $t < T$, is given by the expected value of the payoff with respect to a risk-neutral probability measure under which L_t is a martingale after adjusting for interest and dividends. Accordingly, we consider the function

$$V(L, A, \sigma, t) = \\ e^{-r(T-t)} E\left\{ m\left(\left(\frac{L_T}{L_0}\right)^{\frac{1}{\beta}} e^{A(0,T)} - 1 \right) | L_t = L, A(0,t) = A, \sigma_t = \sigma \right\}, \tag{18}$$

where $E(\cdot)$ denotes expectation with respect to the pricing measure and S_T and $L_T, A(0,T)$ are connected via formula (10). This function corresponds to the "fair value" of a derivative security written on L_t which delivers the return of the underlying ETF at time T (see, for instance, Avellaneda and Laurence (1999)). We show in the Appendix that

$$V(L, A, \sigma, t) = V(L, A, t) = e^{-d(T-t)} m \left(\frac{L}{L_0}\right)^{\frac{1}{\beta}} e^A e^{\frac{f(T-t)}{\beta}} - e^{-r(T-t)} m. \tag{19}$$

Notice that this value depends only on $A(0,t)$ and L_t at time t and not on the current value of the stochastic volatility. This means that theoretical we should be able to replicate the target return on the ETF by dynamic hedging with the LETF, without additional risk due to volatility fluctuations.

Consider the function

$$\Delta(L, A, t) = e^{-d(T-t)} e^{\frac{f(T-t)}{\beta}} e^A \frac{m}{\beta} \left(\frac{L}{L_0}\right)^{1/\beta}. \tag{20}$$

In the Appendix, we show that that a static investment in this "derivative security" and a dynamically adjusted position in LETFs consisting of $\Delta(L_t, A(0,t), t)$ dollars invested in the LETFs at time t will have identical payoffs at time T. This gives us a replicating strategy, under arbitrary stochastic volatility models, for leveraged returns of the underlying ETF over any time horizon T using LETFs.

In Tables 7 and 8 we demonstrate the effectiveness of this dynamic replication method using different rebalancing techniques. We consider dynamic hedging in which we rebalance if the total Delta exceeds a band of 1%, 2%, 5% and 10%, and also hedging with fixed time-steps of 1, 2, 5 or 15 business days. Table 9 indicates the expected number of days between rebalancing for strategies that are price-dependent. The results indicate that rebalancing when the Delta exposure exceeds 5% of notional give reasonable tracking errors. The corresponding average intervals between rebalancings can be large, which means that, in practice, one can achieve reasonable tracking errors without necessarily rebalancing the Delta daily or even weekly.

A strong motivation for using LETFs to target a given level of return is to take advantage of leverage. However, in order to achieve his target return over an extended investment period using LETFs, the investor needs to rebalances his

Average tracking error (%) for dynamic replication of 6-month returns using double-long LETFs

ETF	1 %	2 %	5 %	10 %	1 day	2 day	5 day	15 day
QQQQ	-0.29	-0.71	-1.05	-1.62	-0.56	-0.96	-1.45	-1.74
DIA	-0.99	-0.99	-1.37	-1.45	-0.47	-0.59	-0.84	-0.99
SPY	-0.97	-0.92	-1.19	-1.47	-0.92	-1.17	-1.55	-1.77
IJH	-0.39	-0.37	-0.79	-0.99	-0.53	-0.75	-1.05	-1.09
IJR	-0.56	-0.57	-1.07	-2.68	-0.66	-0.90	-1.44	-1.70
IWM	0.37	0.22	-0.49	-1.44	0.47	0.03	-0.70	-0.93
IWD	-0.03	-0.35	-0.30	-0.64	0.00	-0.15	-0.57	-0.79
IWF	-0.15	-0.25	-0.54	-1.08	-0.12	-0.37	-0.68	-0.81
IWS	0.87	0.22	0.81	0.14	0.69	0.71	0.54	0.24
IWP	-0.16	-0.14	-0.54	-1.41	-0.36	-0.40	-0.82	-0.89
IWN	0.94	0.40	0.56	-0.03	-0.91	0.86	0.36	0.14
IWO	0.23	0.03	-1.00	-1.63	-0.05	-0.44	-1.15	-1.45
IYM	-0.39	-0.51	-0.89	-2.35	-0.24	-0.67	-1.54	-1.91
IYK	0.24	0.13	-0.16	-0.06	0.37	0.34	0.10	0.04
IYC	0.58	0.57	-0.13	-0.76	0.71	0.70	0.04	-0.21
IYF	-0.36	-0.62	0.01	-0.54	-0.30	-0.35	-1.28	-2.17
IYH	0.22	-0.10	-0.14	0.27	0.30	0.19	0.03	0.07
IYJ	0.12	-0.09	-0.36	-0.92	0.14	-0.04	-0.30	-0.61
IYE	-1.44	-2.02	-1.90	-1.76	-1.19	-1.82	-2.21	-2.07
IYR	-0.43	0.58	-0.80	-0.95	-0.61	0.55	-0.74	-1.48
IYW	-0.50	-0.46	-1.67	-1.39	-0.37	-0.85	-1.41	-1.76
IDU	0.75	0.45	0.73	0.11	0.83	0.78	0.46	0.52

Table 7: Average tracking error, in % of notional, for the dynamic replication of ETF returns over 6 months with $m = \beta$. The first four columns correspond to rebalancing when the Delta reaches the edge of a band of $\pm x\%$ around zero. The last four columns correspond to rebalancing at fixed time intervals. The data used to generate this table corresponds, for each ETF, to all overlapping 6-month returns in the year 2008.

Standard deviation of tracking error (%) for dynamic replication of 6-month returns using double-long LETFs

ETF	1 %	2 %	5 %	10 %	1 day	2 day	5 day	15 day
QQQQ	0.75	0.77	0.80	1.20	0.75	0.76	0.84	0.93
DIA	0.35	0.37	0.41	0.36	0.36	0.39	0.47	0.43
SPY	0.27	0.32	0.39	0.69	0.27	0.30	0.39	0.45
IJH	0.48	0.49	0.56	1.19	0.47	0.48	0.62	0.65
IJR	1.19	1.21	1.33	1.39	1.20	1.29	1.22	1.17
IWM	0.66	0.67	0.71	1.60	0.67	0.75	0.71	0.74
IWD	1.38	1.38	1.40	1.52	1.38	1.43	1.41	1.49
IWF	0.93	0.94	1.07	1.21	0.95	0.99	0.94	0.99
IWS	2.05	2.05	2.08	2.29	2.05	2.01	2.07	2.09
IWP	0.83	0.82	0.93	1.24	0.83	0.91	0.84	0.91
IWN	1.71	1.70	1.76	2.09	1.72	1.70	1.80	1.82
IWO	0.80	0.80	0.91	1.32	0.79	0.99	0.84	1.00
IYM	1.05	1.07	1.15	1.24	1.07	1.20	1.29	1.59
IYK	0.57	0.57	0.63	0.67	0.56	0.63	0.61	0.63
IYC	0.80	0.78	0.83	0.95	0.80	0.98	0.91	1.03
IYF	1.12	1.18	1.12	1.88	1.10	1.21	2.01	1.49
IYH	0.56	0.55	0.58	0.73	0.56	0.55	0.57	0.62
IYJ	0.71	0.75	0.82	0.84	0.70	0.70	0.79	0.87
IYE	0.64	0.66	0.77	1.26	0.64	0.65	1.02	1.49
IYR	1.47	1.47	1.62	2.08	1.47	1.60	1.88	1.83
IYW	1.45	1.45	1.59	2.05	1.45	1.39	1.49	1.42
IDU	0.53	0.52	0.54	0.72	0.51	0.53	0.54	0.60

Table 8: Same as the previous table, for the standard deviation of tracking errors.

Average number of business days between portfolio rebalancing for the 6-month dynamic hedging strategy: the effect of changing the Delta band

ETF	1 %	2 %	5 %	10 %
QQQQ	2.03	4.14	24	60
DIA	2.50	5.22	30	120
SPY	2.73	5.22	40	NR
IJH	2.26	4.62	24	NR
IJR	2.03	4.29	20	NR
IWM	1.85	4.62	30	NR
IWD	2.18	5.00	30	120
IWF	2.26	5.00	30	NR
IWS	2.26	6.67	17	NR
IWP	1.85	4.14	30	NR
IWN	2.26	4.62	24	NR
IWO	1.85	4.00	20	60
IYM	1.74	3.08	9	40
IYK	3.16	8.57	30	120
IYC	1.90	3.87	30	NR
IYF	1.45	2.93	9	30
IYH	2.79	10.91	30	120
IYJ	2.35	4.80	17	NR
IYE	1.79	3.43	12	40
IYR	1.62	3.16	17	30
IYW	2.00	3.53	40	NR
IDU	2.67	6.32	20	120

Table 9: Each column shows the average number of days between rebalancing the portfolio, assuming different Delta-bandwidth for portfolio rebalancing. For instance, the column with heading of 1% corresponds to a strategy that rebalances the portfolio each time the net delta exposure exceeds 1% of the notional amount. The expected number of days between rebalancing increases as the bandwidth increases.

portfolio according to his Delta exposure. Because of this, dynamic replication with LETFs may not be suitable to many retail investors. On the other hand, this analysis will be useful to active traders, or traders who manage leveraged ETFs with longer investment horizons, since we have shown that the latter can be "replicated" dynamically with LETFs which are rebalanced daily.

5 Conclusion

This study presents a formula for the value of a leveraged ETF in terms of the value of the underlying index or ETF. The formula is validated empirically using end-of-day data on 56 LETFs of which 44 are double-leveraged and 12 are triple leveraged. This formula validates the fact that on log-scale leveraged ETFs will underperform the nominal returns by a contribution that is primarily due to the realized volatility of the underlying ETF. The formula also takes into account financing costs and shows that for short ETFs, the cost of borrowing the underlying stock may play a role as well, as observed in Avellaneda and Lipkin (2009).

We also demonstrate that LETFs can be used for hedging and replicating unleveraged ETFs, provided that traders engage in dynamic hedging. In this case, the hedge-ratio depends on the realized accumulated variance as well as on the level of the LETF at any point in time. The path-dependence of leveraged ETFs makes them interesting for the professional investor, since they are linked to the realized variance and the financing costs. However, they may not be suitable for buy-and-hold investors which aim at replicating a particular index taking advantage of the leveraged provided, for the reasons explained above.

6 References

Marco Avellaneda and Laurence, Peter, Quantitative Modeling of Derivative Securities: From Theory to Practice, CRC Press Inc., 1999

Avellaneda, Marco and M.D. Lipkin, A Dynamic Model for Hard-to-Borrow Stocks(March 10, 2009). *RISK Magazine*, June 2009. Also available at SSRN: http://ssrn.com/abstract=1357069

Cheng, Minder and Ananth Madhavan, The Dynamics of Leveraged and Inverse Exchange-Traded Funds(April 8, 2009), Barclays Global Investors, Available at: http://www.barclaysglobal.com/secure/repository/publications /usa/ResearchPapers/Leveraged_ETF.pdf

Lu, Lei, Wang, Jun and Zhang, Ge, Long Term Performance of Leveraged ETFs(February 15, 2009). Available at SSRN: http://ssrn.com/abstract=1344133

7 Appendix

7.1 Discrete-time Model (Equation (4))

Set $a_i = (\beta - 1)r + f + \beta\lambda_i$, where λ_i is the cost of borrowing the underlying asset on day i (λ_i is zero if $\beta > 0$.) We assume that

$$R_i^S = \xi_i \sqrt{\Delta t} + \mu \Delta t$$

where $\Delta t = 1/252$ and $\xi_i, i = 1, 2, ...$ is a stationary process such that ξ_i has mean equal to zero and finite absolute moments of order 3. This assumption is consistent with many models of equity returns. Notice that we do not assume that successive returns are uncorrelated.

From Equations (1), (2) and (3), we find using Taylor expansion that

$$\begin{aligned} ln\left(\frac{L_t}{L_0}\right) &= \sum_i ln\left(1 + R_i^L\right) \\ &= \sum_i ln\left(1 + \beta R_i^S - a_i \Delta t\right) \\ &= \sum_i \left(\beta R_i^S - a_i \Delta t - \frac{\beta^2}{2}(R_i^S)^2\right) + \sum_i (O(|R_i^S|^3) + O(|R_i^S|\Delta t)) \end{aligned} \tag{21}$$

By the same token, we have

$$\begin{aligned} \beta ln\left(\frac{S_t}{S_0}\right) &= \beta \sum_i ln\left(1 + R_i^S\right) \\ &= \beta \sum_i \left(R_i^S - \frac{1}{2}(R_i^S)^2\right) + \sum_t O((R_i^S)^3) \end{aligned} \tag{22}$$

Subtracting equation (22) from equation (21), we find that

$$
\begin{aligned}
ln\left(\frac{L_t}{L_0}\right) - \beta ln\left(\frac{S_t}{S_0}\right) &= -\sum_i \left(a_i \Delta t + \frac{\beta^2-\beta}{2}(R_i^S)^2\right) + \sum_i (O(|R_i^S|^3) + O(|R_i^S|\Delta t)) \\
&= -\sum_i \left(a_i \Delta t + \frac{\beta^2-\beta}{2}((R_i^S)^2 - \mu^2(\Delta t)^2)\right) \\
&\quad + \sum_i (O(|R_i^S|^3) + O(|R_i^S|\Delta t) + O((\Delta t)^2)) \\
&= -\sum_i a_i \Delta t - \frac{\beta^2-\beta}{2} V_t + \sum_i (O(|R_i^S|^3) + O(|R_i^S|\Delta t) + O((\Delta t)^2))
\end{aligned}
\tag{23}
$$

We show that the remainder in this last equation is negligible. In fact, we have

$$
\begin{aligned}
\sum_i |R_i^S|^3 &= \sum_i |\xi_i|^3 (\Delta t)^{3/2} \\
&= \left(\frac{\sum_i |\xi_i|^3}{N}\right) t\sqrt{\Delta t} \\
&\approx E(|\xi_1^3|)\, t\sqrt{\Delta t}
\end{aligned}
\tag{24}
$$

and, similarly,

$$
\begin{aligned}
\sum_i |R_i^S|\Delta t &= \sum_i |\xi_i| (\Delta t)^{3/2} \\
&= \left(\frac{\sum_i |\xi_i|}{N}\right) t\sqrt{\Delta t} \\
&\approx E(|\xi_1|)\, t\sqrt{\Delta t}.
\end{aligned}
\tag{25}
$$

Therefore, the contribution of the last sum in (23) is bounded by the first three moments of ξ_1, multiplied by the investment horizon, t and by $\sqrt{\Delta t}$. This means that if we neglect the last terms, for investment horizons of less than 1 year, the error is of the order of the standard deviation of the daily returns of the underlying ETF, which is neglig

7.2 Continuous-time model (Equation (10))

The above reasoning is exact for Ito processes, because it corresponds to $\Delta t \to 0$. To be precise, we consider equations (7), (8) and (9) and apply Itô's Lemma to obtain

$$
\begin{aligned}
dlnS_t &= \frac{dS_t}{S_t} - \frac{\sigma_t^2}{2}dt \qquad (26)\\
dlnL_t &= \beta\frac{dS_t}{S_t} - \frac{\beta^2\sigma_t^2}{2}dt + ((\beta-1)r + f + \beta\lambda_i)dt, \quad (\lambda_i = 0 \ \text{ if } \ \beta > 0) \qquad (27)
\end{aligned}
$$

Multiplying equation (26) by β and subtracting it from equation (27) , we obtain

$$
dlnL_t - \beta dlnS_t = -\frac{(\beta^2-\beta)\sigma_t^2}{2}dt + ((\beta-1)r + f + \beta\lambda_i)dt, \qquad (28)
$$

which implies equation (10).

7.3 Dynamic Replication

We neglect the borrowing costs λ_t, for simplicity. The risk-neutral measure is such that L_t satisfies the stochastic differential equation

$$
\frac{dL_t}{L_t} = \beta\sigma dW_t + (r - \beta d)dt \qquad (29)
$$

where d is the dividend yield of the underlying ETF. The reason for this is that the holder of the LETF receives (pays) the corresponding multiple of the dividend of the underlying index, as in a total-return swap. Notice that the risk-neutral measure does not involve the expense-ratio, f.

Using this last equation, we have

$$
\begin{aligned}
V(L,A,t) &= e^{-r(T-t)}E\left\{m\left(\left(\frac{L_T}{L_0}\right)^{\frac{1}{\beta}}e^{A(0,T)} - 1\right)|L_t = L, A(0,t) = A\right\}\\
&= e^{-r(T-t)}m\left(\frac{L}{L_0}\right)^{\frac{1}{\beta}}e^{A}E\left\{\left(\frac{L_T}{L}\right)^{\frac{1}{\beta}}e^{A(t,T)}|L_t = L, A(0,t) = A\right\} - e^{-r(T-t)}m\\
&= e^{-r(T-t)}m\left(\frac{L}{L_0}\right)^{\frac{1}{\beta}}e^{A}e^{(r-\beta d)\frac{T-t}{\beta} + \frac{(\beta-1)r(T-t)}{\beta} + \frac{f(T-t)}{\beta}} - e^{-r(T-t)}m\\
&= e^{-r(T-t)}m\left(\frac{L}{L_0}\right)^{\frac{1}{\beta}}e^{A}e^{r(T-t)-d(T-t)+\frac{f(T-t)}{\beta}} - e^{-r(T-t)}m\\
&= e^{-d(T-t)}m\left(\frac{L}{L_0}\right)^{\frac{1}{\beta}}e^{A}e^{\frac{f(T-t)}{\beta}} - e^{-r(T-t)}m \qquad (30)
\end{aligned}
$$

where $A(t,T)$ is defined in equation (17). This is a direct consequence of Equation (10). Set $U(L,A,t) = e^{-d(T-t)}m\left(\frac{L}{L_0}\right)^{1/\beta}e^{A}e^{\frac{f(T-t)}{\beta}}$,so that $\Delta(L,A,t) = L\frac{\partial U}{\partial L} = \frac{U}{\beta}$. By Itô's Lemma,

$$
\begin{aligned}
dU(L_t, A(0,t), t) &= \frac{\partial U}{\partial L} dL_t + \frac{1}{2}\frac{\partial^2 U}{\partial L^2}(dL_t)^2 + U dA(0,t) + (d - \frac{f}{\beta})U dt \\
&= \frac{1}{\beta} U \frac{dL_t}{L_t} + \frac{1}{2}\frac{1}{\beta}\left(\frac{1}{\beta} - 1\right) U \frac{(dL_t)^2}{L_t^2} \\
&\quad + U\left[\frac{\beta - 1}{2}\sigma^2 + \frac{\beta - 1}{\beta} r + \frac{f}{\beta}\right] dt + (d - \frac{f}{\beta}) U dt \\
&= \frac{1}{\beta} U \frac{dL_t}{L_t} + \frac{1}{2}\frac{1}{\beta}\left(\frac{1}{\beta} - 1\right) U \sigma^2 \beta^2 dt + (Ud) dt \\
&\quad + U\left[\frac{\beta - 1}{2}\sigma^2 + \frac{\beta - 1}{\beta} r\right] dt \\
&= \frac{1}{\beta} U \frac{dL_t}{L_t} + U\left[d + \frac{\beta - 1}{\beta} r\right] dt \\
&= \frac{1}{\beta} U \frac{dL_t}{L_t} + rU dt + \frac{1}{\beta} U \left(\beta d - r\right) dt. \qquad (31)
\end{aligned}
$$

A financed position consisting of one unit of the derivative security and short $\Delta(L, A, t) = \frac{U}{\beta}$ dollars of the LETF will have a profit-loss of

$$
\Pi = dV - \frac{1}{\beta} U \frac{dL_t}{L_t} - rV dt + (r - \beta d)\frac{1}{\beta} U dt,
$$

where the last two terms correspond to the financing of the derivative and that cost of carry for the hedge. Substituting formula (31), we find that

$$
\begin{aligned}
\Pi &= dU - rme^{-r(T-t)} dt - \frac{1}{\beta} U \frac{dL_t}{L_t} - rV dt + (r - \beta d)\frac{1}{\beta} U dt \\
&= \frac{1}{\beta} U \frac{dL_t}{L_t} + rU dt + \frac{1}{\beta} U \left(\beta d - r\right) dt - rme^{-r(T-t)} dt \\
&- \frac{1}{\beta} U \frac{dL_t}{L_t} - rV dt + (r - \beta d)\frac{1}{\beta} U dt \\
&= rU dt - rme^{-r(T-t)} dt - rV dt \\
&= 0, \qquad (32)
\end{aligned}
$$

which shows us that continuous delta-hedging gives an exact replication of the target return, as suggested by the empirical study.