Statistical Arbitrage and Securities Prices[*]

Oleg Bondarenko[†]

May 2002 (first version: March 2001)

Abstract

This paper introduces the concept of a *statistical arbitrage opportunity* (SAO). In a finite-horizon economy, a SAO is a zero-cost trading strategy for which i) the expected payoff is positive, and ii) the conditional expected payoff in each final state of the economy is nonnegative. Unlike a *pure arbitrage opportunity*, a SAO can have negative payoffs provided that the *average* payoff in each final state is nonnegative.

If the pricing kernel in the economy is path-independent then no SAOs can exist. Furthermore, ruling out SAOs imposes a novel martingale-type restriction on the dynamics of securities prices. The important properties of the restriction are that it 1) is model-free, in the sense that it requires no parametric assumptions about the true equilibrium model, 2) can be tested in samples affected by selection biases, such as the Peso problem, and 3) continues to holds when investors' beliefs are mistaken.

The paper argues that one can use the new restriction to empirically resolve the joint hypothesis problem present in the traditional tests of the Efficient Market Hypothesis.

JEL Classification: D84, G12, G13, G14.

Keywords: Arbitrage, Pricing Kernel, Market Efficiency Hypothesis, Rational Learning, Option Valuation, Risk-Neutral Density, Peso problem

---

[†]Department of Finance (MC 168), University of Illinois at Chicago, 601 S. Morgan St., Chicago, IL 60607. Phone: (312) 996-2362. Email: olegb@uic.edu

1 Introduction

In a fairly general environment, this paper proposes a novel martingale-type restriction on the dynamics of securities prices. This restriction has a number of important properties. Most notably, the restriction may be viewed as model-free because it requires no parametric assumptions about the true equilibrium model. To derive the restriction, we rely on the concept of statistical arbitrage, a generalization of pure arbitrage.

A *pure arbitrage opportunity* (PAO) is a zero-cost trading strategy that offers the possibility of a gain with no possibility of a loss. As well known, the existence of PAOs is incompatible with a competitive equilibrium in asset markets. The fundamental theorem of the financial theory establishes a link between the absence of PAOs and the existence of a positive *pricing kernel* which supports securities prices.

While the absence of PAOs is a necessary condition for any equilibrium model, this condition alone often yields pricing implications that are too weak to be practically useful. For example, when valuing options in incomplete markets, the no-arbitrage bounds on option prices are typically very wide. To strengthen pricing implications, several recent papers have suggested to further restrict the set of available investment opportunities and/or the set of admissible pricing kernels.

Cochrane and Saá-Requejo (2000) proposes to rule out not only PAOs but also "good deals" (GDs), or investment opportunities with high Sharpe ratios. Following Hansen and Jagannathan (1991), Cochrane and Saá-Requejo show that precluding GDs imposes an upper bound on the pricing kernel volatility and yields tighter pricing implications when markets are incomplete. Bernardo and Ledoit (2000) proposes to rule out *approximate arbitrage opportunities* (AAOs), or investment opportunities which offer high gain-loss ratios, where gain (loss) is the expectation of the positive (negative) part of the excess payoff computed under a benchmark risk-neutral measure. They demonstrate that restricting the maximum gain-loss ratio implies, loosely stated, that an admissible pricing kernel cannot deviate too far from the benchmark pricing kernel.

In this paper, we propose a different approach to restrict the set of admissible pricing kernels. In this approach, we do not preclude opportunities whose attractiveness – as measured by the Sharpe ratio, the gain-loss ratio, or other criteria – exceeds some *ad hoc* threshold. Nor do we need to make parametric assumptions about a benchmark pricing kernel. Instead, we impose an arguably weak assumption on a *functional form* of admissible pricing kernels and show that this assumption has striking implications for securities prices. The idea of our approach can be best explained for a simple special case.

Consider a finite-horizon economy with a single asset. The asset is traded in a frictionless market on dates $t = 0, 1, \ldots, T$. The asset's price is v_t, and $I_t = (v_0, \ldots, v_t)$ is the price history through time-t. Let Z_t denote the value of a general derivative security with a payoff $Z_T = Z(I_T)$. The absence of PAOs implies that there exists a positive pricing kernel m_T such that

$$E[Z_s m_s \mid I_t] = Z_t m_t, \qquad t < s \leq T, \tag{1}$$

where $m_t = E[m_T \mid I_t]$ and the risk-free rate is assumed to be zero.

Generally, the pricing kernel m_T may depend on the complete price history, or $m_T = m(I_T)$. Except for the positivity constraint, the function $m(I_T)$ has to satisfy no other conditions. This means that the function $m(I_T)$ could be economically rather "unreasonable." For example, values of the pricing kernel for two "close" price histories are allowed to be arbitrary far apart. Such a pricing kernel, however, is unlikely to describe anyone's marginal utility function.

In this paper we argue that, in many important situations, the economics of the problem imposes an additional structure on admissible pricing kernels. Specifically, suppose that the preferences of the representative investor are given by the utility function $U(v_T)$. Then the pricing kernel is a function of v_T only, or $m_T = m(v_T)$. It turns out that the fact that the pricing kernel is *path-independent* considerably reduces the set of investment opportunities that can exist in the economy.

We show that a pricing kernel $m(v_T) > 0$ exists if and only if no *statistical arbitrage opportunities* (SAOs) are available. Here, a SAO is a zero-cost trading strategy for which 1) the expected payoff is positive, $E[Z_T \mid I_t] > 0$, and 2) for each v_T, the expected payoff *conditional* on the asset's final price being v_T is nonnegative, $E[Z_T \mid I_t^{v_T}] \geq 0$, where $I_t^{v_T} = (v_0, \ldots, v_t; v_T)$ is the *augmented* information set, which in addition to I_t also includes the knowledge of the final price. Unlike a PAO, a SAO is allowed to have negative payoffs, provided that the *average* payoff for each v_T is nonnegative.

The concept of a SAO is useful because ruling out SAOs induces a new powerful restriction on securities prices. To demonstrate it, let $h_t(v_T)$ denote the conditional risk-neutral density of the asset's final price. Suppose that an empiricist observes a price history I_T with the final price $v_T = v$. Then securities prices deflated by the risk-neutral density evaluated at the final price must follow a martingale process:[1]

$$E\left[\frac{Z_s}{h_s(v)} \mid I_t^v\right] = \frac{Z_t}{h_t(v)}, \qquad t < s < T. \qquad (2)$$

Intuitively, the restriction in (2) states the following. Suppose that the empiricist observes many repetitions of the same environment and selects only those price histories that result in the same final price $v_T = v$. Then, in the selected histories the ratio $\frac{Z_t}{h_t(v)}$ must change over time unpredictably.

The unusual feature of the restriction is that it involves conditioning on future information.[2] To test the restriction, the empiricist needs to know the asset's price at time-T. The empiricist cannot conduct testing in "real-time" – she must wait until the final price is revealed. Note also that the restriction assumes that the risk-neutral density $h_t(v_T)$ is available to the empiricist. Despite the fact that the risk-neutral density is not directly observable in financial markets, it is implicit in securities prices. In particular, it can be estimated from prices of traded options, such as standard European calls with different

---

[1] An additional technical assumption will be made to ensure that the term inside the conditional expectation operator in (2) is well-defined for all price histories.

[2] In an important paper, Bossaerts (1999a) demonstrates that conditioning on future price outcomes can be useful in testing asset pricing models. Our paper builds on many of his insights, as will be discussed shortly.

strikes. Consequently, the new restriction in (2) is best suited for applications where liquid option markets exist.

Three important properties of the restriction in (2) deserve mentioning. First and most significantly, the restriction is completely preference-independent. In other words, the utility function $U(v_T)$ can be arbitrary and the restriction in (2) must still hold. This might seem counterintuitive. One would conjecture that no meaningful restriction on securities prices can exist if the utility function is allowed to be arbitrary, because almost any price dynamics can be generated. This paper shows, however, that this conjecture is untrue. Under an arguably weak assumption that preferences are path-independent, there is an easily verifiable "structure" in securities prices, for *all* preferences.

This implies that the restriction in (2) can be used to resolve the joint hypothesis problem ubiquitous in tests of the Efficient Market Hypothesis (EMH). Traditional tests of EMH rely on the restriction in (1) and require one to pre-commit to a specific pricing kernel or, equivalently, an equilibrium model. As a result, if tests are rejected, this could be because the market is truly inefficient or because the incorrect equilibrium model has been assumed. In contrast, the restriction in (2) requires no parametric assumptions about the true equilibrium model. Rejection of the restriction in (2) means that the market cannot be efficient for *any* model with a path-independent pricing kernel.

Second, the restriction in (2) can be used in samples which come with various selection biases. To see this more clearly, suppose that the empiricist has collected a dataset in which not all price histories are present; say, the dataset includes only those histories for which the asset's final price is greater than initial price, $v_T \geq v_0$. Such a deliberate selection bias will generally cause rejection of the restriction in (1), even if the true pricing kernel m_t were known. Interestingly, the selection bias will *not* affect the restriction in (2). This is because the restriction involves conditioning on the final price.

As a more empirically relevant illustration, consider the so-called *Peso problem*. The Peso problem arises when a rare but influential event could have reasonably happened but did not happen in the sample. A typical example of this problem is when investors have correctly anticipated the possibility of a crash, but the sample includes no crash. In the presence of the Peso problem, the restriction in (2) can still be tested, while the restriction in (1) cannot.

Third, the restriction in (2) continues to hold even when investors' beliefs are mistaken. Specifically, suppose that investors have incorrect expectations about the distribution of v_T. However, they continue to update their expectations in a rational way. Then, under a certain condition, the restriction in (2) must still hold.

To summarize, the new restriction permits a nonparametric test of whether investors are rational or not, for a broad class of preferences and beliefs. The test works even in the presence of selection biases.

It should be emphasized that the analogue of the restriction in (2) obtains in a fairly general environment. In particular, there can be multiple assets, trading can take place continuously or at discrete intervals, markets can be incomplete, and the information flow can be represented by general filtrations. A general version of the restriction essentially looks like (2), except that v_t is replaced with a state variable ξ_t that describes uncertainty in the economy. The state variable ξ_t may include prices of

traded assets as well as additional economic factors. Assuming that the pricing kernel m_T is a function of the final state ξ_T only, we show that no SAOs can exist, where a SAO is a zero-cost trading strategy with a positive expected payoff and nonnegative conditional expected payoffs for each ξ_T. We argue that the assumption of a path-independent pricing kernel is satisfied by many popular asset pricing models, including CAPM, the consumption-based models, the multifactor pricing models, the Epstein-Zin-Weil model, the Black-Scholes model, and others. Consequently, in all these models, not only PAOs but also more general SAOs cannot exist.

It is important to point out that a special case of the restriction in (2) have originally been developed in Bossaerts (1999a, 1999b). His motivation is different from ours. While we are primarily interested to learn how securities prices are affected by *risk-preferences*, Bossaerts focuses on the effects of *incorrect beliefs*. He introduces an extension of EMH where investors are rational but may have mistaken beliefs. The extension is termed *Efficiently Learning Market* (ELM). Using the fundamental property of Bayesian beliefs, Bossaerts shows that, in ELM under *risk-neutrality* and a certain condition on beliefs, there exists a set of novel restrictions on securities prices. His restrictions involve conditioning on future information and deflating securities prices by posterior beliefs evaluated at the eventual outcome.[3]

In this paper, we argue that there is an interesting *equivalence* relationship between preferences and beliefs. The equivalence relationship means that, in economies populated by rational investors, the same securities prices can result from either risk-aversion or biased beliefs, or some combination of the two. In particular, for every economy in which investors are risk-averse but have correct beliefs, there is another economy in which investors are risk-neutral but have biased beliefs, such that the two economies support the same securities prices. An outside empiricist, who only uses market data, *cannot* distinguish the two economies.

The remainder of the paper is organized as follows. The theory is presented in Section 2 and Section 3. To develop intuition, Section 2 considers a finite-state economy with discrete trading. This section introduces the concept of a SAO and derives the new restriction on securities prices, first for EMH and then for ELM. Section 3 considers a general continuous-time economy. This section also presents the equivalence relationship between preferences and beliefs. Section 4 discusses empirical implications and Section 5 concludes. The appendix contains proofs of all results.

2   Basic Model

In this section, we develop the theory under a number of simplifying assumptions. These assumptions are relaxed in Section 3.

---

[3]Bossaerts (1999a, 1999b) also propose results for the risk-averse case. Those results, however, assume that investors' preferences are known.

2.1 The Economy

We consider a finite-horizon economy modeled as follows.

Information structure. There are a finite number of trading dates, indexed by $t = 0, 1, \ldots, T$. At time-t, the state of the economy is represented by a random variable ξ_t. The history of states up to time-t determines the market information set $I_t = (\xi_1, \ldots, \xi_t)$. We will distinguish between "elementary" and "final" states. The elementary state $I_T \in \mathcal{I}_T$ provides a complete description of uncertainty from time 1 to T, while the final state $\xi_T \in \Xi_T$ describes the price relevant uncertainty on the final date.

For example, one can interpret ξ_t as the value of the market portfolio at time-t, with I_T representing the complete time-series path. More generally, ξ_t may represent a vector of values of traded assets and other economic factors (such as stochastic volatility or interest rates). The structure for state variables can be very general, but for now we assume that ξ_t is a discrete random variable which takes on a finite number of different values.

Securities market. There is a finite number of primary assets that are traded in a frictionless and competitive market. At time-t, their prices depend on the state ξ_t. For simplicity, we assume that the risk-free rate is zero.

By trading primary assets, investors can generate various payoffs at time-T. Specifically, consider a self-financing trading strategy which pays a random, path-dependent payoff $Z_T = Z(I_T)$. Let Z_t denote the value of such a generic payoff at time-t.[4] (The set of available final payoffs is denoted $\mathcal{Z}$. The set $\mathcal{Z}$ is a linear space, that is, if payoffs $Z_T^{(1)}, Z_T^{(2)} \in \mathcal{Z}$, then the payoff $\alpha Z_T^{(1)} + \beta Z_T^{(2)} \in \mathcal{Z}$ for all constants α and β. If the market is complete then $\mathcal{Z} = \mathcal{R}^N$, where N is the dimension of $\mathcal{I}_T$.)

Alternatively, Z_t can be interpreted as time-t price of a general European-style derivative security with a path-dependent payoff $Z(I_T)$.

Probability distributions. Let $F_t(I_T) = F(I_T|I_t)$ denote the *objective* probability of the elementary state I_T conditional on the market information at time-t. Similarly, $G_t(I_T)$ and $H_t(I_T)$ are the conditional *subjective* and *risk-neutral* probabilities, respectively. Expectations with respect to the three probability measures are denoted as $E[\cdot]$, $E^S[\cdot]$, and $E^N[\cdot]$.

The objective probabilities $F_t(I_T)$ reflect the *true* (or *physical*, or *large-sample*) frequencies with which elementary states I_T occur. The subjective probabilities $G_t(I_T)$ represent the investors' *beliefs* regarding the distribution of I_T. In general, investors may not know the true frequencies of various outcomes. Therefore, we allow for the possibility that $G_t(I_T) \neq F_t(I_T)$.

---

[4]Formally, suppose that there are n primary assets and let $d_t = (d_t^1, \ldots, d_t^n)$ and $p_t = (p_t^1, \ldots, p_t^n)$ denote their dividends and (ex dividend) prices at time-t. One of the assets may represent a risk-free bond. A self-financing trading strategy (dynamic portfolio) is a nonanticipating process $\theta_t = (\theta_t^1, \ldots, \theta_t^n)$, where θ_t^i represents the number of shares of asset i held at time-t, such that $\theta_{t-1} \cdot (p_t + d_t) = \theta_t \cdot p_t$ for all $t \geq 1$. The value process of the strategy is defined as $Z_t = \theta_t \cdot p_t$.

The risk-neutral probabilities are also known as the *state prices* or *Arrow-Debreu prices*. Intuitively, the risk-neutral probability $H_t(I_T)$ is equal to the price of a security that pays one dollar in state I_T and zero in all other states. The risk-neutral probabilities allow securities prices to be expressed as expected values of their payoffs. The price of a security with a payoff $Z(I_T)$ is

$$Z_t = E^N[Z(I_T) \,|\, I_t] = \sum_{I_T} Z(I_T) \, H_t(I_T).$$

As well known, the set of risk-neutral probabilities always exists if the market is arbitrage-free (Harrison and Kreps (1979)). This set is unique if the market is complete (Harrison and Pliska (1981)). However, we do not insist on market completeness.

Besides distributions of *elementary* states, we will also need distributions of *final* states. Therefore, let $f_t(\xi_T)$, $g_t(\xi_T)$, and $h_t(\xi_T)$ denote the conditional objective, subjective, and risk-neutral probabilities of the final state ξ_T. In the continuous-state case, we will refer to $f_t(\cdot)$, $g_t(\cdot)$, and $h_t(\cdot)$ as the objective, subjective and risk-neutral densities.

EMH versus ELM. In what follows, we distinguish between two cases of investors' rationality. The first one is the *Efficient Market Hypothesis* (EMH), which is defined by the following two conditions:

(i) **Rational learning**. This means that, when new information arrives to the market, investors update their beliefs by applying the rules of conditional probability, i.e., Bayes' law. Intuitively, this condition states that investors beliefs change over time unpredictably:

$$G_t(I_T) = E^S[G_s(I_T) \,|\, I_t], \qquad t \leq s \leq T. \tag{3}$$

(ii) **Correct beliefs**. This means that, when making investment decisions, investors weigh potential future outcomes using frequencies with which these outcomes will actually occur. In other words, this condition states that the objective and subjective probabilities coincide:

$$F_t(I_T) = G_t(I_T), \quad \text{and} \quad E[\cdot \,|\, I_t] = E^S[\cdot \,|\, I_t].$$

EMH is a special case of the *Efficiently Learning Market* (ELM), considered in Bossaerts (1999a).[5] He argues that of the two conditions underlying EMH, it is the condition of rational learning that reflects the essence of rationality. In contrast, the possibility of biased expectations is not an indication of irrationality. In ELM, Bossaerts maintains (i), but relaxes (ii).

---

[5]A remark on terminology. The classical definition of EMH in Fama (1970) does not explicitly identify conditions (i) and (ii). Still, most financial economists interpret EMH as meaning the *Rational Expectations Equilibrium*.

Pricing kernel. The risk-neutral probability $H_t(I_T)$ is related to the subjective probability $G_t(I_T)$ via the *pricing kernel*.[6] The pricing kernel is a strictly positive random variable $m_T > 0$ such that:

$$H_t(I_T) = \frac{m_T}{m_t} G_t(I_T), \qquad \text{where} \qquad m_t := E^S[m_T \,|\, I_t].$$

In general, the pricing kernel is a function of the elementary state, or $m_T = m(I_T)$. In this paper, however, we focus on the important class of economies for which the pricing kernel depends on the final state ξ_T but not the complete history I_T. Therefore, we impose

Assumption 1 *The pricing kernel is path-independent, or $m_T = m(\xi_T)$.*

Path-independence of the pricing kernel is the main economic assumption of our analysis. We discuss this assumption in detail in Section 4.1 and argue that it is satisfied by many important asset pricing models, including CAPM, the consumption-based models, the multifactor pricing models, the Epstein-Zin-Weil model, the Black-Scholes model, and others.[7]

A simple setting where Assumption 1 holds is the following. Consider a pure-endowment economy with a single risky asset (the market portfolio). The asset's price v_t follows an exogenous process. Also traded is the risk-free bond, which is in zero net supply. The representative investor maximizes the expected value of a von Neumann-Morgenstern utility function $U(v_T)$. The utility function U is twice continuously differentiable with $U' > 0$ and $U'' \leq 0$. In this case, the pricing kernel $m_T = U'(v_T)$ depends on the final state $\xi_T = v_T$, but not on the complete history $I_T = (v_0, \ldots, v_T)$. We will often use this simple setting for illustration purposes. However, it should be reiterated that Assumption 1 is satisfied in much more general equilibrium frameworks.

In view of Assumption 1, we can write

$$h_t(\xi_T) = \frac{m(\xi_T) g_t(\xi_T)}{m_t}, \qquad \text{where} \qquad m_t = \sum_{\xi_T} m(\xi_T)\, g_t(\xi_T). \tag{4}$$

The pricing kernel $m(\xi_T)$ can be interpreted as the state price per unit probability for a payoff in the state ξ_T. When investors are risk-neutral, $m(\xi_T) \equiv 1$. When they are risk-averse, $m(\xi_T) \not\equiv 1$ reflecting the fact that investors value differently gains in "good" and "bad" states. For example, one dollar when the aggregate wealth is low may be worth to investors more than when the aggregate wealth is high.

---

[6]In the consumption-based models, m_T/m_t is also referred to as the *intertemporal marginal rate of substitution* (IMRS).

[7]In Assumption 1, path-independence is stated with respect to the final date T. If in addition to this assumption it is also assumed that the state ξ_t follows a Markov process (that is, $\text{Prob}(I_T \,|\, I_t) = \text{Prob}(I_T \,|\, \xi_t)$ for all I_T, I_t, ξ_t), then the pricing kernel is path-independent for *all* $t \leq T$, or $m_t = m(\xi_t, t)$. Section 4.1 demonstrates that many pricing models have this property.

2.2   Case of EMH

We first consider the case of EMH when the objective and subjective probabilities coincide. Then in Section 2.3, we allow investors to have subjective beliefs which differ from the objective probabilities.

Under EMH, securities prices satisfy the standard restriction that

$$E[m_s Z_s \,|\, I_t] = m_t Z_t, \quad t < s \leq T. \tag{5}$$

To test this restriction, one needs to know the pricing kernel. In reality, however, the pricing kernel is unobservable and, as a result, tests of EMH based on (5) suffer from a joint hypothesis problem. Rejections may emerge because the market is truly inefficient or because an incorrect pricing kernel has been assumed.

One of the objectives of this paper is to derive a restriction on securities prices which is independent of the pricing kernel. Central to our approach is the idea of statistical arbitrage.

2.2.1   Statistical Arbitrage

A *statistical* arbitrage opportunity (SAO) is a generalization of a *pure* arbitrage opportunity (PAO). A PAO is a zero-cost trading strategy that offers the positive expected payoff with no possibility of a loss.

Definition 1 *A zero-cost trading strategy with a payoff $Z_T = Z(I_T)$ is called a* pure arbitrage opportunity *if*

   i) $E[Z_T \,|\, I_0] > 0$, and

   ii) $Z_T \geq 0$, for all I_T.

From Harrison and Kreps (1979), a pricing kernel $m(I_T) > 0$ exists *if* and *only if* there are no PAOs. This result is often referred to as the First fundamental theorem of asset pricing.

A SAO is a zero-cost trading strategy for which the expected payoff is positive and the conditional expected payoff in each final state ξ_T is nonnegative. Formally, let $I_t^{\xi_T} := (I_t; \xi_T) = (\xi_1, \ldots, \xi_t; \xi_T)$ denote the *augmented* information set which in addition to the market information at time-t also includes the knowledge of the final state of the economy.

Definition 2 *A zero-cost trading strategy with a payoff $Z_T = Z(I_T)$ is called a* statistical arbitrage opportunity *if*

   i) $E[Z_T \,|\, I_0] > 0$, and

   ii) $E[Z_T \,|\, I_0^{\xi_T}] \geq 0$, for all ξ_T.

Unlike a PAO, a SAO can have negative payoffs in some elementary states I_T, as long as the average payoff for each ξ_T is nonnegative. Implicit in the definition of a SAO is the assumption that there are many different histories I_T corresponding to a given final state ξ_T, meaning that a path-dependent strategy may have uncertain payoffs in ξ_T. It is clear that any PAO is a SAO but the reverse is not true.

The concept of a SAO is useful because of the following duality result.

Proposition 1 *A pricing kernel $m(\xi_T) > 0$ exists* if *and* only if *there are no statistical arbitrage opportunities.*

Proposition 1 differs from the First fundamental theorem in that the pricing kernel is restricted to be path-independent. Path-independence implies that not only PAOs but also more general SAOs cannot exist. In the following subsection, we will show that the absence of SAOs has important pricing implications. In particular, we will derive a new restriction on the *dynamics* of securities prices. Before that, however, we want to point out that the concept of a SAO could be useful in a *static* environment as well.

Consider, for example, Hakansson (1978). He studies a single period model with several assets when investors' preferences depend on the market return but not returns of individual assets. In such a model, he shows that state securities on the market portfolio are the only securities needed to achieve the full allocational efficiency. That is, a complete market would lead to exactly the same allocations and prices.

Suppose now that ξ_T denotes the market return and that I_T is a vector of individual returns. In other words, ξ_T represents a "metastate" which includes many elementary states I_T. Then a SAO can be defined as a zero-cost portfolio for which the expected payoff $E[Z_T \,|\, I_0]$ is positive and the expected payoff in each metastate $E[Z_T \,|\, I_0^{\xi_T}]$ is nonnegative. Since preferences only depend on ξ_T, Proposition 1 implies that no SAOs can exist. This confirms the general intuition of Hakansson: conditional on the market return, uncertainty about distribution of individual returns should not be priced.

2.2.2 Preference-Independent Restriction

We first present intuition and then derive the result formally. Consider three dates $t = 0$, 1, and 2. For fixed x, let δ^x denote the Arrow-Debreu security which at $t = 2$ pays \$1 if the final state of the world is $\xi_2 = x$ and zero otherwise. The security's price at time-t is equal to the risk-neutral probability $h_t(x)$. Consider two strategies that both invest one dollar in δ^x: the first strategy buys $\frac{1}{h_0(x)}$ shares at $t = 0$, while the second strategy buys $\frac{1}{h_1(x)}$ shares at $t = 1$. The payoffs of the strategies are:

$$Z^I = \begin{cases} \frac{1}{h_0(x)}, & \xi_2 = x \\ 0, & \xi_2 \neq x, \end{cases} \qquad Z^{II} = \begin{cases} \frac{1}{h_1(x)}, & \xi_2 = x \\ 0, & \xi_2 \neq x. \end{cases}$$

The strategies have the same zero payoff in all states $\xi_2 \neq x$. This means that they must also pay the same expected payoff conditional on $\xi_2 = x$. Because, if not, either $(Z^I - Z^{II})$ or $(Z^{II} - Z^I)$ will be a SAO. Therefore,

$$E[Z^I \,|\, I_0^x] = E[Z^{II} \,|\, I_0^x],$$

9

or
$$E\left[\frac{1}{h_1(x)} \,|\, I_0^x\right] = \frac{1}{h_0(x)}.$$

The last equation states that if we consider an Arrow-Debreu security which eventually matures in-the-money, then the inverse of its price follows a martingale process. The result must hold for *all* pricing kernels $m(\xi_2) > 0$.

To generalize the above argument, we consider another Arrow-Debreu security δ^y, which pays \$1 in state $\xi_2 = y$ and zero in other states. As before, two strategies are compared. The first strategy at $t = 0$ sells one share of δ^y and uses the proceeds to purchase shares of δ^x. The second strategy is similar to the first one, except the trading takes place at the intermediate period $t = 1$. The payoffs of the strategies are

$$Z^I = \begin{cases} \frac{h_0(y)}{h_0(x)}, & \xi_2 = x \\ -1, & \xi_2 = y \\ 0, & \xi_2 \neq x, y, \end{cases} \qquad Z^{II} = \begin{cases} \frac{h_1(y)}{h_1(x)}, & \xi_2 = x \\ -1, & \xi_2 = y \\ 0, & \xi_2 \neq x, y. \end{cases}$$

Again, an important feature of the two strategies is that their payoffs are the same for all states $\xi_2 \neq x$. Therefore, there will be no SAO only if both strategies have the same expected payoff conditional on $\xi_2 = x$, implying that

$$E\left[\frac{h_1(y)}{h_1(x)} \,|\, I_0^x\right] = \frac{h_0(y)}{h_0(x)}.$$

This restriction states that the change in the ratio of the Arrow-Debreu prices $\frac{h_1(y)}{h_1(x)}$ is unpredictable in price histories for which the final state is $\xi_2 = x$.

A similar argument can be used to show that for a general security with a payoff $Z(I_2)$ at $t = 2$:

$$E\left[\frac{Z_1}{h_1(x)} \,|\, I_0^x\right] = \frac{Z_0}{h_0(x)}.$$

We are now ready to state the result formally. Let $x \in \Xi_T$ denote a possible final state and let $T' < T$. We assume that the following condition is satisfied.

Assumption 2 *For all histories $I_{T'}$, the risk-neutral probability $h_{T'}(x) > 0$.*

Proposition 2 *Suppose that EMH and Assumptions 1-2 hold. Then securities prices deflated by the risk-neutral probability of the final state $h_t(x)$ are martingale processes under the objective probability measure and with respect to the augmented information set I_t^x. That is,*

$$E\left[\frac{Z_s}{h_s(x)} \,|\, I_t^x\right] = \frac{Z_t}{h_t(x)}, \qquad t < s \leq T'. \tag{6}$$

Assumption 2 is a technical assumption, which ensures that the ratio inside the conditional expectation operator in (6) is always well-defined. Its purpose is to preclude situations when at some point $s \leq T'$ investors learn that state x cannot possibly happen and thus $h_s(x) = 0$. In the previous analysis of the two trading strategies, Assumption 2 guarantees that the price of the Arrow-Debreu security δ^x is positive for all intermediate states ξ_1 so that the second strategy can always be implemented.[8] Later on, we will illustrate the role of Assumption 2 in the context of specific examples.

Proposition 2 says that, if of many repetitions of the same environment, an empiricist selects histories that result in the same final state $\xi_T = x$, then in those histories the ratio $Z_t/h_t(x)$ must change over time unpredictably. The unusual feature of the new restriction in (6) is that it involves conditioning on future information. The empiricist cannot test the restriction in "real-time" – she must wait until the final state is revealed. Note also that the restriction assumes that the risk-neutral probability $h_t(x)$ is available to the empiricist. Despite the fact that the risk-neutral probabilities are not directly observable in financial markets, they are implicit in prices of derivative securities. In Section 4.2, we will discuss how the risk-neutral probabilities can be estimated from prices of traded call options with different strikes.

As mentioned earlier, an important property of the new restriction is that it makes no reference to the pricing kernel. This means that one can be completely agnostic about the true equilibrium model and still be able to test EMH provided that Assumptions 1-2 are satisfied.

Another important property is that the restriction in (6) can be used in samples which come with various selection biases. Specifically, let $A \subset \Xi_T$ denote a subset of final states. Suppose that the empiricist has a sample of histories I_T^j ($j = 1, \ldots, J$) with final states $x^j \in A$. By Proposition 2, the quantity

$$\frac{1}{J} \sum_{j=1}^{J} \left(\frac{Z_s^j}{h_s^j(x^j)} - \frac{Z_t^j}{h_t^j(x^j)} \right)$$

must be insignificantly different from zero for all $t < s \leq T'$. Note that this must be the case for *all* subsets A, because the expectation in (6) is *conditioned* on the final state.[9] For example, if ξ_T represents the market return, then A can be a subset of returns that fall within specific interval $[r_l, r_h]$. This is useful for the case of the Peso problem. The Peso problem refers to a situation when rare but influential outcomes (say, very extreme returns) have not happened in the sample.[10]

Consider the following illustration of this problem. Suppose that every year there is a 5% chance of a market crash. (Let say, a crash is defined as a 20% one-day decline.) Investors know the probability of a crash and correctly incorporate it in securities prices. If crashes are independent, then on average one crash occurs every twenty years. Suppose

---

[8]We state Assumption 2 slightly differently from the "No Early Exclusion" Assumption in Bossaerts (1999a, 1999b), but the two serve the exactly same purpose.

[9]We implicitly assume that Assumption 2 is satisfied for all $x \in A$.

[10]The Peso problem is analyzed in, for example, Bekaert, Hodrick, and Marshall (1995). Other biases may result from backfilling, censoring, and survivorship. See Brown, Goetzmann, and Ross (1995).

now that the empiricist studies a random 20-year sample of data. Then there is a 36% chance that the sample will include no crash. For such a sample, the empiricist will conclude that investors are too pessimistic (because *ex post* realized returns are too high). On the other hand, with a probability of 26%, the sample may include two or more crashes. In this case, investors will appear to the empiricist as too optimistic. In both cases, the EMH restriction in (5) will be rejected, even if the true pricing kernel were known to the empiricist.

Clearly, the Peso problem is merely a small sample problem. Unfortunately, it may require unrealistically long data series to overcome this problem using the traditional approaches (and assuming that the data generating process is stationary – which may be objectionable as well). However, with the help of Proposition 2, even short series can be analyzed because the Peso problem does not affect the restriction in (6).

In the next subsection, we argue that the new restriction satisfies yet another important property: it continues to hold even when investors' beliefs are mistaken.

2.3   Extension to ELM

Suppose that investors' beliefs may be different from objective probabilities, $F_t(I_T) \neq G_t(I_T)$. In this more general setting, we can still follow the approach of Section 2.2. The notions of a PAO and a SAO, however, must now be redefined in terms of subjective expectations $E^S[\cdot]$ instead of objective expectations $E[\cdot]$. The restriction in (6) becomes

$$E^S \left[\frac{Z_s}{h_s(x)} \mid I_t^x \right] = \frac{Z_t}{h_t(x)}, \qquad t < s \leq T'. \tag{7}$$

The problem with the restriction in (7) is that it involves expectation under the subjective probabilities and, therefore, is of little use to an empiricist who observes securities prices sampled with the objective frequencies. It turns out, however, that there is a broad class of economies for which it is possible to "remove" the superscript from the expectation operator $E^S[\cdot]$ in (7). This class of economies has been introduced in Bossaerts (1999a).

As mentioned earlier, Bossaerts extends EMH to a more general ELM, where investors are rational but may have potentially biased beliefs. He demonstrates that, in ELM under *risk-neutrality* and the assumption of *correct conditional beliefs* (to be formulated shortly), securities prices satisfy a set of novel restrictions.[11]

By following his approach, we can extend Proposition 2 to the case of ELM. Bossaerts restricts the set of possible beliefs in the following way. Suppose that beliefs are partitioned into two components: initial beliefs (priors) and beliefs conditional on the final state ξ_T (likelihood functions). Then he assumes that initial beliefs can be arbitrary but conditional beliefs must be correct. Formally, let $\lambda_t(\xi_{t+1}; \xi_T)$ denote the objective transition probability of state ξ_{t+1} conditional on the history I_t and the final state ξ_T. That is,

$$\lambda_t(\xi_{t+1}; \xi_T) = \text{Prob}(\xi_{t+1} \mid I_t^{\xi_T}).$$

---

[11] For empirical applications based on his theory, see Bondarenko (1997), Bondarenko and Bossaerts (2000), Bossaerts (1999a, 1999b), Bossaerts and Hillion (1998).

The subjective transition probability $\lambda_t^S(\xi_{t+1}; \xi_T)$ is defined in a similar fashion.

Assumption 3 *Investors' conditional beliefs are correct. That is,*

$$\lambda_t(\xi_{t+1}; \xi_T) = \lambda_t^S(\xi_{t+1}; \xi_T), \qquad \text{for all } \xi_{t+1}, \xi_T, \text{ and } I_t.$$

The economic interpretation for Assumption 3 is that intermediate states ξ_t $(t < T)$ are "signals" about the final state ξ_T and that, even though investors may not know the correct distribution of final states, they do understand how the signals are generated for each realization ξ_T.

Assumption 3 is not completely innocuous. It puts a fair amount of structure on possible beliefs. Specifically, it assigns all "incorrectness" of beliefs to priors; conditional beliefs are restricted to be correct. It turns out, however, that many interesting applications exist where it can be natural to partition beliefs into biased priors and correct conditional beliefs. Two examples are provided in Sections 2.5 and 3.3. One should also keep in mind that Assumption 3 is an integral part of the standard EMH. However, EMH requires priors to be correct as well.

Proposition 3 *Suppose that ELM and Assumptions 1-3 hold. Then securities prices deflated by the risk-neutral probability of the realized final state $h_t(x)$ are martingale processes under the objective probability measure and with respect to the augmented information set I_t^x. That is,*

$$E\left[\frac{Z_s}{h_s(x)} \,\Big|\, I_t^x\right] = \frac{Z_t}{h_t(x)}, \qquad t < s \le T'. \tag{8}$$

The only difference between Proposition 2 and Proposition 3 is that in the latter EMH is replaced with more general ELM. Proposition 3 states that biases in initial beliefs do not affect the restriction in (6) provided that the assumption of correct conditional beliefs is satisfied. Clearly, Proposition 2 is a special case of Proposition 3 since under EMH Assumption 3 is satisfied trivially.

Proposition 3 extends the original results obtained in Bossaerts (1999a, 1999b) in two ways. First, for the risk-neutral case, the restriction in (8) is slightly more general than the corresponding restrictions in Bossaerts (1999a, 1999b). In the first paper, Bossaerts considers a security whose liquidation value can take only two values, $v_T \in \{0, v\}$. The security's price is thus $v_t = v\, h_t(v)$. He shows that the *inverse* of the price of the security that matures in-the-money (i.e., for which $v_T = v$) is a martingale:

$$E\left[\frac{1}{v_s} \,\Big|\, I_t^v\right] = \frac{1}{v_t}.$$

To derive this result from Proposition 3, consider a risk-free bond with payoff $Z(v_T) = 1$. Applying the restriction in (8) to the bond's price $Z_t = 1$, we obtain

$$E\left[\frac{1}{h_s(v)} \,\Big|\, I_t^v\right] = \frac{1}{h_t(v)}.$$

In the second paper, Bossaerts considers a general asset whose liquidation value v_T is a continuous random variable. The restriction on the asset's price v_t follows from Proposition 3 by taking the final payoff $Z(v_T) = v_T$:

$$E\left[\frac{v_s}{h_s(v)} \,\middle|\, I_t^v\right] = \frac{v_t}{h_t(v)}, \tag{9}$$

where v is the final value of the asset.

Proposition 3 proves the martingale property not only for the price of the *underlying* asset v_t as in (9) but for the price of a general *derivative* security Z_t. Moreover, Proposition 3 restricts the dynamics of the whole risk-neutral distribution, not just its mean as is the case with (9). Specifically, the restriction in (8) implies that

$$E\left[\frac{h_s(\xi_T)}{h_s(x)} \,\middle|\, I_t^x\right] = \frac{h_t(\xi_T)}{h_t(x)}, \qquad \text{for all } \xi_T. \tag{10}$$

Second and more significantly, Proposition 3 extends the results in Bossaerts (1999a, 1999b) to the risk-averse case.[12] With the help of Proposition 3, one can now test the rationality of asset pricing without worrying that investors' preferences or beliefs are misspecified.

An interesting implication of Proposition 3 is that risk-aversion and biases in beliefs appear to have similar effects on securities prices. In Section 3 we will confirm this intuition. We will show that there exists an "equivalence" relationship between the two, in the sense that same equilibrium prices can be explained by either risk-aversion or biased beliefs.

We conclude this subsection with a comment on why extending EMH to ELM is important. One situation when beliefs may deviate from the objective frequencies occurs when investors deal with unique and nonrepetitive events. The Gulf war, the Asian crisis, the Russian debt default, the Long-Term Capital Management's meltdown are examples of events which had dramatic effects on the world's financial markets and which were very difficult to anticipate in advance. Under EMH, however, market participants are assumed to correctly predict the frequencies of all such historic events, which is arguably unrealistic.

EMH is often defended on the ground that, even if beliefs are *sometimes* biased, it is sufficient if beliefs are correct *on average*. This is not true. One can construct an example of a risk-neutral economy where 1) half of the time investors are too optimistic about assets' returns, 2) half of the time they are too pessimistic, 3) investors' expectations are correct on average, and 4) returns are predictable, in contradiction to EMH. It is not enough that expectations are correct most of the time, or on average – EMH requires expectations to be correct at *all* times. In many cases, this is a strong requirement.

---

[12] As mentioned earlier, Bossaerts (1999a, 1999b) proposes results for the risk-averse case as well. In those, the pricing kernel is assumed to be known. To obtain a martingale restriction, securities prices are first risk-adjusted using the pricing kernel and then deflated by the subjective probability $g_t(x)$. Proposition 3 shows that if $g_t(x)$ is replaced with $h_t(x)$, then any reference to the pricing kernel disappears. An added advantage of the restriction based on the *risk-neutral* probabilities, as opposed to the *subjective* ones, is that the former can be derived from prices of traded securities. In contrast, beliefs are difficult to observe or estimate.

2.4 Two Rationality Restrictions

As follows from the previous subsections, there are two alternative approaches for testing rationality of asset pricing. The first one is based on the standard EMH restriction in (5). However, an empiricist can follow this approach only if 1) the true pricing kernel is known, 2) the sample is free of selection biases, and 3) investors' beliefs are correct.

The second approach is based on the new martingale restriction in (8). In this approach, the empiricist does not have to make parametric assumptions about the pricing kernel or investors' preferences; the sample can be affected by the Peso problem and other selection biases; and investors' beliefs can be incorrect. In fact, preferences and beliefs may even change from one history to another. The second approach works if Assumptions 1-3 are satisfied (recall that Assumption 3 is required for the first approach as well) and if the risk-neutral probabilities $h_t(\xi_T)$ can be estimated from prices of traded securities.

The two rationality restrictions will be violated when investors are irrational, that is, when the condition in (3) does not hold.[13] Recently, it has become fashionable to explain anomalies of EMH through suboptimal forms of learning motivated by the psychology literature. While there is only one way to learn optimally, there are many possible rules of inefficient learning, corresponding to various behavior biases (such as overconfidence, representativeness, conservatism, self-attribution, disposition, limited memory, and framing). In Section 3.3, we will consider an example where investors overreact or underreact to news. By imposing additional structure on the state variable, we will argue that, when investors overreact (underreact), the following *supermartingale* (*submartingale*) restriction holds

$$E\left[\frac{Z_s^p}{h_s(x)} \mid I_t^x\right] \quad < (>) \quad \frac{Z_t^p}{h_t(x)},$$

where $Z_t^p > 0$ denotes the price of a security with a *nonnegative* final payoff $Z_T^p \geq 0$. The restriction is independent of the pricing kernel and obtains under ELM and Assumptions 1-3.

2.5 Binomial Tree Example

In this subsection, we illustrate Propositions 3 with the help of a simple example. The example is the standard binomial tree model of Cox, Ross, and Rubinstein (1979). Consider a pure-endowment economy with a single risky asset. There are three dates $t=0$, 1, and 2. The asset's price v_t follows a binomial process shown in Figure 1. The initial price is $v_0 = 1$ and in each period the price either doubles or halves. Each period,

---

[13]There is another distinction between the two approaches. Under EMH, the existence of a positive pricing kernel m_T that satisfies the standard restriction in (5) is only *necessary* for the condition in (3). This is because, in an incomplete market, a particular choice of m_T may differ from the *true* pricing kernel and be disconnected from marginal rates of substitution in the economy. In contrast, one can show that, if ELM and Assumptions 1-3 hold, then the new restriction in (8) is both *necessary* and *sufficient* for (3).

the objective, subjective, and risk-neutral probabilities of an "up" movement are p, q, and r, respectively. The asset pays no dividends. Also traded is the risk-free bond. The bond is in zero net supply, and the risk-free rate is zero.
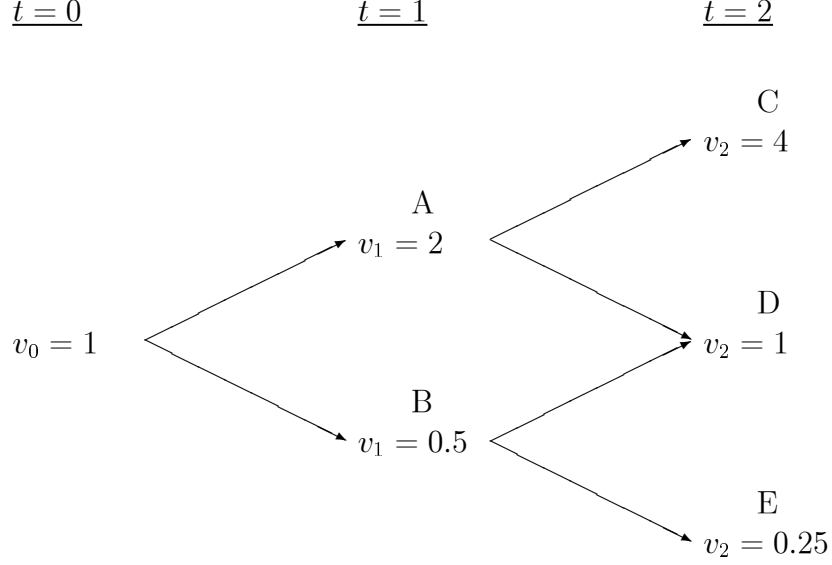

Figure 1: The dynamics of the asset's price v_t in two-step binomial tree.

Given the asset's prices, the risk-neutral probability of an "up" movement is $r = \frac{1}{3}$. Suppose that investors preferences are given by the utility function $U(v) = -\frac{1}{v}$. Then the subjective probability of an "up" movement is $q = \frac{2}{3}$.[14] We first consider the case of EMH and then the more general case of ELM.

2.5.1 Correct beliefs

Under EMH, beliefs are correct and $p = q = \frac{2}{3}$. Let $\delta = \delta^D$ denote an Arrow-Debreu security which at time-2 pays \$1 in state D and nothing in states C and E. The price of the Arrow-Debreu security is

$$\text{at } t = 0: \quad \delta_0 = \frac{4}{9},$$

---

[14]Consider the *Constant Relative Risk Aversion* (CRRA) utility function $U(v) = \frac{v^{1-\gamma}}{1-\gamma}$ with $0 < \gamma \neq 1$. Then the objective probability of the "up" movement satisfies

$$q\, 2^{1-\gamma} + (1-q)\, 2^{\gamma-1} = 1.$$

When $\gamma = 2$, this implies that $q = \frac{2}{3}$.

16

$$\text{at } t = 1: \quad \delta_1 = \begin{cases} \frac{2}{3}, & \text{in state A,} \\ \frac{1}{3}, & \text{in state B.} \end{cases}$$

Consider now two trading strategies which both start with initial endowment of \$1 and invest in the Arrow-Debreu security δ. In the first strategy, we purchase the Arrow-Debreu security at $t = 0$, resulting in total of $n_1 = \frac{9}{4}$ shares of δ. In the second strategy, we wait one period and purchase the Arrow-Debreu security at $t = 1$. In this case, the number of shares of δ depends on the state at $t = 1$: with probability $p = \frac{2}{3}$ the state is A and we buy $n_2 = \frac{3}{2}$ shares; with probability $1 - p = \frac{1}{3}$ the state is B and we buy $n_2 = 3$ shares. Note that the expected number of shares

$$E_0\left[n_2\right] = \text{Prob(A)} \cdot \frac{3}{2} + \text{Prob(B)} \cdot 3 = \frac{2}{3} \cdot \frac{3}{2} + \frac{1}{3} \cdot 3 = 2 < n_1,$$

suggesting that at $t = 1$ the second strategy on average results in a greater number of shares of δ than the first strategy does. However, when comparing the two strategies, the *unconditional* average number of shares is irrelevant. Instead, what *is* relevant is the average number of shares *provided* that state D is eventually realized:

$$E_0\left[n_2 \,|\, \text{D}\right] = \text{Prob(A}\,|\,\text{D)} \cdot \frac{3}{2} + \text{Prob(B}\,|\,\text{D)} \cdot 3 = \frac{1}{2} \cdot \frac{3}{2} + \frac{1}{2} \cdot 3 = \frac{9}{4} = n_1,$$

implying that

$$E_0\left[\frac{1}{\delta_1} \,\Big|\, \text{D}\right] = \frac{1}{\delta_0}. \tag{11}$$

In fact, one can check that for any security with a payoff $Z(v_2)$ at time $t = 2$:

$$E_0\left[\frac{Z_1}{\delta_1} \,\Big|\, \text{D}\right] = \frac{Z_0}{\delta_0}, \tag{12}$$

in agreement with Propositions 3. In particular, for the asset's price v_t:

$$E_0\left[\frac{v_1}{\delta_1} \,\Big|\, \text{D}\right] = \frac{1}{2} \cdot \frac{2}{\frac{2}{3}} + \frac{1}{2} \cdot \frac{\frac{1}{2}}{\frac{1}{3}} = \frac{9}{4} = \frac{v_0}{\delta_0}.$$

Remark 1. In the above example, conditions (11)-(12) will continue to hold for *any* utility function $U(v)$. This is because the choice of the utility function only affects the objective probability $p = q$. The choice does not affect the risk-neutral probability r or securities prices. Furthermore, it is a property of the binomial tree model that frequencies of states A and B conditional on state D being realized are *independent* of p. In particular,

$$\text{Prob(A}\,|\,\text{D)} = \text{Prob(B}\,|\,\text{D)} = \frac{1}{2}. \tag{13}$$

In other words, one can use the restriction in Propositions 3 to test whether securities prices are set under EMH without specifying the utility function. In contrast, the

traditional EMH restriction in (5) does require the knowledge of the objective probability p. For example, the risk-premium for the asset's return is

$$E_0 \left[\frac{v_1 - v_0}{v_0} \right] = \frac{1}{2}(3p - 1).$$

The risk-premium depends on p and thus on the assumed utility function. Only when investors are risk-neutral ($p = q = r = \frac{1}{3}$) will the risk-premiums for all securities be equal to zero.

Remark 2. The binominal tree example allows us to discuss the role of Assumption 2. It is easy to see that Assumption 2 is violated for state C (as well as for state E): if at $t = 1$ the "down" node B is reached, then state C can never occur, contradicting the assumption. In this case, the intuitive approach used in Section 2.2.2 to derive the new restriction breaks down. To be more specific, let δ^C denote the Arrow-Debreu security that pays \$1 in state C. As before, we consider two strategies which invest in δ^C, one at $t = 0$ and the other at $t = 1$. This time, however, we are unable to ensure that the two strategies have the same zero payoff in states D and E: when node B is reached, the second strategy will pay a positive amount either in state D or state E, or both. This means we no longer can rank the two strategies independently of the pricing kernel, and Propositions 3 does not obtain.

Consequently, if an empiricist observes historical repetitions of the same binomial tree environment and wishes to test Propositions 3, then she will not be able to use all collected price histories. She will have to select only those histories for which the final state is D. As for histories with the final state C or E, they are unusable for the purpose of testing the restriction of Propositions 3. Fortunately, Assumption 2 is rarely violated when more realistic models are considered.

Remark 3. It may be instructive to contrast various extensions of arbitrage proposed in the literature. Extensions of arbitrage are useful for deriving sharper pricing implications when markets are *incomplete.* In incomplete markets, securities prices cannot be uniquely determined by replication. By ruling out certain investment opportunities in addition to PAOs, one can tighten bounds on possible values of securities prices.

In the two-step binomial tree, the set of elementary states $\mathcal{I}_T$ consists of four elements corresponding to price histories AC, AD, BD, and BE. Therefore, any payoff at $t = 2$ can be represented by a vector (z_1, z_2, z_3, z_4) of payoffs in these elementary states.

Consider now the following three payoffs:

$$\begin{aligned}
Z^{(1)} &= (0, 0, 0, 1), \\
Z^{(2)} &= (0, 1 + \varepsilon, -1, 0), \\
Z^{(3)} &= (0, 1, 0, -\varepsilon),
\end{aligned}$$

where ε denotes an arbitrary small positive number. When the market is complete, the

values of these payoffs at $t = 0$ are uniquely determined via dynamic replication as:

$$\frac{4}{9}, \qquad \frac{2\varepsilon}{9}, \qquad \frac{2 - 4\varepsilon}{9}.$$

Suppose, however, that the asset can no longer be traded at $t = 1$. In this case, the market is incomplete and many pricing kernels can exist. Different pricing kernels will assign different time-0 values to $Z^{(1)}$, $Z^{(2)}$, and $Z^{(3)}$. In what follows, we assume that it costs nothing to form the three payoffs at $t = 0$ and then check whether this assumption is consistent with the absence of arbitrage and some of its extensions.

Of the three zero-cost investment opportunities, only $Z^{(1)}$ is a PAO. The other two may result in a loss in some elementary states and, therefore, they are not inconsistent with the absence of pure arbitrage. It is easy to check that $Z^{(2)}$ is a SAO, while $Z^{(3)}$ is not a SAO. (For $Z^{(2)}$, expected payoffs conditional on final states C, D, and E are nonnegative. In particular, conditional on the final state being D, histories AD and BD are equally likely and thus $E_0\left[Z^{(2)} \,|\, \mathrm{D}\right] = \varepsilon/2 > 0$.)

Bernardo and Ledoit (2000) introduces an extension of a PAO, called an *approximate arbitrage opportunity* (AAO). An AAO is a zero-cost investment which offers the gain-loss ratio above some prespecified level, where the gain (loss) is the expectation of the positive (negative) part of the payoff computed under the benchmark risk-neutral probabilities. Since PAOs are characterized by an infinite gain-loss ratio, the set of AAOs includes PAOs as a special case. We observe that investment opportunity $Z^{(3)}$ is an AAO, since its gain-loss ratio is high:

$$\frac{E_0^N\left[\max(Z^{(3)}, 0)\right]}{E_0^N\left[\max(-Z^{(3)}, 0)\right]} = \frac{1}{2\varepsilon}.$$

On the other hand, $Z^{(2)}$ is not an AAO, since its gain-loss ratio $(= 1 + \varepsilon)$ is not particularly large.

Cochrane and Saá-Requejo (2000) introduces the notion of "good deals" (GDs), or investment opportunities with high Sharpe ratios. They show that by ruling out GDs one can obtain tighter bounds on securities prices. One limitation of their approach is that not all PAOs qualify as good deals. To demonstrate this, we compare the Sharpe ratio of $Z^{(1)}$ with the Sharpe ratio offered by the asset itself. The latter is given by the Sharpe ratio of zero-cost investment opportunity with the following payoff at $t = 2$

$$Z^{(4)} \quad = \quad (3, 0, 0, -3/4).$$

Note that the payoff of $Z^{(4)}$ is equal to $(v_2 - v_0)$. Direct computations of the two Sharpe ratios yield:

$$\text{SR of } Z^{(1)} = \sqrt{\frac{1}{8}} = 0.35, \qquad \text{SR ratio of } Z^{(4)} = \sqrt{\frac{5}{8}} = 0.79.$$

The Sharpe ratio of $Z^{(1)}$ is low relative to that offered by the market. Thus, zero-cost investment opportunity $Z^{(1)}$ is not a GD, despite being a PAO. Even though $Z^{(1)}$ never results in a loss and is clearly an attractive investment opportunity, its payoff has a

high standard deviation, which implies a low Sharpe ratio. Intuitively, the Sharpe ratio penalizes variability of payoffs even when all payoffs are nonnegative. The following table summarizes the results for the three zero-cost investment opportunities:

	PAO	SAO	AAO	GD
$Z^{(1)}$	Yes	Yes	Yes	No
$Z^{(2)}$	No	Yes	No	No
$Z^{(3)}$	No	No	Yes	No

2.5.2 Biased beliefs

Consider now the case of ELM where the subjective probabilities are different from the objective ones. The price dynamics and utility function are kept as before. This means that the risk-neutral and subjective probabilities also do not change, $r = \frac{1}{3}$ and $q = \frac{2}{3}$. However, we are free to choose arbitrary objective probability p. For example, take $p = \frac{1}{2}$, the situation representing optimistic beliefs. It is easy to see that allowing for arbitrary p does not affect conditions (11)-(12). The reason for that has already been noted in Remark 1: frequencies of states A and B conditional on state D being realized are independent of p. See (13). This implies that Assumption 3 is *automatically* satisfied if investors know that the underlying model is the standard binomial tree (they are not required to know the *probabilities* of the tree).

To summarize, conditions (11)-(12) obtain in a rather general setting. They hold when investors have arbitrary preferences and beliefs. Preferences, beliefs, and the price dynamics can even change across historical repetitions of the same environment. All we need is that the utility function is path-independent and that investors know that the asset's price follows the binomial process.

3   General argument

In this section we state a general version of Proposition 3 for abstract probability spaces. This allows us to relax simplifying assumptions made in the previous section (such as finite number of states, discrete trading, constant risk-free rate, etc.) and to obtain additional theoretical insights. We also propose a more elegant proof of the main result based on the change of measure technique.

3.1   Continuous-Time Setting

Let P, Q, and R be three equivalent probability measures defined on a common measurable space $(\Omega, \mathcal{F}_T)$. The three measures should be interpreted as the objective, subjective, and risk-neutral probability measures, respectively. The information flow to the market is represented by a filtration $(\mathcal{F}_t, t \in \mathcal{T})$, where $\mathcal{T} = [0, T]$. Let $E_t^P[\cdot] := E^P[\cdot \mid \mathcal{F}_t]$, $E_t^Q[\cdot] := E^Q[\cdot \mid \mathcal{F}_t]$, and $E_t^R[\cdot] := E^R[\cdot \mid \mathcal{F}_t]$ denote time-t conditional expectations under the objective, subjective, and risk-neutral probability measures.

The state of the economy at time-T is represented by a $\mathcal{F}_T$-measurable random variable X. In Section 2 we have assumed that the pricing kernel is a function of the final state but not the whole history. To formulate a similar assumption for general probability measures, we use the following definition.

Definition 3 *Let P_1 and P_2 be two probability measures equivalent on $(\Omega, \mathcal{F}_T)$. We say that P_1 and P_2 are X-equivalent if the Radon-Nikodym derivative $\frac{dP_1}{dP_2}$ is $\sigma(X)$-measurable, where $\sigma(X) \subset \mathcal{F}_T$ is the σ-field generated by random variable X.*

Informally, Definition 3 states that the Radon-Nikodym derivative $\frac{dP_1}{dP_2}$ is a function of X. We now impose the following analogue of Assumption 1.

Assumption 4 *The subjective and risk-neutral probability measures Q and R are X-equivalent.*

Let x denote the "outcome" of random variable X. For fixed x, define the *augmented filtration* $(\mathcal{F}_t^x, t \in \mathcal{T})$, where $\mathcal{F}_t^x := \mathcal{F}_t \wedge \{X = x\}$. As before, the augmented filtration in addition to the market information also includes the knowledge of the final state. Note that $\mathcal{F}_t^x \subseteq \mathcal{F}_T$, for all $t \in \mathcal{T}$.

Let $T' < T$ and $\mathcal{T}' := [0, T']$. Consider a new probability measure R^x on the measurable space $(\Omega, \mathcal{F}_{T'})$ constructed from the measure R as follows.

$$R^x(A) := E^R[I_A \,|\, \mathcal{F}_{T'}^x], \qquad \text{all } A \in \mathcal{F}_{T'},$$

where I_A is the indicator function of the event A, defined for each $\omega \in \Omega$ by

$$I_A(\omega) := \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{if } \omega \notin A \end{cases}$$

Intuitively, the measure R^x is the risk-neutral probability measure *conditional* on $X = x$. The measure R^x is *absolutely continuous with respect to* the measure R on $(\Omega, \mathcal{F}_{T'})$. This simply means that, if $R(A) = 0$ for some event $A \in \mathcal{F}_{T'}$, then $R^x(A) = 0$. The following assumption requires that R is also absolutely continuous with respect to R^x, so that R and R^x are equivalent on $(\Omega, \mathcal{F}_{T'})$.

Assumption 5 *The probability measure R is absolutely continuous with respect to the probability measure R^x on $(\Omega, \mathcal{F}_{T'})$.*

Assumption 5 is a generalization of Assumption 2. Intuitively, it says that, at any time $t \leq T'$, it is impossible to observe an event that rules out the final outcome $X = x$. Specifically, for any event $A \in \mathcal{F}_{T'}$ such that $R(A) > 0$, it must be that $R^x(A) > 0$.

Since R and R^x are equivalent measures on $(\Omega, \mathcal{F}_{T'})$, the Radon-Nikodym derivative of R^x with respect to R is a strictly positive random variable. Let $\eta_{T'}$ denote this Radon-Nikodym derivative (which is $\mathcal{F}_{T'}$-measurable):

$$\eta_{T'} := \frac{dR^x}{dR}.^{15}$$

---

[15] As always, all relations involving random variables should actually be interpreted in an *almost sure* sense, but for convenience we suppress the label "a.s." everywhere.

Furthermore, consider any $\mathcal{F}_T$-measurable random variable Z_T and let $Z_t = E_t^R[Z_T]$ and $\eta_t = E_t^R[\eta_{T'}]$. Then the abstract version of Bayes' formula states that

$$E_t^{R^x}[Z_s] = \frac{E_t^R[\eta_s Z_s]}{\eta_t}, \qquad \text{for } t < s \le T'. \tag{14}$$

In a similar fashion, P^x and Q^x are defined as the objective and subjective probability measures *conditional* on $X = x$, respectively. Assumption 5 implies that probability measures P, Q, R, P^x, Q^x, and R^x are all equivalent on $(\Omega, \mathcal{F}_{T'})$.

In Section 2.3 we have allowed investors to have arbitrary priors about possible outcomes but have required conditional beliefs to be correct (Assumption 3). With the help of Definition 3, a general version of this assumption is stated as follows.

Assumption 6 *The objective and subjective probability measures P and Q are X-equivalent.*

It may be not immediately obvious why Assumption 6 for the continuous-time case is analogous to Assumption 3 for the discrete-time case. However, as established in the proof of Proposition 4, if P and Q are X-equivalent then $P^x = Q^x$ for all x. The latter can be interpreted as the condition that beliefs conditional on the eventual outcome are correct. Moreover, comparison with Assumption 4 reveals an interesting alternative interpretation of Assumption 6. The requirement of correct conditional beliefs is equivalent to the requirement that the *belief kernel* $\frac{dQ}{dP}$ is path-*independent*, where the term "belief kernel" has been introduced in Bossaerts (1999b).

Proposition 4 *Suppose that Assumptions 4-6 hold. Consider a random process $(Z_t, \mathcal{F}_t, t \in \mathcal{T}')$ which is R-martingale, that is,*

$$E_t^R[Z_s] = Z_t, \qquad t < s \le T'.$$

Then the process $(\frac{Z_t}{\eta_t}, \mathcal{F}_t, t \in \mathcal{T}')$ is P^x-martingale, that is,

$$E_t^{P^x}\left[\frac{Z_s}{\eta_s}\right] = \frac{Z_t}{\eta_t}, \qquad t < s \le T'. \tag{15}$$

The relation of Proposition 4 to Proposition 3 is clear. Here, Z_t represents the price of a general security. When investors are rational, the process $(Z_t, \mathcal{F}_t, t \in \mathcal{T}')$ is R-martingale. Then, by Proposition 4, the process $(\frac{Z_t}{\eta_t}, \mathcal{F}_t, t \in \mathcal{T}')$ must be P^x-martingale. (Alternatively, we can say that the process $(\frac{Z_t}{\eta_t}, \mathcal{F}_t^x, t \in \mathcal{T}')$ is P-martingale, where the *augmented* filtration $\mathcal{F}_t^x$ is used.)

When X is a *discrete* random variable, the deflator η_t in (15) becomes the risk-neutral probability of the eventual outcome:

$$\eta_t = \text{Prob}^R(X = x \,|\, \mathcal{F}_t) = h_t(x).$$

When X is a *continuous* random variable, η_t is the risk-neutral *density* evaluated at the eventual outcome.

3.2 Equivalent Economies

In this subsection, we argue that there exists an intimate relationship between investors' preferences and beliefs.

Suppose that the measurable space of events $(\Omega, \mathcal{F}_T)$ and the filtration $(\mathcal{F}_t, t \in \mathcal{T})$ are fixed. We use the triplet $M = \{P, Q, R\}$ to represent the original economy, where P, Q, and R are the objective, subjective, and risk-neutral probability measures on $(\Omega, \mathcal{F}_T)$. Suppose that Assumptions 4 and 6 are satisfied in economy M. Consider another economy $M' = \{P, Q', R\}$. In economy M' the objective and risk-neutral measures are the same but the subjective measure is replaced with a new measure Q'. Q' can be an arbitrary measure on $(\Omega, \mathcal{F}_T)$ except for the fact that Q' must be X-equivalent to P (and thus to R). In other words, Assumptions 4 and 6 are satisfied in economy M' as well.

Note that an outside empiricist cannot distinguish M' from the original economy M: in both economies events occur with the same objective frequencies and securities prices are the same. Therefore, we say that M and M' are *equivalent* economies.

There are many economies equivalent to M, but two are of particular interest. Let $M^{RN} = \{P, R, R\}$ and $M^{CB} = \{P, P, R\}$ denote equivalent economies for which the subjective measure is replaced with R and P, respectively. In economy M^{RN} investors are risk-neutral (the subjective measure coincides with the risk-neutral one), while in economy M^{CB} investors have correct beliefs (the subjective measure coincides with the objective one). We refer to M^{RN} and M^{CB} as the *equivalent risk-neutral* and *equivalent correct-beliefs* economies.

Intuitively, in the equivalent risk-neutral economy M^{RN}, preferences are "incorporated" into beliefs. Moreover, beliefs are risk-adjusted in a specific way. Since Assumption 6 is satisfied in all equivalent economies, beliefs *conditional* on the eventual outcome are correct in M^{RN}; only *initial* beliefs, or priors, need to be changed. In contrast, in the equivalent correct-beliefs economy M^{CB}, bias in initial beliefs is "translated" into preferences.

As mentioned earlier, economies M, M^{RN} and M^{CB} are observationally indistinguishable, in the sense that the same securities prices may be supported by any of them (as well as numerous other equivalent economies). This has important implications for empirical studies. Suppose that the empiricist verifies that securities prices satisfy the restriction of Proposition 4 but violate the EMH restriction under *risk-neutrality*:

$$E[Z_s \mid I_t] = Z_t, \quad t < s \leq T. \tag{16}$$

The empiricist will then conclude that investors are rational. However, she will not be able to pinpoint why exactly the restriction in (16) is violated. She may attribute the violation to risk-aversion, or biased priors, or some combination of the two.

There is a simple way to interpret equivalent economies. We refer to the Radon-Nikodym derivative $\frac{dR}{dP}$ as the *empirical kernel* and write

$$\frac{dR}{dP} = \frac{dR}{dQ'} \times \frac{dQ'}{dP},$$

or

$$\text{Empirical kernel} = \text{Pricing kernel} \times \text{Belief kernel}.$$

In all equivalent economies, the empirical kernel is the same. Therefore, different choices of the subjective measure Q' correspond to different decompositions of the empirical kernel into a pricing kernel and a belief kernel. The pricing kernel then determines investors' preferences, while the belief kernel determines their beliefs.[16] In the next section, we illustrate the idea of equivalent economies with a numerical example.

Several papers have addressed the issue of distinguishing the effects of beliefs and preferences on securities prices. In particular, Kraus and Sick (1980) and Cuoco and Zapatero (2000) consider parametric models and ask whether investors' beliefs and preferences can be jointly recovered from market data. Our contribution to this literature is that we identify fairly general conditions (Assumptions 4 and 6) under which the answer is No.

One can think of several situations in which distinguishing beliefs from preferences is relevant. First, it has been documented that the equity premium varies with the business cycle: high during recession, low during expansion. One can explain this effect by shifts in 1) preferences (during recession, investors are more risk-averse), or 2) beliefs (during recession, investors are more pessimistic). However, can one separate 1) from 2) using market data only?

Second, Bondarenko (2000) estimates the risk-neutral densities (RNDs) implicit in the S&P 500 options and documents that shapes of implied RNDs are related to the recent returns of the Index. In particular, on trading days when the Index declines RNDs have higher standard deviation and are more non-lognormal (more skewed and peaked) than when the Index advances. One can again consider two possible interpretations for the time-series relations: when the Index performs poorly, investors become 1) more risk-averse, or 2) more pessimistic. When Assumptions 4 and 6 are satisfied, the two interpretations are empirically indistinguishable.

3.3 Continuous-Time Example

In this subsection we illustrate Proposition 4 with the help of a continuous-time example. There is an asset whose liquidation value at time-T is a random variable v_T. The objective distribution of v_T at time-0 is a normal density with mean v_0 and standard deviation σ_0, that is, $f_0(v_T) = n(v_T; v_0, \sigma_0)$, where for all μ and σ we use the notation

$$n(v_T; \mu, \sigma) := \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(v_T - \mu)^2}{2\sigma^2}\right).^{17}$$

Investors' initial beliefs are represented by a normal density $g_0(v_T) = n(v_T; u_0, \eta_0)$ where u_0 and η_0 are the mean and standard deviation of the *subjective* density. In general, $(v_0, \sigma_0) \neq (u_0, \eta_0)$. Investors maximize the expected utility function with *Constant*

---

[16]Recall that, in order for Proposition 4 to hold, the empirical kernel must be a function of X only, a condition which immediately follows from Assumptions 4 and 6.

[17]In this example, the liquidation value is normally distributed, violating limited liability. However, the example can be reformulated for lognormal densities.

Absolute Risk Aversion (CARA)

$$U(v_T) = \begin{cases} \frac{1}{\rho}(1 - e^{-\rho v_T}), & \text{if } \rho > 0 \\ v_T, & \text{if } \rho = 0. \end{cases}$$

For a CARA utility function, the risk-neutral density is also normal, $h_0(v_T) = n(v_T; w_0, \eta_0)$, with the same standard deviation as that for the subjective density and with risk-adjusted mean

$$w_0 = u_0 - \rho \eta_0^2.$$

As the maturity date approaches, investors learn about the realized liquidation value, denoted by v. Specifically, they observe continuous flow of signals modeled by the process

$$dS_t = v dt + \eta_s dB_t, \qquad S_0 = 0, \tag{17}$$

where B_t is the standard Brownian motion. This means that, at time-t, investors receive an "incremental" signal dS_t which in normally distributed with mean v. The flow of market information is represented by the filtration $(\mathcal{F}_t, 0 \leq t)$ where $\mathcal{F}_t = \sigma(S_t)$. It is easy to check that at any time-t all three probability densities (objective, subjective, and risk-neutral) are normal and that they depend on the cumulative signal S_t but not on the complete path. In particular, for all $t < T$, the objective density is $f_t(v_T) = n(v_T; v_t, \sigma_t)$ where

$$v_t = \sigma_t^2 \left(\frac{v_0}{\sigma_0^2} + \frac{S_t}{\eta_s^2} \right) = v + \frac{\sigma_t^2}{\sigma_0^2}(v_0 - v) + \frac{\sigma_t^2}{\eta_s}B_t,$$

$$\frac{1}{\sigma_t^2} = \frac{1}{\sigma_0^2} + \frac{t}{\eta_s^2}.$$

Similarly, $g_t(v_T) = n(v_T; u_t, \eta_t)$ and $h_t(v_T) = n(v_T; w_t, \eta_t)$ where

$$u_t = \eta_t^2 \left(\frac{u_0}{\eta_0^2} + \frac{S_t}{\eta_s^2} \right) = v + \frac{\eta_t^2}{\eta_0^2}(u_0 - v) + \frac{\eta_t^2}{\eta_s}B_t,$$

$$\frac{1}{\eta_t^2} = \frac{1}{\eta_0^2} + \frac{t}{\eta_s^2}.$$

$$w_t = u_t - \rho \eta_t^2.$$

Since all densities are normal, v_t, u_t, and w_t can be interpreted as the objective, subjective, and risk-neutral time-t expectations of the liquidation value. In particular, w_t is the asset's price.

In what follows we illustrate Proposition 4. Note that Assumptions 4-6 are satisfied in this example. Assumption 4 holds because the utility function is path-independent, Assumption 5 holds because at any time $T' < T$ the probability that the liquidation value will be any given v_T is always positive. Finally, Assumption 6 holds because investors know how signals are generated conditional on liquidation value v, that is, they know equation (17). We first analyze the risk-neutral case ($\rho = 0$) and then we consider the risk-averse case ($\rho > 0$).

3.3.1   Risk-neutrality

When $\rho = 0$, the subjective density $g_t(v_T)$ and the risk-neutral density $h_t(v_T)$ coincide, and we can focus on the former. We observe that, as the maturity date approaches, standard deviations computed from $f_t(v_T)$ and $g_t(v_T)$ decrease and means approach v. Moreover, the initial bias of the subjective density is partly carried forward. Specifically, if $u_0 \neq v_0$, then u_t continues to be a biased estimate of the liquidation value:

$$E_0[u_t - v] = \frac{\eta_t^2}{\eta_0^2}(u_0 - v_0) \neq 0,$$

while, of course, $E_0[v_t - v] = 0$. (Recall that all expectations are computed under the objective probability measure.) Another way to illustrate the biasedness of u_t is to look at the stochastic process that u_t follows. By Ito's lemma,

$$du_t = \frac{\eta_t^2}{\eta_s^2}((v - u_t)dt + \eta_s dB_t), \quad \text{and} \quad E_0[du_t] = \frac{\eta_t^2}{\eta_s^2}(v_0 - u_0)dt.$$

We can also look at the dynamics of investors' beliefs. By Ito's lemma, $g_t(v_T)$ satisfies the following stochastic differential equation:

$$dg_t(v_T) = \frac{g_t(v_T)(v_T - u_t)}{\eta_s^2}((v - u_t)dt + \eta_s dB_t).$$

When beliefs are biased, the EMH restriction in (16) does not generally hold. In particular, beliefs will change predictably unless the mean of the subjective density is unbiased ($E_0[dg_t(v_T)] = 0$ only if $u_0 = v_0$).

It is interesting to note that, in this particular example, the EMH restriction holds even when the subjective standard deviation η_0 is biased, provided that u_0 is unbiased. For example, suppose that investors view their information as too imprecise ($\eta_0 > \sigma_0$), but they are correct about the mean of the distribution of the liquidation value ($u_0 = v_0$). In this case, the EMH restriction will still hold and securities returns will still be unpredictable under the objective density. However, to an empiricist, securities volatilities will appear high relative to the variability of the fundamental uncertainty. (In particular, the diffusion coefficient of the stochastic process for u_t is greater than that for v_t, that is, $\eta_t^2/\eta_s > \sigma_t^2/\eta_s$).

Empirically, it has been well-documented that stock prices exhibit *too much* volatility when compared to time-series variability of the earnings. This finding is frequently cited as an evidence of market irrationality. In our example, however, the excessive volatility is *not* due to investors' irrationality. Instead, it stems entirely from biases in beliefs.

We now turn to Proposition 4. Let $r_t(v_T) = r_t(v_T; v) := g_t(v_T)/g_t(v)$ denote the ratio of the time-t values of the subjective density evaluated at v_T and at the eventual outcome v. Then

$$r_t(v_T) = \exp\left(\frac{(v - v_T)(v + v_T - 2u_t)}{2\eta_t^2}\right).$$

By Ito's lemma,

$$dr_t(v_T) = \frac{r_t(v_T)}{\eta_s}(v_T - v)dB_t. \tag{18}$$

An important feature of the above equation is the absence of the drift term. This confirms Proposition 4: for every realization v, the quantity $r_t(v_T)$ is a martingale process under the objective probability measure with respect to the augmented information filtration, that is, $E[dr_t \,|\, \mathcal{F}_t^v] = 0$.

Furthermore, consider a derivative security with a payoff $Z(v_T)$. If we divide the security's price Z_t by $g_t(v)$, then this will again yield a martingale process with respect to $\mathcal{F}_t^v$, that is, $E\left[d\left(\frac{Z_t}{g_t(v)}\right) \,|\, \mathcal{F}_t^v\right] = 0.$[18] Specifically, one can verify that

$$d\left(\frac{Z_t}{g_t(v)}\right) = \frac{L(v_T)}{\eta_s}dB_t, \quad \text{where} \quad L(v_T) := \int g_t(v_T)(v_T - v)Z(v_T)dv_T.$$

For special cases $Z(v_T) = 1$ and $Z(v_T) = v_T$ we obtain

$$d\left(\frac{1}{g_t(v)}\right) = \frac{u_t - v}{\eta_s}dB_t, \qquad \text{and} \qquad d\left(\frac{u_t}{g_t(v)}\right) = \frac{\eta_t^2 + u_t(u_t - v)}{\eta_s}dB_t.$$

3.3.2 Risk-aversion

When investors are risk-averse ($\rho > 0$), the subjective and risk-neutral densities are no longer the same, $g_t(v_T) \neq h_t(v_T)$. However, the risk-averse case can be reduced to the risk-neutral one. Following the discussion in Section 3.2, let the filtration $(\mathcal{F}_t, 0 \leq t)$ be fixed and let the triplet $M = \{f_0, g_0, h_0\}$ represent the original economy, while the triplet $M^{RN} = \{f_0, h_0, h_0\}$ represent the *equivalent risk-neutral* economy. Economy M^{RN} is the same as M except the subjective density is now $g_0^{RN}(v_T) = h_0(v_T)$.

As explained in Section 3.2, in economy M^{RN} risk-aversion is "incorporated" into initial beliefs. This is done by "risk-adjusting" the original subjective density $g_0(v_T) = n(v_T; u_0, \eta_0)$ to $g_0^{RN}(v_T) = n(v_T; u_0 - \rho\eta_0^2, \eta_0)$. Since securities prices are the same in economies M and M^{RN} and since $g_0^{RN}(v_T)$ is still a normal density, all the previous analysis continues to hold for economy M^{RN}.

In Section 3.2, we have also considered the *equivalent correct-beliefs* economy $M^{CB} = \{f_0, f_0, h_0\}$ for which bias in initial belief is "translated" into preferences. In this respect, it may be instructive to ask the question: What kind of the utility function $U(v_T)$ must investors have in that economy?

Since the objective and risk-neutral densities are normal, $f_0(v_T) = n(v_T; v_0, \sigma_0)$ and $h_0(v_T) = n(v_T; w_0, \eta_0)$, the pricing kernel in economy M^{CB} is given to within a constant by

$$\ln(m(v_T)) = \frac{v_T^2}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\eta_0^2}\right) + v_T\left(\frac{w_0}{\eta_0^2} - \frac{v_0}{\eta_0^2}\right).$$

Following Harrison and Kreps (1979), we note that any economy with a positive pricing kernel can be sustained in a competitive equilibrium with a representative investor,

---

[18] In fact, the same property holds even when the payoff of the derivative security is path-*dependent*.

simply by taking the preferences which are defined by the price functional of Harrison and Kreps. Not all associated preferences, however, will have an expected utility representation.[19]

In this example, preferences will be represented by a concave utility function $U(v_T)$ only if the pricing kernel $m(v_T)$ is monotonically decreasing in v_T. This can only happen if $\sigma_0 = \eta_0$ and $v_0 > w_0$, which corresponds to a CARA utility function with $\rho = (v_0 - w_0)/\sigma_0^2$. In all other cases, preferences admit no expected utility representation.

3.3.3 Irrational learning

We conclude the example by examining what happens when investors' learning is not fully rational. There are many ways how irrationality can be introduced in investors' behavior. The behavioral finance literature often advocates two types of irrational behavior: *underreaction* and *overreaction*. See, for example, De Bondt and Thaler (1985), Chopra, Lakonishok, and Ritter (1992), Lakonishok, Shleifer, and Vishny (1994), Barberis, Shleifer, and Vishny (1998), Daniel, Hirshleifer and Subrahmanyam (1998).

Underreaction and overreaction can also be modeled in a number of ways. For example, Bayesian updating in (3) can be replaced by alternative, suboptimal rules. In our setting, a simple way to introduce underreaction and overreaction is the following. We retain the same information structure as before and focus on the risk-neutral case. Investors continue to update their beliefs in a Bayesian fashion but with one change. They now *misinterpret* the arriving signals. Specifically, we assume that, even though signals continue to be generated by equation (17), investors update their beliefs using an *incorrect* model:

$$dS_t = vdt + \hat{\eta}_s dB_t,$$

where the diffusion coefficient $\hat{\eta}_s$ is different from its true counterpart η_s. Note that such a specification is inconsistent with rationality. This is because investors can easily estimate the true variability of signals η_s but they use the incorrect model with $\hat{\eta}_s$ instead. Note also that Assumption 6 is now violated.

We interpret the case when $\hat{\eta}_s < \eta_s$ as overreaction. In this case, investors (mistakenly) perceive signals as very precise. As a result, investors over-adjust their beliefs. In contrast, when $\hat{\eta}_s > \eta_s$, investors underreact: they understate the informativeness of signals and update beliefs too slowly.

Because investors misinterpret the precision of signals, the equation in (18) must be replaced by

$$\frac{dr_t(v_T)}{r_t(v_T)} = \frac{(v_T - v)^2}{2\hat{\eta}_s^2}\left(\frac{\eta_s^2}{\hat{\eta}_s^2} - 1\right)dt + \frac{\eta_s}{\hat{\eta}_s^2}(v_t - v)dB_t. \qquad (19)$$

When $\hat{\eta}_s = \eta_s$, the drift term disappears and equation (19) reduces to equation (18). However, when $\hat{\eta}_s \neq \eta_s$, the drift is nonzero. This implies that the process $r_t(v_T)$ is no longer a martingale with respect to the augmented filtration $\mathcal{F}_t^v$. In particular,

---

[19]To admit expected utility representation, a preference relation must satisfy a set of certain conditions. For the case of state-independent utility $U(v_T)$, conditions can be found, for example, in Huang and Litzenberger (1988), Chapter 1.

$E[dr_t \,|\, \mathcal{F}_t^v] < 0$ in the case of overreaction, while $E[dr_t \,|\, \mathcal{F}_t^v] > 0$ in the case of underreaction. Furthermore, let $Z_t^p > 0$ denote the price of a security whose final payoff $Z^p(v_T)$ is *nonnegative* for all v_T. (For example, Z_t^p may represent the price of a vanilla call or put option written on v_T.) Then

$$E\left[d\left(\frac{Z_t^p}{h_t(v)} \right) \,|\, \mathcal{F}_t^v \right] < (>) \, 0,$$

when investors overreact (underreact). This result has an interesting implication. Suppose that the empiricist studies securities prices and rejects restriction (15) suggesting that investors' learning cannot be fully rational, for *any* choice of preferences and initial beliefs. She will then be able to verify whether the rejection is due to overreaction, underreaction, or neither by testing the following *supermartingale* and *submartingale* restrictions:

$$\text{Overreaction:} \qquad E\left[\frac{Z_s^p}{h_s(v)} \,|\, \mathcal{F}_t^v \right] < \frac{Z_t^p}{h_t(v)}. \tag{20}$$

$$\text{Underreaction:} \qquad E\left[\frac{Z_s^p}{h_s(v)} \,|\, \mathcal{F}_t^v \right] > \frac{Z_t^p}{h_t(v)}. \tag{21}$$

The above restrictions are still preferences- and beliefs-independent.[20]

4 Discussion

In this section, we first discuss the role of Assumption 1 and then suggest how the new theoretical restriction can be tested empirically.

4.1 Is Assumption 1 Restrictive?

Path-independence of the pricing kernel is a key assumption in our analysis. The assumption may seem restrictive but, in reality, it is not. To demonstrate this, we provide several examples of popular models that satisfy Assumption 1 (Assumption 4 in the general case). In these models, the pricing kernel $m_t = m(\xi_t)$ depends on the state variable ξ_t but not the whole history I_t. Often, the state variable ξ_t is either the aggregate consumption c_t or the value of the market portfolio v_t.

(a) **Black-Scholes model**. In this model, the market portfolio follows a geometric Brownian motion. The objective price process is:

$$\frac{dv_t}{v_t} = \mu dt + \sigma dB_t,$$

---

[20]See also Bossaerts and Hillion (1998). They propose another model of irrational learning, termed the *Variable Reversal Delay* model. To measure overreaction and underreaction, Bossaerts and Hillion use inequalities similar to (20)-(21).

where μ, σ are constant, and B_t is the standard Brownian motion. As well known, prices of derivative securities in this model are supported by the following pricing kernel:

$$m_t = m(v_t) = v_t^{-\gamma}, \tag{22}$$

where $\gamma = (\mu - r_f)/\sigma^2 > 0$ is the coefficient of relative risk-aversion, and r_f is the risk-free rate.[21] The pricing kernel is path-independent. This means that, for a general derivative on the market portfolio v_t (exotic, path-dependent), the new restriction must hold:

$$E\left[\frac{Z_s}{h_s(v)} \mid \mathcal{F}_t^v\right] = \frac{Z_t}{h_t(v)}, \qquad t < s < T,$$

where the risk-neutral density is

$$h_t(v) = h(v, T; v_t, t) = \frac{1}{2\pi\sigma\sqrt{\tau}} \frac{1}{v} \exp\left(-\frac{\left(\ln\frac{v}{v_t} - \left(r_f - \frac{1}{2}\sigma^2\right)\tau\right)^2}{2\sigma^2\tau}\right), \qquad \tau = T - t.$$

(Clearly, Assumption 1 is also satisfied in a discrete time setting, where the price process is modeled by the standard binomial tree.)

(b) **The Bates model**. An interesting extension of the Black-Scholes model is Bates (2001). He studies a jump-diffusion economy with "crash-averse" investors. The pricing kernel in this economy takes the form:

$$m_t = v_t^{-\gamma} e^{Y_c n_t},$$

where $\gamma > 0$, n_t is the number of jumps over $[0, t]$, and $Y_c > 0$ is the parameter of crash-aversion.[22]

Let $\xi_t = (v_t, n_t)$, then the pricing kernel is path-independent and no SAOs can exist. Moreover, the risk-neutral density $h_t(\xi_T)$ can be computed explicitly in this model. Therefore, we can apply the new martingale restriction in Proposition 4 to a general option Z_t written on process ξ_t. For example, Z_t may represent the jump insurance contract considered in Bates (2001).

(c) **Consumption-based models**. Intertemporal asset pricing models often assume that the representative investor maximizes the expectation of a time-separable utility function:

$$\max E_t\left[\sum_{j=t+1}^{\infty} \delta^j U(c_t, t)\right], \tag{23}$$

---

[21] For simplicity, we omit irrelevant constant multiples in all formulas for pricing kernels.

[22] The representative investor's utility function depends on v_T and the number of jumps n_T. Specifically, the utility function is

$$U(v_T, n_T) = e^{Y_c n_T} \begin{cases} \frac{1}{1-\gamma} v_T^{1-\gamma}, & \text{if } \gamma \neq 0 \\ \ln v_T, & \text{if } \gamma = 0, \end{cases}$$

where δ is the time discount factor and $U(\cdot, t)$ is a strictly concave utility function. In these models, path independence of the pricing kernel immediately follows from time-separability. To see this, recall that the pricing kernel can be found from the first order conditions to the portfolio problem:

$$m_t = U_c(c_t, t). \tag{24}$$

Therefore, the pricing kernel m_t depends on c_t but not on consumption in earlier periods. An important special case of the specification in (23) is the Consumption Capital Asset Pricing Model (CCAPM) with the CRRA preferences. For this model, the pricing kernel is simply

$$m_t = c_t^{-\gamma}.$$

Some consumption-based models make additional assumptions that allow to replace c_t in (24) with other variables. One example is Rubinstein (1976). He demonstrates that the pricing kernel in (22) arises in a more general setting. In particular, the market portfolio v_t is not restricted to follow a geometric Brownian motion.

It should be mentioned that the path-independence assumption is also satisfied in various extensions (modifications) of (23). In these specifications,

1) the time horizon can be finite (with a possibility of a terminal bequest);

2) time can be continuous;

3) c_t can represent a *vector* of several consumption goods. (Additional variables, which has been suggested in the literature, include leisure, government spending, and the stock of durable goods).

Another way to extend the specification in (23) is to rewrite the utility function at time-t as $U(c_t, y_t, t)$, where y_t is an additional state variable. One important example is habit formation models in which y_t represents the time-varying "habit" or subsistence level. To be specific, consider two specifications of habit formation of many proposed in the literature. Abel (1990) and Campbell and Cochrane (1999) develop models of *external* habit. In these models, the pricing kernel m_t is path-independent with the state variable $\xi_t = (c_t, y_t)$.[23] In particular, in Campbell and Cochrane (1999),

$$m_t = U_c(c_t, y_t) = (c_t - y_t)^{-\gamma}.$$

(d) **Epstein-Zin-Weil model**. The standard CCAPM with the CRRA preferences has one well-known limitation. In this model, a single parameter γ controls both the elasticity of intertemporal substitution and the coefficient of relative risk aversion. Epstein and Zin (1989, 1991) and Weil (1989) propose a more flexible model which allows

---

[23]Note that since habit y_t is a function of past consumption $c_{t-1}, c_{t-2}, \ldots$, the pricing kernel m_t depends on past consumption through habit y_t. However, m_t depends *not* on the exact sequence of past consumption, but only on a "summary statistics" y_t.

to disentangle intertemporal substitution from risk aversion. Their objective function is defined recursively by

$$U_t = \left\{ (1-\delta)c_t^\rho + \delta \left(E_t[U_{t+1}^{\theta\rho}] \right)^{\frac{1}{\theta}} \right\}^{\frac{1}{\rho}},$$

where $0 \neq \rho < 1$, and $\theta > 0$. The resulting pricing kernel depends both on consumption c_t and the value of the market portfolio v_t:

$$m_t = c_t^{\theta(\rho-1)} v_t^{\theta-1}.$$

The Epstein-Zin-Weil model nests CCAPM with the CRRA preferences ($\theta = 1$) and the log utility CAPM ($\theta = 0$). The pricing kernel m_t depends on c_t for $\theta = 1$, on v_t for $\theta = 0$, and on c_t and v_t for all other values of θ. In any case, the pricing kernel is path-independent with $\xi_t = (c_t, v_t)$.

(e) **Multifactor arbitrage models**. In a general continuous-time multifactor model, the expected return on any security is given by

$$E_t[dp_t/p_t] = \alpha dt + \beta_1 d\lambda_{1t} + \ldots + \beta_k d\lambda_{kt},$$

where $\lambda_{1t}, \ldots, \lambda_{kt}$ represent k common factors. In this model, the pricing kernel is

$$m_t = \exp(a + b_1\lambda_{1t} + \ldots + b_k\lambda_{kt}),$$

where $a, b_1, \ldots, b_k$ are constants. The pricing kernel is path-independent with $\xi_t = (\lambda_{1t}, \ldots, \lambda_{kt})$.

The list of examples can be continued but the message is clear: Many important models in finance fit in our framework. In all examples, there is a state variable with respect to which the pricing kernel is path-independent. This in turn implies that certain investment opportunities (namely, SAOs) cannot exist in equilibrium and that securities prices must satisfy the new martingale restriction.

Our setting is general in several other respects. First, the analysis does not depend on the existence of the representative investor. All the above models assume that the representative investor exists. However, we do not need that preferences and beliefs of heterogeneous investors aggregate in some meaningful way. Recall that a positive pricing kernel always exists provided that securities prices admit no pure arbitrage. In general, this pricing kernel does not have to correspond to a von Neumann-Morgenstern utility function. Nevertheless, if the pricing kernel is path-independent, then more general SAOs must be ruled out as well.

Second, our setting allows for incomplete markets. In incomplete markets, securities prices can be supported by many pricing kernels. If among admissible pricing kernels there is at least one which is path-independent, then SAOs cannot exist and the new restriction holds.

Finally, we impose no specific structure on the state variable ξ_T. This means that ξ_T could, in principle, be redefined to include the complete description of all intermediate states ξ_t. In other words, we could choose ξ_T to contain as much information as the elementary state I_T does. In this extreme case, there would be no difference between a PAO and a SAO, and Proposition 1 would reduce to the standard result by Harrison and Kreps (1979). However, in all situations when ξ_T contains strictly less information than I_T, the concept of a SAO is more general than that of a PAO. Therefore, ruling out SAOs has stronger pricing implications.

There is, of course, a cost to defining the state variable ξ_T too broadly. When ξ_T contains more information, it will be more difficult to recover the risk-neutral probability $h_t(\xi_T)$ from prices of traded securities. Still, the new restriction is useful even when $h_t(\xi_T)$ is not observable.

Suppose that one has in mind a specific parametric model with a path-independent pricing kernel. Then ruling out SAOs imposes additional testable implications that have been largely overlooked in the literature. Moreover, the new restriction has important advantages even in a *parametric* setting. In particular, the new restriction can be used in samples affected by selection biases.

4.2   Empirical Application

We implement the methodology of this paper in Bondarenko (2002) using the S&P 500 Index futures options data. We report that, over the period from 1987 to 2000, a number of option strategies appear to be highly profitable. In particular, selling unhedged put options one month before maturity would have resulted in extraordinary high and statistically significant average excess returns. For at-the-money (ATM) puts, the average return over the studied period is about -40% per month. For deeply out-of-the-money (OTM) puts, the average return approaches -100% per month. In other words, puts appear to be grossly overpriced, at least from the point of view of restriction (16) for EMH under risk-neutrality.

Do these findings indicate that investors are irrational? Are there true inefficiencies in option markets? Before answering these questions affirmatively, three explanations should be investigated:

E1: *Risk-premium.* According to this explanation, high prices of puts are *expected* and reflect normal risk-premium under some equilibrium model. Even though the canonical models (such as CAPM or the Rubinstein model) cannot explain option prices, maybe there is another model which can. In this true model, investors strongly dislike negative returns of the S&P 500 Index and are willing to pay hefty premiums to hedge their portfolios against declines.

E2: *The Peso problem.* To understand this explanation, suppose that market crashes (similar to that of October 1987) occur on average once in 5 years. Suppose also that investors correctly incorporate a probability of another crash in option prices. However, since only one major market crash has actually happened over the 14-year period, the *ex post* realized returns of the Index are different from investors'

33

ex ante beliefs. According to this explanation, puts only *appear* overpriced. The mispricing would have disappeared if data for a much longer period were available.

E3: *Biased beliefs.* According to this explanation, investor's beliefs are mistaken. Investors, however, still process information rationally. This is the case of the more general ELM. OTM puts were expensive because investors assigned too high probabilities to negative returns of the Index. Perhaps, memories of the 1987 stock market crash were still fresh and, even though the *true* probability of another extreme decline was small, investors continued to overstate that probability.

Proposition 4 allows us to test whether E1-E3 can explain the historical put returns. Specifically, we assume that the pricing kernel is of the form $m_t = m(v_t, t)$, where v_t denotes the value of the S&P 500 Index.[24] Recall that, to test the new martingale restriction, we need to know the risk-neutral density (RND) $h_t(v_T)$. We estimate RND from prices of standard call options as follows.

Consider a European-style call option with strike price K and maturity date T written on v_T. The option price $C_t(K)$ is equal to the expected present value of the payoff under the risk-neutral density (RND) $h_t(v_T)$:

$$C_t(K) = e^{-r_f(T-t)} \int_0^\infty \max(v_T - K, 0)\, h_t(v_T)\, dv_T,$$

where r_f is the risk-free rate over period $[t, T]$.

Differentiating the above equation twice with respect to K, we obtain the relationship first discovered in Ross (1976), Breeden and Litzenberger (1978), Banz and Miller (1978):

$$h_t(v_T) = e^{-r_f(T-t)} \left. \frac{\partial^2 C_t(K)}{\partial K^2} \right|_{K=v_T}$$

The relationship states that RND is proportional to the second partial derivative of call price with respect to strike evaluated at $K = v_T$. In liquid option markets, it is not uncommon to observe 30 and more concurrently traded call contracts with different strikes K and the same maturity T. Given a collection of these options, it is possible to accurately estimate the function $C_t(K)$ and its second derivative. In the literature, a number of estimation methods are now available. See Jackwerth (1999) for a recent survey. We estimate RNDs using the method recently developed in Bondarenko (2000).

After RNDs are obtained from option prices, we test the restriction in Proposition 4 and strongly reject it. The rejection means that the put pricing anomaly cannot be explained by *any* asset pricing model with the path-independent pricing kernel $m(v_t, t)$, even if investors have mistaken beliefs and the sample is affected by selection biases. In

---

[24]Note that in this application we focus on a very short horizon (T=1 month). This means that other theory motivated state variables discussed in Section 4.1 are not likely to be important. In particular, time series for the aggregate consumption c_t or habit y_t are very smooth at monthly frequency. Moreover, since the pricing kernel is allowed to explicitly depend on t, the long-term dependence of the pricing kernel on c_t and y_t can be incorporated.

fact, only a very small portion of the put mispricing can be attributed to risk-aversion, the Peso problem, and biased beliefs.[25] Moreover, by testing the restriction in (20), we argue that put prices are consistent with *overreaction*.

5   Conclusion

In this paper, we introduce the concept of a *statistical arbitrage opportunity* (SAO). In a finite-horizon economy, a SAO is a zero-cost trading strategy for which i) the expected payoff is positive, and ii) the conditional expected payoff in each final state of the economy is nonnegative.

We show that when the pricing kernel is path-independent then no SAOs can exist. Furthermore, precluding SAOs imposes a novel martingale-type restriction on the dynamics of securities prices. It states that securities prices deflated by the risk-neutral density evaluated at the eventual outcome must follow a martingale. The restriction extends the results in Bossaerts (1999a, 1999b). Important properties of the restriction are that it

1) requires no parametric assumptions on investors' preferences;

2) can be tested in samples affected by selection biases, such as the Peso problem; and

3) continues to holds even when investors' beliefs are mistaken (provided that conditional beliefs are correct).

The new restriction allows one to resolve the joint hypothesis problem present in the traditional tests of the Efficient Market Hypothesis. In order to test the restriction, one must first estimate the risk-neutral density from market prices of traded options. Consequently, the methodology is best suited for applications where well-developed liquid option markets exist. One such an application is proposed in Bondarenko (2001).

The concept of a SAO can also be useful in a *parametric* setting. Suppose that one has in mind a specific model with a path-independent pricing kernel. Then ruling out SAOs imposes additional testable implications that have been largely overlooked in the literature. Compared to the standard approaches, the new restriction has the important advantage that it is robust to selection biases.

An interesting extension of our approach is securities valuation in incomplete markets. In incomplete markets, many possible pricing kernels can exist and perfect replication of a derivative security with primary assets is impossible. This means that one

---

[25]Our findings do not rule out two other possibilities: (a) the pricing kernel is path-dependent, and (b) the pricing kernel is path-independent but is a function of other state variables in addition v_t (a natural candidate here would be stochastic volatility). Neither possibility, however, has received enough attention in the literature. First, as argued in Section 4.1, standard equilibrium models usually produce path-independent kernels. Second, there is currently no accepted *equilibrium* model in which preferences of the representative investor depend on volatility. To be able to explain the put anomaly, one might need to develop new theoretical models that allow for (a) or (b).

can assign to the derivative security a range of possible values, all consistent with the absence of PAOs. The resulting no-arbitrage bounds, however, are often too wide to be practically useful. One way to obtain tighter pricing bounds is to rule out more general SAOs. We leave this topic for future research.

Appendix

Proof of Proposition 1

1) If there is a pricing kernel $m(\xi_T) > 0$, then no SAOs can exist. Indeed, consider a strategy with a payoff $Z_T = Z(I_T) \in \mathcal{Z}$ that satisfies conditions i) and ii) in Definition 2, that is, $E[Z_T \mid I_0] > 0$ and $E[Z_T \mid I_0^{\xi_T}] \geq 0$, for all ξ_T. Then the initial value of this strategy Z_0 cannot be zero, because

$$Z_0 = E[m(\xi_T)Z_T \mid I_0] = \sum_{\xi_T} E[m(\xi_T)Z_T \mid I_0^{\xi_T}] \operatorname{Prob}(\xi_T \mid I_0)$$

$$= \sum_{\xi_T} m(\xi_T) E[Z_T \mid I_0^{\xi_T}] \operatorname{Prob}(\xi_T \mid I_0) > 0,$$

where the last inequality follows from

$$E[Z_T \mid I_0] = \sum_{\xi_T} E[Z_T \mid I_0^{\xi_T}] \operatorname{Prob}(\xi_T \mid I_0) > 0.$$

2) Conversely, suppose that there are no SAOs. For an arbitrary path-*dependent* trading strategy with an initial value Z_0 and a final payoff $Z_T = Z(I_T) \in \mathcal{Z}$, we construct the "associated" path-*independent* strategy with the same initial value $\tilde{Z}_0 = Z_0$ and the payoff defined as

$$\tilde{Z}_T = \tilde{Z}(\xi_T) := E[Z_T \mid I_0^{\xi_T}].$$

By construction, the two strategies have the same expected payoffs at $t = 0$:

$$E[Z_T \mid I_0] = \sum_{\xi_T} E[Z_T \mid I_0^{\xi_T}] \operatorname{Prob}(\xi_T \mid I_0) = \sum_{\xi_T} \tilde{Z}(\xi_T) \operatorname{Prob}(\xi_T \mid I_0) = E[\tilde{Z}_T \mid I_0].$$

Let $\tilde{\mathcal{Z}}$ denote the set of associated path-*independent* payoffs $\tilde{Z}_T$ corresponding to all path-dependent payoffs $Z_T \in \mathcal{Z}$. While payoffs $Z_T \in \mathcal{Z}$ are defined on the set of elementary states $\mathcal{I}_T$, payoffs $\tilde{Z}_T \in \tilde{\mathcal{Z}}$ are defined on the set of final states Ξ_T.

We now note that, if a path-dependent strategy is a SAO on $\mathcal{I}_T$, then its associated path-independent strategy is a PAO on Ξ_T. The absence of SAOs on $\mathcal{I}_T$, therefore, implies the absence of PAOs on Ξ_T. By the standard argument (see Harrison and Kreps (1979)), there exists a function $m(\xi_T) > 0$ that correctly prices all path-independent payoffs $\tilde{Z}_T \in \tilde{\mathcal{Z}}$

$$E[\tilde{Z}_T m(\xi_T) \mid I_0] = \tilde{Z}_0.$$

This function $m(\xi_T)$ then also supports all path-dependent payoffs $Z_T \in \mathcal{Z}$, because

$$E[m(\xi_T)Z_T \mid I_0] = \sum_{\xi_T} m(\xi_T) E[Z_T \mid I_0^{\xi_T}] \operatorname{Prob}(\xi_T \mid I_0)$$

$$= \sum_{\xi_T} m(\xi_T) \tilde{Z}(\xi_T) \operatorname{Prob}(\xi_T \mid I_0) = E[m(\xi_T)\tilde{Z}_T \mid I_0] = \tilde{Z}_0 = Z_0.$$

Therefore, $m(\xi_T) > 0$ a proper pricing kernel.

$\square$

Proof of Proposition 2

We first consider the case when the security's final payoff is a function of ξ_T only and then the general case when the payoff may depend on the whole history I_T.

1) At time-$(t+1)$, the objective probability will depend on the realized state ξ_{t+1}, that is, $f_{t+1}(\xi_T) = f_{t+1}(\xi_T; \xi_{t+1})$. Let $\lambda_t(\xi_{t+1}; \xi_T)$ denote time-t objective transition probability of state ξ_{t+1} conditional on $I_t^{\xi_T}$ (i.e., conditional on history I_t and the fact that the final state is ξ_T). Then $f_{t+1}(\xi_T; \xi_{t+1})$ can be computed using Bayes' law as follows:

$$f_{t+1}(\xi_T; \xi_{t+1}) = \frac{f_t(\xi_T)\lambda_t(\xi_{t+1}; \xi_T)}{\sum_{\xi_T'} f_t(\xi_T')\lambda_t(\xi_{t+1}; \xi_T')} = \frac{\text{Prob}(\xi_{t+1}, \xi_T \mid I_t)}{\text{Prob}(\xi_{t+1} \mid I_t)}.$$

In view of relationship (4), the risk-neutral probability

$$h_{t+1}(\xi_T; \xi_{t+1}) = \frac{m(\xi_T)}{m_{t+1}} f_{t+1}(\xi_T; \xi_{t+1}).$$

Therefore,

$$\frac{h_{t+1}(\xi_T; \xi_{t+1})}{h_{t+1}(x; \xi_{t+1})} = \frac{m(\xi_T)}{m(x)} \frac{f_t(\xi_T)}{f_t(x)} \frac{\lambda_t(\xi_{t+1}; \xi_T)}{\lambda_t(\xi_{t+1}; x)} = \frac{h_t(\xi_T)}{h_t(x)} \frac{\lambda_t(\xi_{t+1}; \xi_T)}{\lambda_t(\xi_{t+1}; x)}. \tag{25}$$

Note that Assumption 2 ensures that, for all ξ_{t+1}, the quantity $\lambda_t(\xi_{t+1}; x)$ is always greater than 0. Using (25), we obtain that

$$E\left[\frac{h_{t+1}(\xi_T)}{h_{t+1}(x)} \mid I_t^x\right] = \sum_{\xi_{t+1}} \frac{h_{t+1}(\xi_T; \xi_{t+1})}{h_{t+1}(x; \xi_{t+1})} \lambda_t(\xi_{t+1}; x)$$

$$= \sum_{\xi_{t+1}} \frac{h_t(\xi_T)}{h_t(x)} \frac{\lambda_t(\xi_{t+1}; \xi_T)}{\lambda_t(\xi_{t+1}; x)} \lambda_t(\xi_{t+1}; x) = \frac{h_t(\xi_T)}{h_t(x)} \sum_{\xi_{t+1}} \lambda_t(\xi_{t+1}; \xi_T) = \frac{h_{\xi_T}(x)}{h_t(x)}.$$

By the law of iterated expectations, for $t < s < T'$

$$E\left[\frac{h_s(\xi_T)}{h_s(x)} \mid I_t^x\right] = \frac{h_t(\xi_T)}{h_t(x)}. \tag{26}$$

Suppose now that the security's final payoff is a function of ξ_T, i.e., $Z_T = Z(\xi_T)$. Then the security's price at time-t is

$$Z_t = \sum_{\xi_T} Z(\xi_T) h_t(\xi_T).$$

Using relationship (26), we obtain that

$$E\left[\frac{Z_s}{h_s(x)} \mid I_t^x\right] = E\left[\sum_{\xi_T} Z(\xi_T)\frac{h_s(\xi_T)}{h_s(x)} \mid I_t^x\right] = \sum_{\xi_T} Z(\xi_T) E\left[\frac{h_s(\xi_T)}{h_s(x)} \mid I_t^x\right]$$

$$= \sum_{\xi_T} Z(\xi_T) \frac{h_t(\xi_T)}{h_t(x)} = \frac{Z_t}{h_t(x)}.$$

2) When the security's payoff depends on the whole history, i.e., $Z_T = Z(I_T)$, the proof is essentially the same. This time, however, we need to establish an analogue of equation (26) for the elementary state $I_T = (\xi_1, \ldots, \xi_T)$.

Recall that $F_t(I_T)$ and $H_t(I_T)$ denote the conditional objective and risk-neutral probabilities of the elementary state I_T. At time-$(t+1)$, the objective probability $F_{t+1}(I_T) = F_{t+1}(I_T; \xi_{t+1})$ depends on the state ξ_{t+1}:

$$F_{t+1}(I_T; \xi_{t+1}) = \frac{F_t(I_T)\iota_{t+1}(\xi_{t+1}; I_T)}{\sum_{I'_T} F_t(I'_T)\iota_{t+1}(\xi'_{t+1}; I'_T)} = \frac{\text{Prob}(\xi_{t+1}, I_T \mid I_t)}{\text{Prob}(\xi_{t+1} \mid I_t)},$$

where $\iota_t(y; I_T)$ is the indicator function equal to 1 if the tth component of vector I_T is y (i.e., $\xi_t = y$) and zero otherwise. The risk-neutral probability $H_{t+1}(I_T; \xi'_{t+1})$ can be written as

$$\frac{H_{t+1}(I_T; \xi_{t+1})}{h_{t+1}(x; \xi_{t+1})} = \frac{H_t(I_T)}{h_t(x)} \frac{\iota_{t+1}(\xi_{t+1}; I_T)}{\lambda_t(\xi_{t+1}; x)}. \qquad (27)$$

Therefore,

$$E\left[\frac{H_{t+1}(I_T)}{h_{t+1}(x)} \mid I_t^x\right] = \sum_{\xi_{t+1}} \frac{H_{t+1}(I_T; \xi_{t+1})}{h_{t+1}(x; \xi_{t+1})} \lambda_t(\xi_{t+1}; x)$$

$$= \frac{H_t(I_T)}{h_t(x)} \sum_{\xi_{t+1}} \iota_{t+1}(\xi_{t+1}; I_T) = \frac{H_t(I_T)}{h_t(x)}.$$

This implies that, for $t < s < T'$,

$$E\left[\frac{H_s(I_T)}{h_s(x)} \mid I_t^x\right] = \frac{H_t(I_T)}{h_t(x)}.$$

Since the security's time-t price is

$$Z_t = \sum_{I_T} Z(I_T)H_t(I_T),$$

we obtain that

$$E\left[\frac{Z_s}{h_s(x)} \mid I_t^x\right] = \sum_{I_T} Z(I_T)\left[\frac{H_s(I_T)}{h_s(x)} \mid I_t^x\right] = \sum_{I_T} Z(I_T)\frac{H_t(I_T)}{h_t(x)} = \frac{Z_t}{h_t(x)}.$$

$\square$

Proof of Proposition 3

The proof of Proposition 3 is a slight modification of the proof of Proposition 2. In part 1), the ratio of risk-neutral probabilities in equation (25) must now be expressed in terms of the *subjective* transition probabilities instead of the *objective* ones. However, by Assumption 3 we have $\lambda_t^S(\xi_{t+1}; \xi_T) = \lambda_t(\xi_{t+1}; \xi_T)$. In this case, equation (25) becomes

$$\frac{h_{t+1}(\xi_T; \xi_{t+1})}{h_{t+1}(x; \xi_{t+1})} = \frac{h_t(\xi_T)}{h_t(x)} \frac{\lambda_t^S(\xi_{t+1}; \xi_T)}{\lambda_t^S(\xi_{t+1}; x)} = \frac{h_t(\xi_T)}{h_t(x)} \frac{\lambda_t(\xi_{t+1}; \xi_T)}{\lambda_t(\xi_{t+1}; x)},$$

and the rest of the proof is the same.

Similarly, in part 2), equation (27) must now be replaced by

$$\frac{H_{t+1}(I_T; \xi_{t+1})}{h_{t+1}(x; \xi_{t+1})} = \frac{H_t(I_T)}{h_t(x)} \frac{\iota_{t+1}(\xi_{t+1}; I_T)}{\lambda_t^S(\xi_{t+1}; x)} = \frac{H_t(I_T)}{h_t(x)} \frac{\iota_{t+1}(\xi_{t+1}; I_T)}{\lambda_t(\xi_{t+1}; x)}.$$

(Note that the indicator function $\iota_{t+1}(\xi_{t+1}; I_T)$ is the same for both the objective and subjective probabilities.) Then the rest of the proof follows through. $\square$

Proof of Proposition 4

1) We first prove that Assumptions 4 and 6 imply that

$$P^x = Q^x = R^x, \qquad \text{all } x.$$

Recall that the measures P, Q, and R are equivalent on $(\Omega, \mathcal{F}_T)$. Let ϕ_T denote the Radon-Nikodym derivative of R with respect to Q:

$$\phi_T := \frac{dR}{dQ}.$$

For any $\mathcal{F}_{T'}$-measurable random variable Z and for $t \leq T'$ we have

$$E^{R^x}[Z \,|\, \mathcal{F}_t] = E^R[Z \,|\, \mathcal{F}_t^x] = \frac{E^Q[\phi_T Z \,|\, \mathcal{F}_t^x]}{E^Q[\phi_T \,|\, \mathcal{F}_t^x]} = \frac{\phi_T(x) E^Q[Z \,|\, \mathcal{F}_t^x]}{\phi_T(x)}$$

$$= E^Q[Z \,|\, \mathcal{F}_t^x] = E^{Q^x}[Z \,|\, \mathcal{F}_t],$$

where we use the fact that the Radon-Nikodym derivative $\phi_T > 0$ is a *constant* when $X = x$ (by Assumption 4). Since the above equation holds for all $\mathcal{F}_{T'}$-measurable random variables Z, the measures Q^x and R^x must coincide on $(\Omega, \mathcal{F}_{T'})$, or $Q^x = R^x$. In a similar fashion, it follows from Assumption 6 that $P^x = Q^x$.

2) Consider now any random process $(Z_t, \mathcal{F}_t, t \in \mathcal{T}')$ which is R-martingale. Then, for $t < s \leq T'$,

$$E_t^{P^x}\left[\frac{Z_s}{\eta_s}\right] = E_t^{Q^x}\left[\frac{Z_s}{\eta_s}\right] = E_t^{R^x}\left[\frac{Z_s}{\eta_s}\right] = \frac{E_t^R\left[\eta_s \frac{Z_s}{\eta_s}\right]}{E_t^R[\eta_s]} = \frac{Z_t}{\eta_t},$$

where the third equality follows from (14). To complete the proof, we also need to check the integrability condition on $\frac{Z_s}{\eta_s}$. Since $(Z_t, \mathcal{F}_t, t \in \mathcal{T}')$ is R-martingale,

$$E_t^{P^x} \left[\left| \frac{Z_s}{\eta_s} \right| \right] = E_t^{P^x} \left[\frac{|Z_s|}{\eta_s} \right] = \frac{E_t^R [|Z_s|]}{\eta_t} < \infty.$$

The last two conditions imply that the process $(\frac{Z_t}{\eta_t}, \mathcal{F}_t, t \in \mathcal{T}')$ is P^x-martingale.

$\square$

References

Abel, A., 1990, "Asset Prices under Habit Formation and Catching up with the Joneses," *American Economic Review* 80, 38-42.

Banz, R., and M. Miller, 1978, "Prices for State-Contingent Claims: Some Estimates and Applications," *Journal of Business* 51, 653-672.

Bekaert, G., R. Hodrick, and D. Marshall, 1995, "Peso Problem' Explanations For Term Structure Anomalies," Stanford University working paper.

Barberis, N., A. Shleifer, and R. Vishny, 1998, "A Model of Investor Sentiment," *Journal of Financial Economics* 49, 307-343.

Bates, D., 2001, "The Market for Crash Risk," University of Iowa working paper.

Bernardo, A., and O. Ledoit, 2000, "Gain, Loss, and Asset Pricing," *Journal of Political Economy* 108, 144-172.

Bondarenko, O., 1997, "Testing Rationality of Financial Markets: An Application to S&P 500 Index Options," Caltech working paper.

Bondarenko, O., 2000, "Recovering Risk-Neutral Densities: A New Nonparametric Approach," UIC working paper.

Bondarenko, O., 2002, "Why are Put Options So Expensive?" UIC working paper.

Bondarenko, O., and P. Bossaerts, 2000, "Expectations and Learning in Iowa," *Journal of Banking and Finance* 24, 1535-1555.

Bossaerts, P., 1999a, "Filtering Returns For Unspecified Biases In Priors When Testing Asset Pricing Theory" *Review of Economic Studies*, forthcoming.

Bossaerts, P., 1999b, "Learning-Induced Securities Price Volatility," Caltech working paper.

Bossaerts, P., and P. Hillion, 1998, "IPO Post-Issue Markets: Questionable Predilections But Diligent Learners?" Caltech working paper.

Breeden, D., and R. Litzenberger, 1978, "Prices of State Contingent Claims Implicit in Options Prices," *Journal of Business* 51, 621-652.

Brown, S., W. Goetzmann, and S. Ross, 1995, "Survival," *Journal of Finance* 50, 853-874.

Campbell, J., and J. Cochrane, 1999, "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior," *Journal of Political Economy* 107, 205-251.

Chopra, N., J. Lakonishok, and J. Ritter, 1992, "Measuring Abnormal Performance: Do stock overreact?" *Journal of Financial Economics* 31, 235-268.

Cochrane, J., and J. Saá-Requejo, 2000, "Beyond Arbitrage: "Good-Deal" Asset Pricing Bounds in Incomplete Markets," *Journal of Political Economy* 108, 79-119.

Cox, J., S. Ross, and M. Rubinstein, 1979, "Option Pricing: A Simplified Approach," *Journal of Financial Economics* 7, 229-263.

Cuoco, D., and F. Zapatero, 2000, "On the Recoverability of Preferences and Beliefs," *Review of Financial Studies* 13, 417-431.

Daniel, K., D. Hirshleifer, and A. Subrahmanyam, 1998, "Investor Psychology and Security Market Under- and Overreaction," *Journal of Finance* 53, 1839-1886.

Epstein, L., and S. Zin, 1989, "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," *Econometrica* 57, 937-968.

Epstein, L., and S. Zin, 1991, "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: An Empirical Investigation," *Journal of Political Economy* 99, 263-286.

Fama, E., 1970, "Efficient Capital Markets: A Review of Theory and Empirical Work," *Journal of Finance* 25, 383-417.

Hakansson, N., 1978, "Welfare Aspects of Options and Supershares," *Journal of Finance* 33, 759-776.

Hansen, L., and R. Jagannathan, 1991, "Implications of Security Market Data for Models of Dynamic Economies," *Journal of Political Economy* 99, 225-262.

Harrison, J., and D. Kreps, 1979, "Martingales and Arbitrage in Multiperiod Securities Markets," *Journal of Economic Theory* 20, 381-408.

Harrison, M., and S. Pliska, 1981, "Martingales and Stochastic Integrals in the Theory of Continuous Trading," *Stochastic Process and Their Applications* 11, 215-260.

Huang, C., and R. Litzenberger, 1988, *Foundations for Financial Economics*, North-Holland, New York.

Jackwerth, J., 1999, "Option Implied Risk-Neutral Distributions and Implied Binomial Trees: A Literature Review," *Journal of Derivatives* 7, 66-82.

Kraus, A., and G. Sick, 1980, "Distinguishing Beliefs and Preferences in Equilibrium Prices," *Journal of Finance* 35, 335-344.

Lakonishok, J., A. Shleifer, and R. Vishny, 1994, "Contrarian Investment, Extrapolation, and Risk," *Journal of Finance* 49, 1541-1578.

Ross, S., 1976, "Options and Efficiency," *Quarterly Journal of Economics* 90, 75-89.

Rubinstein, M., 1976, "The Valuation of Uncertain Income Streams and the Pricing of Options," *Bell Journal of Economics* 7, 407-425.

Weil, P., 1989, "The Equity Premium Puzzle and the Risk-Free Rate Puzzle," *Journal of Monetary Economics* 24, 401-421.