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Leveraged exchange-traded funds: admissible leverage and risk horizon

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This paper provides a quantitative risk analysis of leveraged exchange-traded funds (LETFs) with a focus on the impact of leverage and investment horizon. From the empirical returns of several major LETFs based on the S&P 500 index, the performance of LETFs generally declines as the investment horizon increases, compared with the unleveraged ETF on the same index. The value erosion is more severe for highly leveraged ETFs. To better understand the risk impact of leverage, we introduce the admissible leverage ratio induced by a risk measure, for example, value-at-risk (VaR) and conditional VaR. This idea can help investors exclude LETFs that are deemed too risky. Moreover, we also discuss the concept of admissible risk horizon so that the investor can control risk exposure by selecting an appropriate holding period. In addition, we also compute the intrahorizon risk, which leads us to evaluate a stop-loss/take-profit strategy for LETFs. Lastly, we investigate the impact of volatility exposure on the return of different LETF portfolios.

1 INTRODUCTION

The market of exchange-traded funds (ETFs) has grown substantially in recent years. As of May 31, 2012, the US ETF industry consists of 1251 funds, with over US$1.1 trillion in assets under management (AUM) (see Mazza (2012)). Within the ETF market, leveraged ETFs (LETFs) have gained popularity among some investors. LETFs are typically designed to replicate multiples of the daily returns of some underlying index or benchmark. For example, the ProShares Ultra S&P 500 (SSO) is supposed to generate twice the daily returns of the S&P 500 index, minus a small expense fee. Moreover, investors can take a bearish position on the underlying index by longing an inverse LETF (with a negative leverage ratio). An example is the ProShares

UltraShort S&P 500 (SDS) on the S&P 500 with leverage ratio of -2. In addition, both long and short triple LETFs are also available for various underlyings. For many investors, LETFs are a highly accessible and liquid instrument. They also tend to be more effective during periods of large momentum and low volatility.

On the other hand, LETFs have been subject to a number of criticisms. Some argue that they tend to underperform over extended (quarterly or annual) investment horizons, as compared with the promised multiple of the underlying index returns (see Figure 1 on page 42 and Figure 2 on page 43). The underperformance has been attributed to ill-timed rebalancing, returns compounding and the use of derivatives to replicate returns. In a discrete-time model, Cheng and Madhavan (2009) illustrate that the LETF value can erode due to its dependence on the realized variance of the underlying index, coupled with daily rebalancing. Avellaneda and Zhang (2010) also discuss the path-dependent performance and potential tracking errors of LETFs under both discrete-time and continuous-time frameworks. In fact, the US Securities and Exchange Commission (SEC) has issued an alert announcement regarding the riskiness of LETFs,[1] and investigated whether LETFs could create a feedback effect and lead to increased market volatility (see Patterson (2011)). From the perspective of an LETF holder, it is crucial to understand the roles of the leverage ratio and the investment horizon in the risk and return of the fund.

In this paper, we provide a quantitative risk analysis of LETFs, with an emphasis on the impact of leverage and the investment horizon. A number of market observations suggest that value erosion is more severe for highly leveraged ETFs. As an example, we compare the empirical returns of several major LETFs based on the S&P 500 index against multiples of the unleveraged ETF. The performance of LETFs generally declines as the investment horizon increases, and this is even more visible for higher leverage ratios. This motivates the analysis on the risk impact of leverage as well as of the risk horizon.

Given an investment horizon, different leverage ratios imply different levels of riskiness. Therefore, we introduce the idea of an admissible range of leverage ratios. These are the leverage ratios for which the associated LETFs satisfy a given risk constraint based on, for example, the value-at-risk (VaR) and conditional VaR. This idea can help investors exclude LETFs that are deemed too risky. Moreover, we also discuss the concept of admissible risk horizon so that the investor can control risk exposure by selecting an appropriate holding period. In addition, we also compute the intrahorizon risk and find that higher leverage can significantly increase the probability of the LETF value hitting a lower level. This leads us to evaluate a stop-loss/take-profit strategy for LETFs. In particular, we determine the optimal take-profit given a

[1] See www.sec.gov/investor/pubs/leveragedetfs-alert.htm.

stop-loss risk constraint. Lastly, we investigate the impact of volatility exposure on the returns of different LETF portfolios.

2 EMPIRICAL RETURNS OF LETFS

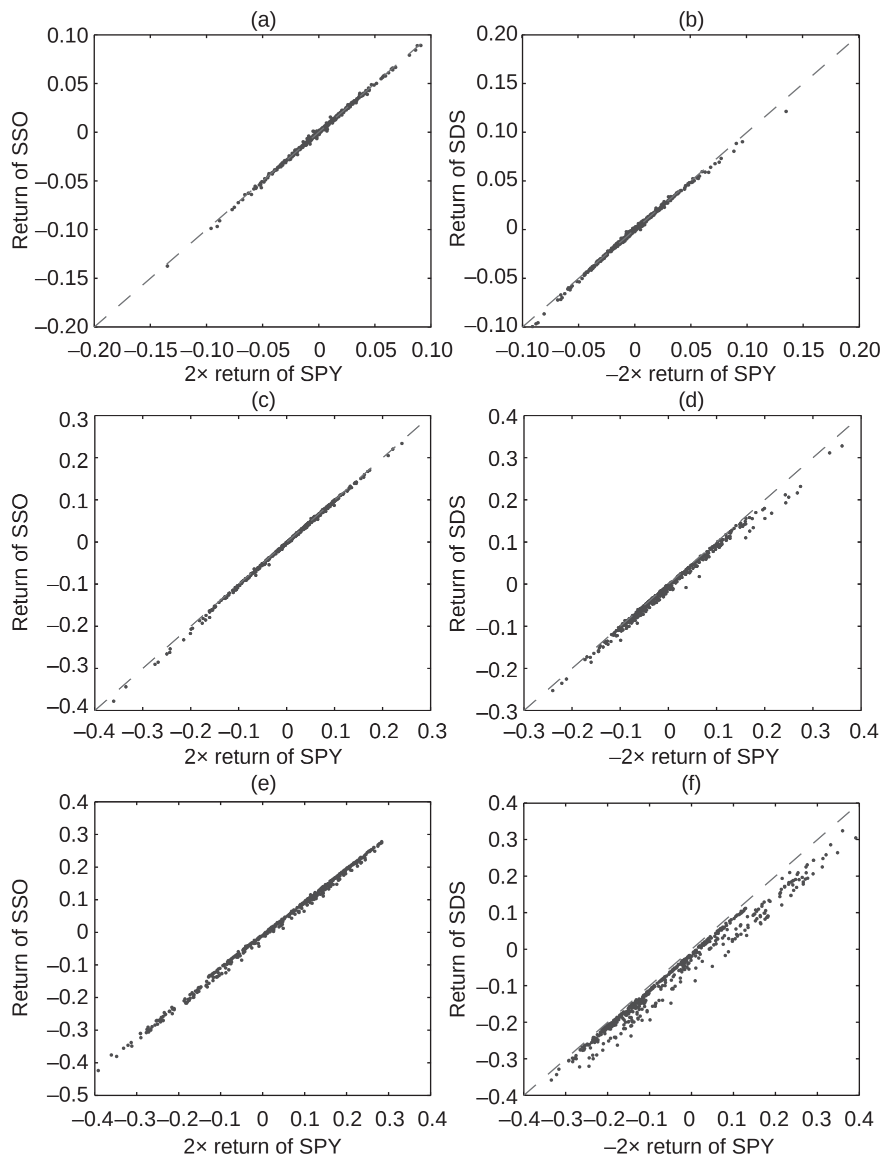
In this section, we first compare the historical returns between the SPDR S&P 500 ETF (SPY) and a number of LETFs that also track the S&P 500 with different leverage ratios. In Figure 1 on the next page, we show the returns of the ProShares Ultra S&P 500 ETF (SSO) with double long leverage and the ProShares UltraShort S&P 500 ETF (SDS) with double short leverage, against ± 2 multiples of the SPY returns. We consider one-day, fourteen-day and sixty-day rolling periods from September 29, 2010 to September 30, 2012.

In parts (a) and (b) of Figure 1 on the next page we observe that the returns fall along the 45° line. This reflects that for both SSO and SDS we are able to replicate, on a daily basis, the relative multiple of the underlying ETF returns. However, when the holding period lengthens to fourteen days and sixty days, return discrepancies start to build (see parts (c)–(f) of Figure 1 on the next page). In these cases, the LETF performance is often inferior to that of the underlying ETF, though the opposite could also happen, typically in a period with strong momentum. In general, a longer horizon also magnifies the erosion effect due to volatility drag. We shall investigate this more closely in subsequent sections.

In Figure 2 on page 43 we present the same analysis between SPY and the triple-leveraged ETFs UPRO ($3\times$) and SPXU ($-3\times$). As we can see, the one-day returns are matched very closely, but longer horizons again lead to higher discrepancies in returns between the triple LETFs and the underlying. Comparing across leverage ratios, the underperformance over a sixty-day period is more pronounced for the triple than the double leverage ratios (see parts (e) and (f) of Figure 1 on the next page and parts (e) and (f) of Figure 2 on page 43). Finally, short LETFs tend to fail to replicate the required returns more often than their long leveraged counterparts.

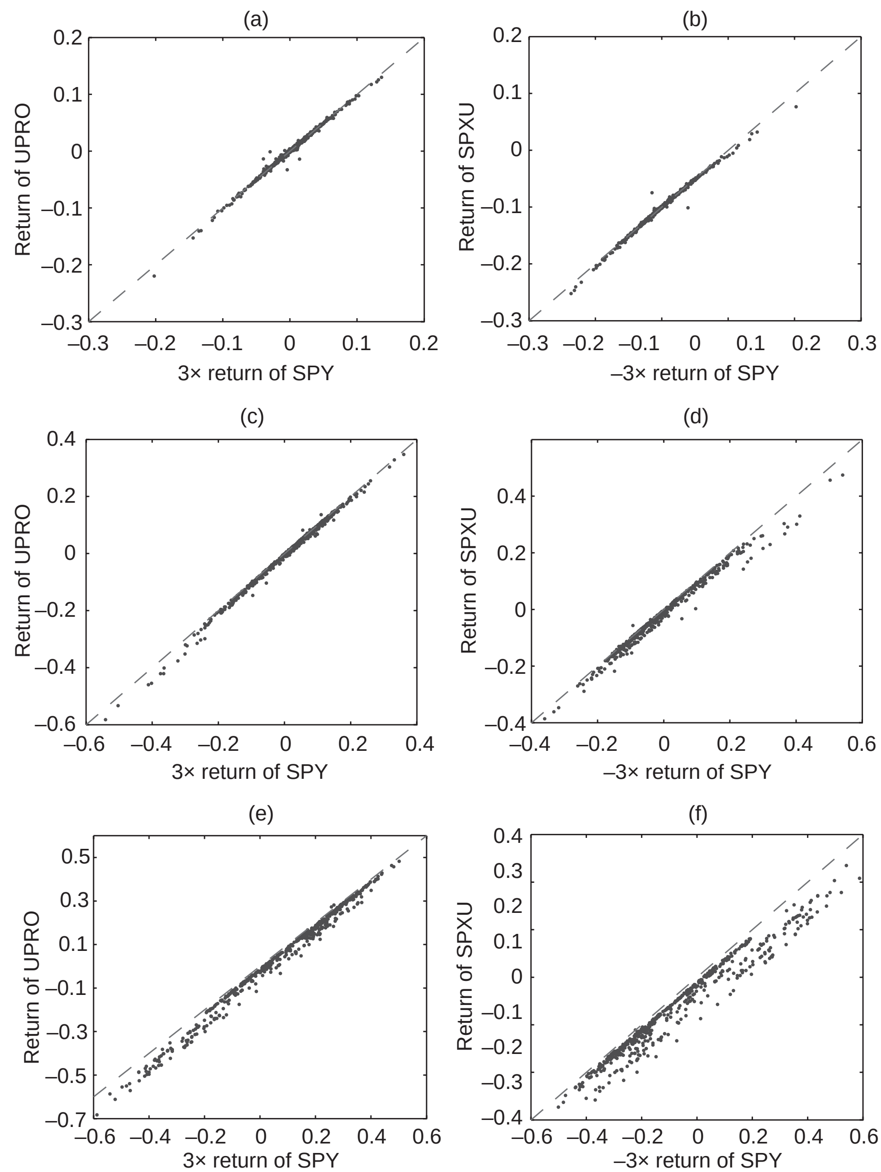
Although long and short LETFs are supposed to move in opposite directions daily by design, it is possible for both to have negative cumulative returns at the same point in time. Figure 3 on page 44 shows the historical cumulative returns of SSO and SDS from December 2010 to November 2011. Over several time intervals in this period, both LETFs have negative cumulative returns. This coincides with the highly volatile period in the market during the second half of 2011. This result, though counterintuitive at first glance, is a natural consequence of the value erosion effect generated by daily tracking of leveraged returns, and is magnified during periods of high volatility.

FIGURE 1 One-day (top), two-week (center) and two-month (bottom) returns of SPY against SSO (left) and SDS (right), in logarithmic scale.



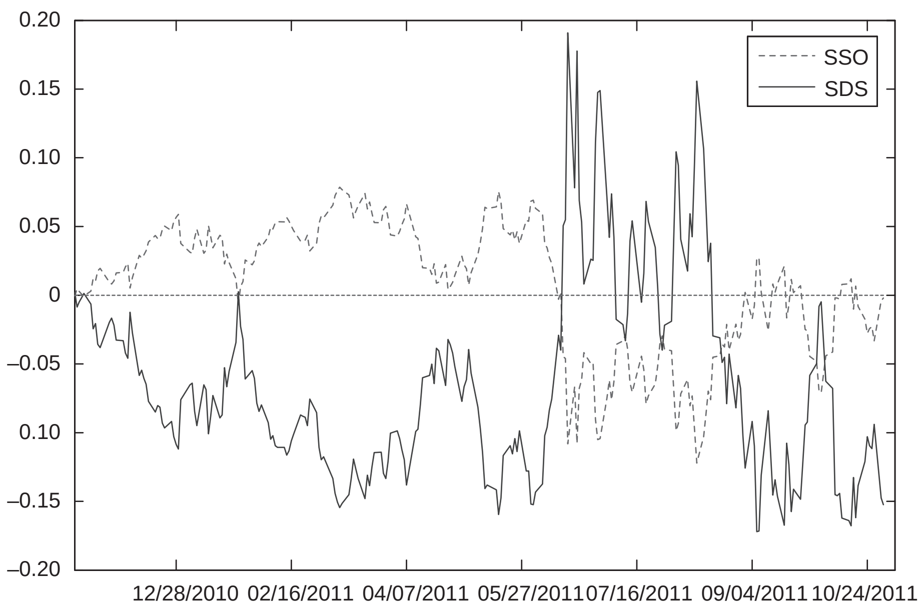
We considered one-day, two-week and two-month rolling periods from September 29, 2010 to September 30, 2012. (a) SPY versus SSO, one-day returns. (b) SPY versus SDS, one-day returns. (c) SPY versus SSO, two-week returns. (d) SPY versus SDS, two-week returns. (e) SPY versus SSO, two-month returns. (f) SPY versus SDS, two-month returns.

FIGURE 2 One-day (top), two-week (center) and two-month (bottom) returns of SPY against UPRO (left) and SPXU (right), in logarithmic scale.



We considered one-day, two-week and two-month rolling periods from September 29, 2010 to September 30, 2012. (a) SPY versus UPRO: one-day returns. (b) SPY versus SPXU: one-day returns. (c) SPY versus UPRO: two-week returns. (d) SPY versus SPXU: two-week returns. (e) SPY versus UPRO: two-month returns. (f) SPY versus SPXU: two-month returns.

FIGURE 3 SSO and SDS cumulative returns from December 2010 to November 2011.



Observe that both SSO and SDS can give negative returns (below the dotted line) simultaneously over several periods in time.

3 PRICE DYNAMICS OF LEVERAGED EXCHANGE-TRADED FUNDS

We model the evolution of the underlying index $(S_t)_{t \geqslant 0}$ by a geometric Brownian motion (GBM):

$$\mathrm{d}S_t = S_t(\mu \,\mathrm{d}t + \sigma \,\mathrm{d}W_t)$$

where W is a standard Brownian motion under the historical measure $\mathbb{P}$. The parameter μ is the ex-dividend annualized growth rate and $\sigma > 0$ is the constant volatility. Based on the reference index S, a long leveraged ETF $(L_t)_{t \geqslant 0}$ with leverage ratio $k \geqslant 1$ is constructed by simultaneously investing the amount kL_t (k times the fund value) in the underlying S, and borrowing the amount $(k-1)L_t$ at the interest rate $r \geqslant 0$. This is essentially a constant proportion trading strategy (see Haugh (2011)). As is typical for all ETFs, a small expense rate $f \geqslant 0$ is incurred. As a result, the k-LETF value evolves according to:

$$\mathrm{d}L_t = L_t k \frac{\mathrm{d}S_t}{S_t} - L_t((k-1)r + f)\,\mathrm{d}t \tag{3.1}$$

On the other hand, a leveraged fund with a negative leverage ratio $k \leqslant -1$ involves taking a short position of amount $|kL_t|$ in S and keeping $(1-k)L_t$ in the money market account. The fund value $(L_t)_{t \geqslant 0}$ also satisfies (3.1) with $k \leqslant -1$. For some short LETFs, it would be appropriate to incorporate the rate of borrowing $\lambda \geqslant 0$ for short-selling S. This can be achieved by replacing μ with $\mu + \lambda$ in (3.1) with $k \leqslant -1$. See Avellaneda and Zhang (2010) for this approach. Theoretically, we can also construct constant proportion portfolios with $k \in (-1, 1)$, but we do not discuss them since the most typical leverage ratios in practice are $k = 2, 3$ (long) and $k = -2, -3$ (short).

For both long and short LETFs, we recognize from (3.1) that L is again a GBM (see also Avellaneda and Zhang (2010)):

$$
\begin{aligned}
L_t &= L_0 \exp((k\mu - (k-1)r - f - \tfrac{1}{2}k^2\sigma^2)t + k\sigma W_t) && (3.2)\\
&= L_0 \left(\frac{S_t}{S_0}\right)^k \exp(-((k-1)r + f + \tfrac{1}{2}k(k-1)\sigma^2)t) && (3.3)
\end{aligned}
$$

Taking the log of both sides, we express the log return of L in terms of that of S, namely:

$$
\log\left(\frac{L_t}{L_0}\right) = k \log\left(\frac{S_t}{S_0}\right) - ((k-1)r + f + \tfrac{1}{2}k(k-1)\sigma^2)t \tag{3.4}
$$

In view of the second term, the long and short LETFs possess asymmetric return characteristics. Due to volatility exposure, for $|k| > 1$, there is an erosion in (log) return proportional to $\sigma^2 t$. Note that this effect is larger for a short LETF than its long leverage counterpart with the same leverage magnitude. In addition, the expense fee also leads to decay in return, as expected.

To motivate our analysis, we first look at the mean and standard deviation of the discounted relative return of LETFs:

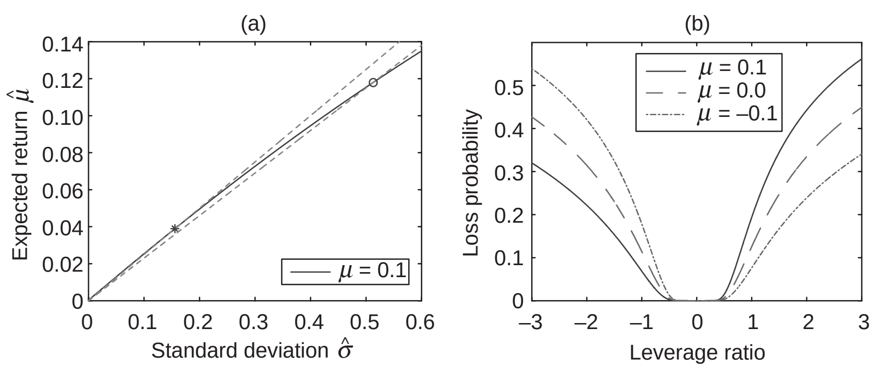
$$
\begin{aligned}
\hat{r}(k) &\equiv \mathbb{E}\{L_t/L_0 - 1\} = \mathrm{e}^{(k(\mu-r)+r-f)T} - 1\\
\hat{\sigma}(k) &\equiv \mathrm{std}\{L_t/L_0 - 1\} = \mathrm{e}^{(k(\mu-r)+r-f)T}\sqrt{\mathrm{e}^{k^2\sigma^2 T} - 1}
\end{aligned}
$$

When selecting a LETF, a risk-sensitive investor may consider the ratio $\hat{r}(k)/\hat{\sigma}(k)$ which, loosely speaking, represents the unit of return that we get for each unit of risk, or the mean–variance trade-off. To choose a range of leverage ratios, the investor can require that:

$$
\frac{\hat{r}(k)}{\hat{\sigma}(k)} \geqslant c \tag{3.5}
$$

where $c > 0$ is the mean–variance trade-off coefficient.

FIGURE 4 Mean-variance frontier and loss probability.



(a) The mean–variance frontier (solid), where each point corresponds to a different leverage ratio within [0, 3]. The "∘" (respectively, "∗") locates the critical leverage ratio k^* satisfying (3.5) in equality with $c = 0.23$ (respectively, $c = 0.25$). (b) Loss probability increases drastically with higher leverage ratios. Parameters: $z = 20\%$, $r = 2\%$, $T = 0.5$, $\sigma = 25\%$ and $f = 0.95\%$.

Figure 4 shows the mean–variance frontier, along which the leverage ratio varies from 0 to 3. With a positive drift $\mu = 10\%$, negative leverage ratios yield inferior expected return for the same standard deviation, and thus, are not shown in this figure. The mark "∘" (respectively, "∗") locates the critical leverage ratio k^* satisfying (3.5) in equality with $c = 0.23$ (respectively, $c = 0.25$). On the right-hand side of the marks, the ratio $\hat{r}(k)/\hat{\sigma}(k)$ falls below the required level c, effectively preventing the investor from selecting the higher leverage ratios.

4 ADMISSIBLE LEVERAGE RATIOS

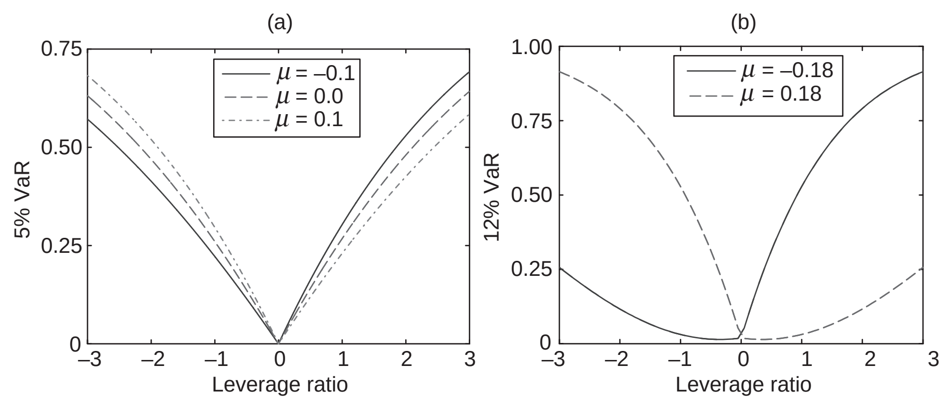
In this section, we investigate the impact of leverage ratio on the risk associated with an LETF. Specifically, we define the admissible leverage ratio based on two risk measures, namely, value-at-risk (VaR) and conditional VaR.

4.1 VaR

Given a fixed investment horizon T, the probability that the LETF will suffer a relative loss greater than $z \in [0, 1]$, is given by:

$$\begin{aligned} p(z, k, T) &= \mathbb{P}\{1 - L_T/L_0 > z\} \\ &= \Phi\left(\frac{\ln(1-z) - \psi(k)T}{|k|\sigma\sqrt{T}}\right) \end{aligned} \tag{4.1}$$

FIGURE 5 $\mathrm{VaR}_\alpha(k)$ is decreasing in k for $k \leqslant k^*$ and increasing for $k \geqslant k^*$.



(a) VaR_α is lowest at $k = 0$, and increases as the absolute value of leverage ratio $|k|$ increases. Parameters: $r = 2\%$, $T = 0.5$, $\sigma = 25\%$ and $f = 0.95\%$. (b) VaR_α achieves a minimum at a nonzero leverage, ie, $k^* \neq 0$ (see (4.5)). Parameters: $r = 0\%$, $T = 2$, $\sigma = 20\%$ and $f = 0.95\%$.

where $\Phi(\cdot)$ is the normal cumulative distribution function and:

$$\psi(k) = k(\mu - r) + r - f - \tfrac{1}{2}k^2\sigma^2 \tag{4.2}$$

Note that $\psi(k)t$ is related to the log return difference between L and S in (3.4).

In view of the continuous distribution of L_T, we define the (relative) value-at-risk, VaR_α, at confidence level $\alpha \in (0, 1)$, via the equation:

$$p(\mathrm{VaR}_\alpha, k, T) = \alpha \tag{4.3}$$

Intuitively, a larger VaR_α means a higher level of risk. For our analysis, $\mathrm{VaR}_\alpha \equiv \mathrm{VaR}_\alpha(k, T)$ is often viewed as a function of leverage ratio k and horizon T. Inverting the loss probability function in (4.3), we obtain an expression for VaR_α.

PROPOSITION 4.1 *Given any investment horizon T and leverage ratio k, the (relative) value-at-risk of holding the LETF is given by:*

$$\mathrm{VaR}_\alpha(k, T) = 1 - \exp(\psi(k)T + |k|\sigma\sqrt{T}\,\Phi^{-1}(\alpha)) \tag{4.4}$$

with $\psi(k)$ defined in (4.2).

To better understand the property of $\mathrm{VaR}_\alpha(k, T)$, we differentiate with respect to k to get:

$$\frac{\partial\,\mathrm{VaR}_\alpha}{\partial k} = ((\mu - r - k\sigma^2)T + \mathrm{sgn}(k)\Phi^{-1}(\alpha)\sigma\sqrt{T})(\mathrm{VaR}_\alpha - 1)$$

Note that this derivative is discontinuous at $k = 0$ and changes sign once. In practical applications, the term $\Phi^{-1}(\alpha)\sigma\sqrt{T}$ is negative, and therefore the jump in $k = 0$ is upward. Given these observations, either the derivative vanishes at some k^*, or the derivative is negative for $k < 0$ and positive for $k > 0$. To summarize, we define:

$$k^* = \begin{cases} \dfrac{\mu - r}{\sigma^2} + \dfrac{\Phi^{-1}(\alpha)}{\sigma\sqrt{T}} & \text{if } \dfrac{\mu - r}{\sigma^2} + \dfrac{\Phi^{-1}(\alpha)}{\sigma\sqrt{T}} > 0 \\ \dfrac{\mu - r}{\sigma^2} - \dfrac{\Phi^{-1}(\alpha)}{\sigma\sqrt{T}} & \text{if } \dfrac{\mu - r}{\sigma^2} - \dfrac{\Phi^{-1}(\alpha)}{\sigma\sqrt{T}} < 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.5}$$

and conclude that $\mathrm{VaR}_\alpha(k)$ is decreasing in k for $k \leqslant k^*$ and increasing for $k \geqslant k^*$. This is illustrated in Figure 5 on the preceding page. Also, note that k^* does not depend on the expense rate f and increases linearly with the excess return.

One way to describe an investor's risk tolerance is to consider the maximum VaR_α threshold $\bar{z}$ which leads to the inequality in k:

$$\mathrm{VaR}_\alpha(k, T) \leqslant \bar{z} \tag{4.6}$$

This will in turn exclude a range of leverage ratios k. Note that at $k = 0$:

$$\mathrm{VaR}_\alpha(0, T) = 1 - \mathrm{e}^{(r-f)T}$$

which is typically a low (positive) point of VaR_α, as discussed above and shown in Figure 5 on the preceding page. In part (a) of Figure 5 on the preceding page, VaR_α is always increasing in $|k|$ and $k^* = 0$. Moreover, VaR_α is not symmetric in k: VaR_α tends to be lower for those leverage ratios k with the same sign as drift μ. In part (b) of Figure 5 on the preceding page, we illustrate that, with a higher α, VaR_α reaches its minimum at $k^* \neq 0$. Therefore, if risk tolerance is very low, or if the expense ratio is high, there might not exist any leverage ratio k for which the investor is willing to invest. Precisely, we have the following result.

PROPOSITION 4.2 *The admissible range of leverage ratios based on criterion (4.6) is given by:*

$$I^- \cup I^+$$

where:

$$I^{\pm} = \begin{cases} [l^{\pm}, u^{\pm}] & \text{if } \Delta^{\pm} \text{ is real} \\ \emptyset & \text{otherwise} \end{cases}$$

with:

$$l_+ = \max\{0, \Gamma^+(\alpha) - \Delta^+(\alpha)\}, \qquad u_+ = \max\{0, \Gamma^+(\alpha) + \Delta^+(\alpha)\} \tag{4.7}$$

$$l_- = \min\{0, \Gamma^-(\alpha) - \Delta^-(\alpha)\}, \qquad u_- = \min\{0, \Gamma^-(\alpha) + \Delta^-(\alpha)\} \tag{4.8}$$

and:

$$\Gamma^{\pm}(\alpha) = \frac{1}{\sigma^2 T}((\mu - r)T \pm \Phi^{-1}(\alpha)\sigma\sqrt{T})$$
$$\Delta^{\pm}(\alpha) = \frac{1}{\sigma^2 T}\sqrt{((\mu - r)T \pm \Phi^{-1}(\alpha)\sigma\sqrt{T})^2 - 2\sigma^2 T((f - r)T + \ln(1 - \bar{z}))}$$

Using Proposition 4.2, we can identify precisely the interval of acceptable leverage ratios. To visualize this, we look at part (a) of Figure 5 on page 47 and consider the leverage ratios whose value-at-risk is lower than a given level. For instance, setting the risk tolerance level $\bar{z} = 0.5$, the leverage ratios 2 and -2 are admissible but 3 and -3 are excluded. In part (b) of Figure 5 on page 47, setting $\bar{z} = 0.25$, the admissible leverage ratio interval for $\mu = -18\%$ is $[l^-, u^-] \cup [l^+, u^+] = [-2.99, 1.21]$, so $k = -3$ is excluded. On the other hand, with $\mu = 18\%$, the admissible interval is $[-1.26, 2.76]$, so $k = 3$ is excluded. The concept of admissible leverage ratio provides a simple recipe for identifying LETFs that are too risky according to a given risk measure.

4.2 CVaR

In addition to value-at-risk, we define the conditional value-at-risk CVaR_α at confidence level α as:

$$\mathrm{CVaR}_\alpha(k, T) := \mathbb{E}\left\{1 - \frac{L_T}{L_0} \,\middle|\, 1 - \frac{L_T}{L_0} > \mathrm{VaR}_\alpha(k, T)\right\}$$

Proposition 4.3 *Given any investment horizon T and leverage ratio k, the CVaR for the LETF is given by:*

$$\mathrm{CVaR}_\alpha(k, T) = 1 - \mathrm{e}^{(k(\mu - r) + r - f)T}\,\frac{\Phi(\Phi^{-1}(\alpha) - |k|\sigma\sqrt{T})}{\alpha} \tag{4.9}$$

The $\mathrm{CVaR}_\alpha(k)$ *is decreasing in k for $k \leqslant k^{**}$ and increasing for $k \geqslant k^{**}$, with the critical leverage k^{**} satisfying:*

$$\frac{\mu - r}{\sigma\sqrt{T}}\Phi(\Phi^{-1}(\alpha) - |k^{**}|\sigma\sqrt{T}) = \mathrm{sgn}(k^{**})\phi(\Phi^{-1}(\alpha) - |k^{**}|\sigma\sqrt{T}) \tag{4.10}$$

Proof For any $k \in \mathbb{R}$, the expected (relative) shortfall is given by:

$$\mathbb{E}\left\{\frac{L_0 - L_T}{L_0} \,\middle|\, \frac{L_0 - L_T}{L_0} > z\right\} = 1 - \mathrm{e}^{(k(\mu - r) + r - f)T}\,\frac{\Phi(d_z - |k|\sigma\sqrt{T})}{\Phi(d_z)} \tag{4.11}$$

where:

$$d_z := \frac{\ln(1 - z) - \psi(k)T}{|k|\sigma\sqrt{T}}$$

Setting $z = \mathrm{VaR}_\alpha$ gives CVaR_α in (4.9). Next, we compute the derivative:

$$\begin{aligned}\frac{\partial \mathrm{CVaR}_\alpha}{\partial k} &= \frac{\mathrm{e}^{(k(\mu-r)+r-f)T}}{\alpha} \\ &\quad \times ((r-\mu)\Phi(\Phi^{-1}(\alpha) - |k|\sigma\sqrt{T}) + \operatorname{sgn}(k)\sigma\sqrt{T}\phi(\Phi^{-1}(\alpha) - |k|\sigma\sqrt{T}))\end{aligned}$$

The sign of the derivative depends on the term in the bracket, and equating this to zero yields the critical value k^{**} in (4.10). □

5 ADMISSIBLE RISK HORIZON

The risk analysis in the previous section sheds light on the choice of leverage ratios. Alternatively, the investor can control risk exposure by appropriately selecting the investment horizon. For risk management purposes, it is important to determine the maximum investment horizon τ such that the risk measure stays under some threshold $C \in (0, 1)$. This idea leads us to study the admissible risk horizon induced by a risk measure.

First, let us consider the value-at-risk $\mathrm{VaR}_\alpha(k, \tau)$ for a k-LETF and horizon τ. The admissible risk horizon $\mathrm{ARH}_\alpha(k, C)$ is defined by:

$$\mathrm{ARH}_\alpha(k, C) = \inf\{\tau \geqslant 0 \colon \mathrm{VaR}_\alpha(k, \tau) = C\} \tag{5.1}$$

and we set $\mathrm{ARH}_\alpha(k, C) = +\infty$ if the equation $\mathrm{VaR}_\alpha(k, \tau) = C$ has no positive root (in τ). Herein, we impose an upper bound of 0.5 on α so that $\Phi(\alpha) < 0$. Using (4.4), we invert (5.1) to get an explicit expression for $\mathrm{ARH}_\alpha(k, C)$.

PROPOSITION 5.1 *Define* $b = |k|\sigma\Phi^{-1}(\alpha)$. *If* $\frac{1}{4}b^2 \geqslant -\psi(k)\ln(1 - C)$, *then the admissible risk horizon for the* k*-LETF with* VaR_α *limited at* C *is given by:*

$$\mathrm{ARH}_\alpha(k, C) = \left(\frac{-\frac{1}{2}b - \sqrt{\frac{1}{4}b^2 + \psi(k)\ln(1 - C)}}{\psi(k)}\right)^2 \tag{5.2}$$

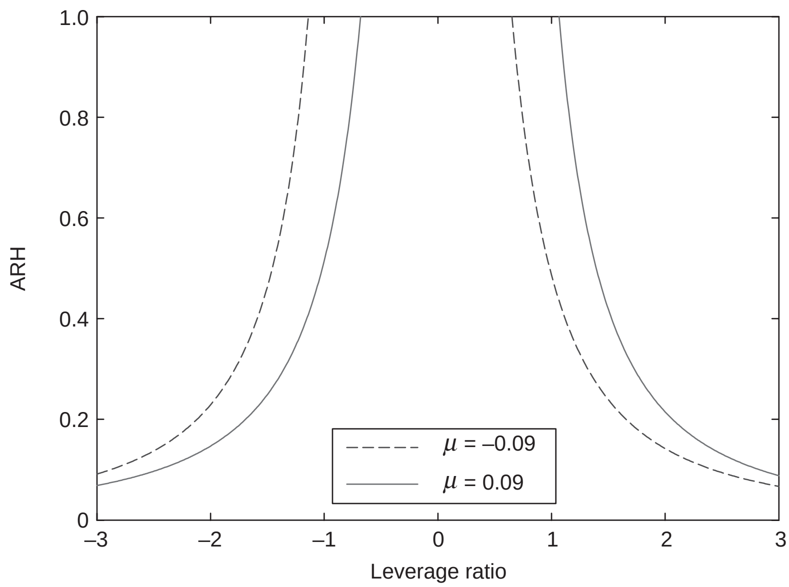
If $\frac{1}{4}b^2 < -\psi(k)\ln(1 - C)$, *then* $\mathrm{ARH}_\alpha(k, C) = +\infty$.

To gain some insight on the conditions in Proposition 5.1, we first recall that:

$$\mathrm{VaR}_\alpha(k, T) = 1 - \exp(\psi(k)T + |k|\sigma\sqrt{T}\Phi^{-1}(\alpha))$$

If, given a certain k, $\psi(k) > 0$, then VaR_α is convex and eventually decreasing in T, and its maximum can potentially lie below the threshold C. In fact, whether this

FIGURE 6 The admissible risk horizon tends to decrease as leverage ratio deviates from zero.



Parameters: $C = 0.25$, $r = 1\%$, $\alpha = 5\%$, $\sigma = 25\%$ and $f = 0.95\%$.

happens or not is determined by the condition $\frac{1}{4}b^2 < -\psi(k)\ln(1-C)$. In reality, we tend to have $\psi(k) > 0$ when k is small or when $\operatorname{sgn}(k)\mu$ is large. The latter means that the investment has a high rate of return, so intuitively VaR_α can stay low.

On the other hand, when $\frac{1}{4}b^2 > -\psi(k)\ln(1-C)$ and $\psi(k) > 0$, the equation $\mathrm{VaR}_\alpha(k,\tau) = C$ admits two positive roots and (5.2) selects only the smallest root, according to (5.1).

In contrast, if $\psi(k) < 0$, then VaR_α is increasing in T. Consequently, the equation $\mathrm{VaR}_\alpha(k,\tau) = C$ always admits a unique strictly positive solution (note that $\mathrm{VaR}_\alpha(k,0) = 0$).

Figure 6 illustrates how ARH_α varies for different values of k and μ. As we can see, the admissible risk horizon increases as $|k|$ decreases. For any fixed positive leverage ratio, the admissible risk horizon tends to increase with drift μ. In addition, it is also preferable to choose a leverage ratio, say $\hat{k}$, with the same sign as that of μ, since the corresponding admissible risk horizon is greater than that of $-\hat{k}$.

Similarly, we can define an admissible risk horizon based on other risk measures. For instance, the admissible risk horizon $\widehat{\mathrm{ARH}}_\alpha(k,C)$ based on the CVaR is determined from the equation:

$$\widehat{\mathrm{ARH}}_\alpha(k, C) = \inf\{\tau \geqslant 0 \colon \mathrm{CVaR}_\alpha(k, \tau) = C\} \tag{5.3}$$

In this case an analytical solution is not available. Nevertheless, we can easily find the zero(s) of the function $g(\tau) := \mathrm{CVaR}_\alpha(k, \tau) - C$.

6 INTRAHORIZON RISK

Both VaR and CVaR concern the loss distribution at a fixed future date, even though the LETF value may experience large losses at intermediate times. In reality, investors may monitor the asset price movement and impose a stop-loss level to limit downside risk. This motivates us to model a stochastic holding period until the LETF falls to a certain threshold.

We define the first passage time that the LETF, starting at L_0, reaches a lower level ℓL_0:

$$\tau_\ell = \inf\{t \geqslant 0 \colon L_t \leqslant \ell L_0\}, \quad \ell \in (0, 1)$$

With this, we define the intrahorizon loss probability:

$$\underline{p}(\ell, k, T) = \mathbb{P}\{\tau_\ell \leqslant T\}$$

This probability is related to the minimum of L over $[0, T]$, denoted by $\underline{L}_T = \min_{0 \leqslant t \leqslant T} L_t$, and admits an explicit expression:

$$\begin{aligned} \underline{p}(\ell, k, T) &= \mathbb{P}\{\underline{L}_T \leqslant \ell L_0\} \\ &= \Phi\left(\frac{\ln(\ell) - \psi(k)T}{|k|\sigma\sqrt{T}}\right) + \ell^{2\psi(k)/(k\sigma)^2}\Phi\left(\frac{\ln(\ell) + \psi(k)T}{|k|\sigma\sqrt{T}}\right) \end{aligned} \tag{6.1}$$

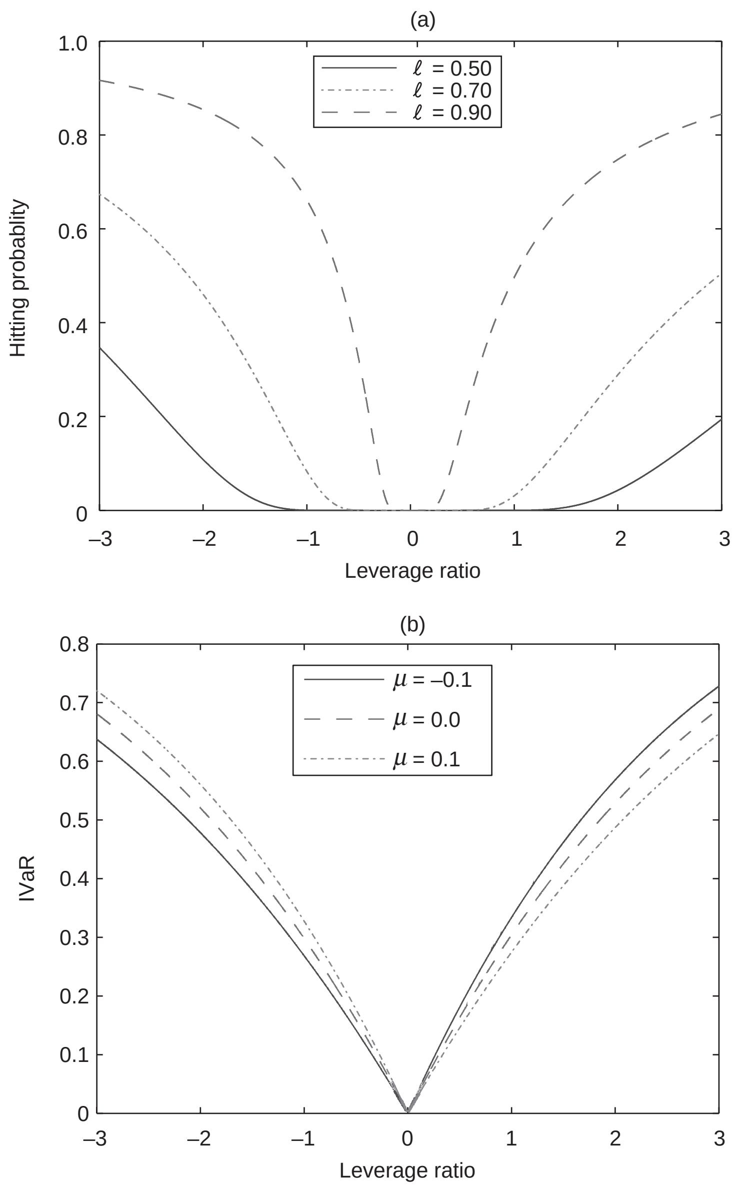
where $\psi(k)$ is defined in (4.2).

Part (a) of Figure 7 on the facing page shows how $\underline{p}(\ell, k)$ varies with respect to k for three different threshold levels. As ℓ or k increases, the intrahorizon loss probability $\underline{p}(\ell, k)$ increases. In addition, for any pair of leverage ratios with the same absolute value, $\underline{p}(\ell, k)$ is lower for the positive leverage ratio.

Given any loss level ℓL_0, it is useful to know whether an LETF will reach it in the long run. This amounts to determining whether the probability $\mathbb{P}\{\tau_\ell < \infty\}$ is equal to or strictly less than 1. Taking $T \uparrow \infty$ in (6.1), direct computation shows that when $\psi(k) \leqslant 0$ we have $\mathbb{P}\{\tau_\ell < \infty\} = 1$. In other words, to ensure $\mathbb{P}\{\tau_\ell < T\} < 1$, we have to restrict the leverage choice so that $\psi(k) > 0$.

To understand the condition in terms of leverage ratio k, we recall from (4.2) that $\psi(k)$ is not always of constant sign, and is quadratic in leverage ratio k. In particular,

FIGURE 7 (a) Probability of hitting a lower level ℓ for different leverage ratios in $[-3, 3]$, with $\mu = 10\%$; (b) intrahorizon VaR versus leverage ratios, with $\alpha = 5\%$.



Common parameters for both plots: $r = 2\%$, $T = 0.5$, $\sigma = 25\%$ and $f = 0.95\%$.

for large $|k|$, the volatility σ will have a dominant negative impact. If $f \leqslant r$, then we have $\psi(k) > 0$ over the interval of leverage ratios:

$$k \in \left(\frac{\mu - r - \sqrt{(\mu - r)^2 + 2\sigma^2(r - f)}}{\sigma^2}, \frac{\mu - r + \sqrt{(\mu - r)^2 + 2\sigma^2(r - f)}}{\sigma^2} \right)$$

Note that the width of the leverage interval is proportional to the excess return $\mu - r$, and inversely proportional to volatility. On the other hand, if $f > r$, then the above interval holds if and only if $(\mu - r)^2 \geqslant 2\sigma^2(f - r)$. Otherwise, we have $\psi(k) \leqslant 0$ and the interval of leverage ratio does not exist. In this case, the hitting time τ_ℓ is finite almost surely.

The intrahorizon loss probability leads us to define the *intrahorizon* value-at-risk, denoted by IVaR_α, via the equation:

$$\underline{p}(1 - \mathrm{IVaR}_\alpha, k, T) = \alpha \tag{6.2}$$

This is a modification of value-at-risk, and it incorporates the possibility that the LETF can fall below a lower level. Due to the complicated form of (6.1), this risk metric does not admit an explicit formula. Nevertheless, the numerical solution for IVaR_α in (6.2) involves a straightforward and instant root finding.

In part (b) of Figure 7 on the preceding page, we illustrate how IVaR_α varies with respect to k for different values of μ. Similar to VaR_α, IVaR_α also increases as the leverage ratio deviates from 0. Note that with common parameters we have $\mathrm{IVaR}_\alpha(k, T) > \mathrm{VaR}_\alpha(k, T)$. This can be inferred from the definitions (4.3) and (6.2), along with the fact that $\underline{p}(1 - z, k, T) > p(z, k, T)$ (see (4.1) and (6.1)).

Proposition 6.1 *The intrahorizon value-at-risk admits the partial derivative:*

$$\begin{aligned} \frac{\partial \mathrm{IVaR}_\alpha}{\partial k} &= (1 - \mathrm{IVaR}_\alpha)|k|\sigma \frac{\partial \underline{p}}{\partial k} \\ &\quad \times \left(\frac{1}{\sqrt{T}} \phi(\hat{d}_-) + \frac{2\psi(k)}{|k|\sigma} (1 - \mathrm{IVaR}_\alpha)^{2\psi(k)/(k\sigma)^2} \Phi(\hat{d}_+) \right. \\ &\qquad \left. + \frac{(1 - \mathrm{IVaR}_\alpha)^{2\psi(k)/(k\sigma)^2}}{\sqrt{T}} \phi(\hat{d}_+) \right)^{-1} \end{aligned} \tag{6.3}$$

where:

$$\hat{d}_\pm := \frac{\ln(1 - \mathrm{IVaR}_\alpha) \pm \psi(k) T}{|k|\sigma\sqrt{T}}$$

We can deduce from (6.3) that the derivative would take the same sign as $\partial \underline{p}(z, k)/\partial k$, unless the drift term $\psi(k)$ is so negative that the denominator also becomes negative.

7 STOP-LOSS EXIT

The intrahorizon risk measure motivates a stop-loss exit strategy in order to limit downside risk during the investment horizon. Incorporating a stop-loss level ℓL_0, we define $\mathcal{R}_T = L_{T\wedge\tau_\ell}/L_0$ and express the expected relative value as:

$$\mathbb{E}\{\mathcal{R}_T\} = \ell\mathbb{P}\{\tau_\ell < T\} + \mathbb{E}\left\{\frac{L_T}{L_0}\mathbf{1}_{\{\tau_\ell > T\}}\right\} \tag{7.1}$$

The first term is given by:

$$\ell\mathbb{P}\{\tau_\ell < T\} = \ell\left[\Phi\left(\frac{\ln(\ell) - \psi(k)T}{|k|\sigma\sqrt{T}}\right) + \ell^{2\psi(k)/(k\sigma)^2}\Phi\left(\frac{\ln(\ell) + \psi(k)T}{|k|\sigma\sqrt{T}}\right)\right]$$

For the second term, we apply standard calculations to get:

$$\begin{aligned}\mathbb{E}\left\{\frac{L_T}{L_0}\mathbf{1}_{\{\tau_\ell > T\}}\right\} &= \exp((\psi(k) + \tfrac{1}{2}k^2\sigma^2)T) \\ &\quad\times\left[\Phi\left(|k|\sigma\sqrt{T} + \frac{\psi(k)\sqrt{T}}{|k|\sigma} - \frac{\ln(\ell)}{|k|\sigma\sqrt{T}}\right)\right. \\ &\qquad\left. - \ell^{2\psi(k)/(k\sigma)^2+2}\Phi\left(|k|\sigma\sqrt{T} + \frac{\psi(k)\sqrt{T}}{|k|\sigma} + \frac{\ln(\ell)}{|k|\sigma\sqrt{T}}\right)\right]\end{aligned} \tag{7.2}$$

Moreover, the variance of the expected relative value with a stop-loss exit also admits a closed-form formula. Precisely, the variance is given by:

$$\operatorname{var}\{\mathcal{R}_T\} = \mathbb{E}\{\mathcal{R}_T^2\} - (\mathbb{E}\{\mathcal{R}_T\})^2$$

with $\mathbb{E}\{\mathcal{R}_T\}$ from (7.1) and:

$$\begin{aligned}\mathbb{E}\{\mathcal{R}_T^2\} &= \frac{1}{L_0^2}\mathbb{E}\{\ell^2\mathbf{1}_{\{\tau_\ell < T\}} + L_T^2\mathbf{1}_{\{\tau_\ell > T\}}\} \\ &= \ell^2\left[\Phi\left(\frac{\ln(\ell) - \psi(k)T}{|k|\sigma\sqrt{T}}\right) + \ell^{2\psi(k)/(k\sigma)^2}\Phi\left(\frac{\ln(\ell) + \psi(k)T}{|k|\sigma\sqrt{T}}\right)\right] \\ &\quad + \exp(2T(\psi(k) + k^2\sigma^2)) \\ &\quad\times\left[\Phi\left(2|k|\sigma\sqrt{T} + \frac{\psi(k)\sqrt{T}}{|k|\sigma} - \frac{\ln(\ell)}{|k|\sigma\sqrt{T}}\right)\right. \\ &\qquad\left. - \ell^{2\psi(k)/(k\sigma)^2+4}\Phi\left(2|k|\sigma\sqrt{T} + \frac{\psi(k)\sqrt{T}}{|k|\sigma} + \frac{\ln(\ell)}{|k|\sigma\sqrt{T}}\right)\right]\end{aligned}$$

Next, suppose the investor seeks to take profit when the asset reaches a sufficiently high level. Then, it is useful to compute the probability that LETF will fall to the stop-loss level before reaching the take-profit level. To fix ideas, we denote τ_ℓ (respectively,

τ_h) as the time for the LETF to hit a lower level ℓL_0 (respectively, upper level hL_0), with $\ell \leqslant 1 \leqslant h$.

PROPOSITION 7.1 *The probability that a k-LETF reaches a lower level ℓL_0 before a higher level hL_0 is given by:*

$$\mathbb{P}\{\tau_\ell < \tau_h\} = \frac{1 - h^{-2\psi(k)/(k\sigma)^2}}{\ell^{-2\psi(k)/(k\sigma)^2} - h^{-2\psi(k)/(k\sigma)^2}} \tag{7.3}$$

The explicit loss probability formula (7.3) can be used to quantify the risk of holding an LETF with a take-profit/stop-loss strategy. A risk-sensitive investor may want to bound it by $q \in (0, 1)$, namely:

$$\mathbb{P}\{\tau_\ell < \tau_h\} \leqslant q \tag{7.4}$$

Let us think of the loss probability bound in (7.4) as exogenously specified. The investor selects the maximum admissible take-profit level $h_{\max}$ in terms of stop-loss level ℓ and probability bound q. From (7.3), we observe that $\mathbb{P}\{\tau_\ell < \tau_h\}$ monotonically increases with h, and we arrive at two cases.

(1) $\psi(k) > 0$ and $q \geqslant \ell^{2\psi(k)/(k\sigma)^2}$. As $h \to \infty$, $\mathbb{P}\{\tau_\ell < \tau_h\}$ is bounded by $\ell^{2\psi(k)/(k\sigma)^2} \leqslant 1$. If $q \geqslant \ell^{2\psi(k)/(k\sigma)^2}$, then the take-profit level can be arbitrarily high by the investor, ie, $h_{\max} = \infty$.

(2) $\psi(k) > 0$ and $q < \ell^{2\psi(k)/(k\sigma)^2}$, or $\psi(k) < 0$. From (7.3), $h = 1 \Rightarrow \mathbb{P}\{\tau_\ell < \tau_h\} = 0$. By monotonicity, there exists a finite upper level $h_{\max} \in [1, \infty)$. By rearranging the inequality (7.4), we obtain the maximum admissible take-profit level:

$$h_{\max} = \left(\frac{1 - \ell^{-2\psi(k)/(k\sigma)^2} q}{1 - q}\right)^{-k^2\sigma^2/2\psi(k)} \tag{7.5}$$

From this result, we also infer that $h_{\max}$ decreases with ℓ and increases with q.

8 LETF PORTFOLIOS AND VOLATILITY EXPOSURE

LETFs can be used in combination to construct various portfolios. Some strategies are designed based on the anticipated tracking errors or value erosion of LETFs. On the other hand, we have seen that volatility exposure will diminish LETF returns. Hence, in this section, we discuss how the performance of some example strategies depends on volatility exposure.

For our analysis, the underlying index S is assumed to follow a general diffusion price dynamics:

$$\mathrm{d}S_t = S_t(\mu \, \mathrm{d}t + \sigma_t \, \mathrm{d}W_t)$$

where $(\sigma_t)_{t \geqslant 0}$ is the stochastic volatility process. Then, for both long and short LETFs, the LETF value is given by:

$$L_t = L_0 \left(\frac{S_t}{S_0} \right)^k \exp((-(k-1)r - f)t - \tfrac{1}{2}k(k-1)V_t) \tag{8.1}$$

where:

$$V_t = \int_0^t \sigma_u^2 \, \mathrm{d}u$$

is the realized variance of S up to time t.

8.1 Short $\pm k$ LETFs

Due to volatility drag, both long and short leveraged ETFs tend to lose value over time. Based on this belief, some investors consider taking short positions in LETFs with both positive and negative leverage ratios. Specifically, let $k > 1$ and consider a portfolio that shorts one dollar of a (positively) k-LETF and simultaneously shorts one dollar of the $-k$-LETF. The two dollars received form short-selling the LETFs is assumed to earn at the risk-free rate $r \geqslant 0$. Without loss of generality, we take $S_0 = 1$. The portfolio value at time t can be expressed in terms of the ETF value S_t and its realized variance V_t:

$$\begin{aligned} \mathcal{E}(S_t, V_t) = {} & -S_t^k \exp(-(r(k-1) + f)t - \tfrac{1}{2}k(k-1)V_t) \\ & - S_t^{-k} \exp((r(k+1) - f)t - \tfrac{1}{2}k(k+1)V_t) + 2\mathrm{e}^{rt}, \quad k > 1 \end{aligned} \tag{8.2}$$

Note that the portfolio value can be positive or negative. While we have chosen to short both $\pm k$-LETFs with identical absolute leverage ratio values, it is straightforward to construct a similar portfolio by shorting LETFs with completely different leverage ratios.

At any fixed $t \geqslant 0$, we determine the set of nonnegative portfolio values:

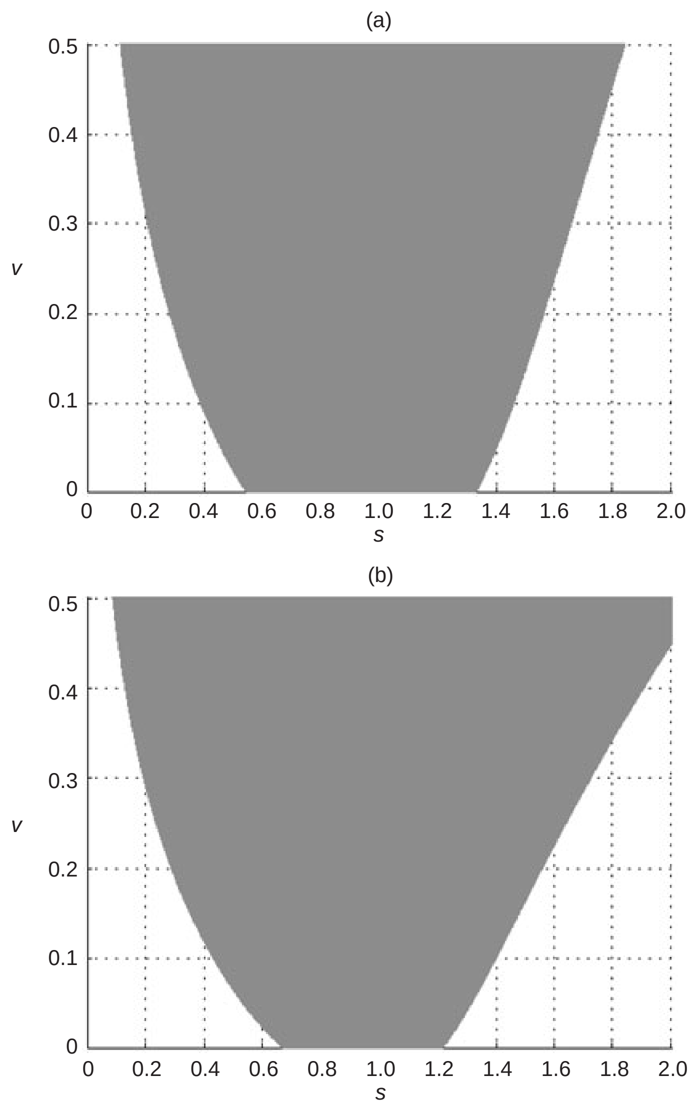
$$\mathcal{S}_k = \{(s, v) \in \mathbb{R}_+^2 : \mathcal{E}(s, v) \geqslant 0\}$$

where:

$$\begin{aligned} \mathcal{E}(s, v) = {} & -s^k \exp(-(r(k-1) + f)t - \tfrac{1}{2}k(k-1)v) \\ & - s^{-k} \exp((r(k+1) - f)t - \tfrac{1}{2}k(k+1)v) + 2\mathrm{e}^{rt} \end{aligned}$$

In other words, we seek to determine all the pairs of ETF value s and realized variance v such that the portfolio value is nonnegative. Interestingly, we do not need to specify the parameters μ and σ in order to study the value of $\mathcal{E}(s, v)$. Instead, we look at the ETF value s and realized variance v.

FIGURE 8 The set $\mathcal{S}_k$ for $k = 2, 3$.



The shaded region indicates where the portfolio value is nonnegative. For a high realized variance v, the region tends to be larger for $k = 3$ (part (a)) than for $k = 2$ (part (b)). Parameters: $T = 1$, $r = 1\%$ and $f = 0.95\%$.

Intuitively, when the future ETF value S_T deviates from 1, that means the ETF has experienced a large directional movement. If the realized variance V_T is large, then there is a lot of fluctuation in the ETF evolution. In order to visualize their effects on the profitability of the investment strategy, we illustrate in Figure 8 on the facing page the set $\mathcal{S}_k$ for $k = 2, 3$. When S_T is close to its initial value, the portfolio has a nonnegative value. This is because the expense ratio and volatility drag will erode the values of both $\pm k$-LETFs, and the portfolio takes a short position on them. Also, for any level of S_T, the portfolio benefits from increasing realized variance due to volatility drag. In other words, the short $\pm k$-LETF strategy works best when there is high realized variance but no strong trend in the underlying. However, it should be noted that a large directional movement in S can result in losses. This is evident from Figure 8 on the facing page, where the portfolio value is negative as long as s is sufficiently far away from 1, at any level of realized variance. In fact, for the double short strategy, the potential loss is unbounded.

8.2 k-LETF versus $k \times$ ETF

Next, we consider a strategy that is based on the difference between a k-LETF and k multiples of the underlying ETF, for any $|k| > 1$. The portfolio involves investing one dollar in a k-LETF (ie, $1/L_0$ shares) and simultaneously shorting k dollars of the underlying ETF. Again, we assume $S_0 = 1$ without loss of generality. The resulting portfolio value is

$$\mathcal{E}(S_t, V_t) = S_t^k \exp(-(r(k-1) + f)t - \tfrac{1}{2}k(k-1)V_t) - kS_t - (1-k)\mathrm{e}^{rt} \qquad k \in \mathbb{R} \setminus [-1, 1] \quad (8.3)$$

This is also discussed by Avellaneda and Zhang (2010), who solved for the breakeven level of S_t with realized variance V_t held fixed for double LETFs. Here, we study the set:

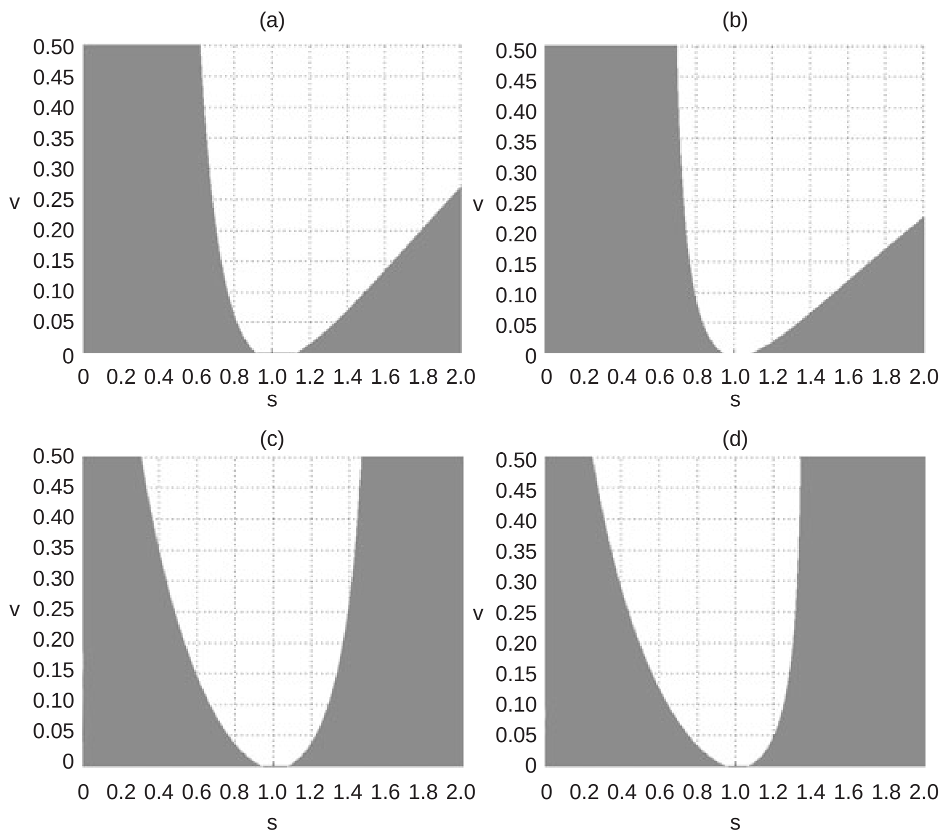
$$\mathcal{S}_k = \{(s, v) \in \mathbb{R}_+^2 : \mathcal{E}(s, v) \geqslant 0\}$$

where:

$$\mathcal{E}(s, v) = s^k \exp(-(r(k-1) + f)t - \tfrac{1}{2}k(k-1)v) - ks - (1-k)\mathrm{e}^{rt}$$

In Figure 9 on the next page, we show the region with nonnegative portfolio value. In contrast to the short $\pm k$-LETFs strategy, the portfolio value here is different for $k = 2$ versus $k = -2$ (see parts (a) and (c)), or $k = 3$ versus $k = -3$ (see parts (b) and (d)). For all choices of leverage ratios, this strategy is not profitable when the underlying does not move in either direction. When the realized variance is high, then a larger directional movement is required to break even. Comparing across leverage ratios, the strategy corresponding to $k = 2$ generates a larger profitable region for large s and a smaller profitable region for small s, as compared with the case with $k = 3$. For inverse LETFs, the strategy with $k = -2$ generates a larger profitable region for $s < 1$ but a smaller profitable region for $s > 1$, as compared with the

FIGURE 9 The shaded region indicates where the portfolio value is nonnegative.



(a) $k = 2$. (b) $k = 3$. (c) $k = -2$. (d) $k = -3$. Parameters: $T = 1$, $r = 1\%$ and $f = 0.95\%$.

case with $k = -3$. Finally, we can reverse the regions simply by taking the opposite position to that in (8.3), ie, shorting the k-LETF and buying the underlying ETF.

9 CONCLUSION

In view of the increasing popularity of ETFs and their leverage counterparts, it is important to better understand and rigorously quantify the risk involved. This paper studies LETFs under various risk measures, and points out the interaction between leverage ratio and investment horizon. Our study offers some guidance on risk control and proper selection of LETFs and associated strategies.

There are a number of directions for future research. This paper looks at examples of equity ETFs, but exchange-traded products are also available for other asset classes, such as commodities, volatility and real estate. Furthermore, there are options written on LETFs. Since different LETFs share similar sources of randomness, one major concern is the price consistency of LETF options across different leverage ratios (see Ahn *et al* (2012) and Leung and Sircar (2012)). From the investor's perspective, it is also important to consider the optimal timing to buy or sell options on ETFs (see, for example, Leung and Ludkovski (2011)). Moreover, models that capture the connection between ETFs and the broader financial market would be very useful. Advances in these directions would be important not only for individual and institutional investors, but also for regulators.

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