# Market Microstructure Knowledge Needed for Controlling an Intra-Day Trading Process

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#### **Abstract**

A great deal of academic and theoretical work has been dedicated to optimal liquidation of large orders these last twenty years. The optimal split of an order through time ('optimal trade scheduling') and space ('smart order routing') is of high interest to practitioners because of the increasing complexity of the market micro structure because of the evolution recently of regulations and liquidity worldwide. This chapter translates into quantitative terms these regulatory issues and, more broadly, current market design.

It relates the recent advances in optimal trading, order-book simulation and optimal liquidity to the reality of trading in an emerging global network of liquidity.

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### 1 Market Microstructure Modeling and Payoff Understanding are Key Elements of Quantitative Trading

As is well known, optimal (or quantitative) trading is about finding the proper balance between providing liquidity in order to minimize the impact of the trades, and consuming liquidity in order to minimize the market risk exposure, while taking profit through potentially instantaneous trading signals, supposed to be triggered by liquidity inefficiencies.

The mathematical framework required to solve this kind of optimization problem needs:

- a model of the consequences of the different ways of interacting with liquidity (such as the market impact model (Almgren et al., 2005; Wyart et al., 2008; Gatheral, 2010));
- a proxy for the 'market risk' (the most natural of them being the high frequency volatility (Aït-Sahalia and Jacod, 2007; Zhang et al., 2005; Robert and Rosenbaum, 2011));
- and a model for quantifying the likelihood of the liquidity state of the market (Bacry et al., 2009; Cont et al., 2010).

A utility function then allows these different effects to be consolidated with respect to the goal of the trader:

- minimizing the impact of large trades under price, duration and volume constraints (typical for brokerage trading (Almgren and Chriss, 2000));
- providing as much liquidity as possible under inventory constraints (typical for market-makers Avellaneda and Stoikov (2008) or Guéant et al. (2011));
- or following a belief about the trajectory of the market (typical of arbitrageurs (Lehalle, 2009)).

Once these key elements have been defined, rigorous mathematical optimization methods can be used to derive the optimal behavior (Bouchard et al., 2011; Predoiu et al., 2011). Since the optimality of the result is strongly dependent on the phenomenon being modeled, some understanding of the market microstructure is a prerequisite for ensuring the applicability of a given theoretical framework.

The *market microstructure* is the ecosystem in which buying and selling interests meet, giving birth to trades. Seen from outside the microstructure, the prices of the traded shares are often uniformly sampled to build time series that are modeled via martingales (Shiryaev, 1999) or studied using econometrics. Seen from the inside of

electronic markets, buy and sell open interests (i.e. passive limit orders) form *limit* order books, where an impatient trader can find two different prices: the highest of the resting buy orders if he needs to sell, and the lowest of the selling ones if he needs to buy (see Figure 1). The buying and selling price are thus not equal. Moreover, the price will monotonically increase (for impatient buy orders) or decrease (for impatient sell orders) with the quantity to trade, following a concave function (Smith et al., 2003): the more you trade, the worse the price you will get.

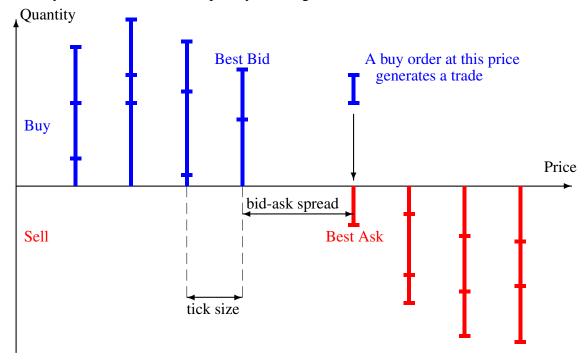


Figure 1: Idealized order-book

The market microstructure is strongly conditioned by the *market design*:

- the set of explicit rules governing the *price formation process* (PFP);
- the type of auction (fixing or continuous ones); the tick size (i.e. the minimum allowed difference between two consecutive prices);
- the interactions between trading platforms (such as 'trade-through rules', pegged orders, interactions between visible and hidden orders, etc.);

are typical elements of the market design.

The market microstructure of an asset class is a mix of the market design, the trading behaviors of trading agents, the regulatory environment, and the availability of correlated instruments (such as Equity Traded Funds, Futures or any kind of derivative

products). Formally, the microstructure of a market can be seen as several sequences of auction mechanisms taking place in parallel, each of them having its own particular characteristics. For instance the German market place is mainly composed (as of 2011) of the Deutsche Börse regulated market, the Xetra mid-point, the Chi-X visible order book, Chi-delta (the Chi-X hidden mid-point), Turquoise Lit and Dark Pools, BATS pools. The regulated market implements a sequence of fixing auctions and continuous auctions (one open fixing, one continuous session, one mid-auction and one closing auction); others implement only continuous auctions, and Turquoise mid-point implements optional random fixing auctions.

To optimize his behavior, a trader has to choose an abstract description of the microstructure of the markets he will interact with: this will be his model of market microstructure. It can be a statistical 'macroscopic' one as in the widely-used Almgren—Chriss framework (Almgren and Chriss, 2000), in which the time is sliced into intervals of 5 or 10 minutes duration during which the interactions with the market combine two statistical phenomena:

- the market impact as a function of the 'participation rate' of the trader;
- and the volatility as a proxy of the market risk.

It can also be a microscopic description of the order book behavior as in the Alfonsi–Schied proposal (Alfonsi et al., 2010) in which the shape of the order book and its resilience to liquidity-consuming orders is modeled.

This chapter will thus describe some relationships between the market design and the market microstructure using European and American examples since they have seen regulatory changes (in 2007 for Europe with the MiFI Directive, and in 2005 for the USA with the NMS regulation) as much as behavioral changes (with the financial crisis of 2008). A detailed description of some important elements of the market microstructure will be conducted:

- dark pools;
- impact of fragmentation on the price formation process;
- tick size;
- auctions, etc.

Key events like the 6 May 2010 flash crash in the US market and some European market outages will also receive attention.

To obtain an optimal trading trajectory, a trader needs to define its payoff. Here also, choices have to be made from a mean-variance criterion (Almgren and Chriss, 2000) to stochastic impulse control (Bouchard et al., 2011) going through stochastic algorithms

(Pagès et al., 2012). This chapter describes the statistical viewpoint of the Almgren–Chriss framework, showing how practitioners can use it to take into account a large variety of effects. It ends with comments on an order-flow oriented view of optimal execution, dedicated to smaller time-scale problems, such as 'Smart Order Routing' (SOR).

## 2 From Market Design to Market Microstructure: Practical Examples

The recent history of the French equity market is archetypal in the sense that it went from a highly concentrated design with only one electronic platform hosted in Paris (Muniesa, 2003) to a fragmented pan-European one with four visible trading pools and more than twelve 'dark ones', located in London, in less than four years.

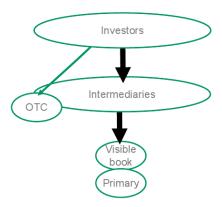


Figure 2: Idealized pre-fragmentation market microstructures

Seen by economists and from outside the microstructure, the equity market is a place where *listed firms* raise capital offering shares for sale. Once shares are available for buying and selling in the market place, the mechanism of balance between offer and demand (in terms of intentions to buy and intentions to sell) forms a *fair price*.

At the microstructure scale, the market place is more sophisticated. Market participants are no longer just listed firms and investors making rational investment decisions; microstructure focuses on the process that allows investors to buy from, or sell to, one another, putting emphasis on the *Price Formation Process*, also known as *Price Discovery*. Moreover, recent regulations promote the use of electronic markets, since they are compatible with the recording and traceability levels such markets provide, leading to *fragmented markets*. It is worthwhile differentiating between two states of the

microstructure: pre- and post-fragmentation, see Figure 2 and 4:

- *Pre-fragmented microstructure*: before Reg NMS in the US and MiFID in Europe, the microstructure can be pictured as three distinct layers:
  - investors, taking buy or sell decisions;
  - intermediaries, giving unbiased advice (through financial analysts or strategists) and providing access to trading pools they are members of; low frequency market makers (or maker-dealers) can be considered to be part of this layer;
  - market operators: hosting the trading platforms, NYSE Euronext, NAS-DAQ, BATS, Chi-X, belong to this layer. They are providing matching engines to other market participants, hosting the *Price Formation Process*.

These three layers are simply connected: intermediaries concentrate a fraction of the buying and selling flows in a (small) *Over the Counter* (OTC) market, the remaining open interests are placed in the order books of the market operators. Facilitators (i.e. low frequency market makers or specialists), localized in the same layer as the intermediaries, provide liquidity, thus minimizing the *Market Impact* of orders from under-coordinated investors (i.e. when a large buyer comes to the market two hours after a large seller, any liquidity provider that is able to sell to the first one and buy to the later will prevent a price oscillation; on the one hand he will be 'rewarded' for this service through the bid—ask spread he will demand of the two investors; on the other hand he will take the risk of a large change in the *fair price* that is in between the two transactions (Gabaix et al., 2006), see Figure 3).

- *Post-fragmented markets*: regulations have evolved with the aim of implementing more competition across each layer of Figure 3 (especially across market operators) and increasing transparency:
  - in the US, Reg NMS decided to keep the competition inside the layer of market operators: it requires an Exchange or an Electronic Communication Network (ECN) to route an order to the platform that offers the best match (it is called the *trade-through rule*). For instance, if a trader sends a buy order at \$10.00 to BATS where the best ask price is \$9.75 and if the best ask for this stock is \$9.50 on NYSE, BATS has to re-route the order to NYSE. This regulation needs two important elements:
    - (1) a way of pushing to all market operators the best bid and ask of any available market with accuracy (it raises concerns linked to the latency of market data);



Figure 3: Idealized kinematics of market impact caused by bad synchronization (A1–A2–A3 sequence) and preservation of the market depth thanks to a market maker agreeing to support market risk (B1–B2-B3 sequence).

(2) that buying at \$9.50 on NYSE is always better for a trader than buying at \$9.75 on BATS, meaning that the other trading costs (especially clearing and settlement costs) are the same.

The data conveying all the best bid and asks is called the *consolidated pre-trade tape* and its best bid and offer is called the *National Best-Bid and Offer* (NBBO).

- in Europe, mainly because of the diversity of the clearing and settlement channels, MiFID allows the competition to be extended to the intermediaries: they are in charge of defining their *Execution Policies* describing how and why they will route and split orders across market operators. The European Commission thus relies on competition between execution policies as the means of selecting the best way of splitting orders, taking into account all trading costs. As a consequence, Europe does not have any officially consolidated pre-trade tape.

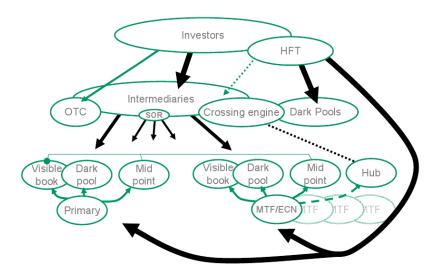


Figure 4: Idealized post-fragmentation market microstructure.

Despite these differences, European and US electronic markets have a lot in common: their microstructures evolved similarly to a state where latency is crucial and *High Frequency Market-Makers* (also called *High Frequency Traders*) became the main liquidity providers of the market.

Figure 4 gives an idealized view of this fragmented microstructure:

- A specific class of investors: the *High Frequency Traders* (HFT) are an essential part of the market; by investing more than other market participants in technology, thus reducing their latency to markets, they have succeeded in:
  - \* implementing market-making-like behaviors at high frequency;
  - \* providing liquidity at the bid and ask prices when the market has low probability of moving (thanks to statistical models);
  - \* being able to cancel very quickly resting orders in order to minimize the market risk exposure of their inventory;

they are said to be feature in 70% of the transactions in US Equity markets,

- 40% in Europe and 30% in Japan in 2010. Their interactions with the market have been intensively studied by Menkveld (2010).
- Because they are the main customers of market operators, HFTs offered new features making it easier to conduct their business: low latency access to matching engines (better quality of service and *co-hosting*; i.e. the ability to locate their computers physically close to the ones of the matching engines), and even *flash orders* (knowing before other market participants that an order is being inserted in the order-book).
- Market participants that were not proprietary high-frequency traders also sought specific features of the order books, mainly to hide their interests from high frequency traders: *Dark Pools*, implementing anonymous auctions (i.e. partially observable), are part of this offer.
- The number of market operators as firms does not increase that much when a market goes from non-fragmented to fragmented, because of high technological costs linked to a fragmented microstructure. On the other hand, each operator offers more products (order books) to clients when fragmentation increases. The BATS and Chi-X Europe merged and the London Stock Exchange–Milan Stock Market–Turquoise trading also formed a single group. Looking at the European order-books offered by NYSE-Euronext in 2011 only, we have:
  - \* several visible (i.e. *Lit*) *order books*: one for Paris–Amsterdam–Brussels stocks, another (NYSE–Arca Europe) for other European names;
  - \* *Mid-points*: an order book with only one queue *pegged* at the mid-price of a reference market (SmartPool);
  - \* *Dark pools*: an anonymous order book (i.e. market participants can send orders as in a Lit book, but no-one can read the state of the book);
  - \* *Fixing auctions*, opening and closing the continuous auctions on visible books.

The result is an interconnected network of liquidity in which each market participant is no longer located in one layer only: HFTs are simultaneously investors and also very close to market operators, intermediaries are offering *Smart Order Routers* to split optimally orders across all available trading pools whilst taking into account the specific liquidity needs of each investor. Thus, market operators are close to technology providers.

The regulatory task is thus more sophisticated in a fragmented market rather than in a concentrated one:

• the *Flash Crash* of 6 May 2010 in US markets raised concerns about the stability of such a microstructure (see Figure 5);

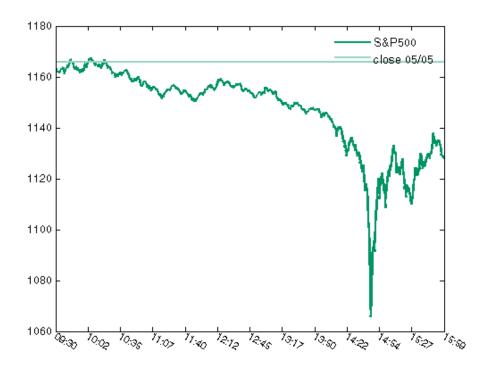


Figure 5: The 'Flash Crash': 6 May 2010, US market rapid down-and-up move by almost 10% was only due to market microstructure effects.

• the cost of surveillance of trading flows across a complex network is higher than in a concentrated one.

Moreover, elements of the market design play many different roles: the *tick size* for instance, is not only the minimum between two consecutive different prices, i.e., a constraint on the bid-ask spread, it is also a key in the competition between market operators. In June 2009, European market operators tried to gain market shares by reducing the tick size on their order books. Each time one of them offered a lower tick than others, it gained around 10% of market shares (see Figure 6). After a few weeks of competition on the tick, they limited this kind of infinitesimal decimation of the tick thanks to a gentleman's agreement obtained under the umbrella of the FESE (Federation of European Security Exchanges): such a decimation had been expensive in CPU and memory demand for their matching engines.

An idealized view of the 'Flash Crash'. The flash crash was accurately described in Kirilenko et al. (2010). The sequence of events that led to a negative jump in price and a huge increase in traded volumes in few minutes, followed by a return to normal in less

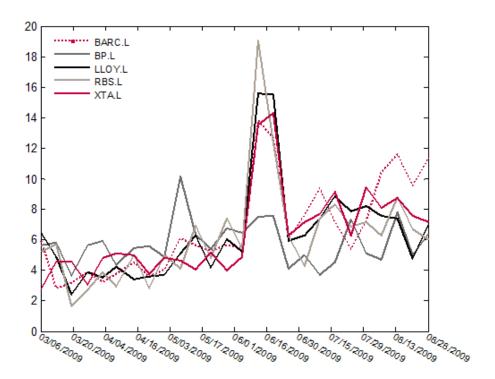


Figure 6: The 'Tick war' in June 2009, in Europe. The increase of market share of Turquoise (an European Multilateral Trading Facility; MTF) on five Stocks listed on the London Stock Exchange following a decrease of the tick size. When other MTFs lowered the tick size, the market share returned to the previous level.

#### than 20 minutes can be pictured as follows:

- (1) A final investor decided to sell a large amount  $v^*$  of shares of the E-Mini future contracts, asking a broker to take care of this sell by electronic means on his behalf.
- (2) The broker decided to use a *PVOL* (i.e. Percentage of Volume) algo, with the instruction to follow almost uniformly 9% of the market volume without regard to price or time. This participation rate is not uncommon (it is usual to see PVOL algos with the instruction to follow 20% of the market volume).
- (3) The trading algorithm could be seen as a trade scheduler splitting the order in slices of one-minute intervals, expecting to see a traded volume  $V_t$  during the tth slice (meaning that  $\mathbb{E}(V_t) \simeq \overline{V}/500$ , where  $\overline{V}$  is the expected daily traded volume).
- (4) For its first step, the algo began to sell on the future market around  $v_0 = \mathbb{E}(V_0) \times 9/(100-9) \simeq \overline{V}/500 \times 0.09$  shares,

- (5) The main buyers of these shares had been intra-day market makers; say that they bought (1-q) of them.
- (6) Because the volatility was quite high on 6 May 2010, the market makers did not feel comfortable with such an imbalanced inventory, and so decided to hedge it on the cash market, selling  $(1-q) \times v_t$  shares of a properly weighted basket of equities.
- (7) Unfortunately the buyers of most of these shares (say (1-q) of them again) were intra-day market makers themselves, who decided in their turn to hedge their risk on the future market.
- (8) It immediately increased the traded volume on the future market by  $(1-q)^2v_0$  shares.
- (9) Assuming that intra-day market makers could play this *hot potato game* (as it was called in the SEC–CFTC report), N times in 1 minute, the volume traded on the future market became  $\sum_{n \le N} (1-q)^{2n} v_0$  larger than expected by the brokerage algo.
- (10) Back to step (4) at t+1, the PVOL algo is now late by  $\sum_{n \le N} (1-q)^{2n} v_0 \times 8/(100-8)$ , and has to sell  $\overline{V}/500 \times 8/(100-8)$  again; i.e. selling

$$v_{t+1} \simeq \left(N \times v_t + \frac{\overline{V}}{500}\right) \times 0.08.$$

Figure 7 shows how explosive the *hot potato game* between intra-day market makers can be, even with not that high a frequency trading rate (here N = 1.1). Most of this trading flow was a selling flow, pushing most US prices to very low levels. For instance Procter and Gamble quoted from \$60 to a low of \$39.37 in approximately 3.5 minutes.

- In reality other effects contributed to the flash crash:
- only a few trading pools implemented circuit breakers that ought to have frozen the matching engines in case of sudden liquidity event;
- most market participants only looked at the consolidated tape for market data, preventing them noticing that trading was frozen on some pools;
- in the US, most retail flow is internalized by market makers. At one point in the
  day these intermediaries decided to hedge their positions on the market on their
  turn, further affecting the prices.

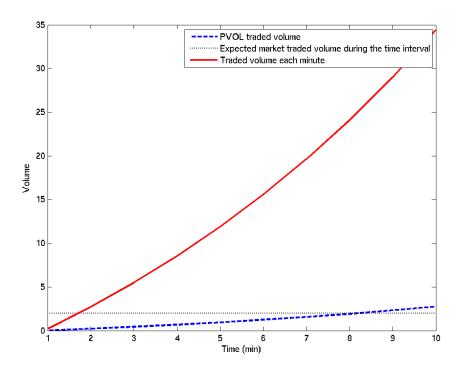


Figure 7: Traded volume of the future market according to the simple idealized model with  $\overline{V} = 100$ , T = 10 and N = 2.

This glitch in the electronic structure of markets is not an isolated case, even if it was the largest one. The combination of a failure in each layer of the market (an issuer of large institutional trades, a broker, HF market-makers, market operators) with a highly uncertain market context is surely a crucial element of this crash. It has moreover shown that most orders do indeed reach the order books only through electronic means.

European markets did not suffer from such *flash crashes*, but they have not seen many months in 2011 without an outage of a matching engine.

**European outages.** Outages are 'simply' bugs in matching engines. In such cases, the matching engines of one or more trading facilities can be frozen, or just stop publishing market data, becoming true *Dark Pools*. From a scientific viewpoint, and because in Europe there is no *consolidated pre-trade tape* (i.e. each member of the trading facilities needs to build by himself his *consolidated view* of the current European best bid and offer), they can provide examples of behavior of market participants when they do not all share the same level of information about the state of the offer and demand.

For instance:

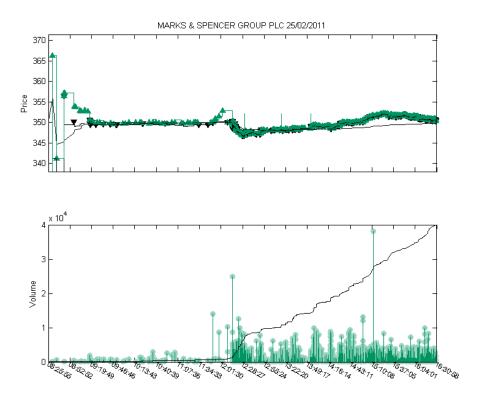


Figure 8: Examples of outages in European equity markets on 25 February 2011. The price (top) and the volumes (bottom) when the primary market opened only after 12:15 (London time). The price did not move much.

- when no information is available on primary markets but trading remains open: two price formation processes can take place in parallel, one for market participants having access to other pools, and the other for participants who just looked at the primary market;
- (Figure 8) when the primary market does not start trading at the very beginning of the day: the price does not really move on alternative markets; no 'real' price formation process takes place during such European outages.

The flash crash in US and the European outages emphasizes the *role of information* in the price formation process. When market participants are confident that they have access to a reliable source of information (during the flash or during some European outages), they continue to *mimic* a price formation process which output can be far from efficient. By contrast, if they do not believe in the information they have, they just freeze their price, observe behavior and trade at the *last confident price*, while waiting for reliable updates.

### 3 Forward and Backward Components of the Price Formation Process

The literature on market microstructure can be split in two generic subsets:

• papers with a *Price Discovery* viewpoint, in which the market participants are injecting into the order book their views on a fair price. In these papers (see for instance Biais et al. (2005); Ho and Stoll (1981); Cohen et al. (1981)), the *fair price* is assumed to exist for fundamental reasons (at least in the mind of investors) and the order books are implementing a Brownian-bridge-like trajectory targeting this evolving fair price. This is a *backward* view of the price dynamics: the investors are updating assumptions on the future value of tradeable instruments, and send orders in the electronic order books according to the distance between the current state of the offer and demand and this value, driving the quoted price to some average of what they expect.

Figure 9 shows a price discovery pattern: the price of the stock changes for fundamental reasons, and the order book dynamics react accordingly generating more volume, more volatility, and a price jump.

• Other papers rely on a *Price Formation Process* viewpoint. For their authors (most of them econophysicists, see for instance Smith et al. (2003); Bouchaud et al. (2002) or Chakraborti et al. (2011) for a review of agent based models of order books) the order books are *building the price* in a forward way. The market participants take decisions with respect to the current orders in the books making assumptions of the future value of their inventory; it is a *forward* process.

Following Lehalle et al. (2010), one can try to crudely model these two dynamics simultaneously. In a framework with an infinity of agents (using a Mean Field Game approach, see Lasry and Lions (2007) for more details), the order book at the bid (respectively at the ask), is a density  $m_B(t,p)$  (resp.  $m_A(t,p)$ ) of agents agreeing at time t to buy (resp. sell) at price p. In such a continuous framework, there is no bid—ask spread and the *trading price*  $p^*(t)$  is such that there is no offer at a price lower than  $p^*(t)$  (and no demand at a price greater then  $p^*(t)$ ). Assuming diffusivity, the two sides of the order book are subject to the following simple partial differential equations:

$$\partial_{t} m_{B}(t,p) - \frac{\varepsilon^{2}}{2} \partial_{pp}^{2} m_{B}(t,p) = \lambda(t) \delta_{p=p^{*}(t)}$$

$$\partial_{t} m_{A}(t,p) - \frac{\varepsilon^{2}}{2} \partial_{pp}^{2} m_{A}(t,p) = \lambda(t) \delta_{p=p^{*}(t)}.$$

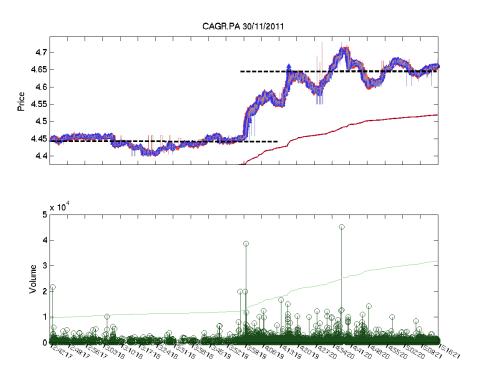


Figure 9: A typical *Price Discovery* exercise: the 30th of November, 2011 on the Crédit Agricole share price (French market). The two stable states of the price are materialized using two dark dotted lines, one before and the other after the announcement by major European central banks a coordinated action to provide liquidity.

Moreover, the trading flow at  $p^*(t)$  is clearly defined as

$$\lambda(t) = -\frac{\varepsilon^2}{2} \partial_p m_B(t, p^*(t)) = \frac{\varepsilon^2}{2} \partial_p m_A(t, p^*(t)).$$

It is then possible to define a regular order book m joining the bid and ask sides by

$$m(t,p) = \begin{cases} m_B(t,p) &, & \text{if } p \le p^*(t) \\ -m_A(t,p) &, & \text{if } p > p^*(t) \end{cases}$$

which satisfies a single parabolic equation:

$$\partial_{t}m(t,p) - \frac{\varepsilon^{2}}{2}\partial_{pp}^{2}m(t,p) = -\frac{\varepsilon^{2}}{2}\partial_{p}m(t,p^{*}(t))\left(\delta_{p=p^{*}(t)-a} - \delta_{p=p^{*}(t)+a}\right) \tag{1}$$

with a limit condition  $m(0,\cdot)$  given on the domain  $[p_{\min},p_{\max}]$  and, for instance, Neumann conditions at  $p_{\min}$  and  $p_{\max}$ .

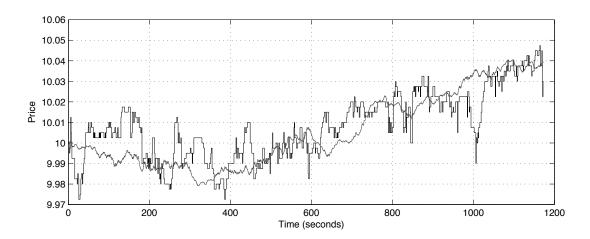


Figure 10: Simulation of the dynamics modeling an order book using a forward–backward approach: the 'fair price' is the continuous grey line and the realized price is the stepwise dark one.

Such a *forward process* describes the order book dynamics without any impact on investors' fundamental views (it is a *price formation process* model).

Lehalle et al. then introduce a more complex source to re-inject the orders in books containing market participants' forward views on the price. For instance, a trend follower with a time horizon of h buying at price  $p^*(t)$  at time t aims to unwind his position at a higher (i.e. 'trend targeted') price and thus insert an order in the book accordingly (around  $p^*(t) + (p^*(t) - p^*(t - h))$ : see the paper for more details). Figure 10 shows an example of such a dynamic.

This is a way of introducing investor-driven views into the model, which are essentially *backward*: a trend follower agrees to be part of a transaction because he believes that the price will continue to move in the same direction over his investment time scale. This future price of the share is at the root of his decision. This is an injection of a *price discovery* component in the model.

## 4 From Statistically Optimal Trade Scheduling to Microscopic Optimization of Order Flows

Modeling the price formation dynamics is of interest for both regulators and policy makers. It enables them to understand the potential effects of a regulatory or rule change on the efficiency of the whole market (see for instance Foucault and Menkveld (2008) for an analysis of the introduction of competition among trading venues on the efficiency of the markets). It thus helps in understanding potential links between market design

and systemic risk.

In terms of risk management inside a firm hosting trading activities, it is more important to understand the trading cost of a position, which can be understood as its *liquidation risk*.

From the viewpoint of one trader versus the whole market, three key phenomena have to be controlled:

- the *market impact* (see Kyle (1985); Lillo et al. (2003); Engle et al. (2012); Almgren et al. (2005); Wyart et al. (2008)) which is the market move generated by selling or buying a large amount of shares (all else being equal); it comes from the forward component of the price formation process, and can be temporary if other market participants (they are part of the backward component of the price discovery dynamics) provide enough liquidity to the market to bring back the price to its previous level;
- *adverse selection*, capturing the fact that providing too much (passive) liquidity via limit orders enables the trader to maintain the price at an artificial level; not a lot of literature is available about this effect, which has been nevertheless identified by practitioners (Altunata et al., 2010);
- and the uncertainty on the fair value of the stock that can move the price during the trading process; it is often referred as the *intra-day market risk*.

#### 4.1 Replacing market impact by statistical costs

A framework now widely used for controling the overall costs of the liquidation of a portfolio was proposed by Almgren and Chriss in the late 1990s Almgren and Chriss (2000). Applied to the trade of a single stock, this framework:

- cuts the trading period into an arbitrary number of intervals N of a chosen duration  $\delta t$ ,
- models the *fair price* moves thanks to a Gaussian random walk:

$$S_{n+1} = S_n + \sigma_{n+1} \sqrt{\delta t} \, \xi_{n+1} \tag{2}$$

• models the *temporary market impact*  $\eta_n$  inside each time bin using a power law of the trading rate (i.e. the ratio of the traded shares  $v_n$  by the trader over the market traded volume during the same period  $V_n$ ):

$$\eta(v_n) = a \psi_n + \kappa \sigma_n \sqrt{\delta t} \left(\frac{v_n}{V_n}\right)^{\gamma}$$
(3)

where a,  $\kappa$  and  $\gamma$  are parameters, and  $\psi$  is the half bid-ask spread;

- assumes the *permanent market impact* is linear in the participation rate;
- uses a mean-variance criterion and minimizes it to obtain the optimal sequence of shares to buy (or sell) through time.

It is important first to notice that there is an implicit relationship between the time interval  $\delta t$  and the temporary market impact function: without changing  $\eta$  and simply by choosing a different time slice, the cost of trading can be changed. It is in fact not possible to choose  $(a, \kappa, \gamma)$  and  $\delta t$  independently; they have to be chosen according to the decay of the market impact on the stock, provided that most of the impact is kept in a time bin of size  $\delta t$ . Not all the decay functions are compatible with this view (see Gatheral and Schied (2012) for details about available market impact models and their interactions with trading). Up to now the terms in  $\sqrt{\delta t}$  have been ignored. Note also that the parameters  $(a, \kappa, \gamma)$  are relevant at this time scale.

One should not regard this framework as if it were based on structural model assumptions (i.e. that the market impact really has this shape, or that the price moves really are Brownian), rather, as if it were a statistical one. With such a viewpoint, any practitioner can use the database of its past executed orders and perform an econometric study of its 'trading costs' on any interval,  $\delta t$ , of time (see Engle et al. (2012) for an analysis of this kind on the whole duration of the order). If a given time scale succeeds in capturing, with enough accuracy, the parameters of a trading cost model, then that model can be used to optimize trading. Formally, the result of such a statistical approach would be the same as that of a structural one, as we will show below. But it is possible to go one step further, and to take into account the statistical properties of the variables (and parameters) of interest.

Going back to the simple case of the liquidation of one stock without any permanent market impact, the value (which is a random variable) of a buy of  $v^*$  shares in N bins of size  $v_1, v_2, \ldots, v_N$  is

$$W(v_{1}, v_{2}, ..., v_{N}) = \sum_{n=1}^{N} v_{n}(S_{n} + \eta_{n}(v_{n}))$$

$$= S_{0}v^{*} + \sum_{n=1}^{N} \sigma_{n}\xi_{n}x_{n}$$
market move
$$+ \sum_{n=1}^{N} a \psi_{n}(x_{n} - x_{n+1}) + \kappa \frac{\sigma_{n}}{V_{n}^{\gamma}}(x_{n} - x_{n+1})^{\gamma+1}, \qquad (4)$$
market impact

using the *remaining quantity to buy*: that is,  $x_n = \sum_{k \ge n} v_k$  instead of the instantaneous volumes  $v_n$ . To obtain an answer in as closed a form as possible,  $\gamma$  will be taken equal to

1 (i.e. linear market impact). (See Bouchard et al. (2011) for a more sophisticated model and more generic utility functions rather than the idealized model which we adopt here in order to obtain clearer illustrations of phenomena of interest.)

To add a practitioner-oriented flavor to our upcoming optimization problems, just introduce a set of independent random variables  $(A_n)_{1 \le n \le N}$  to model the *arbitrage opportunities* during time slices. It will reflect our expectation that the trader will be able to buy shares at price  $S_n - A_n$  during slice n rather than at price  $S_n$ .

Such an effect can be used to inject a statistical arbitrage approach into optimal trading or to take into account the possibility of crossing orders at mid price in Dark Pools or Broker Crossing Networks (meaning that the expected trading costs should be smaller during given time slices). Now the cost of buying  $v^*$  shares is:

$$W(\mathbf{v}) = S_0 v^* + \sum_{n=1}^{N} \sigma_n \xi_n x_n + \sum_{n=1}^{N} (a \psi_n - A_n) v_n + \kappa \frac{\sigma_n}{V_n} v_n^2$$
 (5)

**Conditioned expectation optimization.** The expectation of this cost,

$$\mathbb{E}(W|(V_n,\sigma_n,\psi_n)_{1\leq n\leq N}),$$

given the market state, can be written as

$$C_0 = S_0 v^* + \sum_{n=1}^{N} (a \, \psi_n - \mathbb{E} A_n) v_n + \kappa \frac{\sigma_n}{V_n} v_n^2, \tag{6}$$

A simple optimization under constraint (to ensure  $\sum_{n=1}^{N} v_n = v^*$ ) gives

$$v_n = w_n \left( v^* + \frac{1}{\kappa} \left( \left( \mathbb{E} A_n - \sum_{\ell=1}^N w_\ell \mathbb{E} A_\ell \right) - a \left( \psi_n - \sum_{\ell=1}^N w_\ell \psi_\ell \right) \right) \right), \tag{7}$$

where  $w_n$  are weights proportional to the inverse of the market impact factor:

$$w_n = \frac{V_n}{\sigma_n} \left( \sum_{\ell=1}^N \frac{V_\ell}{\sigma_\ell} \right)^{-1}.$$

Simple effects can be deduced from this first idealization.

(1) Without any arbitrage opportunity and without any bid-ask cost (i.e.  $\mathbb{E}A_n = 0$  for any n and a = 0), the optimal trading rate is proportional to the inverse of the market impact coefficient:  $v_n = w_n \cdot v^*$ . Moreover, when the market impact has no intra-day seasonality,  $w_n = 1/N$  implying that the optimal trading rate is linear.

(2) Following formula (7) it can be seen that the greater the expected arbitrage gain (or the lower the spread cost) on a slice compared to the market-impact-weighted expected arbitrage gain (or spread cost) over the full trading interval, the larger the quantity to trade during this slice. More quantitatively:

$$\frac{\partial v_n}{\partial \mathbb{E} A_n} = \frac{w_n}{2\kappa} (1 - w_n) > 0, \ \frac{\partial v_n}{\partial \psi_n} = -\frac{a}{2\kappa} (1 - w_n) w_n < 0.$$

This result gives the *adequate weight* for applying to the expected arbitrage gain in order to translate it into an adequate trading rate so as to profit on arbitrage opportunities on average. Just note that usually the expected arbitrage gains increase with market volatility, so the  $w_n$ -weighting is consequently of interest to balance this effect optimally.

Conditioned mean-variance optimization. Going back to a mean-variance optimization of the cost of buying progressively  $v^*$  shares, the criterion for minimizing (using a risk aversion parameter  $\lambda$ ) becomes

$$C_{\lambda} = \mathbb{E}(W|(V_{n}, \sigma_{n}, \psi_{n})_{1 \leq n \leq N}) + \lambda \mathbb{V}(W|(V_{n}, \sigma_{n}, \psi_{n})_{1 \leq n \leq N})$$

$$= S_{0}v^{*} + \sum_{n=1}^{N} (a\psi_{n} - \mathbb{E}A_{n})(x_{n} - x_{n+1}) + \left(\kappa \frac{\sigma_{n}}{V_{n}} + \lambda \mathbb{V}A_{n}\right)(x_{n} - x_{n+1})^{2} + \lambda \sigma_{n}^{2}x_{n}^{2}.$$
(8)

To minimize  $C_{\lambda}$  when it is only constrained by terminal conditions on x (i.e.  $x_0 = v^*$  and  $v_{N+1} = 0$ ), it is enough to cancel its derivatives with respect to any  $x_n$ , leading to the recurrence relation

$$\left(\frac{\sigma_{n}}{V_{n}} + \frac{\lambda}{\kappa} \mathbb{V} A_{n}\right) x_{n+1} = \frac{1}{2\kappa} \left(a(\psi_{n-1} - \psi_{n}) - (\mathbb{E} A_{n-1} - \mathbb{E} A_{n})\right) \\
+ \left(\frac{\lambda}{\kappa} \sigma_{n}^{2} + \left(\frac{\sigma_{n}}{V_{n}} + \frac{\lambda}{\kappa} \mathbb{V} A_{n} + \frac{\sigma_{n-1}}{V_{n-1}} + \frac{\lambda}{\kappa} \mathbb{V} A_{n-1}\right)\right) x_{n} \\
- \left(\frac{\sigma_{n-1}}{V_{n-1}} + \frac{\lambda}{\kappa} \mathbb{V} A_{n-1}\right) x_{n-1}.$$
(9)

This shows that the variance of the arbitrage has an effect similar to that of the market impact (through a risk-aversion rescaling), and that the risk-aversion parameter acts as a multiplicative factor on the market impact, meaning that within an arbitrage-free and spread-costs-free framework (i.e. a = 0 and  $\mathbb{E}A_n = 0$  for all n), the market impact model for any constant b has no effect on the final result as long as  $\lambda$  is replaced by  $b\lambda$ .

Figure 11 compares optimal trajectories coming from different criteria and parameter values.

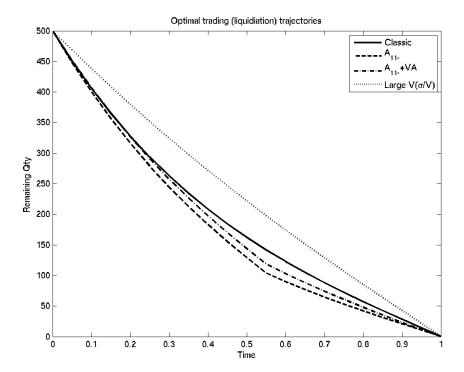


Figure 11: Examples of optimal trading trajectories for mean-variance criteria: the classical result (Almgren-Chriss) is the solid line, the dotted line is for high variance of the variable of interest ( $\sigma/V$ ), the semi-dotted ones for an arbitrage opportunity ( $A_{11+}$  means after the 11th period; and  $A_{11+} + VA$  means adding expected variance to the arbitrage opportunity).

**A statistical viewpoint.** The two previous examples show how easy it is to include effects in this sliced mean-variance framework. The implicit assumptions are:

- within one time-slice, it is possible to capture the market impact (or *trading costs*) using model (3);
- the trader knows the traded volumes and market volatility in advance.

In practical terms, the two assumptions come from statistical modeling:

• The market impact parameters  $a, \kappa$  and  $\gamma$  are estimated on a large database of trades using a maximum likelihood or MSE methods; the reality is consequently that the market model has the following shape:

$$\eta(v_n) = a \psi_n + \kappa \sigma_n \sqrt{\delta t} \left(\frac{v_n}{V_n}\right)^{\gamma} + \varepsilon, \tag{10}$$

where  $\varepsilon$  is an i.i.d. noise.

• Moreover, the market volatility and traded volumes are estimated using historical data and market context assumptions (to take into account at least the scheduled news, such as the impact of the expiry of derivative products on the volume of the cash market; see Figure 12 for typical estimates).

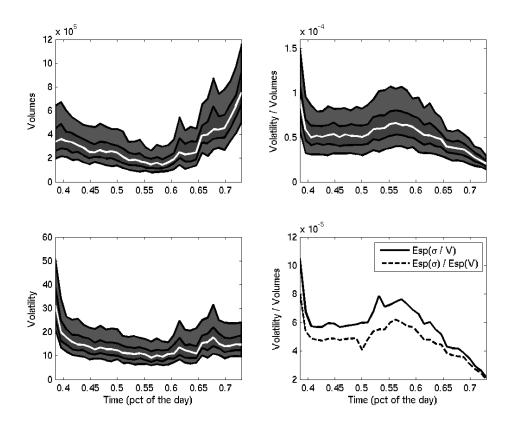


Figure 12: Typical intra-day traded volume (top left) and realized volatility (bottom left) profiles (i.e. intra-day seasonalities on traded volumes and market volatility) with their quantiles of level 25% and 75%. The *x*-axis is time. The top right chart contains the quantiles of the ratio of interest  $\sigma/V$ . The bottom right ones shows the difference between the expectation of the ratio (solid line) and the ratio of the expectations (dotted line).

Taking these statistical modeling steps into account in the classical mean–variance criterion of (8), changes that equation into its unconditioned version:

$$\tilde{C}_{\lambda} = \mathbb{E}(W) + \lambda \mathbb{V}(W) 
= S_{0}v^{*} + \sum_{n=1}^{N} (a\mathbb{E}\psi_{n} - \mathbb{E}A_{n})(x_{n} - x_{n+1}) 
+ \left(\kappa \mathbb{E}\left(\frac{\sigma_{n}}{V_{n}}\right) + \lambda (a^{2}\mathbb{V}\psi_{n} + \mathbb{V}A_{n} + \mathbb{V}\varepsilon)\right)(x_{n} - x_{n+1})^{2} 
+ \lambda \sigma_{n}^{2}x_{n}^{2} + \lambda \kappa^{2}\mathbb{V}\left(\frac{\sigma_{n}}{V_{n}}\right)(x_{n} - x_{n+1})^{4}.$$
(11)

The consequences of using this criterion rather than the conditioned one are clear:

- the simple plug-in of empirical averages of volumes and volatility in criterion (8) instead of the needed expectation of the overall trading costs leads us to use  $(\mathbb{E}\sigma_n)/(\mathbb{E}V_n)$  instead of  $\mathbb{E}(\sigma_n/V_n)$ . Figure 12 shows typical differences between the two quantities.
- If the uncertainty on the market impact is huge (i.e. the  $V\varepsilon$  term dominates all others), then the optimal trading strategy is to trade linearly, which is also the solution of a purely expectation-driven minimization with no specific market behavior linked with time.

Within this new statistical trading framework, the inaccuracy of the models and the variability of the market context are taken into account: the obtained optimal trajectories will no longer have to follow sophisticated paths if the models are not realistic enough.

Moreover, it is not difficult to solve the optimization program associated to this new criterion; the new recurrence equation is a polynomial of degree 3. Figure 11 gives illustrations of the results obtained.

Many other effects can be introduced in the framework, such as auto-correlations on the volume–volatility pair. This statistical framework does not embed recent and worth-while proposals such as the decay of market impact (Gatheral and Schied, 2012) or a set of optimal stopping times that avoid a uniform and a priori sampled time (Bouchard et al., 2011). It is nevertheless simple enough so that most practitioners can use it in order to include their views of the market conditions and the efficiency of their interactions with the market on a given time scale; it can be compared to the Markowitz approach for quantitative portfolio allocation (Markowitz, 1952).

### 4.2 An order-flow oriented view of optimal execution

Though price dynamics in quantitative finance are often modeled using diffusive processes, just looking at prices of transactions in a limit order book convinces one that

a more discrete and event-driven class of model ought to be used; at a time scale of several minutes or more, the assumptions of diffusivity used in equation (2) to model the price are not that bad, but even at this scale, the 'bid–ask bounce' has to be taken into account in order to be able to estimate with enough accuracy the intra-day volatility. The effect on volatility estimates of the rounding of a diffusion process was first studied in Jacod (1996); since then other effects have been taken into account, such as an additive microstructure noise (Zhang et al., 2005), sampling (Aït-Sahalia and Jacod, 2007) or liquidity thresholding – also known as uncertainty zones – (Robert and Rosenbaum, 2011). Thanks to all these models, it is now possible to use high frequency data to estimate the volatility of an underlying diffusive process generating the prices without being polluted by the signature plot effect (i.e. an explosion of the usual empirical estimates of volatility when high frequency data are used).

Similarly, advances have been made in obtaining accurate estimates of the correlations between two underlying prices thereby avoiding the drawback of the *Epps effect* (i.e. a collapse of usual estimates of correlations at small scales (Hayashi and Yoshida, 2005)).

To optimize the interactions of trading strategies with the order-books, it is necessary to zoom in as much as possible and to model most known effects taking place at this time scale (see Wyart et al. (2008); Bouchaud et al. (2002)). Point processes have been successfully used for this purpose, in particular because they can embed short-term memory modeling (Large, 2007; Hewlett, 2006). Hawkes-like processes have most of these interesting properties and exhibit diffusive behavior when the time scale is zoomed out (Bacry et al., 2009). To model the prices of transactions at the bid  $N_t^b$  and at the ask  $N_t^a$ , two coupled Hawkes processes can be used. Their intensities  $\Lambda_t^b$  and  $\Lambda_t^a$  are stochastic and are governed by

$$\Lambda_t^{a/b} = \mu^{a/b} + c \int_{\tau < t} e^{-k(t-\tau)} dN_t^{b/a};$$

here  $\mu^b$  and  $\mu^a$  are constants. In such a model the more transactions at the bid (resp. ask), the more likely will there be one at the opposite price in the near future.

The next qualitative step is to link the prices with the traded volumes. It has recently been shown that under some assumptions that are almost always true for very liquid stocks in a calm market context (a constant bid—ask spread and no dramatic change in the dynamics of liquidity-providing orders), there is a correspondence between a two-dimensional point process of the quantities available at the first limits and the price of the corresponding stock (Cont et al., 2010; Cont and De Larrard, 2011).

To understand the mechanism underlined by such an approach, just notice that the set of stopping times defined by the instants when the quantity at the first ask crosses zero (i.e.  $\mathcal{T}^a = \{\tau: Q^a_\tau = 0\}$ ) exactly maps the increases of prices (if the bid–ask spread is constant). Similarly the set  $\mathcal{T}^b = \{\tau: Q^b_\tau = 0\}$  maps the decreases of the price.

Despite these valuable proposals for modeling the dynamics of the order book at small time scales, they have not yet been directly used in an optimal trading framework. The most sophisticated approaches for optimal trading including order book dynamics are based on continuous and martingale assumptions (Alfonsi and Schied, 2010) or on Poisson-like point processes (Guéant et al., 2011).

On another hand, focusing on the optimality of very-short-term trading strategies (such as *Smart Order Routing*) let us build optimal tactics with assumptions in accordance with recetn high-level views on order book dynamics. Smart Order Routers (SOR) are software devices dedicated to splitting an order across all available trading venues in order to obtain the desired quantity as fast as possible implementing a so-called *liquidity capturing scheme*. With the rise of fragmented electronic equity markets (see Figure 13), it is impossible to access more than 60% of European liquidity without a SOR.

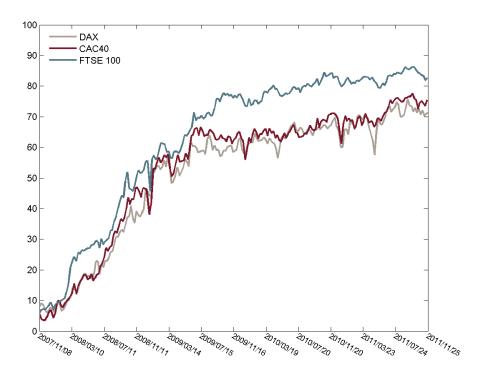


Figure 13: Fragmentation of European markets: the market share of the primary market decreases continuously after the entry in force of the MiFI Directive; the use of an SOR is mandatory for accessing the liquidity of the whole market. This graph monitors the *normalized entropy* of the fragmentation: if the market shares (summing to 1) over K exchanges are  $q_1, \ldots, q_K$ , the indicator is  $-\sum_k q_k \ln q_k$  renormalized so that its maximum is 100% (i.e. divided by  $\ln(K)$ ).

Optimal policies for SOR have been proposed in Pagès et al. (2012) and Ganchev et al. (2010). The latter used censored statistics to estimate liquidity available and to build an optimization framework on top of it; the former built a stochastic algorithm and proved that it asymptotically converges to a state that minimizes a given criterion.

To get a feel for the methodology associated with the stochastic algorithm viewpoint, just consider the following optimization problem.

**Optimal liquidity seeking: the expected fast end criterion.** To define the criterion to be optimized, first assume that K visible order books are available (for instance BATS, Chi-X, Euronext, Turquoise for an European stock). At time t, a buy order of size  $V_t$  has to be split over the order books according to a key  $(r_1, \ldots, r_K)$ , given that on the kth order book:

- the resting quantity 'cheaper' than a given price S is  $I_t^k$  (i.e. the quantity at the bid side posted at a price higher or equals to S, or at a lower price at the ask);
- the incoming flow of sell orders consuming the resting quantity at prices cheaper than S follows a Poisson process  $N_t^k$  of intensity  $\lambda^k$ , i.e.

$$\mathbb{E}(N_{t+\delta t}^k - N_t^k) = \delta t \cdot \lambda^k;$$

• the waiting time on the *k*th trading destination to consume a volume v added on the *k*th trading destination at price S in t is denoted  $\Delta T_t^k(v)$ ; it is implicitly defined by:

$$\Delta T_t^k(v) = \arg\min_{\tau} \{ N_{t+\tau}^k - N_t^k \ge I_t^k + v \}.$$

Assuming that there is no specific toxicity in available trading platforms, a trader would like to split an incoming order at time t of size  $V_t$  according to an allocation key  $(r^1, \ldots, r^K)$  in order to minimize the waiting time criterion. Thus:

$$\mathscr{C}(r^1,\ldots,r^K) = \mathbb{E}\max_k \{\Delta T_t^k(r^k V_t)\}. \tag{12}$$

This means that the trader aims at optimizing the following process:

- (1) an order of size  $V_{\tau(u)}$  is to be split at time  $\tau(u)$ , the set of order arrival times being  $\mathscr{S} = \{\tau(1), \dots, \tau(n), \dots\};$
- (2) it is split over the K available trading venues thanks to an 'allocation key',  $\mathbf{R} = (r^1, \dots, r^K)$ : a portion  $r^k V_{\tau(u)}$  is sent to the kth order book (all the quantity is spread, i.e.  $\sum_{k \le K} r_k = 1$ );
- (3) the trader waits the time needed to consume all the sent orders.

The criterion  $\mathscr{C}(\mathbf{R})$  defined in (12) reflects the fact that the faster the allocation key lets us obtain liquidity, the better: the obtained key is well suited for a *liquidity-seeking* algorithm.

First denote by  $k_t^*(\mathbf{R})$  the last trading destination to consume the order sent at t:

$$k_t^*(\mathbf{R}) = \arg\max_k \{\Delta T_t^k(r^k V_t)\}.$$

A gradient approach to minimizing  $\mathscr{C}(\mathbf{R})$  means we must compute  $\partial \Delta T_t^u(r^uV_t)/\partial r^k$  for any pair (k,u). To respect the constraint, just replace an arbitrary  $r^\ell$  by  $1-\sum_{u\neq\ell}r^u$ . Consequently,

$$\frac{\partial \Delta T_t^u(r^u V_t)}{\partial r^k} = \frac{\partial \Delta T_t^k(r^k)}{\partial r^k} \, 1\!\!1_{k_t^*(\mathbf{R}) = k} + \frac{\partial \Delta T_t^\ell(r^\ell)}{\partial r^k} \, 1\!\!1_{k_t^*(\mathbf{R}) = \ell}$$

where 11 is a delta function. With the notation  $\Delta N_t^k = N_t^k - N_{t-}^k$ , we can write that any allocation key **R** such that, for any pair  $(\ell, k)$ ,

$$\mathbb{E}\left(V_t \cdot D_t^k(r^k) \cdot 11_{k_t^*(\mathbf{R}) = k}\right) = \mathbb{E}\left(V_t \cdot D_t^\ell(r^\ell) \cdot 11_{k_t^*(\mathbf{R}) = \ell}\right)$$
(13)

where

$$D_t^k(r^k) = \frac{1}{\Delta N_{t+\Delta T_t^k(r^k)}^k} \operatorname{11}_{\left(\Delta N_{t+\Delta T_t^k(r^k)}^k > 0\right)}$$

is a potential minimum for the criterion  $\mathscr{C}(\mathbf{R})$  (the proof of this result will not be provided here). Equation (13) can also be written as:

$$\mathbb{E}\left(V_t \cdot D_t^k(r^k) \cdot \mathbb{1}_{l_t^*(\mathbf{R}) = k}\right) = \frac{1}{K} \sum_{\ell=1}^K \mathbb{E}\left(V_t \cdot D_t^{\ell}(r^{\ell}) \cdot \mathbb{1}_{k_t^*(\mathbf{R}) = \ell}\right).$$

It can be shown (see Lelong (2011) for generic results of this kind) that the asymptotic solutions of the following stochastic algorithm on the allocation weights through time

$$\forall k, r^{k}(n+1) = r^{k}(n) - \gamma_{k+1} \left( V_{\tau(n)} \cdot D_{\tau(n)}^{k}(r^{k}(n)) \cdot 11_{k_{\tau(n)}^{*}(\mathbf{R}(n)) = k} - \frac{1}{K} \sum_{\ell=1}^{K} V_{\tau(n)} \cdot D_{\tau(n)}^{\ell}(r^{\ell}(n)) \cdot 11_{k_{\tau(n)}^{*}(\mathbf{R}(n)) = \ell} \right)$$
(14)

minimize the expected fast end criterion  $\mathscr{C}(\mathbf{R})$ , provided there are strong enough ergodicity assumptions on the  $(V,(N^k)_{1\leq k\leq K},(I^k)_{1\leq k\leq K})$ -multidimensional process.

Qualitatively, we read this update rule to mean that if a trading venue k demands more time to execute the fraction of the volume that it receives (taking into account the combination of I and N) than the average waiting time on all venues, then the fraction  $r^k$  of the orders to send to k has to be decreased for future use.

### 5 Perspectives and Future Work

The needs of intra-day trading practitioners are currently focused on optimal execution and trading risk control. Certainly some improvements on what is actually available have been proposed by academics, in particular:

- provide optimal trading trajectories taking into account *multiple trading desti*nations and different type of orders: liquidity-providing (i.e. limit) ones and liquidity-consuming (i.e. market) ones;
- the *analysis of trading performances* is also an important topic; models are needed to understand what part of the performance and risk are due to the planned scheduling, the interactions with order books, the market impact and the market moves;
- stress testing: before executing a trading algorithm in real markets, we must understand its dependence on different market conditions, from volatility or momentum to bid—ask spread or trading frequency. The study of the 'Greeks' of the payoff of a trading algorithm is not straightforward since it is inside a closed loop of liquidity: its 'psi' should be its derivative with respect to the bid—ask spread, its 'phi' with respect to the trading frequency, and its 'lambda' with respect to the liquidity available in the order book.

For the special case of portfolio liquidity studied in this chapter (using the payoff  $\tilde{C}_{\lambda}$  defined by equality (11)), these trading Greeks would be:

$$\Psi = \left(\frac{\partial \tilde{C}_{\lambda}}{\partial \psi_{\ell}}\right)_{1 < \ell < N}, \quad \Phi = \frac{\partial \tilde{C}_{\lambda}}{\partial N}, \quad \Lambda = \frac{\partial \tilde{C}_{\lambda}}{\partial \kappa}.$$

Progress in the above three directions will provide a better understanding of the price formation process and the whole cycle of asset allocation and hedging, taking into account execution costs, closed loops with the markets, and portfolio trajectories at any scales.

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