



TI 2002-068/4

Tinbergen Institute Discussion Paper

Stock Index Volatility Forecasting with High Frequency Data

Eugenie Hol

*Siem Jan Koopman**

*Department of Accounting and Finance, University of Birmingham, and Department of Econometrics,
Free University Amsterdam*

** Tinbergen Institute*

Tinbergen Institute

The Tinbergen Institute is the institute for economic research of the Erasmus Universiteit Rotterdam, Universiteit van Amsterdam and Vrije Universiteit Amsterdam.

Tinbergen Institute Amsterdam

Keizersgracht 482
1017 EG Amsterdam
The Netherlands
Tel.: +31.(0)20.5513500
Fax: +31.(0)20.5513555

Tinbergen Institute Rotterdam

Burg. Oudlaan 50
3062 PA Rotterdam
The Netherlands
Tel.: +31.(0)10.4088900
Fax: +31.(0)10.4089031

Most TI discussion papers can be downloaded at
<http://www.tinbergen.nl>

Stock Index Volatility Forecasting with High Frequency Data

Eugenie Hol and Siem Jan Koopman*

Department of Accounting and Finance, University of Birmingham

Department of Econometrics, Free University Amsterdam

June 25, 2002

Abstract

The increasing availability of financial market data at intraday frequencies has not only led to the development of improved ex-post volatility measurements but has also inspired research into their potential value as an information source for longer horizon volatility forecasts. In this paper we explore the forecasting value of these high frequency series in conjunction with a variety of volatility models for returns on the Standard & Poor's 100 stock index. We consider two so-called realised volatility models in which the cumulative squared intraday returns are modelled directly. We adopt an unobserved components model where actual volatility is modelled as an autoregressive moving average process and an autoregressive fractionally integrated moving average model which allows for long memory in the logarithms of realised volatility. We compare the predictive abilities of these realised volatility models with those of daily time-varying volatility models, such as Stochastic Volatility (SV) and Generalised Autoregressive Conditional Heteroskedasticity (GARCH) models which are both extended to include the intraday volatility measure. For forecasting horizons ranging from one day to one week the most accurate out-of-sample volatility forecasts are obtained with the realised volatility and the extended SV models; all these models contain information inherent in the high frequency returns. In the absence of the intraday volatility information, we find that the SV model outperforms the GARCH model.

JEL classification: C22, C53, G15.

Keywords: ARFIMA, Financial market volatility, GARCH, Realised volatility, Stochastic volatility, Stock index returns, Unobserved ARMA component.

1 Introduction

Spurred by the initial research of Andersen and Bollerslev (1998) and Barndorff-Nielsen and Shephard (2001) high frequency intraday returns are increasingly considered for the purpose of approximating realised volatility. The notion that daily ex-post volatility is better approximated when based on cumulative squared intraday return data is supported by the theory that the measurement noise contained in daily squared returns prevents the observation of the actual volatility process but is reduced as the sampling frequency of the return series from which volatility is calculated is increased¹. As such, it therefore theoretically justifies and extends the earlier work of French, Schwert and Stambaugh (1987), amongst others. Andersen and Bollerslev (1998) also showed that daily Generalised

*Corresponding author: Siem Jan Koopman, Department of Econometrics, Free University, De Boelelaan 1105, NL-1081 HV Amsterdam. Email s.j.koopman@feweb.vu.nl. We would like to thank Marius Ooms for his comments on an earlier draft.

¹See, for example, Andersen, Bollerslev, Diebold and Labys (2001a), Andersen, Bollerslev, Diebold and Labys (2001b), Barndorff-Nielsen and Shephard (2001) and Barndorff-Nielsen and Shephard (2002).

Autoregressive Conditional Heteroskedasticity (GARCH) volatility forecasts of exchange rates, when evaluated against intraday volatility measures, are far more accurate than had been previously assumed. These findings were subsequently confirmed with regard to stock index data by Blair, Poon and Taylor (2001) and Hol and Koopman (2000) who examined the predictive accuracy of out-of-sample volatility forecasts based on GARCH and Stochastic Volatility (SV) models, respectively.

As its value with regard to forecasting evaluation appears to have become generally recognised by now, research attention has shifted more towards the potential gains that might be obtained from using intraday data as an information source for out-of-sample volatility forecasting. Andersen, Bollerslev and Lange (1999) examined whether the modelling of intraday returns could improve daily and perhaps even longer-run volatility forecasts. For forecasting horizons of 1 day, 1 week and 1 month they found that the most accurate GARCH out-of-sample volatility forecasts were always obtained with a 1-hour interval series². Even though their theoretical study had indicated otherwise, empirical results deteriorated as sampling intervals were shortened beyond this 1-hour mark. Rather than increasing the sampling frequency, Blair *et al.* (2001) suggested to incorporate the intraday volatility measure for stock index returns as an explanatory variable in the variance equation of a daily GARCH model. Like Andersen, Bollerslev and Lange (1999) they also observed a considerable improvement in the out-of-sample forecasting performance of the GARCH model. Martens (2001) then compared both GARCH-based methods for two exchange rates. He found that the most accurate intraday GARCH model, which proved to be the model with the highest sampling frequency, could not outperform the daily GARCH model extended with intraday volatility. These studies therefore indicate that intraday return series contain incremental information for longer-run volatility forecasts when used in combination with GARCH models, but thus far the issue has not been researched in the context of SV models. Here we examine whether extension of the SV model with intraday volatility information leads to similar improvements as observed for the GARCH model.

Alternatively, the intraday volatility process can be modelled directly which is reminiscent of the methods adopted for monthly volatility in a number of earlier studies such as those by French *et al.* (1987) and Poon and Taylor (1992). The forecasting performance of these volatility models have been studied by Ebens (1999) and Andersen *et al.* (2001b) who, in order to capture the long memory presumably present in the logarithms of intraday volatility series, prefer to use autoregressive fractionally integrated moving average (ARFIMA-RV) models with which they obtain more accurate forecasts than with daily GARCH models³. Following Barndorff-Nielsen and Shephard (2002) we define an alternative realised volatility model where volatility is modelled as a continuous time series process consisting of independent Ornstein-Uhlenbeck (OU) processes. The resulting model for discrete time intervals is an unobserved components model which consists of independent ARMA components plus an error term and is estimated by casting it into state space form. We refer to this model as the Unobserved Components - Realised Volatility model, or, in short, the UC-RV model.

In this paper we explore the forecasting performance of a number of models for the Standard & Poor's 100 stock index series over the period 6 January 1997 to 29 December 2000. We compare the UC-RV and the ARFIMA-RV model, which we collectively refer to as realised volatility models, with the so-called daily time-varying SV and GARCH volatility models which are also extended to include the intraday volatility measure. Thus far the emphasis in the high frequency volatility literature has been mainly on GARCH and ARFIMA-RV models. Our contribution is that we implement and empirically investigate a wide range of methods for the estimation of volatility and compare the out-of-sample forecasting performance of the various models.

The remainder of this paper is organised as follows. In the next section we introduce the data

²Due to strong periodic intraday patterns the GARCH models are not actually estimated at the higher intraday level, see *e.g.* Andersen and Bollerslev (1997). Instead the parameter estimates of the GARCH models at the higher frequencies are inferred from the temporal aggregation results of Drost and Nijman (1993).

³Fractionally integrated models are also advocated in the context of financial market data by, for example, Andersen, Bollerslev, Diebold and Labys (1999) and Andersen *et al.* (2001a).

together with intraday return based volatility measures. In section 3 we give details of the UC-RV and ARFIMA-RV models and in section 4 the daily time-varying volatility models are described. The forecasting methodology and the evaluation criteria are discussed in section 5 and in section 6 we present both the empirical in-sample and out-of-sample forecasting results. In section 7 we provide a summary and our conclusions.

2 Stock Return Data and Volatility

2.1 Data

The data for our empirical study consists of Standard & Poor's 100 stock index transaction prices during the period 6 January 1997 to 29 December 2000. From the original dataset, which includes prices recorded for every trade, we extract 5-minute interval data as this is the frequency used by Andersen and Bollerslev (1998) to construct their realised intraday volatility measure⁴. For the 5-minute price we take the last transaction price recorded before the relevant time mark and we calculate the 5-minute returns as the difference between successive log prices and express these in percentages, so

$$R_{t,d} = 100(\ln P_{t,d} - \ln P_{t,d-1}), \quad (1)$$

where $R_{t,d}$ denotes the return for intraday period d on trading day t , with $d \geq 1$ and $t = 1, \dots, T$. The New York Stock Exchange (NYSE) opens at 9.30 a.m. and closes at 4.00 p.m. EST. A full trading day therefore consists of 78 intraday returns and one overnight return. The overnight return is then defined in a similar way as the intraday returns, *i.e.*,

$$R_{t,N} = 100(\ln P_{t,0} - \ln P_{t-1,D}), \quad (2)$$

with $D = 78$, so $P_{t-1,D}$ is the 4.00 p.m. price on trading day $t - 1$ and $P_{t,0}$ the 9.30 a.m. price on the following trading day t . For $P_{t,0}$ we always select the first available trading price after 9.29 a.m. Not all 1004 trading days in our sample consist of 79 observations and this is partly attributable to the fact that the NYSE closes early on certain days, such as on Christmas Eve⁵. Other important reasons are lapses in trading and in data reporting. For all these intervals without price quotes we insert zero return values except when the 9.30 a.m. price, which we require for the calculation of the overnight return, is not available⁶. In that case we assume that the first available price observation on day t , denoted by P_{t,d^*} with intraday period $d^* \geq 2$, was observed at 9.30 a.m. and assign a zero return value to R_{t,d^*} .

The daily return series R_t is defined as the first difference between the 4.00 p.m. closing prices on consecutive trading days, again expressed in percentage terms, so

$$R_t = 100(\ln P_t - \ln P_{t-1}), \quad (3)$$

where P_t and P_{t-1} could also be written as $P_{t,78}$ and $P_{t-1,78}$, respectively. In figure 1 we graph the daily return series R_t together with the squared return series R_t^2 over the full sample period and we report the summary statistics of both series in the adjacent table 1. From the graphs we can discern several more volatile periods which occurred towards the end of 1997, during the third quarter of 1998 and at the beginning of 2000. Each of the three largest shocks to the return process took place in

⁴Provided that the asset is sufficiently liquid, the 5-minute frequency is acknowledged as the highest frequency at which the effects of market microstructure biases, such as bid-ask bounces and discrete price observations, are not too distorting. Also see: Andersen *et al.* (2001a), Andersen *et al.* (1999), Ebens (1999) and Andersen and Bollerslev (1997).

⁵Three trading days are missing from our data sample: 11 February 1998, 28 August 1998 and 14 January 1999.

⁶After accounting for early market closures, 559 price notations are "missing" of which 92 are 9.30 a.m. prices.

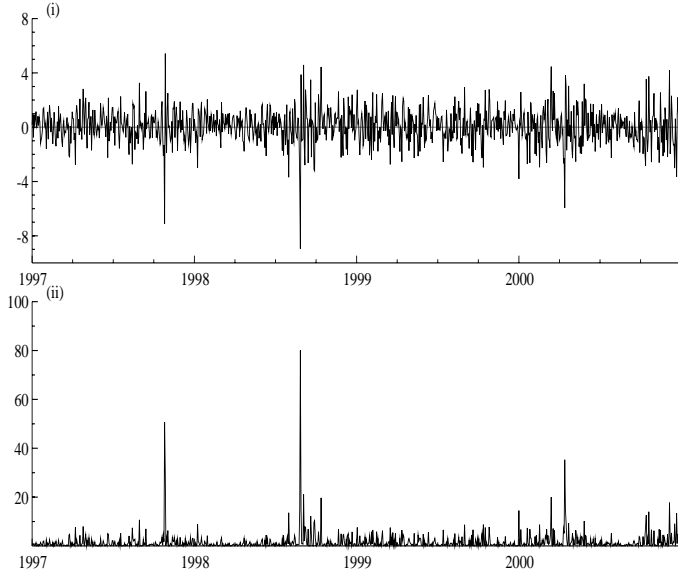


Figure 1: The daily (i) return series R_t and (ii) squared return series R_t^2 of the Standard & Poor's 100 stock index over the period 6 January 1997 to 29 December 2000

Table 1: Summary Statistics R_t and R_t^2

Period T	1997-2000 1004	
Series	R_t	R_t^2
Mean	0.063	1.673
Variance	1.670	16.576
Skewness	-0.447	10.490
Exc.Kurt.	4.024	164.548
Minimum	-8.947	0.000
Maximum	5.427	80.057
$\hat{\rho}_1$	-0.001	0.254
$\hat{\rho}_2$	-0.066	0.059
$\hat{\rho}_3$	-0.037	0.017
$\hat{\rho}_4$	0.010	0.042
$\hat{\rho}_5$	-0.040	0.122
$Q(12)$	21.780	93.224

$\hat{\rho}_\ell$ is the sample autocorrelation coefficient at lag ℓ with asymptotic standard error $1/\sqrt{T}$ and $Q(\ell)$ is the Box-Ljung portmanteau statistic based on ℓ squared autocorrelations.

one of these periods and was negative. This largely contributed to the negative skewness coefficient of -0.447 reported for the return series and the large positive skewness coefficient of 10.490 of the squared returns. We further observe that R_t exhibits excess kurtosis and that none of its first five autocorrelation coefficients is significantly different from zero at the 1% significance level. The Box-Ljung $Q(12)$ statistics indicate that returns are serially uncorrelated, whereas squared returns exhibit a high degree of serial correlation. The highly significant value for the first-order serial correlation coefficient $\hat{\rho}_1$ can be interpreted as an indication of volatility clustering. However, estimates for the subsequent three autocorrelation coefficients are not significant which might well be attributed to the fact that R_t^2 is a noisy estimator of the variance process of R_t .

2.2 Intraday volatility

It has become generally acknowledged that squared daily returns provide a poor approximation of realised daily volatility. It was first pointed out by Andersen and Bollerslev (1998) that more accurate estimates could be obtained with the sum of squared intraday returns⁷. More specifically, they defined realised volatility in the foreign exchange market as the sum of 288 5-minute squared returns. If we were to apply this method directly to the stock market, realised volatility would be defined as the sum of the squared overnight and the cumulative squared 5-minute intraday returns, so

$$\tilde{\sigma}_{t,1}^2 = R_{t,N}^2 + \sum_{d=1}^D R_{t,d}^2, \quad (4)$$

⁷Theoretically, the volatility estimates become free of measurement noise as the sampling frequency interval becomes infinitesimally small; see the references in footnote 1.

with $R_{t,d}$ and $R_{t,N}$ as defined in equations (1) and (2) and with $D = 78$. However, this ignores that the overnight return is a special case. Stock markets, unlike foreign exchange markets, are not opened 24 hours a day and the changes in the stock index price during the hours that the stock market is closed are relatively large compared to the 5-minute returns observed during trading hours. In order to account for the fact that overnight returns are presumably more volatile than intraday 5-minute returns and that a large value for $R_{t,N}$ will have a pronounced and distorting effect on the realised volatility estimate $\tilde{\sigma}_{t,1}^2$, we also use two alternative realised volatility measures which exclude the "noisy" overnight returns. The first of these is simply calculated as

$$\tilde{\sigma}_{t,2}^2 = \sum_{d=1}^D R_{t,d}^2, \quad (5)$$

and therefore only measures the volatility during trading hours as opposed to *daily* volatility⁸. It was suggested by Martens (2002) to use a scaler in order to obtain a *daily* realised volatility measure based on intraday returns only. Furthermore, he found that in the absence of intranight returns the most accurate estimate of daily volatility on Standard & Poor's 500 index futures was obtained with this scaled sum of squared intraday returns, which he defined as

$$\begin{aligned} \tilde{\sigma}_{t,3}^2 &= (1+c) \sum_{d=1}^D R_{t,d}^2 \\ &= \frac{\sigma_{oc}^2 + \sigma_{co}^2}{\sigma_{oc}^2} \sum_{d=1}^D R_{t,d}^2, \end{aligned} \quad (6)$$

where σ_{oc}^2 and σ_{co}^2 are the in-sample open-to-close and close-to-open variances, *i.e.* $\text{var}(\sum_{d=1}^D R_{t,d})$ and $\text{var}(R_{t,N})$, respectively, which implies $c \geq 0$.⁹ For our full Standard & Poor's 100 stock index sample we find an open-to-close variance of 1.447 and a close-to-open variance of 0.092, resulting in a scaling value $(1+c)$ of 1.064 which is considerably lower than the value of 1.205 observed by Martens (2002) for his Standard & Poor's 500 stock index futures series.

In table 2 we provide summary statistics for $\tilde{\sigma}_{t,1}^2$ and $\tilde{\sigma}_{t,3}^2$, together with those of their logarithmic counterparts. We observe that the mean of $\tilde{\sigma}_{t,1}^2$ is slightly higher than that of the scaled trading hours volatility measure $\tilde{\sigma}_{t,3}^2$ with values that translate into annualised standard deviations of 18.6% and 18.5%, respectively. The variance of $\tilde{\sigma}_{t,1}^2$ exceeds that of $\tilde{\sigma}_{t,3}^2$ which is explained by the fact that the latter does not include the noisy overnight return. Comparing the summary statistics of the intraday volatility measures with those of the daily squared return series in table 1 we find that R_t^2 has an average annualised standard deviation of 20.5% combined with a much higher degree of variation and larger skewness and excess kurtosis values. In addition, the reported autocorrelation coefficients for the intraday volatility series slowly decay as the lag length increases and they are always statistically significant, unlike those of the squared return series. As the intraday volatility series are clearly positively skewed and leptokurtic we also report on the logarithmic intraday volatility measures $\ln \tilde{\sigma}_{t,1}^2$ and $\ln \tilde{\sigma}_{t,3}^2$ as advocated in, for example, Andersen *et al.* (2001). The resulting four series are presented in figure 2 together with their histograms. The $\tilde{\sigma}_{t,1}^2$ and $\tilde{\sigma}_{t,3}^2$ series are highly correlated and their distributions are leptokurtic¹⁰. The distribution of the logarithmic intraday volatility measures appears approximately Gaussian, mirroring earlier findings for the stock market by Ebens (1999), Andersen *et al.* (2001) and Areal and Taylor (2002). These conclusions are further supported by

⁸ Andersen, Bollerslev, Diebold and Ebens (2001) and Andersen and Bollerslev (1997) use this definition of realised volatility in their stock market studies.

⁹ Yet a different method is explored by Areal and Taylor (2002) who suggest assigning different weights to the intraday squared returns with weights depending on variance proportions which are calculated for each day of the week.

¹⁰ We find $\text{corr}(\tilde{\sigma}_{t,1}^2, \tilde{\sigma}_{t,3}^2) = 0.973$ and $\text{corr}(\ln \tilde{\sigma}_{t,1}^2, \ln \tilde{\sigma}_{t,3}^2) = 0.985$.

Table 2: *Summary statistics for the realised volatility measures*

Period	1997-2000			
Number of observations T	1004			
Series	$\tilde{\sigma}_{t,1}^2$	$\tilde{\sigma}_{t,3}^2$	$\ln \tilde{\sigma}_{t,1}^2$	$\ln \tilde{\sigma}_{t,3}^2$
Mean	1.372	1.359	0.017	0.020
Variance	2.207	2.053	0.527	0.508
Skewness	5.299	5.185	0.451	0.430
Excess Kurtosis	43.725	40.992	0.559	0.601
Minimum	0.106	0.109	-2.241	-2.215
Maximum	19.164	16.608	2.953	2.810
N	84678	74794	47.154	46.116
$\hat{\rho}_1$	0.588	0.572	0.622	0.609
$\hat{\rho}_2$	0.373	0.392	0.541	0.532
$\hat{\rho}_3$	0.318	0.323	0.502	0.484
$\hat{\rho}_4$	0.300	0.296	0.460	0.443
$\hat{\rho}_5$	0.296	0.261	0.427	0.400
$Q(12)$	1189.1	1155.2	2296.4	2120.8

N is the χ^2 normality test statistic with 2 degrees of freedom and a critical value of 9.21 at the 1% significance level. $\hat{\rho}_\ell$ is the sample autocorrelation coefficient at lag ℓ with asymptotic standard error $1/\sqrt{T}$ and $Q(\ell)$ is the Box-Ljung portmanteau statistic based on ℓ squared autocorrelations. The critical value at the 1% significance level for the $Q(12)$ statistic is 26.22.

the skewness and excess kurtosis coefficients which have standard errors equal to $\sqrt{6/T} = 0.077$ and $\sqrt{24/T} = 0.155$, respectively, and are therefore close to normal but still have values which are different from zero at very high significance levels¹¹. The sample autocorrelation coefficients indicate a highly persistent volatility process and the values for the $Q(12)$ statistic are so high that the hypothesis of zero autocorrelation is convincingly rejected¹².

3 Realised Volatility Models

In this section we discuss the two realised volatility models which are the unobserved components (UC-RV) and the ARFIMA (ARFIMA-RV) models where realised volatility is modelled directly as opposed to the volatility models in the next section which model volatility as the second moment of returns. Unlike the UC-RV model, the ARFIMA-RV model is defined in logarithmic terms.

3.1 Unobserved Components OU type stochastic volatility models

Realised volatility, as defined in subsection 2.2 and denoted by $\tilde{\sigma}_t^2$, can be used as an estimator of volatility σ_t^2 ; see, for example, Andersen and Bollerslev (1998). Barndorff-Nielsen and Shephard (2002)

¹¹The t-statistics for the skewness coefficients are 5.83 and 5.56 and those of the excess kurtosis coefficients are 3.62 and 3.89 for $\ln \tilde{\sigma}_{t,1}^2$ and $\ln \tilde{\sigma}_{t,3}^2$, respectively.

¹²Also see Ebens (1999) and Andersen *et al.* (2001) for similar autocorrelation coefficients and comparable Q statistic values.

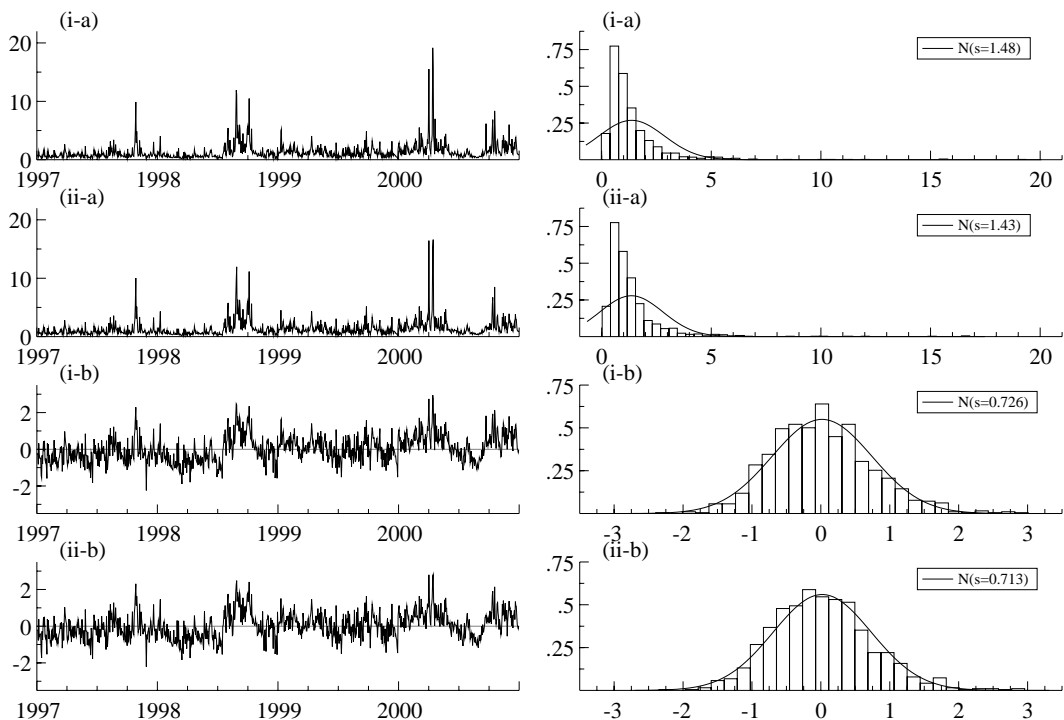


Figure 2: *Time series and histograms with normal approximations for the Standard & Poor's 100 stock index realised volatility measures (i-a) $\tilde{\sigma}_{t,1}^2$ and (ii-a) $\tilde{\sigma}_{t,3}^2$ and their logarithmic counterparts (ii-a) $\ln \tilde{\sigma}_{t,1}^2$ and (ii-b) $\ln \tilde{\sigma}_{t,3}^2$ over the period 6 January 1997 to 29 December 2000*

provide an excellent discussion of the properties of realised volatility. In particular, they investigate the statistical properties of the estimation error $\sigma_t^2 - \tilde{\sigma}_t^2$ and they argue that a more accurate estimator can be obtained when a model is considered for σ_t^2 . Actual volatility can be modelled as a continuous time series process consisting of independent Ornstein-Uhlenbeck (OU) processes, that is

$$\sigma^2(t) = \sum_{j=1}^J \tau^j(t), \quad d\tau^j(t) = -\lambda_j \tau^j(t)dt + dz_j(\lambda_j t),$$

where $\sigma^2(t)$ is the continuous-time process for actual volatility, $z_j(t)$ are independent Lévy processes with non-negative increments and λ_j are unknown parameters, for $j = 1, \dots, J$ and J is typically a small number in practice (say, 3 or 4). Developments of this approach, with special attention to statistical and probabilistic aspects, are reported by Barndorff-Nielsen and Shephard (2001, 2002).

The formulation of the continuous time model for $\sigma^2(t)$ implies that actual volatility in discrete time intervals of length Δ can be modelled as

$$\sigma_t^2 = \sum_{j=1}^J \tau_t^j, \quad t = 1, \dots, T,$$

where each τ_t^j represents an ARMA(1,1) model with the autoregressive and moving average coefficients determined by $\exp(\lambda_j \Delta)$; see Barndorff-Nielsen and Shephard (2002) for further details. A model for realised volatility is then simply given by

$$\tilde{\sigma}_t^2 = \sigma_t^2 + u_t, \quad t = 1, \dots, T,$$

where u_t is white noise, with mean zero and variance depending on Δ , and is uncorrelated with σ_j^2 for $j = 1, \dots, T$. The resulting model is an unobserved components model which consists of J independent ARMA components plus an error component. Linear optimal estimators of this model can be obtained by casting the model into state space form. The Kalman filter can be applied to construct the Gaussian likelihood function. Quasi-maximum likelihood estimates of λ_j are obtained by numerically maximising the Gaussian likelihood with respect $\lambda_1, \dots, \lambda_J$.

In this paper we adopt this model for realised volatility $\tilde{\sigma}_t^2$ with $J = 1$. The resulting ARMA(1,1) plus error model will be referred to as the Unobserved Components - Realised Volatility, or UC-RV, model and is given by

$$\begin{aligned}\tilde{\sigma}_t^2 &= \sigma_t^2 + u_t, \\ \sigma_t^2 &= \mu + \phi(\sigma_{t-1}^2 - \mu) + \theta\eta_{t-1} + \eta_t.\end{aligned}$$

Note that we take the variance of u_t as implied by Δ but we do estimate the variance of η_t , that is σ_η^2 . This leads to a model with four unknown parameters: μ , ϕ , θ and σ_η . The state space form of this model can be written as

$$\begin{aligned}\tilde{\sigma}_t^2 &= Z\alpha_t + u_t, \\ \alpha_{t+1} &= T\alpha_t + R\eta_t,\end{aligned}\tag{7}$$

where

$$Z = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad T = \begin{bmatrix} \phi & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R' = \begin{pmatrix} 1 & \theta & 0 \end{pmatrix},$$

and ϕ , θ and σ_u are the parameters of the ARMA(1,1) process. The first element of the 3×1 state vector α_t is $\tilde{\sigma}_t^2 - \mu$ and its last element is μ .

Estimation

When the ARMA(1,1) plus error model is casted in the state space model (7), the Kalman filter can be used to compute the Gaussian likelihood function and parameter estimates can be obtained by numerically optimising the resulting quasi-likelihood function. We estimate the parameters of the ARMA process without taking allowance for its dependence on Δ , as described in Barndorff-Nielsen and Shephard (2001), in order to get a better fit of the data. However, all four parameters of the unrestricted ARMA(1,1) plus error model are not identified. By restricting the standard deviation σ_u of the unobserved ARMA(1,1) process to unity, the remaining three parameters are identified and can be estimated by maximum likelihood. More details on the identification and estimation of unobserved ARMA components models are given by Harvey (1989, section 4.4).

Forecasting

Forecasting for linear time series models in state space form is relatively straightforward. Firstly, the realised volatility series is artificially extended with missing values at the end. Secondly, the Kalman filter can deal with missing observations and is therefore applied to the new extended series. The estimates of the signal σ_t^2 , corresponding to the missing values at $t = T + 1, T + 2, \dots$, are taken as the forecasts of actual volatility σ_{T+j}^2 , for $j = 1, 2, \dots$. The mean squared errors of the forecasts are also provided by the Kalman filter. Details of this approach to forecasting are discussed in Harvey (1989, section 3.5).

3.2 ARFIMA models

In empirical work on realised volatility it is pointed out that the realised volatility series $\tilde{\sigma}_t^2$ can be regarded as being generated by a Gaussian process after it is transformed by taking logarithms. The dynamic properties of log realised volatility exhibit features known as long memory, that is, the correlogram of such a series decays less than exponentially as the lag length increases. An appropriate model framework to deal with such specific dynamic properties is based on the ARFIMA model.

The ARFIMA(1, d , 1) model with mean μ is given by

$$(1 - \phi L)(1 - L)^d(y_t - \mu) = (1 + \theta L)\varepsilon_t,$$

where L is the lag operator ($Ly_t = y_{t-1}$), coefficients d , ϕ and θ are fixed and unknown and ε_t is Gaussian white noise with mean zero and variance σ^2 . The following restrictions on the parameters apply,

$$0 < d < 0.5, \quad |\phi| < 1, \quad |\theta| < 1, \quad \sigma^2 > 0.$$

In the context of volatility modelling, the ARFIMA model for the logs of realised volatility is empirically investigated by Ebens (1999), Andersen *et al.* (2001b) and Oomen (2001).

Estimation

The parameters of the ARFIMA model, including mean μ , can be estimated by the method of maximum likelihood; for details, see, for example, Sowell (1992). It is, for example, pointed out by Brodsky and Hurvich (1999) and Bos, Franses and Ooms (2002) that standard ARMA(1,1) models can also capture long memory features and that, depending on the sample spectrum of the data, not all parameters of an ARFIMA(1, d ,1) can be identified from the data. This usually applies to the case of realised volatility. In empirical studies one may fix the d parameter to a certain value and estimate the remaining parameters. However, we rather concentrate on the estimation of the crucial d parameter in an ARFIMA(1, d ,0), ARFIMA(0, d ,1) or ARFIMA(0, d ,0) model. Although in prior analyses the three ARFIMA models produced rather similar results, we will consider the ARFIMA(1, d ,0) model in our empirical study. The required computations are implemented using the ARFIMA package of Doornik and Ooms (2001) within the programming environment of Ox; see Doornik (1998).

Forecasting

Forecasting can be carried out by extrapolating the series in which the correlation structure implied by the estimated ARFIMA model is taken into account. Details of how these computations can be implemented elegantly for ARFIMA models are given by Doornik and Ooms (2001).

4 Daily Time-Varying Volatility Models

In this section we discuss the daily time-varying volatility models where volatility is explicitly modelled as the second moment of daily returns. This rather well-established range of volatility models includes the SV and GARCH classes of models. In addition to standard formulations we also consider extensions for both models with intraday volatility incorporated in the variance equation.

4.1 Daily SV model

The daily Stochastic Volatility (SV) model can be defined as

$$\begin{aligned} y_t &= \sigma_t \varepsilon_t, & \varepsilon_t &\sim \text{NID}(0, 1), & t &= 1, \dots, T, \\ \sigma_t^2 &= \sigma^{*2} \exp(h_t), \\ h_t &= \phi h_{t-1} + \sigma_\eta \eta_t, & \eta_t &\sim \text{NID}(0, 1), \end{aligned} \quad (8)$$

where y_t denotes the return series of interest which is the daily series R_t as given in equation (3). The volatility process σ_t^2 is defined as the product of a scaling factor $\sigma^{*2} > 0$ and the exponential of the stochastic process h_t , which in turn is modelled as a first order autoregressive process. The persistence parameter ϕ is restricted to be positive and smaller than one to ensure the stationarity of σ_t^2 . We further assume that ε_t and η_t are mutually uncorrelated, both contemporaneously and at all lags. For reviews of the SV model we refer to Taylor (1994), Ghysels, Harvey and Renault (1996) and Shephard (1996).

Estimation

The parameters of the SV model are estimated by exact maximum likelihood methods using Monte Carlo importance sampling techniques. The likelihood function for the SV model can be constructed using simulation methods developed by Shephard and Pitt (1997) and Durbin and Koopman (1997). We start by considering the standard SV model of equation (8). The non-linear relation between log-volatility h_t and the observation equation of y_t does not allow the computation of the likelihood by linear methods such as the Kalman filter. For the SV model we can express the likelihood function as

$$L(\psi) = p(y|\psi) = \int p(y, \theta|\psi) d\theta = \int p(y|\theta, \psi) p(\theta|\psi) d\theta, \quad (9)$$

where

$$\psi = (\phi, \sigma_\eta, \sigma_\varepsilon)', \quad \theta = (h_1, \dots, h_T)'.$$

An efficient way of evaluating such expressions is by using importance sampling; see Ripley (1987, Chapter 5). A simulation device is required to sample from an importance density $\tilde{p}(\theta|y, \psi)$ which we prefer to be as close as possible to the true density $p(\theta|y, \psi)$. An obvious choice for the importance density is the conditional Gaussian density since in this case it is relatively straightforward to sample from $\tilde{p}(\theta|y, \psi) = g(\theta|y, \psi)$ using simulation smoothers such as the ones developed by de Jong and Shephard (1995) and Durbin and Koopman (2002). Guidelines for the construction of an importance model and the likelihood function for the SV model using this approach are given by Hol and Koopman (2000). The SV models are estimated using programs written in the Ox language of Doornik (1998) using SsfPack by Koopman, Shephard and Doornik (1999). The Ox programs can be obtained from www.econ.vu.nl/koopman/sv/.

Forecasting

For the case that forecasting horizon is one day, the daily volatility forecast for the SV model can be written as

$$E(\sigma_{T+1|T}^2) = \hat{\sigma}^{*2} \exp(\hat{h}_{T+1|T} + 0.5p_{T+1|T}), \quad (10)$$

where $\hat{\sigma}^{*2}$ is the maximum likelihood estimate of σ^{*2} , $\hat{h}_{T+1|T}$ is the estimator of h_{T+1} given all T observations and $p_{T+1|T}$ is its mean square error. When the forecasting horizon spans N days, we have

$$E(\sigma_{T+1,T+N|T}^2) = \hat{\sigma}^{*2} \sum_{j=1}^N \exp(\hat{h}_{T+j|T} + 0.5p_{T+j|T}), \quad (11)$$

The estimator of h_{T+1} given all T observations, and its mean square error $p_{T+1|T}$ are computed with the simulation methods developed by Durbin and Koopman (2000); for $j \geq 2$ the values for $\hat{h}_{T+j|T}$ and $p_{T+j|T}$ are given by

$$\begin{aligned} \hat{h}_{T+j|T} &= \hat{\phi}^{j-1} \hat{h}_{T+1|T}, \\ p_{T+j|T} &= \hat{\phi}^{2(j-1)} p_{T+1|T} + \sum_{i=0}^{N-2} \hat{\phi}^{2i} \hat{\sigma}_\eta^2, \end{aligned}$$

where $\hat{\phi}$ and $\hat{\sigma}_\eta^2$ are the maximum likelihood estimates of ϕ and σ_η^2 , respectively. When these definitions of $\hat{h}_{T+j|T}$ and $p_{T+j|T}$ are considered in conjunction with equation (11) it becomes apparent that as N increases $E(\sigma_{T+N|T}^2)$ will converge to a value which is identical to the unconditional variance given by

$$\hat{\sigma}^{*2} \exp\left(0.5 \frac{\hat{\sigma}_\eta^2}{1 - \hat{\phi}^2}\right),$$

where the rate of convergence depends on the value for the volatility persistence parameter estimate $\hat{\phi}$ which tends to be close to unity for the type of daily financial time series we are studying here. Volatility forecasts for small values of N are therefore mainly determined by the values for $\hat{h}_{T+1|T}$ and $p_{T+1|T}$.

4.2 Daily SV model with intraday volatility

The daily SV model with intraday volatility as explanatory variable in the variance equation is an extension of the SV model as defined in equation (8) where the stochastic process h_t is given by

$$h_t = \phi h_{t-1} + \gamma x_{t-1} + \sigma_\eta \eta_t \quad (12)$$

and where x_t denotes $\ln \tilde{\sigma}_t^2$ as defined in subsection 2.2. The value of ϕ we restrict to be smaller than one in absolute terms, so $-1 < \phi < 1$. Alternative formulations for the h_t process are possible, for example, one could consider the inclusion of a persistence adjustment term. However, we have chosen this one as it is most closely related to other models used in the intraday volatility literature¹³. The SV model with intraday volatility is identical to the SVX model discussed by Hol and Koopman (2000) who used implied volatility instead of intraday volatility as explanatory variable in the variance equation.

¹³See *e.g.* Martens (2001) and Blair *et al.* (2001).

Estimation

The inclusion of realised volatility in the equation of h_t does not affect the non-linear relationship between observation y_t and the unobserved volatility component h_t . Therefore the estimation and forecasting methods for the SV model as described in subsection 4.1 can be applied straightforwardly. More details of maximum likelihood estimation using importance sampling techniques for the model in equation (12) are given by Hol and Koopman (2000).

Forecasting

The daily forecast for the SV model with intraday volatility is obtained with similar methods as the daily forecast for the SV model defined in equation (10) but requires in addition the availability of x_T for the calculation of $\hat{h}_{T+1|T}$ and $p_{T+1|T}$. Because intraday volatility values are not known beyond time T we cannot calculate $\hat{h}_{T+j|T}$ and $p_{T+j|T}$ for $j \geq 2$ as we did in the case of the SV model. We therefore assume that these values do not change during subsequent trading days and define the equivalent of equation (11) as the one-day ahead volatility forecast multiplied by the length of the forecasting horizon N , so

$$\begin{aligned} E(\sigma_{T+1,T+N|T}^2) &= N \hat{\sigma}^{*2} \exp(h_{T+1|T} + 0.5 p_{T+1|T}) \\ &= N E(\sigma_{T+1|T}^2). \end{aligned} \quad (13)$$

4.3 Daily GARCH(1,1) model

The second class of daily time-varying volatility models we consider is the GARCH(1,1) model as given by

$$\begin{aligned} y_t &= \sigma_t \varepsilon_t & \varepsilon_t &\sim \text{NID}(0, 1), & t &= 1, \dots, T, \\ \sigma_t^2 &= \omega + \alpha(\sigma_{t-1} \varepsilon_{t-1})^2 + \beta \sigma_{t-1}^2, \end{aligned} \quad (14)$$

with y_t as defined above in subsection 4.1 and parameter restrictions $\omega > 0$, $\alpha \geq 0$, $\beta \geq 0$ and $\alpha + \beta < 1$.

Estimation

Maximum likelihood methods for GARCH models are well established; the procedures developed by Bollerslev (1986) have been implemented in many standard econometric software packages such as PcGive and EViews¹⁴. Hence we do not give estimation details and refer to GARCH surveys, such as those by Bollerslev, Chou and Kroner (1992), Bera and Higgins (1993) and Bollerslev, Engle and Nelson (1994).

Forecasting

As all information for the one-day ahead volatility forecast is available at time T the daily GARCH(1,1) forecast can be directly calculated as

$$E(\sigma_{T+1|T}^2) = \hat{\omega} + \hat{\alpha} \sigma_T^2 \varepsilon_T^2 + \hat{\beta} \sigma_T^2, \quad (15)$$

¹⁴More specifically, the parameters in the GARCH model were estimated using the G@RCH package of Laurent and Peters (2002).

where $\hat{\omega}$, $\hat{\alpha}$ and $\hat{\beta}$ denote the maximum likelihood estimates of ω , α and β , respectively. The N -period ahead GARCH(1,1) forecast is then obtained by applying the law of iterated expectations and this forecast can be expressed as

$$E(\sigma_{T+1,T+N|T}^2) = \sum_{j=1}^N \frac{\hat{\omega}}{1 - \hat{\alpha} - \hat{\beta}} + (\hat{\alpha} + \hat{\beta})^{j-1} \left(E(\sigma_{T+1|T}^2) - \frac{\hat{\omega}}{1 - \hat{\alpha} - \hat{\beta}} \right). \quad (16)$$

From this equation we can deduce that $E(\sigma_{T+N|T}^2)$ converges to the unconditional variance value

$$\frac{\hat{\omega}}{1 - \hat{\alpha} - \hat{\beta}}$$

as N increases and that the rate of convergence is governed by the sum of $\hat{\alpha}$ and $\hat{\beta}$, which measures the degree of volatility persistence.

4.4 Daily GARCH(1,1) model with intraday volatility

The daily GARCH(1,1) model is extended to include intraday volatility by incorporating this information in the variance equation, so the volatility process σ_t^2 in equation (14) can be rewritten as

$$\sigma_t^2 = \omega + \alpha(\sigma_{t-1}\varepsilon_{t-1})^2 + \beta\sigma_{t-1}^2 + \gamma x_{t-1}, \quad (17)$$

where x_t represents the realised volatility measure $\tilde{\sigma}_t^2$ as defined in subsection 2.2. We refer to this model as the GX model.

Estimation

Standard packages have options to include explanatory variables within the GARCH process as the estimation of the coefficients in equation (17) is relatively straightforward.

Forecasting

As for the standard GARCH model, all information for the calculation of the one-day ahead volatility forecast $E(\sigma_{T+1|T}^2)$ is available at time T as it is given by

$$E(\sigma_{T+1|T}^2) = \hat{\omega} + \hat{\alpha}\sigma_T^2\varepsilon_T^2 + \hat{\beta}\sigma_T^2 + \hat{\gamma}x_T, \quad (18)$$

where $\hat{\gamma}$ is the maximum likelihood estimate of γ . Due to the fact that intraday volatility values are not known beyond time T we define the N -period ahead volatility forecast for the extended GARCH(1,1) model in a similar manner as for the SVX model, *i.e.*, as

$$\begin{aligned} E(\sigma_{T+1,T+N|T}^2) &= N(\hat{\omega} + \hat{\alpha}\sigma_T^2\varepsilon_T^2 + \hat{\beta}\sigma_T^2 + \hat{\gamma}x_T) \\ &= N E(\sigma_{T+1|T}^2). \end{aligned} \quad (19)$$

5 Forecasting Methodology and Evaluation Criteria

5.1 Forecasting methodology

In the second part of the next section we present an out-of-sample forecasting study in which we compare the relative forecasting performance of the six volatility models described in sections 3 and 4. Our full sample consists of 1004 trading days and each model is initially estimated over the first 800

observations of the full sample, *i.e.*, over the period 6 January 1997 to 10 March 2000. As a result, the out-of-sample period is from 13 March to 29 December 2000 providing 204 daily observations. The parameter estimates obtained with the data from the initial in-sample period are inserted in the relevant forecasting formulas given in the same sections and volatility forecasts $E(\sigma_{T+1,T+N|T}^2)$ are calculated given the information available at time T for horizons ranging from 1 day to 1 week, so for $N = 1, \dots, 5$. The sample is then rolled forward by one trading day keeping the size of the sample constant at 800 observations and volatility forecasts are obtained for the subsequent N trading days. This procedure is repeated until we have obtained forecasts for the entire out-of-sample period with $T = 800, \dots, 1003$. As our volatility forecasts are overlapping, we have forecasting samples containing $205 - N$ volatility forecasts for each of the six models.

5.2 Evaluation criteria

As a measure of realised volatility we use $\tilde{\sigma}_{t,3}^2$, which is the scaled intraday volatility measure defined in equation (6). We prefer this realised volatility measure to the highly correlated $\tilde{\sigma}_{t,1}^2$ of equation (4) as the latter includes noisy overnight returns. Realised volatility for forecasting horizons exceeding one trading day are calculated by summing the realised volatility measures of equation (6) over the relevant forecasting horizon N , so

$$\tilde{\sigma}_{(T+1,T+N),3}^2 = \sum_{i=1}^N \tilde{\sigma}_{T+i,3}^2. \quad (20)$$

In order to assess the predictive abilities of the various volatility models we report on the goodness-of-fit coefficient R^2 as calculated from the OLS regression

$$\tilde{\sigma}_{(T+1,T+N),3}^2 = a + b E(\sigma_{T+1,T+N|T}^2) + \xi, \quad (21)$$

where $E(\sigma_{T+1,T+N|T}^2)$ denotes the N -period ahead volatility forecasts obtained with the volatility models defined in sections 3 and 4. If the volatility forecasts are unbiased, then $a = 0$ and $b = 1$. We test these hypotheses using standard regression methods with Newey-West adjustments to account for the special error covariance structure due to the rolling window construction of the forecasts. In addition to the regression-based method, which is by far the most popular post-sample evaluation procedure, we also report on two error statistics which are used by Andersen, Bollerslev and Lange (1999) and Martens (2002). These are the heteroskedasticity adjusted root mean squared error (HRMSE) and the mean absolute error (HMAE) which are calculated as

$$HRMSE = \sqrt{\frac{1}{205 - N} \sum_{T=800}^{1004-N} \left\{ 1 - \frac{E(\sigma_{T+1,T+N|T}^2)}{\tilde{\sigma}_{(T+1,T+N),3}^2} \right\}^2}, \quad (22)$$

and

$$HMAE = \frac{1}{205 - N} \sum_{T=800}^{1004-N} \left| 1 - \frac{E(\sigma_{T+1,T+N|T}^2)}{\tilde{\sigma}_{(T+1,T+N),3}^2} \right|. \quad (23)$$

6 Empirical Results

6.1 In-sample results

In this subsection the results obtained with volatility models described in sections 3 and 4 are presented for the full in-sample period. More specifically, we report on the estimation results of the following models:

$$\begin{aligned} \text{UC-RV Model:} \quad & \tilde{\sigma}_{t,3}^2 = \sigma_t^2 + u_t, \\ & \sigma_t^2 = \mu + \phi(\sigma_{t-1}^2 - \mu) + \theta\eta_{t-1} + \eta_t \end{aligned}$$

$$\text{ARFIMA-RV Model:} \quad (1 - \phi L)(1 - L)^d \ln \tilde{\sigma}_{t,3}^2 = \mu + u_t$$

$$\begin{aligned} \text{SV Model:} \quad & y_t = \sigma_t \varepsilon_t, \\ & \ln \sigma_t^2 = (1 - \phi) \ln \sigma^{*2} + \phi \ln \sigma_{t-1}^2 + \sigma_\eta \eta_t \end{aligned}$$

$$\begin{aligned} \text{SVX Model:} \quad & y_t = \sigma_t \varepsilon_t, \\ & \ln \sigma_t^2 = (1 - \phi) \ln \sigma^{*2} + \phi \ln \sigma_{t-1}^2 + \gamma_i \ln \tilde{\sigma}_{t-1,i}^2 + \sigma_\eta \eta_t \end{aligned}$$

$$\begin{aligned} \text{GARCH}(1,1) \text{ Model:} \quad & y_t = \sigma_t \varepsilon_t, \\ & \sigma_t^2 = \omega + \alpha(\sigma_{t-1}^2 \varepsilon_{t-1}^2) + \beta \sigma_{t-1}^2 \end{aligned}$$

$$\begin{aligned} \text{GX Model:} \quad & y_t = \sigma_t \varepsilon_t, \\ & \sigma_t^2 = \omega + \alpha(\sigma_{t-1}^2 \varepsilon_{t-1}^2) + \beta \sigma_{t-1}^2 + \gamma_i \tilde{\sigma}_{t-1,i}^2, \end{aligned}$$

where the volatility processes of the SV and GARCH classes of models are driven by daily returns with $y_t = R_t$ and $\varepsilon_t \sim \text{NID}(0, 1)$. The SVX and GX models then contain additional intraday information in the variance definition itself with i either equal to 1 or 2, so the realised volatility measures we use in these models are $\tilde{\sigma}_{t,1}$ and $\tilde{\sigma}_{t,2}$ as given in equations (4) and (5), respectively. The realised volatility models UC-RV and ARFIMA-RV, on the other hand, directly model the realised volatility measure $\tilde{\sigma}_{t,3}$ as defined in equation (6).

Table 3 presents the estimation results for the above specified models over the full sample period 6 January 1997 to 29 December 2000. The fractional integration parameter estimate in the ARFIMA-RV model has a value of 0.446, exceeding the d estimates found by Ebens (1999) and Andersen *et al.* (2001b) who report values in the region of 0.4. It should be noted that these studies either employ different ARFIMA model specifications or different estimation methods¹⁵. More importantly however, the logarithmic volatility process may not be covariance-stationary as the estimate of d is close to the boundary value of 0.5¹⁶. For the UC-RV model we observe estimated values for the first-order autoregressive and moving average parameters of 0.747 and -0.160 , respectively, which are both statistically significant.

Volatility persistence estimates for the standard SV and the GARCH models are statistically significant and close to unity with $\hat{\phi} = 0.946$ and $\hat{\alpha} + \hat{\beta}$ equal to 0.938 which confirms earlier findings in the literature with regard to daily stock index return series. The SVX and GX models with realised volatility incorporated in the volatility equations show highly significant estimates for the γ regression parameters with $\hat{\gamma}_2$ slightly higher than $\hat{\gamma}_1$. This is to be expected as $\tilde{\sigma}_{t,1}$ is equal to $\tilde{\sigma}_{t,2}$ plus the

¹⁵Ebens (1999), for example, never estimates the autoregressive parameter in his ARFIMAX(p,d,q) model and Andersen *et al.* (2001b) fix the value of the d parameter prior to the estimation of the other parameters in the ARFIMA model.

¹⁶Also see Oomen (2001) who encounters the same problem with regard to a ten-year sample of FTSE-100 index returns.

Table 3: In-sample estimation results for the Standard & Poor 100 stock index over the period 6 January 1997 to 29 December 2000

Model Series	UC-RV $\tilde{\sigma}_{t,3}^2$	ARFIMA-RV $\ln \tilde{\sigma}_{t,3}^2$	SV Models y_t		GARCH Models y_t
μ	1.357 1.149 1.564	0.013 -1.218 1.244	1.358 1.073 1.719	ω	0.111 0.019 0.202
ϕ	0.747 0.662 0.833	-0.061 -0.151 0.029	0.946 0.876 0.978	α	0.106 0.056 0.155
θ	-0.160 -0.317 -0.003		0.036 0.015 0.087	β	0.832 0.744 0.920
σ_η^2	0.279 0.202 0.386		0.443 0.347 0.538	γ_1	0.680 0.416 0.944
σ_u^2	0.286	0.286	0.465 0.369 0.561	γ_2	0.743 0.468 1.017
d	0.446 0.381 0.512				
$\ln L$	-1581.24	-797.11	-1623.57		-1640.19
$LR(\gamma_1 = 0)$			-1603.85		-1602.02
$LR(\gamma_2 = 0)$			39.44		76.34
AIC	3170.48	1602.21	3253.14		3286.38
$Q(12)$	13.528	11.607	21.265		20.033
N	21.641	32.163	2.549		223.26
			6.305		31.909
			43.14		83.76
			3212.00		3204.61
			22.634		21.300
			5.994		26.780

Parameter estimates are reported together with the asymptotic 95% confidence intervals which are a-symmetric for σ^{*2} , ϕ in the SV models, and σ_η^2 , $LR(\gamma_1 = 0)$ and $LR(\gamma_2 = 0)$ are the likelihood ratio statistics for the hypotheses $\gamma_1 = 0$ and $\gamma_2 = 0$, respectively. Confidence intervals for σ_u^2 in the ARFIMA model are not produced by the ARFIMA package. AIC is the Akaike Information Criterion which is calculated as $-2(\ln L) + 2p$; $Q(\ell)$ is the Box-Ljung portmanteau statistic for the estimated observation errors which is asymptotically χ^2 distributed with $\ell - p$ degrees of freedom where p is the total number of estimated parameters; N is the χ^2 normality test statistic with 2 degrees of freedom.

squared overnight return at time t . In addition we observe that the subsequent log-likelihood values increase compared to those of the SV and the GARCH models. The likelihood ratio statistics for the null hypotheses $\gamma_1 = 0$ and $\gamma_2 = 0$ are 39.44 and 43.14 for the SV and 76.34 and 83.76 for the GARCH models, respectively. They clearly indicate that inclusion of the realised volatility measures $\tilde{\sigma}_{t,1}$ or $\tilde{\sigma}_{t,2}$ significantly improves the fit of the models¹⁷. With regards to the other parameters we find that the estimates for α in the GX models are negative and no longer statistically significant but that the estimates for σ_η^2 in the SVX models have increased considerably in value. This suggests that the SV type models still benefit from inclusion of both daily and 5-minute returns but that for GARCH models there is little to no incremental value in the daily returns once the information contained in the intraday (and overnight) returns is included¹⁸. Although we cannot compare the SV and GARCH models in terms of goodness-of-fit, we can confirm that the distributional assumptions with regard to the error term ε_t are more closely followed by the SV than by the GARCH models.

6.2 Out-of-sample results

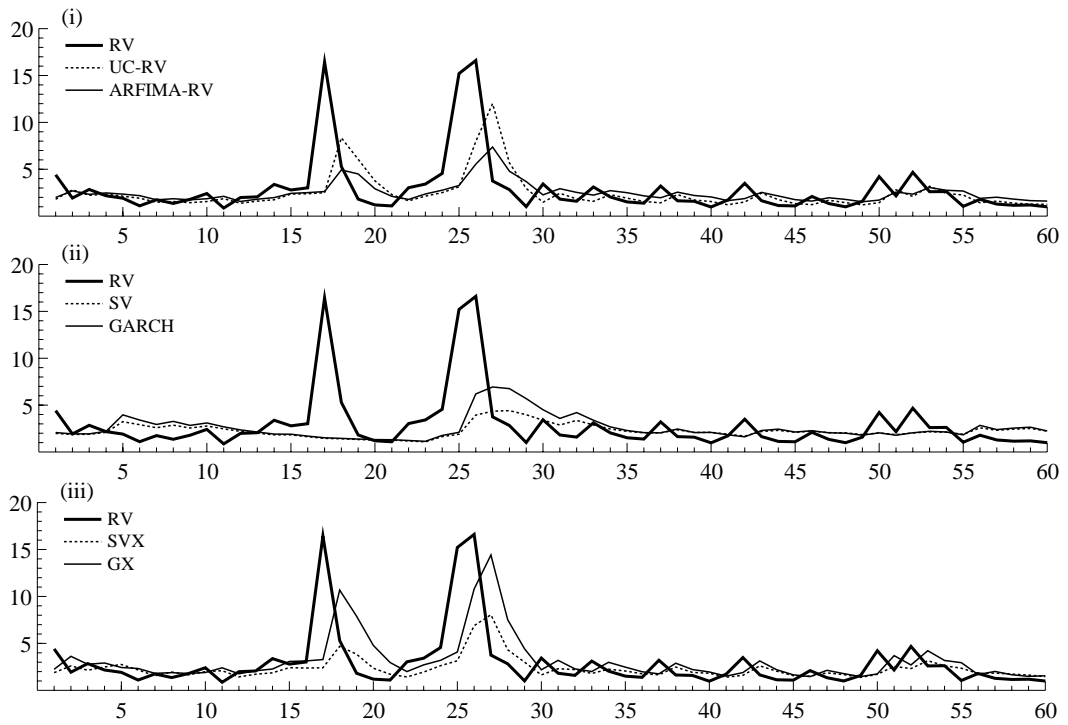


Figure 3: One-day ahead volatility forecasts of the (i) UC-RV and ARFIMA-RV, (ii) SV and GARCH, and the (iii) SVX and GX models against the realised volatility measure $\tilde{\sigma}_{t,3}^2$ (RV) over the period 13 March to 6 June 2000

Out-of-sample volatility forecasts are constructed from the SV, GARCH and realised volatility models for the evaluation period 13 March 2000 to 29 December 2000 and forecasting horizons range

¹⁷Unlike Martens (2002), we find a slightly better in-sample fit when the overnight returns are excluded from the realised volatility measure.

¹⁸These GARCH model findings confirm the empirical results of Blair *et al.* (2001) who examine Standard & Poor's 100 stock index returns over the earlier 1987 to 1992 period and find values for γ similar to ours. In contrast, Martens (2002) reports on much smaller and statistically insignificant γ estimates for returns on Standard & Poor's 500 futures. Also see: Taylor and Xu (1997).

Table 4: *Out-of-sample forecasting results evaluated against $\tilde{\sigma}_{(T+1,T+N),3}^2$ based on the Standard & Poor's 100 6 January 1997 to 28 December 2000 sample and for the evaluation period 13 March to 29 December 2000*

Forecasting Model		Forecasting Horizon				
		$N = 1$	$N = 2$	$N = 3$	$N = 4$	$N = 5$
UC-RV Model	a	0.5173 (1.877)	1.5421 (2.777)	2.7216 (3.076)	3.8066 (3.132)	4.8179 (3.082)
	b	0.7727 (1.263)	0.6312 (2.250)	0.5579 (2.707)	0.5377 (2.889)	0.5370 (2.881)
	R^2	0.2140	0.1573	0.1274	0.1187	0.1157
ARFIMA-RV Model	a	-0.0199 (0.585)	0.0194 (0.032)	0.3667 (0.414)	0.7123 (0.638)	1.0526 (0.769)
	b	1.1677 (0.679)	1.0498 (0.220)	0.9890 (0.051)	0.9604 (0.196)	0.9461 (0.279)
	R^2	0.2579	<u>0.2352</u>	<u>0.2266</u>	<u>0.2279</u>	<u>0.2310</u>
SV Model	a	0.4627 (1.073)	1.4650 (2.075)	2.5123 (2.251)	3.7111 (2.202)	4.7429 (2.088)
	b	0.7680 (0.864)	0.6181 (2.099)	0.5623 (2.723)	0.5143 (2.855)	0.5059 (2.690)
	R^2	0.0732	0.0580	0.0546	0.0497	0.0501
SVX Model	a	-0.1991 (0.597)	0.4693 (0.929)	1.4373 (2.011)	2.5186 (2.671)	3.7026 (2.998)
	b	1.1385 (0.597)	0.8925 (0.614)	0.7553 (1.648)	0.6709 (2.623)	0.6101 (3.349)
	R^2	<u>0.2705</u>	0.2195	0.1934	0.1802	0.1699
GARCH(1,1) Model	a	0.5868 (1.419)	1.7942 (2.744)	3.0419 (2.991)	4.4492 (2.958)	5.7337 (2.834)
	b	0.6346 (1.586)	0.4776 (3.583)	0.4204 (4.778)	0.3722 (5.109)	0.3570 (4.841)
	R^2	0.0894	0.0606	0.0524	0.0439	0.0415
GX Model	a	0.5453 (2.221)	1.6964 (3.767)	3.0473 (4.398)	4.3803 (4.687)	5.7157 (4.880)
	b	0.6061 (2.889)	0.4614 (4.969)	0.3825 (6.362)	0.3450 (7.482)	0.3227 (8.285)
	R^2	0.2323	0.1778	0.1503	0.1444	0.1442

Parameter estimates and goodness-of-fit R^2 statistics for the OLS regressions as defined in equation (21). The t-statistics testing for the null hypotheses $a = 0$ and $b = 1$ are in parentheses and based on standard errors using Newey-West heteroskedasticity and autocorrelation consistent covariance estimates. The highest values for R^2 are underlined.

Table 5: *Out-of-sample forecasting results evaluated against $\tilde{\sigma}_{(T+1,T+N),3}^2$ based on the Standard & Poor's 100 6 January 1997 to 28 December 2000 sample and for the evaluation period 13 March to 29 December 2000*

Forecasting Model		Forecasting Horizon									
		$N = 1$		$N = 2$		$N = 3$		$N = 4$		$N = 5$	
UC-RV Model	HRMSE	0.6546	2	0.6326	3	0.6394	3	0.6108	3	0.6040	3
	HMAE	0.4853	2	0.4572	2	0.4606	3	0.4488	3	0.4511	3
ARFIMA-RV Model	HRMSE	0.6401	1	0.5613	1	0.5383	1	0.5144	1	0.5058	1
	HMAE	0.4807	1	0.4406	1	0.4264	1	0.4146	1	0.4100	1
SV Model	HRMSE	0.8583	4	0.7503	4	0.7070	4	0.6823	4	0.6648	4
	HMAE	0.6561	5	0.5989	5	0.5614	4	0.5402	4	0.5232	4
SVX Model	HRMSE	0.6684	3	0.5984	2	0.5842	2	0.5555	2	0.5432	2
	HMAE	0.5134	3	0.4699	3	0.4530	2	0.4346	2	0.4238	2
GARCH(1,1) Model	HRMSE	1.0093	6	0.8835	6	0.8374	5	0.8154	5	0.7999	6
	HMAE	0.7600	6	0.6997	6	0.6576	6	0.6331	6	0.6166	6
GX Model	HRMSE	0.9031	5	0.8659	5	0.8787	6	0.8248	6	0.7954	5
	HMAE	0.6446	4	0.5951	4	0.5820	5	0.5552	5	0.5419	5

Error statistics HRMSE and HMAE as defined in equations (22) and (23), respectively, together with model rankings for the relevant forecasting horizon to the right of these error statistics.

from one day to one week¹⁹. Tables 4 and 5 present the forecasting performance results of these models with scaled intraday volatility in equation (6) as the measure of realised volatility and forecasts evaluated by means of the regression in equation (21) and the error statistics in equations (22) and (23).

The regression-based results in table 4 show that those models which use intraday information produce more accurate out-of-sample volatility forecasts than those that do not. The SV and the GARCH(1,1) model have the lowest coefficients of determination R^2 and as these daily models depend solely on closing prices, extreme price movements during the trading day do not necessarily show up. For illustrative purposes we plot in figure 3 the one-day ahead volatility forecasts of all six models for the first 60 trading days of our out-of-sample forecasting period during which three so-called high volatility days can be observed. The first of these occurs on 4 April 2000 (out-of-sample day 17) but is not recognised as such by the daily volatility models in graph 3-ii because daily closing prices only showed a relatively moderate drop of 1%. The other four models incorporate the intraday returns and their volatility forecasts therefore increase after 4 April as can be seen in graph 3-i for the realised volatility models and in graph 3-iii for the SVX and GX models. The GX model then reacts the most strongly of the four. The second higher volatility period, on the other hand, which is that of 14 to 17 April 2000 (out-of-sample days 25 and 26) also leads to higher volatility forecasts for the SV and GARCH model as daily returns do change considerably for this period. On the whole it appears from the graphs in figure 3 that the volatility forecasts of the models with intraday return information follow the realised volatility measure much more closely than those without, confirming the regression based results of table 4. More specifically, the goodness-of-fit statistics indicate that the SVX model gives the most accurate one trading day ahead volatility forecasts, whereas the ARFIMA model outperforms the other five models for forecasting horizons $N \geq 2$. Further it has to be noted that the forecasts of the SVX and the ARFIMA-RV model are the least biased for $N = 1$ and $N \geq 2$, respectively. What is more, the hypotheses that the parameter estimates for a and b are equal to 0 and 1, respectively, can never be rejected at the 5% significance level for the ARFIMA-RV model. The worst performing model in this respect is the GX model which has t-statistics that always exceed the critical 5% value of 1.96.

Results change marginally when we evaluate the volatility forecasts against the error statistics in table 5. In terms of the HRMSE and HMAE statistics the ARFIMA-RV model now consistently appears to have the most accurate out-of-sample volatility forecasts, followed by the UC-RV for the shorter and the SVX model for the longer forecasting horizons. The worst performing volatility models however are the GARCH models and even though the SV model does not include the intraday volatility measure it frequently outperforms the GX model. The reason for this appears twofold. Firstly, GARCH models react very strongly to sharp increases in volatility at time T which leads to the overestimation of volatility at time $T + N$ as can be seen in figure 3 for the GX model with $N = 1$. Secondly, GARCH models appear to have a volatility level which is in general too high²⁰. The problem of overestimation is then exacerbated by the choice of error statistics in table 5 as overestimation is penalised more severely by the heteroskedasticity adjusted error statistics than underestimation. The value for the ratio that appears in equations (22) and (23) lies between zero and one when $E(\sigma_{T+1,T+N|T}^2) < \tilde{\sigma}_{(T+1,T+N),3}^2$, whereas the value of this term is upwardly unbounded when $E(\sigma_{T+1,T+N|T}^2) > \tilde{\sigma}_{(T+1,T+N),3}^2$.²¹

We therefore conclude that a relatively simple ARFIMA(1,d,0) model for intraday volatility in

¹⁹The SVX and GX models are based on the intraday volatility measure of equation (5). Adding the squared overnight returns to the intraday volatility measure produces very similar volatility forecasts although they are slightly less accurate; these results can be obtained from the authors on request.

²⁰Also see: Andersen *et al.* (2001b) who make the observation that daily GARCH models are not very good at assessing the current volatility level.

²¹We also calculated the RMSE and MAE error statistics which react symmetrically to under- and overestimations; in terms of model ranking little changed but relative differences between error statistics decreased considerably.

logarithmic form provides more accurate out-of-sample forecasts than other, more elaborate, volatility models even though the UC-RV and the SVX model also perform very well. When intraday data is not available, the SV model is the preferred volatility model as GARCH models perform worse in terms of error statistics due to overestimation.

7 Summary and Conclusions

In this paper we examine the predictive abilities of six volatility models which we evaluate on the basis of a realised volatility measure that is defined as the scaled sum of squared intraday returns. The models we consider can be divided into realised volatility and daily time-varying volatility models. To the first group belong the Unobserved Components (UC-RV) and the Autoregressive Fractionally Integrated Moving Average (ARFIMA-RV) models where intraday volatility is modelled directly. The Stochastic Volatility (SV) and the Generalised Autoregressive Conditional Heteroskedasticity (GARCH) models, together with their intraday extensions, are defined as daily time-varying models. We empirically investigate the out-of-sample forecasting performance of the various methods for the Standard & Poor's 100 stock index over the period 13 March to 29 December 2000 and for forecasting horizons ranging from one day to one week. We conclude that those models which include the intraday information perform better than those that are solely based on daily returns. The most accurate forecasts are then obtained with the ARFIMA-RV model, followed by the SVX and the UC-RV model. Although the GARCH model extended with intraday volatility appears to perform well when its forecasts are evaluated on the basis of regression methods, other evaluation criteria indicate that it tends to overestimate volatility. As the GARCH(1,1) model also suffers from this problem we conclude that in the absence of intraday volatility information the SV model is the preferred model for forecasting volatility.

References

- Andersen, T.G. and T. Bollerslev (1997), Intraday Periodicity and Volatility Persistence in Financial Markets, *Journal of Empirical Finance* **4**, 115–158.
- Andersen, T.G. and T. Bollerslev (1998), Answering the Skeptics: Yes, Standard Volatility Models Do Provide Accurate Forecasts, *International Economic Review* **39**, 885–905.
- Andersen, T.G., T. Bollerslev, F.X. Diebold, and H. Ebens (2001), The Distribution of Realized Stock Return Volatility, *Journal of Financial Economics* **61**, 43–76.
- Andersen, T.G., T. Bollerslev, F.X. Diebold, and P. Labys (1999), (Understanding, Optimizing, Using and Forecasting) Realized Volatility and Correlation, Working Paper, Northwestern University, Duke University, New York University and University of Pennsylvania.
- Andersen, T.G., T. Bollerslev, F.X. Diebold, and P. Labys (2001a), Applications and Case Studies - The Distribution of Exchange Rate Volatility, *Journal of the American Statistical Association* **96**, 42–55.
- Andersen, T.G., T. Bollerslev, F.X. Diebold, and P. Labys (2001b), Modeling and Forecasting Realized Volatility, Working Paper, National Bureau of Economic Research, No. 8160.
- Andersen, T.G., T. Bollerslev, and S. Lange (1999), Forecasting Financial Market Volatility: Sample Frequency vis-à-vis Forecast Horizon, *Journal of Empirical Finance* **6**, 457–477.
- Areal, N.M.P.C. and S.J. Taylor (2002), The Realized Volatility of FTSE-100 Futures Prices, *Journal of Futures Markets* **22**, 627–648.

- Barndorff-Nielsen, O.E. and N. Shephard (2001), Non-Gaussian OU Based Models and Some of Their Uses in Financial Economics (with discussion), *Journal of the Royal Statistical Society, Series B* **63**, 167–241.
- Barndorff-Nielsen, O.E. and N. Shephard (2002), Econometric Analysis of Realised Volatility and Its Use in Estimating Stochastic Volatility Models, *Journal of the Royal Statistical Society, Series B* **64**, 253–280.
- Bera, A.K. and M.L. Higgins (1993), ARCH Models: Properties, Estimation and Testing, *Journal of Economic Surveys* **7**, 305–366.
- Blair, B.J., S. Poon, and S.J. Taylor (2001), Forecasting S&P 100 Volatility: The Incremental Information Content of Implied Volatilities and High Frequency Returns, *Journal of Econometrics* **105**, 5–26.
- Bollerslev, T. (1986), Generalized Autoregressive Conditional Heteroskedasticity, *Journal of Econometrics* **31**, 307–327.
- Bollerslev, T., R.Y. Chou, and K.F. Kroner (1992), ARCH Modeling in Finance: A Review of the Theory and Empirical Evidence, *Journal of Econometrics* **52**, 5–59.
- Bollerslev, T., R.F. Engle, and D.B. Nelson (1994), ARCH Models, In R.F. Engle and D.L. McFadden (Eds.), *Handbook of Econometrics*, Volume 4, pp. 2959–3038, Elsevier Science, Amsterdam.
- Bos, C.S., P.H. Franses, and M. Ooms (2002), Inflation, Forecast Intervals and Long Memory Regression Models, *International Journal of Forecasting* **18**, 243–264.
- Brodsky, J. and C.M. Hurvich (1999), Multi-Step Forecasting for Long-Memory Processes, *Journal of Forecasting* **18**, 59–75.
- de Jong, P. and N. Shephard (1995), The Simulation Smoother for Time Series Models, *Biometrika* **82**, 339–350.
- Doornik, J.A. (1998), *Object-Oriented Matrix Programming using Ox 2.0*, Timberlake Consultants Ltd., London,
<http://www.nuff.ox.ac.uk/Users/Doornik/>.
- Doornik, J.A. and M. Ooms (2001), A Package for Estimating, Forecasting and Simulating Arfima Models: Arfima package 1.01 for Ox, Working Paper, Nuffield College, Oxford.
- Drost, F.C. and T.E. Nijman (1993), Temporal Aggregation of GARCH Processes, *Econometrica* **61**, 909–927.
- Durbin, J. and S.J. Koopman (1997), Monte Carlo Maximum Likelihood Estimation for Non-Gaussian State Space Models, *Biometrika* **84**, 669–684.
- Durbin, J. and S.J. Koopman (2000), Time Series Analysis of Non-Gaussian Observations Based on State Space Models from both Classical and Bayesian Perspectives (with discussion), *Journal of the Royal Statistical Society, Series B* **62**, 3–56.
- Durbin, J. and S.J. Koopman (2002), A Simple and Efficient Simulation Smoother for State Space Time Series Analysis, Discussion Paper, London School of Economics, Department of Statistics, (forthcoming in: *Biometrika*).
- Ebens, H. (1999), Realized Stock Volatility, Working Paper, John Hopkins University.
- French, K.R., G.W. Schwert, and R.F. Stambaugh (1987), Expected Stock Returns and Volatility, *Journal of Financial Economics* **19**, 3–29.
- Ghysels, E., A.C. Harvey, and E. Renault (1996), Stochastic Volatility, In G.S. Maddala and C.R. Rao (Eds.), *Handbook of Statistics*, Volume 14, Statistical Methods in Finance, pp. 119–191, North-Holland, Amsterdam.

- Harvey, A.C. (1989), *Forecasting, Structural Time Series Models and the Kalman Filter*, Cambridge University Press, Cambridge.
- Hendry, D.F. and J.A. Doornik (1996), *Empirical Econometric Modelling Using PcGive for Windows*, Timberlake Consultants Ltd., London.
- Hol, E. and S.J. Koopman (2000), Forecasting the Variability of Stock Index Returns with Stochastic Volatility Models and Implied Volatility, Discussion Paper, Tinbergen Institute, Amsterdam-Rotterdam, The Netherlands,
<http://www.tinbergen.nl>.
- Koopman, S.J., N. Shephard, and J. Doornik (1999), Statistical Algorithms for Models in State Space using SsfPack 2.2, *Econometrics Journal* **2**, 113–166,
<http://www.ssfpack.com>.
- Laurent, S. and J.P. Peters (2002), G@RCH 2.2: An Ox Package for Estimating and Forecasting Various ARCH Models, *Journal of Economic Surveys* **16**, 447–484.
- Martens, M. (2001), Forecasting Daily Exchange Rate Volatility Using Intraday Returns, *Journal of International Money and Finance* **20**, 1–23.
- Martens, M. (2002), Measuring and Forecasting S&P 500 Index-Futures Volatility Using High-Frequency Data, *Journal of Futures Markets* **22**, 497–518.
- Oomen, R.C.A. (2001), Using High Frequency Stock Market Index Data to Calculate, Model & Forecast Realized Return Variance, Working Paper, European University Institute.
- Poon, S. and S.J. Taylor (1992), Stock Returns and Volatility: An Empirical Study of the UK Stock Market, *Journal of Banking and Finance* **16**, 37–59.
- Quantitative Micro Software (1998), *EViews 3 User's Guide*, Quantitative Micro Software.
- Ripley, B.D. (1987), *Stochastic Simulation*, Wiley, New York.
- Shephard, N.G. (1996), Statistical Aspects of ARCH and Stochastic Volatility, In D.R. Cox, D.V. Hinkley, and O.E. Barndorff-Nielsen (Eds.), *Time Series Models in Econometrics, Finance and Other Fields*, Number 65 in Monographs on Statistics and Applied Probability, pp. 1–67, Chapman and Hall, London.
- Shephard, N. and M.K. Pitt (1997), Likelihood Analysis of Non-Gaussian Measurement Time Series, *Biometrika* **84**, 653–667.
- Sowell, F. (1992), Maximum Likelihood Estimation of Stationary Univariate Fractionally Integrated Time Series Models, *Journal of Econometrics* **53**, 165–188.
- Taylor, S.J. (1994), Modeling Stochastic Volatility: A Review and Comparative Study, *Mathematical Finance* **4**, 183–204.
- Taylor, S.J. and X. Xu (1997), The Incremental Volatility Information in One Million Foreign Exchange Quotations, *Journal of Empirical Finance* **4**, 317–340.