FITTING BIVARIATE CUMULATIVE RETURNS WITH COPULAS

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Abstract.

A copula based statistical method for fitting joint cumulative returns between a market index and a single stock to daily data is proposed. Modifying the method of inference functions for margins (IFM method), one performs two separate maximum likelihood estimations of the univariate marginal distributions, assumed to be normal inverse gamma mixtures with kurtosis parameter equal to 6, followed by a minimization of the bivariate chi-square statistic associated to an adequate bivariate version of the usual Pearson goodness-of-fit test. The copula fitting results for daily cumulative returns between the Swiss Market Index and a stock in the index family for an approximate one-year period are quite satisfactory. The best overall fits are obtained for the new linear Spearman copula, as well as for the Frank and Gumbel-Hougaard copulas. Finally, a significant application to covariance estimation for the linear Spearman copula is discussed.

Keywords : copula, normal inverse gamma mixture, IFM method, bivariate chi-square statistic, daily cumulative return, covariance estimation

1. Introduction.

The present paper is a sequel and synthesis of work done in [1]. We link empirical results on fitting univariate daily cumulative returns with the representation of bivariate distributions by copulas to study the statistical fitting of joint cumulative returns between a market index and a single stock to daily data.

In Section 2, the following empirical fact on fitting univariate distributions to daily cumulative returns is recalled. The normal inverse gamma mixture distribution with kurtosis parameter equal to 6, obtained by mixing the variance of a normal distribution with an inverse gamma prior, usually beats the lognormal and the logLaplace under a chi-square goodness-of-fit test with regrouped data. Therefore, our marginal distributions in bivariate fitting of cumulative returns are restricted to this analytically tractable two-parameter family of symmetric location-scale distributions.

Section 3 contains a short review of the representation of bivariate distributions by copulas. For our purposes, a number of attractive one-parameter families of copulas are retained. They include the copulas [2], [3], [4], [5], [6], [7] and [8]. The parameter of these copulas measures the degree of dependence between the margins. In case the widest possible range of dependence should be covered, one is especially interested in one-parameter families of copulas, which are able to model continuously a whole range of dependence between the lower Fréchet bound copula, the independent copula, and the upper Fréchet bound copula.

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Such families are called inclusive or comprehensive, and include the copulas by Frank and Clayton. Another simple copula with this property is the linear Spearman copula described in details in Section 4.

The linear Spearman copula represents a mixture of perfect dependence and independence. If X and Y are uniform(0,1), $Y = X$ with probability $\theta \geq 0$ and Y is independent of X with probability $1-\theta$, then (X,Y) has the linear Spearman copula with the positive dependence structure. In the statistical literature it has been considered in [9] and motivated in [10]. The chosen nomenclature for this copula suggests that it has piecewise *linear* sections and that the parameter of dependence θ coincides with *Spearman*'s grade correlation coefficient. Besides the useful fact that this copula is suitable for analytical calculation, it has many important properties. It satisfies two extremal properties, one of which is related to a conjecture by Hutchinson and Lai in [11]. Of great significance for financial modelling is its simple tail dependence structure. The coefficient of (upper) tail dependence coincides with the dependence parameter θ, which implies asymptotic positive dependence in case $\theta > 0$. This is a desirable property in insurance and financial modelling because data often tend to be dependent in their extreme values. In contrast to this, the ubiquitous Gaussian copula yields always asymptotic independence, unless perfect correlation holds.

The main issue of copula fitting is discussed in Section 5. We apply a method close in spirit to the method of inference function for margins or IFM method studied in [12], [13] and [14], Section 10.1. This estimation method proceeds by doing two separate maximum likelihood estimations of the univariate marginal distributions, followed by an optimization of the bivariate likelihood as a function of the dependence parameter. In our proposal, we do not maximize the bivariate likelihood. Instead, we determine the dependence parameter, which maximizes the p-value (respectively minimizes the bivariate chi-square statistic) of a bivariate version of the usual Pearson goodness-of-fit test. The reason for considering a modified method lies in the observation that the IFM method reduces the p-value in some cases rather drastically, leading eventually to a rejection of the model. Our copula fitting results for daily cumulative returns between the Swiss Market Index and a stock in the index family for an approximate one-year period are quite satisfactory. In particular, the analytically tractable linear Spearman copula does very well. The Frank and Gumbel-Hougaard copulas provide competitive best overall fits.

Finally, Section 6 illustrates our estimation results at a significant application. First, based on a general covariance formula for the linear Spearman copula, derived in Theorem 6.1, we show that for the linear Spearman copula model with the chosen normal inverse gamma mixture margins, the Spearman grade correlation coefficient coincides with the Pearson linear correlation coefficient. This allows one to compare the standard product-moment correlation estimator with the estimated dependence parameter from the linear Spearman copula fitting. We observe a considerable discrepancy between the absolute values of both estimators, but on a relative scale both estimators rank the strength of dependence quite similarly.

2. Fitting univariate cumulative returns.

There exist many distributions, which are able to fit the daily cumulative returns on individual stocks and market indices. As the application of the penalized likelihood scoring method in [15] suggests (so-called Schwartz Bayesian Criterion), it is reasonable to restrict the attention to two-parameter distributions, as shown in [1]. The only analytically tractable distributions we retain for comparative purposes are the lognormal, the logLaplace, and a normal inverse gamma mixture with fixed kurtosis equal to 6.

Let X represent the daily cumulative return of a market index or a stock in the index family. If $\ln X \sim N(\mu,\sigma)$ follows a normal distribution, then X has the lognormal distribution $F_X(x) = \Phi\big((\ln x - \mu)/\sigma\big)$, with $\Phi(x)$ the standard normal distribution. If the random variable $\ln X \sim L(\mu,\sigma)$ follows a Laplace distribution, one obtains the logLaplace

$$F_X(x) = \begin{cases} \frac{1}{2}(e^{-\mu}x)^{\frac{\sqrt{2}}{\sigma}}, & x \le e^{\mu}, \\ 1 - \frac{1}{2}(e^{-\mu}x)^{-\frac{\sqrt{2}}{\sigma}}, & x \ge e^{\mu}. \end{cases} \tag{2.1}$$

The use of a logLaplace distribution in Finance has been somewhat motivated in [16].

The normal inverse gamma distribution (NIΓ) is a special element of the class of generalized hyperbolic distributions (GH), introduced by [17] in connection with the "sand project" (investigation of the physics of wind-blown sand). Following [18], Section 4, the GH distributions are the variance-mean mixtures of normal distributions constructed as follows. Consider the density of a generalized inverse Gaussian distribution (GIG) with parameters (λ,δ,γ) given by

$$f_{GIG}(x;\lambda,\delta,\gamma) = \left(\tfrac{\gamma}{\delta}\right)^{\lambda} \frac{1}{2K_{\lambda}(\delta\gamma)} x^{\lambda-1} \exp\!\left(-\tfrac{1}{2}\left(\tfrac{\delta^2}{x} + \gamma^2 x\right)\right), \quad x > 0, \tag{2.2}$$

where K_{λ} denotes the modified Bessel function of the third kind with index λ. The parameter space is described by the conditions

$$\begin{aligned} &\delta \ge 0, \quad \gamma > 0, \quad \textit{if} \quad \lambda > 0, \\ &\delta > 0, \quad \gamma > 0, \quad \textit{if} \quad \lambda = 0, \\ &\delta > 0, \quad \gamma \ge 0, \quad \textit{if} \quad \lambda < 0. \end{aligned} \tag{2.3}$$

Let $\varphi(x) = \Phi'(x)$ be the density of the standard normal distribution. Then the density of the generalized hyperbolic distribution with parameters $(\lambda,\alpha,\beta,\delta,\mu)$ is given by

$$f_{GH}(x;\lambda,\alpha,\beta,\delta,\mu) = \int_0^{\infty} \tfrac{1}{\sqrt{y}}\, \varphi\!\left(\tfrac{x-\mu-\beta y}{\sqrt{y}}\right) f_{GIG}(y;\lambda,\delta,\sqrt{\alpha^2 - \beta^2})\,dy. \tag{2.4}$$

Since GH distributions are known to be infinitely divisible by [19], every member of this family generates a Lévy process with stationary independent increments for use in a dynamic context. In financial modelling, the GH distributions are further considered in [20], [21], [22] and [23]. Several special cases have been studied in the literature.

For $\lambda = 1$ one gets the subclass of hyperbolic distributions (HD) introduced in finance by [24] (see also [25], [26], [27], [28] and [29]). For $\lambda = -\frac{1}{2}$ one gets the subclass of normal inverse Gaussian distributions (NIG) introduced in finance by [30].

For $\lambda < 0, \delta > 0, \gamma = 0,$ the GIG distribution (2.2) represents the inverse gamma distribution (IΓ) with density

$$f_{I\Gamma}(x;\lambda,\delta) = \left(\frac{2}{\delta^2}\right)^{\lambda} \frac{x^{\lambda-1}}{\Gamma(-\lambda)} \exp\!\left(-\frac{1}{2}\frac{\delta^2}{x}\right), \quad x > 0. \tag{2.5}$$

Setting $\alpha = \beta = 0, \lambda = -\alpha > 0, \delta = c > 0$ in (2.4), one gets the normal inverse gamma density. In the following, a random variable X with a NIΓ distribution is denoted by $X \sim NI\Gamma(\mu, c, \alpha)$. Its density is given by

$$f_X(x) = \frac{1}{B(\frac{1}{2},\alpha)} \cdot \frac{1}{c}\left[\frac{c^2}{c^2 + (x-\mu)^2}\right]^{\alpha+\frac{1}{2}}, \tag{2.6}$$

where $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is a beta coefficient. Recall that the location-scale transform $Z = (X-\mu)/c$ has a Pearson type VII density

$$f_Z(z) = \frac{1}{B(\frac{1}{2},\alpha)\cdot(1+z^2)^{\alpha+\frac{1}{2}}}. \tag{2.7}$$

If $\alpha = \upsilon/2$, $\upsilon = 1,2,3,...,$ is an integer, the random variable $\sqrt{\upsilon}\cdot Z$ has a Student t-distribution with υ degrees of freedom. The substitution $t = z^2/(1+z^2)$ shows the identity

$$\int_0^x \frac{dz}{(1+z^2)^{\alpha+\frac{1}{2}}} = \frac{1}{2}\int_0^{\frac{x^2}{1+x^2}} t^{-\frac{1}{2}}(1-t)^{\alpha-1}dt, \tag{2.8}$$

from which one derives the distribution function

$$F_Z(z) = \begin{cases} \frac{1}{2}\left[1 - \beta(\frac{1}{2},\alpha;\frac{z^2}{1+z^2})\right], & z \le 0, \\ \frac{1}{2}\left[1 + \beta(\frac{1}{2},\alpha;\frac{z^2}{1+z^2})\right], & z \ge 0, \end{cases} \tag{2.9}$$

where $\beta(a,b;x) = \int_0^x t^{a-1}(1-t)^{b-1}dt / B(a,b)$ is a beta density. The variance (if $\alpha > 1$) and kurtosis (if $\alpha > 2$) are equal to

$$\sigma_Z^2 = Var[Z] = \frac{1}{2(\alpha-1)}, \quad \gamma_{2,Z} = \frac{E[Z^4]}{Var[Z]^2} = 3\cdot\left(\frac{\alpha-1}{\alpha-2}\right). \tag{2.10}$$

The kurtosis, which takes values in $[3,\infty)$, is able to model leptokurtic data, a typical feature of observed financial market returns. This model, first suggested by [31], has been fitted to stock returns by [32] and [33] (see [34], Section 2.8). These authors find maximum likelihood estimates for α between $3/2$ and 3. With the suggested restriction to two-parameter distributions our proposal is $\alpha = 3$, which yields a kurtosis parameter $\gamma_{2,Z} = 6$. This choice is motivated by the fact that it yields the less dangerous model among those models for which $\alpha \in [3/2, 3]$. For comparison, the kurtosis parameter of the logLaplace is also equal to 6, an additional argument for this choice.

As a real-life application we have fitted the daily cumulative returns of the SMI (Swiss Market Index) as well as some typical stocks using the maximum likelihood method. In our calculations we use the so-called scoring method, where the observed data is grouped into a finite number of classes ([35], chap. 3.7 and 4.3, [36]). The model $NI\Gamma(\hat{\mu},\hat{c},3)$ beats the lognormal and the logLaplace under a chi-square goodness-of-fit test with regrouped data as in [1], Table 6.5. In view of this comparison result, our marginal distributions in bivariate fitting of cumulative returns are throughout $NI\Gamma(\mu,c,3)$ distributions. Though the obtained fit is not best possible in the wide class of GH distributions, in particular it is limited to symmetric distributions, our simple two-parameter model is parsimonious and analytically more tractable than the more sophisticated four-parameter NIG distribution and other models derived from the five-parameter GH distributions.

3. Bivariate cumulative returns from copulas.

Though copulas have been introduced since [37] their use in insurance and finance is more recent (e.g. [38], [39], [40], [41], [42], [43], [44]). A lot of background on copulas can be found in [45], [11], [46], [47], [48], [14] and [49].

In the present paper, we are only interested in the joint cumulative returns between a market index and a single stock, that is in the most important bivariate case. [14] and [49] provide extensive lists of one-parameter families of copulas $C_\theta(u,v)$, where θ is a measure of dependence, from which we retain the following candidates.

a) Cuadras-Augé [2]

$$C_\theta(u,v) = \left[\min(u,v)\right]^\theta \left[uv\right]^{1-\theta}, \quad 0 \le \theta \le 1. \tag{3.1}$$

b) Gumbel-Hougaard ([3], [4], [11])

$$C_\theta(u,v) = \exp\left\{ -\left[(-\ln u)^\theta + (-\ln v)^\theta\right]^{\frac{1}{\theta}} \right\}, \quad \theta \ge 1. \tag{3.2}$$

c) Galambos [5] ([14], p. 142)

$$C_\theta(u,v) = \exp\left\{ \left[(-\ln u)^{-\theta} + (-\ln v)^{-\theta}\right]^{-\frac{1}{\theta}} \right\}, \quad \theta \ge 0. \tag{3.3}$$

d) Frank [6]

$$C_\theta(u,v) = -\frac{1}{\theta}\ln\left\{ 1 + \frac{\left(e^{-\theta u}-1\right)\left(e^{-\theta v}-1\right)}{e^{-\theta}-1} \right\}, \quad \theta \ne 0. \tag{3.4}$$

e) Hüsler-Reiss [7] ([14], p. 143)

$$C_\theta(u,v) = \exp\left\{ \ln u \cdot \Phi\left[\tfrac{1}{\theta} + \tfrac{1}{2}\theta \ln\left(\tfrac{\ln u}{\ln v}\right)\right] + \ln v \cdot \Phi\left[\tfrac{1}{\theta} + \tfrac{1}{2}\theta \ln\left(\tfrac{\ln v}{\ln u}\right)\right] \right\}, \quad \theta \ge 0, \tag{3.5}$$

with $\Phi(x)$ the standard normal distribution.

f) Clayton [8] ([50], [51])

$$C_\theta(u,v) = \max\left(\left[u^{-\theta} + v^{-\theta} - 1\right]^{-\frac{1}{\theta}}, 0\right), \quad \theta \geq 0. \tag{3.6}$$

In all these copula families, the parameter θ of the joint cumulative distribution $F(x,y) = C_\theta[F_X(x), F_Y(y)]$ associated to a random couple (X,Y) with marginal distributions $F_X(x), F_Y(y)$, measures the degree of dependence between X and Y. The larger θ is in absolute value, the stronger the dependence. A positive value of θ indicates a positive dependence. Often one considers one-parameter families of copulas, which are able to model continuously a whole range of dependence between the lower Fréchet bound copula $\max(u+v-1,0)$, the independent copula uv, and the upper Fréchet bound copula $\min(u,v)$. Such families are called *inclusive* or *comprehensive*. The extensive list by [49], p. 96, contains only two one-parameter inclusive families of copulas, namely [6] and [8]. Another one is described in the next Section.

4. The linear Spearman copula.

Consider the *linear Spearman copula*, which is defined as follows. For $\theta \in [0,1]$ one has

$$C_\theta(u,v) = \begin{cases} [u + \theta(1-u)] \cdot v, & v \leq u, \\ [v + \theta(1-v)] \cdot u, & v > u, \end{cases} \tag{4.1}$$

and for $\theta \in [-1,0]$ one has

$$C_\theta(u,v) = \begin{cases} (1+\theta) \cdot uv, & u+v < 1, \\ uv + \theta \cdot (1-u) \cdot (1-v), & u+v \geq 1. \end{cases} \tag{4.2}$$

For $\theta \in [0,1]$ this copula is family B11 in [14], p. 148. It represents a mixture of perfect dependence and independence. If X and Y are uniform(0,1), $Y = X$ with probability θ and Y is independent of X with probability $1-\theta$, then (X,Y) has the linear Spearman copula. This distribution has been first considered by [9] and motivated in [10] along Cohen's kappa statistic (see [11], Section 10.9). For the extended copula, the chosen nomenclature *linear* refers to the piecewise linear sections of this copula, and *Spearman* refers to the fact that the *grade correlation coefficient* ρ_S by [52] coincides with the parameter θ. This follows from the calculation

$$\rho_S = 12 \cdot \int_0^1 \int_0^1 [C_\theta(u,v) - uv] du dv = \theta, \tag{4.3}$$

where a proof of the integral representation is given in [53]. The linear Spearman copula, which leads to the so-called *linear Spearman bivariate distribution*, has a singular component, which according to Joe should limit its field of applicability. Despite of this it has many interesting and important properties and is suitable for analytical computation.

For the reader's convenience, let us describe first two extremal properties. Kendall's τ for this copula equals using [53] :

$$\tau = 1 - 4 \cdot \int_0^1 \int_0^1 \frac{\partial}{\partial u} C_\theta(u,v) \cdot \frac{\partial}{\partial v} C_\theta(u,v) du dv = \frac{1}{3} \rho_S \cdot \left[2 + \mathrm{sgn}(\rho_S) \rho_S \right]. \tag{4.4}$$

Invert this to get

$$\rho_S = \begin{cases} -1 + \sqrt{1+3\tau}, & \tau \geq 0, \\ 1 - \sqrt{1-3\tau}, & \tau \leq 0. \end{cases} \tag{4.5}$$

Relate this to the convex two-parameter copula in [54] defined by

$$C_{\alpha,\beta}(u,v) = \beta \cdot C_{-1}(u,v) + (1-\alpha-\beta) \cdot C_0(u,v) + \alpha \cdot C_1(u,v), \quad \alpha,\beta \geq 0, \quad \alpha+\beta \leq 1, \tag{4.6}$$

where $C_\theta(u,v)$ has been defined in (4.1)-(4.2). Since $\rho_S = \alpha - \beta$ and $\tau = (\alpha-\beta)(2+\alpha+\beta)/3$ for this copula, one has the inequalities

$$\begin{aligned} &\tau \leq \rho_S \leq -1 + \sqrt{1+3\tau}, \quad \tau \geq 0, \\ &1 - \sqrt{1-3\tau} \leq \rho_S \leq \tau, \quad \tau \leq 0. \end{aligned} \tag{4.7}$$

The linear Spearman copula satisfies the following extremal property. For $\tau \geq 0$ the upper bound for ρ_S in Fréchet's copula is attained by the linear Spearman copula, and for $\tau \leq 0$ it is the lower bound, which is attained.

In case $\tau \geq 0$ a second more important extremal property holds, which is related to a conjectural statement. Recall that Y is *stochastically increasing* on X, written $SI(Y|X)$, if $\Pr(Y > y | X = x)$ is a nondecreasing function of x for all y. Similarly, X is *stochastically increasing* on Y, written $SI(X|Y)$, if $\Pr(X > x | Y = y)$ is a nondecreasing function of y for all x. (Note that [55] speaks instead of *positive regression dependence*). If X and Y are continuous random variables with copula $C(u,v)$, then one has the equivalences ([49], Theorem 5.2.10) :

$$SI(Y|X) \quad \Leftrightarrow \quad \frac{\partial}{\partial u} C(u,v) \text{ is nonincreasing in } u \text{ for all } v, \tag{4.8}$$

$$SI(X|Y) \quad \Leftrightarrow \quad \frac{\partial}{\partial v} C(u,v) \text{ is nonincreasing in } v \text{ for all } u. \tag{4.9}$$

The *Hutchinson-Lai conjecture* consists of the following statement. If (X,Y) satisfies the properties (4.8) and (4.9), then Spearman's ρ_S satisfies the inequalities

$$-1 + \sqrt{1+3\tau} \leq \rho_S \leq \min\left\{ \tfrac{3}{2}\tau, 2\tau - \tau^2 \right\}. \tag{4.10}$$

The upper bound $2\tau - \tau^2$ is attained for the one-parameter copula in [56] (see also [11], Section 13.7). The lower bound is attained by the linear Spearman copula ([9], p. 277). Alternatively, if the conjecture holds, the maximum value of Kendall's τ by given ρ_S is attained for the linear Spearman copula. Note that the upper bound $\rho_S \leq (3/2)\tau$ has been disproved recently in [49], Exercise 5.36. The remaining conjecture

$-1+\sqrt{1+3\tau} \le \rho_S \le 2\tau - \tau^2$ is still unsettled (however, see [57] for the case of bivariate extreme value copulas).

As an important modelling characteristic, let us show that the linear Spearman copula leads to a simple tail dependence structure, which is of interest when extreme values are involved. Recall the *coefficient of (upper) tail dependence* of a couple (X,Y) defined by

$$\lambda = \lambda(X,Y) = \lim_{\alpha \to 1^-} \Pr(Y > Q_Y(\alpha) | X > Q_X(\alpha)), \tag{4.11}$$

provided a limit λ in $[0,1]$ exists ($Q_X(u) = \inf\{x | \Pr(X \le x) \ge u\}$ denotes a quantile function of *X*). If $\lambda \in (0,1]$ then the couple (X,Y) is called *asymptotically dependent* (in the upper tail) while if $\lambda = 0$ one speaks of *asymptotic independence*. Tail dependence is an asymptotic property of the copula. Its calculation follows easily from the relation

$$\begin{aligned} &\Pr(Y > Q_Y(\alpha) | X > Q_X(\alpha)) \\ &= \frac{1-\Pr(X \le Q_X(\alpha)) - \Pr(Y \le Q_Y(\alpha)) + \Pr(X \le Q_X(\alpha), Y \le Q_Y(\alpha))}{1-\Pr(X \le Q_X(\alpha))}. \end{aligned} \tag{4.12}$$

For a linear Spearman couple one obtains

$$\lambda(X,Y) = \lim_{\alpha \to 1^-} \frac{1-2\alpha + C_\theta(\alpha,\alpha)}{1-\alpha} = \lim_{\alpha \to 1^-} (1-\alpha+\theta\alpha) = \theta. \tag{4.13}$$

Therefore, unless X and Y are independent, a linear Spearman couple is always asymptotically dependent. This is a desirable property in insurance and financial modelling, where data tend to be dependent in their extreme values. In contrast to this, the ubiquitous Gaussian copula yields always asymptotic independence, unless perfect correlation holds ([58], [59], Chap. 5, [44], Section 4.4).

5. Bivariate copula fitting.

Our view of multivariate statistical modelling is that in [14], Section 1.7. In particular, our bivariate models satisfy the desirable properties suggested in [14], Section 4.1, and [42].

Firstly, modelling bivariate cumulative returns with the copulas of Sections 3 and 4 yields a closed-form representation for the bivariate distribution, which can be numerically evaluated provided the margins follow $N\Gamma(\mu,c,3)$ distributions. Secondly, the parameter θ in our copulas, which models the dependence, covers the relevant range of positive dependence lying between the independent copula and the upper Fréchet bound copula. Thirdly, it remains to analyze the effective fitting of the chosen copulas to actual data. Statistical inference is done by specifying the estimation method and a bivariate goodness-of-fit test.

To estimate the 5 parameters $(\mu_X, c_X, \mu_Y, c_Y, \theta)$ of a copula-based bivariate model $F(x,y) = C_\theta[F_X(x), F_Y(y)]$, where $F_X(x)$, $F_Y(x)$ are $N\Gamma(\mu_X, c_X, 3)$, $N\Gamma(\mu_Y, c_Y, 3)$ distributed, we apply a method close in spirit to the method of *inference function for margins* or *IFM method* studied in [12], [13], and [14], Section 10.1. We perform two separate maximum likelihood estimations of the univariate margins, followed by an estimation of the dependence parameter. However, we do not maximize the bivariate likelihood, except for comparison purposes in the Table 5.1. Instead, we determine the dependence parameter, which maximizes the p-value (respectively minimizes the bivariate chi-square statistic) of a bivariate version of the usual Pearson goodness-of-fit test. We note that an application of the uniform test by [60] for testing Y given X and X given Y, as proposed in [42], Section 5, is only satisfactory in one way. Testing the stock return Y given the index return X is accepted, while testing the index return X given the stock return Y is rejected.

Let us describe in more details our estimation method. Given $n+1$ daily observations I_i of a market index and $n+1$ daily prices S_i of a stock in the index family, let $X_i = I_{i+1}/I_i$, $Y_i = S_{i+1}/S_i$, $i=1,...,n$, be the daily cumulative returns used for statistical fitting.

We validate the estimation of the marginal distributions using two separate univariate Pearson chi-square tests. Following [1], Section 6, the raw data X_i is regrouped into 6 classes $(v_0,v_1], (v_1,v_2], ..., (v_5,v_6]$, where the boundaries v_i's are chosen such that the number of observations $\lambda_1, \lambda_2, ..., \lambda_6$ in the corresponding classes are as symmetrically distributed as possible ($v_3=1$ appears adequate). The data Y_i is regrouped into 6 similar classes $(w_0,w_1], (w_1,w_2], ..., (w_5,w_6]$ with observations $\eta_1, \eta_2, ..., \eta_6$ in each class. One obtains two separate chi-square statistics

$$\begin{aligned} \chi_X^2 &= \sum_{i=1}^{6} \frac{(\lambda_i - nf_i)^2}{nf_i}, \quad \chi_Y^2 = \sum_{i=1}^{6} \frac{(\eta_i - ng_i)^2}{ng_i}, \\ f_i &= \frac{F_X(v_i) - F_X(v_{i-1})}{1 - F_X(v_0)}, \quad g_i = \frac{F_Y(w_i) - F_Y(w_{i-1})}{1 - F_Y(w_0)}. \end{aligned} \tag{5.1}$$

We report the p-values p_X, p_Y corresponding to χ_X^2, χ_Y^2 for a chi-square distribution with 3 degrees of freedom.

Next, to create a meaningful bivariate chi-square statistic, we look at the number of observations $z_{i,j}$ in the 36 two-dimensional intervals $(v_{i-1},v_i]x(w_{j-1},w_j]$, $i,j=1,...,6$. By [61], [62] ([36], p. 121) regroup these intervals in 10 larger rectangular interval classes as follows. Recommended is an expected frequency of at least 1% in each class and a 5% expected frequency in 80% of the classes. With $n=250$ daily observations over an approximate one-year period the following rectangular regrouping in 10 classes C_k, $k=1,...,10$, appears adequate (at least in our examples) :

	w_0	w_1	w_2	w_3	w_4	w_5	w_6
v_0							
	C_1	C_1	C_1	C_1	C_1	C_1	
v_1							
	C_1	C_3	C_5	C_5	C_5	C_2	
v_2							
	C_1	C_6	C_7	C_8	C_5	C_2	
v_3							
	C_1	C_6	C_9	C_{10}	C_5	C_2	
v_4							

v_5	C_1	C_6	C_6	C_6	C_4	C_2
v_6	C_2	C_2	C_2	C_2	C_2	C_2

Then, consider the fitted or expected number of observations $f_{i,j}$ in each of the 36 two-dimensional intervals $(v_{i-1}, v_i]x(w_{j-1}, w_j]$ given by

$$\begin{aligned} f_{i,j} &= n \cdot \lfloor F(v_i, w_j) - F(v_{i-1}, w_j) - F(v_i, w_{j-1}) + F(v_{i-1}, w_{j-1}) \rfloor, \\ i, j &= 1,\dots,6, \quad F(x, y) = C_\theta [F_X(x), F_Y(y)]. \end{aligned} \tag{5.2}$$

Through summation of $z_{i,j}$'s, respectively $f_{i,j}$'s, one obtains the number of observations O_k, respectively the expected number of observations E_k, in each class C_k, $k = 1,\dots,10$. The bivariate chi-square statistic is then defined by

$$\chi^2 = \sum_{k=1}^{10} \frac{(O_k - E_k)^2}{E_k}. \tag{5.3}$$

For each copula one obtains numerical values of θ, which minimize χ^2 respectively maximize the bivariate p-value corresponding to χ^2 for a chi-square distribution with 4 degrees of freedom. For comparisons, the value of the bivariate negative log-likelihood is also of interest. It is defined by

$$\begin{aligned} -\ln L &= n \cdot \ln[1 - F_X(v_0) - F_Y(v_0) + F(v_0, w_0)] \\ &- \sum_{i=1}^{6} \sum_{j=1}^{6} z_{i,j} \cdot \ln[F(v_i, w_j) - F(v_{i-1}, w_j) - F(v_i, w_{j-1}) + F(v_{i-1}, w_{j-1})]. \end{aligned} \tag{5.4}$$

Following the IFM method described above, one obtains numerical values of θ, which minimize $-\ln L$ under a p-value of at least 5%. Our examples show that the IFM method reduces the p-value of χ^2 sometimes rather drastically. For this reason, we prefer the proposed bivariate minimum chi-square or maximum p-value estimation method.

To illustrate, we start with a comparison of fit based on the IFM method. Table 5.1 reports copula fitting results for daily cumulative returns between the SMI index and a single stock for the one-year period between September 29, 1998 and September 24, 1999. For the Credit Suisse Group stock the Gumbel-Hougaard copula maximizes the bivariate log-likelihood, the linear Spearman copula yields the highest p-value. For the Novartis stock the Clayton copula maximizes the bivariate log-likelihood, the Frank copula yields the highest p-value.

Table 5.1 : IFM method for Credit Suisse Group and Novartis stock

Credit Suisse parameters :	$\mu_X = 1.000656016$, $c_X = 0.030403482$, $\mu_Y = 1.001624799$, $c_Y = 0.051511031$.				
model	θ	p-value	$(-\ln L)_{\min}$	p_X	p_Y
Gumbel-Hougaard	2.65	13.5	599.4	74.9	84.9

Galambos	1.97	13.1	600.3	74.9	84.9
Hüsler-Reiss	2.48	5	602.9	74.9	84.7
Frank	9.25	16.7	604.4	75.6	82.9
Cuadras-Augé	0.665	31.1	610.8	75.9	83.5
Linear Spearman	0.52	66.7	614.2	76.6	83.4
Clayton	3.58	5	629.5	76.4	85.1
Novartis parameters :	$\mu_X = 1.000656016,\quad c_X = 0.030403482,$ $\mu_Y = 1.000225799,\quad c_Y = 0.035275623.$				
model	θ	p-value	$(-\ln L)_{\min}$	p_X	p_Y
Clayton	2.25	30.7	592.8	75.4	93.8
Frank	6.9	32.5	593	75.6	93.4
Gumbel-Hougaard	2.23	11.6	594.9	75.1	93.4
Galambos	1.57	11	595.1	75	93.3
Hüsler-Reiss	2.03	9.5	596	75.1	93.3
Cuadras-Augé	0.619	5.5	611.5	75.9	93.5
Linear Spearman	0.474	25.7	612.7	76.7	93.6

The Table 5.2 reports comparisons of fit using the bivariate minimum chi-square estimation method for 6 pairs of daily cumulative returns between the SMI index and a single stock for the same one-year period. Except for the Nestlé stock, whose fitted marginal distribution is rejected, we find in most cases satisfactory fits, in particular for the SMI stocks Credit Suisse Group, Sulzer, Novartis and UBS. The rejected fit of the Nestlé marginal distribution may be due to the fact that the extreme values of the daily returns were rather moderate between -6% and 6%. The fit of the Swisscom stock is not rejected but less satisfactory. Rather surprisingly, the highest bivariate p-value is obtained for the very tractable linear Spearman copula (except for the Swisscom stock). Except for the Cuadras-Augé (UBS) and Clayton copulas (Credit Suisse Group and Swisscom), the p-values seem sufficiently high. In our view, the linear Spearman, Frank and Gumbel-Hougaard copulas provide the best overall fits for the analyzed pairs.

Despite of the obtained good fit for the linear Spearman copula, we do not claim that the proposed model explains the joint distribution inferred by the data. Indeed, from a modelling viewpoint, the linear Spearman copula with positive dependence is a mixture of the independent copula and the Fréchet upper bound. This implies that the data can be split into two populations, one independent and the other with perfect positive dependence. Therefore, a scatter plot should reveal a sub-population of data, where one variable is a monotonically increasing function of the other. This property does not seem to be exhibited easily if at all.

Table 5.2 : Bivariate maximum p-value method for some stocks in the SMI index

Credit Suisse Group parameters :	$\mu_X = 1.000656016,\quad c_X = 0.030403482,$ $\mu_Y = 1.001624799,\quad c_Y = 0.051511031.$				
model	θ	$(p-value)_{\max}$	$-\ln L$	p_X	p_Y
Linear Spearman	0.495	71.8	614.5	76.6	83.4
Cuadras-Augé	0.64	36.9	611.3	76	83.3
Gumbel-Hougaard	3.05	28.8	601.4	74.7	85.4
Galambos	2.37	28.2	602.8	74.6	85.5
Frank	10.6	26.4	605.7	75.5	83

Hüsler-Reiss	3.15	26.4	609.5	74.5	85.5
Clayton	3.83	5.7	632.2	76.4	85.1

Novartis parameters : $\mu_X = 1.000656016, \quad c_X = 0.030403482,$
$\mu_Y = 1.000225799, \quad c_Y = 0.035275623.$

model	θ	$(p-value)_{\max}$	$-\ln L$	p_X	p_Y
Linear Spearman	0.399	56.4	614.3	76.6	93.5
Frank	6.9	32.5	593	75.6	93.4
Clayton	2.18	31.1	592.8	75.3	93.8
Cuadras-Augé	0.547	15.6	612.9	76	93.4
Gumbel-Hougaard	2.23	11.6	594.9	75.1	93.4
Galambos	1.56	11.1	595.1	75	93.3
Hüsler-Reiss	2.17	10.3	596.5	75	93.3

UBS parameters : $\mu_X = 1.000656016, \quad c_X = 0.030403482,$
$\mu_Y = 1.0016621, \quad c_Y = 0.0477182.$

model	θ	$(p-value)_{\max}$	$-\ln L$	p_X	p_Y
Linear Spearman	0.403	38	629.6	77.1	52.3
Clayton	2.42	25.3	648.2	76.7	49.6
Frank	7.43	14.5	633	76.7	54.2
Gumbel-Hougaard	2.38	14.1	626.9	76.8	52.6
Galambos	1.69	14	629	76.8	52.6
Hüsler-Reiss	2.34	13.8	638.6	76.8	52.5
Cuadras-Augé	0.55	5.5	634	76.7	54

Nestlé parameters : $\mu_X = 1.000656016, \quad c_X = 0.030403482,$
$\mu_Y = 1.00096518, \quad c_Y = 0.03362619.$

model	θ	$(p-value)_{\max}$	$-\ln L$	p_X	p_Y
Cuadras-Augé	0.5	34	604.1	79.1	0
Gumbel-Hougaard	2.11	32.1	579.9	80.8	0
Linear Spearman	0.377	31.5	606.1	78.8	0
Galambos	1.42	31.2	579.5	80.8	0
Hüsler-Reiss	2.01	29.7	578.9	80.9	0
Frank	6.34	29.3	584.9	75.6	0.1
Clayton	2.03	2.5	592.2	75.8	0

Sulzer parameters : $\mu_X = 1.000656016, \quad c_X = 0.030403482,$
$\mu_Y = 1.0014062, \quad c_Y = 0.0466025.$

model	θ	$(p-value)_{\max}$	$-\ln L$	p_X	p_Y
Linear Spearman	0.264	69.7	649.1	76.8	46.7
Gumbel-Hougaard	1.77	64.9	636.9	77.1	47.9
Galambos	1.09	64.9	637.8	77.1	47.9
Hüsler-Reiss	1.62	64.8	639.9	77	48
Frank	4.72	50.8	641.8	76.3	48.7
Cuadras-Augé	0.382	46.2	649.9	76.7	47.8
Clayton	1.45	24.3	634.9	76.2	42.7

Swiscom parameters : $\mu_X = 1.000656016, \quad c_X = 0.030403482,$
$\mu_Y = 1.00071313, \quad c_Y = 0.05098643.$

model	θ	$(p-value)_{\max}$	$-\ln L$	p_X	p_Y
Cuadras-Augé	0.151	15.9	700.6	75.6	18.2
Frank	2.09	12.8	695.7	75.3	18.1
Linear Spearman	0.094	9.1	703.7	75.2	18.3
Gumbel-Hougaard	1.22	8.5	696.3	75.7	18
Galambos	0.47	7.9	696.6	75.9	18
Hüsler-Reiss	0.84	7.7	697	75.9	17.9
Clayton	0.36	2.5	701.9	72	18.4

6. Covariance estimation with the linear Spearman copula.

As an application of our estimation results, let us compare standard empirical values of the linear correlation coefficient with the estimated linear correlation coefficient from a fitted copula. This is of interest because for heavy tailed data the standard product-moment correlation estimator has a very bad performance, and there is a need for more robust estimators (e.g. [63]). In view of the importance of correlation measures in modern finance (portfolio theory, CAPM models), this is a highly relevant issue.

Extreme, synchronized rises and falls of indices and stocks occur infrequently, but more often than what is predicted by a bivariate normal model. Since the linear Spearman copula has a simple tail dependence structure (end of Section 4) and our copula fitting was quite satisfactory, we focus on the evaluation of the covariance for this copula.

Theorem 6.1. Let (X,Y) be distributed as $F(x,y)=C_\theta[F_X(x),F_Y(y)]$, where $C_\theta(u,v)$ is the linear Spearman copula, and the continuous and strictly increasing marginal distributions are defined on the open supports (a_X,b_X), (a_Y,b_Y). For an arbitrary differentiable function $\psi(y)$, assume the following regularity assumption holds :

$$\begin{aligned}&\lim_{y\to a_Y}\psi(y)F_Y(y)=0,\\ &\lim_{y\to b_Y}\psi(y)\left(E[X]F_Y(y)-\int_{a_Y}^{y}F_X^{-1}\left[F_Y^\theta(t)\right]dF_Y(t)\right)=0.\end{aligned} \tag{RA}$$

Then one has the covariance formula

$$Cov[X,\psi(Y)]=\operatorname{sgn}(\theta)\theta\cdot E\left[\left(F_X^{-1}\left[F_Y^\theta(Y)\right]-E[X]\right)\cdot\psi(Y)\right], \tag{6.1}$$

where one sets

$$F_Y^\theta(y)=\begin{cases}F_Y(y), & \theta\geq 0,\\ \overline{F}_Y(y), & \theta<0,\end{cases} \tag{6.2}$$

with the abbreviation $\overline{F}_Y(y)=1-F_Y(y)$.

Proof. Let us first derive the regression function $E[X|Y=y]$. The conditional distribution of X given $Y=y$ equals for $\theta\geq 0$

$$F(x|y) = \frac{\partial C_\theta}{\partial v}[F_X(x), F_Y(y)] = \begin{cases} F_X(x) + \theta \overline{F}_X(x), & x \geq F_X^{-1}[F_Y(y)], \\ (1-\theta)F_X(x), & x < F_X^{-1}[F_Y(y)], \end{cases} \tag{6.3}$$

and for $\theta < 0$

$$F(x|y) = \begin{cases} (1+\theta)F_X(x), & x < F_X^{-1}[\overline{F}_Y(y)], \\ F_X(x) - \theta \overline{F}_X(x), & x \geq F_X^{-1}[\overline{F}_Y(y)]. \end{cases} \tag{6.4}$$

Through calculation one obtains the regression formula

$$E[X|y] = \int_0^\infty [1 - F(x|y)]dx - \int_{-\infty}^0 F(x|y)dx \tag{6.5}$$

$$= \begin{cases} E[X] - \theta \cdot (E[X] - F_X^{-1}[F_Y(y)]), & \theta \geq 0, \\ E[X] + \theta \cdot (E[X] - F_X^{-1}[\overline{F}_Y(y)]), & \theta \leq 0, \end{cases} \tag{6.6}$$

which is a weighted average of the mean $E[X]$ and the quantile $F_X^{-1}[F_Y(y)]$ respectively $F_X^{-1}[\overline{F}_Y(y)]$. To obtain the stated covariance formula, one uses the well-known formula in [64] and [55], Lemma 2, to get the expression

$$\begin{aligned} Cov[X, \psi(Y)] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [F(x,y) - F_X(x)F_Y(y)]\psi'(y)dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_Y(y)[F(x|Y \leq y) - F_X(x)]\psi'(y)dxdy \\ &= \int_{-\infty}^{\infty} F_Y(y)\left[\int_{-\infty}^{0} F(x|Y \leq y)dx - \int_{-\infty}^{0} F_X(x)dx + \int_0^\infty \overline{F}_X(x)dx - \int_0^\infty [1 - F(x|Y \leq y)dx]\right]\psi'(y)dy \\ &= \int_{-\infty}^{\infty} (E[X] - E[X|Y \leq y])F_Y(y)\psi'(y)dy. \end{aligned} \tag{6.7}$$

Furthermore, from (6.6) one obtains

$$E[X] - E[X|Y \leq y] = \mathrm{sgn}(\theta)\theta \cdot (E[X] - E[F_X^{-1}[F_Y^\theta(Y)]|Y \leq y]). \tag{6.8}$$

Inserted in (6.7), a partial integration yields

$$\begin{aligned} Cov[X, \psi(Y)] &= \mathrm{sgn}(\theta)\theta \cdot \int_{a_Y}^{b_Y} \left(E[X]F_Y(y) - \int_{a_Y}^{y} F_X^{-1}[F_Y^\theta(t)]dF_Y(t) \right)\psi'(y)dy \\ &= \mathrm{sgn}(\theta)\theta \cdot \begin{pmatrix} \psi(y)\left[E[X]F_Y(y) - \int_{a_Y}^{y} F_X^{-1}[F_Y^\theta(t)]dF_Y(t) \right]_{a_Y}^{b_Y} \\ + \int_{a_Y}^{b_Y} \psi(y)(F_X^{-1}[F_Y^\theta(y)] - E[X])dF_Y(y) \end{pmatrix}, \end{aligned} \tag{6.9}$$

which implies (6.1) by the regularity assumption. ◊

The application of this result to margins from a symmetric location-scale family is simple.

Corollary 6.1. Under the assumptions from Theorem 6.1 suppose that $F_X(x) = F_Z\left(\frac{x-\mu_X}{c_X}\right)$, $F_Y(x) = F_Z\left(\frac{x-\mu_Y}{c_Y}\right)$, $\mu_X = E[X], \mu_Y = E[Y]$, and $\overline{F}_Z(-z) = F_Z(z)$. Then one has

$$Cov[X, \psi(Y)] = \theta \frac{c_X}{c_Y} Cov[Y, \psi(Y)]. \qquad (6.10)$$

Proof. The result follows from (6.1) noting that $F_X^{-1}\left[F_Y^{\theta}(y)\right] = \mu_X + \mathrm{sgn}(\theta)\frac{c_X}{c_Y}(y - \mu_Y)$. ◊

Example 6.1.

In case $X \sim N\Pi\Gamma(\mu_X, c_X, \alpha)$, $Y \sim N\Pi\Gamma(\mu_Y, c_Y, \alpha)$, $\alpha > 1$, one has $c_X = \sqrt{2(\alpha-1)Var[X]}$, $c_Y = \sqrt{2(\alpha-1)Var[Y]}$. For $\psi(y) = y$ the regularity assumption (RA) holds, hence $Cov[X, Y] = \theta\sqrt{Var[X]Var[Y]}$. In this special situation Spearman's correlation coefficient θ coincides with Pearson's linear correlation coefficient. It is therefore possible to compare the standard product-moment correlation estimator with the estimated θ obtained from the linear Spearman copula fitting. Results for some stocks are found in Table 6.1.

Table 6.1 : linear correlation estimators for the linear Spearman copula

Stock	Product-moment estimator	Fitted estimator
Credit Suisse Group	0.829	0.495
UBS	0.708	0.403
Nestlé	0.777	0.377
Novartis	0.798	0.399
Sulzer	0.663	0.264
Swisscom	0.285	0.094

The discrepancy between both estimators is considerable. However, on a relative scale both estimators rank the strength of dependence similarly.

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