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SMILE ARBITRAGE: ANALYSIS AND VALUING

Master's Thesis

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Abstract

The thesis studies the implied volatility, how it is recognized, modeled, and the ways used by practitioners in order to benefit from an arbitrage opportunity when compared to the realized volatility. Prediction power of implied volatility is examined and findings of previous studies are supported, that it has the best prediction power of all existing volatility models. When regressed on implied volatility, realized volatility shows a high beta of 0.88, which contradicts previous studies that found lower betas. Moment swaps are discussed and the ways to use them in the context of volatility trading, the payoff of variance swaps shows a significant negative variance premium which supports previous findings. An algorithm to find a fair value of a structured product aiming to profit from skew arbitrage is presented and the trade is found to be profitable in some circumstances. Different suggestions to implement moment swaps in the context of portfolio optimization are discussed.

Keywords: Implied volatility, realized volatility, moment swaps, variance swaps, dispersion trading, skew trading, derivatives, volatility models

Language: English

Contents

Abbreviations and Acronyms

ATM	At The Money
ITM	In The Money
OTM	Out of The Money
B/S	Black-Scholes
GARCH	Generalized Autoregressive Conditional Heteroscedasticity
ART	Alternative Risk Transfer
GBM	Geometric Brownian Motion
CBOE	Chicago Board Options Exchange
VIX	Volatility Index on S&P500 Index
VSMI	Volatility index on Swiss Market Index
SMI	Swiss Market Index
S&P500	Standard & Poor's 500
RV	Realized Volatility
IV	Implied Volatility
MLE	Maximum Likelihood Estimation

APC	Average Portfolio Correlation
FX	Foreign Exchange

List of Tables

List of Figures

1. Introduction

1.1. Initial situation

Currently a lot of financial products were developed in order to deal with uncertainty in financial markets. Volatility as a parameter that is plugged into financial mathematical models in order to value financial products is difficult to model, the easiest way to deal with it is just to assume it to be constant. This is exactly what Black-Scholes (B/S) and Merton have done, using some other assumptions they developed the B/S model in order to be able to value derivatives.

This step was considered to be the initiation and step-stone for the whole process to make modeling volatility more realistic and near to real market data. Due to the fact that the observed volatility implied in options prices deviate from the volatility parameter plugged initially into B/S models, a whole new plethora of volatility products have evolved, variance swaps, volatility swaps, skew swaps, kurtosis swaps, correlation swaps, gamma swaps, etc. When there is no ready answer to fulfill a specific need; or even for marketing and profit purposes, new exotic and structured products were developed. This behavior led to a complexity that is difficult to look through. Now B/S model is used mainly as a proxy and a quoting mechanism for derivatives prices, other models are needed to count for other influences imposed on prices in real markets. Implied volatility, which represents the opinion of the market participants about the future developments in a certain asset, is considered to be a good prediction of the future realized volatility as outlined by Christensen [1998] [14] and Fleming [1998] [21]. Huang [2003] studied the capability of different models and came to the conclusion that implied volatility perform best compared to stochastic and Jump-diffusion models only GARCH models can perform similarly [41].

Having a view about the future developments of a certain underlying, observing the market opinion about that development; hedger, trader or speculator can take relevant positions to profit from the discrepancy. To achieve that, as mentioned before, a whole plethora of derivates and even dedicated products that have volatility components can be employed.

1.2. Motivation and goals of the thesis

I analyze the volatility smile, concentrating on the equities market, listing the different models that aim to capture and count for the implied volatility when pricing financial products, as well as the products evolved to deal with the implied volatility, the moments and cross-moments swaps, starting from their building blocks including the Log Contract, listing the different variations, their valuing and the way they can be replicated using other instruments including plain vanilla options and even lower moments.

I use market data of the S&P500 and the VIX indices from january 1999 up to March 2009 to examine the behavior of variance swaps, to test whether a negative variance premium exists and whether the implied volatility represented by the VIX index can be a good estimator for the S&P500 realized volatility.

Using implied volatility data of the SMI index and implied volatility of all its constituents from December 2007 up to March 2009 to examine the behavior of the implied correlation in the SMI index, which can reveal the view of the market about how the constituents of the index would correlate.

An exotic structured product which aim is to capture the third moment of a specific distribution (the skewness) is examined, I use data of the SMI constituents prices and implied volatilities from December 2007 up to March 2009 in order to calculate the fair value of that product and test whether the assumptions made about its payoff profile are realistically met, also testing in which circumstances such product can be profitable.

Finally I examine the integration of volatility products in the context of portfolio management and asset allocation, whether it is meaningful to use them and how they can be integrated in a mean-variance framework to find an optimal portfolio.

1.3. Structure of the thesis

Starting by the second chapter, I go systematically through dealing with the fact of existing volatility smile, starting from B/S model with its unrealistic assumptions, to the attempts to count for the fact that volatility changes over time, depending on stochastic influences. Local volatility, stochastic volatility and jump-diffusion models are handled.

The next section of chapter two deals with analyzing the volatility smile, some rules of thumb are examined, their relation to existing volatility models and some researches that test their relevance, after that predictability of volatility and the implied dividends and interest rate are discussed.

After analyzing volatility, chapter three deals with trading volatility where the point comes at which all instruments needed to further analyze ways to exploit a trading chance are avialable, and an arbitrage oportunity can be identified. The concept of delta-neutrality is mentioned, afterwards, I analyze the known ways to profit from the fact of the smile, going through using plain vanilla options strategies to dedicated financial instruments which were developed to directly trade volatility as an asset class, without the cumbersome methods of options strategies and the related high transaction costs. The role of the Log Contract is discussed because of its importance as a building block in constructing moment swaps, volatility and correlation indices are discussed briefly.

Moment swaps are handled in a general manner and specifically according to the various moments starting from variance swap, which gets more focus because of its importance. I show how a variance swap is valued and some results from its payoff calculation are presented and discussed. Third generation volatility swaps, correlation, skew and kurtosis swaps are handled after that.

In chapter four, two specific known trading strategies are discussed, dispersion and skew trading. All before-mentioned volatility instruments can be integrated in a way or the other in these trades, these are handled with some detail starting by dispersion trading through plain vanilla options strategies then through variance, correlation and gamma swaps.

In the second section of chapter four, the skew trading is discussed, at the beginning the skew characteristics are discussed related to time, perception of the market and the regime where the market is existing. The different ways to conduct skew trading are mentioned at next, first through plain vanilla options strategies, then through gamma/variance swaps and through skew swaps.

At the last part of chapter four, the focus is set on a chosen structured product which aim is to exploit a view on the future realized skew. A description of the product is provided, I use algorithms that lead to a fair value for the product using Monte Carlo simulation of a multidimensional GBM. After that the results are presented and discussed.

In chapter five, I examine under which circumstances volatility products can be integrated in the context of a portfolio optimization framework, with all the problems that may occur.

Finally, all developed Matlab codes with some explanation is provided as appendices.

2. Volatility

2.1. Volatility in the Black-Scholes world

To price derivatives, B/S formula is considered to be the most recognized and widely accepted model, despite its shortcomings that resulted from its oversimplified assumptions. This oversimplification and almost unrealistic assumptions were necessary in order to come to a solution to the partial differential equation; hence be able to come to a fair price for a derivative contract. In reality, almost all of these assumptions are violated, most of the time:

- The price of the underlying instrument S_t follows a Wiener process W_t with constant drift μ and volatility σ, and the price changes are log-normally distributed:

$$\begin{aligned} dS_t &= \mu S_t \, dt + \sigma S_t \, dW_t \\ \frac{dS_t}{S_t} &= \mu \, dt + \sigma \, dW_t \end{aligned} \tag{2.1}$$

- Short selling the underlying stock is possible
- No arbitrage opportunities
- Continuous Trading in the stock is possible
- No transaction costs or taxes
- All securities are perfectly divisible, e.g. it is possible to buy any fraction of a share
- It is possible to borrow and lend cash at a constant risk-free interest rate
- The stock does not pay a dividend
- Events that would end up options life early are ignored, e.g. because of takeovers

Black [1988] listed these assumptions and commented them [4], he argues how the values would be otherwise, if these assumptions were substituted by more realistic ones [3].

While the market price for the call before expiration must satisfy the equation,

$$c = \mathbb{E}\left[\max(S_T - K, 0)\right] \tag{2.2}$$

where $\mathbb{E}$ represents the expectation, S_T the price of the underlying at expiration time T and K is the strike price of the call option c,

it should not be possible to achieve an arbitrage opportunity through initiating a portfolio of the options and assets that underlie them. Through adding dynamics in a portfolio consisting of the option, the underlying asset and a cash position, using some mathematical manipulations B/S were able to reach a solution which makes the expected return of the portfolio equal to the expected return on a risk-free asset; hence reach a fair derivative price which is arbitrage-free in a risk-neutral world, in which risk-preferences of the market participants does not count [5].

$$c = S\Phi(d1) - Ke^{-r(T-t)}\Phi(d2) \qquad \text{for a call option} \tag{2.3}$$

$$p = Ke^{-r(T-t)}\Phi(-d2) - S\Phi(-d1) \qquad \text{for a put option} \tag{2.4}$$

where c represents the call price, p the put price, S the price of the underlying, $Ke^{-r(T-t)}$ a cash position discounted using the relevant interest rate r for the time to maturity $(T-t)$, $\Phi(x)$ represents cumulative normal distribution, $d1$ and $d2$ are defined as

$$d1 = \frac{ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \qquad \text{and} \qquad d2 = d1 - \sigma\sqrt{T-t}$$

Figure 2.1 shows the sensitivity of the option's price to the level of volatility as well as the price of the underlying assuming that B/S formula were the right model to use in order to explain derivatives behavior.

In reality, if volatility were allowed to change, OTM options price would be more sensitive to changes in the underlying assets, stock price declines implies that volatility would strongly increase and vice versa, increasing volatility for OTM options would lead their prices to increase compared to ATM options, this is mostly due to a broken interest to write these options, which may lead to increasing their prices resulting from the power of supply and demand.

The Cox-Ross formula accounts for the fact that volatility is not constant over time, OTM options would be less valued relative to ATM and ITM options, when compared

Figure 2.1.: Call and put option prices plotted against price of the underlying and volatility

to prices resulting from the B/S Formula. Accounting for jumps in the underlying price processes, OTM and ITM options relative value would increase, ATM options on the other hand, as outlined by Merton, would decrease in their relative value [38]. The fact that volatility changes over time negates the illusion of risk-free hedge and leaves the portfolio risky in some extent.

Thinking in discrete dynamics, defining Δ, the change of the price of an option due to changes in the price of the underlying as

$$\Delta = \frac{f_u - f_d}{S_0 u - S_0 d} \tag{2.5}$$

where $(u > 1), (d < 1)$, $(u-1)$ represents the percent increase and $(1-d)$ the percent decrease in the underlying asset price. A portfolio consisting of short one option and long (Δ) amount of the underlying asset $(S_0 \, \Delta - f)$ should earn the risk-free rate through delta-hedging $(S_0 u \, \Delta - f_u = S_0 d \, \Delta - f_d)$; hence achieving a risk-less portfolio that should be determinant.

The sensitivity of a portfolio of derivatives to some parameters is called the Greeks, the major Greeks under the B/S model are shown in table 2.1. Understanding and managing these sensitivities is vital when dealing with derivatives for trading and risk management purposes.

Assuming that the volatility parameter σ is stochastic and that the value of the portfolio depends on S, t and σ in a specific time period Δt, using taylor's series expansion

$$\Delta\Pi = \frac{\partial\Pi}{\partial S}\Delta S + \frac{\partial\Pi}{\partial\sigma}\Delta\sigma + \frac{\partial\Pi}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2\Pi}{\partial S^2}\Delta S^2 + \frac{1}{2}\frac{\partial^2\Pi}{\partial\sigma^2}\Delta\sigma^2 \tag{2.6}$$

traders usually make the portfolio insensitive to a particular parameter through making it neutral, i.e. eliminating the relating element in equation 2.6 [27]. Figure 2.2 shows the major Greeks in a more realistic behavior.

2.2. Implied risk-neutral distribution

If the B/S formula were correct, if we consider the probability distribution of any stock prices, it should be log-normal at any time, which means that the implied volatility should be constant. In reality the implied volatility is not constant, it is often observed to change with different levels of strikes and the considered time horizon. In the B/S world the process governing the evolution of the stock price is considered to be a random walk, what should be done to count for the implied volatility, but still be able to exist in the B/S world? Derman et al. [1994] suggest modifying the random walk process to deal with the fact of existing volatility smiles. They suggest considering volatility to be a function of the underlying price level and time to maturity, $\sigma(S, t)$. They suggest using the market prices of options on the underlying to induce the real random walk process moving the prices of the underlying, implied binomial trees can be employed to reach future prices; hence reach a probability distribution of the underlying prices implied in the current option

The Greek	Sign and Sensitivity
delta	$\Delta = \frac{\partial\Pi}{\partial S}$
gamma	$\Gamma = \frac{\partial\Delta}{\partial S}$
vega	$\nu = \frac{\partial\Pi}{\partial\sigma}$
theta	$\Theta = \frac{\partial\Pi}{\partial t}$
rho	$\rho = \frac{\partial\Pi}{\partial r}$

Table 2.1.: The Black-Scholes sensitivities: The Greeks

Figure 2.2.: Sensitivities of call price to changes in delta, volatility, time-to-maturity and interest rate. The fourth dimension, the color, represents delta.

prices [33].

The main idea behind B/S approach is the none-existence of a risk-preference which makes it possible to use risk-neutral valuation. The underlying asset should earn the riskless rate of return; hence to achieve a riskless portfolio options can be hedged through an amount of that riskless underlying asset. Breeden and Litzenberger [1978] argue that using the B/S formula it would be possible to retrieve the risk-neutral probability distribution implied in the market place. The second derivative of the B/S price with respect to the strike price is the continuous discounted risk-neutral probability distribution of the price of the underlying at expiration [8].

$$\frac{\partial^2 C}{\partial K^2} = e^{-rT} f^Q(K) \tag{2.7}$$

Through using the implied volatility, we can reach an implied probability distribution, which is risk-neutral for future underlying prices and is won through the volatility smile of options on that underlying maturing at the future time considered. The most important characteristics of the implied probability distribution is that it has even heavier tails than the log-normal distribution, which means that it has even more extreme prices of the underlying, which justifies the existence of the volatility smile [27]. Having allowed for volatility to be time-variant, we give OTM options –which have higher volatility compared to constant volatility in B/S model– higher probability to move to be ITM. The most accurate Information about the implied volatility and the implied risk-neutral distribution can be extracted from the ATM near expiration options on the relevant underlying.

2.3. Volatility smile

Volatility "the wrong number in the wrong formula to get the right price" [43].

In the B/S world volatility is the only parameter that can not be directly observed in the market and is supposed to be constant as an important assumption of the model. This volatility measure must be calculated or predicted using historical data in order to be used in the context of B/S model. All options on the same underlying must have the same volatility according to B/S, which does not hold in reality, the market does not believe in the same assumptions as the B/S model and therefore the market believes that the price process of the underlying has other attributes than the ones assumed by B/S, especially that their processes follow geometric brownian motions (GBM).

2.3.1. Implied volatility

As market price of an option is not exactly the value that we get when using an option pricing model like B/S, we are interested in the volatility implied in such prices because it can give us information about how the market participants anticipate the behavior of the process of the underlying. The anticipation of the market participants introduces the reality link to the theoretical pricing models in which there are factors that are neither assumed -as in B/S model- nor quantifiable. The long term level of the implied volatility is observed to be often higher than the short term level, higher strike implied volatilities are observed to be lower than low strike implied volatilities, OTM put options have higher implied volatilities than OTM call options. If this were not the case, OTM put options would be considered underpriced compared to OTM call options.

Interested in the implied volatility, we can not just inverse e.g. the B/S formula in order to get a value for the implied volatility using

$$\text{Implied Volatility} = f(C, K, S, \tau, r). \tag{2.8}$$

In order to find that value, iterative numerical methods, like Newton-Raphson or binomial trees for american options, are needed.

Implied volatility surface is a graph which plots the implied volatility against ATM strike prices and time to maturity in order to visualize implied volatility levels and its behavior. Implied volatility, when considered with regard to time to maturity, is recognized to be mean reverting, it converges to the long-term volatility level, also implying volatility clusters. The volatility smile surface $\sigma(K, t)$ must be distinguished from the volatility surface $\sigma(S, t)$.

When the price of the underlying used in the derivatives pricing model is different from the ones that the market is using, the implied volatility of call options can vary from the ones for put options, which is the case when future prices are considered against theoretical future prices used by the model.

The observed implied volatility plotted against ATM strike prices is not flat, especially after the 1987 financial markets crash, and for that reason, this phenomena is often called crashophobia:

- For equities implied volatility tends to skew higher downward at low strike prices and lower downwards at higher strike prices, this is due to risk-averse investors giving higher probability of prices going down than up, demanding more OTM put options to protect their positions, also after large price decrease, the market expects the underlying asset price volatility to increase, therefore giving higher probability that the OTM call would land ITM soon, these options get more expensive implying higher future volatilities.

- Commodities are reversal to equities, at ATM strike prices implied volatility is lowest and skews upwards at higher strike prices and downwards at lower strike prices (skewed to the right, an aversion to increasing commodities prices).
- For FX implied volatility it is lowest at ATM strike prices and skews upwards at higher and lower strike prices, where falling prices are good news and increasing prices are bad news. The shape of the skew is dependent on the relative strength of each currency.

Figure 2.3 shows the implied volatility skew for both the SMI and the S&P500 indices.

On accepting the B/S model of option prices, the volatility smile behavior has to be modeled in order to realistically cope with the trading environment in the market-place. Traders express option prices in terms of implied volatility as relative value because the price of the option may increase, but the option may still considered to be cheap, if implied volatility got lower, this would mean that the price of the underlying has got higher, in this case, the option is considered to be relatively cheap because in order to hedge the long call position, the trader can sell, with gain, a portion of the underlying position, which became more expensive. The relationship between the volatility parameter σ_{BS} used to calculate the option price in the B/S formula and the implied volatility σ_i observed in the market place is now to be examined closer to study its behavior.

When mentioning implied volatility, then this must be in the context of a theoretical model, in our case mentioned above, it is the B/S model which employs a geometric brownian motion process which has a B/S volatility, implied volatility by itself has a nonlinear relationship to the B/S option's price, specifically, ATM options are the most sensitive to volatility.

A statistical model is employed to estimate statistical volatility measure out of historical data, mostly a time series model is applied (especially GARCH) that can also be used to forecast future volatility. Which opinion about the future volatility is correct, the market opinion or the statistical opinion? Comparing both over the lifetime of an option can give information whether the option is over or underpriced. Lower and upper statistical confidence limits are defined, the market opinion is then compared with these limits, if it is exceeding the upper limit, then it is a sign that the market

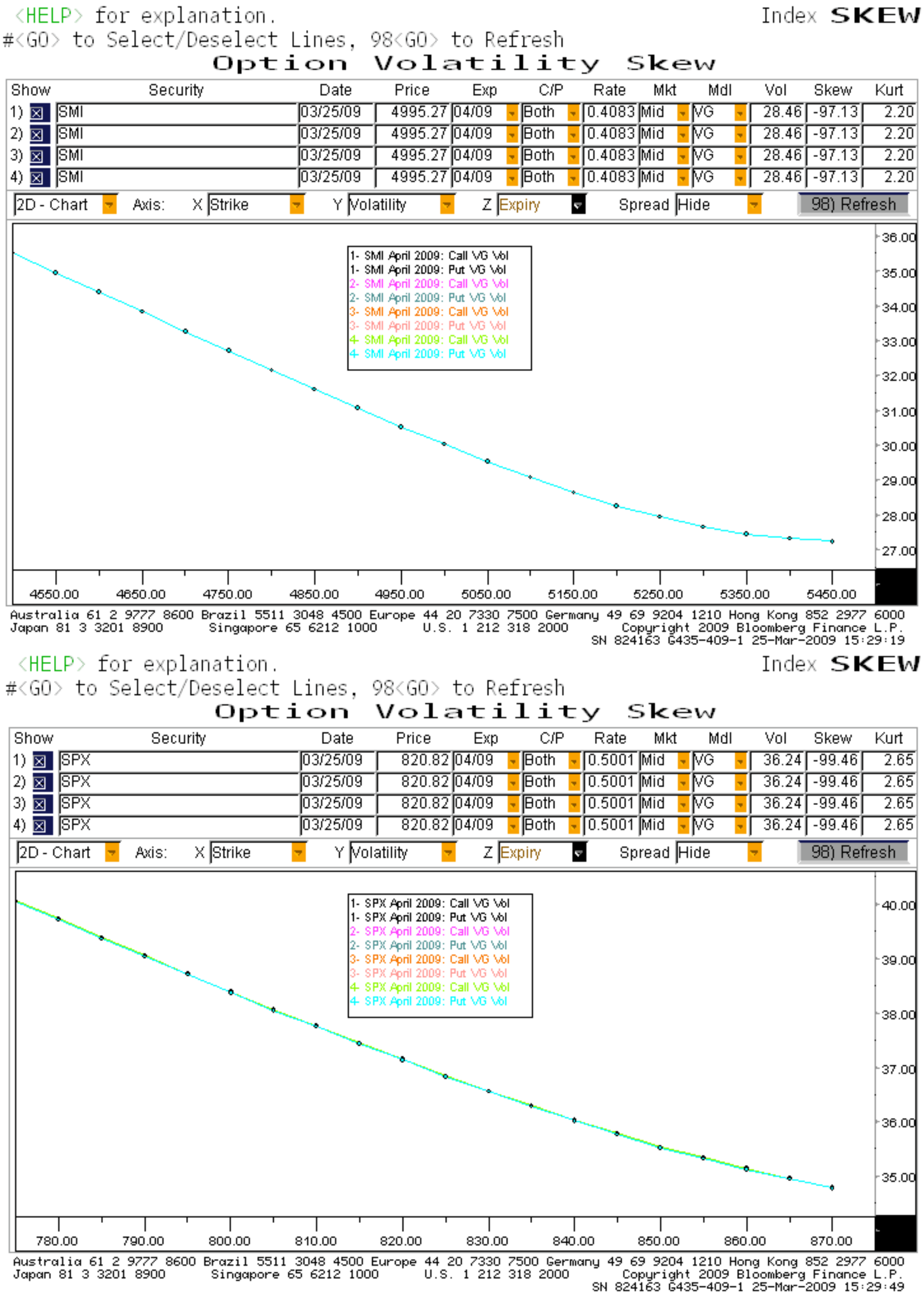

Figure 2.3.: The S&P500 and SMI Skews, Source: Bloomberg

is overestimating, if market opinion is lower than the lower limit, then it is supposed to be underestimating volatility; hence underpricing the option. These observations might represent a trading opportunity, but sensitivity to changes in volatility cannot be hedged away, that's why traders have different views about volatility, which give rise to bid-ask spreads.

Realized volatility is an ex-post estimate of the volatility actually realized of the price process and is to be estimated according to the applied model, if it were a GARCH model, the realized volatility should be estimated using historical data with the GARCH model. Other models which partly extend the B/S model and assume a semi-stochastic or a fully stochastic volatility of the price process are shortly explained in the next section.

2.3.2. Modeling volatility

To model the volatility skew different approaches are followed. These approaches are quasi-deterministic (local volatility models) and stochastic (the stochastic volatility model), others include the jump models that take jumps in the price process into consideration.

Jump diffusion

The market is not completely liquid and imbalances between the power of supply and demand may occur, there are jumps in prices, which is not allowed in the B/S model. To count for this fact, Merton developed a model that takes jumps besides the continuos processes into account; hence an option which takes into account the jumps in the underlying should be more valuable than another option that does not [38]. Through taking jumps into account when pricing options fat tails in the log-normal distribution are accentuated; hence the B/S model can be made more realistic.

Unfortunately taking jumps into account when forming hedging strategies is not feasible, it requires having as much as hedging instruments as there are possible occurring jumps in all future probable states of the world; knowledge of the price process of the replicating instruments is also required, which is practically impossible. Implementing the jump-diffusion model leaves the market arbitrage-free but not complete, different persons in the market place would price the same derivative differently; hence no unique price can be defined. Black suggested another simple method in order to still be able to use the standard B/S formula but take jumps into consideration, he suggests estimating two volatilities one for high and one for low regimes, then weighting them with their respective probabilities of occurrence [3].

The jump process dynamic when considering a sum of jumps per unit time, and percent change of the price of the underlying, conditional on the occurrence of jumps, is represented by:

$$dS_t = (\mu - \mu_{(j-1)}\lambda)S_t dt + \sigma_t S_t dW_t + (e^j - 1)S_t dN_t \tag{2.9}$$

Where S_t denotes price, μ the drift, σ_t the standard deviation given no jumps, dW_t the increment of a brownian motion, λ is the amount of jumps that occur per time unit, $\mu_{(j-1)}$ is the discrete percentage change of the price of the underlying S_t due to the jump, dN_t is a change of a Poisson process which represents the jumps, according to the occurrence of the jump it can have a value of one or zero, $(e^j - 1)$ is the intensity of the jump, where j represents the jump's continuous rate of return.

The stock price according to the jump-diffusion model is considered to be continuous between jumps, when jumps with a random sign, magnitude as well as deterministic frequency of λ occur, they become discontinuous [38].

The jump-diffusion model can be used to model the volatility smile as close as possible; therefore the actual market data is needed to compare it with the output of a jump-diffusion model in order to find the best possible representation of the actual smile. The main disadvantage of using jump-diffusion models to model the smile is that it can produce smiles which are more accentuated for short-term maturities and more flat for long maturities [31].

Local volatility

Using the different variations to count for the smile is not straightforward. The possibility of being able to hedge the options positions directly through an amount of the riskless underlying asset is no longer available, preferences of the investors to risk, which is taken into consideration in the different variations of B/S, leaves the underlying asset risky and the option must be hedged using more complex instruments. Moreover, variations of B/S, such as these of Merton [38], which takes into consideration the jumps in the asset prices, and stochastic models, which consider volatility to be stochastic, require parameters that need to be estimated [33]. Local volatility models were developed simultaneously by the different researchers, Derman-Kani, Dupire and Rubinstein. Rubinstein [44] as well as Derman and Kani developed the time discrete; Dupire the time continuous version.

At a specific time t, and at a specific price of an underlying S_t, there exists a value for volatility called local volatility $\sigma(S_t, t)$, which would justify the current value of the implied volatility, calculated from the existing T maturity options on the same underlying.

The idea of using implied trees to value options is straightforward it guarantees a price that is arbitrage-free, risk-preference free, no extra parameters are needed and required data are obtained through observable options prices. In this way, smile effect is taken into consideration when pricing options that are path-dependent, american or relatively illiquid, also other exotic options and structured products can be more easily priced.

The model, as outlined by Derman et al. [1994], is considered to be deterministic, the main difference to B/S is that volatility is a function in time and price of the underlying and not constant as assumed in B/S; hence the smile can be accentuated making the model more realistic. The price process still follows brownian geometric motion with the volatility term adjusted to show the smile effect. The B/S model is extended by $\sigma(S, t)$ so that the Derman-Kani model can be described as [19]

$$dS_t = \mu S_t dt + \sigma(S_t, t) S_t dW_t \tag{2.10}$$

where $\sigma(S, t)$ is numerically deduced from the smile. Interpolation must be used to get prices for all options for all strikes and expirations from the known traded options prices. Using this approach allows the model to incorporate higher volatility when the price decreases and lower when it increases. We can think of local volatility as the forward volatility, analogue to the forward rate calculated from the bond yield curve.

Dupire [22] derives a continuous-time formula for the instantaneous volatility of the underlying

$$\frac{\partial C}{\partial T} = \frac{1}{2}\sigma^2(K, T; S_0) K^2 \frac{\partial^2 C}{\partial K^2} \tag{2.11}$$

where C the price of a European call option, T its time to maturity, $\sigma^2(K, T)$ the instantaneous variance and K is the strike price.

The instantaneous variance $\sigma^2(K, T)$ can be obtained by

$$\sigma^2(K, T) = \frac{2\frac{dC}{dT}}{K^2 \frac{d^2C}{dK2}} \tag{2.12}$$

The instantaneous volatilities produced through the local volatility models can be used to produce options prices that are in line with those observed in the market.

Main difficulty to implement local volatility models is the assumption that options prices for all strikes and maturities are available; hence implied volatility, which is not necessarily true in reality. In order to implement these models, interpolating –e.g. through constant piecewise– maturities and strikes is a must.

The main advantage of local volatility models is that they are easy to calibrate, since randomness follows mainly from changes of the price of the underlying itself, main disadvantage is that they do not capture the dynamics of the prices good enough and may lead to unrealistic implied volatility surfaces.

Stochastic volatility

One other way to model the volatility surface that counts for the existing volatility smile is through assuming that the volatility itself is following a stochastic process rather than being constant. This way it is more accurate to deal with the volatility factor when applying it to financial applications.

Considering the process dynamics followed by the underlying asset prices S

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{2.13}$$

where μ the return on the asset S at time t, σ is its volatility and dW_t a Wiener process

The solution to that stochastic differential equation is

$$S_T = S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}N(0,1)} \tag{2.14}$$

where S_T the price of the underlying at expiration time T, S_0 the price of the underlying at initiation and r the relevant interest rate for the time T.

When considering the joint risk-neutral dynamics between the stock price and its volatility as suggested by Heston [26], a stochastic volatility model assumes that volatility ν_t is also stochastic and follows a Geometric brownian motion by its own, but is still correlated with the process of the underlying through correlated Wiener processes

$$dS_t = \mu_t S_t dt + \sqrt{\nu_t} S_t dZ_1 \tag{2.15}$$

$$d\nu_t = \alpha_{S,t} dt + \beta_{S,t} dZ_2 \tag{2.16}$$

where $\alpha_{S,t}$ and $\beta_{S,t}$ are functions of ν the instantaneous volatility, and dZ_2 is another Wiener process that is correlated with the first Wiener process dZ_1 with constant correlation factor ρ.

One example of stochastic volatility models is the Heston model which can be expressed as

$$dS_t = \mu_t S_t dt + \sqrt{\nu_t} S_t dZ_1 \tag{2.17}$$

$$d\nu_t = \kappa(\theta - \nu_t)dt + \sigma\sqrt{\nu_t} dZ_2 \tag{2.18}$$

where θ is the average long-term volatility, κ is the mean reversion factor, σ is the volatility of volatility.

The volatility is tending with the passage of time to revert to a long-term average volatility value θ, the acceleration of this process is determined by the factor κ, as t tends to infinity the value of ν would be equal to θ [26].

When considering the risk-neutral measure in the Heston model, the set of equivalent martingale measures has the dimension of one, this is due to having one asset and two different Wiener processes, hence no unique risk-free measure exists. In the case of B/S, there were just one Wiener process that left the set of equivalent martingale measures having the value of zero, meaning that there is one single value for the drift, which can be interpreted as a unique equivalent martingale measure. This measure, when used with a discounted asset, will leave it following a martingale. In this case there is no arbitrage, all information from the past is already priced in the price of the underlying at the present time t.

Because of not having a unique risk-neutral measure, the Heston model is problematic. While having several risk-neutral measures, each of them would lead to a different price, but one of them is the one that is acceptable and compatible to the market. The solution would be to get an additional volatility dependent asset which cancels out the one dimensional set of equivalent martingales; hence leaving the portfolio with one risk-neutral measure which is acceptable by the market and can be used to price derivatives and other exotics.

GARCH model

Generalized AutoRegressive Conditional Heteroskedacity (GARCH) is a stochastic model which is used to estimate volatility. The most important assumption of the model is that the variance process is also random and varies as the variance changes, this is different in the Heston model, where the variance process varies as the volatility, represented by the standard deviation, changes. A GARCH(1, 1) model is expressed by

$$\sigma_t^2 = \omega + \alpha\epsilon_{t-p}^2 + \beta\sigma_{t-1}^2 \tag{2.19}$$

where ω is the unconditional variance, σ_{t-1}^2 is the conditional variance at time $(t-1)$, ϵ is a random error. The parameters α and β are responsible for the short-term changes. If beta value were large this would mean that conditional variance shocks would take longer to vanish, a large value of α would suggest that the intensity at which volatility reacts due to moves of market prices is high; hence the variance does not only depend on the time series of the data but also on the time series of the variance itself and how it behaved in the past. The parameters of the GARCH(1, 1) model are pushing the variance in the direction of a long term variance level [6].

There are other variants of the GARCH model that have been developed, such as EGARCH, GJR-GARCH, NGARCH and LGARCH.

GARCH models can be used to fit the market smile, through estimating options prices through plugging the relevant volatility value into the B/S formula; hence produce GARCH implied volatilities which can be compared to the actual implied volatility observed in the market place and iteratively adjusting the GARCH parameters to fit the data.

Model calibration

When applying a specific stochastic volatility model, a calibration of its parameters should be undertaken, e.g. using the MLE method, which tends to discover the model parameters that are most likely to exist for the collected data. In an MLE model the algorithm starts by having an estimate of the initial value of a parameter, using that parameter in the context of the chosen model, regressing the outcome to the collected actual data, the residuals are then minimized by iteratively choosing different values for the parameter, a re-calibration is normally undertaken afterwards.

2.3.3. Analyzing the implied volatility and some rules of thumb

Derman [1999] asked the question: How does volatility skew change when market prices change? This gave a chance for a new debate in financial theory, instead of explaining how the volatility surface in a certain time and strike price would look like, this debate goes a step further to discuss the behavior (evolution) of the implied volatility as the price of the underlying asset moves. Observing that the volatility skew follows a different pattern at a certain stage or period, some rules of thumb were developed which may help predict the behavior of volatility in different circumstances. Embedded information in the options markets can be processed and extracted, these information may include: The regimes that an index moves follow, whether range-bounded, trending or jumping; how the realized volatility that would be used to replicate an option may behave; the risk premium for hedging errors and liquidity; as well as the effect that jumps have on the future volatility [18].

The following empirical formula can parametrize the skew as it is observed that the implied volatility tends to be approximatively linear as the ATM value moves away from the strike price

$$\Sigma(k,t) = \Sigma_{atm}(t) - b(t)(k - S_0) \tag{2.20}$$

where $\Sigma()$ is B/S implied volatility of an option, K is its strike price, t its expiration, S_0 is the current price level and $b(t)$ is the slope of the skew and represents annual percentage points of the volatility per strike point, it has a positive value when the skew is negative, but this can vary dramatically in turbulent times. This formula is handling the case of $\Sigma()$ depending on the level of S_0 not the general price level S.

After showing how the three-month implied volatility of S&P500 options with different strikes behave along the evolution of the index, Derman emphasizes the observation of volatility skew which is always negative and widens in dropping markets.

Relating the previous observations to B/S model, Derman denotes that the instantaneous future index volatility $\Sigma(S,t)$ is considered to be a good proxy for the implied volatility in the B/S world, they would decrease when the index level increases and vice versa.

Changes in the skew through sticky (invariant) quantities as the index evolves can be examined and observed, the main purpose is to understand the influence on options prices invoked by the behavior of the relationship between the implied volatility and the index level in a specific period and for different price levels. This understanding can help to modify the implied volatility in a way that takes into consideration the embedded information mentioned earlier and to react accordingly to achieve more reliable hedge ratios, risk management and more accurate options pricing.

Derman [1999] lists different patterns that describe the sticky quantities and the way to react accordingly as follows:

Sticky-Strike Options' implied volatility of a certain strike remains the same independent from the moves of the price of the underlying asset, in this case the B/S model is correct, the implied volatility is constant and the option value depends only on its moneyness $\left(\frac{K}{S}\right)$; the exposure delta is also the same as the B/S delta.

$$\Sigma(S, K, t) = \Sigma_{atm}(t) - b(t)(K - S_0) \tag{2.21}$$

Implied volatility is independent from moves of the price of the underlying but may be influenced by some other stochastic variables [16]. Balland [2002] shows that the sticky strike rule is actually representing the B/S volatility[2]. When considering ATM volatility, it is supposed to increase as the index price decreases and vice versa. If the ATM implied volatility decreases as the price of the underlying increases, it might keep on decreasing and even tend to converge to reach zero, which is irrational.

When having a view of the world which considers the realized volatility to be stable and jumps-free and the level of the index to be bound to a range, risk premium and costs of hedging are stable, the most relevant action would be to preserve the implied volatility of each option, hence keep the current smile.

Sticky-Monyness Options' implied volatility that have the same moneyness stays the same after a movement in the price of the underlying asset.

$$\Sigma(S, K, t) = \Sigma_{atm}(t) - b(t)(\frac{K}{S} - 1)S_0 \tag{2.22}$$

Sticky-Delta This rule of thumb implies that the level of implied volatility depends only on the level of moneyness; hence if considering the B/S delta, its dependence on the strike price and the price level of the index is indirectly invoked through the moneyness effect, this leads to considering both sticky moneyness and sticky delta to be equivalent. Equation 2.22 can be approximated –if S and K are considered to be close to the price of the index at the initial observation of the skew S_0– through

$$\Sigma(S, K, t) = \Sigma_{atm}(t) - b(t)(K - S) \tag{2.23}$$

According to equation 2.23, options which have a strike of K would have higher implied volatility as the underlying price increases, the delta of this exposure must be greater than the B/S delta, hence ATM volatility is supposed to be constant. This pattern in volatility is best characterized by the stochastic volatility model.

When having a view of the world which considers the realized volatility to be stable but the level of the index is going through significant changes, estimating ATM options would be best achieved through using realized volatility as a proxy for implied volatility, remarking the ATM options at the same realized volatility as the period before the shift in the price level of the index. 50-delta options are constant, subtracting it from the OTM options would lead to the same risk premium which implicitly means that the sticky-delta pattern is been followed.

Sticky implied tree The implied tree can be seen as a combination of all option prices available on the same index, which forms future instantaneous volatilities for the index that are consistent with current implied volatilities and the prevailing views about the evolution of the future volatility. These volatilities are called local volatilities if they were tending to match the current volatility skew. The volatility skew can be attributed to the market expectations of higher implied and realized instantaneous volatilities, which can also be interpreted as the market having an aversion to jumps.

Considering implied trees to be an instrument to isolate sub-trees, having a view and a scenario of future market behavior, options' prices can be more accurately calculated using averages of relevant local volatilities between the current price level S and option's strike K, considering the price evolution of the underlying to follow the process described by 2.10. These local volatilities can be extracted from the implied volatility skew currently observed for a specific price level.

There is a linear dependency of the implied volatility on the price level, induced by the linear dependency of the implied volatility on the strike price. Implied volatility decreases as the price and strike level rise, this can be expressed as

$$\Sigma(S, K, t) = \Sigma_{atm}(t) - b(t)(K - S) \tag{2.24}$$

Despite the difference between the future instantaneous implied tree and the B/S implied volatility, the average future instantaneous volatility in the implied tree context is considered to be a good proxy for the B/S implied volatility; hence B/S implied volatility may increase when the index price decreases and vice versa. On the other hand, implied tree delta presented by Derman would differ than the B/S delta, even when both the implied tree and the B/S models agree on the same option's value, this is due to the fact that the implied volatility in the context of implied trees is allowed to vary, which is not the case regarding the B/S volatility. As for ATM implied volatility, it decreases twice as fast as other implied volatilities, which induces a lower exposure delta than the B/S delta for the same volatility. This pattern is best characterized by the local volatility model.

When having a view of the world which considers the realized volatility to increase and jumps to occur –much probably downward jumps– the individual volatility should be adjusted accordingly.

Figure 2.4 summarizes these patterns. The implied tree model is considered to be the most accurate of all three mentioned above, it keeps the boundaries of arbitrage-free options prices, which does not necessarily hold for the others.

The previous cases are considering the short term behavior and may not hold when considering long term behavior, the combination of both the local volatility and the implied volatility because of being mean-reverting tends towards the same ATM implied volatility level.

Daglish et al. [2007] studied the various rules of thumb just mentioned before. They developed a no-arbitrage condition and examined whether they hold and are consistent with the developed no-arbitrage condition, they added another rule of thumb often used by practitioners which is called the square root of time rule (a relationship between options volatility that have different strike prices and expiration time at a specific time) [16]. They agree with Derman that the sticky strike rule gives rise to arbitrage opportunities because the instantaneous volatility of the asset price is only dependent on time t while the strike K is considered to be constant, which makes it compatible with Merton's model [37]. Trading different options on the same underlying and same maturity, with a price level that is independent from volatility level at different prices, gives rise to arbitrage opportunities. The same argumentation holds in the case of sticky delta rule when volatility is considered to be a deterministic function, on the other side, the no-arbitrage condition can be met in the case of a sticky delta rule as volatility is considered to be stochastic (relative sticky delta) which also holds for the square root of time rule, which can be attributed to the occurrence of mean-reverting characteristic, in case the reversion coefficient were large enough.

2.3.4. Predictability of volatility

Various studies [14] [21] show that ATM options have the most information content about future volatility, even if B/S model were not the correct formula to be used to value options, moreover, these studies conclude that implied volatility outperforms historical volatility in predicting future volatility. Granger and Poon [2005] found out that the implied volatility provides the most accurate forecast of volatility, historical

Figure 2.4.: Implied volatility regimes according to Derman[18]

volatility as well as GARCH models can also perform similarly [41].

Using MLE for the expected σ^2 we get $E\left[\hat{\sigma}^2\right] = \frac{n-1}{n}\sigma^2$ which is used in deterministic models like B/S and Cox-Ross-Rubinstein.
If we shift our view to correlation, another type of volatility, we find two types of parameters that can be employed:

unconditional correlation: When correlation is considered to be time-independent

$$\rho_{12} = \frac{\sigma_{12}}{\sigma_1 \sigma_2}$$

conditional correlation: When the correlation is considered to be time-dependent

$$\rho_{12t} = \frac{\sigma_{12t}}{\sigma_{1t}\sigma_{2t}}$$

When valuing a three month option one should look at the unconditional three month volatility, if they were significantly different from the long term average conditional time-varying-models should be used to estimate volatility. The same is true for correlation, a bivariate GARCH model can be used to estimate and forecast the conditional correlation, but it is often unstable over longer time, because the assumption of risk neutral probability does not hold any longer (B/S delta and vega are not appropriate any more to hedge with).

Characteristics of volatiltiy:

- Mean reverting
- Volatility clustering

- High volatility of volatility
- Negative correlation with equity returns

Understanding volatility enables to achieve the following:

- Hedge against equity market exposure
- Hedge against market crashes
- Trading volatility spreads in diffrent markets
- Trading changes in volatility term structure (speculation on directional changes in volatility)

Implied dividends and interest rate

Actually, using the same argumentation as applied to retrieve the implied volatility, it would be possible to retrieve the implied dividends and interest rates [25].

Considering the future price of the underlying to be

$$F_0 = S_0 e(r - q)T \tag{2.25}$$

Where S_0 is the price at $t = 0$, r the interest rate and q the dividend yield.

If we also consider the put/call parity:

$$C + Xe^{rT} = P + S_0 e^{-qT} \tag{2.26}$$

where C is the call price, P the put price, X is a cash position discounted by the relevant interest rate r for the relevant maturity T and S_0 is the price of the stock at initiation time 0 discounted by the relevant dividend yield q for the relevant time considered T.

Simultaneously solving this system of equations it would be theoretically possible to solve for the dividend yield and the interest rate. In this case, a call and a put option as well as a futures contract that have the same expiration on the same underlying, can be used to build a dividend yield and an interest rate curve. These curves can be used to extract the implied dividend and interest rates for any time-to-maturity. This method is limited because all three instruments must be traded at the same time which can be quite difficult.

2.3.5. Implied correlation

To calculate an implied correlation of a portfolio of assets, e.g. a stock-index, the implied volatility of both the stock-index and its constituent stocks can be used. The result is an average pairwise implied correlation, or average portfolio correlation (APC). In order to extract the implied volatility from options prices, it is most efficient to use the ATM options because, as mentioned above, they contain the most information about the future volatility. Another definition to implied correlation is that it is the way that the market thinks the constituents of the index are correlated to each other.

If we consider an index or a basket of stocks, which has a volatility of σ_I, in order to replicate the basket we need a portfolio of its n constituents with each ith stock having volatility of σ_i and a weight of w_i in the basket, these stocks have correlation of ρ_{ij} between stock i and stock j. The volatility of the replicating basket is:

$$\sigma_\Pi^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2 \sum_{i=1}^{N} \sum_{j>i}^{N} \rho_{ij} w_i w_j \sigma_i \sigma_j \tag{2.27}$$

We can use a pairwise volatility weighted average portfolio correlation to come to a value for the basket correlation, which not only takes into consideration the proportion of each constituent to the basket but also its volatility; this value can also be regarded as a measure for the level of the portfolio diversification

$$\text{VWAC} = \frac{\sum_{i=1}^{N} \sum_{j>i}^{N} \rho_{i,j} w_i w_j \sigma_i \sigma_j}{\sum_{i=1}^{N} \sum_{j>i}^{N} w_i w_j \sigma_i \sigma_j} \tag{2.28}$$

This formula can also be used to calculate the implied correlation, using implied volatility available from options prices. One other possibility to calculate implied correlation is through using the following formula, assuming constant pairwise implied correlation, where ρ_{ij} becomes ρ, solving for it in equation 2.27; we get the implied basket correlation[35]:

$$\begin{aligned} \rho_{imp} &= \frac{\sigma_I^2 - \sum_{i=1}^{N} w_i^2 \sigma_i^2}{2 \sum_{i=1}^{N} \sum_{j>i}^{N} w_i w_j \sigma_i \sigma_j} \\ &= \frac{\sigma_I^2 - \sum_{i=1}^{N} w_i^2 \sigma_i^2}{(\sum_{i=1}^{N} w_i \sigma_i)^2 - \sum_{i=1}^{N} w_i^2 \sigma_i^2} \end{aligned} \tag{2.29}$$

Practitioners often use a proxy to facilitate the calculation of the implied correlation;

presumed that the correlation is not close to zero, number of constituents is more than twenty and weights of the constituents are relatively small[7]

$$\rho_{imp} = \frac{\sigma_I^2}{(\sum_{i=1}^{N}\sum_{i=1}^{N} w_i\sigma_i)^2} \tag{2.30}$$

This implies that the index implied volatility can be approximated using

$$\sigma_I = \sqrt{\rho_{imp}}\left(\sum_{i=1}^{N}\sum_{i=1}^{N} w_i\sigma_i\right) \tag{2.31}$$

SMI data were employed to calculate its implied correlation. Three-month implied volatility of the index and each of its constituents as well as their weights in the index were used with a Matlab code which can be found in appendix C. Figure 2.5 shows the resulting calculations putting the calculated implied correlation in relation to both the VSMI and the SMI indices, a positive correlation with the implied volatility represented by the VSMI index and a negative correlation with the market index represented by the SMI index can be observed. In turbulent times, the implied correlation is quite accentuated, market participants anticipate high correlation between the index constituents.

Figure 2.5.: The relationship between the implied correlation and both the VSMI and the SMI indexes

3. Trading volatility

The arbitrage concept describes a risk free gain through using financial instruments, starting with a portfolio that is worth nothing or negative value today but has a none-zero probability to be of positive value and a zero probability of having negative value in the future. If recognized, one can take advantage of that arbitrage opportunity through going into relevant positions. Exhausting the arbitrage opportunity leaves the market efficient, and that is the fundamental idea of the efficient market hypothesis. In financial mathematics, it is often assumed that there is no arbitrage, through mathematical tools a price is found in the context of no-arbitrage arguments, then searching for an opportunity comparing theoretical price to the observed market price [31].

In order to study the different ways to trade volatility, it is necessary to mention the concept of delta-neutrality which is fundamental to having a position in volatility using plain vanilla options strategies or any other position built upon portfolios of plain vanilla options. In the last part of this chapter I'll show how volatility is traded as an independent asset class, without the need to keep a portfolio of plain vanilla options delta-neutral, avoiding the workload that is needed to achieve it and the inaccuracy it is exposed to.

On accepting the fact that the theoretical volatility in a model like the B/S can differ from the actual implied volatility, traders can profit from any arbitrage opportunity that may occur. With regard to standard options, traders can calculate the theoretical volatility and deduce the implied volatility anticipated by market sentiment and the power of supply and demand in order to take relevant positions to achieve profit. In the context of exotic options, traders can use this approach for arbitrage or pricing purposes –through risk-neutral-dynamics– exotic options are otherwise quite difficult to price and hedge.

3.1. Delta-neutrality

A portfolio containing positions in options with offsetting deltas brings the net delta of the whole portfolio to zero. In this case it is called a delta-neutral portfolio. Delta hedging is used to bring the sensitivity of the option's price change to its underlying price change, in the context of a portfolio, as close to zero as possible. When the portfolio is instantaneously delta-neutral, this means that the instantaneous change in the fair value of the portfolio, resulting from an infinitesimal change of the price of the underlying $\frac{\partial V}{\partial S}$ is equal to zero and the fair value of the portfolio is insensitive to changes of the underlying. This is accomplished through buying or selling an amount of the underlying corresponding to the delta of the portfolio. In some cases for example if the underlying stock cannot be borrowed and hence cannot be sold short, a portfolio of options' deltas can be used to make the portfolio delta-neutral. To keep a portfolio delta-neutral, hedging must be done regularly because the required position is changing according to the market. Making the portfolio having zero-delta involves static hedging, while regularly keeping delta close to zero involves dynamic hedging.

3.2. Using plain vanilla options

Volatility arbitrage is one type of statistical arbitrage with the main difference that volatility arbitrage is involved in options not equities. This type of arbitrage can be implemented by taking relevant positions in the implied volatility of an option against forecasted realized volatility of its underlying. A delta-neutral portfolio of an option and its underlying can be traded to achieve this arbitrage so that volatility is bought when it is low and sold when it is high. A trader is long volatility when options contracts are bought building a delta-neutral portfolio, the trader bets that the future realized volatility of the underlying is getting higher, on the other side, when options contracts are bought, the trader is short volatility and is betting that the future realized volatility of the underlying is getting lower. The main condition is that the portfolio is kept delta-neutral. Because of the put-call parity it does not make a difference whether to use puts or calls to implement a volatility arbitrage trade. The trader conducts volatility forecasts from the historical realized volatility of the underlying and makes some adjustments according to probable forthcoming events that may influence these forecasts.

A volatility arbitrage trading can be initiated after winning a forecast of future realized volatility and calculating the implied volatility from options prices won from the current market data using a reliable options pricing model. If the trader notices that the calculated implied volatility σ_i is significantly lower than the forecasted volatility, the trader goes long the option and delta-hedges it with the underlying asset to make a delta-neutral portfolio. If the implied volatility were higher than the realized forecasted volatility, the trader goes short the option and delta-hedges it. The trader would realize profit, if his volatility forecast were closer to the realized one than the market's anticipated implied volatility. Keeping this position delta-neutral through continuously hedging it, produces the actual profit of the trade.

In Black's own words, after wondering why people still use the B/S model despite its oversimplified assumptions, he stresses on the importance of the oversimplified assumptions in order to have a solid ground to obtain a proxy to reality, if the shortcomings of these assumptions are properly observed and understood, a better approximation can be achieved through adjusting the assumptions to reach more accuracy and sophistication accordingly. Black proceeds, after mentioning ways to handle the shortcomings resulting from the assumptions of constant volatility and continuous, smooth changes in stock price (no jumps), he stresses the role of supply and demand powers in options markets which result from transaction costs and restrictions on short selling as well as expensive cash positions.

Having a better forecast of future realized volatility than the one implied in options prices available in the market, a relevant position in options buying or selling volatility can be taken. Options strategies such as straddles (call and put options which have the same strike and time to maturity) can be employed to get into a portfolio which is insensitive to price changes but still sensitive to volatility moves. Using ATM straddles in the context of a trend-following strategy and rolling it over so that the exposure is increased when the market tends upwards and decreased when the market is trending downwards, the trader pays implied volatility and gets exposure to realized volatility throughout the lifetime of the straddle. Another way to achieve the same effect in some circumstances might be through building a trend-following strategy through managed futures, as outlined in Fung and Hsieh [1997] where they show that there is a high correlation between both strategies, with managed futures strategies generating more alpha. They show that it is possible through reducing the implied volatility in the long-volatility strategy to achieve the same payoff profile, they argue that this occurs because the long-volatility strategy gets more gamma exposure than theta, due to having more exposure to changes in the underlying asset [24].

On the short side caution is advised when considering using short straddle positions, alternatively, covered calls can be sold or going long bond and short puts. Black comments [4]:

> "Because the formula is so popular, because so many traders and investors use it, option prices tend to fit the model even when they shouldn't. You can gain by looking for cases where the formula should be used only with modified inputs."

Trading volatility using plain vanilla options strategies leaves a chance for being strongly dependent on changes in the volatility of the underlying assets. Trading delta-hedged options does not necessarily hedge against volatility in the volatility itself (volga, vega-gamma or vega convexity), this is due to the fact that theoretical models, in which the volatility parameter is assumed to be constant, are only approximations to reality. In reality, volatility changes over time and therefore is hard to predict and manage. There is always a rest risk that remains after delta is been hedged, which is mainly due to volatility risk.

In the context of the B/S model, if we were writing a call option, we would have to hedge it against the risk of changes that may occur to the underlying asset (delta-hedging), taking a long position according to the B/S delta to make the portfolio delta-hedged, doing that does not eliminate all risk the position is exposed to. Neuberger [1994] emphasizes this point while introducing his work on the Log Contract, having a delta-hedged position still leaves the position with other risks and hedge-errors, the rest risk is due mainly to the probability that volatility may change. So after having a delta-hedged options portfolio, the position must be attempted to be hedged against the risk of changing volatility, which can be quite difficult, the position afterwards is still dependent on the underlying price and time to maturity [40].

To benefit from volatility independent of the direction of the underlying, one is interested in building a quadratic position $(\Delta S)^2$ instead of ΔS. In this case, a delta-hedged position is profitable if realized volatility exeeds implied volatility, otherwise when they are equal, the time decay reduces options value; hence eating away the hedge benefit, most volatility options trades are short volatility. The definitions of volatility spreads by Natenberg [1994] are quite insightful in this context, he points out that these spreads are approximately delta-neutral, sensitive to changes in the price of the underlying, sensitive to changes in the implied volatility and sensitive to the passage of time (theta sensitive) [39]. Other studies, e.g. Chaput and Ederington

[2005] found that the typical design of these volatility spreads reflects three objectives [13]:

1. The desire to maximize the combination's gamma and vega, (straddles have the highest gamma and vega from all options strategies)
2. The desire to minimize the delta
3. The desire to minimize transaction costs

A general trading rule is to place a long straddle in a low volatility environment and a short straddle in a high volatility environment.

If we stay in the B/S world, considering a delta-neutral portfolio Π, consisting of long call and short the amount Δ of the underlying S

$$\Pi = C - \Delta S \tag{3.1}$$

In the passage of time the position would change to be

$$\Delta\Pi = \Delta C - \Delta(\Delta S) \tag{3.2}$$

Theoretically, this portfolio, which is self-financing, should have zero-payoff; hence

$$\Delta C = \Delta(\Delta S) \tag{3.3}$$

using taylor's expansion, we get a better approximation for changes in the portfolio, due to changes in the price of the underlying, the delta of the option and time to maturity, expressed by the Greeks

$$\Delta C = \Delta(\Delta S) + \Theta\Delta t + \frac{1}{2}\Gamma(\Delta S)^2 \tag{3.4}$$

substituting 3.4 in 3.2, we get

$$\Pi = \Theta\Delta t + \frac{1}{2}\Gamma(\Delta S)^2 \tag{3.5}$$

this portfolio still must have the same value, zero, as in 3.2; hence

$$\Theta\Delta t = -\frac{1}{2}\Gamma(\Delta S)^2 \tag{3.6}$$

If we consider ΔS to be approximately equal to $\sigma S\sqrt{t}$ –as can be shown from a GBM process– and if we consider σ to be the volatility implied in the market price of the option, denoting the implied volatility by σ_i

$$\Theta \Delta t = -\frac{1}{2}\Gamma \sigma_i^2 S^2 \Delta t \tag{3.7}$$

substituting 3.7 in 3.5

$$\Delta \Pi = -\frac{1}{2}\Gamma \sigma_i^2 S^2 \Delta t + \frac{1}{2}\Gamma (\Delta S)^2 \tag{3.8}$$

rearranging 3.8 we get

$$\Delta \Pi = \frac{1}{2}\Gamma S^2 \left[\left(\frac{\Delta S}{S} \right)^2 - \sigma_i^2 \Delta t \right] \tag{3.9}$$

Considering the term $(\frac{\Delta S}{S})^2$ to be the realized variance, because the daily average is quite small enough, when considering big enough periods to maturity, dropping it would not make a big difference on the resulting variance value.

According to B/S the delta-neutral portfolio in

$$\Delta \Pi = \frac{1}{2}\sum_{t=0}^{N} \Gamma S^2 \left[\left(\frac{\Delta S}{S} \right)^2 - \sigma_i^2 \Delta t \right] \tag{3.10}$$

should also be equal to zero, but changes in the delta-hedged portfolio within a time period of time N show path dependency. The difference of the value between implied and realized variance is weighted by both gamma and the price level of the underlying –also known as dollar-gamma– traders and speculators using this strategy are not considered to have a pure exposure to volatility, since the value of the portfolio in 3.10 is also dependent on the price of the underlying as well as on gamma [34]. The ideal case would have been a position which is insensitive to price and gamma and only sensitive to volatility.

3.3. Trading volatility as an independent asset class

In the passage of time the way of thinking of volatility changed, instead of managing and hedging it, an increased need for trading it evolved. Banks doing business with structured products, developing, packaging and selling it, are left with a lot of risk that they need to manage and hedge, therefore treating volatility as an independent asset class has evolved over the last few years. Independent means that a position in volatility is not influenced by changes in assets and the derivative instruments they underlie. Volatility trading itself is been active since long time now, but the increased need for having direct positions in volatility is due to the inaccuracy that trading volatility indirectly through other derivatives, e.g. plain vanilla options strategies as mentioned above, suffer.

The behavior of volatility related to an index, e.g. the SMI or the S&P500, as illustrated in figure 3.3, is negatively correlated. In times when the index price goes up the volatility normally tends to decrease, in bear markets, where prices tend to decrease, the volatility normally tends to increase. This phenomena is the key point why portfolio managers tend to use options on indices to insure their portfolios.

Some institutions act as risk provider, others as risk taker, the first having written a lot of securities or sold structured products; hence need to hedge and protect themselves against crashes or sudden moves in the market (jumps), the latter trading or speculating, both of them need some handy instruments to fulfill their needs, here comes a moment swap as a pure and practical way to achieve this. As mentioned below, in order to keep only an exposure to volatility –not also to the price of the underlying as well as the volatility of volatility–, having positions through options strategies is not the optimal choice, a lot of administration and transaction costs are required to keep the position delta-neutral.

Since late nineties, trading moments swaps, especially volatility and variance swaps, has taken a large step into being a standard trade in financial markets, variance swaps being quite liquid and practical have proven to be the financial institutions first choice. Hedge funds, being exposed to volatility, when executing their strategies, e.g. long-short equities, relative and statistical trades, need to deal with the extra volatility driven from the market conditions, in order to keep their returns and growth targets. Portfolio managers also need to secure their positions, while exposed to market volatility, and being benchmarked through tracking error, they need to continuously hedge their positions, theoretically, moment swaps should help achieve their objectives at the least effort and costs possible.

Using futures to hedge portfolios might lead to unpredictable system risk through creating gaps between supply and demand which in turbulent times may make things worse. Using moment swaps might ease these effects over a time span; hence avoid crashes.

Figure 3.1.: The relationship between the SMI and the VSMI, as well as between the SPX and the VIX

3.3.1. The Log Contract

Options may loose its "characteristics" in the passage of time, if the option is deep ITM, then it looses these characteristics and gets similar to the underlying itself. If the option gets deep OTM, then it also looses options characteristics because it is almost worthless. Only options that are ATM have and keep option-like characteristics.

Neuberger [1994] developed the idea of using the Log Contract to produce an instrument that keeps the characteristics of options –because of its being nonlinear– independent of the moves in the underlying price with a constant gamma. The Log Contract is thought to be a hedging instrument for volatility, it is a futures contract which has the settlement price equal to the log of the spot price and has a payoff which is a function of the difference between the realized and the implied volatility of the underlying [40].

If we consider F_T to be the future's settlement price at expiration and L_T to be the settlement price of the Log Contract, using B/S assumptions –especially that the forward price follows a GBM process with constant volatility– it can be shown that the fair price of the Log Contract at time t, is given by

$$L_t = LogF_t - \frac{1}{2}\sigma^2(T-t) \tag{3.11}$$

where F_t is the future's price at time t. If we set $t = T$ the result would be exactly the continuous payoff of the underlying asset.

With the Log Contract, the investor does not need to forecast volatility in order to hedge his portfolio correctly. The Log Contract only depends on realized volatility, table 3.1 lists the main Greeks of the Log Contract. Delta is independent of the level of volatility and is equal to $\frac{1}{F_t}$, one short underlying asset would hedge one long Log Contract [40]. The present value of profit & loss of the strategy of being short one Log Contract and delta-hedging by going long one underlying, over the life of the volatility contract, is approximately

$$\frac{1}{2}\left(\sigma_r^2 - \sigma_i^2\right)(T-t) \tag{3.12}$$

where σ_r is the realized volatility, and σ_i is the implied volatility in the Log Contract. The resulting delta-hedged position is a pure volatility position and the type of price process of the underlying is insignificant.

Consider the function: $f(S,T) = Ln(S)$

The Greek	Sign and Sensitivity
delta	$\Delta = \frac{\partial L}{\partial S}$
gamma	$\Gamma = \frac{\partial^2 L}{\partial S^2}$
vega	$\nu = \frac{\partial L}{\partial \sigma}$
theta	$\Theta = \frac{\partial \tau}{\partial t}$

Table 3.1.: The greeks of a Log Contract

The partial differential equation for the Log Contract is

$$rL_{S,t} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rs\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} \tag{3.13}$$

The dynamics of the Log Contract can be found using Itô's lemma

$$L_{S,t} = \left\{ Ln(S) - \left(\frac{1}{2}\sigma^2 - r\right)(T-t) \right\} e^{-r(T-t)} \tag{3.14}$$

Now consider the function: $f(F,T) = Ln(F)$

$$L_{F,t} = Ln(F) - \frac{1}{2}\sigma^2(T-t) \tag{3.15}$$

Where $F = Se^{rT}$ is the forward price, S the spot price and T is time to maturity.

The Log Contract is a function of the volatility of the underlying, $\Delta = \left(\frac{1}{F}\right), \Gamma = \left(\frac{1}{F^2}\right)$. Gamma hedging and volatility trading are much more stable through the Log Contract than it is with options

- The delta-hedged Log Contract provides a near-perfect volatility hedge (no assumptions, even with fat tails).
- Liquid options well be illiquid in the passage of time, but the Log Contract stays liquid over time.
- The Log Contract is an instrument specially designed to hedge volatility.

The Δ-risk is removed and the rest of the remaining risk is due to volatility.

The Log Contract can be achieved through a continuum of plain vanilla options, weighted with $\left(\frac{1}{K^2}\right)$ the strike price of the underlying asset [22]:

$$ln\left(\frac{S_T}{S_*}\right) = \frac{S_T - S_*}{S_*} - \int_0^{S_*} \frac{P(K,T)}{K^2} dK - \int_{S_*}^{\infty} \frac{C(K,T)}{K^2} dK \tag{3.16}$$

where S is the price of the underlying, P(K, T) is a European put option with strike K and time to maturity T and S_* is ATM forward price and is considered to be the boundary between liquid puts and liquid calls [17]. Accordingly, the Log Contract can be replicated through:

- Long $\frac{1}{S_*}$ forward contracts (to emphasize the discrete payoff of returns on the underlying)
- Short a static position in a put-portfolio, with strike prices starting from 0 up to S_*, each put option is weighted by $\frac{1}{K^2}$
- Short a static position in a call-portfolio, with strike prices starting from S_* up to ∞, each call option is weighted by $\frac{1}{K^2}$

This idea of replicating the Log Contract is essential for pricing volatility swaps.

Carr-Madan [12] developed the idea, assuming any investor can take positions in a future price $f(F_T)$ through going into positions in options but having no assumptions about the stochastic process governing the future price; hence the payoff of the Log Contract can be decomposed into different components, they show how any payoff function, being twice differentiable, can be decomposed as follows

$$\begin{aligned} f(F_T) &= f(k) + f'(k)\left[(F_T - k)^+ - (k - F_T)^+\right] \\ &\quad + \int_0^k f''(K)(K - F_T)^+ dK + \int_k^{\infty} f''(K)(F_T - K)^+ dK. \end{aligned} \tag{3.17}$$

which includes terms that can be interpreted as:

- $f(k)$ static position in zero-coupon bond, where k represents the strike price of an ATM option
- $f'(k)$ call options with strike $= k$ minus $f'(k)$ put options with strike $= k$
- Static position in $f''(K)dK$ OTM put options at all strikes that are less than k
- Static position in $f''(K)dK$ OTM call options at all strikes that are greater than k

Applying this argumentation, the no-arbitrage condition requires that

$$\begin{aligned} V_0^f &= f(k)B_0 + f'(k)\left[C_0(k) - P_0(k)\right] \\ &\quad + \int_0^k f''(K)P_0(K)dK + \int_k^\infty f''(K)C_0(K)dK. \end{aligned} \tag{3.18}$$

Where B_0 denotes the value of the zero-coupon bond, V_0^f the value of the payoff, both at initial time.

The party that is short a Log Contract is long options, it is forced to buy more OTM options with lower strikes in order to hedge its position; hence it is implicitly long skew.

Why hedge volatility?

Writing naked options is risky, banks mostly hedge their positions using the B/S formula which is inaccurate. The portfolios are not continuously replicated, price changes are not really log-normally distributed, jumps exist and volatility is not constantly known. There still are remaining risks after having the option portfolio delta-neutral according to B/S formula. About eighty percent of hedging errors after a portfolio is been delta-hedged are due to wrongly predicting volatility over the life of the option. Through hedging volatility, hedge errors could be reduced by the factor of five, if a bank is not able to forecast volatility better than the market, it should reduce its exposure to risk as far as possible through hedging volatility using cheap accurate means, hence its capability to write more contracts would increase [40].

Delta-hedged option can be used as a position on volatility, because the payoff to a delta-hedged option is quite well correlated with volatility. Portfolio insurers who want to hedge their volatility exposure normally create synthetic put options e.g. through trading futures on an index or they can buy short-dated options and delta-hedge them to hedge against volatility shocks. If there were an instrument that could precisely lock in volatility, it would be possible to lock in the cost of the insurance.

Trader who expects the volatility of the underlying to be fifteen percent but notices that the options on that underlying imply only ten percent can take a position accordingly to lock in a profit; the trader will be using the options market to take a volatility exposure position and delta-hedge it. Options writers can immunize their positions through gamma-neutral options positions. It would have been better and more efficient if these parties would have been able to take their positions in volatility directly.

3.3.2. Volatility and correlation indices

Volatility indices

The Log Contract contributed to develop the establishment of a volatility index, the CBO VIX is basically built upon its fundaments. When managing the risk of an options portfolio, it may be more efficient to use futures and options on a market implied volatility index like the VIX on the S&P500, than using plain vanilla index options.

In 1993 the CBOE introduced the VIX, its aim is to measure the thirty day implied volatility of ATM S&P100 Index option prices. In 2003 CBOE introduced the new VIX which still measure thirty days implied volatility but on the S&P500 Index option prices and not only form the ATM options but also from all other options with different strike prices.

Later on, other volatility indices were developed for the Nasdaq-100 (VXNSM), the DJIA (VXDSM), the Russell 2000 (RVXSM) and the S&P500 3-month (VXVSM). In 2008, CBOE expanded using the same VIX methodology to some commodities and foreign currencies, Crude Oil Volatility Index (OVXSM), Gold Volatility Index (GVZSM) and EuroCurrency Volatility Index.

In March 2004 the CBOE issued futures, in February 2006 options on the VIX to trade the S&P500 index volatility. Futures on the VXD and the RVX are also available. Other instruments are also traded like S&P500 BuyWrite Index Futures, S&P500 three-month variance futures, and S&P500 12-month variance futures. In appendix A is an excerption of the white paper published by the CBOE to show how the VIX is calculated [23].

Correlation indices

CBOE initiated on June 4, 2008 its Implied Correlation Index on the S&P500 index and tied it to two SPX options maturities, January 2009 and January 2010. The purpose of this index is to calculate the expected average correlation of the S&P500 constituents, using options prices on both the largest fifty constituents and the index itself. A relative comparison between the implied volatility of the index and its constituents can be easily done. Data back to 2003 are also available. CBOE is also planning to issue new products based upon this index.

3.3.3. Moment swaps

The concept of moment swaps

Moment swaps are not real swaps as known in the classical sense, they are considered to be forward contracts on higher moments of the realized distribution. Assuming constant volatility is considered to be a starting point for more realistic models which take into consideration the real return volatility in the market place. The B/S model assumed the volatility to be given and constant; hence the moments of the returns distribution are limited to the second moment –the variance– which is unrealistically assumed to be constant. To count for the higher moments in a realistic manner, it is important not only to count for the second moment, but it is then also important to care for the higher moments (skewness and kurtosis). There are different ways to deal with these higher moments. The moment swap is putting a fixed leg against a variable leg (the forward realized moment (variance-, skew- and kurtosis swaps)), which is payed at the expiration date.

The fair price of a moment swap is the risk-neutral discounted expected payoff at contract expiration date, in this context, the fixed leg value cannot be defined before the fair value of the variable leg has been defined, which makes these instruments more likely to be called forward, rather than swap instruments.

A general form to express the payoff of a moment swap can be written as [1]

$$ln\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)^k = k!\left[\left(\frac{\Delta S_{t_i}}{S_{t_{i-1}}}\right) - ln\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right) - \sum_{j=2}^{k-1}\frac{1}{j!}\left[ln\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)\right]^j + \mathcal{O}\left[ln\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)\right]^{k+1}\right] \tag{3.19}$$

Ignoring the higher moments, represented by the Landau notation ($\mathcal{O}$), this swap can be replicated through:

- short position in an instrument that pays out the continuous return of the underlying (The Log Contract)

- long position in the underlying, dynamic position which pays out the discrete return on the underlying (delta-hedge of the Log Contract)
- short a series of positions in the higher moment swaps, beginning from $j = 2$ to $k - 1$, short sell $\frac{j!}{(j-1)!}$ each moment swap

An existing hedging error resulting from not being able to hedge the Log Contract continuously made it possible for moment swaps to evolve and being found appealing. The Log Contract is considered to be the main block when building a moment swap, a delta-hedged Log Contract position which periodically pays off the hedging error is considered to be the helping instrument to be able to replicate the payoff of a moment swap.

As can be seen from equation 3.19, ignoring the higher moments leaves us with an error when replicating a moment swap, this error is of the order $(k+1)$, when buying a moment swap, because of that replication error, the buyer actually pays for all the upper moments, while getting only the realized moment the buyer explicitly wanted an exposure to. The difference is supposed to be the "moment-risk premium" that has to be paid, in order to compensate the market for bearing the risk of the other moments [45].

Do moment swaps complete the market?

Corcuera et al. [2005] suggest that under an incomplete Lévy market (all moments are allowed to deviate from the normal distribution in a stochastic manner), when trading all higher moments is allowed it would lead to a complete market [15].

Hedging moment swaps

To replicate a long moment swap, a short position in Log Contracts, accompanied by dynamically delta-hedging it, plus a static position in lower order moment swaps is needed [15]. The payoff of the moment swap, as outlined before, is the Log Contract hedging error. The Log Contract of its own can be hedged using a dynamic position in futures and static positions in bonds and options; hence the payoff of a moment swap is the payments stream resulting from the hedge error at each time period between inception and maturity plus the payment resulting from the options portfolio at maturity.

3.3.4. Variance swaps

Considering the general case of deriving a price for moment swaps, as shown in 3.19, the variance swap can be priced following

$$ln\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)^2 = 2\left[\left(\frac{\Delta S_{t_i}}{S_{t_{i-1}}}\right) - ln\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right) + \mathcal{O}\left[ln\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)\right]^3\right] \tag{3.20}$$

hence a variance swap can be replicated through:

- short two Log Contracts
- a position in delta-Hedge dynamic of a Log Contract

which represents a dynamically adjusted delta-neutral portfolio of plain vanilla options.

Variance swaps are effectively forward contracts on the realized variance of the returns of a specific asset over the life of the swap, its payoff, if considered for a long volatility position, is calculated as the difference between the annualized realized variance and a predefined variance strike price, over a predefined period of time. This payoff is then converted into dollar terms through multiplying it by a variance notional, which is specified in terms of vega. Variance swaps makes it possible to have an exposure to the volatility of a specific asset without the need to establish an options strategy on it and being forced to keeping it delta-neutral, as mentioned above.

The payoff of a variance swap is given by

$$N_{var}(\sigma_r^2 - \sigma_k^2) \tag{3.21}$$

where

$$\sigma_r^2 = \text{annualization factor}\left[\frac{\sum ln\left(\frac{S_t}{S_{t-1}}\right)^2}{\text{return calculation period}}\right]10000 \tag{3.22}$$

In practice the number of contract units is defined as $N_{var} = \frac{Nvega}{2\sigma_k}$ where N_{var} is the variance notional, σ_r^2 the annualized realized variance, σ_k^2 the variance strike, N_{vega} the average profit or loss resulting from one percent move in the volatility (this also enables variance and volatility swaps to be comparable).

The realized variance is calculated ignoring the daily average returns, having a very small value that can be ignored, the fact that this is possible, makes the payoff of the variance swaps additive.

Figure 3.2 shows the forecasting power that implied volatility represented by the volatility index can give, with a high beta as high as 0.88 when related to the realized volatility. The distribution characteristics can also be observed to be log-normal for the realized volatility, positively skewed with higher kurtosis and log-normal for the implied volatility, also positively skewed but having a lower kurtosis. The implied volatility distribution is leptokurtic (more flat), which can be interpreted as a typical characteristic for an opinion about the future realized volatility of a certain asset, which tends to prefer yes or no (high/low) and not much in between. The existence of heteroskedastic error terms observed in the same figure is another proof that they are resulting from different distributions.

In order to lock in the variance risk premium (Implied variance - realized variance) an investor, e.g. hedge funds, can sell variance swaps. Buyers of variance swaps, e.g. banks, need to hedge their positions by buying back Log Contracts –or sell a spectrum of options with different maturities and strike prices– this way, this can lead to a higher supply for options on the different maturities, which can lower the level of the volatility skew.

Valuing variance swaps

The variance swap can be valued using 3.18, considering the variance swap to be a forward contract and taking the risk-neutral expectation at time $t = T$, the strike price can be calculated as:

$$K = \mathbb{E}[\sigma_T^2] \tag{3.23}$$

Due to considering the mean to be equal to zero, it is possible to gain the additivity characteristic for the variance swap. This characteristic enables using a semi-static hedging strategy, which makes it practical to trade variance swaps in a liquid market.

One way to look at the fair value of a variance swap is through considering the time discrete, arithmetic brownian motion return process of the underlying S

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t \tag{3.24}$$

solving the stochastic differential equation 3.24, using Itô's lemma, we get the continuous time return rate

$$d(lnS_t) = \left(\mu_t - \frac{1}{2}\sigma_t^2\right) dt + \sigma_t dW_t \tag{3.25}$$

Figure 3.2.: Annualized S&P500 monthly realized volatility, compared to the VIX

we get a difference between both processes equal to $\frac{1}{2}(\sigma_t^2 dt)$, so that

$$\sigma_t^2 dt = 2\left(\frac{dS_t}{S_t} - d(lnS_t)\right) \tag{3.26}$$

integrating over a time period $[0,T]$, using the fact that the continuous time variance equals $\left(\frac{1}{T}\int_0^T \sigma_t^2 dt\right)$, we get

$$\sigma_T^2 = \frac{2}{T}\left(\int_0^T \frac{dS_t}{S_t} - ln\left(\frac{S_T}{S_0}\right)\right) \tag{3.27}$$

as shown in 3.16, the term $ln\left(\frac{S_T}{S_0}\right)$ represents a Log Contract which is continuously hedged using $\int_0^T \frac{dS_t}{S_t}$.

The risk-neutral expected variance at $t = T$ should correspond to K, the fair strike price of the variance swap

$$K = \mathbb{E}[\sigma_T^2] = \frac{2}{T}\mathbb{E}^Q\left(\int_0^T \frac{dS_t}{S_t} - ln\left(\frac{S_T}{S_0}\right)\right) \tag{3.28}$$

using the risk-neutral process for the underlying

$$\mathbb{E}^Q\left(\int_0^T \frac{dS_t}{S_t}\right) = \mathbb{E}^Q\left[\int_0^T rdt + \sigma_t d\tilde{W}_t\right] = rT \tag{3.29}$$

$$K = \frac{2}{T}\left[rT - \mathbb{E}^Q\left(ln\left(\frac{S_T}{S_0}\right)\right)\right] \tag{3.30}$$

in this equation, the term rT cancels out both forward as well as the dynamic hedging position in equation 3.16 for the Log Contract $ln\left(\frac{S_T}{S_0}\right)$, when assuming S_* to be equal to the forward price Se^{rT}. The remaining value is

$$K = e^{rT}\frac{2}{T}\left[\int_0^{S_*} \frac{P(K,T)}{K^2}dK - \int_{S*}^{\infty} \frac{C(K,T)}{K^2}dK\right] \tag{3.31}$$

which represents a continuum of call and put options, this method represents the model-free fair value of the future variance.

Using the Carr-Madan method [12], we get the following solution

$$K = \left[\int_0^{S_*} \frac{2\left[1 + ln\left(\frac{K}{S_*}\right)\right]}{K^2} P(K,T)dK + \int_{S*}^{\infty} \frac{2\left[1 - ln\left(\frac{K}{S_*}\right)\right]}{K^2} C(K,T)dK\right] \tag{3.32}$$

Figure 3.3 shows the variance exposure V of different portfolios of call options of different strikes as a function of the strike price K, each figure on the left side shows the individual V_i contributions for each option of strike K_i, the corresponding figure on the right shows the sum of the contributions, weighted two different ways; the dotted line corresponds to an equally-weighted portfolio of options; the solid line corresponds to a portfolio of options weighted inversely proportional to K^2, we can notice how in the second case the variance exposure becomes totally independent from strike price K inside the strike range [17].

Having positions in variance swaps can be quite tricky, it requires the investor, whether it is a hedge fund or a trader, to manage its risks efficiently, in May 2006 a lot of investors suffered huge losses due to a sudden spike in realized volatility caused by inflationary and macroeconomic effects. Investors who were short variance swaps, who should have hedged their positions, as mentioned above, did so but obviously it was neither sufficient nor effective [32]. As shown in figure 3.4, this pattern has repeated itself but in a massive manner because of the sub-prime crisis and the global financial crisis that followed it.

Micro-events are events that occur with regard to individual assets, other than macro-events, which influence the whole economy; hence all of the assets together, micro-events influence the value of the variance swap more than macro-events.

Why not volatility swaps?

A volatility swap would be quite difficult to replicate for the party with the short position, it is exposed to model risk, as it must calculate volatility using most probably stochastic volatility models. It would also need to employ a dynamic hedge strategy using none-linear instruments such as variance swaps in order to hedge the exposure to volatility of volatility (sometimes practitioners call it vomma, vega convexity, vega gamma or volga). Moreover in order to mark the volatility swap to market, it would not be possible to add the pay off of the different periods to come up with a fair value for it.

In order to come to an approximate value for the volatility swap, a convexity correction can be calculated and subtracted from the variance swap. Carr and Lee [2005] show that the ATM implied volatility is a good approximation to come to a value for volatility swaps –the ATM volatility is considered to be a good proxy for the realized volatility– they consider ATM implied volatility to be a lower bound for the fair value of a volatility swap and the variance swap value to be the higher bound [10].

Figure 3.3.: Volatility of options portfolios. Source: Derman [1999] [17]

Figure 3.4.: Payoffs of a rolling forward strategy, holding long S&P500 one month variance swap

Third generation volatility swaps

To meet the needs of the market, some other kinds of volatility oriented swaps evolved, which are known as third generation volatility swaps. Below I mention these swaps briefly.

Corridor variance swaps are cheaper than the general form variance swaps, they are limited with a cap and a floor, as long as the volatility is been within the limits, the realized leg will be calculated, other moves outside these limits do not count. On the other side, the party which is long a corridor swap should pay the fixed leg K, the strike price, independent from the developments of the realized leg; however, the value of K compared to the one for a generalized variance swap is lower accordingly, this is due to the fact that in order to replicate the corridor variance swap, less options would be needed, namely the ones that have strike prices corresponding to the limits of the corridor. Other variations of corridor swaps, such as corridor up and down variance swaps are also available, in a corridor up variance swap, the corridor is defined at a floor but open to unlimited upwards, for a corridor down variance swap, there is a cap for the corridor but unlimited downwards.

The payoff of a corridor variance swap is represented by [11]

$$\mathrm{Var}_{\mathrm{Corr}}(L,H) = N_{\mathrm{Corr}}(\sigma_T^2(L,H) - K_{\mathrm{Corr}}) \tag{3.33}$$

where $\sigma_T^2(L,H)$ the realized variance in a pre-specified range of returns –other returns would be considered to be equal to zero–, N_{Corr} the nominal of the contract and K_{Corr} is the strike price.

Corridor variance swaps can be used as skew trading instrument and as a risk management for other derivatives portfolios.

Conditional variance swaps are similar to corridor variance swaps, the main difference is that by conditional variance swaps, the fixed leg is also dependent on the times the up and down limits were hit.

The payoff of the conditional variance swap is given by

$$\mathrm{Var}_{\mathrm{Con}}(L,H) = N_{\mathrm{Con}}\left(\sigma_T^2(L,H) - \left(\frac{N_{\mathrm{in}}}{N}\right) K_{\mathrm{Con}}\right) \tag{3.34}$$

Conditional variance swaps can be used to have a position while having a view about the future development of the skewness or kurtosis of a specific index (or basket of assets).

Forward starting variance swaps are contracts that are closed at $t = 0$, which enables a variance swap to take place, according to the variance implied in the market at $t = T^*$, which would be realized at $t = T$, where $T^* < T$

The payoff of a forward starting variance swap is

$$\text{Var}_{\text{fwdSVS}} = N_{\text{fwdSVS}}(\sigma^2_{T-T^*} - K_{\text{fwdSVS}}) \tag{3.35}$$

The forward starting realized variance at expiration is

$$\sigma^2 T - T^* = \frac{1}{T - T^*} \int_{T^*}^{T} \sigma_t^2 dt \tag{3.36}$$

Forward starting variance swaps allow hedgers and traders to express an opinion about the term-structure of volatility. Because of the additivity characteristic of variance swaps, forward starting variance swaps were possible. This swap can be achieved through:

- long variance swap that starts at $t = 0$ and ends at $t = T$
- short variance swap that starts at $t = 0$ and ends at $t = T^*$

The advantage of having the short variance swap position is that it reduces the premium which is paid, involuntarily, to have a position in the skew, while the main purpose is to have a pure variance exposure. These positions are similar to the plain vanilla options time-spread strategy, the main difference is that through forward variance swaps, the position does not need to be kept delta-neutral, because gamma is being kept constant.

Using Carr-Madan formula, forward starting variance swaps can be valued using

$$\begin{aligned} \mathbb{E}[\sigma^2_{T-T^*}] = {} & \frac{2e^{rT}}{T}\left[\int_0^{S_*} \frac{1}{K^2} P_0(K,T)dk + \int_{S_*}^{\infty} \frac{1}{K^2} C_0(K,T)dk\right] \\ & - \frac{2e^{rT}}{T^*}\left[\int_0^{S_*} \frac{1}{K^2} P_0(K,T^*)dk + \int_{S_*}^{\infty} \frac{1}{K^2} C_0(K,T^*)dk\right] \end{aligned} \tag{3.37}$$

Gamma swaps are similar to variance swaps but have the important characteristic that they are linked to the movements in the underlying through weighting the realized variance with the price level of the underlying. The payoff of gamma swaps is [28]

$$p\&l = N_{\text{gamma}} \left(\frac{1}{T} \int_0^T \sigma_t^2 \left(\frac{S_t}{S_0} \right) dt - K_{\text{gamma}} \right) \tag{3.38}$$

where N_{gamma} the nominal, S_t the spot price at time t, S_0 the spot price at initiation and K_{gamma} the strike price.

The replication of gamma swaps can be achieved using Itô's lemma and Carr-Madan formula. The resulted replication of a long position is

- long position in a portfolio of calls and puts
- futures position $= -2ln\left(\frac{F_t}{F_0}\right)$ rolled over
- a zero-coupon bond $= -2r\left[F_t\, ln\left(\frac{F_t}{F_0}\right) - F_t + F_0\right]$

where F_t is the future price at time t and F_0 is the future price at initiation

Considering $C(S_0, K, T)$ and $P(S_0, K, T)$ to be a call and put written on the stock S, with strike K and time to maturity T, the fair price of a gamma swap with strike price of $K_\Gamma^{0,T}$ at $t = 0$ is therefore:

$$\begin{aligned} K_\Gamma^{0,T} &= \mathbb{E}_0 \left(\frac{1}{T} \int_0^T \sigma_t^2 \left(\frac{S_t}{S_0} \right) dt \right) \\ &= \frac{2e^{2rT}}{TS_0} \left[\int_0^{F_0} \frac{1}{K} P(S_0, K, T) dK + \int_{F_0}^{+\infty} \frac{1}{K} C(S_0, K, T) dK \right] \end{aligned} \tag{3.39}$$

Gamma swaps do not need dynamic re-hedging, like variance swaps, because the movements of the underlying have been taken into consideration. Moreover, gamma swaps are not capped because jumps are taken into consideration, other than variance swaps that are capped in some of its variants, as mentioned before.

3.3.5. Correlation swaps

Correlation is a cross-moment which occurs when focusing on the relation between different assets together. When considering being long a portfolio of assets, there is an implied position in correlation, in this case a short one. If we consider a bank which is dealing with structured products, in this case, the bank is short correlation. Earlier the bank kept the correlation risk and has done nothing to eliminate it, mainly because it did not reach a critical level to justify its management. in the new market circumstances the question is how would the bank dealing with structured products would reduce correlation risk? This must be done through taking an opposite position to hedge against being short correlation. Which instruments are available to be able to go long correlation?

Correlation and variance swaps are the main ways to deal with correlation. Potential counter-parties are hedge funds (being involved in strategies such as vanilla options dispersion trading), reinsurers and other institutional investors, which are interested in hedging their natural long correlation positions.

Trying to gain an exposure to correlation through plain vanilla options requires huge amount of them, which needs administration as well as delta- and vega-hedging. That is why a direct pure exposure for these parties is quite interesting, here comes the role of correlation and covariance swaps.

The long leg of a correlation swap receives the Notional multiplied by the realized correlation points, the short one gets the notional multiplied by the fixed strike points. Normally these swaps are settled on baskets not indices which are often spread globally [30].

One way to gain exposure to correlation can be done through variance swaps, but this has a main downside, it is more expensive than using plain vanilla options, and in the context of dispersion trading, the correlation skew is also implied in the exposure, which was not necessarily desired.

Correlation swaps may help to achieve the desired exposure. The payoff of a Correlation swap is

$$P\&L = N_C(\widehat{WAC} - K_C) \tag{3.40}$$

Where N_C is notional amount per correlation point, K_C is the strike price, $\widehat{WAC}$ represents the weighted average correlation and is calculated by:

$$\widehat{WAC} = \frac{\sum_{i=1}^{N} \sum_{j>i}^{N} w_i w_j \hat{\rho}_{ij}}{\sum_{i=1}^{N} \sum_{j>i}^{N} w_i w_j}$$

where $\hat{\rho}_{ij}$ represents future realized average correlation; w_i and w_j are weights of the assets i and j respectively.

The correlation swap, if considered for only the two assets X and Y, can be valued through using a stochastic multi-factor volatility model, such as the Heston model to find a value for

$$K_{corr} = \mathbb{E}^Q \left[\rho_T^{XY}\right] \tag{3.41}$$

Main disadvantages of correlation swaps are their being illiquid –mostly because of the difficulty of marking them to market– and the absence of an arbitrage-freemodel that can replicate a correlation swap.

The value of the correlation swap is influenced more by macro-events than micro-events.

3.3.6. Skew swaps

One main advantage of the Log Contract is that it takes into consideration the third moment of the return distribution, the skewness. This is given by the Log Contract construction. Being weighted by $\left(\frac{1}{K^2}\right)$, higher amount of low strike options have to be sold or bought, according to the Log Contract being short or long, this leads to the Log Contract to be short or long skew respectively; moreover, the amount of put options, according to the same reasoning, is supposed to be less than call options, which strengthens the skew effect; hence skewness is determined by the correlation between price and volatility.

The skew swap is a forward contract on the realized skewness of the underlying returns distribution, its payoff is calculated by

$$SkewS = N_{\text{skew}}(X - K_{\text{skew}}) \tag{3.42}$$

where N_{skew} the nominal, X the annualized realized skew and K_{skew} is the annualized skew strike price.

Similar to variance swaps, the skew swap can be decomposed using

$$ln\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)^3 = 6\left[\left(\frac{\Delta S_{t_i}}{S_{t_{i-1}}}\right) - ln\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right) - \frac{1}{2}\left[ln\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)\right]^2 + \mathcal{O}\left[ln\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)\right]^3\right] \tag{3.43}$$

the fair value of a skew swap can be replicated through

- short 6 Log Contracts
- a position in 6 delta-Hedge dynamics of a Log Contract
- short three variance swaps

3.3.7. Kurtosis swaps

Jumps in financial markets leaves the probability distribution of the returns of a specific asset, contradicted to the theoretical assumptions, normally distributed, these jumps lead to an excess kurtosis which is higher than the one for normal distributions which is equal to three. If an empirical distribution had this characteristic, it would mean that fat tails exist –rare events have higher probability to occur than supposed by normal distribution– individual options volatility skews would have higher ends. If volatility skew were flat, the effect on the options price from the kurtosis would be higher than that from the skew, since if higher kurtosis exists, the difference between OTM calls and puts would decline, generally OTM options would be more expensive than ATM options.

The kurtosis swap is a forward contract on the realized kurtosis of the underlying returns distribution, its payoff is calculated by

$$\mathrm{KurtS} = N_{\mathrm{kurt}}(X - K_{\mathrm{kurt}}) \tag{3.44}$$

where N_{kurt} the nominal, X the annualized realized kurtosis and K_{kurt} is the annualized kurtosis strike price.

The quadratic relationship between asset price and volatility determines the kurtosis. Weighting should be cubic, the OTM options would be weighted more than it was the case by variance swaps.

Similar to variance swaps, the kurtosis swap can be decomposed using

$$ln\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)^4 = 24\left[\left(\frac{\Delta S_{t_i}}{S_{t_{i-1}}}\right) - ln\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right) - \frac{1}{2}\left[ln\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)\right]^2 - \frac{1}{6}\left[ln\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)\right]^3\right] \tag{3.45}$$

The fair value of a kurtosis swap can be replicated through

- short 24 Log Contracts
- a position in 24 delta-hedge dynamics of a Log Contract
- short 12 variance swaps
- short 4 skew swaps

4. Dispersion and skew trading

4.1. Dispersion trading

When considering an index and its constituents, Bakshi et al. [2003] found that the risk-neutral distributions of the individual constituents have a lower negative skewness and much higher volatility compared to the risk-neutral distribution of the index [1].

Driessen et al. (2006) studied the correlation of the index and its constituents and found that implied index variance is higher than the realized one, which was not the case when considering its individual constituents where the differences between implied and realized variance was not significant. They consider this to be an evidence that the correlation risk must be priced in the options on indices and that the implied correlation is higher than the realized one. This implies the existence of a large correlation risk premium; hence there is a price for taking the correlation risk that is independent from stock market risk. As mentioned in section 3.3.5, there exists no direct way to observe and quantify correlation, only the fact that correlation is changing over time due to market dynamics, therefore an arbitrage possibility may exist, trading realized against implied correlation can make use of that possibility [20]. Normally, the market does not take the correlation risk into consideration when pricing individual options on the constituents. If correlation were <1, there would be a spread between the index implied volatility and the implied volatility of the basket constituents.

Dispersion trading is also called volatility or index-option arbitrage and is an exposure to correlation –being long dispersion means being short correlation and vice versa– it is implicitly trading the average volatility of the constituents and allows to have a lot of gamma exposure on the relative underlying assets. It is about how the constituents of a basket disperse themselves around a mean of zero. Before the year 2000, dispersion trading was more profitable because there was a long dispersion bias (on average, more realized dispersion than implied). Currently, dispersion trading is not used for speculation but for liquidity and risk management purposes.

If we consider a basket of stocks

$$S = \sum_{i=1}^{n} w_i S_i \tag{4.1}$$

where

$$w_i = \frac{n_i S_i}{\sum_{i=1}^{N} n_i S_i}$$

which have the variance:

$$\sigma_I^2 = \sum_{i=1}^{n} w_i^2 \sigma_i^2 + 2 \sum_{i=1, J \neq i}^{n} w_i w_j \sigma_i \sigma_j \rho_{ij} \tag{4.2}$$

persuming the correlation factor ρ to be equal to one, we get the following average variance for the basket:

$$\bar{\sigma}_I^2 = \sum_{i=1}^{n} w_i \sigma_i^2 \tag{4.3}$$

calculating the difference between that value as an upper limit of the actual basket variance and the average variance of the constituents, we get a dispersion spread

$$D = \sqrt{\sum_{i=1}^{n} w_i \sigma_i^2 - \sigma_I^2} \tag{4.4}$$

A relative trading position is normally built upon the resulting spread. If the trader is long dispersion, then the position is long volatility of the constituents and short volatility of the basket. This position would be profitable if the volatility of the constituents were greater than that of the basket, in this case it is desirable to have higher daily returns for the constituents but lower daily returns for the basket. Delta-hedging the positions on the constituents makes the trade gamma exposed on the constituents side, but less exposed on the basket side while it did not move as much as the constituents. But on the short side, the basket, the position earns theta (the position gets nearer to expiration, hence winning time value), these theta earnings compensate for the theta losses on the other long side of the position, the constituents (bleeding through theta). In order for this strategy to be profitable, the constituents should not move together in the same direction, they should disperse.

The long dispersion bias, mentioned above, can happen due to several reasons:

- higher demand for put options on the basket (Index)

- more short positions on the constituents derivatives side, than it is on the basket side
- futures on the basket are much more liquid than the constituents
- underestimating extreme events

In order to identify a trading opportunity and its best timing, the main three correlation values; theoretical, realized and implied are compared to each other. Theoretical value is the value that can be calculated through a theoretical model; realized is the value that can be calculated from historical market data of the constituents and the implied correlation, as described in section 2.3.5, is won through available options data on both the constituents and the index. As rule of thumb, if implied is higher (lower) than realized volatility, a long (short) dispersion strategy is undertaken.

4.1.1. Through options strategies

The common way to trade dispersion is through taking relevant positions in ATM options straddles or strangles. Straddles have a positive and increasing pay off when prices tend to be deep in or out of the money. Strangles are also used because of being less expensive than straddles, having a range where there is no exposure to price movements.

A dispersion trade using options can be carried out in the following order:

1. Initially buy underpriced options or strategies, or sell overpriced options or strategies.
2. Offset the option position by taking an opposing position in the underlying asset, making it delta-neutral.
3. Periodically buy or sell an appropriate amount of the underlying asset to remain delta-neutral over the life of the strategy (dynamic hedging).
4. Liquidate the entire position at expiration.

In theory, when the position is closed out, the total profit (or loss) should be approximately equal to the amount by which the options were originally misprized. Delta of this position is equal to zero upon initiation, but unless hedged continuously varies in the passage of time according to the movements in the price of the underlying, which leaves the whole position exposed to directional movements (being no longer delta-neutral) and not only to volatility as originally intended. In this case the trader is being long gamma and profits from the more frequent moves in the price of the underlying, opposed to another, that is short gamma. The trader must roll the long gamma position forward to avoid having too high gamma exposure at expiry of ATM options (individual gammas of the options are highest by ATM options at expiration as shown in figure 2.2).

One drawback of using this strategy is that it does not have a pure exposure to volatility and correlation. The options must be kept delta-neutral in order to keep the volatility and correlation exposure constant or at least stable, this leads to increasing hedging inaccuracy in addition to the increasing transaction costs. Moreover, options on some of the constituents of a basket (e.g. an index) may lack liquidity which hinders carrying out the trade. There are some ways to overcome this issue, but it is not accurate, e.g. employing options on a subset of the constituents which makes out the majority of the weighted market capitalization of the basket, but such methods leave the trade less reliable; therefore in order to achieve a pure exposure to volatility, the attractiveness of employing variance swaps in the trade arose; hence trying to be only dependent on correlation and not also on the other unwanted factors.

4.1.2. Through variance swaps

Variance swaps are used to execute dispersion trades in a more strait forward way, they do not need to be kept delta-neutral opposed to options strategies, such as straddles and strangles. Another advantage of using a variance swaps strategy is that variance swaps are quite liquid because they can be replicated more easily; hence marking them to market is strait forward. After identifying a trading opportunity or realizing the need to hedge, a dispersion trade can be initiated. The main target of this strategy is to be less correlated to the underlying assets dynamics and different regimes that may occur in the market place and only stay exposed to the aimed correlation exposure.

In order to implement a variance dispersion strategy an implied correlation weighted scheme is often used by practitioners to define the needed weight of each variance swap to be included

$$\alpha_i = \mathrm{VWAC}\left(\frac{\omega_i\, K_{var}^i}{K_{var}^I}\right) \tag{4.5}$$

where VWAC is the volatility weighted average correlation as shown in 2.29, ω_i the weight of the constituent i in the basket, K_{var}^i the strike of a variance swap on the constituent i and K_{var}^I is the strike of a variance swap on the index itself. Using this weighting scheme aims to achieve a payoff that is more exposed to correlation than to volatility [7].

There are disadvantages to using variance swaps for dispersion trading:

- using variance swaps is more expensive than using options strategies
- correlation skew risk is inevitably included in variance swaps due to the index implied correlation risk being priced differently in the different options
- unwanted position in the correlation skew are included while using second generation variance swaps, because of being capped/floored

4.1.3. Through correlation swaps

Using correlation swaps is much relevant to get involved in dispersion trading. Using this swap does not require delta-hedging the position and gives a direct exposure to the future realized correlation. Anyhow, using these instruments is limited, mainly because they are considered to be more expensive than it is when using the other strategies, moreover they are very hard to replicate and hedge because of being relatively illiquid.

4.1.4. Through gamma swaps

As mentioned in section 3.3.4 gamma swaps are currently observed to be relatively illiquid, these instruments are considered to be much suitable for conducting dispersion trading because of the fact that an exposure through gamma swaps changes as the price of the underlying changes. While using variance swaps fixes the exposure to the different constituents weightings at inception, using gamma swaps allows for the weighting to change as prices of the constituents changes; for this reason weightings remains constant. This allows the dispersion trading position to be more accurate; weighting schemes such as the one mentioned in 4.5 is not needed anymore.

4.2. Skew Trading

Providers of structured products (such as worst-of-basket) as well as sellers of OTM calls and puts are short skew risk which is very difficult or impossile to hedge in tradional markets like derivatives. They can use alternative risk transfer (ART) in order to transfer risk premium, e.g. through skew trading buying back the skew risk from other parties like hedge funds and reinsurers. This way, they free up more risk capacity; therefore be able to develop and sell more structured products and conduct more risky transactions. Nevertheless, the most crucial part is to identify the portion of their books which is exposed to skew risk and understand its kind, after that they have to dynamically assess their exposure in order to be able to manage it accordingly –as the prices move, volatility changes; therefore exposure to vega and to the skew changes– whether through using direct volatility instruments or plain vanilla options strategies. At the current time banks are more concerned about other volatilities such as volatility of volatility and correlation volatility.

An interesting observation can be seen in figure 4.1 which shows the SMI three-month implied volatility and the VSMI three-month values and the difference between them, as well as the relationship between that difference and the implied correlation on the SMI. The skew can be observed through the difference between the index implied volatility and the volatility index which is model-free including theoretically an infinite amount of options on the index with all strikes, this enables the volatility index to capture the skew effect, which is not possible in the case of using the implied volatility on the index, which takes only ATM options on the index into consideration.

4.2.1. Skew characteristics

In time

As time to maturity gets smaller the skew gets higher, one explanation might be that when considering the prices of an asset to be independent random variables, the central limit theorem would suggest that in the long-run its distribution would tend to a normal distribution, which implies a flat skew; on the other side when considering short-run, the assumption of the central limit theorem would not be met, the prices may be dependent; hence the distribution might be different than the normal distribution.

Perception of the market

For a given strike, the term structure shows perception differences between short term and long term risk, scrutinizing both skew and volatility can provide a leading signal on the market direction. The skew curve remains unchanged when volatility perception is stable; when the downside risk perception changes, market gains/loses

Figure 4.1.: The relationship between the implied correlation and the difference between both the SMI 3-months implied volatility and the 3-months VSMI index

momentum; when the risk perception changes, market price switches to a new level; once the risk perception has changed, market is by then settled in a new regime.

Market regimes

If volatility goes to infinity, options would all have the same value, if there were a market crash the skew would flatten. The skew would steepen for the same reasoning, if there were rallies in the market.

Sometimes practitioners use the following rule of thumb to come to a value for the skew steepness, which is independent of options' maturity

- 10 delta put / ATM volatility
- 30 delta put / ATM volatilit
- 30 delta call / ATM volatility
- 10 delta call / ATM volatility

4.2.2. Through options strategies

If a symmetric distribution in the historical data, but a skewness in the implied data is observed, risk reversals options strategies can be used. If the implied skew is considered to be higher (lower) than the historical one, a long (short) risk reversal strategy can be followed, this can be achieved through long (short) OTM call, short (long) OTM put and hedging dynamically. Using this strategy, a long (short) skew position is taken (it is presumed that at expiration, the realized skew is going to be higher (lower) than the implied one) to speculate that the skew would be flatter (steeper).

Considering a long risk reversals position

$$\Pi = C(S_t, K_C) - \frac{\nu_C}{\nu_P} P(S_t, K_p) - \left(\Delta C - \frac{\nu_C}{\nu_P}\Delta_P\right) S_t \tag{4.6}$$

where ν is the vega and Δ the delta of the respective option [29].

Through using options strategies, the position must be kept delta and vega neutral which is exposed to high transaction costs, even after doing that, there would still be exposure to the underlying assets' prices, volatilities and correlation; if volatility of volatility increases, there would be a parallel shift in the skew upwards (downwards) which leads to the long (short) position to loose. Because of the drawbacks of using options, moment swaps are a better choice to apply a view on the future realized skew.

4.2.3. Through gamma and variance swaps

The advantage of skew trading through gamma swaps is that a correlation between the underlying price and the variance is made possible which enables taking positions in the skew.

One way to trade the skew using a combination of variance and gamma swaps can be achieved through taking a short position in a gamma swap and a long position in a variance swap. The nominal value of the variance swap stays constant while the gamma swap's nominal value varies with the level of the index. If the index level increases this would implicitly mean that the implied skew would decrease and the nominal value of the gamma exposure would increase; hence the gamma swap would loose proportionally more than the variance swap, this is similar to the effect achieved using risk-reversals plain vanilla options strategy and the payoff can be positive.

If the implied skew is considered to be higher (lower) than the historical one, a long (short) risk reversal strategy using variance swaps can be followed through going long (short) variance swaps on some of the constituents and short (long) variance swap on the others.

Conditional variance swaps opened the possibility to have a relatively direct investment in the skew. Structured products providers are long volatility when stock prices rise and short volatility when stock prices fall, they initially used up-variance swaps to offset their short skew positions. One way to benefit from the fact that the skew is not realized is to isolate the skew by going into a combined position of both up-variance swaps and down-variance swaps, the investors should benefit if the stock price stayed between the two barriers. The risk of such strategy is; if the price of the stock would exceed the two barriers in a market sell-off, the strategy would become unprofitable. Other possibilities are conditional variance swaps spread between; e.g. two indices, e.g. S&P500 index and the Nikkei 225 index.

Practitioners use a 95/105 strategy to have a position in the skew of an index, long up-variance swap at 95 percent strike and short a down-variance swap at 105 percent strike– which payoff is the difference between both swaps. When prices rise it turns to be a long variance swap position and when they drop it turns to be a short variance swap position. Banks may use this strategy to buy back the skew [46].

4.2.4. Through skew swaps

Skew swaps (see section 3.3.6) are illiquid, sellers of these swaps need to have a good hedge against jumps risk which can be quite expensive, some banks trade these swaps with a predefined upper limit (cap) to protect them from extreme jumps which renders them less attractive for investors to take positions in them.

4.2.5. Through relative smile swaps

Description

The possibility of having a position on the skew through a structured product that is recently offered by different investment banks is now to be examined, the product has the following characteristics:

- It is based on a basket of stocks and in some cases also indices which are ranked in order of performance based on three-month observation periods.
- The portfolio is split into 3 groups, the best, the average and the worst performing stocks.
- The relative smile is then observed by calculating the difference between upside and downside performance (the portfolio of best-performing stocks minus the portfolio of middle-performing stocks on the upside, and the portfolio of middle-performing stocks minus the portfolio of worst-performing stocks on the downside).
- To profit from this none-realisation of the implied smile in the short term, the structure comprises a series of swaps with three-month maturities (a two-year structure would incorporate 22 swaps of three-month maturities launched each month).
- The two-year structure may also have a global floor.
- There should be a robust payoff whatever the basket.
- The payoff is very skew sensitive and has the characteristic to be short skew and long volatility.
- The product performs well during the spikes of volatility.
- The bank selling this product is hedged through its existing long book of options of all maturities on the underlying stocks included; hence being short implied

volatility smile (short skew).

Primarily hedge fund investors would buy that product as a hedge for their portfolios because it is long volatility and is negatively correlated to the market. The payoff is calculated as

Payoff of swap$_i$

$$
\begin{aligned}
Swap_i = & Max(Floor; \frac{\text{PerfUpBasket}_i + \text{PerfDownBasket}_i}{2} \\
& - \text{PerfMedianBasket}_i - Strike)
\end{aligned} \tag{4.7}
$$

Payoff of the global product

$$
\text{Payoff} = \text{Max}\left[0, \sum_{i=1}^{n} \text{Swap}_i\right] \tag{4.8}
$$

Value

Bloomberg data of the VSMI index, the SMI Index as well as its constituent companies, from March 2004 up to March 2009 are used. I use a numerical procedure which employs Monte Carlo simulations to find a fair price for the strike price; hence win a payoff profile for the product which makes it possible to discuss the before-mentioned presumed characteristics.

1. Use the following algorithm to calculate the realized leg:
   a) use the realized prices to calculate three months returns
   b) sort these returns to get the three values needed to calculate the realized skew
   c) calculate the realized skew returns at $t = T$
   d) the bank selling this product is hedged through its existing long book of options of all maturities on the underlying stocks included; hence being short implied volatility smile (short skew)
2. use the following algorithm to come to prices that can be used to calculate the skew strike:
   a) Use a multidimensional Monte Carlo simulation to obtain estimated asset prices for the next period (three months, 60 trading days) for each stock

and each time step:

i. Calculate the average returns of each asset S_j for the last three trading months (each trading month is presumed to contain 20 trading days).

ii. Calculate a covariance matrix from the returns of the last three trading months. In this case, the components of the Brownian motions are independent

$$dX_t = \mu X_t dt + \sigma X_t dW_t \tag{4.9}$$

μ is a diagonal matrix representing returns of the different constituents, X_t are the prices of the constituents which are included in a matrix along the diagonal and σ a diagonal matrix representing standard deviations of the different constituents.

iii. Use Cholesky decomposition to invoke correlation between the Wiener processes, finding A (Using the covariance matrix between asset S_i and asset S_j) subject to

$$C = AA^t \tag{4.10}$$

where A is a lower triangular matrix.

iv. Draw $Z = nk$ random independent standard normal variables, where n the number of assets, k the number of time steps, in this case it is sixty days.

v. Set $W = AZ$ which creates correlated random numbers.

vi. W_j is then used to simulate the behavior of stock j

$$S_{j,T-t} = S_{j,0}e^{\left(r-\frac{1}{2}\sigma_j^2\right)(T-t)+\sigma_j\sqrt{T-t}W_j} \tag{4.11}$$

where $S_{j,T-t}$ is the price of stock j after one time step $T-t$, $S_{j,0}$ is the price of stock j at time $t = 0$, r is the risk free rate for the period $(T-t)$, σ_i is the volatility of the stock S_i for the previous period and $\sqrt{T-t}W_j$ is the correlated Brownian motion as described earlier.

vii. repeat for each time step $(T-t)$ and each stock S_j.

Figure 4.2 shows one of the simulated paths for correlated geometric brownian motions for the twenty constituent stocks of the SMI index.

b) Use these forecasted prices to calculate expected three months returns.

c) Sort these returns to get the three values needed to calculate the implied skew, analogous to these needed to calculate the realized skew.

d) Calculate the skew strike returns.

e) Discount skew strike returns using the relevant interest rate, I use 0.40167 percent three month LIBOR.

f) The result is a fair skew which if subtracted from the realized skew at $t = 0$ would result in a zero payoff that defines the fair value of the product.

3. Use the following algorithm to calculate the payoff of each Swap_i and the global payoff of the product:

   a) calculate Swap_i's payoff using 4.7

   b) calculate the global payoff using 4.8

In appendix D a Matlab code is provided which implements these algorithms. The resulted payoff profile can be seen in figure 4.3, the next section analyzes these results.

Analysis

The party that is short skew position would do better if there would be an instrument (not a proxy) that would enable it to hedge its positions in a more direct, accurate and effective way.

This structured product aims to exploit the dispersion that may occur during uncertainty periods, if the market is in normal conditions –no market crash or global crisis– it is supposed to perform less if in a bear market, because assets correlate

Figure 4.2.: Sample correlated geometric brownian motions

and their volatilities tend to move together, which leads to the dispersion not taking place. The relative smile implies that high-performing stocks are less volatile than low-performing ones. It is hard to extract trading opportunity from the relative implied volatility smile, no instrument is clearly capable of achieving this purpose.

This structured product is based upon the idea, that the implied volatility smile would not take place in the short-run for the following reasons:

1. On the upside of the smile, it is most probable that the implied volatility does not occur in the short run, realized volatility would be higher because of take-over, speculation, or unexpected good news related to some of the stocks, investors are quick to buy.
2. On the downside of the smile, it is most probable that the implied volatility does not occur in the short run, realized volatility would be lower, the reason is that the market tends to be more frequently bullish than bearish. If a bull market happened to have high correlation, the dispersion would not take place and volatility is low, investors are slow to sell.

The structure aims to profit from the dispersion between implied –where the level of the skew is thought to be hardly realized– and realized volatility smile among predefined basket of stocks on short-term basis (3 months); hence have a bit on the skew. The product is structured in cliquets form (contracts that have local/global caps and floors), which is a natural way to take the position being sensitive to the skew. Cliquets can be seen as forward starting options that are considered to depend on the skew rolled over; hence a series of risk reversals strategies as mentioned in section 4.2.2 is used.

From the description of that structured product, the investor which have the view that the skew is not occurring in the short term would go long the upside by being long the best performing and short the downside by being short the worst performing stocks. A floor is added so that the maximum loss for the investor is limited. Investors are systematically going long the upside performance of the best-performing stocks and short the downside performance of the worst performers. They will profit as long as the best-performing stocks' volatility is greater than implied by the smile and volatility on the down-side is less than what is implied.

From figure 4.3 which shows the relationship between the Swap$_i$ payoff, the VSMI and the implied correlation, we can observe the following:

1. In periods when implied correlation and implied volatility are quite high, the payoff of $Swap_i$ is reaching its floor, which is negative.

2. In periods of high dispersion between relatively high implied correlation and high implied volatility, the payoff of $Swap_i$ is negative.
3. In periods when implied correlation is low and the level of the implied volatility is also low and both are close to each other, conditional on recovering from very high regimes, the $Swap_i$ achieves the highest positive payoffs.
4. Sometimes when implied correlation moves from a higher regime to a lower one, while implied volatility stayed at a low level, and the dispersion between both is considered to be high, the $Swap_i$ achieves positive payoffs.
5. In periods when implied correlation is considered to be relatively low but the implied volatility is staying at a high level, the payoff of $Swap_i$ is negative.
6. In periods when correlation was considered to be relatively low e.g. the end of the fourth quarter 2008, the financial crisis was in the middle, some dispersion occurred in market expectations on development of the different constituents; hence this could explain the high payoff at that period, e.g. the UBS, Credit Suisse as well as the insurance companies and almost all financial institutions.

Considering the description as well as the hypothetical behavior of the relative skew structured product, compared to the above observations:

1. A similarity to the behavior of risk reversals options strategies as well as variance swaps strategies is observed.
2. The seller of such structured products must own a huge options book in order to be able to hedge its exposures in the basket's constituents.
3. The payoff of the product is negatively correlated with correlation level.
4. The product works well in bull markets and short term uncertainties and worse in fully bearish markets.
5. In the context of portfolio management, this product can contribute to a better diversification because of its sensitivity to dispersion.
6. This product would work better with a less liquid basket of stocks, as correlation would not be much effective.
7. This product may work better in emerging markets where dispersion and idiosyncratic risk may occur much often.
8. If a stock on the upside of the skew would perform well without another one on the downside that performs less, the product would be profitable.
9. If a stock on the downside of the skew would perform badly without another

one on the upside that performs better, the product's payoff is negative.

10. In order to construct such products, a specific number of stocks must be found, which achieves an optimal dispersion.
11. Marking the product to market must be examined carefully, the product is still exposed to different sensitivities such as counter-party, interest rate, underlying, volatility and correlation.
12. Liquidity must be guaranteed in order to be able to close out the position if necessary.

Figure 4.3.: The relationship between the Swap_i, the SMI 3-month implied correlation and the 3-month VSMI index

5. Volatility assets in the portfolio optimization

The classical Markowitz portfolio selection method presumes that volatility and correlation are constant and not stochastic, which empirically proved to be wrong. A portfolio of stocks is also exposed to volatility, diffusion and jump risks.

Due to the fact that volatility assets correlate negatively to the index level, portfolio and fund managers tended to use it to achieve more efficient diversification. $\frac{90}{10}$ is a rule of thumb that is used sometimes by portfolio and fund managers, it suggests to invest ninety per cent of capital in classical assets and ten per cent in volatility, but this strategy was proven to be expensive.

The existence of the negative variance premium as can be seen in figure 3.2 and 3.4 –which magnitude is influenced by the level of the implied volatility, the time to maturity and buying pressure for protective puts on the index– made it attractive for investors like hedge funds to work on capturing this premium through selling implied volatility having to decide for a trade-off between achieving higher Sharpe ratios or being exposed to skewness and kurtosis risks. Hedge funds used variance swaps because of the advantages mentioned in section 3.3.4. In the context of portfolio management and asset allocation, the other way around is the most suitable way to enhance the efficient frontier, this is due to the attractive characteristics of variance swaps being negatively correlated to the price level of the index; hence achieving a better diversification which maximizes the portfolio Sharpe ratio.

Going long variance swaps should not be regarded as a normal asset class which can be integrated in a traditional buy and hold strategy as shown in figure 5, this would lead to continuous and massive losses. Investing in long variance swaps or using them as a part of trading strategy should be accompanied by an efficient and complete risk management process. Variance swaps are most adequate for financial institutions that already has a derivatives book, that can be used to hedge a position in variance swaps, accompanied by a proven functioning and accurate pricing model.

Variance swaps are not an asset which can be included in the context of a portfolio

Figure 5.1.: Cumulative payoff of a rolling forward strategy, holding S&P500 one month variance swap, compared to another, holding a rolling forward strategy holding long the S&P500 index

optimization process because of the fact that it is actually a forward contract which does not require cash flows at initiation, to avoid that obstacle, as suggested by Hafner et al. [2007], variance swaps can be considered as if they were a risk-free asset; hence a weight can be initiated and the mean-variance portfolio optimization process can be executed. One way to include variance swaps is to define its investments as

$$f(e^{-rT}K_{VarS}) \tag{5.1}$$

which is the discounted strike price K_{VarS} of a variance swap that have to be paid upfront in order to receive one monetary unit, f guarantees that the volatility of the variance swap returns stays the same as the volatility of the index to make sure that no extreme fluctuations occur and that variance swaps weights and assets' weights stay comparable [36].

Because of the variance swap payoffs being negatively correlated to the index level, a short position in the index can be included in a portfolio to profit from that fact. Another way to look at the returns from a rolling short position in variance swaps is to consider it as an "equity-insurance" premium, this premium is not fully correlated to neither a bond nor an equity position; therefore if included in a portfolio it can provide more diversification and push the efficient frontier out and even as mentioned before, the bond position can be replaced.

Two important issues must be addressed when holding short variance swaps positions:

The payoff distribution of such strategy is negatively skewed, which is the probability of the occurrence of an unexpected high realized volatility, which may lead to severely high losses. This risk should be managed through applying conditional value-at-risk (CVar) which quantifies the highest probable loss and its severity given a specific level of confidence in a predefined period of time, not only the traditional Value-at-risk (Var) which does not take into consideration the severity of the incurring loss. The second issue is liquidity, because of possible unlimited losses from the short position in variance swaps, liquidity must be available to meet the margin calls that may occur.

Reinhold et al. [2007] argue that reaching a good trade off between a high Sharpe ratio and skewness is seldom possible, aiming to profit from the negative variance premium can lead to a negatively skewed payoff structure. Investors tend to have extreme positions which are either Sharpe ratio oriented, skewness oriented or kurtosis oriented [36].

6. Conclusions

Plain vanilla options are used to express an opinion about the future realized volatility, trying to profit from the implied volatility being over/under estimated. Recently some other innovations were developed to match the needs of market participants to take direct positions in volatility. These innovations contain the idea of moment swaps which are based on the Log Contract as a building block.

The following aspects were studied and discussed: Implied volatility and its modeling; trading volatility as well as dispersion and skewness using options strategies, moment swaps and structured products. I conclude that main advantages of using moment swaps is avoiding the administrative overload in order to keep the portfolio delta-neutral and transaction costs resulting when using the traditional plain vanilla options strategies.

My findings support the existence of a negative variance premium on the S&P500 index, which systematically has been used by some investors like hedge funds through going short positions in volatility products, e.g. variance swaps. The data show that the implied volatility is considered to be a good prediction of future realized volatility which support other studies like Christensen [1998] [14], Huang [2003] [42] and Fleming [1998] [21] and contradicts findings of some other studies, e.g. Canina [1993] [9] that implied volatility is a poor prediction of the future realized volatility.
.

The implied correlation is calculated for the SMI stock index which is used when analyzing the performance of the relative skew swaps product.

A fair value can be found for the chosen structured product which aim is to have a position in the skew of a specific basket of assets. This can be achieved through using a numerical method such as monte carlo simulation to simulate a multidimensional GBM that takes into consideration path dependency and can impose correlation between the wiener processes of the different constituents prices making the generated prices more realistic. The relative skew trading structured product can capture the skew effect in a good way, it has similar characteristics to using plain vanilla options strategies such as risk reversals.

The product can achieve a positive payoff during periods when:

- both low implied correlation and implied volatility which are close to each other recovering from a very high regime period, e.g. the end of the fourth quarter 2008, where dispersion occurred in the market expectations about the development of the different constituents.
- implied correlation moves from a higher to a lower regime while implied volatility stayed at a low level and the dispersion between both is considered to be high

The product can achieve a negative payoff during periods when:

- implied correlation and implied volatility are quite high
- high dispersion between relatively high implied correlation and high implied volatility

The analyzed relative skew structured product has the following characteristics:

1. It behaves similar to risk reversals options strategies.
2. The seller must own a huge options book to be able to hedge it.
3. Its payoff is negatively correlated with correlation level.
4. Performs well in bull markets and short term uncertainties and worse in fully bearish markets.
5. It would work better with a less liquid basket of stocks.
6. It would work better in emerging markets.
7. The product is exposed to different sensitivities such as counter-party, interest rate, underlying, volatility and correlation.
8. Liquidity must be guaranteed in order to be able to close out the position if necessary.

Using moment swaps in the context of portfolio optimization can be useful; nevertheless this must be done with caution because the traditional VaR which is often used for risk management ignores the severity of losses; therefore other risk management concepts that take loss severity into consideration must be applied, e.g. CVaR or the shortfall principle.

A. VIX Index calculation

$\sigma^2 = \frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2} e^{RT} Q(K_i) - \frac{1}{T}\left[\frac{F}{K_0} - 1\right]^2$

Where:

σ	$= \frac{\text{VIX}}{100}$
F	Forward index level derived from index options prices
K_0	First strike below the forward index level, F
K_i	Strike price of i^{th} out-of-the-money option; a call if $K_i > K_0$ and a put if $K_i < K_0$; both put and call if $K_i = K_0$.
ΔK_i	Interval between strike prices - half the difference between the strike on either side of K_i: $\Delta K_i = \frac{K_{i+1} - K_{i-1}}{2}$ (Note: ΔK for the lowest strike is simply the difference between the lowest strike and the next higher strike. Likewise, ΔK for the highest strike is the difference between the highest strike and the next lower strike.)
R	Risk free interest rate to expiration
$Q(K_i)$	The midpoint of the bid-ask spread for each option with strike K_i.

B. Matlab code to calculate variance swaps payoff

```
% Matlab code to calculate variance swaps payoff
% Author: Alaa El-Din Hammam
% Date: 15 May 2009
% University of St. Gallen, MBF
% Master's thesis: Smile Arbitrage: Analysis and valuing
% Notice: Matlab financial tool box must be used for some functions to work
% ———————————————————————————
% Data included (from january 1999 up to March 2009)
% Date, VIX, SPX
% Load data
load SPX_VIX.mat
MDate = x2mdate(data(:,1))
%% —–calculate S&P500 daily return
returns= price2ret(data(:,3));
%% —–calculate squared returns
returns2=returns.^2
%% —–calculate anualized monthly realized variance
count=size(data(2:end-20,2)); %to define needed number of iterations
for i=1:count
  AnuMonthRVar(i,1)=10000*252*mean(returns2(i:i+20,1));
end
%retcount=size(returns2(1:end-21,1))
%% —–calculate anualized monthly realized volatility
AnuMonthRVola=sqrt(AnuMonthRVar)
%% —–calculate payoff of 1 month variance swap
VIX2=data(2:end-20,2).^2
VarSwapPayoff=AnuMonthRVar-VIX2(:,1)
```

C. Matlab code to calculate the implied correlation

```
% Describtion: Matlab code to calculate the implied correlation
% Author: Alaa El-Din Hammam
% Date: 15 May 2009
% University of St. Gallen, MBF
% Master's thesis: Smile Arbitrage: Analysis and valuing
% Notice: Matlab financial tool box must be used for some functions to work
% ————————————————————————————
%% data included (from December 2007 up to March 2009)
% Date, SMI, VSMI, SMI3MTHIV,
% ABBNVX, ABBN3MTHIV, ATLNVX, ATLN3MTHIV,
% ADENVX, ADEN3MTHIV, BALNVX, BALN3MTHIV,
% CFRVX, CFR3MTHIV, CSGNVX,
% CSGN3MTHIV, HOLNVX, HOLN3MTHIV, NESNVX,
% NESN3MTHIV, BAERVX, BAER3MTHIV,
% NOBNVX, NOBN3MTHIV, NOVNVX, NOVN3MTHIV,
% ROGVX, ROG3MTHIV, UHRVX, UHR3MTHIV,
% SLHNVX, SLHN3MTHIV, RUKNVX, RUKN3MTHIV,
% SCMNVX, SCMN3MTHIV, SYNNVX, SYNN3MTHIV,
% SYSTVX, SYST3MTHIV, UBSNVX, UBSN3MTHIV,
% ZURNVX, ZURN3MTHIV, VSMI3M
%% load data
load data.mat
%% Build a matrix with all prices, format date for matlab and clculate returns
 prices = [data(:,5) data(:,7) data(:,9) data(:,11)...
 data(:,13) data(:,15) data(:,17) data(:,19) data(:,21) ...
 data(:,23) data(:,25) data(:,27) data(:,29) data(:,31)...
 data(:,33) data(:,35) data(:,37) data(:,39) data(:,41) ...
 data(:,43)]
 Date=x2mdate(data(:,1))
 returns=price2ret(prices);
 %% Build a matrix with all implied volatilities
 IVolasData = [data(:,3) data(:,6) data(:,8) data(:,10)...
 data(:,12) data(:,14) data(:,16) data(:,18) data(:,20) ...
 data(:,22) data(:,24) data(:,26) data(:,28) data(:,30)...
 data(:,32) data(:,34) data(:,36) data(:,38) data(:,40) ...
 data(:,42)]
```

```
IVolas = IVolasData./100
%% The SMI is a capitalization weighted index, weights are caclulated
SharesInIndex=[2322.792835 111.648884 121.374686 50 522 1184.635653 ...
209.840886 3637.734 211.034256 124.316530 2643.623 702.5627 30.84 29.611364...
335.779763 23.946912 96.914857 60.451161 2932.580549 142.124819]
%bsxfun() makes it possible for row-wise multiplication
 IndexSum=sum(bsxfun(@times,prices,SharesInIndex), 2)
 Weights = bsxfun(@times, bsxfun(@times,prices,SharesInIndex),(1./IndexSum))
 %% Calcualte the implied correlation matrix
 % IC =[sigma_i^{2}-sum_{i=1}^{N}(w_{i}^{2} * sigma_{i}^{2})]/
 % [sum(w_{i}*sigma_{i})^2 - sum_{i}^{N} w_{i}^2*sigma_{i}^{2}})]
 [m,n]=size(data)
 for i=1:m
  IC(i,1)= [(data(i,45)/100)^2 - sum(Weights(i,:).^2 .*IVolas(i,:).^2)].* ...
  [1/(sum((Weights(i,:) .* IVolas(i,:))))^2]
 end
```

D. Matlab code to calculate the relative skew trade

```
% Describtion: Matlab code to calculate the relative skew trade
% Author: Alaa El-Din Hammam
% Date: 15 May 2009
% University of St. Gallen, MBF
% Master's thesis: Smile Arbitrage: Analysis and valuing
% Notice: Matlab financial tool box must be used for some functions to work
% ————————————————————————————————
%% Data included
% Date, SMI, VSMI, SMI3MTHIV, ABBNVX, ABBN3MTHIV,
% ATLNVX, ATLN3MTHIV, ADENVX, ADEN3MTHIV, BALNVX,
% BALN3MTHIV, CFRVX, CFR3MTHIV, CSGNVX, CSGN3MTHIV,
% HOLNVX, HOLN3MTHIV, NESNVX, NESN3MTHIV, BAERVX,
% BAER3MTHIV, NOBNVX, NOBN3MTHIV, NOVNVX, NOVN3MTHIV,
% ROGVX, ROG3MTHIV, UHRVX, UHR3MTHIV, SLHNVX, SLHN3MTHIV,
% RUKNVX, RUKN3MTHIV, SCMNVX, SCMN3MTHIV, SYNNVX,
% SYNN3MTHIV, SYSTVX, SYST3MTHIV, UBSNVX, UBSN3MTHIV,
% ZURNVX, ZURN3MTHIV, VSMI3M
%% load data
load data.mat
%% Build a matrix with all prices and format date properly
prices = [data(:,5) data(:,7) data(:,9) data(:,11)...
data(:,13) data(:,15) data(:,17) data(:,19) data(:,21) ...
data(:,23) data(:,25) data(:,27) data(:,29) data(:,31)...
data(:,33) data(:,35) data(:,37) data(:,39) data(:,41) ...
data(:,43)]
% Format date for matlab and clculate returns
 Date=x2mdate(data(:,1))
 returns=price2ret(prices);
 %% calculate 3 months returns as required by the product
 [m,n]=size(prices)
 ThreeMReturns=zeros(m-60,n);
 for i=1:m-60
  for j=1:n
   ThreeMReturns(i,j)=(prices(i+60,j)/prices(i,j))-1;
  end
 end
```

```
%% Realized performance of 3 Months: Create highest, lowest and middle perform-
ing baskets    % value the realized leg of the swap_i
[m,n]=size(ThreeMReturns)
realized=zeros(m,n);
for i= 1:m
  sorted(i,:)=sort(ThreeMReturns(i,:));
  for j=1:7
    H(i,j)=sorted(i,21-j);
    L(i,j)=sorted(i,j);
  end
  for j=1:6
    M(i,j)=sorted(i,7+j);
  end
  realized(i)=(sum(H(i,:))+sum(L(i,:)))*0.5-sum(M(i,:));
end
%% Inducing dependence and correlation using cholesky decomposition
[m,n]=size(returns);
for i=1:m-60
  expReturn = diag(mean(returns(i:i+59,:)));
  covariance = cov(returns(i:i+59,:));
  sigma = cholcov(covariance)';
  GBM = gbm(expReturn, sigma);
  randn('state', 0)
  [X,T] = GBM.simByEuler(60);
  SimPrices(i,:)=X(61,:).*prices(i+59,:);
end
%% The strike leg
%calculate simulated three months returns
[m,n]=size(SimPrices)
SimThreeMReturns=zeros(m-60,n);
for i=1:m-60
  for j=1:n
    SimThreeMReturns(i,j)=(SimPrices(i+60,j)/SimPrices(i,j))-1;
  end
end
%% Simulated performance of 3 Months: Create highest, lowest and middle per-
forming baskets
% value the fixed leg of the swap_i
[m,n]=size(SimThreeMReturns)
simulated=zeros(m,n);
for i= 1:m
  sorted(i,:)=sort(SimThreeMReturns(i,:));
  for j=1:7
    H(i,j)=sorted(i,21-j);
    L(i,j)=sorted(i,j);
  end
```

```
  for j=1:6
    M(i,j)=sorted(i,7+j);
  end
  simulated(i)=(sum(H(i,:))+sum(L(i,:)))*0.5-sum(M(i,:));
end
% Discount the payoff of the strike side
Discounted=simulated.*exp(-0.25*0.0040167) % using 0.40167% three month LI-
BOR rate
%% Calculate swap_i payoff: max[{realized - strike}, 0.01]
% where realized =(0.5*(Highest+lowest)-middle)
[m,n]=size(Discounted)
for i=1:m
  if (realized(i+60, 1)-Discounted(i,1))>0;
    Swapi(i,1)=realized(i+60, 1)-Discounted(i,1);
  else
    Swapi(i,1)=-0.01;
    end;
  end
  %% Calcualte the global payoff of the product
  [m,n]=size(Swapi);
  for i=1:8
    for j=1:20:m-9 % Number of days that form full trading months of 20 days
      GlPayoff(j,:)=Swapi(i+j-1,1);
    end
    GlobalPayoff(i,:)=sum(GlPayoff);
  end
```

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Declaration of authorship

I hereby declare

- that I have written this thesis without any help from others and without the use of documents and aids other than those stated above,

- that I have mentioned all used sources and that I have cited them correctly according to established academic citation rules.

Dietikon, 15^{th} May 2009

Alaa El Din Hammam