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Pricing of Spreads and Other Options in Cointegrated Markets

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This master's thesis is submitted under the master's programme *Modelling and Data Analysis*, with programme option *Finance, Insurance and Risk*, at the Department of Mathematics, University of Oslo. The scope of the thesis is 60 credits.

The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Abstract

In this thesis we analyze spread functions in the cointegrated market, with dynamics based on different combinations of Brownian motions and Ornstein-Uhlenbeck processes, and their structural differences. Theoretical and computational methods for pricing these options are explored, and we will discuss the parameters and dynamics of the price functions by examining analytical expressions and numerical simulations. The extention of cointegrated spreads to quanto options is also touched upon. We use the mean-reversal, stationary dynamic of Ornstein-Uhlenbeck spreads to suggest a model on the form of an European put-option to approximate quanto options in cointegrated markets, which are based on a product of spreads.

Keywords: Spread options, Ornstein-Uhlenbeck processes, Cointegrated markets, Quanto options

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Chapter 1

Introduction

The use of spread options are widespread, whether the motivation stems from risk mitigation, asset valuation or speculation. In most models and markets, futures and forward contract prices are simply the spot price of the stock corrected for growth at the current interest rate, and uses arbitrage arguments to price derivatives. However, in energy markets this relationship between spot and forward markets does not hold, mainly due to seasonality and mean reversion. The energy markets differ from other commodity markets in characteristics such as limited storeability of the spot or seasonally dependent prices with spikes.

Spread options models the difference between two assets, and are used in the energy market as as a way to hedge price differencies between energies. Examples of spread options in the energy market are spark spread options, which are call and put options written on the difference between electricity and gas prices, and crack spreads, that price the difference between crude oil and a refined product.

This thesis will explore the dynamics of cointegrated spread options, and also include a suggestion at extending these applications to quanto options. While typically an option used in currency markets with a regular call-put payoff structure, in energy markets quanto options have a payoff structure similar to a product of call-put options. In the later parts of this thesis, we will briefly touch upon the expansion of quanto options into a product of spread call-put options.

1.1 Outline of the Thesis

In this long master thesis we will focus on European spread options in cointegrated markets. We will also extend this focus to an outlook on quanto options featuring cointegrated spreads in it's structure.

We will start by examining spread options with simple Brownian motion dynamics in Chapter 3, before extending the applications to spreads based on Ornstein-Uhlenbeck processes in Chapter 4. These chapters will feature methods for finding the price European put-options based on the respective spreads, and simulate and discuss results and dynamics of the models. Chapter 5 will not look at spread options, but at options based on both Brownian motion and Ornstein-Uhlenbeck options.

In Chapter 6 we will extend our previous results to the case of quanto options, where we examine options on the form of a product of the options analyzed in Chapter 4 and Chapter 5. We will focus on the structural dynamics found in previous chapters to discuss approximations for quanto options.

The Appendix will feature some basic probability concepts used in calculations, as well as R code for simulation and plots.

Chapter 2

Preliminaries

Bibliographical notes

The main references in this chapter are: Øksendal (2010) [8], and Benth, Benth and Koekebakker (2008) [5]

2.1 Ornstein-Uhlenbeck processes

The *Ornstein-Uhlenbeck* (OU) process $X(t)$ is a stochastic process described by the following stochastic differential equation (SDE):

$$dX_t = \alpha(\mu - X_t)dt + \sigma dB_t \tag{2.1.1}$$

where α, μ, $\sigma > 0$, and B_t is a standard Brownian motion (BM).

The process is the only non-trivial continuous Markov process that has a stationary Gaussian distribution.

Unlike Brownian motion, the Ornstein-Uhlenbeck process is mean-reverting, meaning the process will drift towards its mean μ in the long term, with σ describing the volatility of the Brownian motion and the parameter α measuring the extent of the mean reversion. As with Brownian motion, μ is called the *drift* and σ the *infinitesimal variance.* α is called the *speed of mean reversion.*

By applying Itô's formula with $g(t, X_t) = X_t e^{\alpha t}$, the SDE can easily be solved by:

$$\begin{aligned} d(X_t e^{\alpha t}) &= \alpha X_t e^{\alpha t} + e^{\alpha t} dX_t \\ &= \alpha \mu e^{\alpha t} + \sigma e^{\alpha t} dB_t \\ X_t e^{\alpha t} &= X_0 + \mu \int_0^t e^{\alpha s} ds + \sigma \int_0^t e^{\alpha s} Bs. \end{aligned}$$

Thus the Ornstein-Uhlenbeck process can equivalently be expressed as:

$$X_t = X_0 e^{-\alpha t} + \mu(1 - e^{-\alpha t}) + \sigma \int_0^t e^{-\alpha(t-s)} dB_s \tag{2.1.2}$$

Since B is Gaussian with $\mathbb{E}[B_t] = 0$, it follows that X has the same distribution, and the expectation is found to be:

$$\mathbb{E}[X_t] = X_0 e^{-\alpha t} + \mu(1 - e^{-\alpha t}) \tag{2.1.3}$$

Applying the Itô isometry gives the variance:

$$\mathrm{Var}(X_t) = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}) \tag{2.1.4}$$

So the distribution of the Ornstein-Uhlenbeck process is:

$$X_t \sim \mathcal{N}(X_0 e^{-\alpha t} + \mu(1 - e^{-\alpha t}), \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}))$$

Similarly to Brownian motion, the variance of the Ornstein-Uhlenbeck process does not depend on the initial value of the process X_0, and as t approaches infinity the distribution converges to:

$$\lim_{t \to \infty} X_t \sim \mathcal{N}(\mu, \frac{\sigma^2}{2\alpha})$$

which confirms that the process is stationary in the long term.

The relation between Brownian motion and the Ornstein-Uhlenbeck process becomes apparent if $\alpha = 0$, as the expression becomes:

$$X_t = X_0 + \sigma \int_0^t dB_s$$

which is simply a Brownian motion process with expectation X_0 and variance σ.

2.2 Stationary stochastic process

A stochastic process $X(t)$ is *stationary* if the distribution of the process is unaffected by the course of time t, i.e. the probability distribution of $X(t)$ is equivalent to the distribution of $X(t+\tau)$ for any fixed value of τ.

There is a weaker form of stationarity known as *weak-sense stationary*, which by definition only requires that the first moment, i.e. the mean, and the auto-covariance do not vary with respect to time. All strictly stationary processes are also stationary in the weak sense.

In mathematical terms, a stochastic process $X(t)$ is stationary if, for all $\tau \in \mathbb{R}$, the following holds for the mean:

$$\mathbb{E}[X(t)] = \mu_t = \mu_{t+\tau} = \mathbb{E}[X(t+\tau)] \tag{2.2.1}$$

and the autocovariance is:

$$\begin{aligned} \mathbb{E}[(X_t - \mu_t)(X_{t+\tau} - \mu_{t+\tau})] &= C_X(t, t+\tau) \\ &= C_X(0, \tau) \\ &= \mathbb{E}[(X_0 - \mu_0)(X_\tau - \mu_\tau)] \end{aligned} \tag{2.2.2}$$

Equation (2.2.1) implies that the expectation is constant, while Equation (2.2.2) implies that the covariance function is only dependent on the time difference τ.

For this thesis, the property that Ornstein-Uhlenbeck processes are stationary is used.

2.3 Cointegration

If a collection of commodities is *cointegrated*, then there exists a linear combination of the commodity prices that becomes stationary.

The *order of integration* of a process, denoted $I(d)$, is a summary statistic that reports the minimum number of differences required to obtain a covariance stationary series. When a time series or process X is integrated of order d, we say that X is $I(d)$.

For the formal definition of cointegration, if two stocks Y and Z each are $I(1)$ and there exists coefficients a and b such that $aY + bZ$ is $I(0)$, then Y and Z are cointegrated.

In practice, this means that if the commodities X and Y are cointegrated, the individual commodities might be non-stationary on their own, but there exists a stationary linear relationship between Y and Z in form of $aY + bZ$.

For this thesis, when commodities are cointegrated, there exists a linear combination of logarithmic prices which becomes stationary in the long term. To construct cointegrated commodities, we use the logarithmic sum of a Geometric Brownian motion process and Ornstein-Uhlenbeck processes.

2.4 Spread options

A spread option is an option where the payoff is based on the difference between two assets, e.g. $S = S_2 - S_1$ where S_1 and S_2 are the prices of two individual assets. For a spread put, the payoff can be written as

$$P = \max(K - (S_2 - S_1), 0) \tag{2.4.1}$$

where K is a constant called the strike price.

In energy markets, spread options are designed to mitigate adverse movements of several indices, i.e. to hedge price differentials. An example is the difference between the power price and a price of a fuel like coal or gas.

2.5 Quanto options

Quanto is short for *quantity adjusting option*, and is traditionally a type of derivatives in which the underlying asset is denominated in a currency different from the currency in which the option is settled. In the currency market, a quanto option is a cross option in which the exchange rate is fixed as a strike price at the outset of the trade, and the payoff is the difference between the underlying and fixed strike, paid out in another currency. In energy and weather markets, a quanto option is contingent on a weather volume variable, usually temperature or precipitation. Energy quanto options are mainly used to hedge exposure to the joint price and volume risk in the power market. For this thesis, the term *quanto options* will be referring to energy quanto options.

The payoff for energy quanto options is triggered by a strike value for the weather variable, and depends on the market price of a commodity such as electricity or gas. It is typically given on the form of a product of an

European option on the energy price and an European option on a volume index, like temperature.

A quanto option based on put options will thus have payoff written as

$$Q = \max(K_2 - S_2, 0) \times \max(K_1 - S_1, 0) \tag{2.5.1}$$

Chapter 3

Spread Options Based on Brownian Motion Processes

Bibliographical notes

The main reference in this chapter is: Carmona and Durrleman (2003) [6]

3.1 Price of Spread Options

In this chapter, the arbitrage-free price of a European spread option with dynamics based on geometric Brownian motion (GBM) processes is estimated.

Let $S^B(t)$ denote the value of the spread between the logarithmic prices of two options with GBM-dynamics at time t:

$$S^B(t) = X_2(t) - X_1(t), \qquad \text{for } t \geq 0$$

where X_1, X_2 are described by:

$$\begin{aligned} X_1(t) &= \log x_1(t) \qquad & x_1(t) &= x_1(0)\exp(\mu_Y t + \sigma_Y B_Y(t)) \\ X_2(t) &= \log x_2(t) \qquad & x_2(t) &= x_2(0)\exp(\mu_Z t + \sigma_Z B_Z(t)) \end{aligned}$$

The correlated Brownian motions $B_Y(t)$ and $B_Z(t)$ are defined on a complete filtered probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$, with $\mu_Z > \mu_Y > 0$.

Without loss of generality, we assume that

$$x_1(0) = x_2(0) = 1$$

The relation between $B_Y(t)$ and $B_Z(t)$ are given as follows, with $U(t)$ and $W(t)$ being independent Brownian motion processes on $(\Omega, \mathcal{F}_t, P)$:

$$\begin{aligned} dB_Y(t) &= \rho dU(t) + \sqrt{1-\rho^2}dW(t) & dB_Z(t) &= dU(t) \\ B_Y(t) &= \rho U(t) + \sqrt{1-\rho^2}W(t) & B_Z(t) &= U(t) \end{aligned}$$

where $\mathrm{corr}(B_Y, B_Z) = \rho$.

The expression for $S^B(t)$ is:

Proposition 3.1.1. *The value of the BM-spread at time t is:*

$$S^B(t) = (\mu_Z - \mu_Y)t + (\sigma_Z - \sigma_Y\rho)U(t) + \sigma_Y\sqrt{1-\rho^2}W(t) \tag{3.1.1}$$

Proof. For $S^B(t)$:

$$\begin{aligned} S^B(t) &= X_2(t) - X_1(t) \\ &= \mu_Z t + \sigma_Z B_Z(t) - \mu_Y t - \sigma_Y B_Y(t) \\ &= (\mu_Z - \mu_Y)t + (\sigma_Z - \sigma_Y\rho)U(t) + \sigma_Y\sqrt{1-\rho^2}W(t) \end{aligned}$$

□

The purpose for this chapter is to estimate the value of the forward contract $f^B(T)$ at time T, where $f^B(T) = \max(K_B - S^B(T), 0)$ denotes the spot price of a put option with the spread $S^B(T)$.

Proposition 3.1.2. *The arbitrage-free forward price of a put spread option, i.e. the price at current time $t = 0$ for a contract with payoff at a future time T, is:*

$$P^B(T) = e^{-rT}\mathbb{E}[(K_B - S^B(T))_+] \tag{3.1.2}$$

where r is a constant interest rate and K_B is the strike price.

For this chapter and the rest of this thesis, we will use $(K_B - S^B(T))_+$ as shorthand for $\max(K_B - S^B(T), 0)$.

3.2 Expectation and Variance for Brownian motion spreads

Since x_1 and x_2 are geometric Brownian Motion processes and log-normally distributed, it follows that X_1 and X_2 are normally distributed processes. $S^B(t)$, the joint distribution of the stochastic variables $X_1(t)$ and $X_2(t)$, then has a bivariate normal distribution. The distribution of $S^B(t)$ can then be determined by finding the expectation, co-variance and variance.

Proposition 3.2.1. *The Brownian motion spread $S^B(t)$ has a bivariate normal distribution with*

$$S^B(t) \sim \mathcal{N}(\boldsymbol{\mu}_B(t), \boldsymbol{\sigma}_B(t))$$

where

$$\boldsymbol{\mu}_B(t) = \begin{pmatrix} \mu_Y t \\ \mu_Z t \end{pmatrix} \tag{3.2.1}$$

$$\boldsymbol{\sigma}_B(t) = \begin{pmatrix} \sigma_Y^2 t & \sigma_Y \sigma_Z \rho t \\ \sigma_Y \sigma_Z \rho t & \sigma_Z^2 t \end{pmatrix} \tag{3.2.2}$$

Proof. The expectation and variance for the spread option $S^B(t)$ are:

Expectation of $\boldsymbol{S}^B(t)$:

$$\begin{aligned} \mathbb{E}[S^B(t)] &= \mathbb{E}[X_2(t)] - \mathbb{E}[X_1(t)] \\ &= (\mu_Z - \mu_Y)t \end{aligned} \tag{3.2.3}$$

Variance of $\boldsymbol{S}^B(t)$:

$$\begin{aligned} \mathrm{Var}(S^B(t)) &= \mathrm{Var}(X_1(t)) - 2\mathrm{Cov}(X_1(t), X_2(t)) + \mathrm{Var}(X_2(t)) \\ &= (\sigma_Y^2 - 2\sigma_Y \sigma_Z \rho + \sigma_Z^2)t \end{aligned} \tag{3.2.4}$$

Expectation:

$$\begin{aligned} \mathbb{E}[X_1(t)] &= \mathbb{E}[\mu_Y t + \sigma_Y B_Y(t)] \\ &= \mu_Y t \\ \mathbb{E}[X_2(t)] &= \mathbb{E}[\mu_Z t + \sigma_Z B_Z(t)] \\ &= \mu_Z t \end{aligned}$$

Variance:

$$
\begin{aligned}
\mathrm{Var}(X_1(t)) &= \mathbb{E}[X_1^2(t)] - \mathbb{E}[X_1(t)]^2 \\
&= \mathbb{E}[(\mu_Y t + \sigma_Y B_Y(t))^2] - \mu_Y^2 t^2 \\
&= \sigma_Y^2 t \\
\mathrm{Var}(X_2(t)) &= \mathbb{E}[X_2^2(t)] - \mathbb{E}[X_2(t)]^2 \\
&= \mathbb{E}[(\mu_Z t + \sigma_Z B_Z(t))^2] - \mu_Z^2 t^2 \\
&= \sigma_Z^2 t
\end{aligned}
$$

Covariance:

$$
\begin{aligned}
\mathrm{Cov}(X_1(t), X_2(t)) &= \mathbb{E}[X_1(t)X_2(t)] - \mathbb{E}[X_1(t)]\mathbb{E}[X_2(t)] \\
&= \mathbb{E}[(\mu_Y t + \sigma_Y B_Y(t))(\mu_Z t + \sigma_Z B_Z(t))] - \mu_Y \mu_Z t^2 \\
&= \sigma_Y \sigma_Z \mathbb{E}[B_Y(t) B_Z(t)] \\
&= \sigma_Y \sigma_Z \mathbb{E}[(\rho U(t) + \sqrt{1-\rho^2} W(t)) U(t)] \\
&= \sigma_Y \sigma_Z \rho t
\end{aligned}
$$

Correlation:

$$
\begin{aligned}
\mathrm{corr}(X_1, X_2) &= \frac{\mathrm{Cov}(X_1(t), X_2(t))}{\sqrt{\mathrm{Var}(X_1)\mathrm{Var}(X_2)}} \\
&= \rho
\end{aligned}
$$

$\square$

3.3 Price of a Brownian Motion European Spread Option

Given that the arbitrage-free forward price of a put spread option with payoff at time T is $P^B(T) = e^{-rT}\mathbb{E}[(K_B - S^B(T))_+]$, we need to find a method for calculating the expected value of the spread option:

$$
\mathbb{E}[f^B(T)] = \mathbb{E}[\max(K_B - S^B(T), 0)]
$$

As $S^B(T)$ is bivariate normal, the expectation can be calculated using double integrals [6].

3.3.1 Computing the Price of a BM Spread Option by Double Integrals

For an European put option, the Black-Scholes formula gives a value for the risk-neutral price when $S^B(T)$ is a geometric Brownian motion, or equivalently, a log-normal random variable:

$$P^B(T) = e^{-rT}\mathbb{E}[(K_B - S^B(T))_+]$$

Proposition 3.3.1. *For the drifted Brownian motion processes X_1, X_2 with correlation ρ, given by*

$$\begin{aligned} X_1(t) &= \mu_Y t + \sigma_Y B_Y(t) \\ X_2(t) &= \mu_Z t + \sigma_Z B_Z(t) \end{aligned}$$

the price of the spread option $f^B(T) = \max(K_B - (X_2(t) - X_1(t)), 0)$ at exercise time T is

$$\begin{aligned} P^B(T) = e^{-rT} \Bigg(&(K_B - (\mu_Z - \mu_Y)T \int_{-\infty}^{\infty} \phi(w)\Phi(c_B(w))dw \\ &- \sigma_y\sqrt{T(1-\rho^2)} \int_{-\infty}^{\infty} w\phi(w)\Phi(c_B(w))dw \\ &+ (\sigma_Z - \sigma_Y\rho)\sqrt{T} \int_{-\infty}^{\infty} \phi(c_B(w))\phi(w)dw \Bigg) \end{aligned}$$

where (u, w) is standard bivariate normal, and

$$c_B = \frac{K_B - (\mu_Z - \mu_Y)T - \sigma_Y\sqrt{T(1-\rho^2)}w}{(\sigma_Z - \sigma_Y\rho)\sqrt{T}}$$

Proof. As the spread $S^B(T)$ is a linear combination of two Brownian Motion Processes, we extend the Black-Scholes pricing paradigm to apply to the spread option.

This pricing paradigm yields the following expression for the risk-neutral price $P^B(T)$:

$$\begin{aligned} P^B(T) &= e^{-rT}\mathbb{E}[(K_B - S^B(T))_+] \\ &= e^{-rT}\mathbb{E}[(K_B - (\mu_Z - \mu_Y)T \\ &\qquad - (\sigma_Z - \sigma_Y\rho)U(T) - \sigma_Y\sqrt{1-\rho^2}W(T))_+] \end{aligned}$$

The price of a put spread option can be written as the double integral

$$e^{-rT}\mathbb{E}[(K_B-(X_2(T)-X_1(T)))_+] = e^{-rT}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(K_B-(x_2-x_1))_+ f_T(x_1,x_2)dx_1dx_2 \quad (3.3.1)$$

where $f_T(x_1,x_2)$ denotes the joint density of the random variables X_1 and X_2, i.e. the bivariate normal distribution.

As X_1 and X_2 are correlated, we compute P^B with regards to U and W instead. Standardizing U and W gives us:

$$\tilde{U}_T = \frac{U(T)}{\sqrt{T}} \sim \mathcal{N}(0,1)$$

$$\tilde{W}_T = \frac{W(T)}{\sqrt{T}} \sim \mathcal{N}(0,1).$$

with $(\tilde{U}_T, \tilde{U}_T)$ having a standard normal bivariate distribution.

So the expectation in the price function becomes:

$$\begin{aligned}\mathbb{E}[(K_B - S^B(T))_+] &= \mathbb{E}[(K_B-(\mu_Z-\mu_Y)T-(\sigma_Z-\sigma_Y\rho)U(T) \\ &\quad -\sigma_Y\sqrt{1-\rho^2}W(T))_+] \\ &= \mathbb{E}\Big[(K_B-(\mu_Z-\mu_Y)T-(\sigma_Z-\sigma_Y\rho)\sqrt{T}\tilde{U}_T \\ &\quad -\sigma_Y\sqrt{T(1-\rho^2)}\tilde{W}_T)_+\Big]\end{aligned}$$

Observe that the random variable inside the expectation is zero when $\tilde{U}_T$ is such that

$$K_B > (\mu_Z-\mu_Y)T + (\sigma_Z-\sigma_Y\rho)\sqrt{T}\tilde{U}_T + \sigma_Y\sqrt{T(1-\rho^2)}\tilde{W}_T$$

or when $\tilde{U}_T < c_B$, where

$$\begin{aligned}c_B(w) &= \frac{K_B-(\mu_Z-\mu_Y)T-\sigma_Y\sqrt{T(1-\rho^2)}w}{(\sigma_Z-\sigma_Y\rho)\sqrt{T}} \\ &= \frac{K_B-(\mu_Z-\mu_Y)T}{(\sigma_Z-\sigma_Y\rho)\sqrt{T}} - \frac{\sigma_Y\sqrt{1-\rho^2}}{(\sigma_Z-\sigma_Y\rho)}w \\ &= c_{B1}-c_{B2}w\end{aligned}$$

with the condition that $\tilde{W}_T = w$.

Computing the expectation by conditioning on $\tilde{W}_T$, with $f_{u,w}(w)$ being the probability density for $\tilde{W}_T$ at time T, we get:

$$\begin{aligned}
\mathbb{E}[(K_B - S^B(T))_+] &= \mathbb{E}[(K_B - S^B(T))_+] \\
&= \mathbb{E}[\mathbb{E}[(K_B - (X_2(T) - X_1(T)))_+ | \tilde{W}_T = w]] \\
&= \int_{-\infty}^{\infty} \mathbb{E}[(K_B - (\mu_Z - \mu_Y)T - (\sigma_Z - \sigma_Y \rho)\sqrt{T}\tilde{U}_T \\
&\qquad - \sigma_Y \sqrt{T(1-\rho^2)} w)_+] f_{T|\tilde{W}_T = w}(w) dw \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{c_B(w)} \Big(K_B - (\mu_Z - \mu_Y)T - (\sigma_Z - \sigma_Y \rho)\sqrt{T} u \right. \\
&\qquad \left. - \sigma_Y \sqrt{T(1-\rho^2)} w \Big) \phi(u) du \right) f_{u,w}(w) dw \\
&= \int_{-\infty}^{\infty} \left(\left(K_B - (\mu_Z - \mu_Y)T - \sigma_Y \sqrt{T(1-\rho^2)} w \right) \Phi(c_B(w)) \right. \\
&\qquad \left. - (\sigma_Z - \sigma_Y \rho)\sqrt{T} \int_{-\infty}^{c_B(w)} u \phi(u) du \right) dw \\
&= \int_{-\infty}^{\infty} \Big((K_B - (\mu_Z - \mu_Y)T - \sigma_Y \sqrt{T(1-\rho^2)} w) \, \Phi(c_B(w)) \\
&\qquad + (\sigma_Z - \sigma_Y \rho)\sqrt{T} \phi(c_B(w)) \Big) \phi(w) dw \\
&= (K_B - (\mu_Z - \mu_Y)T) \int_{-\infty}^{\infty} \phi(w) \Phi(c_B(w)) dw \\
&\qquad - \sigma_Y \sqrt{T(1-\rho^2)} \int_{-\infty}^{\infty} w \phi(w) \Phi(c_B(w)) dw \\
&\qquad + (\sigma_Z - \sigma_Y \rho)\sqrt{T} \int_{-\infty}^{\infty} \phi(w) \phi(c_B(w)) dw
\end{aligned}$$

The only integral in the expression with an analytic solution is $\int_{-\infty}^{\infty} \phi(w)\phi(c_B(w))dw$, but the other two can also be expressed in terms of the error function erf.

The integrals in question can be expressed as follows:

$$\int_{-\infty}^{\infty} \phi(w)\Phi(c_B(w))dw = \frac{1}{2} + \frac{1}{2}\int \text{erf}(\frac{c(w)}{\sqrt{2}})\phi(w)dw$$

$$\int_{-\infty}^{\infty} w\phi(w)\Phi(c_B(w))dw = \frac{1}{2}\int w\phi(w)\text{erf}(\frac{c(w)}{\sqrt{2}})dw$$

$$\int_{-\infty}^{\infty} \phi(w)\phi(c_B(w))dw = \frac{1}{\pi\sqrt{2(c_{B2}^2+1)}}e^{-\frac{c_{B1}^2}{2(c_{B2}^2+1)}}$$

Detailed calculations for the integrals are shown in Section 3.5. □

3.4 Value of the BM Spread Option in the Long Term

The spread option in the long term becomes:

$$\begin{aligned}\lim_{T\to\infty} S^B(T) &= \lim_{T\to\infty}(X_2(T) - X_1(T)) \\ &= \lim_{T\to\infty}\Big((\mu_Z - \mu_Y)T + (\sigma_Z - \sigma_Y\rho)U(T) + \sigma_Y\sqrt{1-\rho^2}W(T)\Big)\end{aligned}$$

As $U(T)$ and $W(T)$ are Brownian motions, they do not converge. However, since $\lim_{T\to\infty}(\mu_Z - \mu_Y)T = \infty$, it will dominate the expression in the long term, as Figure 3.1 suggests.

While not a convergent stochastical process, a standard Brownian motion process $B(t)$ converges in mean to 0, as

$$\lim_{T\to\infty}\mathbb{E}[|B(T) - 0|] = 0$$

Thus, it follows that the expectation of the spread term in the long term also diverges:

$$\begin{aligned}\lim_{T\to\infty}\mathbb{E}[S^B(T)] &= \lim_{T\to\infty}\mathbb{E}[X_2(T)] - \mathbb{E}[X_1(T)] \\ &= \lim_{T\to\infty}(\mu_Z - \mu_Y)T \\ &= \infty\end{aligned}$$

In other words, $\mathbb{E}[S^B(T)]$ diverges in the long term, and the difference between $\mathbb{E}[X_1(T)]$ and $\mathbb{E}[X_2(T)]$ expands with time. This structure is

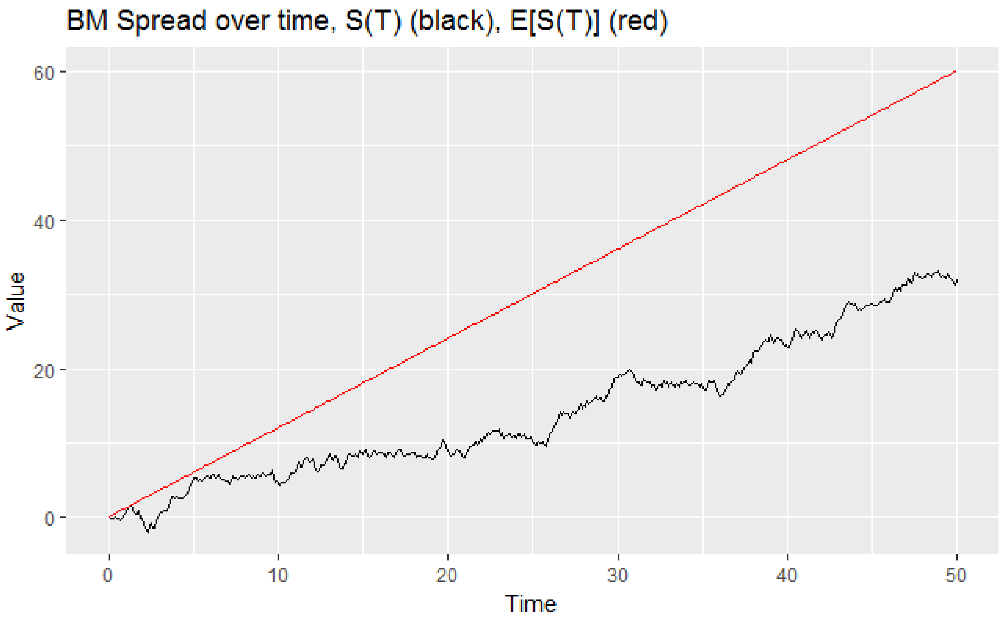

Figure 3.1: **BM Spread.** Parameters: $K = 15$, $\mu_Y = 0.3$, $\mu_Z = 0.9$, $\sigma_Y = 0.8$, $\sigma_Z = 1.3$

consistent with the fact that $S^B(T)$ is not stationary. $\lim_{T\to\infty} S^B(T)$ is still moving towards infinity at a slower rate than $\mathbb{E}[S^B(T)]$.

For the long-term forward price estimate, we look at what happens to the different components in the expression as T approaches infinity.

$$\begin{aligned}
\lim_{T\to\infty} c_B(w) &= \lim_{T\to\infty} \frac{K_B - (\mu_Z - \mu_Y)T}{(\sigma_Z - \sigma_Y\rho)\sqrt{T}} - \frac{\sigma_Y\sqrt{1-\rho^2}}{(\sigma_Z - \sigma_Y\rho)} w \\
&= \begin{cases} \infty & \text{if } \mu_Y > \mu_Z \\ -\frac{\sigma_Y\sqrt{1-\rho^2}}{(\sigma_Z-\sigma_Y\rho)} w & \text{if } \mu_Y = \mu_Z \\ -\infty & \text{if } \mu_Y < \mu_Z \end{cases}
\end{aligned}$$

Thus, we get the following expression for the expectation of the spread option:

$$
\begin{aligned}
\lim_{T\to\infty} \mathbb{E}[(K_B - S^B(T))_+] = \lim_{T\to\infty} \Bigg(& (K_B - (\mu_Z - \mu_Y)T) \int_{-\infty}^{\infty} \phi(w)\Phi(c_B(w))dw \\
& - \sigma_Y \sqrt{T(1-\rho^2)} \int_{-\infty}^{\infty} w\phi(w)\Phi(c_B(w))dw \\
& + (\sigma_Z - \sigma_Y \rho)\sqrt{T} \int_{-\infty}^{\infty} \phi(c_B(w))\phi(w)dw \Bigg)
\end{aligned}
$$

The long-term value of $\mathbb{E}[(K_B - S^B(T))_+]$ depends on the value for $\lim_{T\to\infty} c_{B(w)}$.

We can derive expressions for the different values of $\mathbb{E}[(K_B - S^B(T))_+]$ by using the following properties of the PDF and the CDF for the standard normal distribution:

$$
\begin{aligned}
\Phi(\infty) &= 1 \\
\Phi(-\infty) &= 1 - \Phi(\infty) = 0 \\
\phi(\infty) &= \phi(-\infty) = 0
\end{aligned}
$$

where $\phi(x)$ and $\Phi(x)$ denotes the PDF and CDF of the standard normal distribution respectively.

3.4.1 $\lim_{T\to\infty} \mathbb{E}[(K_B - S^B(T))_+]$ when $\mu_Y > \mu_Z$

When $\mu_Y > \mu_Z$, the expectation of the spread S^B is negative, and $\lim_{T\to\infty} c_B(w) = \infty$.

$$
\begin{aligned}
\lim_{T\to\infty} \mathbb{E}[(K_B - S^B(T))_+] &= \lim_{T\to\infty} \Bigg((K_B - (\mu_Z - \mu_Y)T) \int_{-\infty}^{\infty} \phi(w)dw \\
& \qquad - \sigma_Y \sqrt{T(1-\rho^2)} \int_{-\infty}^{\infty} w\phi(w)dw \Bigg) \\
&= \infty
\end{aligned}
$$

Figure 3.2 show that $\mathbb{E}[(K_B - S^B(T))_+]$ starts at the strike price K_B and grows with time. This is reasonable, as $K_B - S^B(T)$ will on average be positive.

3.4.2 $\lim_{T\to\infty} \mathbb{E}[(K_B - S^B(T))_+]$ when $\mu_Y = \mu_Z$

In the case that $\mu_Y = \mu_Z$, the expectation of the spread S^B is 0, and the value of the spread option $(K_B - S^B(T))_+$ would on average be around K_B.

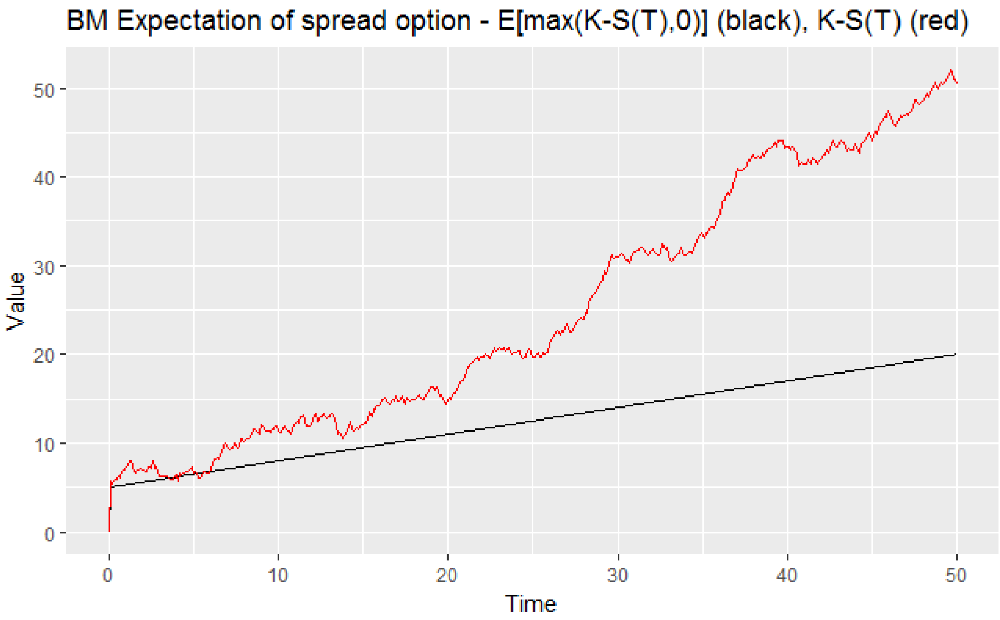

Figure 3.2: $\boldsymbol{\mu}_Y > \boldsymbol{\mu}_Z$: $\mathbb{E}[(K_B - S^B(T))_+]$ over time. Parameters: $K_B = 5$, $\mu_Y = 0.8$, $\mu_Z = 0.5$, $\sigma_Y = 0.8$, $\sigma_Z = 1.3$

We also get that $\lim_{T\to\infty} c_B(w) = -\frac{\sigma_Y\sqrt{1-\rho^2}}{(\sigma_Z - \sigma_Y\rho)} w$

$$
\begin{aligned}
\lim_{T\to\infty} \mathbb{E}[(K_B - S^B(T))_+] = K_B \int_{-\infty}^{\infty} \phi(w)dw - \lim_{T\to\infty} \Bigg(&\sigma_Y\sqrt{1-\rho^2} \int_{-\infty}^{\infty} w\phi(w)dw \\
&-\sigma_Y\sqrt{1-\rho^2} \int_{-\infty}^{\infty} w\phi(w)\Phi\left(\frac{\sigma_Y\sqrt{1-\rho^2}}{(\sigma_Z - \sigma_Y\rho)} w\right) dw \\
&-(\sigma_Z - \sigma_Y\rho) \int_{-\infty}^{\infty} \phi\left(\frac{\sigma_Y\sqrt{1-\rho^2}}{(\sigma_Z - \sigma_Y\rho)} w\right) \phi(w)dw \Bigg) \sqrt{T} \\
= K_B + \sigma_Y\sqrt{1-\rho^2} &\int_{-\infty}^{\infty} w\phi(w)\Phi\left(\frac{\sigma_Y\sqrt{1-\rho^2}}{(\sigma_Z - \sigma_Y\rho)} w\right) dw \\
&+ (\sigma_Z - \sigma_Y\rho) \int_{-\infty}^{\infty} \phi\left(\frac{\sigma_Y\sqrt{1-\rho^2}}{(\sigma_Z - \sigma_Y\rho)} w\right) \phi(w)dw \lim_{T\to\infty} \sqrt{T} \\
= \lim_{T\to\infty} K_B + (A_{B1} + A_{B2})\sqrt{T} \\
= \begin{cases} \infty & \text{if } A_{B1} < A_{B2} \\ 0 & \text{if } A_{B1} = A_{B2} \\ -\infty & \text{if } A_{B1} > A_{B2} \end{cases}
\end{aligned}
$$

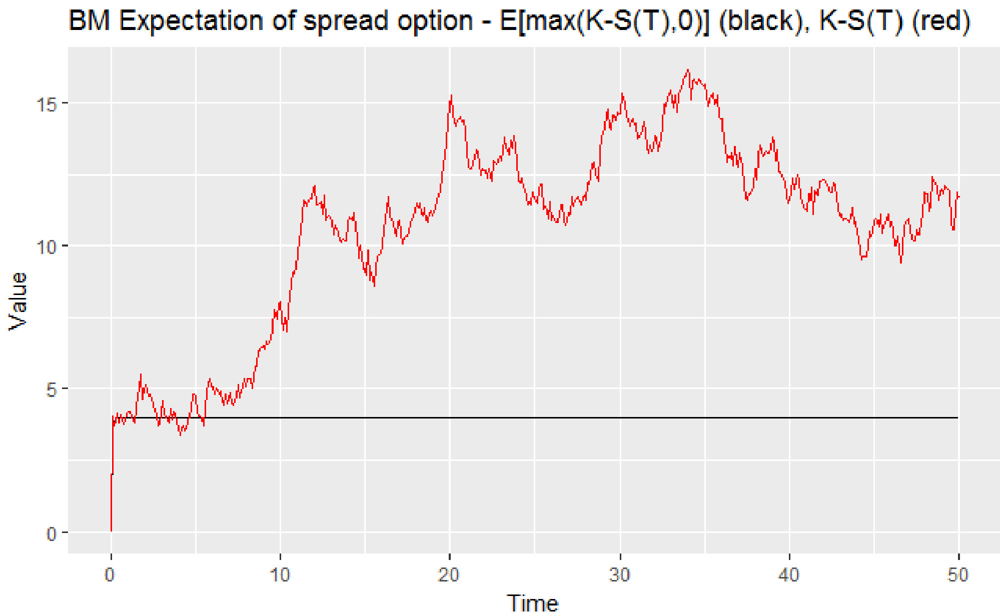

Figure 3.3: $\boldsymbol{\mu}_Y = \boldsymbol{\mu}_Z$: $\mathbb{E}[(K_B - S^B(T))_+]$ over time. Parameters: $K_B = 4$, $\mu_Y = 0.5$, $\mu_Z = 0.5$, $\sigma_Y = 0.8$, $\sigma_Z = 1.3$

Where A_{B1} and A_{B2} are short for

$$A_{B1} = \sigma_Y \sqrt{1-\rho^2} \int_{-\infty}^{\infty} w\phi(w)\Phi\left(\frac{\sigma_Y \sqrt{1-\rho^2}}{(\sigma_Z - \sigma_Y \rho)})w\right) dw$$

$$A_{B2} = (\sigma_Z - \sigma_Y \rho) \int_{-\infty}^{\infty} \phi\left(\frac{\sigma_Y \sqrt{1-\rho^2}}{(\sigma_Z - \sigma_Y \rho)} w\right) \phi(w) dw$$

3.4.3 $\lim_{T\to\infty} \mathbb{E}[(K_B - S^B(T))_+]$ when $\mu_Y < \mu_Z$

With $\mu_Y < \mu_Z$, the expectation of $S^B(T)$ is positive. Thus $K_B - S^B(T)$ will on average become negative. The value $c_B(w)$ becomes $\lim_{T\to\infty} c_B(w) = -\infty$

$$\lim_{T\to\infty} \mathbb{E}[(K_B - S^B(T))_+] = 0$$

In Figure 3.4 we have that $\mu_Y < \mu_Z$, and the price does indeed approach 0 in the long term. By definition, $(K_B - S^B(T))_+$ will never be negative, and thus takes the value 0 when $K_B < S^B(T)$.

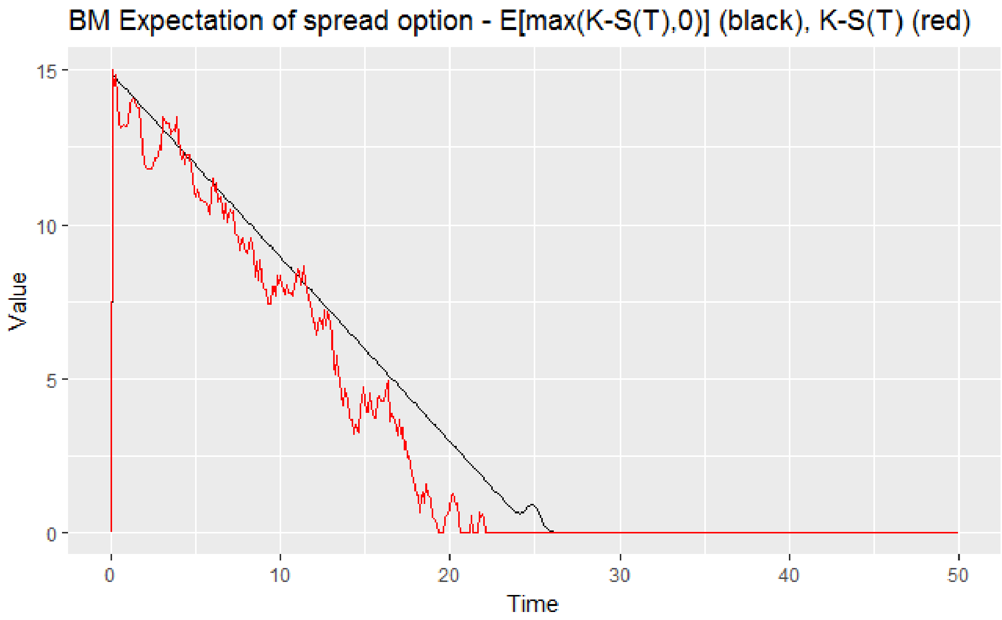

Figure 3.4: $\boldsymbol{\mu}_Y < \boldsymbol{\mu}_Z$: $\mathbb{E}[(K_B - S^B(T))_+]$ over time. Parameters: $K_B = 15$, $\mu_Y = 0.3$, $\mu_Z = 0.9$, $\sigma_Y = 0.8$, $\sigma_Z = 1.3$

In an arbitrage-free market, only when $\mu_Y < \mu_Z$ does $\mathbb{E}[(K_B - S^B(T))_+]$ resemble the traditional European put option. For the other two cases, when $\mu_Y = \mu_Z$, the undiscounted price is an option on the constant K_B, and for $\mu_Y > \mu_Z$, the expectation $\mathbb{E}[(K_B - S^B(T))_+]$ is positive and growing with time, and thus there is no risk to hedge against.

3.5 Calculation of integrals

This section will feature the step-by-step calculations of the following integrals, where we are given (u, w) as standard bivariate normal random variables, and $c(w) = c_1 + c_2 w$ where c_1, c_2 are constants.

$$\int_{-\infty}^{\infty} \phi(w)\Phi(c(w))dw \tag{3.5.1}$$

$$\int_{-\infty}^{\infty} w\phi(w)\Phi(c(w))dw \tag{3.5.2}$$

$$\int_{-\infty}^{\infty} \phi(w)\phi(c(w))dw \tag{3.5.3}$$

ϕ denotes the probability density of the standard normal distribution and Φ is the cumulative distribution function for the standard normal distribution.

For this purpose, we will make use of the integrals $\int_{-\infty}^{\infty} \Phi(c(w))dw$ and $\int_{-\infty}^{\infty} y\Phi(c(w))dw$ during our calculation steps for the main integrals in question.

Calculation of $\int_{-\infty}^{\infty} \Phi(c(w))dw$ by substitution:

$$\begin{aligned} c(w) &= c_1 + c_2 w & d(c(w)) &= c_2 dw \\ \frac{d(c(w))}{dw} &= c_2 & dw &= \frac{d(c(w))}{c_2} \end{aligned}$$

$$\begin{aligned} \int_{-c_Y}^{\infty} \Phi(c(w))dw &= \frac{1}{c_2} \int_{-\infty}^{\infty} \Phi(c(w))d(c(w)) \\ &= \frac{1}{c_2} \Big[c(w)\Phi(c(w)) + \phi(c(w)) \Big]_{-\infty}^{\infty} \end{aligned}$$

Calculation of $\int_{-\infty}^{\infty} y\Phi(c(w))dw$ by integration by parts:

$$\begin{aligned} a(w) &= w & b(y) &= \frac{1}{c_2} \Big[c(w)\Phi(c(w)) + \phi(c(w)) \Big]_{-\infty}^{\infty} \\ \frac{da}{dw} &= 1 & \frac{db}{dw} &= \Phi(c(w)) \end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{\infty} w\Phi(c(w))dw &= \frac{w}{c_2}\Big[c(w)\Phi(c(w)) + \phi(c(w))\Big]_{-\infty}^{\infty} \\ &\quad - \int_{-\infty}^{\infty} \frac{1}{c_2}\Big(c(w)\Phi(c(w)) + \phi(c(w))\Big)dw \\ \int_{-\infty}^{\infty} w\Phi(c(w))dw &= \frac{y}{c_2}\Big[c(w)\Phi(c(w)) + \phi(c(w))\Big]_{-\infty}^{\infty} \\ &\quad - \frac{1}{c_2}\Big[c(w)\Phi(c(w)) + \phi(c(w))\Big]_{-\infty}^{\infty}\end{aligned}$$

3.5.1 The Integral in Equation (3.5.1)

Calculation of $\int_{-\infty}^{\infty} \phi(w)\Phi(c(w))dw$:

$$\begin{aligned}\int_{-\infty}^{\infty} \Phi(c(w))\phi(w)dw &= \int_{-\infty}^{\infty} \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{c(w)}{\sqrt{2}}\right)\right)\phi(w)dw \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \phi(w)dw + \frac{1}{2}\int_{-\infty}^{\infty} \operatorname{erf}\left(\frac{c_1 + c_2 w}{\sqrt{2}}\right)\phi(w)dw \\ &= \frac{1}{2} + \frac{1}{2}\int_{-\infty}^{\infty} \operatorname{erf}\left(\frac{c(w)}{\sqrt{2}}\right)\phi(w)dw\end{aligned}$$

3.5.2 The Integral in Equation (3.5.2)

Calculation of $\int_{-\infty}^{\infty} w\phi(w)\Phi(c(w))dw$:

$$\begin{aligned}\int_{-\infty}^{\infty} w\Phi(c(w))\phi(w)dw &= \frac{1}{2}\int_{-\infty}^{\infty} w\left(1 + \operatorname{erf}\left(\frac{c(w)}{\sqrt{2}}\right)\right)\phi(w)dw \\ &= \frac{1}{2}\int_{-\infty}^{\infty} w\phi(w)dw + \frac{1}{2}\int_{-\infty}^{\infty} \operatorname{erf}\left(\frac{c_1 + c_2 w}{\sqrt{2}}\right)w\phi(w)dw\end{aligned}$$

Since we have:

$$\frac{d\phi(w)}{dw} = -w\phi(w)$$

$$\begin{aligned}\int_{-\infty}^{\infty} w\phi(w)dw &= \big[-\phi(w)\big]_{-\infty}^{\infty} \\ &= 0\end{aligned}$$

we get:

$$\int_{-\infty}^{\infty} w\Phi(c(w))\phi(w)dw = \frac{1}{2}\int_{-\infty}^{\infty} \operatorname{erf}\left(\frac{c_U(w)}{\sqrt{2}}\right) w\phi(w)dw$$

3.5.3 The Integral in Equation (3.5.3)

Calculation of $\int_{-\infty}^{\infty}\phi(w)\phi(c(w))dw$:

$$\begin{aligned}
\int_{-\infty}^{\infty} \phi(c(w))\phi(w)dw &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{(c_1+c_2w)^2}{2}} e^{-\frac{1}{2}w^2}dw \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{(c_1+c_2w)^2+w^2}{2}}dw \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{(a+bw)^2+w^2}{2}}dw
\end{aligned}$$

Substitution:

$$\begin{aligned}
t^2 &= \frac{(ab+(b^2+1)w)^2}{2(b^2+1)} & \frac{dt}{dw} &= \frac{\sqrt{b^2+1}}{\sqrt{2}} \\
t &= \frac{ab+(b^2+1)w}{\sqrt{2(b^2+1)}} & dw &= \frac{\sqrt{2}}{\sqrt{b^2+1}}dt \\
y &= -\infty & t &= -\infty \\
y &= \infty & t &= \infty
\end{aligned}$$

Where we have:

$$\begin{aligned}
\frac{a^2}{2(b^2+1)} + t^2 &= \frac{a^2+(ab+(b^2+1)w)^2}{2(b^2+1)} \\
&= \frac{a^2(b^2+1)+2ab(b^2+1)w+(b^2+1)^2w^2}{2(b^2+1)} \\
&= \frac{a^2+2abw+(b^2+1)w^2}{2} \\
&= \frac{(a+bw)^2+w^2}{2}
\end{aligned}$$

So we get:

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-\frac{(a+bw)^2+w^2}{2})dw &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-\frac{a^2}{2(b^2+1)}+t^2)dw \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-\frac{a^2}{2(b^2+1)}+t^2)\frac{\sqrt{2}}{\sqrt{b^2+1}}dt \\
&= \frac{1}{\pi\sqrt{2(b^2+1)}}e^{-\frac{a^2}{2(b^2+1)}}\int_{-\infty}^{\infty}e^{-t^2}dt \\
&= \frac{1}{\pi\sqrt{2(b^2+1)}}e^{-\frac{a^2}{2(b^2+1)}}\operatorname{erf}(\infty) \\
&= \frac{1}{\pi\sqrt{2(b^2+1)}}e^{-\frac{a^2}{2(b^2+1)}} \\
&= \frac{1}{\pi\sqrt{2(c_2^2+1)}}e^{-\frac{c_1^2}{2(c_2^2+1)}}
\end{aligned}$$

Chapter 4

Cointegrated Spread Options Based on Ornstein-Uhlenbeck Processes

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Bibliographical notes

The main references of this chapter are: Benth and Koekebakker (2015)[2], Benth and Benth (2012)[4], and Benth (2016)[1].

4.1 Price of Cointegrated Spread Options

Our goal for this chapter is to estimate the arbitrage-free price of a European spread option with dynamics based on Ornstein-Uhlenbeck processes. Unlike the geometric Brownian motion spreads in the previous chapter, Ornstein-Uhlenbeck spreads are cointegrated and mean-reverting.

Let $S(t)$ be the value of the spread option:

$$S(t) = S_2(t) - S_1(t) \tag{4.1.1}$$

for $t \geq 0$, where $S_1(t)$, $S_2(t)$ are described by:

$$S_1(t) = \log s_1(t) = X(t) + Y(t), \quad s_1(t) = s_1(0)\exp(X(t) + Y(t)) \tag{4.1.2}$$

$$S_2(t) = \log s_2(t) = X(t) + Z(t), \quad s_2(t) = s_2(0)\exp(X(t) + Z(t)) \tag{4.1.3}$$

Without loss of generalization, it is assumed that

$$s_1(0) = s_2(0) = 1$$

The expressions for $X(t)$, $Y(t)$ and $Z(t)$ are:

$$X(t) = \mu t + \sigma B(t) \tag{4.1.4}$$

$$Y(t) = Y(0)e^{-\alpha_Y t} + \mu_Y(1 - e^{-\alpha_Y t}) + \sigma_Y \int_0^t e^{-\alpha_Y(t-s)} dB_Y(s) \tag{4.1.5}$$

$$Z(t) = Z(0)e^{-\alpha_Z t} + \mu_Z(1 - e^{-\alpha_Z t}) + \sigma_Z \int_0^t e^{-\alpha_Z(t-s)} dB_Z(s) \tag{4.1.6}$$

where $B(t)$, $B_Y(t)$ and $B_Z(t)$ are Brownian motions defined on a complete filtered probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$, with $\mu_Z > \mu_Y > 0$.

$B_Y(t)$ and $B_Z(t)$ have the same relationship as defined in chapter 3, that is to say:

$$dB_Y(t) = \rho dU(t) + \sqrt{1-\rho^2} dW(t) \qquad dB_Z(t) = dU(t)$$
$$B_Y(t) = \rho U(t) + \sqrt{1-\rho^2} W(t) \qquad B_Z(t) = U(t)$$

where $\rho \in [0, 1]$ is the correlation between $B_Y(t)$ and $B_Z(t)$. $U(t)$ and $W(t)$ are independent Browninan motions processes on $(\Omega, \mathcal{F}_t, P)$.

We also define the correlation between $B(t)$ and $B_Y(t)$, $B_Z(t)$ to be:

$$\text{corr}(B, B_Y) = \rho_Y$$
$$\text{corr}(B, B_Z) = \rho_Z$$

In this thesis, we will assume B to be independent from B_Y and B_Z, so $\rho_Y = \rho_Z = 0$.

$X(t)$ is a Brownian motion with drift μ and infinitesimal variance σ, which represents the non-stationary long-term variations of $S(t)$. $Y(t)$ and $Z(t)$ are Ornstein-Uhlenbeck processes, and their terms in $S(t)$ models the stationary short-term variations, as the processes will drift towards their mean in the long term.

We assume $S_1(t)$ and $S_2(t)$ to be cointegrated, meaning there exists a linear combination $aS_1(t) + bS_2(t)$ with $a, b > 0$, that becomes stationary for $n \to \infty$.

The stochastic processes $X(t)$, $Y(t)$, and $Z(t)$ can also be expressed as:

$$\begin{aligned}
dX(t) &= \mu dt + \sigma dB(t) \\
dZ(t) &= -\alpha_Z(Z(t) - \mu_Z)dt + \sigma_Z dB_Z(t) \\
dY(t) &= -\alpha_Y(Y(t) - \mu_Y)dt + \sigma_Y dB_Y(t) \\
&= -\alpha_Y(Y(t) - \mu_Y)dt + \sigma_Y(\rho dB_Z(t) + \sqrt{1-\rho^2} dW(t))
\end{aligned}$$

So the expression for $S(t) = S_1(t) - S_2(t) = Z(t) - Y(t)$ becomes:

Proposition 4.1.1. *The value of the Ornstein-Uhlenbeck spread at time t is:*

$$\begin{aligned} S(t) &= S_2(t) - S_1(t) \\ &= Z_0 e^{-\alpha_Z t} + \mu_Z(1 - e^{-\alpha_Z t}) + \int_0^t \sigma_Z e^{-\alpha_Z(t-s)} - \sigma_Y \rho e^{-\alpha_Y(t-s)} dU(s) \\ &\quad - Y_0 e^{-\alpha_Y t} - \mu_Y(1 - e^{-\alpha_Y t}) - \sigma_Y \sqrt{1-\rho^2} \int_0^t e^{-\alpha_Y(t-s)} dW(s) \end{aligned} \tag{4.1.7}$$

In this chapter, our aim is to estimate the future value of a contract with spot price $f^O(T) = K - S(T)$.

Proposition 4.1.2. *The arbitrage-free price of a put spread option, i.e. the price at current time $t = 0$ for a contract with payoff at a future time T, is:*

$$P^O(T) = e^{-rT}\mathbb{E}[(K - S(T))_+] \tag{4.1.8}$$

where r is a constant interest rate and K is the strike price.

4.2 Expectation and Variance for Ornstein-Uhlenbeck Spreads:

As $X(t)$, $Y(t)$ and $Z(t)$ are all linear combinations of Brownian motions, the processes are normally distributed. It follows that $S(t) = Z(t) - Y(t)$ have a bivariate normal distribution. The expectation, variance and covariance for the Ornstein-Uhlenbeck processes are presented below. Detailed calculations for the expressions can be found in Section 4.6.2.

Expectation:

$$\begin{aligned} \mathbb{E}[Y(t)] &= \mathbb{E}[Y(0)e^{-\alpha_Y t} + \mu_Y(1 - e^{-\alpha_Y t}) + \sigma_Y \int_0^t e^{-\alpha_Y(t-s)} \left(\rho dU(s) + \sqrt{1-\rho^2} dW(s)\right)] \\ &= Y(0)e^{-\alpha_Y t} + \mu_Y(1 - e^{-\alpha_Y t}) \end{aligned}$$

$$\begin{aligned} \mathbb{E}[Z(t)] &= \mathbb{E}[Z(0)e^{-\alpha_Z t} + \mu_Z(1 - e^{-\alpha_Z t}) + \sigma_Z \int_0^t e^{-\alpha_Z(t-s)} dU(s)] \\ &= Z(0)e^{-\alpha_Z t} + \mu_Z(1 - e^{-\alpha_Z t}) \end{aligned}$$

Variance:

$$\begin{aligned}\mathrm{Var}(Y(t)) &= \mathbb{E}[Y^2(t)] - \mathbb{E}[Y(t)]^2 \\ &= \frac{\sigma_Y^2}{2\alpha_Y}\left(1 - e^{-2\alpha_Y t}\right)\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(Z(t)) &= \mathbb{E}[Z^2(t)] - \mathbb{E}[Z(t)]^2 \\ &= \frac{\sigma_Z^2}{2\alpha_Z}\left(1 - e^{-2\alpha_Z t}\right)\end{aligned}$$

Covariance:

$$\begin{aligned}\mathrm{Cov}(Y(t), Z(t)) &= \mathbb{E}\left[(Y(t) - \mathbb{E}[Y(t)])(Z(t) - \mathbb{E}[Z(t)])\right] \\ &= \frac{\rho\sigma_Y\sigma_Z}{\alpha_Y + \alpha_Z}\left(1 - e^{-t(\alpha_Y + \alpha_Z)}\right)\end{aligned}$$

Correlation:

$$\begin{aligned}\mathrm{corr}(Y(t), Z(t)) &= \frac{\mathrm{Cov}(Y(t), Z(t))}{\mathrm{Var}(Y(t))\mathrm{Var}(Z(t))} \\ &= \frac{\rho}{\alpha_Y + \alpha_Z}\left(1 - e^{-t(\alpha_Y + \alpha_Z)}\right)\end{aligned}$$

Hence, it follows that for $S(t)$:

Proposition 4.2.1. *The Ornstein-Uhlenbeck spread $S(t) = Z(t) - Y(t)$ has a bivariate normal distribution, with*

$$S(t) \sim \mathcal{N}(\boldsymbol{\mu}(t), \boldsymbol{\sigma}(t))$$

where

$$\boldsymbol{\mu}(t) = \begin{pmatrix} Y(0)e^{-\alpha_Y t} + \mu_Y(1 - e^{-\alpha_Y t}) \\ Z(0)e^{-\alpha_Z t} + \mu_Z(1 - e^{-\alpha_Z t}) \end{pmatrix} \tag{4.2.1}$$

$$\boldsymbol{\sigma}(t) = \begin{pmatrix} \frac{\sigma_Y^2}{2\alpha_Y}\left(1 - e^{-2\alpha_Y t}\right) \\ \frac{\rho\sigma_Y\sigma_Z}{\alpha_Y+\alpha_Z}\left(1 - e^{-t(\alpha_Y+\alpha_Z)}\right) \\ \frac{\rho\sigma_Y\sigma_Z}{\alpha_Y+\alpha_Z}\left(1 - e^{-t(\alpha_Y+\alpha_Z)}\right) \\ \frac{\sigma_Z^2}{2\alpha_Z}\left(1 - e^{-2\alpha_Z t}\right) \end{pmatrix} \tag{4.2.2}$$

$$\tag{4.2.3}$$

The expectation and variance for the spread option $S(t)$ are:

Expectation of $\boldsymbol{S}(t)$:

$$\begin{aligned}\mathbb{E}[S(t)] &= \mathbb{E}[Z(t)] - \mathbb{E}[Y(t)] \\ &= Z(0)e^{-\alpha_Z t} - Y(0)e^{-\alpha_Y t} + \mu_Z(1 - e^{-\alpha_Z t}) - \mu_Y(1 - e^{-\alpha_Y t})\end{aligned}$$

Variance of $\boldsymbol{S}(t)$:

$$\begin{aligned}\mathrm{Var}(S(t)) &= \mathrm{Var}(Y(t)) - 2\mathrm{Cov}(Y(t), Z(t)) + \mathrm{Var}(Z(t)) \\ &= \frac{\sigma_Z^2}{2\alpha_Z}\left(1 - e^{-2\alpha_Z t}\right) - 2\frac{\rho\sigma_Y\sigma_Z}{\alpha_Y + \alpha_Z}\left(1 - e^{-t(\alpha_Y + \alpha_Z)}\right) \\ &\quad + \frac{\sigma_Y^2}{2\alpha_Y}\left(1 - e^{-2\alpha_Y t}\right)\end{aligned}$$

4.3 Price of an Ornstein-Uhlenbek European Spread Option

Similarly to the BM spread option examined in Chapter 3, we wish to model the price of the put spread option, in this case for an Ornstein-Uhlenbeck spread. The price $P^O(T)$ at current time $t = 0$ for a contract with payoff at a future time T is:

$$P^O(T) = e^{-rT}\mathbb{E}[(K - S(T))_+] \tag{4.3.1}$$

where r denotes the constant interest rate and K is the strike price. $S(T)$ is bivariate normal, and like the case with Brownian motion, the expectation can be calculated by double integrals.

4.3.1 Computing the Price of an OU Spread Option by Double Integrals

For the risk-neutral price of an European option, we compute $\mathbb{E}\left[(K - S(T))_+\right]$:

$$\begin{aligned}
S(T) &= S_2(T) - S_1(T) \\
&= Z(T) - Y(T) \\
&= Z_0 e^{-\alpha_Z T} - Y_0 e^{-\alpha_Y T} + \mu_Z(1 - e^{-\alpha_Z T}) - \mu_Y(1 - e^{-\alpha_Y T}) \\
&\quad + \int_0^T \sigma_Z e^{-\alpha_Z(T-s)} - \sigma_Y \rho e^{-\alpha_Y(T-s)} dU(s) \\
&\quad - \sigma_Y \int_0^T e^{-\alpha_Y(T-s)} \sqrt{1-\rho^2} dW(s)
\end{aligned}$$

Which gives us:

$$\begin{aligned}
\mathbb{E}[(K - S(T))_+] &= \mathbb{E}[(K - (Z_0 e^{-\alpha_Z T} - Y_0 e^{-\alpha_Y T} + \mu_Z(1 - e^{-\alpha_Z T}) \\
&\quad - \mu_Y(1 - e^{-\alpha_Y T}) \\
&\quad + \int_0^T \sigma_Z e^{-\alpha_Z(T-s)} - \sigma_Y \rho e^{-\alpha_Y(T-s)} dU(s) \\
&\quad - \sigma_Y \int_0^T e^{-\alpha_Y(T-s)} \sqrt{1-\rho^2} dW(s)))_+] \\
&= \mathbb{E}[(K - Z_0 e^{-\alpha_Z T} + Y_0 e^{-\alpha_Y T} - \mu_Z(1 - e^{-\alpha_Z T}) \\
&\quad + \mu_Y(1 - e^{-\alpha_Y T}) \\
&\quad - \int_0^T \sigma_Z e^{-\alpha_Z(T-s)} - \sigma_Y \rho e^{-\alpha_Y(T-s)} dU(s) \\
&\quad + \sigma_Y \int_0^T e^{-\alpha_Y(T-s)} \sqrt{1-\rho^2} dW(s))_+] \\
&= \mathbb{E}\left[(A_T - U_T^* - W_T^*, 0)_+\right]
\end{aligned}$$

where

$$A_T = K - Z_0 e^{-\alpha_Z T} - \mu_Z(1 - e^{-\alpha_Z T}) + Y_0 e^{-\alpha_Y T} + \mu_Y(1 - e^{-\alpha_Y T})$$

$$U_T^* = \int_0^T \sigma_Z e^{-\alpha_Z(T-s)} - \sigma_Y \rho e^{-\alpha_Y(T-s)} dU(s)$$

$$W_T^* = \sigma_Y \sqrt{1-\rho^2} \int_0^T e^{-\alpha_Y(T-s)} dW(s)$$

As in Chapter 3, we start by standardizing the stochastic processes:

$$
\begin{aligned}
\mathbb{E}[U_T^*] &= \mathbb{E}[\int_0^T \sigma_Z e^{-\alpha_Z(T-s)} - \sigma_Y \rho e^{-\alpha_Y(T-s)} dU(s)] \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}(U_T^*) &= \mathbb{E}[\int_0^T (\sigma_Z e^{-\alpha_Z(T-s)} - \sigma_Y \rho e^{-\alpha_Y(T-s)})^2 ds] \\
&= \frac{\sigma_Z^2}{2\alpha_Z}(1 - e^{-2\alpha_Z T}) - \frac{2\sigma_Y\sigma_Z\rho}{\alpha_Y + \alpha_Z}(1 - e^{-(\alpha_Y+\alpha_Z)T}) + \frac{\sigma_Y^2\rho^2}{2\alpha_Y}(1 - e^{-2\alpha_Y T})
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[W_T^*] &= \mathbb{E}[\sigma_Y\sqrt{1-\rho^2}\int_0^T e^{-\alpha_Y(T-s)} dW(s)] \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}(W_T^*) &= \mathbb{E}[\int_0^T (\sigma_Y\sqrt{1-\rho^2} e^{-\alpha_Y(T-s)})^2 ds] \\
&= \frac{\sigma_Y^2(1-\rho^2)}{2\alpha_Y}(1 - e^{-2\alpha_Y T})
\end{aligned}
$$

Hence, the standardized stochastic processes $\tilde{U}_T, \tilde{W}_T \sim \mathcal{N}(0,1)$ are:

$$
\begin{aligned}
\tilde{U}_T &= \frac{U_T^*}{\sqrt{\mathrm{Var}(U_T^*)}} \\
&= \frac{\int_0^T \sigma_Z e^{-\alpha_Z(T-s)} - \sigma_Y \rho e^{-\alpha_Y(T-s)} dU(s)}{\sqrt{\frac{\sigma_Z^2}{2\alpha_Z}(1 - e^{-2\alpha_Z T}) - \frac{2\sigma_Y\sigma_Z\rho}{\alpha_Y+\alpha_Z}(1 - e^{-(\alpha_Y+\alpha_Z)T}) + \frac{\sigma_Y^2\rho^2}{2\alpha_Y}(1 - e^{-2\alpha_Y T})}}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{W}_T &= \frac{W_T^*}{\sqrt{\mathrm{Var}(W_T^*)}} \\
&= \frac{\sqrt{2\alpha_Y}\int_0^T e^{-\alpha_Y(T-s)} dW(s)}{\sqrt{1 - e^{-2\alpha_Y T}}}
\end{aligned}
$$

Then the standardized price function becomes:

$$
\begin{aligned}
Price(T) &= e^{-rT}\mathbb{E}[(A_T - U_T^* - W_T^*)_+] \\
&= e^{-rT}\mathbb{E}[(A_T - \mathrm{Var}(U_T^*)\tilde{U}_T - \mathrm{Var}(W_T^*)\tilde{W}_T)_+] \\
&= e^{-rT}\mathbb{E}[(K - Z_0e^{-\alpha_Z T} - \mu_Z(1-e^{-\alpha_Z T}) + Y_0e^{-\alpha_Y T} \\
&\quad + \mu_Y(1-e^{-\alpha_Y T}) - (\frac{\sigma_Z^2}{2\alpha_Z}(1-e^{-2\alpha_Z T}) \\
&\quad - \frac{2\sigma_Y\sigma_Z\rho}{\alpha_Y+\alpha_Z}(1-e^{-(\alpha_Y+\alpha_Z)T}) \\
&\quad + \frac{\sigma_Y^2\rho^2}{2\alpha_Y}(1-e^{-2\alpha_Y T}))\tilde{U}_T \\
&\quad - \frac{\sigma_Y^2(1-\rho^2)}{2\alpha_Y}(1-e^{-2\alpha_Y T})\tilde{W}_T)_+]
\end{aligned}
$$

The random variable inside the expectation is zero when

$$
\begin{aligned}
K < {} & Z_0e^{-\alpha_Z T} + \mu_Z(1-e^{-\alpha_Z T}) - Y_0e^{-\alpha_Y T} - \mu_Y(1-e^{-\alpha_Y T}) \\
& + \left(\frac{\sigma_Z^2}{2\alpha_Z}\left(1-e^{-2\alpha_Z T}\right) - \frac{2\sigma_Y\sigma_Z\rho}{\alpha_Y+\alpha_Z}\left(1-e^{-(\alpha_Y+\alpha_Z)T}\right)\right. \\
& \left. + \frac{\sigma_Y^2\rho^2}{2\alpha_Y}\left(1-e^{-2\alpha_Y T}\right)\right)\tilde{U}_T + \frac{\sigma_Y^2(1-\rho^2)}{2\alpha_Y}\left(1-e^{-2\alpha_Y T}\right)\tilde{W}_T
\end{aligned}
$$

$$
\tilde{U}_T > \frac{A_T - \mathrm{Var}(W_T^*)\tilde{W}_T}{\mathrm{Var}(U_T^*)}
$$

So the boundary is:

$$
\begin{aligned}
&\tilde{U}_T < c_U \\
c_U(w) &= \frac{K - Z_0e^{-\alpha_Z T} - \mu_Z(1-e^{-\alpha_Z T}) + Y_0e^{-\alpha_Y T} + \mu_Y(1-e^{-\alpha_Y T})}{\frac{\sigma_Z^2}{2\alpha_Z}\left(1-e^{-2\alpha_Z T}\right) - \frac{2\sigma_Y\sigma_Z\rho}{\alpha_Y+\alpha_Z}\left(1-e^{-(\alpha_Y+\alpha_Z)T}\right) + \frac{\sigma_Y^2\rho^2}{2\alpha_Y}\left(1-e^{-2\alpha_Y T}\right)} \\
&\quad - \frac{\frac{\sigma_Y^2(1-\rho^2)}{2\alpha_Y}\left(1-e^{-2\alpha_Y T}\right)w}{\frac{\sigma_Z^2}{2\alpha_Z}\left(1-e^{-2\alpha_Z T}\right) - \frac{2\sigma_Y\sigma_Z\rho}{\alpha_Y+\alpha_Z}\left(1-e^{-(\alpha_Y+\alpha_Z)T}\right) + \frac{\sigma_Y^2\rho^2}{2\alpha_Y}\left(1-e^{-2\alpha_Y T}\right)} \\
&= \frac{A_T}{\mathrm{Var}(U_T^*)} - \frac{\mathrm{Var}(W_T^*)}{\mathrm{Var}(U_T^*)}w \\
&= c_{U1} + c_{U2}w
\end{aligned}
$$

with the condition that $\tilde{W}_T = w$.

We can then find the price by computing a double integral:

$$\begin{aligned} P^O(T) &= e^{-rT}\mathbb{E}\left[(K-(S_2(T)-S_1(T)))_+\right] \\ &= e^{-rT}\mathbb{E}\left[(K-(Z(T)-Y(T)))_+\right] \\ &= e^{-rT}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(K-z+y)_+ f_T(y,z)dudz \end{aligned}$$

where $f_T(y,z)$ is the probability density of the bivariate normal distribution.

For $\tilde{U}_T$ and $\tilde{W}_T$ standardized:

$$\begin{aligned} P^O(T) &= e^{-rT}\mathbb{E}\left[(K-(S_2(T)-S_1(T)))_+\right] \\ &= e^{-rT}\mathbb{E}[(K-Z_0e^{-\alpha_Z T}-\mu_Z(1-e^{-\alpha_Z T})+Y_0e^{-\alpha_Y T} \\ &\quad +\mu_Y(1-e^{-\alpha_Y T})-(\frac{\sigma_Z^2}{2\alpha_Z}(1-e^{-2\alpha_Z T})-\frac{2\sigma_Y\sigma_Z\rho}{\alpha_Y+\alpha_Z}(1-e^{-(\alpha_Y+\alpha_Z)T}) \\ &\quad +\frac{\sigma_Y^2\rho^2}{2\alpha_Y}(1-e^{-2\alpha_Y T}))\tilde{U}_T-\frac{\sigma_Y^2(1-\rho^2)}{2\alpha_Y}(1-e^{-2\alpha_Y T})\tilde{W}_T)_+] \end{aligned}$$

With $f_{u,w}(w)$ being the probability density for $\tilde{W}_T$ at time T, and $f_{u,w}(u)$ the density for $\tilde{U}_T$ at time T, we calculate:

$$\begin{aligned} \mathbb{E}\left[(K-S(T))_+\right] &= \mathbb{E}\left[\mathbb{E}[(A_T-U_T^*-W_T^*)_+|\tilde{W}_T]\right] \\ &= \int_{-\infty}^{\infty}\mathbb{E}[(A_T-U_T^*-W_T^*)_+|\tilde{W}_T=w]f_{u,w}(w)du \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}(A_T-\mathrm{Var}(U_T^*)u\right. \\ &\qquad \left.-\mathrm{Var}(W_T^*)w)_+ f_{F,G|\tilde{W}_T=w}(u)du\right)f_{u,w}(w)dw \end{aligned}$$

$$\begin{aligned} &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{c_U(w)}(A_T-\mathrm{Var}(U_T^*)u-\mathrm{Var}(W_T^*)w)\,\phi(u)du\right)f_{u,w}(w)dw \\ &= \int_{-\infty}^{\infty}\left((A_T-\mathrm{Var}(W_T^*)w)\,\Phi(c_U(w))+\mathrm{Var}(U_T^*)\phi(c_U(w))\right)f_{u,w}(w)dw \\ &= \int_{-\infty}^{\infty}\left((A_T-\mathrm{Var}(W_T^*)w)\,\Phi(c_U(w))+\mathrm{Var}(U_T^*)\phi(c_U(w))\right)\phi(w)dw \\ &= A_T\int_{-\infty}^{\infty}\Phi(c_U(w))\phi(w)dw-\mathrm{Var}(W_T^*)\int_{-\infty}^{\infty}w\Phi(c_U(w))\phi(w)dw \\ &\quad +\mathrm{Var}(U_T^*)\int_{-\infty}^{\infty}\phi(c_U(w))\phi(w)dw \end{aligned}$$

where

$$A_T = K - Z_0 e^{-\alpha_Z T} - \mu_Z(1 - e^{-\alpha_Z T}) + Y_0 e^{-\alpha_Y T} + \mu_Y(1 - e^{-\alpha_Y T})$$

$$\begin{aligned}\mathrm{Var}(U_T^*) &= \frac{\sigma_Z^2}{2\alpha_Z}\left(1 - e^{-2\alpha_Z T}\right) - \frac{2\sigma_Y\sigma_Z\rho}{\alpha_Y + \alpha_Z}\left(1 - e^{-(\alpha_Y+\alpha_Z)T}\right) \\ &\quad + \frac{\sigma_Y^2\rho^2}{2\alpha_Y}\left(1 - e^{-2\alpha_Y T}\right)\end{aligned}$$

$$\mathrm{Var}(W_T^*) = \frac{\sigma_Y^2(1-\rho^2)}{2\alpha_Y}\left(1 - e^{-2\alpha_Y T}\right)$$

$$c_U(w) = \frac{A_T - \mathrm{Var}(W_T^*)}{\mathrm{Var}(U_T^*)} w$$

$$\int_{-\infty}^{\infty} \Phi(c_U(w))\phi(w)dw = \frac{1}{2} + \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{erf}\left(\frac{c_U(w)}{\sqrt{2}}\right)\phi(w)dw$$

$$\int_{-\infty}^{\infty} w\Phi(c_U(w))\phi(w)dw = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{erf}\left(\frac{c_U(w)}{\sqrt{2}}\right)w\phi(w)dw$$

$$\int_{-\infty}^{\infty} \phi(c_U(w))\phi(w)dw = \frac{1}{\pi\sqrt{2(c_{U2}^2+1)}} e^{-\frac{c_{U1}^2}{2(c_{U2}^2+1)}}$$

The detailed calculations of the integral terms are described in Section 3.5.

Considering the resulting integral terms cannot be solved analytically, we turn to numerical simulation to compute the price and analyse the structure of the Ornstein-Uhlenbeck spread option further. See Appendix section B.4 for the code for the numerical simulation.

4.4 Value of the OU Spread Option in the Long Term

We further examine what happens to the expectation in the price function as T approaches infinity:

$$\lim_{T\to\infty} \mathbb{E}\left[(K - S(T))_+\right]$$

4.4.1 Value of $\lim_{T\to\infty} S(T)$ and $\lim_{T\to\infty} \mathbb{E}[S(T)]$

The limit of the OU spread $S(T)$ is undefined, as the BM processes $U(t)$, $W(t)$ does not converge. However, the OU spread has a constant term $\mu_Z - \mu_Y$ when T approaches infinity.

$$\begin{aligned}
\lim_{T\to\infty} S(T) &= \lim_{T\to\infty} (Z(T) - Y(T)) \\
&= \lim_{T\to\infty} \Big(Z_0 e^{-\alpha_Z T} + \mu_Z(1 - e^{-\alpha_Z T}) \\
&\quad + \sigma_Z \int_0^T e^{-\alpha_Z(T-s)} - \sigma_Y \rho e^{-\alpha_Y(T-s)} dU(s) - Y(0)e^{-\alpha_Y T} \\
&\quad + \mu_Y(1 - e^{-\alpha_Y T}) - \sigma_Y \sqrt{1-\rho^2} \int_0^t e^{-\alpha_Y(T-s)} dW(s) \Big) \\
&= \mu_Z - \mu_Y + \lim_{T\to\infty} \Big(\sigma_Z e^{-\alpha_Z T} \int_0^T e^{\alpha_Z s} dU(s) \\
&\quad - \sigma_Y \rho e^{-\alpha_Y T} \int_0^T e^{\alpha_Y s} dU(s) - \sigma_Y \sqrt{1-\rho^2} e^{-\alpha_Y T} \int_0^T e^{\alpha_Y s} dW(s) \Big)
\end{aligned}$$

While $S(T)$ does not converge to a constant limit, $\mathbb{E}[S(T)]$ does:

$$\lim_{T\to\infty} \mathbb{E}[S(T)] = \mu_Z - \mu_Y$$

From Figure 4.1 we see that the OU spread is indeed stationary and mean-reverting around its stationary part, i.e the limit of $\mathbb{E}[S(T)]$. In this case $\mu_Z - \mu_Y = 0.6$, and we see that $S(T)$ does not seem to stray far the mean, unlike the BM spread.

4.4.2 Value of $\lim_{T\to\infty} \mathbb{E}\left[(K - S(T))_+\right]$

For $\lim_{T\to\infty} \mathbb{E}\left[(K - S(T))_+\right]$, breaking down the terms of the expectation and examining the limits results in the following stationary terms:

$$\begin{aligned}
A_T &= K - Z_0 e^{-\alpha_Z T} - \mu_Z(1 - e^{-\alpha_Z T}) + Y_0 e^{-\alpha_Y T} + \mu_Y(1 - e^{-\alpha_Y T}) \\
\lim_{T\to\infty} A_T &= K - \mu_Z + \mu_Y \\
&= A_{T\lim}
\end{aligned}$$

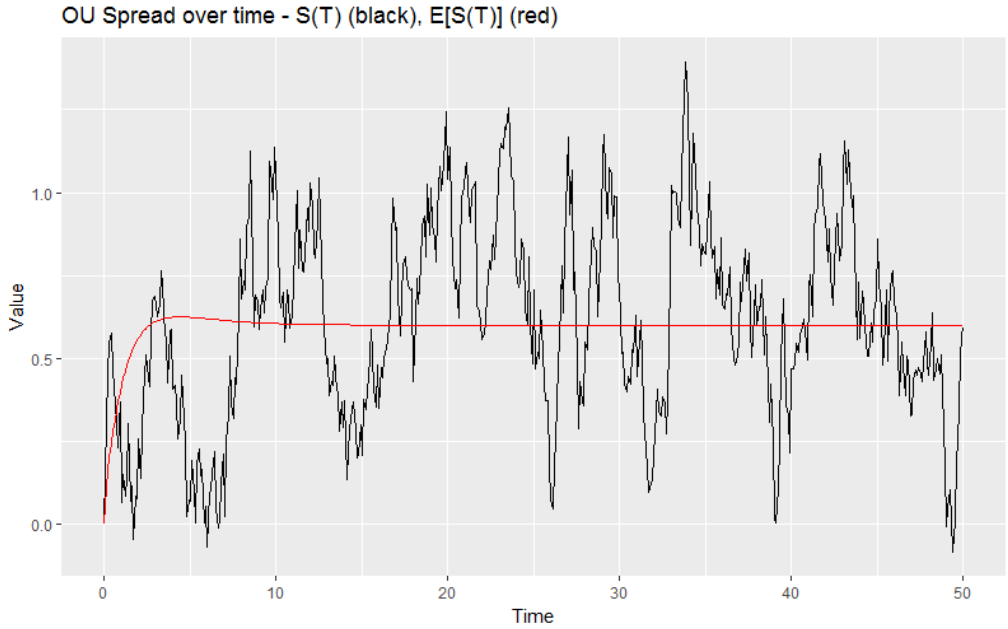

Figure 4.1: **OU Spread Option.** Parameters: $K = 15$, $\alpha_Y = 0.4$, $\alpha_Z = 0.8$, $\mu_Y = 0.3$, $\mu_Z = 0.9$, $\sigma_Y = 0.8$, $\sigma_Z = 1.3$

$$
\begin{aligned}
\mathrm{Var}(U_T^*) &= \frac{\sigma_Z^2}{2\alpha_Z}\left(1-e^{-2\alpha_Z T}\right) - \frac{2\sigma_Y\sigma_Z\rho}{\alpha_Y+\alpha_Z}\left(1-e^{-(\alpha_Y+\alpha_Z)T}\right) + \frac{\sigma_Y^2\rho^2}{2\alpha_Y}\left(1-e^{-2\alpha_Y T}\right) \\
\lim_{T\to\infty}\mathrm{Var}(U_T^*) &= \frac{\sigma_Z^2}{2\alpha_Z} - \frac{2\sigma_Y\sigma_Z\rho}{\alpha_Y+\alpha_Z} + \frac{\sigma_Y^2\rho^2}{2\alpha_Y} \\
&= \mathrm{Var}(U_T^*)_{\lim}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}(W_T^*) &= \frac{\sigma_Y^2(1-\rho^2)}{2\alpha_Y}\left(1-e^{-2\alpha_Y T}\right) \\
\lim_{T\to\infty}\mathrm{Var}(W_T^*) &= \frac{\sigma_Y^2(1-\rho^2)}{2\alpha_Y} \\
&= \mathrm{Var}(W_T^*)_{\lim}
\end{aligned}
$$

$$
\begin{aligned}
c_U(w) &= \frac{A_T - \mathrm{Var}(W_T^*)}{\mathrm{Var}(U_T^*)} w \\
\lim_{T\to\infty} c_U(w) &= \frac{K - \mu_Z + \mu_Y - \frac{\sigma_Y^2(1-\rho^2)}{2\alpha_Y}}{\frac{\sigma_Z^2}{2\alpha_Z} - \frac{2\sigma_Y\sigma_Z\rho}{\alpha_Y+\alpha_Z} + \frac{\sigma_Y^2\rho^2}{2\alpha_Y}} \\
&= c_{U\,\mathrm{lim}}(w)
\end{aligned}
$$

$$
\begin{aligned}
c_{U1} &= \frac{A_T}{\mathrm{Var}(U_T^*)} \\
\lim_{T\to\infty} c_{U1} &= \frac{K - \mu_Z + \mu_Y}{\frac{\sigma_Z^2}{2\alpha_Z} - \frac{2\sigma_Y\sigma_Z\rho}{\alpha_Y+\alpha_Z} + \frac{\sigma_Y^2\rho^2}{2\alpha_Y}} \\
&= c_{U1\,\mathrm{lim}}
\end{aligned}
$$

$$
\begin{aligned}
c_{U2} &= -\frac{\mathrm{Var}(W_T^*)}{\mathrm{Var}(U_T^*)} \\
\lim_{T\to\infty} c_{U2} &= -\frac{\sigma_Y^2(1-\rho^2)}{\frac{\alpha_Y\sigma_Z^2}{\alpha_Z} - \frac{4\alpha_Y\sigma_Y\sigma_Z\rho}{\alpha_Y+\alpha_Z} + \sigma_Y^2\rho^2} \\
&= c_{U2\,\mathrm{lim}}
\end{aligned}
$$

The previous terms all become constants when T approaches infinity. As a consequence, the integral terms also converge to constants:

$$
\begin{aligned}
\int_{-\infty}^{\infty} \Phi(c_U(w))\phi(w)dw &= \frac{1}{2} + \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{erf}\left(\frac{c_U(w)}{\sqrt{2}}\right)\phi(w)dw \\
\lim_{T\to\infty}\int_{-\infty}^{\infty} \Phi(c_U(w))\phi(w)dw &= \frac{1}{2} + \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{erf}\left(\frac{c_{U\,\mathrm{lim}}(w)}{\sqrt{2}}\right)\phi(w)dw
\end{aligned}
$$

$$
\begin{aligned}
\int_{-\infty}^{\infty} w\Phi(c_U(w))\phi(w)dw &= \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{erf}\left(\frac{c_U(w)}{\sqrt{2}}\right)w\phi(w)dw \\
\lim_{T\to\infty}\int_{-\infty}^{\infty} w\Phi(c_U(w))\phi(w)dw &= \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{erf}\left(\frac{c_{U\,\mathrm{lim}}(w)}{\sqrt{2}}\right)w\phi(w)dw
\end{aligned}
$$

$$\int_{-\infty}^{\infty} \phi(c_U(w))\phi(w)dw = \frac{1}{\pi\sqrt{2(c_{U2}^2+1)}} e^{-\frac{c_{U1}^2}{2(c_{U2}^2+1)}}$$

$$\lim_{T\to\infty} \int_{-\infty}^{\infty} \phi(c_U(w))\phi(w)dw = \frac{1}{\pi\sqrt{2(c_{U2\lim}^2+1)}} e^{-\frac{c_{U1\lim}^2}{2(c_{U2\lim}^2+1)}}$$

The price of the spread option in the long run then converges to:

$$\begin{aligned}\lim_{T\to\infty} \mathbb{E}\left[(K-S(T))_+\right] = \lim_{T\to\infty} \Bigg(& A_T \int_{-\infty}^{\infty} \Phi(c_U(w))\phi(w)dw \\ & - \mathrm{Var}(W_T^*) \int_{-\infty}^{\infty} w\Phi(c_U(w))\phi(w)dw \\ & + \mathrm{Var}(U_T^*) \int_{-\infty}^{\infty} \phi(c_U(w))\phi(w)dw \Bigg) \\ = A_{T\lim} & \left(\frac{1}{2} + \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{erf}\left(\frac{c_{U\lim}(w)}{\sqrt{2}}\right)\phi(w)dw\right) \\ & - \mathrm{Var}(W_T^*)_{\lim} \left(\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{erf}\left(\frac{c_{U\lim}(w)}{\sqrt{2}}\right) w\phi(w)dw\right) \\ & + \mathrm{Var}(U_T^*)_{\lim} \left(\frac{1}{\pi\sqrt{2(c_{U2\lim}^2+1)}} e^{-\frac{c_{U1\lim}^2}{2(c_{U2\lim}^2+1)}}\right)\end{aligned} \tag{4.4.1}$$

which is a stationary constant. Figure 4.2 confirms that, for the given parameters, $\lim_{T\to\infty} \mathbb{E}\left[(K-S(T))_+\right]$ converges to a constant $L_{\mathbb{E}[S]}$.

We see that while α and μ both affect the value of $\mathbb{E}\left[(K-S(T))_+\right]$, σ only affects when the price stabilizes, and not the long-term value itself. Only when $\alpha_Y = \alpha_Z = 0$ does the price function diverge, which is consistent with our previous results as $S(T)$ then becomes a BM spread rather than an OU spread.

4.5 Analysis of OU Spread Parameters

As briefly mentioned in Section 2.1, in an Ornstein-Uhlenbeck process μ is called the drift, σ the infinitesimal variance and α the speed of mean reversion. For Ornstein-Uhlenbeck spreads, the effect of these parameters are less direct, which will be discussed in this section.

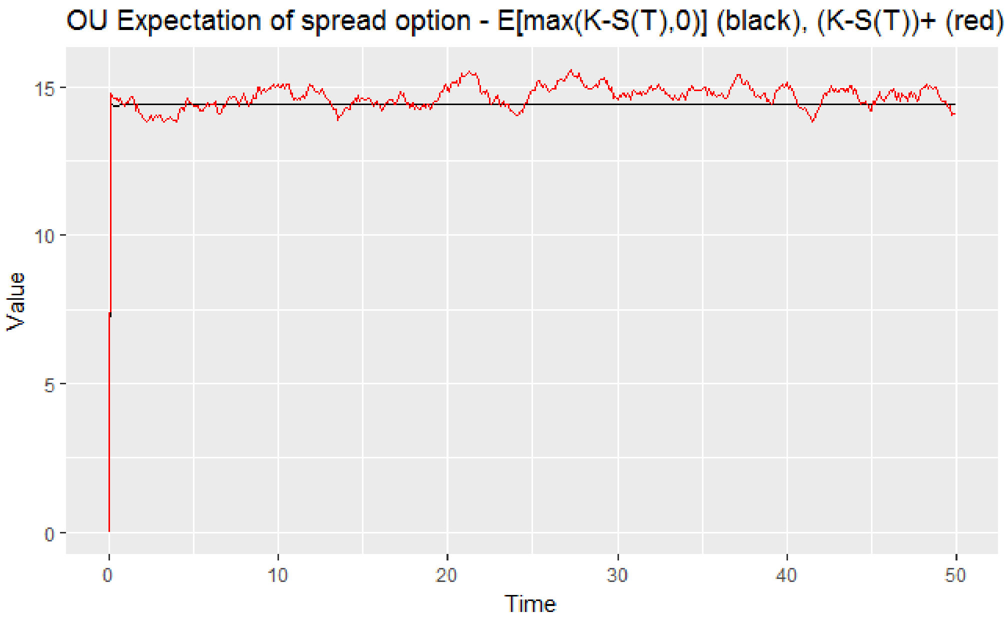

Figure 4.2: $\mathbb{E}[(K_B - S^B(T))_+]$ over time. Parameters: $K = 15$, $\alpha_Y = 0.4$, $\alpha_Z = 0.8$, $\mu_Y = 0.3$, $\mu_Z = 0.9$, $\sigma_Y = 0.8$, $\sigma_Z = 1.3$

4.5.1 Parameters: α_Y and α_Z

When the α parameter in the Ornstein-Uhlenbeck process is equal to 0, the rate of mean-reversal is 0 and the Ornstein-Uhlenbeck process becomes a Brownian motion process. Thus, when $\lim\limits_{\alpha_Y,\alpha_Z \to 0}$, the Ornstein-Uhlenbeck processes $Y(t)$, $Z(t)$ become Brownian motion processes:

$$\begin{aligned}
\lim_{\alpha_Y \to 0} Y(t) &= \lim_{\alpha_Y \to 0} \Bigg(Y_0 e^{-\alpha_Y t} + \mu_Y (1 - e^{-\alpha_Y t}) \\
&\qquad + \sigma_Y \int_0^t e^{-\alpha_Y (t-s)} \left(\rho dU(s) + \sqrt{1-\rho^2} dW(s) \right) \Bigg) \\
&= Y_0 + \sigma_Y \rho U(t) + \sqrt{1-\rho^2} W(t)
\end{aligned}$$

$$\begin{aligned}
\lim_{\alpha_Z \to 0} Z(t) &= \lim_{\alpha_Z \to 0} \left(Z_0 e^{-\alpha_Z t} + \mu_Z (1 - e^{-\alpha_Z t}) + \sigma_Z \int_0^t e^{-\alpha_Z (t-s)} dU(s) \right) \\
&= Z_0 + \sigma_Z U(t)
\end{aligned}$$

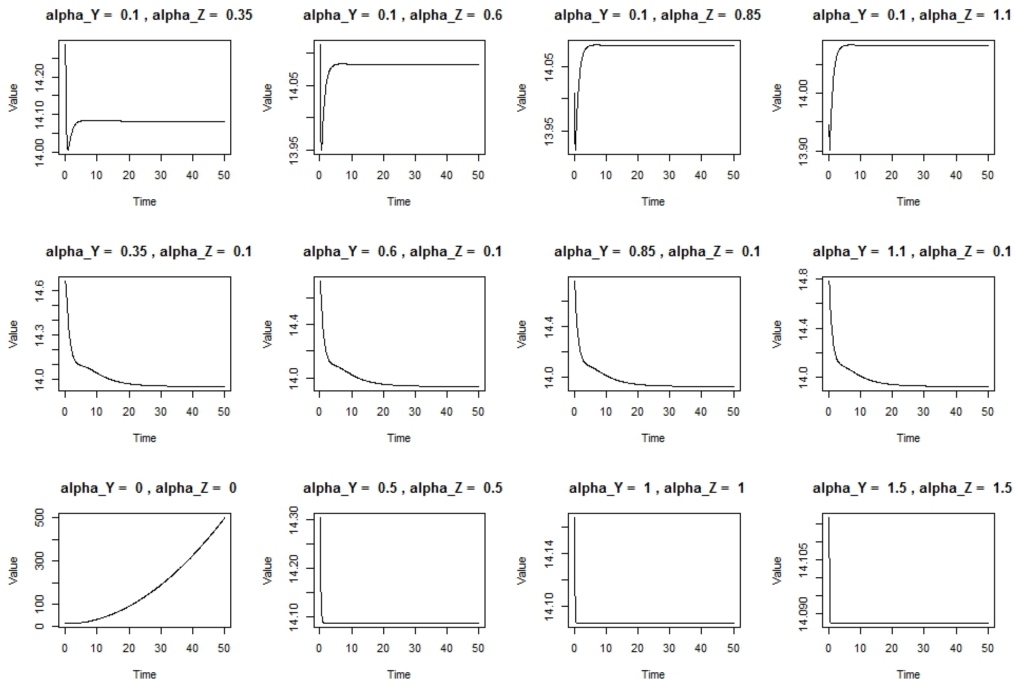

Figure 4.3: $\mathbb{E}\left[(K - S(T))_+\right]$: Comparisons with different α-values. Parameters: $K = 15$, $\mu_Y = 0.3$, $\mu_Z = 0.9$, $\sigma_Y = 0.8$, $\sigma_Z = 1.3$

Subsequently, the standardized stochastic variables are:

$$\lim_{\alpha_Y, \alpha_Z \to 0} U_t^* = (\sigma_Z - \sigma_Y \rho) U(t)$$

$$\lim_{\alpha_Y, \alpha_Z \to 0} W_t^* = \sigma_Y \sqrt{1 - \rho^2} W(t)$$

For variance, we get:

$$
\begin{aligned}
\lim_{\alpha_Y,\alpha_Z\to 0} \mathrm{Var}(U_T^*) &= \lim_{\alpha_Y\to 0} \frac{\sigma_Z^2}{2\alpha_Z}(1-e^{-2\alpha_Z T}) \\
&\quad - \lim_{\alpha_Z\to 0}\left(\lim_{\alpha_Y\to 0} \frac{2\sigma_Y\sigma_Z\rho}{\alpha_Y+\alpha_Z}(1-e^{-(\alpha_Y+\alpha_Z)T})\right) \\
&\quad + \lim_{\alpha_Z\to 0} \frac{\sigma_Y^2\rho^2}{2\alpha_Y}(1-e^{-2\alpha_Y T}) \\
&= \sigma_Z^2 T + 2\sigma_Y\sigma_Z\rho T^2 + \sigma_Y^2\rho^2 T \\
\lim_{\alpha_Y,\alpha_Z\to 0} \mathrm{Var}(W_T^*) &= \lim_{\alpha_Y\to 0} \frac{\sigma_Y^2(1-\rho^2)}{2\alpha_Y}(1-e^{-2\alpha_Y T}) \\
&= \sigma_Y^2 T(1-\rho^2)
\end{aligned}
$$

The expression for the OU spread then becomes:

$$
\begin{aligned}
\lim_{\alpha_Y,\alpha_Z\to 0} S(t) &= \lim_{\alpha_Y,\alpha_Z\to 0} (Z(t)-Y(t)) \\
&= Z_0 - Y_0 + (\sigma_Z - \sigma_Y\rho)U(t) - \sigma_Y\sqrt{1-\rho^2}W(t)
\end{aligned}
$$

Which is simply a BM spread with no drift, starting at $Z_0 - Y_0$. Figure 4.3 confirms that in the case of $\alpha_Y = \alpha_Z = 0$, the undiscounted price is simply a Brownian motion, and not stationary.

In Figure 4.3, we also see that the different values of α_Y and α_Z affect the spread option mainly through the difference $\alpha_Z - \alpha_Y$. When this difference is positive, the price dips before returning to a stationary price that appears to be higher the greater α_Z is relative to α_Y. This dip seemingly does not appear when $\alpha_Z - \alpha_Y \leq 0$, but as long as the difference is negative, greater magnitude of the difference does not have seem to have an impact.

4.5.2 Parameters: μ_Y and μ_Z

The μ-values are the drift, and are the parameters with the biggest impact on the mean value of the spread. This also affects the price $\mathbb{E}\left[(K - S(T))_+\right]$, as shown in Figure 4.4. In addition, it appears that higher values for μ_Y, or both μ_Y and μ_Z leads to a higher stationary price, while a high μ_Z value alone with μ_Y being small leads to a lower price.

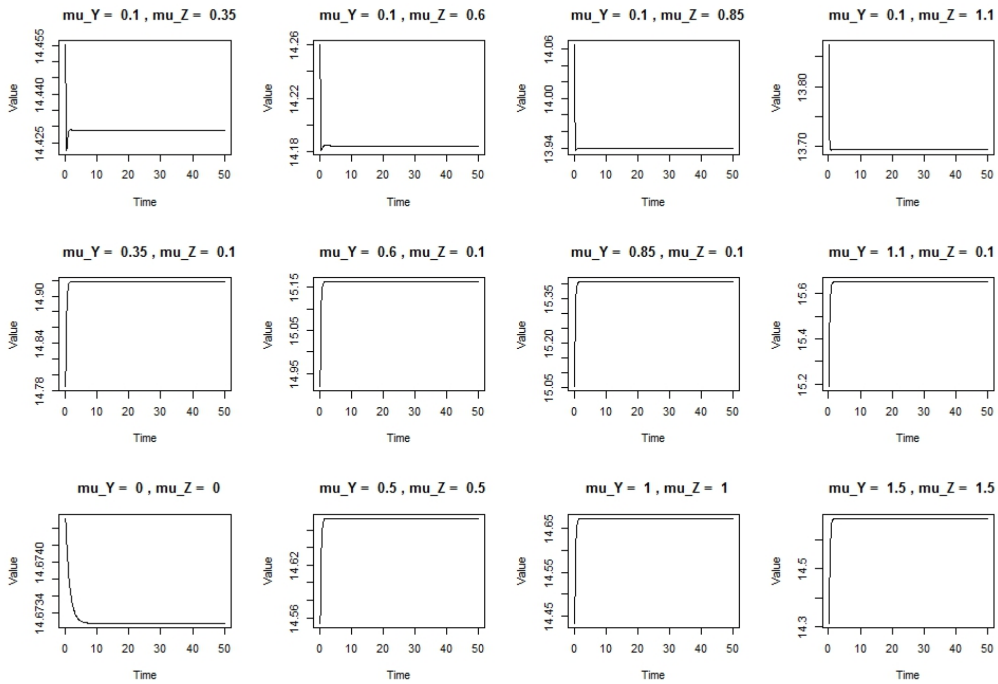

Figure 4.4: $\mathbb{E}\left[(K - S(T))_+\right]$: Comparisons with different μ-values. Parameters: $K = 15$, $\alpha_Y = 0.4$, $\alpha_Z = 0.8$, $\sigma_Y = 0.8$, $\sigma_Z = 1.3$

4.5.3 Parameters: σ_Y and σ_Z

When $\lim\limits_{\sigma_Y, \sigma_Z \to 0}$, the variance of the standardized process $\tilde{U}_T$ becomes:

$$
\begin{aligned}
\lim_{\sigma_Y, \sigma_Z \to 0} \tilde{U}_T &= \lim_{\sigma_Y, \sigma_Z \to 0} \frac{U_T^*}{\sqrt{\operatorname{Var}(U_T^*)}} \\
&= \lim_{\sigma \to 0} \frac{\sigma \int_0^T e^{-\alpha_Z(T-s)} - \rho e^{-\alpha_Y(T-s)} dU(s)}{\sigma \sqrt{\frac{1}{2\alpha_Z}\left(1 - e^{-2\alpha_Z T}\right) - \frac{2\rho}{\alpha_Y + \alpha_Z}\left(1 - e^{-(\alpha_Y + \alpha_Z)T}\right) + \frac{\rho^2}{2\alpha_Y}\left(1 - e^{-2\alpha_Y T}\right)}} \\
&= \frac{\int_0^T e^{-\alpha_Z(T-s)} - \rho e^{-\alpha_Y(T-s)} dU(s)}{\sqrt{\frac{1}{2\alpha_Z}\left(1 - e^{-2\alpha_Z T}\right) - \frac{2\rho}{\alpha_Y + \alpha_Z}\left(1 - e^{-(\alpha_Y + \alpha_Z)T}\right) + \frac{\rho^2}{2\alpha_Y}\left(1 - e^{-2\alpha_Y T}\right)}}
\end{aligned}
$$

From Figure 4.5 it becomes apparent that neither σ-value has any long-term impact on the price. For higher σ-values, the price will vary slightly early on before converging to a constant.

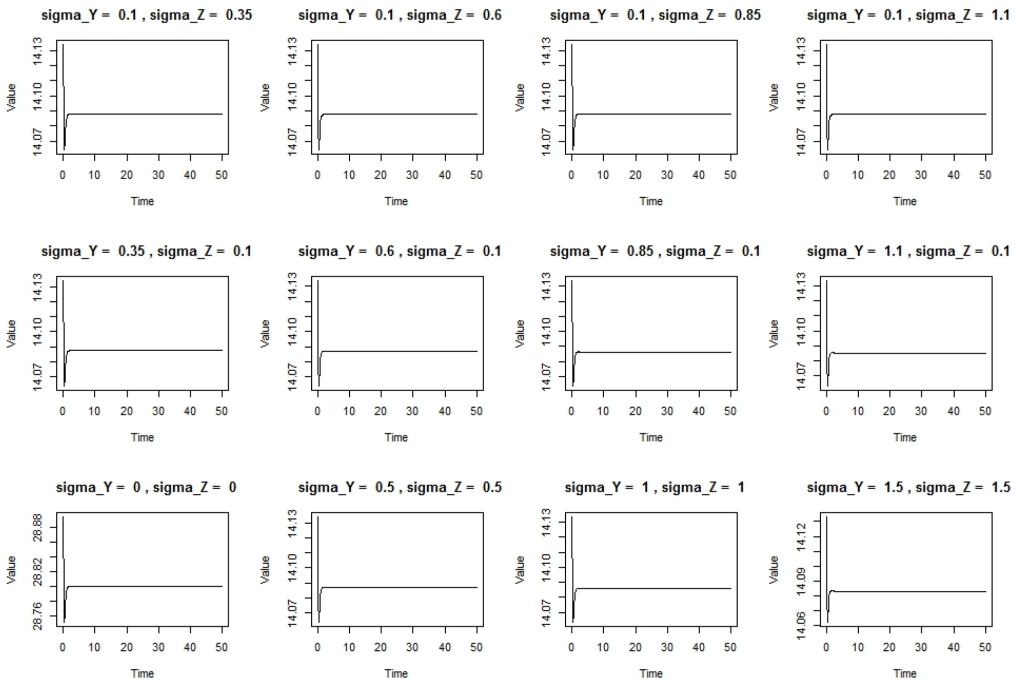

Figure 4.5: $\mathbb{E}\left[(K - S(T))_+\right]$: Comparisons with different σ-values. Parameters: $K = 15$, $\alpha_Y = 0.4$, $\alpha_Z = 0.8$, $\mu_Y = 0.3$, $\mu_Z = 0.9$

4.5.4 Parameters and effect: Variance

We take a look at how the different α and σ parameters affect the variance. The variance for the independent stochastic processes are:

$$\begin{aligned}
\mathrm{Var}(U_T^*) &= \frac{\sigma_Z^2}{2\alpha_Z}\left(1 - e^{-2\alpha_Z T}\right) - \frac{2\sigma_Y\sigma_Z\rho}{\alpha_Y + \alpha_Z}\left(1 - e^{-(\alpha_Y+\alpha_Z)T}\right) + \frac{\sigma_Y^2\rho^2}{2\alpha_Y}\left(1 - e^{-2\alpha_Y T}\right)\\
\lim_{T\to\infty}\mathrm{Var}(U_T^*) &= \frac{\sigma_Z^2}{2\alpha_Z} - \frac{2\sigma_Y\sigma_Z\rho}{\alpha_Y + \alpha_Z} + \frac{\sigma_Y^2\rho^2}{2\alpha_Y}\\
&= \mathrm{Var}(U_T^*)_{\mathrm{lim}}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(W_T^*) &= \frac{\sigma_Y^2(1-\rho^2)}{2\alpha_Y}\left(1 - e^{-2\alpha_Y T}\right)\\
\lim_{T\to\infty}\mathrm{Var}(W_T^*) &= \frac{\sigma_Y^2(1-\rho^2)}{2\alpha_Y}\\
&= \mathrm{Var}(W_T^*)_{\mathrm{lim}}
\end{aligned}$$

We are mainly interested in the long-term values for variance, i.e. $\mathrm{Var}(U_T^*)_{\mathrm{lim}}$ and $\mathrm{Var}(W_T^*)_{\mathrm{lim}}$.

Proposition 4.5.1. *For the parameters α_Y and σ_Y, variance for W_T^* is kept constant if the relation between α_Y and σ_Y are:*

$$\begin{aligned} \alpha_1 &= k_1 \alpha_Y \\ \sigma_1 &= \sqrt{k_1}\sigma_Y \end{aligned}$$

for

$$\begin{aligned} \mathrm{Var}(W_T^*)_{\mathrm{lim}} &= \frac{\sigma_Y^2(1-\rho^2)}{2\alpha_Y} \\ &= \frac{\sigma_1^2(1-\rho^2)}{2\alpha_1} \end{aligned}$$

Proof. Direct calculation:

$$\frac{(\sqrt{k_1}\sigma_Y)^2(1-\rho^2)}{2k_1\alpha_Y} = \frac{\sigma_Y^2(1-\rho^2)}{2\alpha_Y}$$

□

Keeping $\mathrm{Var}(W_T^*)_{\mathrm{lim}}$ constant, if we also wish to keep $\mathrm{Var}(U_T^*)_{\mathrm{lim}}$ constant we get:

Proposition 4.5.2. *Given the Proposition 4.5.1 and the parameters α_Z and σ_Z, variance for U_T^* is kept constant if the relation between α_Y and σ_Y are:*

$$\begin{aligned} \alpha_2 &= k_2 \alpha_Z \\ \sigma_2 &= \frac{-b \pm \sqrt{b^2 - 4c}}{2} \end{aligned}$$

where

$$\begin{aligned} b &= \frac{4\sqrt{k_1}k_2\alpha_Z\sigma_Y\rho}{k_1\alpha_Y + k_2\alpha_Z} \\ c &= k_2\sigma_Z^2(k_1\alpha_Y + k_2\alpha_Z) - \frac{4k_2\alpha_Z\sigma_Y\sigma_Z\rho}{\alpha_Y + \alpha_Z}(k_1\alpha_Y + k_2\alpha_Z) \end{aligned}$$

$$b^2 \geq 4c$$

for

$$\begin{aligned}\mathrm{Var}(U_T^*)_{\mathrm{lim}} &= \frac{\sigma_Z^2}{2\alpha_Z} - \frac{2\sigma_Y\sigma_Z\rho}{\alpha_Y+\alpha_Z} + \frac{\sigma_Y^2\rho^2}{2\alpha_Y} \\ &= \frac{\sigma_2^2}{2\alpha_2} - \frac{2\sigma_1\sigma_2\rho}{\alpha_1+\alpha_2} + \frac{\sigma_1^2\rho^2}{2\alpha_1}\end{aligned}$$

Proof. Direct calculation:

$$\begin{aligned}\frac{\sigma_2^2}{2\alpha_2} - \frac{2\sigma_1\sigma_2\rho}{\alpha_1+\alpha_2} + \frac{\sigma_1^2\rho^2}{2\alpha_1} &= \frac{(\frac{-b\pm\sqrt{b^2-4c}}{2})^2}{2k_2\alpha_Z} - \frac{2\sqrt{k_1}\sigma_Y\frac{-b\pm\sqrt{b^2-4c}}{2}\rho}{k_1\alpha_Y+k_2\alpha_Z} + \frac{(\sqrt{k_1}\sigma_Y)^2\rho^2}{2k_1\alpha_Y} \\ &= \frac{\sigma_Z^2}{2\alpha_Z} - \frac{2\sigma_Y\sigma_Z\rho}{\alpha_Y+\alpha_Z} + \frac{\sigma_Y^2\rho^2}{2\alpha_Y}\end{aligned}$$

□

So changing a parameter by a factor of k gives us the following change in variance:

Parameters for Y: Changing α_Y with a factor k such that $\hat{\alpha} = k\alpha_Y$:

$$\begin{aligned}\mathrm{Var}(U_T^*)_{\hat{\alpha}\,\mathrm{lim}} &= \frac{\sigma_Z^2}{2\alpha_Z} - \frac{2\sigma_Y\sigma_Z\rho}{\hat{\alpha}+\alpha_Z} + \frac{\sigma_Y^2\rho^2}{2\hat{\alpha}} \\ &= \frac{\sigma_Z^2}{2\alpha_Z} - \frac{2\sigma_Y\sigma_Z\rho}{k\alpha_Y+\alpha_Z} + \frac{\sigma_Y^2\rho^2}{2k\alpha_Y} \\ &\begin{cases}\geq \mathrm{Var}(U_T^*)_{\mathrm{lim}} & \text{if } k \leq \frac{-\alpha_Y^2-\alpha_Z^2\pm\sqrt{(\alpha_Y^2+\alpha_Z^2)^2+4(\alpha_Y\alpha_Z-\alpha_Y^2)(\alpha_Z^2+\alpha_Y\alpha_Z)}}{2(\alpha_Y\alpha_Z-\alpha_Y^2)} \\ < \mathrm{Var}(U_T^*)_{\mathrm{lim}} & \text{if } k > \frac{-\alpha_Y^2-\alpha_Z^2\pm\sqrt{(\alpha_Y^2+\alpha_Z^2)^2+4(\alpha_Y\alpha_Z-\alpha_Y^2)(\alpha_Z^2+\alpha_Y\alpha_Z)}}{2(\alpha_Y\alpha_Z-\alpha_Y^2)}\end{cases} \\ \mathrm{Var}(W_T^*)_{\hat{\alpha}\,\mathrm{lim}} &= \frac{\sigma_Y^2(1-\rho^2)}{2\hat{\alpha}} \\ &= \frac{\sigma_Y^2(1-\rho^2)}{2k\alpha_Y} \\ &= \frac{1}{k}\mathrm{Var}(W_T^*)_{\mathrm{lim}}\end{aligned}$$

$$
\begin{aligned}
\hat{\sigma} &= k\sigma_Y \\
\mathrm{Var}(U_T^*)_{\hat{\sigma}\,\mathrm{lim}} &= \frac{\sigma_Z^2}{2\alpha_Z} - \frac{2\hat{\sigma}\sigma_Z\rho}{\alpha_Y+\alpha_Z} + \frac{\hat{\sigma}^2\rho^2}{2\alpha_Y} \\
&= \frac{\sigma_Z^2}{2\alpha_Z} - \frac{2k\sigma_Y\sigma_Z\rho}{\alpha_Y+\alpha_Z} + \frac{k^2\sigma_Y^2\rho^2}{2\alpha_Y} \\
&\begin{cases} \geq \mathrm{Var}(U_T^*)_{\mathrm{lim}} & \text{if } k \geq 1 \\ < \mathrm{Var}(U_T^*)_{\mathrm{lim}} & \text{if } k < 1 \end{cases} \\
\mathrm{Var}(W_T^*)_{\hat{\sigma}\,\mathrm{lim}} &= \frac{\hat{\sigma}^2(1-\rho^2)}{2\alpha_Y} \\
&= \frac{k\sigma_Y^2(1-\rho^2)}{2\alpha_Y} \\
&= k\mathrm{Var}(W_T^*)_{\mathrm{lim}}
\end{aligned}
$$

Parameters for Z: Changing α_Y with a factor k such that $\hat{\alpha} = k\alpha_Z$:

$$
\begin{aligned}
\mathrm{Var}(U_T^*)_{\tilde{\alpha}\,\mathrm{lim}} &= \frac{\sigma_Z^2}{2\tilde{\alpha}} - \frac{2\sigma_Y\sigma_Z\rho}{\alpha_Y+\tilde{\alpha}} + \frac{\sigma_Y^2\rho^2}{2\alpha_Y} \\
&= \frac{\sigma_Z^2}{2k\alpha_Z} - \frac{2\sigma_Y\sigma_Z\rho}{\alpha_Y+k\alpha_Z} + \frac{\sigma_Y^2\rho^2}{2\alpha_Y} \\
&\begin{cases} \geq \mathrm{Var}(U_T^*)_{\mathrm{lim}} & \text{if } k \leq \frac{-\alpha_Y^2-\alpha_Z^2\pm\sqrt{(\alpha_Y^2+\alpha_Z^2)^2+4(\alpha_Y\alpha_Z-\alpha_Z^2)(\alpha_Y^2+\alpha_Y\alpha_Z)}}{2(\alpha_Y\alpha_Z-\alpha_Z^2)} \\ < \mathrm{Var}(U_T^*)_{\mathrm{lim}} & \text{if } k > \frac{-\alpha_Y^2-\alpha_Z^2\pm\sqrt{(\alpha_Y^2+\alpha_Z^2)^2+4(\alpha_Y\alpha_Z-\alpha_Z^2)(\alpha_Y^2+\alpha_Y\alpha_Z)}}{2(\alpha_Y\alpha_Z-\alpha_Z^2)} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\sigma} &= k\sigma_Z \\
\mathrm{Var}(U_T^*)_{\tilde{\sigma}\,\mathrm{lim}} &= \frac{\tilde{\sigma}^2}{2\alpha_Z} - \frac{2\sigma_Y\tilde{\sigma}\rho}{\alpha_Y+\alpha_Z} + \frac{\sigma_Y^2\rho^2}{2\alpha_Y} \\
&\begin{cases} \geq \mathrm{Var}(U_T^*)_{\mathrm{lim}} & \text{if } k \geq 1 \\ < \mathrm{Var}(U_T^*)_{\mathrm{lim}} & \text{if } k < 1 \end{cases}
\end{aligned}
$$

The calculations are featured in detail below.

4.5.5 Calculations for the Change in Variance

For $\mathrm{Var}(U_T^*)_{\hat{\alpha}\,\mathrm{lim}}$: If

$$
\begin{aligned}
\frac{1}{2\alpha_Y} - \frac{1}{\alpha_Y+\alpha_Z} &> \frac{1}{2\hat{\alpha}} - \frac{1}{\hat{\alpha}+\alpha_Z} \\
&> \frac{1}{2k\alpha_Y} - \frac{1}{k\alpha_Y+\alpha_Z}
\end{aligned}
$$

Then the calculation is:

$$\begin{aligned}
\frac{1}{2\alpha_Y} - \frac{1}{\alpha_Y + \alpha_Z} &> \frac{1}{2k\alpha_Y} - \frac{1}{k\alpha_Y + \alpha_Z} \\
\alpha_Z - \alpha_Y &> \frac{\alpha_Y + \alpha_Z}{k} - \frac{2\alpha_Y(\alpha_Y + \alpha_Z)}{k\alpha_Y + \alpha_Z} \\
k(\alpha_Z - \alpha_Y)(k\alpha_Y + \alpha_Z) &> (\alpha_Y + \alpha_Z)(k\alpha_Y + \alpha_Z) - 2k\alpha_Y(\alpha_Y + \alpha_Z) \\
0 &< (\alpha_Y\alpha_Z - \alpha_Y^2)k^2 + (\alpha_Y^2 + \alpha_Z^2)k - \alpha_Z^2 - \alpha_Y\alpha_Z \\
k &> \frac{-\alpha_Y^2 - \alpha_Z^2 \pm \sqrt{(\alpha_Y^2 + \alpha_Z^2)^2 + 4(\alpha_Y\alpha_Z - \alpha_Y^2)(\alpha_Z^2 + \alpha_Y\alpha_Z)}}{2(\alpha_Y\alpha_Z - \alpha_Y^2)}
\end{aligned}$$

For $\mathrm{Var}(U_T^*)_{\tilde{\alpha}\lim}$: If

$$\begin{aligned}
\frac{1}{2\alpha_Z} - \frac{1}{\alpha_Y + \alpha_Z} &> \frac{1}{2\tilde{\alpha}} - \frac{1}{\alpha_Y + \tilde{\alpha}} \\
&> \frac{1}{2k\alpha_Z} - \frac{1}{\alpha_Y + k\alpha_Z}
\end{aligned}$$

The calculation becomes:

$$\begin{aligned}
\frac{1}{2\alpha_Z} - \frac{1}{\alpha_Y + \alpha_Z} &> \frac{1}{2k\alpha_Z} - \frac{1}{\alpha_Y + k\alpha_Z} \\
\alpha_Y - \alpha_Z &> \frac{\alpha_Y + \alpha_Z}{k} - \frac{2\alpha_Z(\alpha_Y + \alpha_Z)}{\alpha_Y + k\alpha_Z} \\
k(\alpha_Y - \alpha_Z)(\alpha_Y + k\alpha_Z) &> (\alpha_Y + \alpha_Z)(\alpha_Y + k\alpha_Z) - 2k\alpha_Z(\alpha_Y + \alpha_Z) \\
0 &< (\alpha_Y\alpha_Z - \alpha_Z^2)k^2 + (\alpha_Y^2 + \alpha_Z^2)k - \alpha_Y^2 - \alpha_Y\alpha_Z \\
k &> \frac{-\alpha_Y^2 - \alpha_Z^2 \pm \sqrt{(\alpha_Y^2 + \alpha_Z^2)^2 + 4(\alpha_Y\alpha_Z - \alpha_Z^2)(\alpha_Y^2 + \alpha_Y\alpha_Z)}}{2(\alpha_Y\alpha_Z - \alpha_Z^2)}
\end{aligned}$$

4.6 Calculations of related terms

While not directly utilized in our analysis, this section features other properties of the Ornstein-Uhlenbeck process and how they affect spreads.

4.6.1 Calculating half-life for $S(T)$

The half-life process is explained in Benth and Benth (2012), p. 103-106 [4], as the speed of mean reversion in an Ornstein-Uhlenbeck process.

Definition 4.6.1. *The half-life of the mean-reversion, $t_{\frac{1}{2}}$, is the average time it will take the process to return half-way back to its mean.*

For an Ornstein-Uhlenbeck process X with drift μ, the half-life is the time $t_{\frac{1}{2}}$ such that the following equation holds:

$$\mathbb{E}[X(t_{\frac{1}{2}})] = \mu + \frac{X_0 - \mu_X}{2}$$

Put in mathematical terms, for the Ornstein-Uhlenbeck spread process S in Equation (4.1.1), we're looking for:

$$\mathbb{E}[S(t_{\frac{1}{2}})] = \mu_Z - \mu_Y + \frac{S_0 - (\mu_Z - \mu_Y)}{2}$$

where S_0 is the value of the process at the start when $t = 0$, with $S(0) = Z_0 - Y_0$.

Thus, for the Ornstein-Uhlenbeck spread process S we get:

$$\begin{aligned} \mathbb{E}[S(t_{\frac{1}{2}})] &= \mu_Z - \mu_Y + \frac{S_0 - (\mu_Z - \mu_Y)}{2} \\ e^{-\alpha_Z t_{\frac{1}{2}}}(Z_0 - \mu_Z) - e^{-\alpha_Y t_{\frac{1}{2}}}(Y_0 - \mu_Y) &= \frac{Z_0 - Y_0 - \mu_Z + \mu_Y}{2} \end{aligned}$$

It is difficult to find an analytical solution for $t_{\frac{1}{2}}$, but not hard to simulate. One can still deduce that the half-life is independent of the state of the process $S(t)$, and is only dependent on the start position of the process and the parameters μ_Y, μ_Z, α_Y and α_Z. The code for finding the half-life $t_{\frac{1}{2}}$ for S can be found in Appendix section B.5.

The half-life of the spread differs from the half-life of a regular Ornstein-Uhlenbeck process.

Proposition 4.6.2. *For an Ornstein-Uhlenbeck process Y with dynamics given in Equation (4.1.5), the half-life $t_{\frac{1}{2}}$ of the process is*

$$t_{\frac{1}{2}} = \frac{\log 2}{\alpha_Y} \tag{4.6.1}$$

where α_Y is the speed of mean reversal.

Proof.

$$\begin{aligned}
\mathbb{E}[Y(t_{\frac{1}{2}})] &= \mu_Y + \frac{Y_0 - \mu_Y}{2} \\
e^{-\alpha_Y t_{\frac{1}{2}}}(Y_0 - \mu_Y) + \mu_Y &= \mu_Y + \frac{Y_0 - \mu_Y}{2} \\
t_{\frac{1}{2}} &= \frac{\log 2}{\alpha_Y}
\end{aligned}$$

□

In this case, the half-time depends only on the parameter α_Y, which is why we call α_Y the *speed of mean reversion* for Y. If α_Y is small, the half-life is large, meaning the process takes longer time in reverting to the mean, while a large α_Y value means smaller half-life and shorter time for the process to revert.

4.6.2 Calculations of expectation, covariance and correlation:

Expectation:

$$\begin{aligned}
\mathbb{E}\left[S(t)\right] &= \mathbb{E}[S_2(t)] - \mathbb{E}[S_1(t)] \\
&= \mathbb{E}[Z_0 e^{-\alpha_Z t} + \mu_Z(1 - e^{-\alpha_Z t}) + \sigma_Z \int_0^t e^{-\alpha_Z(t-s)} dB_Z(s)] \\
&\quad - \mathbb{E}[Y_0 e^{-\alpha_Y t} - \mu_Y(1 - e^{-\alpha_Y t}) - \sigma_Y \int_0^t e^{-\alpha_Y(t-s)} dB_Y(s)] \\
&= Z_0 e^{-\alpha_Z t} + \mu_Z(1 - e^{-\alpha_Z t}) - Y_0 e^{-\alpha_Y t} - \mu_Y(1 - e^{-\alpha_Y t})
\end{aligned}$$

Covariance: Y, Z We have applied the Itô isometry in our calculations.

$$
\begin{aligned}
\mathbb{E}[Y(t)Z(t)] &= \mathbb{E}\left[\left(Y(0)e^{-\alpha_Y t} + \mu_Y(1-e^{-\alpha_Y t}) + \sigma_Y \int_0^t e^{-\alpha_Y(t-s)} dB_Y(s)\right)\right.\\
&\qquad \left.\times \left(Z(0)e^{-\alpha_Z t} + \mu_Z(1-e^{-\alpha_Z t}) + \sigma_Z \int_0^t e^{-\alpha_Z(t-s)} dB_Z(s)\right)\right]\\
&= Y(0)Z(0)e^{-t(\alpha_Y+\alpha_Z)} + Y(0)\mu_Z e^{-\alpha_Y t}(1-e^{-\alpha_Z t})\\
&\qquad + Z(0)\mu_Y e^{-\alpha_Z t}(1-e^{-\alpha_Y t}) + \mu_Y\mu_Z(1-e^{-\alpha_Z t})\\
&\qquad + \sigma_Y\sigma_Z \mathbb{E}\left[\int_0^t e^{-\alpha_Y(t-s)} dB_Y(s) \int_0^t e^{-\alpha_Z(t-s)} dB_Z(s)\right]\\
&= \big(Y(0)e^{-\alpha_Y t} + \mu_Y(1-e^{-\alpha_Y t})\big) \times \big(Z(0)e^{-\alpha_Z t} + \mu_Z(1-e^{-\alpha_Z t})\big)\\
&\qquad + \sigma_Y\sigma_Z \mathbb{E}\left[\int_0^t e^{-\alpha_Y(t-s)} dB_Y(s) \int_0^t e^{-\alpha_Z(t-s)} dB_Z(s)\right]
\end{aligned}
$$

$$
\mathbb{E}[Y(t)]\mathbb{E}[Z(t)] = \big(Y(0)e^{-\alpha_Y t} + \mu_Y(1-e^{-\alpha_Y t})\big) \times \big(Z(0)e^{-\alpha_Z t} + \mu_Z(1-e^{-\alpha_Z t})\big)
$$

$$
\begin{aligned}
\mathrm{Cov}(Y(t), Z(t)) &= \mathbb{E}[Y(t)Z(t)] - \mathbb{E}[Y(t)]\mathbb{E}[Z(t)]\\
&= \sigma_Y\sigma_Z \mathbb{E}\left[\int_0^t e^{-\alpha_Y(t-s)} dB_Y(s) \int_0^t e^{-\alpha_Z(t-s)} dB_Z(s)\right]\\
&= \sigma_Y\sigma_Z \mathbb{E}\left[\int_0^t e^{-\alpha_Y(t-s)} (\rho dU(s) + \sqrt{1-\rho^2} dW(s)) \int_0^t e^{-\alpha_Z(t-s)} dU(s)\right]\\
&= \rho\sigma_Y\sigma_Z e^{-t(\alpha_Y+\alpha_Z)} \left(\frac{1}{\alpha_Y+\alpha_Z}(e^{t(\alpha_Y+\alpha_Z)} - 1)\right)\\
&= \frac{\rho\sigma_Y\sigma_Z}{\alpha_Y+\alpha_Z}\left(1 - e^{-t(\alpha_Y+\alpha_Z)}\right)
\end{aligned}
$$

Calculation of correlation ρ: For the relation between B_Y and B_Z, suppose $U(t)$ and $W(t)$ are independent Brownian motions:

$$
\begin{aligned}
B_Y(t) &= \rho B_Z(t) + \sqrt{1-\rho^2} W(t) \qquad & B_Z(t) &= U(t)\\
dB_Y(t) &= \rho dB_Z(t) + \sqrt{1-\rho^2} dW(t) \qquad & dB_Z(t) &= dU(t)
\end{aligned}
$$

$$
\mathrm{corr}(dB_Y, dB_Z) = \rho dt
$$

$$
\begin{aligned}
\operatorname{corr}(Y(t), Z(t)) &= \mathbb{E}\left[\int_0^t e^{-\alpha_Y(t-s)} dB_Y(s) \int_0^t e^{-\alpha_Z(t-s)} dB_Z(s)\right] \\
\rho &= \mathbb{E}\left[\int_0^t e^{-\alpha_Y(t-s)} dB_Y(s) \int_0^t e^{-\alpha_Z(t-s)} dB_Z(s)\right] \\
&= \mathbb{E}\left[\left(\int_0^t e^{-\alpha_Y(t-s)} (\rho dB_Z(s) + \sqrt{1-\rho^2} dW(s))\right)\right. \\
&\qquad \left. \times \left(\int_0^t e^{-\alpha_Z(t-s)} dB_Z(s)\right)\right] \\
&= \rho \mathbb{E}\left[\left(\int_0^t e^{-\alpha_Y(t-s)} dB_Z(s)\right)\left(\int_0^t e^{-\alpha_Z(t-s)} dB_Z(s)\right)\right] \\
&\qquad + \sqrt{1-\rho^2} \mathbb{E}\left[\int_0^t e^{-\alpha_Y(t-s)} dW(s)\right] \mathbb{E}\left[\int_0^t e^{-\alpha_Z(t-s)} dB_Z(s)\right] \\
&= \rho \mathbb{E}\left[\int_0^t e^{-(\alpha_Y+\alpha_Z)(t-s)} ds\right] \\
&= \rho \mathbb{E}\left[\frac{1}{\alpha_Y+\alpha_Z}\left(1-e^{-t(\alpha_Y+\alpha_Z)}\right)\right] \\
&= \frac{\rho}{\alpha_Y+\alpha_Z}\left(1-e^{-t(\alpha_Y+\alpha_Z)}\right)
\end{aligned}
$$

Variance:

$$
\begin{aligned}
\operatorname{Var}(Y(t)) &= \mathbb{E}[Y(t)^2] - \mathbb{E}[Y(t)]^2 \\
&= \mathbb{E}\left[\left(Y(0)e^{-\alpha_Y t} + \mu_Y(1-e^{-\alpha_Y t}) + \sigma_Y \int_0^t e^{-\alpha_Y(t-s)} dB_Y(s)\right)^2\right] \\
&\qquad - \left(Y(0)e^{-\alpha_Y t} + \mu_Y(1-e^{-\alpha_Y t})\right)^2 \\
&= \sigma_Y^2 \mathbb{E}\left[\left(\int_0^t e^{-\alpha_Y(t-s)} dB_Y(s)\right)^2\right] \\
&= \sigma_Y^2 \mathbb{E}\left[\int_0^t e^{-2\alpha_Y(t-s)} ds\right] \\
&= \frac{\sigma_Y^2}{2\alpha_Y}\left(1-e^{-2\alpha_Y(t-s)}\right)
\end{aligned}
$$

Expectation and covariance:

$$\mu = \begin{pmatrix} Y(0)e^{-\alpha_Y t} + \mu_Y(1 - e^{-\alpha_Y t}) \\ Z(0)e^{-\alpha_Z t} + \mu_Z(1 - e^{-\alpha_Z t}) \end{pmatrix}$$

$$\mathbf{\Sigma} = \begin{pmatrix} \frac{\sigma_Y^2}{2\alpha_Y}\left(1 - e^{-2\alpha_Y t}\right) & \frac{\rho\sigma_Y\sigma_Z}{\alpha_Y+\alpha_Z}\left(1 - e^{-t(\alpha_Y+\alpha_Z)}\right) \\ \frac{\rho\sigma_Y\sigma_Z}{\alpha_Y+\alpha_Z}\left(1 - e^{-t(\alpha_Y+\alpha_Z)}\right) & \frac{\sigma_Z^2}{2\alpha_Z}\left(1 - e^{-2\alpha_Z t}\right) \end{pmatrix}$$

Expectation and covariance in the long term:

$$\lim_{t\to\infty} \mu = \begin{pmatrix} \mu_Y \\ \mu_Z \end{pmatrix}$$

$$\lim_{t\to\infty} \mathbf{\Sigma} = \begin{pmatrix} \frac{\sigma_Y^2}{2\alpha_Y} & \frac{\rho\sigma_Y\sigma_Z}{\alpha_Y+\alpha_Z} \\ \frac{\rho\sigma_Y\sigma_Z}{\alpha_Y+\alpha_Z} & \frac{\sigma_Z^2}{2\alpha_Z} \end{pmatrix}$$

Chapter 5

Options based on BM and OU-processes

Bibliographical notes

The main reference for this chapter is Benth and Koekebakker (2015) [2].

In this chapter we look at options based on a sum of Brownian motion and Ornstein-Uhlenbeck processes. While not spread options, they are a different linear combination of assets we wish to look at further.

We have X, Y and Y from the earlier chapters, with

$$S_1(t) = X(t) + Y(t)$$
$$S_2(t) = X(t) + Z(t)$$

While S_1 and S_2 have the same structure and properties, we will calculate expressions for both and examine them in detail, with focus on S_2 for the purposes it serves in the next chapter.

While it has been established that both Y and Z are stationary and mean-reverting, X is not, thus S_1 and S_2 does not have these properties either.

5.1 Expectation and Variance

Both S_1 and S_2 have bivariate normal distributions.

Proposition 5.1.1. *S_1 and S_2 are bivariate normally distributed,*

$$S^1(t) \sim \mathcal{N}(\boldsymbol{\mu}_1(t), \boldsymbol{\sigma}_1(t))$$
$$S^2(t) \sim \mathcal{N}(\boldsymbol{\mu}_2(t), \boldsymbol{\sigma}_2(t))$$

with mean and covariance matrix:

$$\boldsymbol{\mu}_1(t) = \begin{pmatrix} \mu t \\ Y(0)e^{-\alpha_Y t} + \mu_Y(1 - e^{-\alpha_Y t}) \end{pmatrix} \tag{5.1.1}$$

$$\boldsymbol{\sigma}_1(t) = \begin{pmatrix} \sigma^2 t & 0 \\ 0 & \frac{\sigma_Y^2}{2\alpha_Y}(1 - e^{-2\alpha_Y t}) \end{pmatrix} \tag{5.1.2}$$

and

$$\boldsymbol{\mu}_2(t) = \begin{pmatrix} \mu t \\ Z(0)e^{-\alpha_Z t} + \mu_Z(1 - e^{-\alpha_Z t}) \end{pmatrix} \tag{5.1.3}$$

$$\boldsymbol{\sigma}_2(t) = \begin{pmatrix} \sigma^2 t & 0 \\ 0 & \frac{\sigma_Z^2}{2\alpha_Z}(1 - e^{-2\alpha_Z t}) \end{pmatrix} \tag{5.1.4}$$

Proof. See Section 3.2 and Section 4.2 for mean and variance for X, Y and Z. $\square$

For the random variables $S_1(T)$, $S_2(T)$, expectation and variance are as follows:

Expectation:

$$\begin{aligned} \mathbb{E}[S_1(T)] &= \mathbb{E}[X(T) + Y(T)] \\ &= \mu T + Y(0)e^{-\alpha_Y T} + \mu_Y(1 - e^{-\alpha_Y T}) \end{aligned}$$

$$\begin{aligned} \mathbb{E}[S_2(T)] &= \mathbb{E}[X(T) + Z(T)] \\ &= \mu T + Z(0)e^{-\alpha_Z T} + \mu_Y(1 - e^{-\alpha_Y T}) \end{aligned}$$

Covariance:

$$\operatorname{Cov}(X(T), Y(T)) = 0$$

$$\operatorname{Cov}(X(T), Z(T)) = 0$$

Variance:

$$
\begin{aligned}
\mathrm{Var}(S_1(T)) &= \mathrm{Var}(X(T) + Y(T)) \\
&= \mathrm{Var}(X(T)) + \mathrm{Var}(Y(T)) + 2\mathrm{Cov}(X(T), Y(T)) \\
&= \sigma T + \frac{\sigma_Y^2}{2\alpha_Y}(1 - e^{-2\alpha_Y T})
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}(S_2(T)) &= \mathrm{Var}(X(T) + Z(T)) \\
&= \mathrm{Var}(X(T)) + \mathrm{Var}(Z(T)) + 2\mathrm{Cov}(X(T), Z(T)) \\
&= \sigma T + \frac{\sigma_Z^2}{2\alpha_Z}(1 - e^{-2\alpha_Z T})
\end{aligned}
$$

5.2 The Option $\max(K_1 - S_1(T), 0)$

S_1 is a combination of X, a Brownian motion with drift, and Y, an Ornstein-Uhlenbeck process.

$$
\begin{aligned}
S_1(t) &= \mu t + Y_0 e^{-\alpha_Y t} + \mu_Y(1 - e^{-\alpha_Y t}) + \sigma B(t) + \sigma_Y e^{-\alpha_Y t} \int_0^t e^{\alpha_Y s} dB_Y(s) \\
&= A_1(t) + B^*(t) + Y^*(t)
\end{aligned}
$$

where B and B_Y are independent, standard BM-processes and $A_1(t)$, $B^*(t)$ and $Y^*(t)$ are given as:

$$
\begin{aligned}
A_1(t) &= Y_0 e^{-\alpha_Y t} + \mu_Y(1 - e^{-\alpha_Y t}) + \mu t \\
B^*(t) &= \sigma B(t) \\
Y^*(t) &= \sigma_Y e^{-\alpha_Y t} \int_0^t e^{\alpha_Y s} dB_Y(s)
\end{aligned}
$$

So to standardize B^* and Y^*, we find expectation and variance:

$$
\begin{aligned}
\mathbb{E}[B^*(t)] &= 0 \\
\mathbb{E}[Y^*(t)] &= 0
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}(B^*(t)) &= \sigma^2 t \\
\mathrm{Var}(Y^*(t)) &= \frac{\sigma_Y^2}{2\alpha_Y}(1 - e^{-2\alpha_Y t}) \\
\mathrm{Cov}(B^*(t), Y^*(t)) &= 0
\end{aligned}
$$

This gives us $(\tilde{B}(t), \tilde{B_Y}(t))$ as bivariate standard normal random variables, with

$$\begin{aligned}\tilde{B}(t) &= \frac{B^*(t)}{\sqrt{\mathrm{Var}(B^*(t))}}\\ &= \frac{B(t)}{\sqrt{t}}\end{aligned}$$

$$\begin{aligned}\tilde{B_Y}(t) &= \frac{Y^*(t)}{\sqrt{\mathrm{Var}(Y^*(t))}}\\ &= \frac{\sqrt{2\alpha_Y}}{\sqrt{e^{2\alpha_Y t}-1}}\int_0^t e^{\alpha_Y s}dB_Y(s)\end{aligned}$$

We can then use the same method as in earlier chapters, and calculate the expectation of the spread option $K_1 - S_1(T))_+$:

$$\begin{aligned}\mathbb{E}[K_1 - S_1(T))_+] &= \mathbb{E}[(K_1 - A_1(T) - B^*(T) - Y^*(T))_+]\\ &= \mathbb{E}[(K_1 - A_1(T) - \mathrm{Var}(B^*(T))\tilde{B}(T) - \mathrm{Var}(Y^*(T))\tilde{B_Y}(T))_+]\\ &= \mathbb{E}\left[\left(K_1 - A_1(T) - \sigma\sqrt{T}\tilde{B}(T) - \frac{\sigma_Y}{\sqrt{2\alpha_Y}}\sqrt{1-e^{-2\alpha_Y T}}\tilde{B_Y}(t), 0\right)_+\right]\end{aligned}$$

With the condition that $\tilde{B}(T) = b$, the random variable inside the expectation is zero when

$$\begin{aligned}K_1 &> A_1(T) + B^*(T) + Y^*(T)\\ K_1 - A_1(T) - \sigma\sqrt{T}b &> \frac{\sigma_Y}{\sqrt{2\alpha_Y}}\sqrt{1-e^{-2\alpha_Y T}}\tilde{B_Y}(T)\end{aligned}$$

i.e. the boundary is

$$\begin{aligned}\tilde{B_Y}(T) &< \frac{\sqrt{2\alpha_Y}(K_1 - A_1(T) - \sigma\sqrt{T}b)}{\sigma_Y\sqrt{1-e^{-2\alpha_Y T}}}\\ c_1(b) &= \frac{\sqrt{2\alpha_Y}(K_1 - A_1(T) - \sigma\sqrt{T}b)}{\sigma_Y\sqrt{1-e^{-2\alpha_Y T}}}\\ &= \frac{\sqrt{2\alpha_Y}(K_1 - A_1(T))}{\sigma_Y\sqrt{1-e^{-2\alpha_Y T}}} - \frac{\sigma\sqrt{2\alpha_Y T}}{\sigma_Y\sqrt{1-e^{-2\alpha_Y T}}}\\ &= \frac{K_1 - A_1(T)}{\sqrt{\mathrm{Var}(Y^*(T))}} - \sqrt{\frac{\mathrm{Var}(B^*(T))}{\mathrm{Var}(Y^*(T))}}b\\ &= c_{1,1} - c_{1,2}b\end{aligned}$$

As $(\tilde{B}(t), \tilde{B_Y}(t))$ are bivariate standard normal, the price can be computed by the following double integral:

$$
\begin{aligned}
&\mathbb{E}[(K_1 - S_1(T))_+] \\
&\quad = \mathbb{E}[(K_1 - A_1(T) - \sigma\sqrt{T}\tilde{B}(T) - \frac{\sigma_Y}{\sqrt{2\alpha_Y}}\sqrt{1 - e^{-2\alpha_Y T}}\tilde{Y}_1(t))_+] \\
&\quad = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(K_1 - A_1(T) - \sigma\sqrt{T}b - \frac{\sigma_Y}{\sqrt{2\alpha_Y}}\sqrt{1 - e^{-2\alpha_Y T}}u)_+ f_T(b,u) \\
&\quad = \int_{-\infty}^{\infty}\int_{-\infty}^{c_1(b)}(K_1 - A_1(T) - \sigma\sqrt{T}b - \frac{\sigma_Y}{\sqrt{2\alpha_Y}}\sqrt{1 - e^{-2\alpha_Y T}}u) f_T(b,u)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{c_1(b)}(K_1 - A_1(T) - \sigma\sqrt{T}b - \frac{\sigma_Y}{\sqrt{2\alpha_Y}}\sqrt{1 - e^{-2\alpha_Y T}}u)\phi(u)du\right)\phi(b)db \\
&= \int_{-\infty}^{\infty}\left((K_1 - A_1(T) - \sigma\sqrt{T}b)\Phi(c_1(b)) + \frac{\sigma_Y}{\sqrt{2\alpha_Y}}\sqrt{1 - e^{-2\alpha_Y T}}\phi(c_1(b))\right)\phi(b)db \\
&= (K_1 - A_1(T))\int_{-\infty}^{\infty}\Phi(c_1(b))\phi(b)db + \sigma\sqrt{T}\int_{-\infty}^{\infty}b\Phi(c_1(b))\phi(b)db \\
&\quad + \frac{\sigma_Y}{\sqrt{2\alpha_Y}}\sqrt{1 - e^{-2\alpha_Y T}}\int_{-\infty}^{\infty}\phi(c_1(b))\phi(b)db
\end{aligned}
$$

This expression has integrals similar in structure to the Ornstein-Uhlenbeck spread option that was analyzed in Chapter 4. As such, the calculation for the integrals can be found in Section 3.5 and can be written as:

$$\int_{-\infty}^{\infty}\Phi(c_1(b))\phi(b)db = \frac{1}{2} + \frac{1}{2}\int_{-\infty}^{\infty}\mathrm{erf}\left(\frac{c_1(b)}{\sqrt{2}}\right)\phi(b)db$$

$$\int_{-\infty}^{\infty}b\Phi(c_1(b))\phi(b)db = \frac{1}{2}\int_{-\infty}^{\infty}\mathrm{erf}\left(\frac{c_1(b)}{\sqrt{2}}\right)b\phi(b)db$$

$$\int_{-\infty}^{\infty}\phi(c_1(b))\phi(b)db = \frac{1}{\pi\sqrt{2(c_{2,2}^2 + 1)}}e^{-\frac{c_{2,1}^2}{2(c_{2,2}^2+1)}}$$

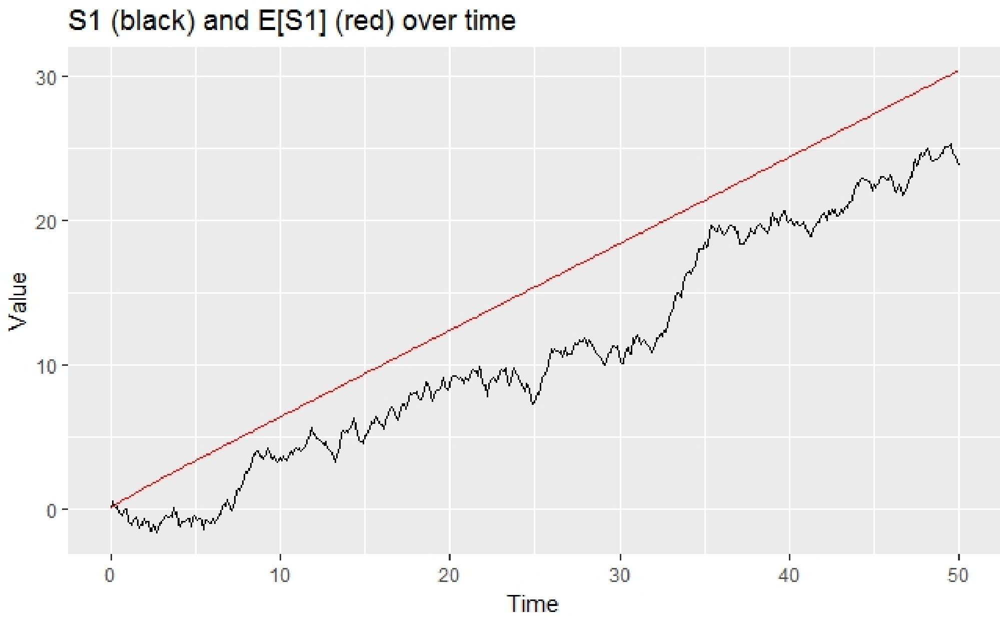

Figure 5.1: Parameters: $K_1 = 5$, $\alpha_Y = 0.4$, $\mu_Y = 0.3$ $\sigma_Y = 0.8$

5.3 The Option $\max(K_2 - S_2(T), 0)$

By following the same method as for the option $\max(K_1 - S_1(T), 0)$, we find the expectation for the S_2-option $\mathbb{E}[(K_2 - S_2(T))_+]$ to be

$$
\begin{aligned}
\mathbb{E}[(K_2 - S_2(T))_+] &= \mathbb{E}[(K_2 - A_2(T) - \sigma\sqrt{T}\tilde{B}(T) - \frac{\sigma_Z}{\sqrt{2\alpha_Z}}\sqrt{1 - e^{-2\alpha_Z T}}\tilde{B}_Y(t))_+] \\
&= (K_2 - A_2(T)) \int_{-\infty}^{\infty} \Phi(c_2(b))\phi(b)db + \sigma\sqrt{T} \int_{-\infty}^{\infty} b\Phi(c_2(b))\phi(b)db \\
&\quad - \frac{\sigma_Z}{\sqrt{2\alpha_Z}}\sqrt{1 - e^{-2\alpha_Z T}} \int_{-\infty}^{\infty} \phi(c_2(b))\phi(b)db
\end{aligned}
$$

with

$$\int_{-\infty}^{\infty} \Phi(c_2(b))\phi(b)db = \frac{1}{2} + \frac{1}{2}\int_{-\infty}^{\infty} \text{erf}\left(\frac{c_1(b)}{\sqrt{2}}\right)\phi(b)db$$

$$\int_{-\infty}^{\infty} b\Phi(c_1(b))\phi(b)db = \frac{1}{2}\int_{-\infty}^{\infty} \text{erf}\left(\frac{c_1(b)}{\sqrt{2}}\right)b\phi(b)db$$

$$\int_{-\infty}^{\infty} \phi(c_1(b))\phi(b)db = \frac{1}{\pi\sqrt{2(c_{2,2}^2+1)}}e^{-\frac{c_{2,1}^2}{2(c_{2,2}^2+1)}}$$

where B and B_Z are independent, standard BM-processes, and the boundary $c_2(b)$ defined as

$$\begin{aligned} c_2(b) &= \frac{\sqrt{2\alpha_Z}(K_2 - A_2(T) - \sigma\sqrt{T}b)}{\sigma_Z\sqrt{1-e^{-2\alpha_Z T}}} \\ &= \frac{\sqrt{2\alpha_Z}(K_2 - A_2(T))}{\sigma_Z\sqrt{1-e^{-2\alpha_Z T}}} - \frac{\sigma\sqrt{2\alpha_Z T}}{\sigma_Z\sqrt{1-e^{-2\alpha_Y T}}} \\ &= \frac{K_2 - A_2(T)}{\sqrt{\text{Var}(Z^*(T))}} + \sqrt{\frac{\text{Var}(B^*(T))}{\text{Var}(Z^*(T))}}b \\ &= c_{2,1} - c_{2,2}b \end{aligned}$$

$A_2(t)$, $B^*(t)$ and $Z^*(t)$ are

$$\begin{aligned} A_2(t) &= Z_0 e^{-\alpha_Z t} + \mu_Z(1-e^{-\alpha_Z t}) + \mu t \\ B^*(t) &= \sigma B(t) \\ Z^*(t) &= \sigma_Z e^{-\alpha_Z t}\int_0^t e^{\alpha_Z s}dB_Z(s) \end{aligned}$$

The expectation and variation of $B^*(t)$ and $Z^*(t)$ are

$$\begin{aligned} \mathbb{E}[B^*(t)] &= 0 \\ \mathbb{E}[Z^*(t)] &= 0 \end{aligned}$$

$$\begin{aligned} \text{Var}(B^*(t)) &= \sigma^2 t \\ \text{Var}(Z^*(t)) &= \frac{\sigma_Z^2}{2\alpha_Z}(1-e^{-2\alpha_Z t}) \end{aligned}$$

$$\text{Cov}(B^*(t), Z^*(t)) = 0$$

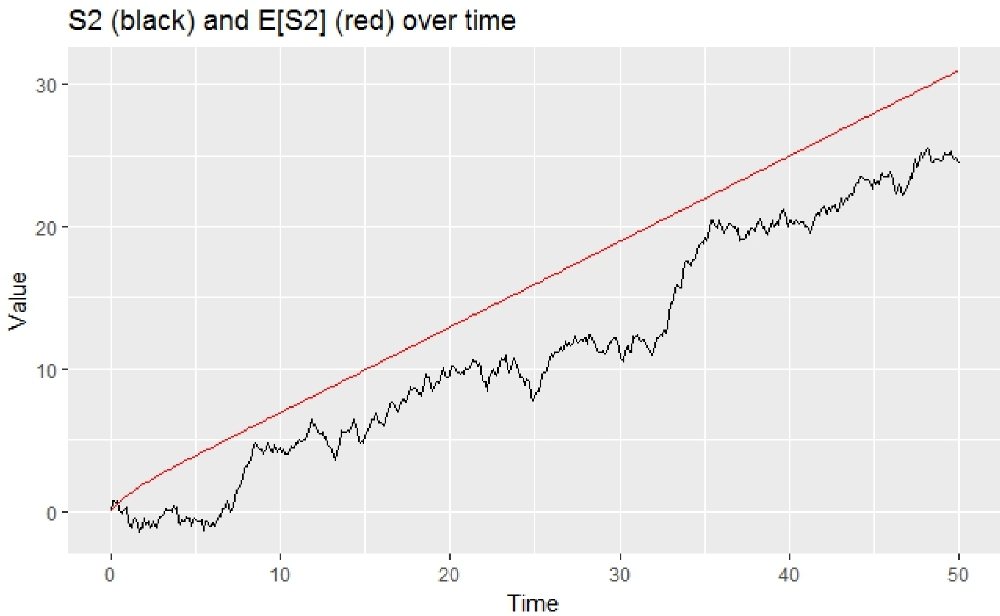

Figure 5.2: Parameters: $K_2 = 10$, $\alpha_Z = 0.8$, $\mu_Z = 0.9$, $\sigma_Z = 1.3$

The standardized processes $(\tilde{B}, \tilde{B_Z})$ are bivariate standard normal, with

$$\begin{aligned}\tilde{B}(t) &= \frac{B^*(t)}{\sqrt{\mathrm{Var}(B^*(t))}} \\ &= \frac{B(t)}{\sqrt{t}}\end{aligned}$$

$$\begin{aligned}\tilde{B_Z}(t) &= \frac{Z^*(t)}{\sqrt{\mathrm{Var}(Z^*(t))}} \\ &= \frac{\sqrt{2\alpha_Z}}{\sqrt{e^{2\alpha_Z t} - 1}} \int_0^t e^{\alpha_Z s} dB_Z(s)\end{aligned}$$

Chapter 6

An Overview of Quanto Options Based on Spreads

Bibliographical notes

The main references of this chapter are: Benth (2016) [1], Benth and Koekebakker (2015)[2] and Benth, Lange, and Myklebust (2015) [3], and Dineen (2005) [7].

6.1 Price of Quanto Options

This section features an outline of quanto options based on combinations of Brownian motion and Ornstein-Uhlenbeck spreads. The aim is to use the cointegrated property of Ornstein-Uhlenbeck processes to simplify the expression and calculation of quanto options based on OU spreads.

The energy quanto options we study have a payoff structure similar to a *product of call-put options*, and are therefore mainly used to hedge exposure to the joint price and volume risk.
Thus the logarithmic spot price f of the quanto option based on Brownian motion and Ornstein-Uhlenbeck processes can be expressed as:

$$\begin{aligned} f(S_1(T), S_2(T)) &= g(S_2(T) - S_1(T), S_2(T)) \\ &= (K_1 - S(T))_+ \times (K_2 - S_2(T), 0)_+ \end{aligned} \tag{6.1.1}$$

where $S_1(T)$, $S_2(T)$ and $S(T)$ are the same as Equation (4.1.2) and Equation (4.1.3), i.e.

$$\begin{aligned} S_2(T) &= X(T) + Z(T) \\ &= \mu_Z + \mu T + Z(0)e^{-\alpha_Z T} - \mu_Z e^{-\alpha_Z T} + \sigma B(T) + \sigma_Z e^{-\alpha_Z T} \int_0^T e^{\alpha_Z s} dU(s) \end{aligned}$$

$$\begin{aligned} S(T) &= Z(T) - Y(T) \\ &= Z(0)e^{-\alpha_Z T} + \mu_Z(1 - e^{-\alpha_Z T}) + \int_0^T \sigma_Z e^{-\alpha_Z (T-s)} - \sigma_Y \rho e^{-\alpha_Y (T-s)} dU(s) \\ &\quad - Y(0)e^{-\alpha_Y T} - \mu_Y(1 - e^{-\alpha_Y T}) - \sigma_Y \sqrt{1-\rho^2} \int_0^T e^{-\alpha_Y (T-s)} dW(s) \end{aligned}$$

A quanto option based on a product of different spread options would usually be too complex to solve analytically. We wish to use the cointegration property of S to simplify the structure of the quanto option.

The equation for $f(S_1(T), S_2(T))$ is:

$$\begin{aligned} f(S_1(T), S_2(T)) &= (K_1 - S(T))_+ \times (K_2 - S_2(T), 0)_+ \\ &= (K_1 - S(T))(K_2 - S_2(T)) \mathbf{1}_{\{K_1 > S(T)\}} \mathbf{1}_{\{K_2 > S_2(T)\}} \\ &= (K_1 K_2 - K_1 S_2(T) - K_2 S(T) + S(T) S_2(T)) \mathbf{1}_{\{K_1 > S(T)\}} \mathbf{1}_{\{K_2 > S_2(T)\}} \end{aligned}$$

$Y(T)$ and $Z(T)$ are stationary when $\lim\limits_{T \to \infty}$, as is $S(T) = Z(T) - Y(T)$. From Section 4.4.1 it follows that $(K_1 - S(T))_+$ is stationary, and converges in the long run to a constant.

We define the stationary limits of $Y(T)$, $Z(T)$ and $S(T)$ to be

$$\begin{aligned} \lim_{T \to \infty} Y(T) &= K_Y \\ \lim_{T \to \infty} Z(T) &= K_Z \\ \lim_{T \to \infty} S(T) &= K_S \end{aligned}$$

Thus, the theory given in this overview is that in the long run, an approximation of f would become:

$$\begin{aligned}
f(S_1(T), S_2(T)) &= (K_1K_2 - K_1S_2(T) - K_2S(T) + S(T)S_2(T))\mathbf{1}_{\{K_1>S(T)\}}\mathbf{1}_{\{K_2>S_2(T)\}} \\
&= (K_1K_2 - K_1(X(T) + Z(T)) - K_2S(T) + S(T)(X(T) + Z(T))) \\
&\quad \mathbf{1}_{\{K_1>K_S\}}\mathbf{1}_{\{K_2-K_Z>X(T)\}} \\
&\approx (K_1K_2 - K_1K_Z - K_2K_S + K_SK_Z - X(T)(K_1 - K_S)) \\
&\quad \mathbf{1}_{\{K_1>K_S\}}\mathbf{1}_{\{K_2>X(T)+K_Z\}} \\
&= (K_1 - K_S)\,(-X(T)) \\
&\quad \mathbf{1}_{\{K_S<K_1\}}\mathbf{1}_{\{X(T)<K_2-K_Z\}} \\
&= k_2(k_1 - X(T))\mathbf{1}_{\{K_S<K_1\}}\mathbf{1}_{\{X(T)<K_2-K_Z\}}
\end{aligned}$$

where

$$\begin{aligned}
k_1 &= \frac{K_1K_2 - K_2K_S}{K_1 - K_S} - K_Z \\
k_2 &= K_1 - K_S
\end{aligned}$$

We define $q(S_1(T), S_2(T))$ to be the approximation:

Definition 6.1.1. *Given S_1 and S_2 as sums of Brownian motion and Ornstein-Uhlenbeck processes, an approximation for the Quanto option based on the product of European options featuring S_1 and S_2 can be constructed as*

$$q(S_1(T), S_2(T)) = k_2(k_1 - X(T))\mathbf{1}_{\{K_S<K_1\}}\mathbf{1}_{\{X(T)<K_2-K_Z\}}$$

where

$$\begin{aligned}
k_1 &= \frac{K_1K_2 - K_2K_S}{K_1 - K_S} - K_Z \\
k_2 &= K_1 - K_S
\end{aligned}$$

The remainder of this chapter will focus on this approximation.

6.2 Expectation and Variance for Quanto Options

The main advantage the approximation q has over the original quanto option f is the reduction of complexity. While f has a multivariate normal distribution, dependent on X, Y and Z, q is normally distributed and only dependent on the stochastic process X.

Proposition 6.2.1. *The quanto option $f(S_1(t), S_2(t))$ based on on X, Y and Z, and has a multivariate normal distribution with*

$$f(S_1(T), S_2(T)) \sim \mathcal{N}(\boldsymbol{\mu}(t), \boldsymbol{\sigma}(t))$$

where

$$\boldsymbol{\mu}(t) = \begin{pmatrix} \mu t \\ Y(0)e^{-\alpha_Y t} + \mu_Y(1 - e^{-\alpha_Y t}) \\ Z(0)e^{-\alpha_Z t} + \mu_Z(1 - e^{-\alpha_Z t}) \end{pmatrix} \tag{6.2.1}$$

$$\boldsymbol{\sigma}(t) = \begin{pmatrix} \sigma^2 t & 0 & 0 \\ 0 & \frac{\sigma_Y^2}{2\alpha_Y}\left(1 - e^{-2\alpha_Y t}\right) & \frac{\rho\sigma_Y\sigma_Z}{\alpha_Y+\alpha_Z}\left(1 - e^{-t(\alpha_Y+\alpha_Z)}\right) \\ 0 & \frac{\rho\sigma_Y\sigma_Z}{\alpha_Y+\alpha_Z}\left(1 - e^{-t(\alpha_Y+\alpha_Z)}\right) & \frac{\sigma_Z^2}{2\alpha_Z}\left(1 - e^{-2\alpha_Z t}\right) \end{pmatrix} \tag{6.2.2}$$

The expectation and variance for $S_1(T)$ and $S_2(T)$ are listed below.

Expectation:

$$\begin{aligned} \mathbb{E}[S_2(T)] &= \mathbb{E}[X(T) + Z(T)] \\ &= \mu T + Z(0)e^{-\alpha_Z T} + \mu_Z(1 - e^{-\alpha_Z T}) \end{aligned}$$

$$\begin{aligned} \mathbb{E}[S(T)] &= \mathbb{E}[Z(T) - Y(T)] \\ &= Z(0)e^{-\alpha_Z T} - Y(0)e^{-\alpha_Y T} + \mu_Z(1 - e^{-\alpha_Z T}) - \mu_Y(1 - e^{-\alpha_Y T}) \end{aligned}$$

Covariance:

$$\mathrm{Cov}(X(T), Y(T)) = 0$$

$$\mathrm{Cov}(X(T), Z(T)) = 0$$

$$\mathrm{Cov}(Y(t), Z(t)) = \frac{\rho\sigma_Y\sigma_Z}{\alpha_Y + \alpha_Z}\left(1 - e^{-t(\alpha_Y+\alpha_Z)}\right)$$

Variance:

$$\begin{aligned}
\mathrm{Var}(S_2(T)) &= \mathrm{Var}(X(T) + Z(T)) \\
&= \mathrm{Var}(X(T)) + \mathrm{Var}(Z(T)) + 2\mathrm{Cov}(X(T), Z(T)) \\
&= \sigma T + \frac{\sigma_Z^2}{2\alpha_Z}(1 - e^{-2\alpha_Z T})
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(S(t)) &= \mathrm{Var}(Y(t)) - 2\mathrm{Cov}(Y(t), Z(t)) + \mathrm{Var}(Z(t)) \\
&= \frac{\sigma_Z^2}{2\alpha_Z}\left(1 - e^{-2\alpha_Z t}\right) - 2\frac{\rho\sigma_Y\sigma_Z}{\alpha_Y + \alpha_Z}\left(1 - e^{-t(\alpha_Y + \alpha_Z)}\right) \\
&\quad + \frac{\sigma_Y^2}{2\alpha_Y}\left(1 - e^{-2\alpha_Y t}\right)
\end{aligned}$$

6.3 The Approximation $\mathbb{E}[q(S_1(T), S_2(T))]$

For the calculation of the approximated price q, we assume that Y, Z and subsequently $S = Z - Y$ are stationary. As such, we define

$$\begin{aligned}
\lim_{T\to\infty} Y(T) &= K_Y \\
\lim_{T\to\infty} Z(T) &= K_Z \\
\lim_{T\to\infty} S(T) &= K_S
\end{aligned}$$

So we have

$$\begin{aligned}
q(S_1(T), S_2(T)) &= k_2(k_1 - X(T))\mathbf{1}_{\{K_S < K_1\}}\mathbf{1}_{\{X(T) < K_2 - K_Z\}} \\
&= k_2(k_1 - \mu T - \sigma B(T))\mathbf{1}_{\{K_S < K_1\}}\mathbf{1}_{\{\mu T + \sigma B(T) < K_2 - K_Z\}}
\end{aligned}$$

where $X(T)$ is a BM with drift. k_1 and k_2 are the constants

$$\begin{aligned}
k_1 &= \frac{K_1K_2 - K_2K_S}{K_1 - K_S} - K_Z \\
k_2 &= K_1 - K_S
\end{aligned}$$

As we take K_1 and K_s to be constants, we assume $K_1 > K_S$, so $\mathbf{1}_{\{K_S < K_1\}} = 1$.
For $K_1 \le K_S$, the trivial result is $q(S_1(T), S_2(T)) = 0$.
Standardizing $B(T) \sim \mathcal{N}(0, T)$, we get $x \sim \mathcal{N}(0, 1)$ where x is given as:

$$x = \frac{B(T)}{\sqrt{T}}$$

The boundary $x < c_1$ is

$$c_q = \frac{K_2 - K_Z - \mu T}{\sigma\sqrt{T}}$$

Assuming $K_1 > K_S$, the expectation becomes

Expectation:

$$\begin{aligned}
\mathbb{E}[q(S_1(T), S_2(T))] &= \mathbb{E}[k_2(k_1 - X(T))\mathbf{1}_{\{K_1 > K_S\}}\mathbf{1}_{\{K_2 - K_Z > X(T)\}}] \\
&= k_1k_2\mathbb{E}[\mathbf{1}_{\{B(T) < \frac{K_2 - K_Z - \mu T}{\sigma}\}}] - k_2\mathbb{E}[(\mu T + \sigma B(T))\mathbf{1}_{\{B(T) < \frac{K_2 - K_Z - \mu T}{\sigma}\}}] \\
&= k_1k_2\mathbb{E}[\mathbf{1}_{\{x < c_q\}}] - k_2\mathbb{E}[(\mu T + \sigma\sqrt{T}x)\mathbf{1}_{\{x < c_q\}}] \\
&= k_1k_2\int_{-\infty}^{c_q}\phi(x)dx - k_2\int_{-\infty}^{c_q}(\mu T + \sigma x)\phi(x)dx \\
&= k_2\Phi(c_q)(k_1 - \mu T) - k_2\sigma\int_{-\infty}^{c_q}x\phi(x)dx \\
&= k_2\Phi(c_q)(k_1 - \mu T) - \frac{k_2\sigma}{\sqrt{2\pi}}\int_{-\infty}^{c_q}xe^{-\frac{1}{2}x^2}dx \\
&= k_2\Phi(c_q)(k_1 - \mu T) + \frac{k_2\sigma}{\sqrt{2\pi}}e^{-\frac{1}{2}c_q^2}
\end{aligned}$$

By approximating $f(S_1(T), S_2(T))$ with $q(S_1(T), S_2(T))$, f has been reduced in complexity, from a pricing function with three stochastic processes to an approximation where only X remains as a stochastic process. This enables us to find a closed form for q that can be solved analytically.

6.3.1 Simulation

For simulation purposes, we approximate K_Y, K_Z and K_S by using averages

$$\frac{\sum_{i=0}^{T} Y(i)}{T} = K_Y$$
$$\frac{\sum_{i=0}^{T} Z(i)}{T} = K_Z$$
$$\frac{\sum_{i=0}^{T} S(i)}{T} = K_S$$

Figure 6.1 is a plot of a comparison between q and f with $T = 50$.

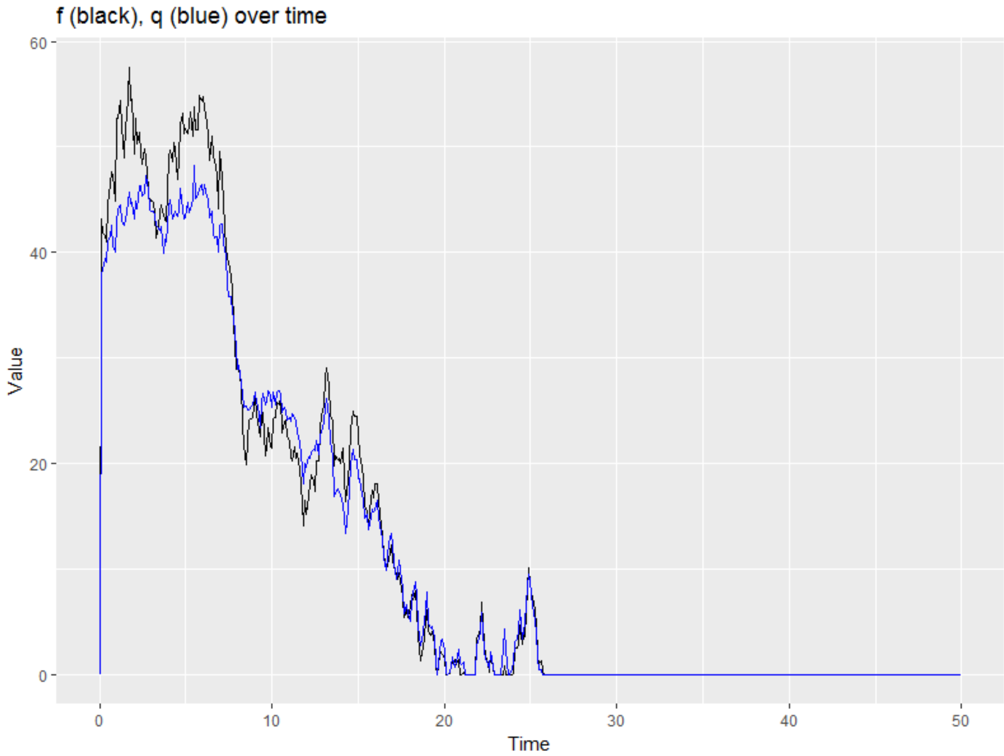

Figure 6.1: Parameters: $K = 15$, $\alpha_Y = 0.4$, $\alpha_Z = 0.8$, $\mu_Y = 0.3$, $\mu_Z = 0.9$, $\sigma_Y = 0.8$, $\sigma_Z = 1.3$

6.4 Comparison between f and q

To estimate how good of an approximation q is to f at time (T), we take a look at

$$\begin{aligned}
&|f(S_1(T), S_2(T)) - q(S_1(T), S_2(T))| = \\
&\qquad = |(K_1K_2 - K_1S_2(T) - K_2S(T) + S(T)S_2(T))\mathbf{1}_{\{K_1 > S(T)\}}\mathbf{1}_{\{K_2 > S_2(T)\}} \\
&\qquad\quad - k_2(k_1 - X(T))\mathbf{1}_{\{K_S < K_1\}}\mathbf{1}_{\{X(T) < K_2 - K_Z\}}| \\
&\qquad = |(K_1K_2 - K_1(X(T) + Z(T)) - K_2S(T) + S(T)(X(T) + Z(T)) \\
&\qquad\qquad \mathbf{1}_{\{K_1 > S(T)\}}\mathbf{1}_{\{K_2 > X(T) + Z(T)\}} \\
&\qquad\quad - (K_1K_2 - K_1K_Z - K_2K_S + K_SK_Z - X(T)(K_1 - K_S)) \\
&\qquad\qquad \mathbf{1}_{\{K_1 > K_S\}}\mathbf{1}_{\{K_2 > X(T) + K_Z\}}|
\end{aligned}$$

Naturally, this expression only gives any indication of how close q is to f at time (T), and nothing about the fit of the process in general. For that estimation, we look at the expectation:

$$
\begin{aligned}
&\mathbb{E}[|f(S_1,S_2)-q(S_1,S_2)|] = \\
&\qquad = \mathbb{E}[|(K_1K_2 - K_1S_2(T) - K_2S(T) + S(T)S_2(T))\mathbf{1}_{\{K_1>S(T)\}}\mathbf{1}_{\{K_2>S_2(T)\}} \\
&\qquad\quad - k_2(k_1 - X(T))\mathbf{1}_{\{K_S<K_1\}}\mathbf{1}_{\{X(T)<K_2-K_Z\}}|] \\
&\qquad = \mathbb{E}[|(K_1K_2 - K_1(X(T)+Z(T)) - K_2S(T) + S(T)(X(T)+Z(T)) \\
&\qquad\qquad\quad \mathbf{1}_{\{K_1>S(T)\}}\mathbf{1}_{\{K_2>X(T)+Z(T)\}} \\
&\qquad\qquad - (K_1K_2 - K_1K_Z - K_2K_S + K_SK_Z - X(T)(K_1-K_S)) \\
&\qquad\qquad\quad \mathbf{1}_{\{K_1>K_S\}}\mathbf{1}_{\{K_2>X(T)+K_Z\}}|] \\
&\qquad \le |\mathbb{E}[(K_1K_2 - K_1(X(T)+Z(T)) - K_2S(T) + S(T)(X(T)+Z(T)) \\
&\qquad\qquad\quad \mathbf{1}_{\{K_1>S(T)\}}\mathbf{1}_{\{K_2>X(T)+Z(T)\}} \\
&\qquad\qquad - (K_1K_2 - K_1K_Z - K_2K_S + K_SK_Z - X(T)(K_1-K_S)) \\
&\qquad\qquad\quad \mathbf{1}_{\{K_1>K_S\}}\mathbf{1}_{\{K_2>X(T)+K_Z\}}]|
\end{aligned}
$$

The last part is due to *Jensen's inequality*[7], from which it follows that

$$|\mathbb{E}[X]| \le \mathbb{E}[|X|]$$

So we have

$$|\mathbb{E}[f(S_1,S_2)-q(S_1,S_2)]| \le \mathbb{E}[|f(S_1,S_2)-q(S_1,S_2)|]$$

Unfortunately, the expression for $\mathbb{E}[f(S_1,S_2)]$ is not easy to find, which is part of the reason why one would wish to find an approximation to f in the first place. We will present an outline of how to find $\mathbb{E}[f(S_1,S_2)]$.

6.5 The Quanto Option $\mathbb{E}[f(S_1(T),S_2(T))]$

The methods for calculating $\mathbb{E}[f(S_1,S_2)]$ is similar to the ones used in the previous chapters.

For

$$
\begin{aligned}
f(S_1(T),S_2(T)) &= (K_1 - S(T))_+ \times (K_2 - S_2(T))_+ \\
&= (K_1 - S(T))(K_2 - S_2(T))\mathbf{1}_{\{S(T)<K_1\}}\mathbf{1}_{\{S_2(T)<K_2\}} \\
&= (K_1 - S(T))(K_2 - S_2(T))\mathbf{1}_{\{Z(T)-Y(T)<K_1\}}\mathbf{1}_{\{X(T)+Z(T)<K_2\}}
\end{aligned}
$$

we wish to solve

$$\mathbb{E}[f(S_1(T), S_2(T))] = \mathbb{E}[K_1K_2\mathbf{1}_{\{Z(T)-Y(T)<S(T)\}}\mathbf{1}_{\{X(T)+Z(T)<K_2\}}] \tag{6.5.1}$$

$$- \mathbb{E}[K_1S_2(T)\mathbf{1}_{\{Z(T)-Y(T)<K_1\}}\mathbf{1}_{\{X(T)+Z(T)<K_2\}} \tag{6.5.2}$$

$$- \mathbb{E}[K_2S(T)\mathbf{1}_{\{Z(T)-Y(T)<K_1\}}\mathbf{1}_{\{X(T)+Z(T)<K_2\}}] \tag{6.5.3}$$

$$+ \mathbb{E}[S(T)S_2(T)\mathbf{1}_{\{Z(T)-Y(T)<K_1\}}\mathbf{1}_{\{X(T)+Z(T)<K_2\}}] \tag{6.5.4}$$

We rewrite the pdf of the bivariate normal distribution in terms of the marginal pdf of the first variable times the conditional pdf of the second variable given the first variable. The expression is broken down and analyzed term by term, following the methods presented in Appendix A of Benth, Lange, and Myklebust (2015) [3].

6.5.1 First Expectation Term: Equation (6.5.1)

We know that $S(T)$ and $S_2(T)$ are normally distributed with expectation and variance as calculated in a previous section. As in Section 4.3, we standardize the random variables.

$S(T)$ and $S_2(T)$ can be written as

$$S(T) = K - (A(T) - \mathrm{Var}(U_T^*)u - \mathrm{Var}(W_T^*)w)$$

$$S_2(T) = A_2(T) - \mathrm{Var}(B_2^*(T))b - \mathrm{Var}(U_2^*(T))u_2$$

Looking at each expectation term, starting with the first term:

$$\begin{aligned}
&\mathbb{E}[K_1K_2\mathbf{1}_{\{K_1>S(T)\}}\mathbf{1}_{\{K_2>S_2(T)\}}] \\
&\quad = K_1K_2\mathbb{E}[\mathbf{1}_{\{Z(T)-Y(T)<K_1\}}\mathbf{1}_{\{X(T)+Z(T)<K_2\}}] \\
&\quad = K_1K_2\mathbb{P}(\{Z(T)-Y(T)<K_1\}\cap\{X(T)+Z(T)<K_2\}) \\
&\quad = K_1K_2\mathbb{P}(\{A^*(T)+\mathrm{Var}(U_T^*)u_1-\mathrm{Var}(W_T^*)w<K_1\}\cap \\
&\qquad\quad \{A_2(T)-\mathrm{Var}(B_2^*(T))b-\mathrm{Var}(U_2^*(T))u_2<K_2\})
\end{aligned}$$

The last probability is calculated with the CDF for the multivariate standard normal distribution, with 4 random variables.

6.5.2 Second Expectation Term: Equation (6.5.2)

Next, we consider the second expectation term:

$$
\begin{aligned}
&\mathbb{E}[K_1 S_2(T)\mathbf{1}_{\{K_1>S(T)\}}\mathbf{1}_{\{K_2>S_2(T)\}}] \\
&\quad = K_1\mathbb{E}[S_2(T)\mathbf{1}_{\{K_1>S(T)\}}\mathbf{1}_{\{K_2>S_2(T)\}}] \\
&\quad = K_1\mathbb{E}[(A_2(T)-\mathrm{Var}(B_2^*(T))b-\mathrm{Var}(U_2^*(T))u_2)\mathbf{1}_{\{K_1>A^*(T)+\mathrm{Var}(U_T^*)u_1-\mathrm{Var}(W_T^*)w\}} \\
&\qquad \mathbf{1}_{\{K_2>A_2(T)-\mathrm{Var}(B_2^*(T))b-\mathrm{Var}(U_2^*(T))u_2\}}] \\
&\quad = K_1A_2(T)F(c_u(w),c_2(b);\rho_{u_1,u_2}) - K_1\mathrm{Var}(B_2^*(T))\mathbb{E}[b\mathbf{1}_{\{u_1<c_u(w)\}}\mathbf{1}_{\{-b<c_b(u_2)\}}] \\
&\qquad - K_1\mathrm{Var}(U_2^*(T))\mathbb{E}[u_2\mathbf{1}_{\{u_1<c_u(w)\}}\mathbf{1}_{\{u_2<c_2(b)\}}]
\end{aligned}
$$

where (see Section 5.3):

$$c_b(u_2) = \frac{\sqrt{T}}{\sigma}\left(K_2 - A_2(T) + \frac{\sigma_Z\sqrt{1-e^{-2\alpha_Z T}}}{\sqrt{2\alpha_Z}}u_2\right)$$

and (from Section 4.3.1)

$$c_U(w) = \frac{A(T)}{\mathrm{Var}(U_T^*)} - \frac{\mathrm{Var}(W_T^*)}{\mathrm{Var}(U_T^*)}w$$

For this term, both the expectation terms turn out complicated, with a 4-dimensional multivariate normal distribution.

6.5.3 Third Expectation Term: Equation (6.5.3)

The third expectation term is essentially the same as the second expectation term.

$$
\begin{aligned}
&\mathbb{E}[K_2 S(T)\mathbf{1}_{\{K_1>S(T)\}}\mathbf{1}_{\{K_2>S_2(T)\}}] \\
&\quad = K_2\mathbb{E}[S(T)\mathbf{1}_{\{K_1>S(T)\}}\mathbf{1}_{\{K_2>S_2(T)\}}] \\
&\quad = K_2\mathbb{E}[(K-(A(T)-\mathrm{Var}(U_T^*)u-\mathrm{Var}(W_T^*)w))\mathbf{1}_{\{K_1>A^*(T)+\mathrm{Var}(U_T^*)u_1-\mathrm{Var}(W_T^*)w\}} \\
&\qquad \mathbf{1}_{\{K_2>A_2(T)-\mathrm{Var}(B_2^*(T))b-\mathrm{Var}(U_2^*(T))u_2\}}] \\
&\quad = K_2(K-A(T))F(c_u(w),c_2(b);\rho_{u_1,u_2}) - K_2\mathrm{Var}(U_T^*)\mathbb{E}[u\mathbf{1}_{\{u_1<c_u(w)\}}\mathbf{1}_{\{-b<c_b(u_2)\}}] \\
&\qquad - K_2\mathrm{Var}(W_T^*)\mathbb{E}[w\mathbf{1}_{\{u_1<c_u(w)\}}\mathbf{1}_{\{u_2<c_2(b)\}}]
\end{aligned}
$$

6.5.4 Fourth Expectation Term: Equation (6.5.4)

The fourth expectation term is

$$\mathbb{E}[S(T)S_2(T)\mathbf{1}_{\{K_1>S(T)\}}\mathbf{1}_{\{K_2>S_2(T)\}}]$$

which is even more complex than the previous terms. The term is still 4-dimensional multivariate normally distributed, and the product $S(T)S_2(T)$ makes the term even more difficult to solve.

6.6 Further Work

In this chapter, we have covered a rudimentary analysis of a quanto option based on cointegrated spreads, and it's approximation by an European put option. Simple simulations show that due to the stationary nature of Ornstein-Uhlenbeck spreads, one can treat the Ornstein-Uhlenbeck processes as stationary constants.

Further work would continue analyzing the approximation q, and discussed how to find an expression for f. The complexity of f suggests that regardless of the existence of a closed expression for the expectation and price, it will be inconvenient to work with. In all likelihood one will not find an analytical expression, but one that needs to be solved through numerical simulation.

If further study confirms that q is an adequate approximation, one has an approximate, closed model that serves as a useful tool for the study of quanto options in cointegrated markets, and it opens the possibility for simplifying even more complex quanto options.

Appendix A

Probability Theory

In this appendix, some basic concepts used in the thesis are listed for reference.

A.1 Standard Normal Distribution

Probability density of the normal distribution:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$$

The **standard normal distribution** is the normal distribution with $\mu = 0$ and $\sigma = 1$, and has probability density:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

with the derivative being:

$$\frac{d\phi(x)}{dx} = -x\phi(x)$$

$$\int_{-\infty}^{x} t\phi(t) = -\phi(x)$$

and with the cumulative distribution function being:

$$\begin{aligned}\Phi(x) &= \int_{-\infty}^{x} \phi(t)dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-x}^{\infty} e^{-\frac{1}{2}t^2} dt\end{aligned}$$

and the antiderivate being:

$$\int \Phi(x)dx = x\Phi(x) + \phi(x)$$

We define the error function as the probability of a random variable with normal distribution of mean 0 and variance $\frac{1}{2}$ in the range$[-x, x]$, that is:

$$\begin{aligned}\mathrm{erf}(x) &= \frac{1}{\sqrt{\pi}} \int_{-x}^{x} e^{-t^2} dt \\ &= \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2} dt\end{aligned}$$

with the complementary error function defined as:

$$\begin{aligned}\mathrm{erfc}(x) &= 1 - \mathrm{erf}(x) \\ &= \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2} dt\end{aligned}$$

So the cumulative distribution function can also be written as:

$$\Phi(x) = \frac{1}{2}\left[1 + \mathrm{erf}(\frac{x}{\sqrt{2}})\right]$$

A.2 Bivariate Normal Distribution

Probability density of the bivariate normal distribution:

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp(-\frac{z}{2(1-\rho^2)})$$

where

$$z = \frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}$$

and ρ is the correlation between x_1 and x_2, i.e.

$$\rho = \frac{\mathrm{Cov}(x_1, x_2)}{\sigma_1\sigma_2}$$

The standard bivariate normal distribution, with $\mu_i = 0$ and $\sigma_i = 1$ for $i = 1, 2$, is then

$$\phi(x_1, x_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp(-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1-\rho^2)})$$

The *marginal probabilities* are

$$\begin{aligned} f_{x_1}(x_1) &= \int_{-\infty}^{\infty} f(x_1, x_2) dx_2 \\ &= \frac{1}{\sigma_1\sqrt{2\pi}} \exp(-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}) \end{aligned}$$

and

$$\begin{aligned} f_{x_2}(x_2) &= \int_{-\infty}^{\infty} f(x_1, x_2) dx_1 \\ &= \frac{1}{\sigma_1\sqrt{2\pi}} \exp(-\frac{(x_2-\mu_2)^2}{2\sigma_2^2}) \end{aligned}$$

A.3 Probability: $P(X < g(Y))$

Let $X : \Omega \to U_X$, $Y : \Omega \to U_Y$ be random variables with the joint probability density function $f_{X,Y}(x, y)$ for $x, y \in (-\infty, \infty)$, and $g : \Omega \to \mathbb{R}$ a function. The probability $P(X < g(Y)$ is then:

$$P(X < g(Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{g(y)} f(x, y)_{X,Y} dxdy$$

The joint probability density function $f_{X,Y}(x, y)$ is equal to

$$\begin{aligned} f_{X,Y}(x, y) &= f_{Y|X}(y|x) f_X(x) \\ &= f_{X|Y}(x|y) f_Y(y) \end{aligned}$$

where $f_{Y|X}(y|x)$ and $f_{X|Y}(x|y)$ are the **conditional probability distributions** of Y given $X = x$ and of X given $Y = y$ respectively, and $f_X(x)$, $f_Y(y)$ are the **marginal probability distributions** for X and Y respectively.

Incorporating this definition, the probability $P(X < g(Y)$ can be calculated by

$$P(X < g(Y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{g(y)} f_{X|Y}(x|y) f_Y(y) dxdy$$

Likewise, for the opposite probability $P(g(X) < Y)$, the probability can be written as

$$P(g(X) < Y) = \int_{-\infty}^{\infty} \int_{g(x)}^{\infty} f_{Y|X}(y|x) f_X(x) dydx$$

Appendix B

R Code

B.1 R Code: Setup

```
###############
# Set-up and parameters
######################

K <- 15  #Strike price
K1 <- 4
K2 <- 10

#Common parameters for BM and OU-processes
mu_Y <- 0.3  # Drift for Y
mu_Z <- 0.7  # Drift for Z
sigma_Y <- 1.1  # Diffusion for Y
sigma_Z <- 1.3 # Diffusion for Z
rho <- 0.4   # Relation between Y and Z

#Parameters for BM-process X
mu <- 0.4
sigma <- 1

X <- array(0.0, dim = c(Nt,Np))   # X
E_X <- array(0.0, dim = c(Nt,Np)) # expectation of X, E[X]
for( k in 2:Nt){
  X[k,] <- X[k-1,] + mu*Dt + sigma*B[k,]
  E_X[k,] <- mu*Dt*k
}


#Parameters for OU-processes Y, Z
Y0 = 0
Z0 = 0
alpha_Y = 0.4   # Rate of mean reversal, Y
alpha_Z = 0.8  # Rate of mean reversal, Z

# BY = rho*U + sqrt(1-rho^2)*W and BZ = U
BY <- rho*U + sqrt(1-rho^2)*W

#Ornstein-Uhlenbeck processes
```

```
Y <- array(Y0, dim = c(Nt,Np))  # Y
Z <- array(Z0, dim = c(Nt,Np))  # Z
E_Y <- array(Y0, dim = c(Nt,Np))  # E[Y]
E_Z <- array(Z0, dim = c(Nt,Np))  # E[Z]

int_Y <- array(0.0, dim = c(Nt,Np))
int_Z <- array(0.0, dim = c(Nt,Np))

for (k in 2:Nt){
  int_Y[k,] <- int_Y[k-1,] + sqrt(Dt)*e_Y(-k*Dt)*(rho*U[k,] + sqrt
      (1-rho^2)*W[k,])
  Y[k,] <- Y[1,]*e_Y(k*Dt) + mu_Y*(1-e_Y(k*Dt)) + sigma_Y*e_Y(k*Dt)
      *int_Y[k,]
  E_Y[k,] <- Y0*e_Y(k*Dt) + mu_Y*(1-e_Y(k*Dt))

  int_Z[k,] <- int_Z[k-1,] + sqrt(Dt)*e_Z(-k*Dt)*U[k,]
  Z[k,] <- Z[1,]*e_Z(k*Dt) + mu_Z*(1-e_Z(k*Dt)) + sigma_Z*e_Z(k*Dt)
      *int_Z[k,]
  E_Z[k,] <- Z0*e_Z(k*Dt) + mu_Z*(1-e_Z(k*Dt))
}

######
# S1(T), S2(T), S(T)
###############
# S1(T) = X(T) + Y(T)
S_1 <- X + Y

# S2(T) = X(T) + Z(T)
S_2 <- X + Z

# S(T) = Z(T) - Y(T)
S_OU <- Z - Y

#Time parameters:
T <- 50  # Time interval
Dt <- 0.1  # time step
# sampling times
times <- seq(0.0, T, by = Dt)

Nt <- T/Dt + 1      # number of sampling points
Np <- 1             # number of paths

#Integral parameters:
up <- 2           # upper integral boundary
low <- -2         # lower integral boundary
N <- 100          # number of integral steps

#Construct three independent Brownian motion processes
B <- array(0.0, dim = c(Nt,Np))
U <- array(0.0, dim = c(Nt,Np))
W <- array(0.0, dim = c(Nt,Np))
for (k in 2:Nt){
  B[k,] <- rnorm(Np, mean=0, sd=sqrt(Dt))
  U[k,] <- rnorm(Np, mean=0, sd=sqrt(Dt))
  W[k,] <- rnorm(Np, mean=0, sd=sqrt(Dt))
}

e_Y <- function(t) {
  return(exp(-alpha_Y*t))
```

```
}

e_Z <- function(t) {
  return(exp(-alpha_Z*t))
}

A <- function(t) {
  a = K - Z0*e_Z(t) - mu_Z*(1-e_Z(t)) + Y0*e_Y(t)+ mu_Y*(1-e_Y(t))
  return(a)
}
```

R/1_Setup.R

B.2 R Code: Functions

```
#indicator function
indicator<-function(condition) ifelse(condition,1,0)

#error function
erf <- function(x) {
  return(2*pnorm(x*sqrt(2)) -1)
}

#trapezoidal rule for integration
tr <- function(f, a, b, n) {    # function, lower limit, upper
    limit, number of steps
  h = (b-a)/n
  c = 0
  for(k in 1:n) {
    c = c + f(a + k*h)
  }
  integ = h*((f(a)+f(b))/2 + c)
  return(integ)
}

#################
# Integral functions for pricing
##########################

# Phi(C)*phi(w)
integral_1 <- function(k, a, b, n, c) {
  it <- function(w) {
    return( erf( c[k,]/sqrt(2) )*dnorm(w) )
  }
  return( 0.5 + 0.5*tr( it, a, b, n) )
}

# w*Phi(C)*phi(w)
integral_2 <- function(k, a, b, n, c) {
  it <- function(w) {
    return( erf( c[k,]/sqrt(2) )*w*dnorm(w) )
  }
  return( 0.5*tr( it, a, b, n) )
}

# phi(C)*phi(w)
```

```
integral_3 <- function(k, c1, c2) {
  return( exp(-c1[k,]^2/(2*(c2[k,]^2 + 1))) / (pi*sqrt(2*(c2[k,]^2 + 1))) )
}



#PrintPaths: Plot function, removes t=0
printPaths <- function(f, title = "", yl = "", xl="")
{
  plot(times[2:Nt], f[2:Nt,], type="l", main=title, ylab=yl, xlab=xl)
  if(Np >= 2) {
    for(ip in 2:Np){lines(times, f[2:Nt,ip])}
  }
}
```

R/1_Functions.R

B.3 R Code: BM Spread Option

```
###############
# Browninan motion - spread option
#####################

# BM Options
X_1 <- array(0.0, dim = c(Nt,Np))
X_2 <- array(0.0, dim = c(Nt,Np))
for( k in 2:Nt){
  X_1[k,] <- X_1[k-1,] + mu_Y*Dt + sigma_Y*(rho*U[k,] + sqrt(1-rho^2)*W[k,])
  X_2[k,] <- X_2[k-1,] + mu_Z*Dt + sigma_Z*U[k,]
}

#BM Spread option
S_BM <- X_2 - X_1

#Expectation of spread option S(T), BM
E_BM <- array(0.0, dim = c(Nt,Np))
for( k in 2:Nt){
  E_BM[k,] <- (mu_Y + mu_Z)*Dt*k
}


# Standardizing U and W: (U~ and W~)
U_st <- array(0.0, dim = c(Nt,Np))
W_st <- array(0.0, dim = c(Nt,Np))
for( k in 2:Nt){
  U_st[k,] <- U[k,]/sqrt(Dt*k)
  W_st[k,] <- W[k,]/sqrt(Dt*k)
}

#Boundary, BM
CB1 <- array(0.0, dim = c(Nt, Np))
CB2 <- array(0.0, dim = c(Nt, Np))
for( k in 2:Nt) {
```

```
  CB1[k,] <- (K-(mu_Z - mu_Y)*k*Dt)/(sigma_Z-sigma_Y*rho)*sqrt(k*Dt
      )
  CB2[k,] <- (sigma_Y*sqrt(1-rho^2))/(sigma_Z-sigma_Y*rho)
}
C_BM <- CB1 - CB2*W_st

# Spread option: max(K - S(T),0)
K_BM <- array(0.0, dim = c(Nt, Np))
for( k in 2:Nt) {
  K_BM[k,] <- max(K-S_BM[k,],0)
}

#Expectation of spread option, BM

ES_BM <- function(w, c1, c2, c, k, a, b, n){
  if (sigma_Y==0 && sigma_Z==0){
    return(max(K-(mu_Z-mu_Y)*k*Dt, 0) )
  } else {
    return( (K-(mu_Z - mu_Y)*k*Dt)*integral_1(k, a, b, n, c)
            - sigma_Y*sqrt(k*Dt*(1-rho^2))*integral_2(k, a, b, n, c
              )
            + (sigma_Z-sigma_Y*rho)*sqrt(k*Dt)*integral_3(k, c1, c2
              ))
  }
}
ET_BM <- array(0.0, dim = c(Nt,Np))
for (k in 2:Nt){ ET_BM[k,] <- ES_BM( W_st[k,], CB1, CB2, C_BM, k,
    low, up, N) }



BM_data <- data.frame(times, X_1, X_2, S_BM, E_BM, K_BM, ET_BM)
```

R/2_BrownianMotion.R

B.4 R Code: OU Spread Option

```
###############
# Ornstein-Uhlenbeck - spread option
####################

#Ornstein-Uhlenbeck processes: use Y and Z

#Separating into independent processes U* and W*
U_star <- array(0.0, dim = c(Nt,Np))
W_star <- array(0.0, dim = c(Nt,Np))

int_U1 <- array(0.0, dim = c(Nt,Np))
int_U2 <- array(0.0, dim = c(Nt,Np))
int_W <- array(0.0, dim = c(Nt,Np))
for (k in 2:Nt){
  int_U1[k,] <- int_U1[k-1,] + e_Y(-k*Dt)*sqrt(Dt)*U[k,]
  int_U2[k,] <- int_U2[k-1,] + e_Z(-k*Dt)*sqrt(Dt)*U[k,]
  U_star[k,] <- sigma_Z*e_Z(k*Dt)*int_U2[k,] - sigma_Y*rho*e_Y(k*Dt
      )*int_U1[k,]

```

```
  int_W[k,] <- int_W[k-1,] + e_Y(-k*Dt)*sqrt(Dt)*W[k,]
  W_star <- sigma_Y*sqrt(1-rho^2)*e_Y(k*Dt)*int_W[k,]
}


# Variance of U* and W*
V_U <- array(0.0, dim =c(Nt,1))
V_W <- array(0.0, dim =c(Nt,1))

# Check if alpha_Y or alpha_Z are 0
if (alpha_Y==0 && alpha_Z==0){
  for (k in 2:Nt){
    V_U[k,] <- sigma_Z^2*k*Dt + 2*sigma_Y*sigma_Z*rho*(k*Dt)^2 + sigma_Y^2*rho^2*k*Dt
    V_W[k,] <- sigma_Y^2*k*Dt*(1-rho^2)
  }
} else if (alpha_Y==0 && alpha_Z!=0){
  for (k in 2:Nt){
    V_U[k,] <- (sigma_Z^2/(2*alpha_Z))*(1-e_Z(2*k*Dt)) - ((2*sigma_Y*sigma_Z*rho)/(alpha_Z))*(1-exp(-alpha_Z*k*Dt)) + sigma_Y^2*rho^2*k*Dt
    V_W[k,] <- sigma_Y^2*k*Dt*(1-rho^2)
  }
} else if (alpha_Y!=0 && alpha_Z==0){
  for (k in 2:Nt){
    V_U[k,] <- sigma_Z^2*k*Dt - ((2*sigma_Y*sigma_Z*rho)/alpha_Y)*(1-exp(-alpha_Y*k*Dt)) + (sigma_Y^2*rho^2)/(2*alpha_Y)*(1-e_Y(2*k*Dt))
    V_W[k,] <- (sigma_Y^2*(1-rho^2))/(2*alpha_Y) * (1 - e_Y(2*k*Dt))
  }
} else {
  for (k in 2:Nt){
    V_U[k,] <- (sigma_Z^2/(2*alpha_Z))*(1-e_Z(2*k*Dt)) - ((2*sigma_Y*sigma_Z*rho)/(alpha_Z + alpha_Y))*(1-exp(-(alpha_Z+alpha_Y)*k*Dt)) + (sigma_Y^2*rho^2)/(2*alpha_Y)*(1-e_Y(2*k*Dt))
    V_W[k,] <- (sigma_Y^2*(1-rho^2))/(2*alpha_Y) * (1 - e_Y(2*k*Dt))
  }
}


# Standardizing U* and W*:
U_tild <- array(0.0, dim =c(Nt,1))
int_tild1 <- array(0.0, dim = c(Nt,Np))
int_tild2 <- array(0.0, dim = c(Nt,Np))
int_tild3 <- array(0.0, dim = c(Nt,Np))
if (sigma_Y==0 && sigma_Z==0){
  for (k in 2:Nt) {
    int_tild1[k,] <- int_tild1[k-1,] + e_Y(-k*Dt)*sqrt(Dt)*U[k,]
    int_tild2[k,] <- int_tild2[k-1,] + e_Z(-k*Dt)*sqrt(Dt)*U[k,]
    int_tild3[k,] <- int_tild3[k-1,] + e_Y(-k*Dt)*sqrt(Dt)*W[k,]

    U_tild[k,] <- (e_Z(k*Dt)*int_tild2[k,] - rho*e_Y(k*Dt)*int_tild1[k,]) / sqrt((1/(2*alpha_Z))*(1-e_Z(2*k*Dt)) - ((2*rho)/(alpha_Z + alpha_Y))*(1-exp(-(alpha_Z+alpha_Y)*k*Dt)) + (rho^2)/(2*alpha_Y)*(1-e_Y(2*k*Dt)))
    W_tild[k,] <- sqrt(2*alpha_Y) * e_Y(k*Dt)*int_tild3[k,]/sqrt(1-e_Y(2*k*Dt))
```

```
  }
} else if (sigma_Y==0){
  U_tild <- U_star/V_U
  for (k in 2:Nt) {
    int_tild3[k,] <- int_tild3[k-1,] + e_Y(-k*Dt)*sqrt(Dt)*W[k,]
    W_tild[k,] <- sqrt(2*alpha_Y) * e_Y(k*Dt)*int_tild3[k,]/sqrt(1-e_Y(2*k*Dt))
  }
} else {
  U_tild <- U_star/V_U   # U~
  W_tild <- W_star/V_W  # W~
}
U_tild[1,]  <- 0
W_tild[1,]  <- 0

#Boundary for the call option
C1 <- array(0.0, dim = c(Nt, Np))
C2 <- array(0.0, dim = c(Nt, Np))
for( k in 2:Nt ){
  C1[k,] <- A(k*Dt)/V_U[k,]
  C2[k,] <- -V_W[k,]/V_U[k,]
}
#If both sigma_Y, sigma_Z = 0, then C_OU = inf
C_OU <- C1 + C2*W_tild

# Phi(C)*phi(w)
integral_1 <- function(k, a, b, n, c) {
  it <- function(w) {
    return( erf( c[k,]/sqrt(2) )*dnorm(w) )
  }
  return( 0.5 + 0.5*tr( it, a, b, n) )
}

# w*Phi(C)*phi(w)
integral_2 <- function(k, a, b, n, c) {
  it <- function(w) {
    return( erf( c[k,]/sqrt(2) )*w*dnorm(w) )
  }
  return( 0.5*tr( it, a, b, n) )
}

# phi(C)*phi(w)
integral_3 <- function(k, c1, c2) {
  return( exp(-c1[k,]^2/(2*(c2[k,]^2 + 1))) / (pi*sqrt(2*(c2[k,]^2 + 1))) )
}


#Expectatation of spread
ES <- array(0.0, dim = c(Nt, Np))
for(k in 2:Nt){
  ES[k,] <- Z0*e_Z(k*Dt) + mu_Z*(1-e_Z(k*Dt)) - Y0*e_Y(k*Dt) - mu_Y*(1-e_Y(k*Dt))
}

#Expectation of spread option, OU
ES_OU <- function(k, a, b, n){
  if (sigma_Y==0 && sigma_Z==0){
    return(2*A(k))
```

```
  } else if (sigma_Y==0) {
    return(A(k)*integral_1(k, a, b, n, C_OU) + V_U[k,]*integral_3(k
        , C1, C2))
  } else {
  return( A(k)*integral_1(k, a, b, n, C_OU) - V_W[k,]*integral_2(k,
       a, b, n, C_OU)
          + V_U[k,]*integral_3(k, C1, C2) )
  }
}
ET_OU <- array(0.0, dim = c(Nt,Np))
for (k in 2:Nt){ ET_OU[k,] <- ES_OU( k, low, up, N) }

#data frames
UW_data <- data.frame(times, U, U_st, U_star, U_tild, W, W_st, W_
    star, W_tild)
OU_data <- data.frame(times, Y, Z, ES, S_OU, ET_OU)
```

R/3_OrnsteinUhlenbeck.R

B.5 R Code: Functions

```
###############
# Half-life of OU spread
######################

# Function for calculating the half-life of an OU process
halflife <- function(S, m) {
  k0 <- which(S==max(S))
  half <- m + (S[k0] - m)/2
  for (k in k0:Nt) {
    if (S[k,] < half) {
     return(c(k0*Dt,k*Dt,(k-k0)*Dt))    #time of: max value, half,
         half-life rate
    }
  }
}
```

R/halflife.R

B.6 R Code: BM and OU Option

```
###############
# S_1(T) = X(T) + Y(T) and S_2(T) = X(T) + Z(T)
#####################

# Separate:
# S_1 = K1 - A_1 + B_st + B_Yst
# S_2 = K2 - A_2 + B_st + B_Zst

A_1 <- function(t) {
  a = Y0*e_Y(t) + mu_Y*(1-e_Y(t)) + mu*t
  return(a)
}
A_2 <- function(t) {
```

```
  a = Z0*e_Z(t) + mu_Z*(1-e_Z(t)) + mu*t
  return(a)
}

B_st <- sigma*B

B_Yst <- array(0.0, dim = c(Nt,Np))
B_Zst <- array(0.0, dim = c(Nt,Np))
for (k in 2:Nt){
  B_Yst[k,] <- sigma_Y*e_Y(k*Dt)*(rho*int_U1[k,] + sqrt(1-rho^2)*
      int_W[k,])
  B_Zst[k,] <- sigma_Z*e_Z(k*Dt)*int_U2[k,]
}

# Variance of B_st, B_Yst, B_Zst
V_B <- array(0.0, dim =c(Nt,1))
V_Y <- array(0.0, dim =c(Nt,1))
V_Z <- array(0.0, dim =c(Nt,1))

#Check if alpha_Y or alpha_Z is 0
if (alpha_Z && alpha_Z==0){
  for (k in 2:Nt){
    V_B[k,] <- sigma^2*k*Dt
    V_Y[k,] <- (sigma_Y^2)*k*Dt
    V_Z[k,] <- (sigma_Z^2)*k*Dt
  }
} else if (alpha_Y==0 && alpha_Z!=0){
  for (k in 2:Nt){
    V_B[k,] <- sigma^2*k*Dt
    V_Y[k,] <- (sigma_Y^2)*k*Dt
    V_Z[k,] <- (sigma_Z^2)/(2*alpha_Z) * (1 - e_Z(2*k*Dt))
  }
} else if (alpha_Y!=0 && alpha_Z==0){
  for (k in 2:Nt){
    V_B[k,] <- sigma^2*k*Dt
    V_Y[k,] <- (sigma_Y^2)/(2*alpha_Y) * (1 - e_Y(2*k*Dt))
    V_Z[k,] <- (sigma_Z^2)*k*Dt
  }
} else {
  for (k in 2:Nt){
    V_B[k,] <- sigma^2*k*Dt
    V_Y[k,] <- (sigma_Y^2)/(2*alpha_Y) * (1 - e_Y(2*k*Dt))
    V_Z[k,] <- (sigma_Z^2)/(2*alpha_Z) * (1 - e_Z(2*k*Dt))
  }
}


# Standardize
B_tild <- B_st/sqrt(V_B)
Y_tild <- B_Yst/sqrt(V_Y)
Z_tild <- B_Zst/sqrt(V_Z)


# Boundary
c11 <- array(0.0, dim = c(Nt, Np))
c12 <- array(0.0, dim = c(Nt, Np))

c21 <- array(0.0, dim = c(Nt, Np))
c22 <- array(0.0, dim = c(Nt, Np))
```

```
for (k in 2:Nt){
  c11[k,] <- (K1 - A_1(k*Dt))/sqrt(V_Y[k,])
  c12[k,] <- sqrt(V_B[k,]/V_Y[k,])

  c21[k,] <- (K2 - A_2(k*Dt))/sqrt(V_Z[k,])
  c22[k,] <- sqrt(V_B[k,]/V_Z[k,])
}

c_1 <- c11 - c12*B_tild
c_2 <- c21 - c22*B_tild


#########################
#Expectation of BM+OU option, E[S1] and E[S2]
################################
E_1 <- array(0.0, dim = c(Nt, Np))
E_2 <- array(0.0, dim = c(Nt, Np))
for(k in 2:Nt){
  E_1[k,] <- mu*k*Dt + Y0*e_Y(k*Dt) + mu_Y*(1-e_Y(k*Dt))
  E_2[k,] <- mu*k*Dt + Z0*e_Z(k*Dt) + mu_Z*(1-e_Z(k*Dt))
}


#################
# E[max(Ki - Si(T),0)] for i = 1, 2
#######################
#Integrals can be reused from ch. 4

# Phi(c_1))*phi(b)
# Phi(c_2))*phi(b)
# integral_1(k, a1, a2, n, c_1/c_2)

# b*Phi(c_1)*phi(b)
# b*Phi(c_2)*phi(b)
# integral_2(k, a1, a2, n, c_1/c_2)

# phi(c_1)*phi(b)
# phi(c_2)*phi(b)
# integral_3(k, c11/c21, c12/c22)


# E[max(K1 - S1, 0)]
ES_1 <- function(k, a1, a2, n){
  if (sigma_Y==0){
    return(mu*k + Y0 + mu_Y + sigma*B[k,] + sigma_Y*BY[k,])
  } else {
    return( (K1 - A_1(k))*integral_1(k, a1, a2, n, c_1) - V_B[k,]*
        integral_2(k, a1, a2, n, c_1)
            - V_Y[k,]*integral_3(k, c11, c12) )
  }
}

# E[max(K2 - S2, 0)]
ES_2 <- function(k, a1, a2, n){
  if (sigma_Z==0){
    return(mu*k + Z0 + mu_Z + sigma*B[k,] + sigma_Z*U[k,])
  } else {
```

```
    return( (K2 - A_2(k))*integral_1(k, a1, a2, n, c_2) - V_B[k,]*
        integral_2(k, a1, a2, n, c_2)
            - V_Z[k,]*integral_3(k, c21, c22) )
  }
}




ET_1 <- array(0.0, dim = c(Nt,Np))
ET_2 <- array(0.0, dim = c(Nt,Np))
for (k in 2:Nt){
  ET_1[k,] <- ES_1(k, low, up, N)
  ET_2[k,] <- ES_2(k, low, up, N)
}


BM_OU_data <- data.frame(times, X, Y, Z, S_1, S_2, E_1, E_2, ET_1,
    ET_2 )
```

R/5_BrownianMotion_plus_OrnsteinUhlenbeck.R

B.7 R Code: Quanto Approximation

```
#################
# Approximation of Quanto Option
##############################+++++-----

########################
# Actual quanto option
########################

f <- function(s1, s2){
  return(max(K1 - s1,0)*max(K2 - s2,0) )
}

F_S <- array(0.0, dim = c(Nt,Np))
for( k in 2:Nt){
  F_S[k,] <- f(S_OU[k,] , S_2[k,])
}


########################
# APPROXIMATION: Q
########################

K_Y = mean(Y)
K_Z = mean(Z)
K_S = mean(S_OU)

k1 <- (K1*K2 - K2*K_S)/(K1 - K_S) - K2
k2 <- K1 - K_S

# Boundary
c_q <- array(0.0, dim = c(Nt, Np))

```

```
for (k in 2:Nt){
  c_q[k,] <- (K2 - K_Z + mu*Dt*k)/(sigma*sqrt(Dt*k))
}

# Approximation of spread
q <- function(k1, k2, x, ky, kz) {
  ks <- kz - ky
  a <- k1 - ks
  b <- k2 + (kz*(ks-k1))/(k1-ks)

  return(a*(b-x)*indicator(x < K2 - kz))
}

Q_S <- array(0.0, dim = c(Nt,Np))
for( k in 2:Nt){
  Q_S[k,] <- q(K1, K2, X[k,], K_Y, K_Z)
}

# Expectation of approximated spread
E_Q <- array(0.0, dim = c(Nt,Np))
for( k in 2:Nt){
  E_Q[k,] <- k2*pnorm(c_q[k,])*(k1 - mu*k*Dt) + k2*sigma/sqrt(2*pi)
      *exp(-0.5*c_q[k,]^2)
}


F_data <- data.frame(times, X, Y, Z, S_1, S_2, S_OU, E_1, E_2, ET_
    1, ET_2, M_1, M_2, M_S, F_S, G_S, Q_S)
```

R/6_q_approximation.R

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