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How Likely is it to Hit a Barrier? Thoretical and Emperical Estimates

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How Likely is it to Hit a Barrier? Theoretical and Empirical Estimates

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Abstract

We study probabilities which determine the payoff of barrier options: the probability that an asset hits a barrier before maturity, the probability that the asset is below the barrier at maturity, and the ratio of both probabilities. The correct estimation of these probabilities has crucial effects on the valuation of barrier options and related structured products. We show, however, that it is not easy to bring empirical measurements and theoretical predictions for these probabilities into agreement and that this problem can be used as a test for asset return models.

Keywords: Barrier options, probability of hitting a barrier, structured products, ARMA, GARCH, EGARCH, GJR.

1 Introduction

A barrier option is a financial derivative contract that is activated (knocked in) or extinguished (knocked out) when the price of the underlying asset crosses a certain level, called barrier. Barrier options are probably the most popular path-dependent options and an important ingredient of popular types of structured financial products.

The literature on the pricing of barrier options is too huge to attempt an overview. But besides theoretical works, there are recently also a few empirical studies on the actual market prices of structured products that incorporate barrier options [21, 12, 17, 22, 20]. Relatively little, however, is

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known on the utility gain of investing in these options. There are only some first studies in the context of structured products [18, 13, 6, 7, 2, 14, 8].

When evaluating barrier options, or structured products with barrier features, it is crucial to estimate the probability that the barrier is touched before maturity. In this paper, we focus on two probabilities, namely the probability that an asset hits the barrier and the probability that the asset ends below (above) the barrier, where the latter determines the comparable attractiveness of a direct investment in the underlying [16]. We also compute the quotient of these two probabilities determining the *relative* attractiveness of a barrier option. How the success of structured products with barrier features can be explained by the misestimation of those probabilities has been studied in [16].

There are basically two possible approaches to compute the probability of hitting the barrier and the probability of an asset to be below the barrier before maturity. One natural approach is the so called "historical probability": Start with the first data point, and check whether the barrier is hit before maturity or not, proceed to the second data point etc. Adding the number of all periods where the barrier is hit and dividing by the total number of periods will give the probability of an asset to hit the barrier.

The second approach is to divide all non-overlapping (independent) time intervals of the same length (equal to the maturity of the option) over the whole period. Dividing all those independent time intervals where the barrier is hit by the total number of periods gives the hit probability. In this approach the time intervals we considered are independent, consequently we call this approach "independent probability". The probability of ending below or above the barrier can also be computed in these two ways. We will see that these two seemingly similar approaches lead to very different probability estimates.

We start our investigation by looking at simple, analytically solvable models, like the Bachelier and the Black-Scholes model. The quotient of the probability that the barrier is touched and the probability of ending below the barrier turns out to be either constant or increasing in the barrier level which seems to contradict the empirical results that we derive for various market indices and single stocks. This "large quotient puzzle" cannot be resolved by using more sophisticated return processes, like ARMA, GARCH, EGARCH or GJR. However, distinguishing between the two above-mentioned definitions of probabilities turns out to be the (somehow surprising) answer that solves the puzzle.

Noticing how sensitive the estimates are to the model choice, we apply probability estimates for hitting a barrier as a test to a number of classical models, like ARMA, GARCH, EGARCH and GJR. In summary, the GJR model seems to be the most appropriate model for describing those probabilities.

2 First theoretical estimates

We begin our discussion with the classical models by Bachelier and Black-Scholes. This section mainly deals with the quotient of the probability that an asset hits the barrier and the probability that the asset is below (above) the barrier at maturity estimated by these models, since this is a good measure for the relative attractiveness of barriers. We call this quotient the "barrier quotient".

2.1 Bachelier model

First we show that the Bachelier model without drift predicts a constant barrier quotient of 2 where we mainly follow the computations in [15]:

Assume that the underlying asset price follows a stochastic process $\{S_t\}_{t\geq 0}$. To simplify the computation, we standardize the price such that $S_0 = 1$. $H \in [0, \infty)$ denotes the barrier. H measures how much down or up the stock price should be in order to touch the barrier. $H = 1$ would correspond to a barrier set at the initial price. When $H < 1$, it is a down barrier level, while $H > 1$ represents an up barrier. T is the maturity of the option.

In the following, we consider mainly down barriers. The results for up barriers are analogous. We define τ as the first t that the underlying asset price touches the barrier H,

$$\tau = \min_{0\leq t\leq T}\{t|S_t \leq H\}.$$

τ is a stopping time.

Assume that S_t follows the Bachelier model without drift – the simplest possible case, i.e.

$$\frac{dS_t}{S_0} = \sigma dW_t.$$

We recall the strong Markov property of Brownian motion.

Theorem 2.1 ***Strong Markov Property*** *Let $\{B(t)\}_{t\geq 0}$ be a standard Brownian motion with filtration$\{\mathcal{F}_t\}$, let s be a stopping time relative to this filtration, with associated stopping σ-algebra $\{\mathcal{F}_s\}$. For $t \geq 0$, define*

$$B^*(t) = B(t+s) - B(s),$$

and let $\{\mathcal{F}_t^*\}_{t\geq 0}$ be the filtration of the process $\{B^*(t)\}_{t\geq 0}$. Then
(a)$\{B^*(t)\}_{t\geq 0}$ is a standard Brownian motion; and
(b) For each $t > 0$, the σ-algebra $\mathcal{F}_t^*$ is independent of $\mathcal{F}_s$

This theorem implies that $\{S(t+\tau) - S(\tau)\}$ is also a Brownian motion conditioned on $\tau \leq T$. With the reflection principal of Brownian motion, we can deduce the following property.

Theorem 2.2 *Given that $S(t)$ follows a standard Brownian motion, for every barrier level* $0 \leq H \leq 1$

$$P(\tau \leq T) = 2P(S(T) \leq H) = 2\frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{H} e^{-\frac{x^2}{2t}}\, dx$$

Proof. See appendix A.1. This implies in particular that the barrier quotient is 2.

2.2 Black-Scholes model

To study a more realistic model we assume now that S_t follows a geometric Brownian motion with drift, i.e $dS_t = \mu S_t dt + \sigma S_t dw_t$. We have $d\ln S_t = \frac{dS_t}{S_t} = \mu dt + \sigma dw_t$, so $\ln S_t = \mu t + \sigma w_t, \ln S_0 = 0$. A short computation gives:

$$\begin{aligned} P(S_T \leq H) &= P(\ln S_T \leq \ln(H)) \\ &= P(\mu T + \sigma w_T \leq \ln H) \\ &= P(w_T \leq \frac{\ln H - \mu T}{\sigma}). \end{aligned}$$

Since $w_T \sim N(0,T)$, we get $P(S_T \leq H) = \Phi(\frac{\ln H - \mu T}{\sigma\sqrt{T}})$.
From this we see that the barrier quotient of the two probabilities is

$$\begin{aligned} \frac{P(\tau \leq T)}{P(S_T \leq H)} &= 1 + \frac{P(\tau \leq T \cap S_T \geq H)}{P(S_T \leq H)} \\ &= 1 + \frac{\exp\left(\frac{2\mu \ln H}{\sigma^2}\right)\Phi(\frac{\ln H + \mu T}{\sigma\sqrt{T}})}{\Phi(\frac{\ln H - \mu T}{\sigma\sqrt{T}})}. \end{aligned}$$

For the detailed proof, we refer to Appendix A.2. Now the quotient is not constant anymore. However, for $H \to 0$ it still converges to 2:

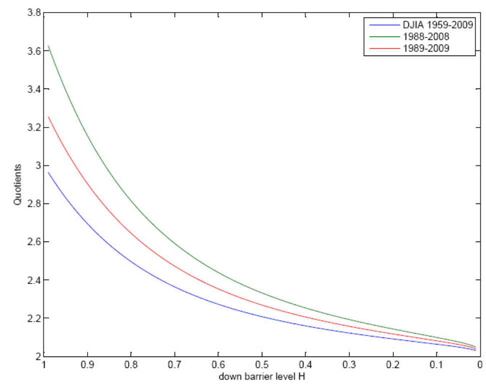


Figure 1: Barrier Quotient of the two probabilities based on a geometric Brownian motion with volatility and drift as specified by the returns of the DJIA in three time periods.

Lemma 2.1 *If the stock price S_t follows a geometric Brownian motion and $T = 1$, then*

$$\lim_{H \to 0} \frac{P(\tau \leq T)}{P(S_T \leq (H))} = 2$$

.

Proof. see Appendix A.3.

The dependence of the quotient on the barrier can be shown when we choose typical numbers for μ and σ. Here we take the μ and σ as derived from DJIA for the time periods 1988-2008, 1989-2009, 1959-2009.

1959-2009	**1988-2008**	**1989-2009**
μ=0.072	μ=0.1105	μ=0.092
σ=0.167	σ=0.181	σ=0.177

Table 1: List of returns and volatilities for Dow &Jones Industrials Average 1959-2009, 1988-2008 and 1989-2009

It can be seen from Figure 4.1 that the quotient is decreasing and converges to 2 when the barrier goes to 0.

3 Empirical estimation and the "large quotient puzzle"

We now compare the predictions by the Bachelier model and the Black-Scholes model with empirical data.

3.1 Methods

For a series of stock prices S_t, the computation of the historical probability is as follows. Assume there are N total observations, i.e. S_1, S_2,...,S_N. We consider options with a time to maturity T. For the probability to hit a down barrier B, we start with S_1 and check whether there exists a t, $1 \leq t \leq T$ such that $S_t \leq B \cdot S_1$. Then we proceed to S_2 and check whether there exists a t, $2 \leq t \leq (T+1)$ such that $S_t \leq B \cdot S_2$. This computation can be done until S_{N-T+1}. Summing up all periods where the barrier has been hit and dividing by $N - T + 1$ gives the historical probability of hitting a barrier.

The way to compute the probability to end below the barrier is similar. Instead of checking whether the barrier is hit or not, we check whether the asset price is below the barrier at maturity.

The computation for an up barrier is analogous.

3.2 Data

For a first empirical test we use the Dow& Jones Industrials Average (DJIA) price index data from January 1st 1988 to January 1st 2008 which consists of 5219 observations and the data from January 1st 1989 to January 1st 2009 of 5218 observations (all data provided by Datastream). The well-known fact that 2008 was a "special" year made it necessary to test two time series. Moreover, we consider a long (50 years) time series starting on January 1st, 1959.

For simplicity, we make the assumption that the number of trading days in each year is the same, namely 261 trading days (the average of the time period from January 1988 till January 2008).

3.3 Empirical measurements of historical barrier quotients

The quotients of the DJIA 1988-2008, 1989-2009 and 1959-2009 up barriers are reported in figure 2. The quotients for the 20 years series are between 1 to 4, but the quotient for 1959-2009 is quite large for some large barrier levels. This figure suggests that the quotients predicted by the Bachelier and the Black-Scholes model are not very consistent with empirical data.

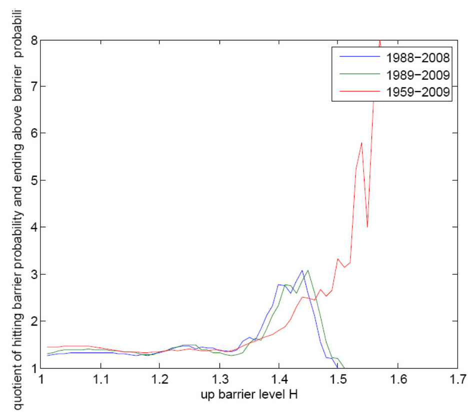


Figure 2: Up barrier quotients for DJIA 1988-2008, 1989-2009 and 1959-2009

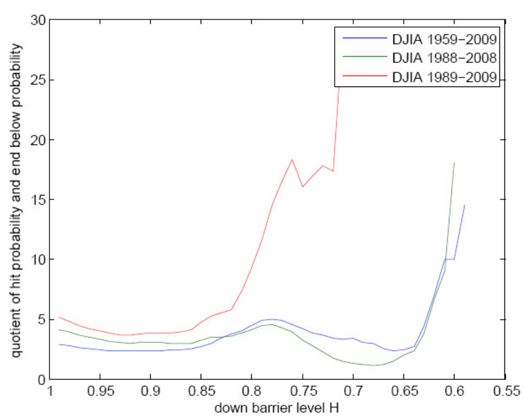


Figure 3: Down barrier quotients for DJIA 1959-2009,1988-2008 and 1989-2009

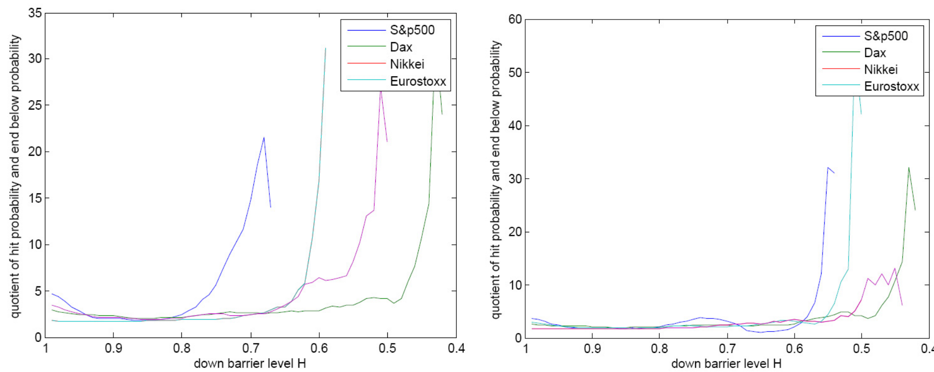


Figure 4: Quotient for other major stock indices, 1988-2008(left) and 1989-2009 (right)

The analysis becomes more surprising as we come to down barrier quotients which are reported in Figure 3.

A quick look at those tables for the down barriers allows us to spot two immediate facts: the down barrier quotients are much higher than 2 for all periods and, more importantly, for low barriers, the quotient tends to increase to infinity. Compared to the quotients derived from the models described in Section 2, the values and limits of empirical data are strikingly different. We call this the *large quotient puzzle*.

Table 3 indicates that the crash of the stock market in 2008 only has limited effect. Even though the ratio for DJIA 1988-2008 is higher than DJIA 1989-2009, both ratios tend to move into the same direction. The large quotient puzzle still exists after the 2008 crash.

Is this just a feature of the DJIA? To test this, we consider other stock indices.

Surprisingly, the large quotient effect prevails in these major stock markets. Figure 4 shows the quotients for S&P 500, Dax, Nikkei, and Eurostoxx. All the ratios in the figure show the same pattern as the DJIA. The ratio increases above certain barrier levels and jumps to infinity. The quotient for 1988-2008 reaches infinity earlier than 1989-2009.

We extended the analysis to 46 other stock indices (compare Appendix B for a complete list). 37 exhibit a quotient that goes to infinity for low barriers. This fact allows us to draw the conclusion that the large quotient effect seems to be a general phenomenon.

Finally, we tested all stocks of the S&P 100 index to show that the large quotient effect also exists in single stocks. The results again confirm the

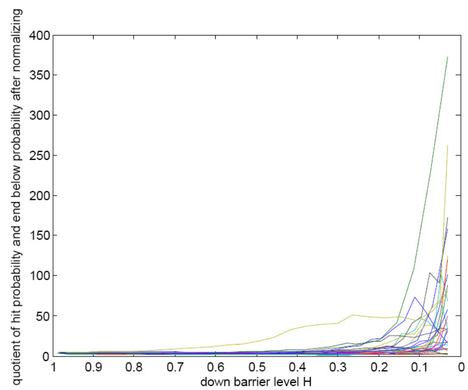


Figure 5: Barrier ratio for the stocks in S&P 100

large quotient effect:

- 88% have a ratio that goes to infinity.
- 92% have values larger than 3 for the lowest three barrier levels.
- 43% have ratios that are mostly larger than 3.

Those facts can be considered as evidence that the large quotient puzzle also holds true for single stocks. To compare the quotient, we normalized all barrier levels according to the last non-infinity quotient (Figure 5.1). In summary: the large quotient puzzle is a robust empirical finding.

But maybe the "puzzle" is just a result of the well-known limitation of the Black-Scholes model? Maybe fat tails or non-constant volatility could model this effect?

4 How to solve the puzzle?

We have seen that large barrier quotients regularly occur in the data, but simple models like Bachelier or Black-Scholes cannot predict them. But maybe this "large quotient puzzle" is just a result of the well-known limitations of these models? It would be natural to conjecture that fat tails or non-constant volatility could cause this effect in the data.

In order to test this explanation we extend the analysis to more sophisticated models: ARMA(1,1), GARCH(1,1), EGARCH(1,1) and GJR(1,1), compare [1, 10, 3, 5, 4, 9, 11, 19]. For all computations we used the GARCH

Toolbox of Matlab. We fitted the parameters of each model to the full time series data (DJIA for the same time windows as above) and then generated 1000 simulations of daily returns for one year for each of the processes in order to estimate the relevant probabilities and the barrier quotient accordingly.

The results are unexpected: in all of these models the quotients are nearly always between 1 and 4. None of them predicts a large quotient effect. (We will present data from these simulations on the barrier hit probabilities later.)

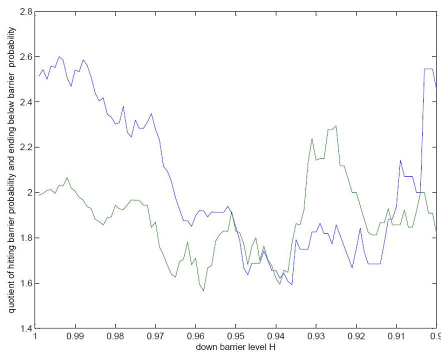
What is wrong? Why do we still have such a big discrepancy between the empirical data of the historical probabilities and the estimates?

It turns out that the solution of the puzzle lies not in the selection of the stochastic process, but elsewhere:

When computing historical probabilities, a decrease of the stock price affects the results for a whole number of starting dates and has substantial influence on the total probability. It turns out that this effect is the underlying reason for the large quotients observed in empirical data. Consider the case when there is only one observation which lies below the barrier. For all starting dates which are close enough to this observation, the barrier is hit. Therefore the historical probabilities as estimated from empirical data so far are not suitable for a comparison with probabilities predicted by the models. To get rid of this problem we can use independent intervals instead of overlapping intervals to compute the barrier hit probability. For a series of data, the computation of this probability (which we call "independent probability") proceeds as follows: consider a time span of N observations of stock prices, i.e. S_1, S_2, ..., S_N. We consider an option with maturity T, so there are N/T independent time intervals. We define the independent probability to hit the barrier as the number of time intervals where the barrier is hit, divided by the total number of time intervals N/T. Similarly, we can define the probability of ending below the barrier.

This new definition needs more data for less samples, thus we mainly consider 1 month and 3 months options for the 50 years return sample of the DJIA. For 1 month options 600 intervals can be used, while 200 intervals can be obtained for 3 months options. For simplicity we mostly focus on the results for 1 month options in the following discussion.

For the 1 month options, Figure 6 plots the quotient for the DJIA 1959 up barrier and down barrier. As we can see, the quotient for an up barrier is between 1.1 to 2, while it is mostly between 1.4 to 2.8 for a down barrier. Compared to the large quotient generated from the historical approach, this is a striking difference – and even not far from the predictions of the simple



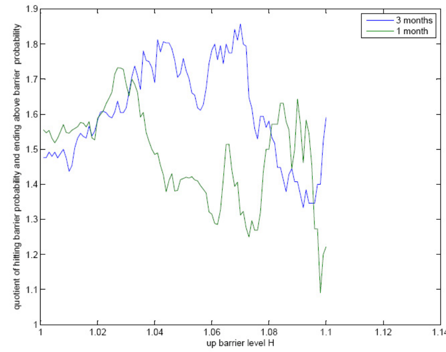


Figure 6: Quotients by independent probability of DJIA 1959-2009 down barrier (left) and up barrier (right)

Bachelier model. For independent probabilities, large quotients disappear.

This is not only the case for the DJIA, but also for the other indices under consideration, as well as for 3 months options. We conclude therefore that the "large quotient puzzle" disappears when choosing a more appropriate method of probability estimation from empirical data.

5 Barrier probabilities as a model benchmark

In this section, we use barrier hit probabilities as a new benchmark to test the performance of various models for stock returns. Here we can use both, historical probabilities and independent probabilities. For each time series model, we apply 1000 simulations. We take the 5th percentile and the 95th percentile of the barrier hit probabilities generated by each simulation and compare them with the actual barrier hit probability for the DJIA (and further indices). If the barrier hit probability lies between the 5th percentile and the 95th percentile of one model, the barrier hit probability is well estimated in this model. Otherwise, the hitting barrier probability is considered as either overestimated or underestimated.

5.1 Historical probability as a benchmark

For historical probability, we focus on DJIA 1988-2008 1 year barrier options. The percentiles and the real barrier hit probability for up barriers and down barriers are displayed separately in figure 7 and figure 8.

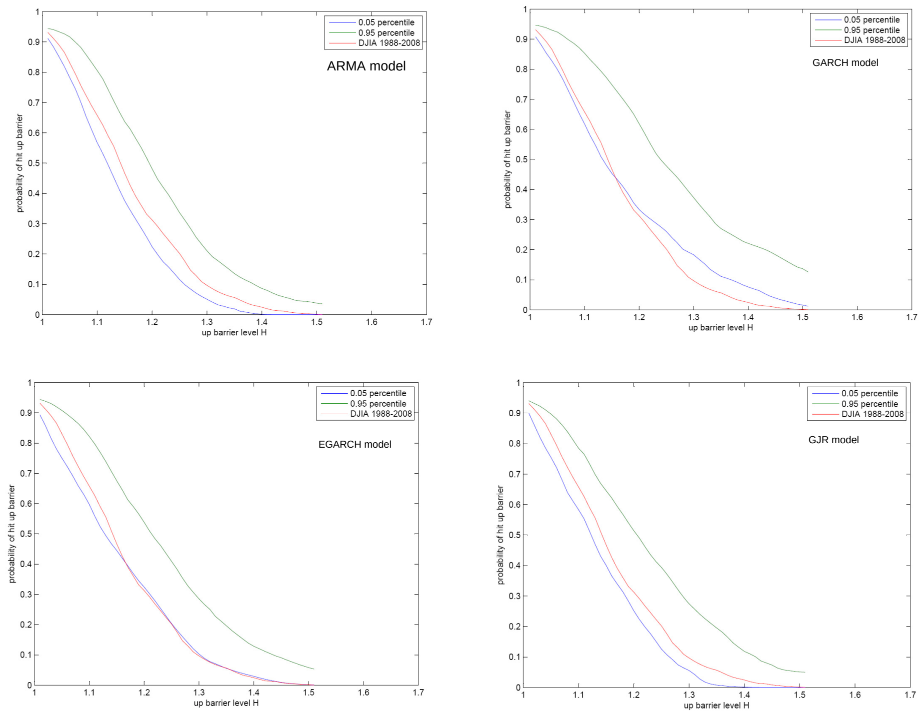


Figure 7: Probability of touching up barrier for DJIA 1988-2008 and estimation from ARMA, GARCH, EGARCH and GJR

As shown in figure 7, the ARMA model predicts the up barrier hit probability well since the barrier hit probability line of the DJIA lies between the 5th and the 95th percentiles. But for down barrier, the ARMA model shows its limitations. For barriers lower than 87%, in total 95% of the simulations underestimate the probability to hit the barrier.

From figure 7 it is clear that 95% of the GARCH simulations estimate the barrier hit probability higher than DJIA for barrier levels above 117%. This overestimation continues as the barrier gets larger. However, the down barrier hit probability estimated by the GARCH model fits the DJIA well.

The EGARCH model in this case, overestimated both, up barriers and down barriers. The DJIA line does not lie within the percentile lines for large up barriers and large down barriers.

The GJR model can be considered as the most appropriate model, since the DJIA line lies between the 5th and the 95th percentile lines for all barriers.

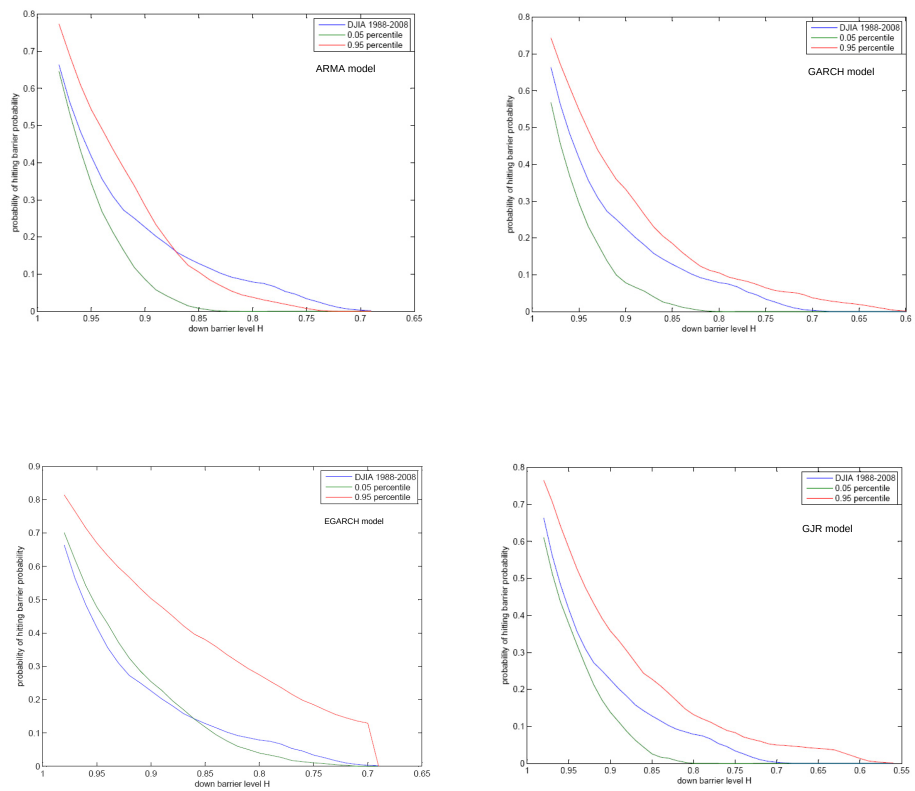
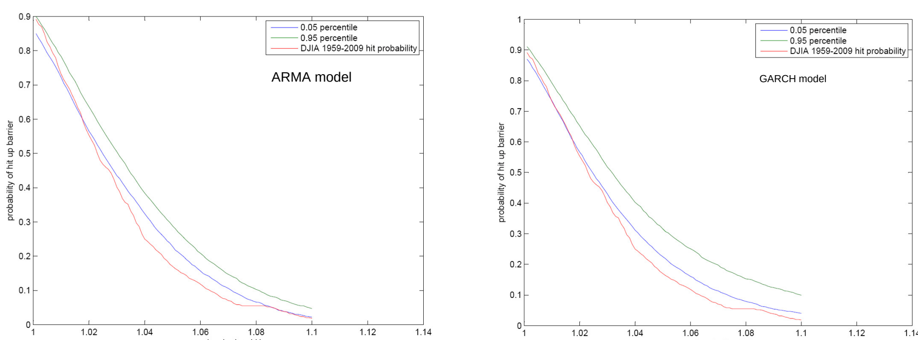


Figure 8: Probability of touching down barrier for DJIA 1988-2008 and estimation from ARMA, GARCH, EGARCH and GJR



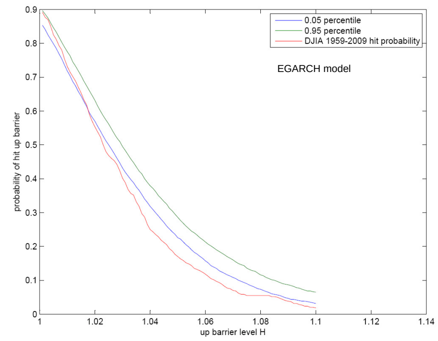


Figure 9: Probability of touching up barrier for DJIA 1959-2009 and estimation from ARMA, GARCH, EGARCH and GJR

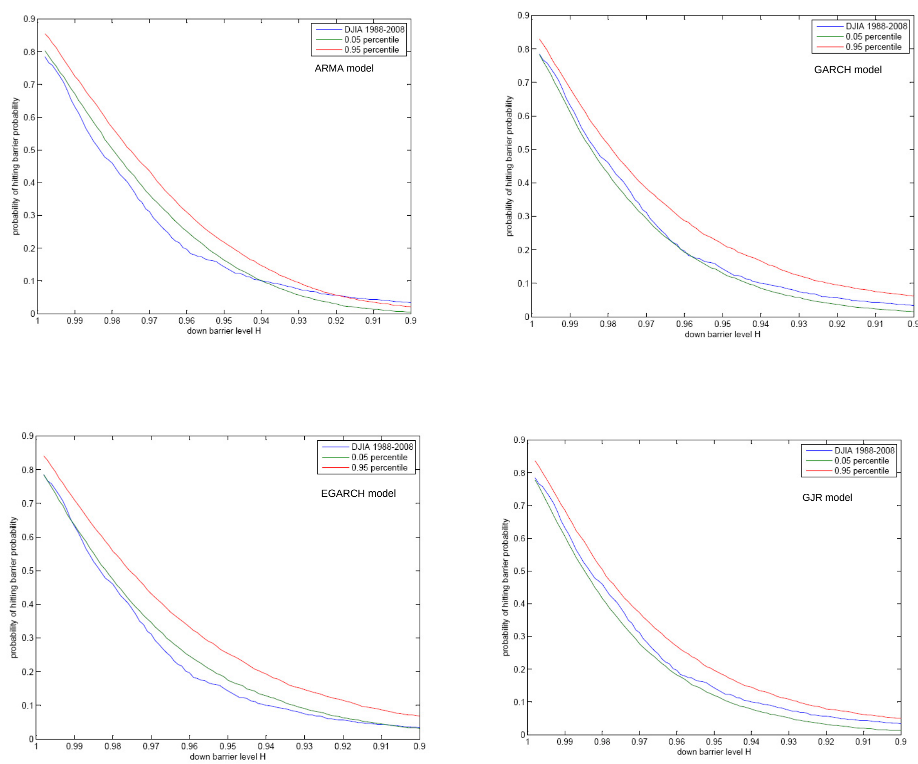


Figure 10: Probability of touching down barrier for DJIA 1959-2009 and estimation from ARMA, GARCH, EGARCH and GJR

Index	Estimates within ±5% confidence interval
Nikkei (49 years)	99%
Topix (49 years)	82%
S&P 500 (45 years)	99%
Dax (44 years)	90%
DOW JONES COMPOSITE 65 STOCK AVE	97%
DOW JONES UTIILITIES	68%
NYSE COMPOSITE	98%
BANGKOK S.E.T.	82%
Hangseng	49%
KLCI COMPOSITE	83%
KOSPI	88%
MSCI EAFE U$	91%
MSCI WORLD U$	49%
Nasdaq	20%
S&PTSX COMPOSITE INDEX	98%
Taiwan	70%
Average	78.9%

Table 2: Quality of GJR predictions of down barrier quotients (100% to 90% for one month options) for all indices with at least 30 years of data. A maximum of 50 years of data has been used.

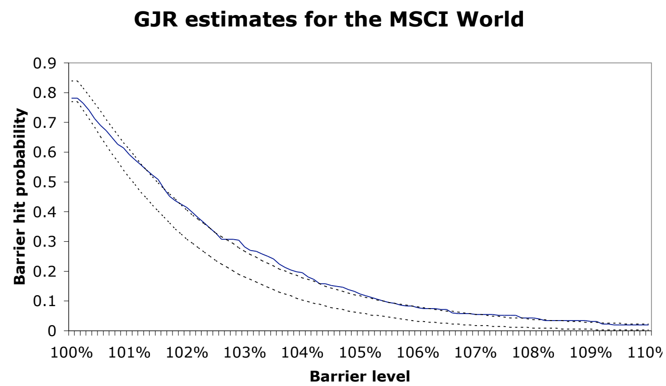


Figure 11: Down barrier hit probabilities for the MSCI World (1959-2009) as compared to the predictions by GJR (±5% confidence band as dotted lines).

5.2 Independent probability as benchmark

For independent probabilities, as mentioned above, our main focus is on 1 month options for the DJIA 1959-2009. We plot the percentiles of simulations and the real barrier hit probability line in figure 9 and figure 10.

For independent probabilities, the ARMA model produces too optimistic probabilities of touching the barrier for large down barriers and for almost all up barriers.

The GARCH model overestimates the probability for almost all up barriers, but predicts the probability for down barriers very well.

The probability to hit a barrier generated from EGARCH is too large for barriers higher than 102%. For barriers lower than 99%, EGARCH also overestimates the probability to hit a barrier.

The GJR model is also the most appropriate model for independent probabilities. The DJIA line is between 5th and 95th percentile lines for down barriers. As for up barriers, the DJIA line is almost identical to the 5th percentile of the GJR simulations. It is relatively hard to see in figure 9 whether GJR fits the up barrier well or not.

Even though the GJR model predicts the probability to hit a down barrier for the DJIA well, further tests of the GJR model for other market indices in the case of down barriers show that the model does not always perform as well, but does deviate rarely much: on average 78.9% of the empirical data for down barriers between 90% and 100% (1 month-option) lies inside the $\pm 5\%$ confidence intervals, compare Table 2 for details. However, sometimes there are discrepancies, as in the case of the MSCI World where the barrier hit probability is often underestimated by GJR(compare Figure 11).

6 Conclusions

When considering aspects of barrier options rather than their pricing, in particular their attractiveness for investors, we need to estimate the probability with which a barrier is hit before maturity of the option. Misestimating this probability is likely a cause why many retail investors find structured products with barrier features attractive. Particularly important is the ratio between the probability to hit a barrier and the probability to still be below this barrier at maturity, the "barrier quotient".

Estimating historical probabilities and barrier quotients for various stock indices reveals a strong increase of the quotient for low barrier levels, particularly in down barriers. This contradicts theoretical computations based on Bachelier or Black-Scholes models, but also simulations based on models like ARMA, GARCH, EGARCH and GJR. The solution of this at first glance puzzling discrepancy lies in the definition of historical probabilities. It turns out that by distinguishing between two seemingly similar definitions the puzzle can be resolved.

Having these two approaches of computing the probability to hit a barrier at hand, we can use them as a new benchmark to test time series models. The results indicate that ARMA, GARCH and EGARCH models are appropriate to estimate barrier hit probabilities for the DJIA in some cases, but fail in other situations. The GJR model is the most appropriate model, but even its performance is limited when tested with data from some other indices. In conclusion, none of the studied models can predict all probabilities well. This demonstrates that considering the probabilities of hitting a barrier could be used as a powerful test of the performance of stochastic models for stock market returns.

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A Mathematical Proofs

A.1 Proof of Theorem 2.2

Theorem 2.2 Given that $S(t)$ follows a standard Brownian motion, for every barrier level $H \geq 0$

$$P(\tau \leq T) = 2P(S(T) \leq H) = 2\frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{H} e^{-\frac{x^2}{2t}}\, dx.$$

Proof. We know that the conditional distribution of $S(T-\tau+\tau)-S(\tau)$ given that $\tau = s$ is Gaussian with mean 0 and variance $T-s$. By the symmetry of the Gaussian distribution,

$$P(S(T-\tau+\tau)-S(\tau) \geq 0|\tau = s) = P(S(T-\tau+\tau)-S(\tau) \leq 0|\tau = s) = 1/2.$$

Integrate over $0 \leq s \leq T$ against the distribution of τ, to obtain

$$P(S(T) - S(\tau) \leq 0 \text{ or } \tau \leq T) = \frac{1}{2}P(\tau \leq T).$$

Note that the event $\{S(T) - S(\tau) \leq 0 \text{ or } \tau \leq T\}$ coincides with the event $\{S(T) \leq H \text{ or } \tau \leq T\}$. Since the event $\{S(T) \leq H\}$ belongs to the event $\{\tau \leq T\}$, we have

$$P(S(T) \leq H \text{ or } \tau \leq T) = P(S(T) \leq H) = \frac{1}{2}P(\tau \leq T).$$

Hence,

$$\frac{P(\tau \leq T)}{P(S_T \leq H)} = 2.$$

□

A.2 Probability of touching a barrier

This appendix shows the computation of the probability to touch a down barrier. The computation of the probability to hit a up barrier is analogous. We know that

$P(\tau \leq T) = P(\tau \leq T \text{ and } S_T \geq H) + P(\tau \leq T \text{ and } S_T \leq H) = P(\tau \leq T \text{ and } S_T \geq H) + P(S_T \leq H)$. Furthermore the first term can be written in terms of indicator function,

$$\begin{aligned} P(\tau \leq T \text{ and } S_T \geq H) &= E^P[1_{\{S_T \geq H\}} 1_{\{\tau \leq T\}}] \\ &= E^P[1_{\{S_T \geq H\}} 1_{\{\tau \leq T\}} \exp{(-\theta w_T - \frac{1}{2}\theta^2 T)} \exp{(\theta w_T + \frac{1}{2}\theta^2 T)}] \end{aligned}$$

If we define a new Brownian motion $\bar{w}_t = w_t + \theta t$, where $\theta = \frac{\mu}{\sigma}$, by Girsanov's Theorem, there exists a new measure $P^Q(A) = E^P[1_A \exp(-\theta w_t - \frac{1}{2}\theta^2 t)]$. Under the new probability measure, the underlying price S_t follows $dS_t = \sigma S_t d\bar{w}_t$. Thus $d\ln S_t = \sigma d\bar{w}_t$ and therefore

$$
\begin{aligned}
P(\tau \le T \text{ and } S_T \ge H) &= E^Q[1_{\{S_T \ge H\}} 1_{\{\tau \le T\}} \exp(\theta \bar{w}_T - \frac{1}{2}\theta^2 T)] \\
&= E^Q[1_{\{\ln(S_T) \ge \ln(H)\}} 1_{\{\tau \le T\}} \exp(\theta \bar{w}_T - \frac{1}{2}\theta^2 T)] \\
&= E^Q[1_{\{\bar{w}_T \ge \frac{\ln(H)}{\sigma}\}} 1_{\{\tau \le T\}} \exp(\theta \bar{w}_T - \frac{1}{2}\theta^2 T)].
\end{aligned}
$$

By reflection principle, if we define $\tilde{H} = \frac{\ln(H)}{\sigma}$, we obtain

$$
\begin{aligned}
P(\tau \le T \text{ and} S_T \ge H) &= E^Q[1_{\{\bar{w}_T \le \tilde{H}\}} \exp(\theta(2\tilde{H} - \bar{w}_T) - \frac{1}{2}\theta^2 T)] \\
&= \frac{e^{2\theta\tilde{H}}}{\sqrt{2\pi T}} \int_{-\infty}^{\tilde{H}} \exp(-\theta \bar{w}_T - \frac{1}{2}\theta^2 T) \exp(-\frac{\bar{w}_T^2}{2T}) d\bar{w}_T \\
&= \frac{e^{2\theta\tilde{H}}}{\sqrt{2\pi T}} \int_{-\infty}^{\tilde{H}} \exp\{-\frac{(\bar{w}_t + \theta T)^2}{2T}\} d\bar{w}_T \\
&= e^{2\theta\tilde{H}} \Phi(\frac{\tilde{H} + \theta T}{\sqrt{T}}) \\
&= \exp(\frac{2\mu \ln(H)}{\sigma^2}) \Phi(\frac{\ln(H) + \mu T}{\sigma\sqrt{T}}).
\end{aligned}
$$

Combined with the probability that stock price ends below barrier, we have,

$$
\begin{aligned}
\frac{P(\tau \le T)}{P(S_T \le H)} &= 1 + \frac{P(\tau \le T \cap S_T \ge H)}{P(S_T \le H)} \\
&= 1 + \frac{\exp(\frac{2\mu\ln(H)}{\sigma^2}) \Phi(\frac{\ln H + \mu T}{\sigma\sqrt{T}})}{\Phi(\frac{\ln(H) - \mu T}{\sigma\sqrt{T}})}.
\end{aligned}
$$

□

A.3 Proof of lemma 2.1

From the expression of the ratio we derived in appendix A.2, we know that to prove lemma 2.1 is equivalent to prove that

$$\lim_{H\to 0} \frac{\exp\left(\frac{2\mu\ln(H)}{\sigma^2}\right)\Phi\left(\frac{\ln(H)+\mu T}{\sigma}\right)}{\Phi\left(\frac{\ln(H)-\mu T}{\sigma}\right)} = 1 \tag{1}$$

If we define $x = -\ln(H)$, equation (1) becomes

$$\lim_{x\to\infty} \frac{\exp\left(\frac{-2\mu x}{\sigma^2}\right)\Phi\left(\frac{-x+\mu T}{\sigma\sqrt{T}}\right)}{\Phi\left(\frac{-x-\mu T}{\sigma\sqrt{T}}\right)} = 1 \tag{2}$$

As $x \to \infty$, both the denominator and numerator go to 0. Applying L'Hospital Principle, we obtain

$$\begin{aligned}
\lim_{x\to\infty} \frac{\exp\left(\frac{-2\mu x}{\sigma^2}\right)\Phi\left(\frac{-x+\mu T}{\sigma\sqrt{T}}\right)}{\Phi\left(\frac{-x-\mu T}{\sigma\sqrt{T}}\right)} &= \lim_{x\to\infty} \frac{-\frac{2\mu}{\sigma^2}e^{\left(\frac{-2\mu x}{\sigma^2}\right)}\Phi\left(\frac{-x+\mu T}{\sigma\sqrt{T}}\right) - \frac{1}{\sigma\sqrt{2\pi T}}e^{\left(\frac{-2\mu x}{\sigma^2}\right)}\exp\left(-\frac{(-x+\mu T)^2}{2\sigma^2 T}\right)}{-\frac{1}{\sigma\sqrt{2\pi T}}\exp\left(-\frac{(-x-\mu T)^2}{2\sigma^2 T}\right)} \\
&= \lim_{x\to\infty} \frac{\frac{2\sqrt{2\pi T}\mu}{\sigma}\Phi\left(\frac{-x+\mu T}{\sigma\sqrt{T}}\right)}{e^{-\frac{(-x+\mu T)^2}{2\sigma^2 T}}} + 1.
\end{aligned}$$

Denoting $s = \frac{-x+\mu T}{\sigma\sqrt{T}}$, the first term becomes

$$\begin{aligned}
\lim_{s\to-\infty} \frac{\frac{2\mu}{\sigma\sqrt{2\pi}}\int_{-\infty}^{s} e^{-\frac{t^2}{2}}\,dt}{\frac{1}{\sqrt{2\pi T}}e^{-\frac{s^2}{2}}} &= \lim_{s\to-\infty} \frac{\frac{2\mu\sqrt{T}}{\sigma}e^{-\frac{s^2}{2}}}{-se^{-\frac{s^2}{2}}} \\
&= \lim_{s\to-\infty} \frac{\frac{2\mu\sqrt{T}}{\sigma}}{-s} = 0.
\end{aligned}$$

This completes the proof. □

B List of Indices

RUSSELL 2000	DJ EURO STOXX 50	DJ COMPOSITE 65 STOCK AVE
MSCI WORLD	FTSE ALL SHARE	DAX 30 PERFORMANCE
NASDAQ 100	SP 500 COMPOSITE	DJ UTIILITIES
FTSE 100	NYSE COMPOSITE	DJ WILSHIRE 5000 COMPOSITE
BANGKOK S.E.T.	NASDAQ COMPOSITE	TAIWAN SE WEIGHTED
OMXS30	FRANCE CAC 40	DOW JONES INDUSTRIALS
IBEX 35	COLOMBO SE ALL SHARE	MADRID SE GENERA
OMXS	PHILIPPINE SE I	MSCI PACIFIC U
S&P EURO	MEXICO IPC(BOLSA)	IRELAND SE OVERALL
TOPIX	ISRAEL TA 100	KOREA SE COMPOSITE
AEX INDEX	KLCI COMPOSITE	NIKKEI 225 STOCK AVERAGE
HANG SENG	TOPIX SECOND MARKET	INDIA BSE (100) NATIONAL
CHILE GENERAL	DJ STOXX 600 E	JAKARTA SE COMPOSITE
OMX HELSINKI	MDAX FRANKFURT	AUSTRIAN TRADED INDEX
MSCI EAFE U	S&P/TSX 60 INDEX	S&P/TSX COMPOSITE INDEX
MSCI EUROPE U		

Table 3: List of indices which have been tested in Section 3.3.