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VOLATILITY TRADING

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A contract based on the volatility of an asset could provide a useful addition to the range of derivative securities available to traders. This paper describes how such a contract could be designed which does not require the explicit calculation of actual or implied volatility. It makes use of a contingent claim whose pay-off depends on the logarithm of the terminal price of the asset. By buying and delta-hedging such a claim, a trader can achieve a pay-off which is directly related to actual volatility. This strategy is shown to be superior to buying and delta hedging a conventional call option.

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The thesis of this paper is that in many markets there is demand for a contract whose pay-off is related directly to the volatility prices. Existing derivative products, notably options, provide an imperfect means of hedging volatility. A contract whose pay-off is directly calculated on the basis of actual elapsed volatility could be designed, but it may face certain technical problems. An alternative is proposed which does not suffer from these difficulties.

There are many circumstances in which a forward contract on volatility might be of value to traders in a market. Brenner and Galai (1989) discuss the demand for a contract on volatility, and suggest a number of ways in which it could be useful to traders. A trader who writes options on an asset can hedge out the risk of changes in the value of the asset by delta hedging. He is then indifferent to the direction in which the asset price moves, but he has sold volatility. If volatility turns out higher than expected, he will generally lose money. A volatility forward contract would enable him to immunise his portfolio against unpredicted changes in volatility. Similarly, a fund manager who wishes to insure some floor value to an equity portfolio can delta hedge the portfolio using index futures. But the effectiveness of the insurance depends on correctly predicting volatility. Again, a contract on volatility would enable him to hedge this risk. A speculator who has a view about volatility can take a position by buying an option and delta hedging it. His gain or loss will depend in part on the degree to which the actual volatility differs from the implied volatility in the option price. However, this is an imperfect way of betting on volatility since the profit or loss will also be affected by jumps in the price of the asset, and on the relation between the time path of volatility and the asset price.

There may well then be some demand for a contract whose pay-off is directly related to actual volatility. One might ask why no such contract currently trades on an organised exchange. [Evidence on what has actually been tried]. In principle, it would not be hard to design a volatility future or forward contract. For example, one could have a forward contract where the asset to be delivered is a sum of money equal to the annualised standard deviation of daily log returns over the following three months.

Returns would have to be calculated according to some daily reference price of the asset concerned, such as the closing price.

The problem with such a contract is two-fold. First, it would involve a specific definition of volatility, based in this case on daily close-to-close returns. This may restrict interest, excluding traders who are concerned about volatility calculated on the basis of shorter or longer period returns, or who do not or cannot trade at closing prices. A second problem concerns the degree to which such a contract could be manipulated.

All financial markets are open to manipulation to some degree. But for markets which are not overly susceptible to rumour, the main way open to a trader to move prices is to trade. If a trader takes a long position in an asset and then wants to close out at an artificially high price, he would have to buy more of the asset. Unless he can find some way of buying so that he moves the price up, and then selling without depressing the price by a similar amount, he will gain nothing from his attempt to manipulate the market. The same analysis holds with forward and futures contracts provided that the derivative and spot markets are reasonably well integrated, so that a trade in the forward market affects the underlying price in much the same way as a trade in the spot market. However the analysis does not hold true for volatility contracts. A trader who is long a volatility contract can buy the underlying near the close on a day when it has risen, and sell the following day, thus exaggerating the size of the day's move. He may face some cost in doing this, but provided that he has a sufficiently large volatility position, he will make money. [A model formalising this argument is at Annex A].

These problems may explain why no contract on actual volatility currently exists. In this paper, an alternative approach is put forward. It requires the creation of a claim which has a pay-off which is fully determined by the terminal price of the asset. By buying this claim and hedging suitably, a trader can ensure a pay-off which is linear in the squared out-turn volatility. To understand the nature of this claim and why it works, it is useful to begin by considering why it is not possible to get the same result by buying and delta-hedging a conventional call option.

So in the first section of this paper the behaviour of a portfolio consisting of a delta-hedged contingent claim is described. It is shown that such a portfolio does not in general provide a satisfactory hedge against volatility because the sensitivity of the portfolio to volatility varies in a stochastic fashion. This suggests that what is needed is a contingent claim whose sensitivity to volatility is constant. We characterise such a claim and show that, when delta-hedged, it has a pay-off which depends only on actual volatility. We then go on in Section II to consider how the strategy is affected if rebalancing is done at discrete intervals. Some empirical evidence on the behaviour of the strategy is presented in section III.

I. Delta-hedging a Claim

Consider a trader who wishes to take a long position in volatility, so that his profit is positively related to the future level of volatility. One thing he could do is to buy a contingent claim at time 0 with maturity T and delta hedge his position by dynamically trading in the forward market in an appropriate fashion. The claim could for example be a call option, but our analysis will hold for any derivative claim whose value depends solely on the asset price at time T.

It simplifies the analysis considerably if all cash flows take place at one point in time, at time T. So we assume that the claim is paid for at time T, and the trader hedges by entering into forward contracts on the risky asset where delivery and payment take place at time T. The volatility to be hedged is the volatility of the forward price of the risky asset. If the riskless interest rate is non-stochastic, the volatility of the forward price and of the spot price are identical. For most assets one is likely to be interested in, the volatility of the forward price and the volatility of the spot price are unlikely to differ greatly. In any case, for many purposes, it is the volatility of the forward price rather than of the spot price which one would wish to hedge (see for example Merton's analysis of option pricing in the presence of stochastic interest rates).

Denote the forward price of the underlying asset at time t by S(t). We assume that the forward price follows a diffusion process:

$$\frac{d\tilde{S}}{S} = \mu(t)\,dt + \sigma(t)\,d\tilde{z} \qquad (1)$$

where dz is a standard Brownian increment. Clearly, if the trader knows the parameters of the diffusion process then there is no need for a volatility contract. So we do not assume that the functions μ and σ are known to the trader. All we assume is that the volatility is strictly positive, bounded away from zero and that the integral of $\sigma^2(t)$ exists.

Note that our assumption about the price process is considerably weaker than the traditional assumption about known and constant volatility. Even if the actual asset price $\{S(t)|0<t<T\}$ were fully specified, the only way in which our assumption could be falsified would be if the asset price were to jump or if the limit of the sum of squared log returns did not exist.

The trader does not know the true path of volatility. But to delta hedge his position, he needs to make some assumption about the path of volatility. He assumes that at time t volatility is $\hat{\sigma}(t)$. He buys a contingent claim. The contingent claim is defined by its terminal value, which is some given function F(S(T)) which depends only on the spot asset price at maturity. If the claim is, for example, an at-the-money call option, F(S) = Max{S-S(0),0}. On the assumption that the trader has correctly predicted the level of volatility, the claim can be valued using arbitrage arguments in the conventional way. Its value at time t is given by C(S(t),t), where C satisfies the partial differential equation:

$$\frac{1}{2}\hat{\sigma}^2(t)S^2C_{ss} + C_t = 0 \qquad (2)$$

subject to the boundary condition that:

$$C(S,T) = F(S)$$

(there is no term involving the riskless interest rate since all prices are forward prices).

The delta hedging strategy requires the trader to short $C_s(S,t)$ forward contracts at time t. If the trader starts with zero wealth, then at time T his terminal wealth W is.

$$\tilde{W} = -C_0 + F(\tilde{S}(T)) + \tilde{W}_H \tag{3}$$

where C_0 is the price of the claim, F(S(T)) is the pay-out on the claim, and W_H is the profit on the delta hedge.

At time t, the trader is short C_s forward contracts. If the forward price rises by dS, the trader loses C_sdS. So:

$$\tilde{W}_H = \int_0^T -C_s dS \tag{4}$$

But S satisfies the stochastic differential equation (1), so by Ito's Lemma:

$$d\tilde{C} = \left(\frac{1}{2}\sigma^2(t)S^2C_{ss} + C_t\right)dt + C_s d\tilde{S} \tag{5}$$

Substituting equation (5) into equation (4):

$$\begin{aligned} \tilde{W}_H &= \int_0^T \left\{\left(\frac{1}{2}\sigma^2(t)S^2C_{ss} + C_t\right)dt - d\tilde{C}\right\} \\ &= \int_0^T \left\{\frac{1}{2}S^2C_{ss}\left(\sigma^2(t) - \hat{\sigma}^2(t)\right)dt\right\} + C(S(0),0) - C(S(T),T) \end{aligned} \tag{6}$$

since C satisfies the partial differential equation (2). The terminal wealth from equation (3) is:

$$\begin{aligned} \tilde{W} &= \{C(S(0),0) - C_0\} + \int_0^T \{X(S,t)(\sigma^2(t) - \hat{\sigma}^2(t))dt\} \\ &\text{where:} \\ X(S,t) &= \frac{1}{2}S^2C_{ss}(S,t) \end{aligned} \tag{7}$$

The terminal wealth therefore has two components. The first is the difference between what the trader pays for the claim, and what he believes it is worth. The second component is proportional to the weighted mean difference between the out-turn squared volatility and the trader's forecast. The weighting is given by the function X(S,t), which in general depends on both S and t. The terminal wealth depends not only on whether out-turn differs from forecast volatility, but also on the path of volatility, and the path of the forward price.

It is worth noting parenthetically that in the special case where $\hat{\sigma} = \sigma =$ constant, $X(S,t) = -C_t/\sigma^2$. X must therefore satisfy the partial differential equation (2). It is therefore a martingale.

Now consider the case where the claim being hedged is an at the money call option, and the trader is forecasting constant volatility. Then:

$$X(S,t) = \frac{1}{2}S^2C_{ss} = \frac{SZ(d_1)}{2\hat{\sigma}\sqrt{T-t}}$$

where Z(.) is the standard normal density function, and:

$$d_1 = \frac{\mathrm{Log}(S/S(0)) + \frac{1}{2}\hat{\sigma}^2(T-t)}{\hat{\sigma}\sqrt{T-t}}$$

As t tends to T, X(S,t) tends to infinity or zero depending on whether the option ends up at or away from the exercise price. If, for example, the option ends up close to the money, the pay-off to the trader will be highly sensitive to actual volatility over the later part of the period. If it ends up away from the money, the volatility in the early part of the period will have a more important influence. One might conjecture that a delta hedged call option is an unsatisfactory way of betting on volatility, and evidence presented later in this paper confirms the intuition.

To get a good volatility hedge, we need to find a contingent claim where $X(S,t)$ is constant. One such claim[1], which we call the Log Claim, has:

$$F(S) = \text{Log}\left(\frac{S(0)}{S}\right) \tag{8}$$

Solving equation (2) for the option value at time t:

$$C(S,t) = \frac{1}{2}\int_t^T \hat{\sigma}^2(x)\,dx + \text{Log}\left(\frac{S(0)}{S}\right) \tag{9}$$

The terminal wealth in a delta-hedged Log Contract portfolio is:

$$\tilde{W} = \frac{1}{2}\int_0^T \sigma^2(t)dt - C_0 \tag{10}$$

The terminal wealth depends only on the out-turn volatility and the price of the Log Contract. The existence of a Log Contract would enable a trader to construct a synthetic forward contract on volatility. The market price of the Log Contract can be seen as being akin to a forward price of volatility.

The Log Contract has a pay-off which is proportional to the continuously compounded rate of return on the forward price from time 0 to time T. This can be contrasted with an ordinary forward contract whose pay-off is proportional to the total (uncompounded) return. The Log Contract's value at time 0 must be positive, but subsequently its value can go negative if the forward price is sufficiently high, as can be seen from equation (9). If it were desirable to have a claim whose value was never negative, the Log Contract could be packaged with $1/S(0)$ forward contracts. The relation between the Log Contrcact, the Forward Contract and the package is shown in Graph 1. As can be seen, the package has a terminal pay-off which is reminiscent of the pay-off on a straddle. This is appealing since the simplest way of betting on volatility

[1] It is in fact unique in that any other claim which satisfies this condition is a linear combination of the log claim, a forward contract and a discount bond.

with a static portfolio strategy is by buying a straddle.

To give further insight into the reason the Log Contract is effective at hedging volatility, note from equation (9) that the hedge ratio, C_s, is just -1/S. This is independent of the trader's beliefs about volatility. The hedge works perfectly because the trader delta hedges in precisely the same way whether he predicts volatility correctly or not.

The existence of a Log Contract enables a trader to construct a dynamic portfolio strategy whose pay-off is linearly related to the mean squared out-turn volatility over the life of the contract. But this analysis has all been in continuous time. It has assumed that the asset price does not jump. In the following section, the behaviour of the volatility hedge with discrete rebalancing is investigated. It is shown that the delta-hedged Log Contract has a pay-off which is closely related to actual volatility as measured by returns over each rebalancing interval.

II. Discrete Time Model

Partition the period [0,T] into N equal periods of length δt (where δt=T/N). Let the log return in the i'th period be denoted by r_i, so:

$$r_i = \mathrm{Log}\left(\frac{S(i\,\delta t)}{S((i-1)\,\delta t)}\right)$$

The hedge ratio is C_s, which is -1/S. So to delta hedge his position, the trader must be long 1/S forward contracts. The profit from trading in the forward market over the entire period is therefore:

$$\begin{aligned} W_H &= \sum_{i=1}^{N} \frac{S(i\,\delta t) - S((i-1)\,\delta t)}{S((i-1)\,\delta t)} \\ &= \sum_{i=1}^{N} \left(e^{r_i} - 1\right) \end{aligned}$$

The pay-out to a long position in the Log Contract is:

$$F(S(T)) = \mathrm{Log}\left(\frac{S(0)}{S(T)}\right)$$
$$= -\sum_{i=1}^{N} r_i$$

From equation (3), the terminal wealth in the portfolio is given by:

$$W = \sum_{i=1}^{N}\left(e^{r_i} - 1 - r_i\right) - C_0$$

This can be rewritten as:

$$W = \frac{T}{2}s^2 - C_0 + \sum_{i=1}^{N}\left(e^{r_i} - 1 - r_i - \frac{1}{2}r_i^2\right) \tag{11}$$

where s^2 is the second moment of log returns around the origin expressed at an annualised rate.

If the forward price process is Brownian, it is straightforward to demonstrate that the term in curly brackets in equation (11) vanishes with probability 1 as the differencing interval tends to zero, and the terminal wealth becomes equal to the mean squared volatility times T/2, less the initial price of the Log Claim. But our concern now is with the behaviour of the hedge in practice, with real forward prices and discrete rebalancing.

The definition of out-turn volatility given returns over a finite number of periods now needs to be addressed. One natural candidate is the sample standard deviation of log returns. This is the maximum likelihood estimate of the population standard deviation, given that log returns are normally distributed with unknown mean[2]. In what

[2] The sample variance is a biassed estimate of the population variance. One might therefore wish to correct for degrees of freedom, but since the bias corrected estimate is simply proportional to the uncorrected estimate, the results presented below are not significantly affected by which is chosen.

follows, we will take as one of our definitions of elapsed volatility the population standard deviation of log returns, expressed as an annualised rate, and this we refer to as σ.

However an argument can be made for using the second moment around the origin, s, as a better definition of volatility. Two reasons can be given for this. First, we do have strong priors about the expected log return (recalling that returns are measured on the forward price, and hence represent excess returns on the spot asset over the riskless rate). If for example we observed log returns over just two successive days, and they were 2% on each day, it would be more plausible to suppose that they are distributed with zero mean and a standard deviation of 2% than assuming that the mean is 2% daily and the standard deviation is zero. Taking an extreme view, and supposing we knew that the mean were zero, the maximum likelihood estimate of the volatility would then be s, the second moment around the origin.

A second reason for being interested in s rather than σ is that in the derivation of the partial differential equation (2) which the option price satisfies, the term which is approximated by $\sigma^2 \delta t$ is $(\delta S/S)^2$, so it could be argued that this is the quantity which should be hedged.

We take no view on which definition of elapsed volatility, σ or s, is more appropriate. In the next section, we present some evidence to show how effective a delta-hedged Log Contract would be in practice in creating a synthetic elapsed volatility contract, using both definitions of elapsed volatility.

III. The Volatility Contract in Practice

To evaluate the effectiveness of the volatility contract we run the following experiment: we use as our risky asset the basket of shares in the UK FT-SE 100 index. We envisage a trader with zero initial wealth. He buys Log Contracts on the index with a maturity of 30 trading days. He hedges his position by going long the appropriate amount of the index. Every trading day, he rebalances his hedge at the closing price.

After 30 trading days, the position is closed out and the terminal value of the position is calculated. The experiment is run over 32 consecutive 30 day periods running from 6 January 1986 to 31 October 1989.

To provide some basis for comparison, we also examined the performance of a delta hedged option position. Here the trader, starting with zero wealth, buys an at-the-money thirty day call option. He then delta hedges (using the Black-Scholes formula to calculate the appropriate hedge ratio) by going short the appropriate number of forward contracts[3].

In order to calculate the initial price of the two contracts, and to calculate the appropriate hedge ratios for the call strategy, we must make some assumption about the market's view of future volatility. For simplicity we assume that the market assumes that volatility is constant at a level equal to the average over the period of 13%.

If the strategies work perfectly, the terminal pay-off should be proportional to the difference between the squared out-turn volatility, and the predicted volatility. To ensure that the two strategies are directly comparable, the size of the initial purchase of the contingent claim in each case is chosen to make the constant of proportionality one. This means going long 2/T Log Contracts, and $\frac{2\hat{\sigma}}{S(0)Z(d_1)\sqrt{T}}$ calls (making use of the result that X(S,t) is a martingale if volatility is known and constant).

So one would expect the terminal wealth in the portfolio to be:

[3] Note that both the Log Contract and a normal call option have a positive second derivative with respect to the forward price of the asset, so in both cases going long a claim constitutes buying volatility. But the Log Contract has a negative first derivative whereas the call has a positive first derivative with respect to the forward price, so delta hedging a long position in the Log Contract requires the trader to go long the forward contract, whereas he has to short the forward contract to delta hedge a call.

$$\tilde{W} = \sigma^2 - \hat{\sigma}^2 + \tilde{\varepsilon}$$

where σ is the out-turn volatility, $\hat{\sigma}$ is the predicted volatility and $\tilde{\varepsilon}$ is an error term. The larger the actual volatility, the larger the likely error term. To avoid heteroscedasticity, the regression equation we estimate is:

$$\frac{\tilde{W}_i}{\sigma_i^2} = \alpha + \frac{\beta}{\sigma_i^2} + \tilde{\eta}_i$$

where the i subscripts refer to the period. Ignoring small sample effects, α should be equal to unity, and β should be equal to $-\hat{\sigma}^2$, or -0.0169. The standard error of the regression is the hedge error expressed as a proportion of out-turn volatility.

The results are presented in Table 1. The out-turn volatility is defined to be the sample standard deviation of log returns. Looking at Panel A, it can be seen that, for both strategies, the slope and intercept are consistent with our priors. It is also clear that the Log Contract easily dominates the call contract with an R-bar squared of over 0.94 against 0.58, and a residual error of 11% against 29%.

The regression results can be interpreted in the following way. Suppose that the implied volatility over the next month as reflected in the prices of Log Contracts and call options is 13%, but the trader believes that it will turn out to be higher. He decides to set up a position whose pay-off is designed to be $\$(100\sigma)^2$-169. So if for example actual volatility turns out to be 20%, he would make \$(400-169), or \$231. Using the log contract, the standard error on the strategy would be 11% of \$400, or \$44, whereas using a call option, the standard error would be 29% of \$400, or \$116.

The sample period covers the crash of October 1987. The two periods which include and immediately follow the crash show unusually high volatility. Separate regressions (not reported here) were run excluding these two periods, but the results were not significantly altered.

Further insight into the causes of the error in the Log Contract hedge can be gained from Panel B where the definition of actual volatility is based on the second moment around the origin (s) rather than the sample standard deviation (σ). As can be seen, the performance of the Log Contract against this benchmark shows a dramatic improvement, with the standard error reduced to 0.4%, even when the Crash periods are included. There is no significant change in the performance of the call strategy.

The tests so far could be argued to be biassed against the call option strategy because it assumes that the best forecast of volatility is some long-term average. If volatility can be forecast more accurately, then the trader could calculate better delta hedge ratios, and the performance of the call option strategy would improve. In the extreme case where volatility is predicted perfectly, the call option strategy should start to approach the efficacy of the Log Contract strategy.

We therefore rerun the experiment, but this time assume that the market can forecast the actual volatility perfectly. The claims are priced, and the call option is delta-hedged, using the actual out-turn volatility (σ) for the period. One would now expect the terminal wealth to be uncorrelated with actual volatility, and the intercept to be zero. The results in Table 2 show that this is indeed the case. The results should be compared with the top panel of Table 1. The standard error on the Log Contract strategy is not affected (unsurprisingly since the trader's strategy is not determined by his forecast of volatility), but the Call option strategy is improved, with the standard error down from 29% to 15%.

IV Conclusions

In this paper we have shown how the creation of a new contingent claim, the Log Claim, whose pay-off depends only on the terminal value of the asset, would enable a trader to construct a portfolio whose performance is directly related to out-turn volatility. The terminal value depends on the sum of squared log returns over the period of the

Log Contract. The same Log Contract can be used to hedge volatility of daily, or weekly returns, or returns over whatever period the trader chooses.

The same approach could be used to hedge volatility if volatility is defined in other ways. Suppose for example that the underlying model of the forward price is that it follows an arithmetic rather than a geometric Brownian process. The appropriate definition of volatility would then be the standard deviation of actual (rather than log) returns. It can readily be seen that the existence of a Squared Contract, whose terminal pay-off is proportional to the square of the asset price at maturity, would enable the trader to construct a portfolio whose terminal value is proportional to the out-turn arithmetic volatility of returns.

The Log Contract's value at inception is directly proportional to squared volatility. If Log Contracts were traded, it would then be possible to construct forwards and options on implied volatility. At time t one could buy a forward Log Contract which starts at time T_1 (>t) and matures at T_2. The pay-off to the Log Contract would be $Log\{S(T_1)/S(T_2)\}$. The implied volatility being traded is the implied volatility at time T_1 for the period (T_1,T_2). Similarly, an option on the forward contract would provide an option on implied volatility.

FORECAST VOLATILITY = 13%

$$\frac{\tilde{W}_i}{\sigma_i^2} = \alpha + \frac{\beta}{\sigma_i^2} + \tilde{\eta}_i$$

Heteroscedastic-consistent T - Statistics in parentheses against null hypothesis that $\alpha = 1$, and $\beta = -0.0169$

A. Out-turn volatility = Sample Standard Deviation

	α	β	$\bar{R}^2$	S.E.
Log Contract	1.038 (1.06)	-.0165 (0.64)	0.943	10.6%
Call Option	0.724 (-1.38)	-.0131 (1.03)	0.575	29.1%

B. Out-turn volatility = Sample Second Moment

	α	β	$\bar{R}^2$	S.E.
Log Contract	0.0994 (-1.62)	-.0168 (1.64)	0.999	0.4%
Call Option	0.710 (-1.42)	-0.136 (0.84)	0.600	28.3%

TABLE 2

Forecast Volatility = Out-turn Volatility

Out-turn Volatility = Sample Standard Deviation

Heteroscedastic-consistent T - Statistics in parentheses against null hypothesis that $\alpha = \beta = 0$

	α	β	$\bar{R}^2$	S.E.
Log Contract	0.0386 (1.07)	-0.0007 (0.63)	-.0217	10.6%
Call Option	0.0670 (0.65)	-0.0006 (-0.41)	-.0224	16.6%

REFERENCES

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