High-dimensional statistical arbitrage with factor models and stochastic control

Jorge Guijarro-Ordonez*

October 2019

Abstract

The present paper provides a study of high-dimensional statistical arbitrage that combines factor models with the tools from stochastic control, obtaining closed-form optimal strategies which are both interpretable and computationally implementable in a high-dimensional setting. Our setup is based on a general statistically-constructed factor model with mean-reverting residuals, in which we show how to construct analytically market-neutral portfolios and we analyze the problem of investing optimally in continuous time and finite horizon under exponential and mean-variance utilities. We also extend our model to incorporate constraints on the investor's portfolio like dollar-neutrality and market frictions in the form of temporary quadratic transaction costs, provide extensive Monte Carlo simulations of the previous strategies with 100 assets, and describe further possible extensions of our work.

Keywords: statistical arbitrage, factor models, algorithmic trading, Ornstein-Uhlenbeck process, mean reversion, stochastic control.

Word count: 10,318 words.

^{*}Department of Mathematics, Stanford University. Email address: jguiord@stanford.edu.

1 Introduction

Modeling of pairs trading based on stochastic control has been an active research topic in mathematical finance for the last few years. After the papers by Jurek and Yang (2007) and Mudchanatongsuk et al. (2008), an increasing number of models have been proposed in this framework (see, for example, Chiu and Wong (2011), Tourin and Yan (2013), and Liu and Timmerman (2013)), in which generally they assume that some statistically-designed relation between the prices of two assets is a mean-reverting stochastic process and find a dynamic optimal allocation in continuous time in some version of the classical Merton framework. More recently, a number of papers have also studied the optimal entry and exit points when trading a couple of cointegrated assets, such as Leung and Li (2015), Lei and Xu (2015), Ngo and Pham (2016), and Kitapbayev and Leung (2018).

In the high-dimensional case, however, relatively little model-based research has been conducted. Cartea and Jaimungal (2016) and Lintilhac and Tourin (2016) investigate a multidimensional generalization of the model in Tourin and Yan (2013) and apply stochastic control to solve a Merton-like problem in continuous time on a collection of cointegrated assets, with exponential utility and finite horizon. In a different direction which is not exactly statistical arbitrage, Cartea et al. (2018) address an optimal execution problem with transaction costs on a basket of multiple cointegrated assets, which they also solve with control techniques. Finally, without using stochastic control, the paper by Avellaneda and Lee (2010) carries out a data-based study of statistical arbitrage in the US equity market by proposing a factor model with mean-reverting residuals and a threshold-based bang-bang strategy. This model is further analyzed and extended by Papanicolaou and Yeo (2017), who discuss risk control and develop an optimization method to allocate the investments given the trading signals.

The previous papers in this high-dimensional framework thus either apply stochastic control to a mean-reverting process they already have or use a factor model to construct this process and then choose the trading signals based on residuals, but none of them considers the combination of these two powerful techniques. The present paper aims to fill this gap by providing a study of statistical arbitrage in a high-dimensional setting that combines factor models and the tools from stochastic control, extending the previous studies and obtaining closed-form optimal strategies which are interpretable and easy to implement computationally.

More precisely, in our framework an investor observes the returns of a high-dimensional collection of risky assets and, similar to Avellaneda and Lee (2010) and Papanicolaou and Yeo (2017), uses historical data to statistically construct a factor model such that the cumulative residuals are assumed to be mean-reverting and following an Ornstein-Uhlenbeck process. However, unlike the previous literature, these residuals may be correlated and interdependent and, based on their behavior, the investor must decide how to optimally allocate her wealth in the risky assets and a riskless security so that the expected utility of her terminal wealth is maximized and she is market-neutral. There are three main results in this paper:

First, for a big class of statistically-constructed factor models that includes PCA we show how the investor may theoretically construct market-neutral portfolios without solving any optimization problem (unlike the approach followed in Papanicolaou and Yeo (2017) or Boyd et al. (2017), for example) provided that the factor model holds, and we show how this makes the optimal allocation problem analytically tractable and guarantees market-neutrality by construction. These portfolios are explicitly computable and depend quadratically on the factor model loadings and, to the best of our knowledge, using this construction to connect factor models and stochastic control theory in statistical arbitrage is new.

Second, using these explicit market-neutral portfolios as control variables, we show how the investor should trade optimally in continuous time to maximize either an exponential utility or a Markowitz-inspired mean-variance objective, obtaining explicit analytic forms of the optimal strategies in both cases in this high-dimensional setting. The structure of these optimal strategies is related to the classical solution of the Merton problem and is affine in the deviation of the residuals from their statistical mean, thus giving a precise estimate of how much we should buy when the assets are underpriced and how much we should sell when they are overpriced, as in classical pairs trading. The coefficients are given by the solution of matrix Riccati differential equations and depend quadratically on the factor model loadings, and the strategies in both the exponential and the mean-variance case are surprisingly similar except for a non-myopic correction term that does not appear in the classical framework under a geometric Brownian motion. This arises from the fact that in our case the drift of the underlying Ornstein-Uhlenbeck process is stochastic.

The structure and the techniques to find these affine strategies are thus similar in spirit to those in the affine process literature in finance (see Duffie et al. (2003) for a broad survey), to the more recent affine control literature in algorithmic trading (see, for example, Cartea et al. (2015) and the references therein), and to the literature on extensions of the Merton problem incorporating an Ornstein-Uhlenbeck process (see, for example, Benth and Karlsen (2005), Liang et al. (2011), Fouque et al. (2015) and Moutari et al (2017), which deal with a single risky asset in the context of the Schwarz model or in geometric Brownian motion with stochastic drift or volatility; and Brendle (2006) and Bismuth et al. (2019), which consider the multiasset case again in the setting of geometric Brownian motion with stochastic drift). While the techniques that we use to find the optimal strategies are therefore classical, the framework is new because the mean-reverting behavior of the underlying stochastic process arises from the residuals of a factor model and in the context of statistical arbitrage, and we consider the general case of an arbitrary number of assets with a market-neutrality restriction. Moreover, the explicit solutions allow us to understand the dependence of the optimal strategies on specific elements of a statistical arbitrage strategy (such as the factor model, its loadings matrix and its connection with market-neutrality, and the mean-reversion speed of the residuals and their correlation structure), and to compare arbitrageurs with exponential and mean-variance utilities.

Third and finally, we extend the previous results in two directions by discussing how to incorporate into the model soft constraints frequently imposed by arbitrageurs (such as dollar-neutrality, limitations on the money spent on each asset, leverage restrictions, etc.) and also market frictions in the form of quadratic transaction costs, inspired by the papers of Garleanu and Pedersen (2013, 2016) and also by the more general quadratic transaction cost and linear price impact literature in portfolio theory (see, for example, Moreau et al. (2017) and Muhle-Karbe et al. (2017) for some new research directions and Obizhaeva and Wang (2013), Rogers and Singh (2010), Almgren and Chriss (2001) and Bertsimas and Lo (1998) for some classical papers). In both extensions, we again find explicit analytic strategies which are easily interpretable, and which quantitatively correspond to quadratic corrections in the structure of the original optimal strategies (when adding soft constraints like dollar neutrality) or to "tracking" averages of the future original optimal portfolios (when adding quadratic transactions costs). Moreover, in both cases these new strategies depend quadratically on the loadings of the factor model. Again, the novelty of the results comes from the study of these questions (dollar neutrality, transaction costs, etc.) in a new context in which they are crucial (statistical arbitrage with an arbitrary number of assets and a market-neutrality restriction, in particular using control techniques and a factor model), and this framework and the strategies that we find are new to the best of our knowledge.

To conclude the paper with a more empirical analysis, we also perform extensive numerical simulations with a high-dimensional number of assets (100). This gives further insights about the behavior of the previous strategies that are not obvious when looking at the corresponding equations, and allows us to understand the sensitivity of the model parameters and the dependence on the underlying factor model. This high-dimensional numerical study is also new with respect to the existing literature, and the main conclusions are that (1) the exponential-utility strategies are more profitable than the mean-variance strategies (and they also take more extreme positions), (2) after some initial up and downs the sample paths of the different wealth processes progressively stabilize due to the asymptotic properties of the Ornstein-Uhlenbeck process, (3) increasing the risk-control parameters (like the parameter controlling dollar neutrality) consistently produces a concentration of the distribution of the terminal wealth around smaller values, and (4) imposing market neutrality when the loadings of the factor model get bigger leads to more aggressive strategies whose terminal wealth has a higher variance.

The remainder of the article is organized as follows. In section 2 we introduce our model, construct the market-neutral portfolios that make the problem analytically tractable, and formulate the control problems. Next, section 3 presents the basic results under the exponential and the mean-variance frameworks, whereas section 4 extends these results by considering the addition of soft constraints and of quadratic transaction costs. Section 5 then allows us to understand in greater depth the behavior of the previous strategies and models by performing some Monte Carlo simulations, and section 6 presents the main conclusions and proposes future new directions of research. Finally, an appendix contains all the proofs.

2 The model

2.1 Set-up and assumptions

In the remainder of this paper we will consider the following general framework. We will assume that an investor observes the returns of a large number N of risky assets and, like in classical portfolio theory based on stochastic control, she must decide how to dynamically allocate her wealth by investing in them or in a riskless asset with constant interest rate r so that the expected utility of her wealth at a finite terminal time T is maximized. However, unlike the classical framework and the existing literature, to do so she will execute a statistical arbitrage strategy based on a factor model, in which instead of trading depending on the state of the original returns she will trade depending on the behavior of the residuals, which will be the trading signals. For example, in the case of two assets, this is equivalent to classical pairs trading, in which the investor may perform a simple linear regression on the returns of two historically correlated securities and, depending on how far the oscillation of the residual is from its historical average, she decides if there is a mispricing and opens and closes long and short positions in the original assets in a market-neutral way. In this paper, we will study the generalization of this to the high-dimensional case of an arbitrary number of assets, in which we substitute the simple linear regression by a statistical factor model and we study the optimal allocations under the framework of stochastic control, assuming a mean-reverting stochastic model for the behavior of the residuals.

More precisely, we make the following three general assumptions on how the investor will generate these residuals and what dynamics they will have:

(1) **Assumption 1:** The investor has computed a factor model for the returns of the risky

assets, which will hold during the the investment (finite) horizon and is given by

$$dR_t = \Lambda dF_t + dX_t, \tag{2.1}$$

where R_t is the cumulative asset returns process, Λ is the (constant-in-time) loadings matrix, F_t is the cumulative factors process, and X_t is the cumulative residuals process¹.

(2) **Assumption 2:** This factor model has been computed statistically by using some version of PCA², so the rows of Λ are the largest eigenvectors of some square matrix and the discrete-time version of dF_t (i.e., the daily, hourly, etc. factors returns) is then computed by linearly regressing the discrete-time version of dR_t (i.e., the daily, hourly, etc. assets returns) on some rescaling of Λ , so

$$dF_t = \tilde{\Lambda} dR_t \tag{2.2}$$

for some rescaling $\tilde{\Lambda}$ of Λ . (In fact, the only thing we need about this assumption is that (2.2) holds for some matrix $\tilde{\Lambda}$, which allows for a bigger class of factor models than classical PCA).

(3) Assumption 3: The process X_t given by the cumulative residuals is mean-reverting. In particular, for analytic tractability we assume that it is a matrix N-dimensional Ornstein-Uhlenbeck process satisfying the following stochastic differential equation with known parameters

$$dX_t = A(\mu - X_t)dt + \sigma dB_t,$$

where A is a constant N-dimensional square matrix whose eigenvalues have positive real parts so that there is mean-reversion, μ is a constant N-dimensional vector, B_t is a vector of m independent Brownian motions in the usual complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{0 \le t \le T})$, and σ is a constant $N \times m$ matrix such that the instantaneous covariance matrix $\sigma \sigma'$ is invertible.

The previous framework thus combines high-dimensional statistical arbitrage, factor models and stochastic control in a way which is new to the best of our knowledge, and it extends several models in the existing literature. For example, statistical arbitrage models based on a more particular case of Assumptions 1, 2, 3 (in which the residuals are assumed to be independent one-dimensional Ornstein-Uhlenbeck processes, so A and σ are diagonal) and in which no stochastic control methods are applied have been studied empirically in the US equity market by Avellaneda and Lee (2010) and Papanicolaou and Yeo (2017). In a different direction, if we consider the particular case of removing the factor model by making $\Lambda=0$, we have the situation in which the returns themselves are globally mean-reverting following a matrix Ornstein-Uhlenbeck process, which has also been studied empirically and analytically using stochastic control techniques in the context of optimal execution in Cartea, Gan, and Jaimungal (2018), and in the context of statistical arbitrage in Cartea and Jaimungal (2016) in the particular case in which A has rank one.

¹Here we have decided to write the factor model in a somewhat unusual differential, continuous-time form in terms of the cumulative residuals and returns because of notational simplicity for this section of the paper. In practice, however, the factor model will be estimated in discrete time, by replacing the differentials by the corresponding discrete increments (so, for instance, dR_t should be replaced by the daily, hourly, etc. asset returns, dF_t would be just the corresponding daily, hourly, etc. factors returns, and so forth). In any case we will only use this notation and framework in this section of the paper, and the reader may look at Avellaneda and Lee (2010) for essentially the same continuous/discrete time framework and some estimation techniques. Note also that the constant loadings assumption is realistic in short time horizons.

²See Letteau and Pelger (2018) and Pelger and Xiong (2018) for some new versions of high-dimensional PCA that might be particularly interesting for this problem.

2.2 Making the model tractable for stochastic control and imposing marketneutrality

Unlike the classical literature on portfolio choice based on stochastic control, choosing as control variables the amount of capital that the agent invests in each of the N risky assets of the previous framework might make the optimal allocation problem intractable. Indeed, since we only have information about the dynamics of the residuals (and not directly about the returns like in the classical framework), these residuals are not independent, and the factors themselves depend on the returns, the classical approach would lead to complicated interdependencies. Moreover, since the investor is executing a statistical arbitrage strategy, we would need to incorporate additional market neutrality constraints so that the returns of the strategy do not depend on the model factors, but just on the idiosyncratic component of the model given by the residuals. This would complicate the problem further, and might require numerical optimization methods as done in Papanicolaou and Yeo (2017) and Boyd et al. (2017).

In this paper, on the contrary, we deal with both problems simultaneously and we solve them analytically by following a new approach. This is based on the following proposition, which shows that, by using the N risky assets at our disposal, it is actually possible to construct analytically N market-neutral portfolios whose returns only depend on *one* coordinate of X, which greatly simplifies the complexity of the problem and makes it analytically tractable:

Proposition 2.1. Under the previous assumptions, it is possible to construct explicitly N market-neutral portfolios such that investing any real number π_{it} of dollars in the i-th one at time t yields an instantaneous return of $\pi_{it}dX_{ti}$ (and hence a combined return of $\pi_t dX_t$).

Moreover, the total amount of capital invested at time t by doing so is $\pi_t \cdot p$ for an explicit constant-in-time vector $p \in \mathbb{R}^N$, which depends quadratically on the factor model loadings.

Proof. The mathematical construction of the market-neutral portfolios under the given assumptions is surprinsingly straightforward and involves just a linear projection. Indeed, (2.1) implies that

$$dR_{ti} = \sum_{j} \Lambda_{ij} dF_{tj} + dX_{ti},$$

whereas (2.2) yields

$$dF_{tj} = \sum_{k} \tilde{\Lambda}_{jk} dR_{tk}.$$

Combining the two previous equations we find that, for $c_{ik} := \sum_{j} \tilde{\Lambda}_{jk} \Lambda_{ij}$,

$$dR_{ti} = \sum_{k} \left(\sum_{j} \tilde{\Lambda}_{jk} \Lambda_{ij} \right) dR_{tk} + dX_{ti} = \sum_{k} c_{ik} dR_{tk} + dX_{ti}.$$

Thus, if at time t we hold the (explicitly constructible) constant-in-time portfolio given by

$$\tilde{p}_i := (-c_{i1}, -c_{i2}, \dots, -c_{i,i-1}, 1 - c_{ii}, -c_{i,i+1}, \dots, -c_{iN})$$

(i.e., we invest $-c_{i1}$ dollars in the first asset, $-c_{i2}$ dollars in the second one, and so on), we automatically obtain an instantaneous return of dX_{ti} , which is market neutral and depends only on the *i*th coordinate of the process X_t . Further, from the above equations it is also obvious that for any real number π_{it} , $\pi_{it}\tilde{p}_i$ will also be market-neutral and yielding a return of $\pi_{it}dX_{ti}$, and the same applies to $\sum_i \pi_{it}\tilde{p}_i$, which will have a return of $\sum_i \pi_{it}dX_{ti} = \pi_t \cdot dX_t$.

Finally, regarding the last part of the statement just observe that the total amount of capital invested in the strategy $\pi_t = (\pi_{it})_{1 \leq i \leq N}$ at time t is simply

$$\sum_{i} (\pi_{it} \tilde{p}_i) \cdot \mathbb{1} = \sum_{i} \pi_{it} (\tilde{p}_i \cdot \mathbb{1}) = \pi_t \cdot p$$

where $p := (\tilde{p}_i \cdot \mathbb{1})_{1 \le i \le N}$, which concludes our proof.

Remark 2.1. Note in particular that, if Λ or $\tilde{\Lambda}$ are sparse matrices, then most of the c_{ik} in the above construction will be 0, so the investor will be investing in a few number of assets in each market-neutral portfolio and this could significantly reduce his transaction costs while rebalancing his positions. In particular, Pelger and Xiong (2018) discuss a way of obtaining this kind of sparse factor model.

The key consequence of the above proposition is that, if we choose as control variables the amount of capital π_t that we wish to invest in these N market-neutral portfolios (instead of directly in the original assets) at time t, the dynamics of the problem get remarkably simpler, they *only* depend separately on the coordinates of X, and we have market-neutrality by construction. This solves simultaneously the two problems we discussed before and allows us to connect stochastic control and the factor model in a simple way, and it is therefore the approach which we will adopt in the remainder of this paper.

Note also that, under these new control variables, all the information about the factor model and in particular about its loadings matrix is now encoded in the vector p, which will play an important role in the remaining sections. Moreover, some statements about the strategies must be rewritten in terms of it within this new framework. For instance, in the new setting a strategy $(\pi_t)_{0 \le t \le T}$ is dollar-neutral at t if $p \cdot \pi_t = 0$, since as we mentioned before $p \cdot \pi_t$ is the total capital spent at time t.

2.3 Formulation of the control problems

Under the previous framework, now we formulate rigorously the control problems we will study in the paper. We suppose that the investor executes the following trading strategy: at each time $t \in [0,T]$, she invests π_t dollars in the risky market-neutral portfolios we constructed in Proposition 2.1, and she invests her remaining wealth (or borrow money if the remaining wealth is negative) in the risk-free asset with constant interest rate r, so that the resulting strategy is self-financing. Thus, assuming for the moment no market frictions or other constraints (which will be both considered in section 4), the evolution of her wealth is given by the equation

$$dW_t = \pi_t \cdot dX_t + (W_t - \pi_t \cdot p)rdt \tag{2.3}$$

and she aims to choose π_t to maximize the expected utility of her terminal wealth (that is, $u(W_T)$ for a given utility function u).

Supposing further that she trades continuously in time, this means that mathematically she must solve the high-dimensional non-linear stochastic optimization problem given by

$$H(t, x, w) = \sup_{\pi \in \mathcal{A}_{[t, T]}} \mathbb{E}_{t, x, w} \left[u(W_T) \right]$$
(2.4)

subject to

$$dW_t = \left(\pi_t' A(\mu - X_t) + (W_t - \pi_t' p)r\right) dt + \pi_t' \sigma dB_t$$

$$dX_t = A(\mu - X_t)dt + \sigma dB_t,$$

where the admissible set $\mathcal{A}_{[t,T]}$ is the set of all the \mathcal{F}_s -predictable and adapted processes $(\pi_s)_{s\in[t,T]}$ in \mathbb{R}^N with the minimal technical restrictions that $\mathbb{E}[\int_t^T ||\pi_s||^2 ds] < \infty$ (so Ito's formula may be applied and doubling strategies are excluded) and the above SDEs have an unique (strong) solution, and ' indicates transposition.

Finally, the associated dynamic programming equation of the problem is non-linear and (N+2)-dimensional, and is given by

$$0 = \partial_t H + (\mu - x)' A' \nabla_x H + \frac{1}{2} \text{Tr}(\sigma \sigma' \nabla_{xx} H) + \sup_{\pi} \left(\left(\pi' A(\mu - x) + (w - \pi' p)r \right) \partial_w H + \frac{1}{2} \pi' \sigma \sigma' \pi \partial_{ww} H + \pi' \sigma \sigma' \nabla_{xw} H \right)$$
(2.5)

with terminal condition H(T, x, w) = u(w).

The problem is therefore formally related to the classical Merton framework, but instead of a geometric Brownian motion there is a multidimensional Ornstein-Uhlenbeck process which makes it impossible to combine the dynamics of W and X into a single equation and to get rid of the N-dimensional state variable x. Moreover, unlike the previous studies on extensions of the Merton problem with an Ornstein-Uhlenbeck process discussed in section 1, in (2.4) and (2.5) the mean-reverting behavior of the underlying stochastic process arises in the context of statistical arbitrage and from the residuals of a factor model (which is encoded in the vector p of the equations above and which will play an important role in the following sections), and we consider the general case of an arbitrary number of assets with a market neutral restriction. Furthermore, the model will be extended in section 4 to incorporate other important features of statistical arbitrage strategies, like dollar neutrality restrictions and transaction costs, and we will analyze the impact of the factor model on these extensions.

Quite surprisingly, the previous problems admit interpretable closed-form solutions – which is computationally useful in this high-dimensional setting, and which allows us to understand the influence of the model parameters and especially of the factor model – in the cases in which the utility is exponential or of a Markowitz-inspired mean-variance type (but not for other usual choices of utility functions, like the HARA family). This is what we will show in the following two sections, first for the simple setup of (2.4) and (2.5) in section 3, and then extending the model in section 4 to incorporate soft constraints on the investor's portfolio and quadratic transaction costs.

3 The frictionless results

In this section we therefore present the closed-form, optimal strategies for the problem given by (2.4) and (2.5) in the cases in which the utility is exponential or of a mean-variance type, discussing the former in the first subsection and the latter in the second one. While the techniques that we use are classical, in both cases the explicit solutions allow us to gain insight on the new framework of statistical arbitrage with a factor model and will be the basis for the extensions of section 4.

3.1 The exponential utility case

In this first setting, the complete, explicit description of the optimal strategy is given by the following main theorem (see Cartea and Jaimungal (2016) and Lintilhac and Tourin (2016) for related results with an exponential utility):

Theorem 3.1. Under an exponential utility (so $u(w) = -e^{-\gamma w}$ for some $\gamma > 0$) and the technical condition described in our verification theorem (Proposition 3.2 below), the optimal portfolio to have at time t according to (2.4) is explicitly computable and given by

$$\pi_t^* = (\sigma \sigma')^{-1} \frac{A(\mu - X_t) - pr}{\gamma e^{r(T-t)}} + \frac{A'(\sigma \sigma')^{-1}}{\gamma e^{r(T-t)}} \left((A(\mu - X_t) - pr)(T-t) - Apr \frac{(T-t)^2}{2} \right).$$

The result follows from the following two propositions, whose proof is given in Appendix A.1 using classical stochastic control techniques:

Proposition 3.1 (Solving the PDE). The value function H associated to (2.4) and (2.5) when $u(w) = -e^{-\gamma w}$ is explicitly computable and admits the probabilistic representation $H(t, x, w) = -\exp(-\gamma(we^{r(T-t)} + h(t, x)))$ where

$$h(t,x) = \mathbb{E}_{t,x}^* \left[\int_t^T \frac{1}{2\gamma} (A(\mu - Y_s) - pr)'(\sigma \sigma')^{-1} (A(\mu - Y_s) - pr) \ ds \right]$$

and $dY_t = rpdt + \sigma dB_t^*$ for a new Brownian motion B^* under a new probability law \mathbb{P}^* . The associated optimal control in feedback form is then

$$\pi^* = -(\sigma\sigma')^{-1} \frac{\mathcal{D}H}{\partial_{ww}H} \tag{3.1}$$

where $\mathcal{D}H = (A(\mu - x) - pr) \partial_w H + \sigma \sigma' \nabla_{xw} H$.

Proposition 3.2 (Verification). The strategy given in Theorem 3.1. is indeed optimal if

$$4 \max_{0 \le s \le T} ||\Lambda_0(s)|| < 1$$
 and $32 \max_{0 \le s \le T} ||\Lambda_1(s)|| < 1$,

where $\Lambda_0(s)$ and $\Lambda_1(s)$ are the diagonal matrices containing, respectively, the eigenvalues of $\Omega^{1/2}(C_0 + C_0')\Omega^{1/2}(s)$ and $\Omega^{1/2}C_1C_1'\Omega^{1/2}(s)$, for

$$C_0(s) = A'(\sigma\sigma')^{-1}A(I_N + A(T - s)), \quad C_1(s) = A'(\sigma\sigma')^{-1}(I_N + A(T - s))\sigma$$

$$\Omega(s) = \int_0^s e^{-A(s-u)}\sigma\sigma'e^{-A'(s-u)}du.$$

Besides being a closed-form strategy which is easily implementable in our high-dimensional setting, the above optimal portfolio is also interpretable. Indeed, the first term of the optimal policy is Merton-like in that it represents the drift of the underlying process (which here is stochastic unlike in the classical geometric Brownian motion) minus the adjusted risk-free rate (which here depends on the loadings of the factor model via p), and divided by a measure of the volatility (which is given by $\sigma\sigma'$, the instantaneous quadratic covariation of X) and the Arrow-Pratt coefficient of absolute risk-aversion of the value function with respect to the wealth w (i.e., $-\partial_{ww}H/\partial_wH$), which is the product $\gamma e^{r(T-t)}$, where γ is the risk aversion parameter of our utility function and the factor $e^{r(T-t)}$ measures the gains from interest between t and T.

On the other hand, the second summand is a non-myopic correction term which again depends linearly on the drift of X, and whose effect vanishes when we approach the terminal time T. Moreover, note that, while the first term does not depend explicitly on the terminal time T, this correction term does, reflecting the fact that, since there are non-zero interest rates and moreover the behavior of the residuals is oscillating, the investor must keep in mind the final horizon to decide if she bets on the mean-reversion cycle before that time. Finally, observe that, quite naturally, in both terms as the risk-aversion parameter γ , the instantaneous volatility $\sigma\sigma'$, or the interest rate r increase, the optimal portfolio vector π_t^* gets closer to 0, implying that the investor will simply invest most of her wealth in the riskless asset.

The above strategy also seems intuitive within our particular framework of statistical arbitrage with a factor model, and sheds further light on the problem. Indeed, note that the current state of the residual process X_t only appears in the strategy through the terms in $A(\mu - X_t)$, which essentially tells us to invest more in the risky assets the further their residuals are from their historical mean μ and in a way proportional to the historical mean reversion speed given by A, like in classical pairs trading. Moreover, all the remaining terms depend jointly on the factor model and the interest rate through the term pr, which reflects the cost of the leverage associated to imposing market-neutrality through the loadings of the factor model. In particular, note that, the bigger the loadings of the factor model are (and hence the bigger p is), the more we will need to invest to achieve market neutrality (again like in pairs trading with a big beta) and the bigger our leverage will be, and this will affect the optimal strategy depending on the interest rate r.

Finally, regarding the technical optimality conditions, intuitively they arise from the fact that $H(t, X_t, W_t^*)$, the value function evaluated at the wealth process W_t^* corresponding to the optimal strategy, may blow up because of the exponential function coming from the exponential utility. In particular, since W_t^* ends up being an Ito process depending quadratically on X_t and X_t is Gaussian, the term $\exp(-\gamma W_t^* e^{r(T-t)})$ is related to the moment generating function of a chi-squared distribution, which blows up far away from 0. Thus, these technical conditions are just ensuring that the corresponding functions are integrable. Interestingly, this does not depend on the risk-aversion parameter γ , the interest rate r, or the factor model used (captured by p), but just on the parameters of X and the terminal time T.

3.2 The mean-variance case

In the second, Markowitz-inspired mean-variance framework, the investor tries to maximize her expected terminal wealth but at the same time she continuously penalizes at each instant the instantaneous variance (i.e., the volatility) of her wealth process according to a volatility-aversion function $\gamma(t)$. The optimal strategy in this case is again available in closed form and interpretable and, quite remarkably, for an appropriate choice of this volatility-aversion function, we obtain exactly the same optimal policy as in the exponential case but without the correction term. This is shown in the following theorem, whose proof is given in Appendix A.2. using classical control techniques:

Theorem 3.2. If $\gamma(t)$ is continuous and positive on [0,T], the problem in (2.4) with the following mean-variance objective function

$$H(t, x, w) = \sup_{\pi \in \mathcal{A}_{t, T}} \mathbb{E}_{t, x, w} \left[W_T - \int_t^T \frac{\gamma(s)}{2} \frac{d}{d\tau} \operatorname{Var}_s(W_\tau)|_{\tau = s} ds \right]$$

has explicit optimal portfolio at t given by

$$\pi_t^* = (\gamma(t)\sigma\sigma')^{-1} \left(A(\mu - X_t) - pr \right) e^{r(T-t)}.$$

In particular, for $\gamma(t) = \gamma e^{2r(T-t)}$, the above optimal policy is the same as the first term of the corresponding one in Theorem 3.1.

This unexpected connection between the mean-variance and the exponential utility cases may in fact be explained heuristically. Indeed, supposing that r=0 for the sake of simplicity and considering a second order approximation of the exponential we get $-\exp(-\gamma W_T) \approx -1 + \gamma W_T - \gamma^2 W_T^2/2$, and for maximization purposes when conditioned on t this behaves essentially as $\mathbb{E}_{t,x,w}[\gamma W_T - \gamma^2 \operatorname{Var}_t(W_T)/2]$. Rewriting this variance as the integral of the corresponding instantaneous variances and dividing by γ we obtain exactly the previous objective function, showing moreover that the correction term of Theorem 3.1 that does not appear in this case is heuristically associated to the moments of order higher than 2 of the exponential utility.

Regarding the interpretation of the mean-variance strategy within our context of statistical arbitrage and its connection with the exponential-utility arbitrageur, there are two important remarks.

First, as we mentioned, the optimal strategy here is the same as the myopic part of the exponential case modulo the value of $\gamma(t)$. In particular, this means that, unlike the exponential arbitrageur, the mean-variance arbitrageur will not take into account the expected number of mean-reversion cycles until the terminal time T. Moreover, for a non-zero interest rate and a constant volatility aversion $\gamma(t)$, the mean-variance arbitrageur is bolder than the corresponding exponential investor with the same γ , since she will invest significantly more money (quantitatively, by a factor of $e^{2r(T-t)}$) in going long or short, taking more aggresive positions the higher the interest rate is and the sooner it is with respect to the terminal date.

Second, the optimal strategy has two components like in section 3.1: one term in $A(\mu - X_t)$ which measures how far the residuals are from their historical mean and how fast they will mean-revert (like in classical pairs trading), and a second term in pr linked to the factor model, which measures the cost of the leverage associated to imposing market neutrality. In particular, note that, the bigger the loadings of the factor model are (and hence the bigger p is), the more aggressive the positions will be and the more leverage the investor will have if $r \neq 0$.

4 Two extensions

In this final theoretical section of the paper, we complete the picture described in the previous two sections by considering two important and new extensions within the context of statistical arbitrage with a factor model. In the first subsection, we show how to incorporate in the above strategies soft constraints frequently imposed by arbitrageurs with the example of dollar-neutrality, whereas in the second one we introduce market frictions in the form of quadratic transaction costs. In both cases, we obtain again closed-form analytic solutions which are interpretable, convenient from a computational perspective in our high-dimensional setting, and which shed further light on the influence of the factor model and its connection with market neutrality.

4.1 Incorporating soft constraints on the admissible portfolios

While imposing restrictions on the portfolios by introducing hard constraints directly on the admissible set $\mathcal{A}_{t,T}$ leads in general to problems that must be solved numerically (and hence potentially unfeasible in a high-dimensional setting), it is still possible to impose many additional soft constraints in the two frameworks of section 3 without increasing significantly the difficulty of the problems, by just adding a carefully chosen penalty term to the corresponding objective function.

As an illustration of this, we give in the next corollary the corresponding optimal strategies when a dollar-neutrality restriction is softly enforced. To do so, recall that, within the framework of section 2 that imposed market-neutrality within the factor model, a strategy π_t is dollar neutral if $p \cdot \pi_t = 0$, which means that the total amount of capital invested at time t is 0. Hence, we can softly enforce dollar neutrality by replacing the wealth process of Theorems 3.1 and 3.2 by the penalized wealth process defined by $d\tilde{W}_t := dW_t - \alpha(t)(\pi_t \cdot p)^2/2dt$ for a certain general time-dependent penalty function $\alpha(t)$. This penalizes non dollar-neutrality (i.e., $\pi_t \cdot p \neq 0$) at each time and is quadratic to simplify the optimization process.

The proof follows exactly the same lines as in the previous two cases and is obtained from them by small modifications, so we will omit it for the sake of brevity.

Corollary 4.1. Suppose that dollar neutrality is softly enforced by replacing the wealth process of Theorems 3.1 and 3.2 by the penalized wealth process defined by $d\tilde{W}_t := dW_t - \alpha(t)(\pi_t \cdot p)^2/2dt$. Then

(1) The problem with mean-variance utility has optimal portfolio at t given by

$$\pi_t^* = (\gamma(t)\sigma\sigma' + \alpha(t)pp')^{-1} (A(\mu - X_t) - pr) e^{r(T-t)}.$$

(2) The problem with exponential utility has optimal portfolio at t given by

$$\pi_t^* = (\gamma e^{r(T-t)} \sigma \sigma' + \alpha(t) p p')^{-1} \left(A(\mu - X_t) - pr - \gamma \sigma \sigma' (b(t) + c(t) X_t) \right).$$

where c(t) is an $N \times N$ symmetric matrix and b(t) is an N-dimensional vector, vanishing when $t \to T$, and with coordinates depending on A, σ, rp, γ and $\alpha(t)$. In particular, c(t) is given by the solution of the matrix Riccati ODE

$$0 = \partial_t c - A'c - cA - \gamma c\sigma\sigma'c + e^{r(T-t)}(A + \gamma\sigma\sigma'c)'M(t)(A + \gamma\sigma\sigma'c)$$

and b(t) is the solution of the linear system of ODEs

$$0 = \partial_t b - A'b + cA\mu - e^{r(T-t)}(A + \gamma\sigma\sigma'c)M(t)(A\mu - pr - \gamma\sigma\sigma'b) - \gamma c\sigma\sigma'b,$$

both with terminal conditions
$$b(T) = c(T) = 0$$
 and where $M(t) = (\gamma \sigma \sigma' e^{r(T-t)} + \alpha(t)pp')^{-1}$.

The resulting optimal policies have therefore the same structure as the two previous strategies of section 3, but now the additional term $\alpha(t)pp'$ has been introduced in the inverse to enforce the dollar-neutrality condition. This again depends on the factor model via p and is related to how extreme the capital positions will be because of the market-neutrality restriction, which depends directly on the loadings matrix and hence on p. Note in particular that, the bigger the loadings of the factor model are, the bigger $\alpha(t)pp'$ will be and hence the bigger the impact of the dollar neutrality restriction will be.

4.2 Incorporating quadratic transaction costs

In this subsection, we finally extend our model to incorporate market frictions in the form of transaction costs, which play a crucial role when executing statistical arbitrage strategies. We consider in particular quadratic transaction costs, which are in general a measure of price impact or illiquidity and which make the model analytically tractable.

To do so, rather than looking directly at the amount of capital π_t invested in the risky assets at time t as the control variables, we consider instead the trading intensity I_t at which these investements will be made at time t, which is therefore given by $d\pi_t = I_t dt$. We can now adapt to our setting the model for temporary transaction costs introduced in Garleanu and Pedersen (2016), who posit (providing a market microstructural justification and referring to empirical research) that these transaction costs at time t may be represented quadratically as I'_tCI_t for a certain symmetric positive-definite matrix C^3 , which essentially comes from the assumption that the price impact of the investor's actions is linear on its trading intensity I_t .

Under this framework, we can then rewrite the performance criteria of Theorem 3.2 (for the sake of brevity, we will just deal with the mean-variance case) by incorporating the adverse effect caused by these transaction costs on the investor's wealth as a running penalty, obtaining the stochastic optimization problem given by

$$H(t, x, w, \pi) = \sup_{I \in \mathcal{A}^*} \mathbb{E}_{t, x, w, \pi} \left[W_T - \int_t^T \frac{\gamma(s)}{2} \frac{d}{d\tau} |\operatorname{Var}_s(W_\tau)|_{\tau = s} ds - \frac{1}{2} \int_t^T I_s' C I_s ds \right]$$
(4.1)

in which as we mentioned the new control variable is I; t, x, w, π are now state variables; and \mathcal{A}^* is the set of all \mathcal{F} -adapted predictable processes I_t such that the corresponding SDEs have a unique (strong) solution for any initial data and both I_t and the resulting π_t given by $d\pi_t = I_t dt$ are again in $L^2(\Omega \times [0,T])$. Thus, the investor aims to maximize her terminal wealth, but penalizing at each instant both for the risk of her strategy (measured by the volatily of her wealth process) and for the price impact caused by her actions (reflected in the quadratic transaction costs).

In this new setting, it is again possible to find explicitly the optimal strategy that the investor should follow, which is described in detail in the next theorem:

Theorem 4.1. If $\gamma(t) \geq 0$ (i.e., non-negative volatility aversion) and is continuous, the optimal strategy in the above problem "tracks" a moving aim portfolio $Aim(t, X_t)$ with a tracking speed of Rate(t), according to the following ODE describing the evolution of the optimal trading intensity $I_t = d\pi_t/dt$

$$I_t = \operatorname{Aim}(t, X_t) + \operatorname{Rate}(t)\pi_t$$

where Rate(t) is a $N \times N$ negative-definite matrix tending to 0 when $t \to T^4$, and $Aim(t, X_t)$ admits the probabilistic representation

$$Aim(t, x) = \int_{t}^{T} f(s) \mathbb{E}_{t, x}[Frictionless(s)] ds$$

where Frictionless(s) is the optimal portfolio at time s in the frictionless case of section 3.2. and f(s) is a certain averaging function given in Proposition 4.3 below.

³The assumption that C is symmetric is without loss of generality, since if the transaction costs are given by $I'_t \tilde{C} I_t$ for a non-symmetric \tilde{C} , then one can see that by considering the *symmetric* part of \tilde{C} (given by $C := (\tilde{C} + \tilde{C}')/2$) we have that $I'_t \tilde{C} I_t = I'_t C I_t$.

⁴ and given by the solution of a matrix Riccati ODE specified in the Porposition 4.2 below.

Furthermore, the optimal portfolio is then

$$\pi_s^* = \pi_t + \int_t^s : \exp\left(\int_u^s \operatorname{Rate}(v)dv\right) : \operatorname{Aim}(u, X_u)du,$$

where the notation : $\exp(\int_u^t \cdot ds)$: represents the time-ordered exponential⁵.

Remark 4.1. If in particular the investor has constant volatility aversion (so $\gamma(t) = \gamma$), the matrix Riccati ODE is explicitly solvable and

$$Rate(t) = C^{-1/2}D \tanh(D(t-T))C^{1/2}$$

for $D:=(\gamma C^{-1/2}\sigma\sigma'C^{-1/2})^{1/2}$. Moreover, if the transaction costs are proportional to the volatility (i.e., $C=\lambda\sigma\sigma'$ for $\lambda>0$, see Garleanu and Pedersen (2013, 2016) for a market microstructural justification) then this rate is indeed a scalar given by $\sqrt{\frac{\gamma}{\lambda}} \tanh\left(\sqrt{\frac{\gamma}{\lambda}}(t-T)\right)$ and $\exp\left(\int_u^s \mathrm{Rate}(v)dv\right) := \cosh\left(\sqrt{\frac{\gamma}{\lambda}}(s-T)\right)/\cosh\left(\sqrt{\frac{\gamma}{\lambda}}(u-T)\right)$.

The result follows from the following sequence of three propositions, which are proved in Appendix A.3:

Proposition 4.1 (Conjectured solution). The solution of the HJB equation associated to the problem is $H(t, x, w, \pi) = e^{r(T-t)}w + \frac{1}{2}\pi'a(t)\pi + \pi'b(t, x) + d(t, x)$ if there exist a $N \times N$ symmetric matrix a(t), a N-dimensional vector b(t, x) and a scalar function d(t, x) satisfying

(1) The matrix Riccati ODE

$$\partial_t a - \gamma(t)\sigma\sigma' + aC^{-1}a = 0 \tag{4.2}$$

with terminal condition a(T) = 0.

(2) The vector-valued and the scalar linear parabolic PDEs

$$(\partial_t + \mathcal{L}_X)b + e^{r(T-t)}(A(\mu - x) - rp) + a'C^{-1}b = 0$$
(4.3)

$$(\partial_t + \mathcal{L}_X)d + \frac{1}{2}b'C^{-1}b = 0$$
 (4.4)

with terminal conditions b(T, x) = d(T, x) = 0 and where \mathcal{L}_X is the infinitesimal generator of X, acting coordinatewise.

The hypothesized optimal trading intensity at (t, x, w, π) is then $I^* = C^{-1}(a(t)\pi + b(t, x))$.

Proposition 4.2 (Existence of solutions). .

(1) If $\gamma(t) \geq 0$ (non-negative volatility-aversion) and is continuous, then the Riccati equation (4.2) has a symmetric, bounded and negative definite solution on all [0,T]. In particular, for $\gamma(t) = \gamma$, the solution is

$$a(t) = C^{1/2}D\tanh(D(t-T))C^{1/2}$$

for
$$D := (\gamma C^{-1/2} \sigma \sigma' C^{-1/2})^{1/2}$$
.

⁵Recall that the time-ordered exponential of a time-dependent matrix A(s) is defined as $: \exp(\int_u^t A(s)ds) := \lim_{\|\mathcal{P}\|\downarrow 0} \prod_{i=1}^{n_{\mathcal{P}}} \exp(A(t_i)\Delta t_i)$, where $\mathcal{P} := \{u = t_0, t_1, \dots, t_n = t\}$ is a partition of [u, t], $\Delta t_i := t_i - t_{i-1}$, and the product is ordered increasingly in time. If A(s) is a scalar, then obviously $: \exp(\int_u^t A(s)ds) := \exp(\int_u^t A(s)ds)$

(2) Moreover, under this condition the parabolic PDEs (4.3) and (4.4) have an unique solution satisfying a polynomial growth condition in x, and this solution admits the probabilistic representation

$$b(t,x) = \mathbb{E}_{t,x} \left[\int_t^T : \exp\left(\int_t^s a'(u) C^{-1} du \right) : e^{r(T-s)} (A(\mu - X_s) - rp) ds \right]$$
(4.5)

$$d(t,x) = \frac{1}{2} \mathbb{E}_{t,x} \left[\int_{t}^{T} b(t,X_s)' C^{-1} b(t,X_s) ds \right]$$
 (4.6)

Furthermore, b has linear growth in x u and d has quadratic growth in x, both uniformly in t.

Proposition 4.3 (Verification). Under the assumptions of the previous proposition, the trading intensity given in Theorem 4.1 is indeed optimal with the choices

$$\operatorname{Aim}(t,x) = C^{-1}b(t,x), \quad \operatorname{Rate}(t) = C^{-1}a(t), \quad f(s) = C^{-1} : \exp\left(\int_t^s \operatorname{Rate}(u)' du\right) : \gamma(s)\sigma\sigma'.$$

The interpretation of the above strategy (which is again explicit and hence easily implementable in practice) is again intuitive and complementary to the infinite-horizon model of Garleanu and Pedersen: in the presence of temporary quadratic transactions costs, the investor trades with a certain decreasing rate Rate(t) towards an aim portfolio $Aim(t, X_t)$ depending on the time and the mean-reversion state of the signals X_t . This aim portfolio is given by a weighted average of the future optimal strategies in the frictionless case, reflecting the fact that now trading is not free and thus to enter a trade the investor must weight the future outcomes derived from the strategy. Moreover, as shown in the above remark, the trading rate is bounded by 1 because of the properties of tanh, depends on t unlike the infinite-horizon case, and is naturally decreasing in the transaction cost parameter λ (or in general in C) and increasing in the volatility aversion parameter γ .

Finally, regarding the influence of the factor model and the imposition of market neutrality in this new setting, note that Rate(t) is insensitive to it (since it only depends on the risk aversion parameter γ , the volatility of the residual process $\sigma\sigma'$, and the transaction cost matrix C) and in Aim(t,x) it only appears through the term $\mathbb{E}_{t,x}[\text{Frictionless}(s)]$ and hence only when considering the future optimal strategies in the frictionless case, which has been described in section 3. Likewise, the residual process X_s only affects the strategy through this same term and hence, as seen in section 3 when studying these frictionless cases, only through the distance between this residual and its historical mean, like in classical pairs trading.

5 Monte Carlo simulations

We finally conclude the paper by providing some high-dimensional numerical simulations that give new insights about the behavior of the previously discussed strategies and their sensitivity to the different parameters, emphasizing in a separate simulation the role of the factor model and its connection with market-neutrality. To this end, we first simulate a large number of paths of X in a high-dimensional setting (in particular, we choose N=100) by using exact Monte Carlo sampling along a discrete time grid, and we then execute the previous strategies for some parametric choices of X and some values of p to compute sample paths of π_t and W_t and

histograms of the terminal Profit & Loss (P&L). We have therefore opted to defer systematic (out-of-sample) experiments with real data to a separate paper, since examining carefully the delicate issues of asset selection, rebalance frequency, construction of the factor models and obtention of X, high-dimensional parameter estimation and updating, risk control, etc. that the problem requires would be impossible to consider here without prohibitively extending the length of the paper.

During all this section, we therefore fix the following parameters for our model:

$$N = 100, \quad \mu = 0, \quad X_0 = \mu,$$

A is diagonal with entries drawn i.i.d from a normal distribution of mean 0.5 and standard deviation 0.1, and the coefficients of σ are drawn i.i.d from a uniform distribution in [-0.3, 0.3], except for the diagonal elements which are drawn from a uniform distribution in [0, 0.5]. Furthermore p=1 for the first simulations, and we will also perturb it later to study different factor model regimes and the impact of imposing market neutrality. We also fix a finite horizon of T=20 and a temporal grid $0=t_0< t_1< \dots t_L=T$ obtained by discretizing [0,T] with constant $\Delta t=T/L=20/400=0.5$.

From a financial perspective, the above choice of parameters means that the 100 coordinates of X are correlated and mean-revert with similar speeds (given by the eigenvalues of A) to an equilibrium of 0, describing an average number of approximately 10 oscillation cycles of ups and downs before the terminal time (given by the product of T and the average mean-reversion speed). The choice of p=1 arises when the asset returns themselves are mean-reverting and may be modeled directly by X so we can take $\Lambda=0$ in our factor model, while the perturbations of p will imply departing from this assumption to factor models with heavier loadings, in which imposing market neutrality leads to more leveraged positions. As an illustration, the reader may look at sample paths of the first four coordinates of such a process X in Figure 1 below.

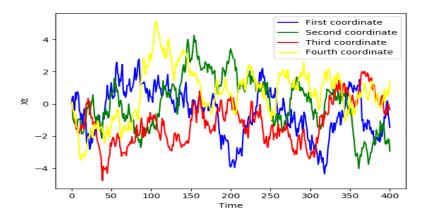


Figure 1: Sample paths of the first four coordinates of X in [0,T]

We then sample M=1,000 paths of X on this grid exactly with standard Monte Carlo techniques by using the fact that, since

$$X_{t+\Delta t} = e^{-A\Delta t} X_t + (I - e^{-A\Delta t})\mu + \int_t^{t+\Delta t} e^{-A(\Delta t + t - s)} \sigma dB_s,$$

 $X_{t+\Delta t}|X_t \sim N(\mu(X_t, \Delta t), \Sigma(\Delta t)), \text{ where}$

$$\mu(X_t, \Delta t) = e^{-A\Delta t} X_t + (I - e^{-A\Delta t}) \mu, \quad \Sigma(\Delta t) = \int_0^{\Delta t} e^{-A(\Delta t - s)} \sigma \sigma' e^{-A'(\Delta t - s)} ds,$$

and execute the following strategies⁶ at the corresponding times t_l 's, with $W_0 = \pi_0 = 0$ and π_t constant between consecutive times:

- (1) The exponential utility strategy of Theorem 3.1 with $\gamma(t) = 1, 2, 3, 4$ and r = 0, 2%.
- (2) The mean-variance utility strategy with dollar-neutrality penalty of Corollary 4.1.1 with $\gamma(t) = 1, 2, 3, 4, \alpha(t) = 0, 20, 50$ and r = 2%.
- (3) The mean-variance utility strategy with quadratic transaction costs of Theorem 4.1 with $\gamma(t) = 1, 2, 3, 4, r = 2\%$, and $C = \lambda \sigma \sigma'$ for $\lambda = 0.1, 0.5, 1$.

Moreover, to study the result of imposing market-neutrality through the market-neutral portfolios constructed in section 2 under different factor model regimes, we perform the following additional simulation in which we experiment with the parameter p, which encapsulates all the factor model information and which we perturb to simulate the effect of going away from the case where the returns themselves are mean-reverting (which corresponds to the previous case p = 1) and of having progressively more leveraged (and more extreme) market-neutral portfolios:

(4) The three strategies above with $\gamma(t) = 1$, $\alpha(t) = 0$ and r = 2% (and $\lambda = 1$ for the third strategy) for $p = 1 + \epsilon_a$ and a = 1, 2, 4, 8, where ϵ_a is a N-dimensional vector whose components are drawn i.i.d. from a uniform distribution in [-a, a].

We finally present the simulated path of a sample wealth process $(W_t)_{t\in[0,T]}$, the simulated path of the first coordinate of a sample allocation process $(\pi_t)_{t\in[0,T]}$, and the histogram for the terminal wealth W_T for each of the above cases in the following four subsections, along with a final analysis:

⁶We have just simulated some simple cases of the previously discussed strategies for space limitation reasons, but of course it would also be possible to include further constraints (like a leverage restriction, for instance) or time-varying hyperparameters with no additional effort, by just using the previously derived formulae. It would be interesting as well to execute the strategies with some perturbations of the real parameters to simulate possible microstructural noise and imperfect estimation.

5.1 Simulations of the exponential-utility strategy

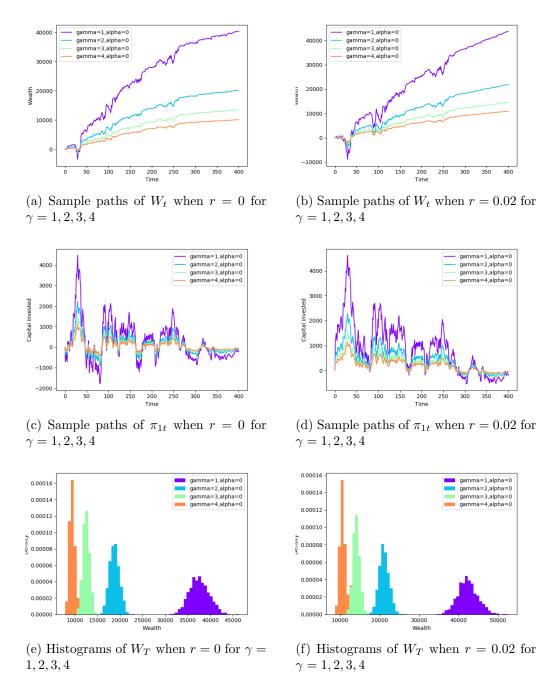


Figure 2: Results for the exponential utility

5.2 Simulations of the mean-variance strategy with dollar neutrality

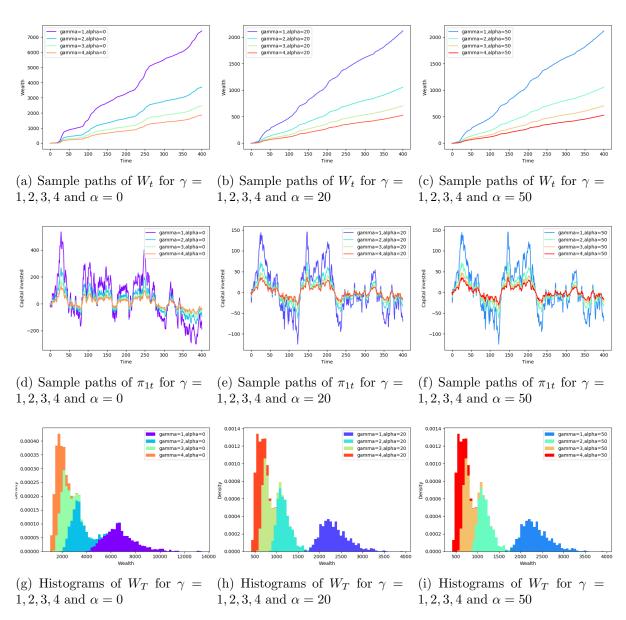


Figure 3: Results for the mean-variance utility when r=0.02 with different dollar-neutrality restrictions

5.3 Simulations of the mean-variance strategy with quadratic transaction costs

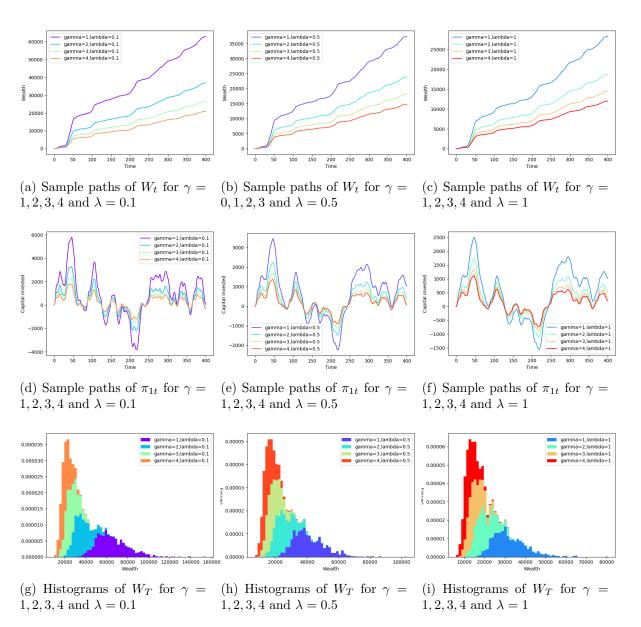


Figure 4: Results for the mean-variance utility when r=0.02 with different quadratic transaction costs $C=\lambda\sigma\sigma'$

5.4 Simulations for different factor model and market-neutrality regimes

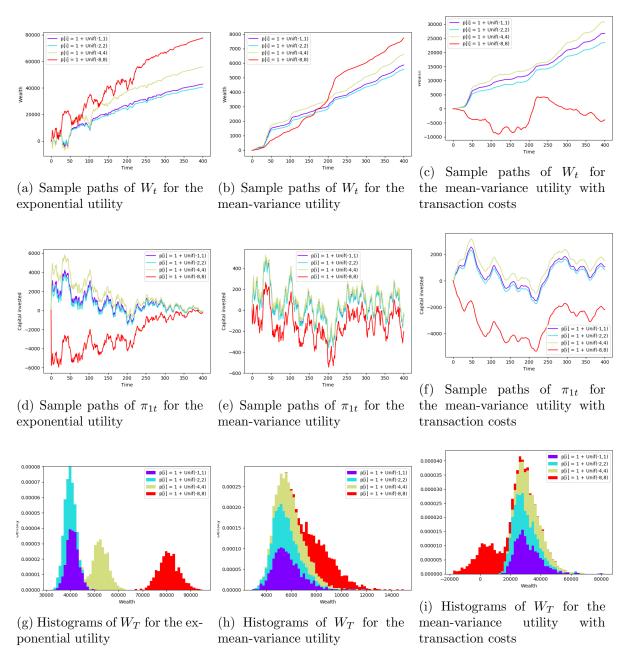


Figure 5: Results for the three strategies above with different p's, when $r=0.02, \alpha=0, \gamma=1, \lambda=1$

5.5 Comparison of the simulated strategies

We now present our main conclusions after observing the previous plots, analyzing the behavior of the histograms of the final wealth W_T , the sample paths of the wealth process $(W_t)_{t\in[0,T]}$, and the sample paths of the positions $(\pi_{1t})_{t\in[0,T]}$, with a final subsubsection discussing the effect of imposing market-neutrality under different p's.

5.5.1 Histograms of the final wealth

Looking first at the above histograms (Figures 3-4 (g)-(i), and 2 (e)-(f)), we see that, for our parametric choice and our setting in which X is effectively a multidimensional Ornstein-Uhlenbeck process with known parameters,

- (1) The most profitable strategy is the one derived from the exponential utility (Figure 2, (e) and (f)) with the lowest risk-aversion parameter γ , even in the most adverse scenarios of the histogram and both with and without zero interest rates. Moreover, even for bigger values of γ this strategy significantly performs better under any regime of r and α than the mean-variance strategy (Figure 3, (g)-(i)).
- (2) We observe the following outcomes when changing one of the parameters for each of the three strategies (Figures 3-4 (g)-(i), and 2 (e)-(f)):
 - Increasing the value of the risk-aversion parameter γ produces a concentration of the density of W_T around smaller values, i.e., the expected wealth decreases and so does the dispersion around it.
 - Increasing the dollar-neutrality penalty α has this same negative effect, but makes little difference unless the increments in α are considerable.
 - Increasing the value of the interest-rate r has an overall positive effect, which is more pronounced in the mean-variance case (since as we mentioned at the end of section 2.2 the investor is then bolder than the exponential agent).
 - Increasing the transaction cost parameter λ decreases the expected terminal wealth, but it also skews its distribution producing a considerable right-tail (whereas all the other distributions are essentially symmetric).

These outcomes have a natural interpretation: since the model is perfectly specified and the parameters are known, the derived strategies will always produce benefits by construction, and they will be bigger the fewer additional constraints we impose (such as risk-aversion, dollar-neutrality, and transaction costs) and the more we can take advantage of previous success (by increasing r). This situation, however, might not apply under parameter misspecification, where the additional constrains would help the investor mitigate the model risk.

5.5.2 Sample paths of the wealth process W_t

Examining next the sample paths for the particular simulation which is plotted (Figures 3,4 (a)-(c), and 2 (a)-(b)), we can observe exactly the same patterns as discussed in the previous paragraph when modifying the parameters γ , α , r and γ . There are, however, two new and interesting remarks:

- (1) In the three strategies, after an initial period of ups and downs and similarity between the different strategies, there is a tendency towards stabilization because of the asymptotic properties of the Ornstein-Uhlenbeck process, and of differentiation depending on the parametric choices.
- (2) This phenomenon is especially pronounced with the exponential utility and with bigger values of r (Figure 2 (a)-(b)) since it takes more aggressive positions, reflecting the fact that sometimes the agent will invest more money than what she will make at that moment

(and sometimes even having temporary negative wealth and borrowing aggressively) to continue executing the strategy.

5.5.3 Sample paths of the positions π_{1t}

Considering now the plots of the sample paths of the positions π_{1t} (Figures 3,4 (d)-(f), and 2 (c)-(d)), we similarly notice that

- (1) The positions become more extreme when decreasing γ , α and λ (i.e., the risk-aversion parameter, the non-dollar-neutrality penalty and the transaction cost parameter) and when increasing r (the interest rate). The greatest overall impact is produced by γ and λ and then r, especially in the mean-variance case for the same reasons as before.
- (2) The exponential utility strategy takes much more extreme positions than the meanvariance strategies, which in this idealized setting of perfect estimation partially explains why the exponential agent obtains a greater wealth at the terminal time.
- (3) The cycles in the positions π_{1t} match the oscillations of X_{1t} depicted in Figure 1, as described theoretically in the corresponding equations.

5.5.4 Effects of imposing market neutrality

Finally, looking separately at the effect of imposing market neutrality under various factor model regimes depending on p (which, as we mentioned, depends quadratically on the factor model loadings), we observe the following (Figure 5):

- (1) As the parameter p gets bigger, the market neutral portfolios of section 2 become more extreme and the adopted positions π_t also become more aggressive, especially in the exponential utility case (Figure 5, (d)-(f)).
- (2) Since the strategy is more aggressive but we have perfect calibration and estimation, with bigger p the mean-variance and especially the exponential strategy become more profitable. However, the wealth process also has more ups and downs (Figure 6, (a)-(c)), the standard deviation of the terminal wealth increases considerably (Figure 6, (g)-(h)), especially in the mean-variance case, and with the biggest p there are also heavy losses when transaction costs are incorporated (Figure 6, (c),(i)). The strategies are therefore riskier, but a relatively large value of p is needed to appreciate its effect.
- (3) Lastly, note that the influence of p on the strategies also depends most of the time on the value of r, since they normally appear combined as a factor of rp in the equations describing the strategies. In particular, when r=0 there is no theoretical effect associated to p (apart from possible model risk and high leverage in a real-world setting) unless the dollar-neutrality parameter $\alpha(t) \neq 0$.

6 Conclusions and further research

In this paper we have aimed to provide a systematic study of high-dimensional statistical arbitrage combining both stochastic control and factor models. To this end, we have first proposed a general framework based on a statistically-constructed factor model, and then shown how

to obtain analytically explicit market-neutral portfolios and rephrase our problem in terms of them to make it tractable and get market neutrality by construction. Using this insight, we have then been able to study the question of optimizing the expected utility of the investor's terminal wealth in continuous time under both an exponential and a mean-variance objective. In both cases, we have obtained explicit closed-form solutions ready for numerical implementation, analyzed the corresponding strategies from the perspective of statistical arbitrage and the underlying factor model, and discussed extensions involving the addition of soft constraints on the admissible portfolios (like dollar-neutrality) and the presence of temporary quadratic transaction costs. Finally, we have run some high-dimensional Monte Carlo simulations to explore the behavior of the previous strategies, and analyzed their qualitative aspects and their sensitivity to the relevant parameters and the underlying factor model.

There are four natural extensions to our work, on which we are conducting research at the moment and which we intend to publish in separate papers. First, one could investigate a more realistic version of the problem in which, rather than in continuous time, the investor may only trade more realistically at an increasing sequence of optimally chosen stopping times, generalizing in multiple directions the literature initiated by the influential work of Leung and Li (2015) and developing robust and efficient numerical methods. Second, it would be interesting to study more realistic modelizations of market frictions, illiquidity, and transaction costs, or to develop a model considering issues of parameter misspecification. Third, on a more empirical side and as we mentioned at the start of the section 5, one should consider in this setting the problems of construction of the factor models, high-dimensional parameter estimation, and risk control, along with (out-of-sample) experiments with real market data under the strategies developed in this paper. Fourth and finally, one could study a more data-driven version of the problem, where the fixed stochastic model is replaced by new tools from reinforcement learning.

7 Acknowledgements

The author would like to thank George Papanicolaou for suggesting the topic of the previous research and for many insightful discussions about the problem and the presentation of the results, and the editor and an anonymous reviewer for their very helpful suggestions to improve the quality of the paper.

A Appendix. Proofs

A.1 Proof of Theorem 3.1

Proof of Proposition 3.1:

The dynamic programming principle suggests that the value function H should satisfy the dynamic programming equation (2.5) with terminal condition $H(T, x, w) = -e^{-\gamma w}$, and the optimal control may then be found in feedback form by looking at the first order condition of the term inside the supremum since the corresponding function is quadratic and concave in π (if $\partial_{ww} H < 0$, i.e., if there is risk aversion). The first order condition gives that

$$0 = \sigma \sigma' \partial_{ww} H \pi + (A(\mu - x) - pr) \partial_w H + \sigma \sigma' \nabla_{xw} H,$$

and solving for π we find the control given in the proposition's statement. Putting it back into (2.5) we get the following non-linear and (N+2)-dimensional PDE

$$0 = \partial_t H + (A(\mu - x))' \nabla_x H + \frac{1}{2} \text{Tr}(\sigma \sigma' \nabla_{xx} H) + wr \partial_w H - \frac{\mathcal{D}H'(\sigma \sigma')^{-1} \mathcal{D}H}{2\partial_{ww} H}.$$
(A.1)

Now, looking at the terminal condition, we guess that the solution of this PDE will be of the form $H(t, x, w) = -\exp(-\gamma(we^{r(T-t)} + h(t, x)))$ for some function h(t, x) to be determined and such that h(T, x) = 0. Some easy computations then show that

$$\partial_t H = -\gamma H(-rwe^{r(T-t)} + \partial_t h) \quad \partial_w H = -\gamma e^{r(T-t)} H \quad \partial_{ww} H = \gamma^2 e^{2r(T-t)} H \quad \nabla_{xw} H = \gamma^2 e^{r(T-t)} H \nabla_x h H = -\gamma e^{r(T-t)} H + -\gamma e^{r(T-t)} H \nabla_x h H = -\gamma e^{r(T-t)} H \nabla_x h$$

$$\nabla_x H = -\gamma H \nabla_x h \qquad \nabla_{xx} H = -\gamma H (\nabla_{xx} h - \gamma \nabla_x h \nabla_x h') \qquad \mathcal{D}H = -\gamma e^{r(T-t)} H (A(\mu - x) - pr - \gamma \sigma \sigma' \nabla_x h).$$

Plugging all this into (A.1), dividing everything by $-\gamma H$, and doing some simple algebra to expand the last term yields

$$0 = -rwe^{r(T-t)} + \partial_t h + (A(\mu - x))'\nabla_x h + \frac{1}{2}\operatorname{Tr}(\sigma\sigma'(\nabla_{xx}h - \gamma\nabla_x h\nabla_x h')) + wre^{r(T-t)} + \frac{1}{2\gamma}(A(\mu - x) - pr)'(\sigma\sigma')^{-1}(A(\mu - x) - pr) + \frac{\gamma}{2}\nabla_x h'\sigma\sigma'\nabla_x h - (A(\mu - x) - pr)'\nabla_x h$$

and we can see that almost miraculously the non-linear terms in h, the terms in w, and the third and part of the last term of the PDE get cancelled and the equation gets dramatically simplified, obtaining

$$0 = \partial_t h + \frac{1}{2} \operatorname{Tr}(\sigma \sigma' \nabla_{xx} h) + r p' \nabla_x h + \frac{1}{2\gamma} (A(\mu - x) - pr)' (\sigma \sigma')^{-1} (A(\mu - x) - pr).$$

This is now a parabolic linear PDE in h and we can find explicitly its solution by using the Feynman-Kac formula. Indeed, if we consider the stochastic process given by

$$dY_t = rpdt + \sigma dB_t^* \tag{A.2}$$

we can rewrite the above equation in terms of the infinitesimal generator \mathcal{L}^* of Y as

$$0 = (\partial_t + \mathcal{L}^*)h + \frac{1}{2\gamma}(A(\mu - x) - pr)'(\sigma\sigma')^{-1}(A(\mu - x) - pr)$$

and then we can express its solution via the following conditional expectation, which is the probabilistic representation given in the proposition's statement:

$$h(t,x) = \mathbb{E}_{t,x}^* \left[\int_t^T \frac{1}{2\gamma} (A(\mu - Y_s) - pr)'(\sigma\sigma')^{-1} (A(\mu - Y_s) - pr) \ ds \right]$$

$$= \frac{1}{2\gamma} (A\mu - pr)'(\sigma\sigma')^{-1} (A\mu - pr) (T - t)$$

$$- \frac{1}{\gamma} (A\mu - pr)'(\sigma\sigma')^{-1} A \mathbb{E}_{t,x}^* \left[\int_t^T Y_s ds \right] + \frac{1}{2\gamma} \mathbb{E}_{t,x}^* \left[\int_t^T Y_s' A'(\sigma\sigma')^{-1} A Y_s ds \right].$$

Finally, to find h explicitly, notice that we can easily solve the SDE (A.2), obtaining, for $s \geq t$,

$$Y_s = x + rp(s - t) + \sigma(B_s^* - B_t^*).$$

and this allows us to compute the two expectations in our expression for h above. Indeed, Fubini's theorem and elementary facts about the Brownian motion immediately yield

$$\mathbb{E}_{t,x}^* \left[\int_t^T Y_s ds \right] = \int_t^T \mathbb{E}_{t,y}^* \left[Y_s \right] ds = x(T-t) + rp \frac{(T-t)^2}{2}$$

and, interchanging integral and expectation again and noticing that

$$\mathbb{E}^*[(B_s^* - B_t^*)'\sigma' A'(\sigma\sigma')^{-1} A \sigma (B_s^* - B_t^*)] = (s - t) \text{Tr}(\sigma' A'(\sigma\sigma')^{-1} A \sigma),$$

we similarly find out that

$$\mathbb{E}_{t,x}^{*} \left[\int_{t}^{T} Y_{s}' A'(\sigma \sigma')^{-1} A Y_{s} ds \right] = \int_{t}^{T} \left(x + r p(s-t) \right)' A'(\sigma \sigma')^{-1} A \left(y + r p(s-t) \right) + (s-t) \operatorname{Tr}(\sigma' A'(\sigma \sigma')^{-1} A \sigma) ds$$

$$= x' A'(\sigma \sigma')^{-1} A x (T-t) + \left(2x' A'(\sigma \sigma')^{-1} A r p + \operatorname{Tr}(\sigma' A'(\sigma \sigma')^{-1} A \sigma) \right) \frac{(T-t)^{2}}{2} + r^{2} p' A'(\sigma \sigma')^{-1} A p \frac{(T-t)^{3}}{3},$$

which gives us the complete explicit solution of the DPE, and hence the explicit form of the optimal strategy π^* by using equation (3.1).

Proof of Proposition 3.2:

Since in the previous proof we have found explicitly the classical smooth solution H of the dynamic programming equation, we just have to check that $\pi^* \in \mathcal{A}_{[0,T]}$ and that the usual regularity conditions hold for the classical proof to apply. More precisely, this means that the local martingale $dH - \mathcal{L}_{t,x,w}^{\pi}Hdt$ is a supermartingale for any admissible π and a true martingale for π^* , where $\mathcal{L}_{t,x,w}^{\pi}$ is the infinitesimal generator of the controlled process (X, W^{π}) , or some sufficient condition for this like the one we stated in Proposition 3.2 in terms of the model parameters, which is what we will show here.

As for the first issue, it is easy to see that $\pi^* \in \mathcal{A}_{[0,T]}$. Indeed, it is obviously \mathcal{F}_t -adapted and predictable (in fact, it has continuous paths) and, using the trivial inequalities $||x+y||^2 \le 2||x||^2 + 2||y||^2$ and $||Ax|| \le ||A||||x||$ and the fact that X_t is a Gaussian process, it is easy to see that $\int_0^T \mathbb{E}[||\pi_s^*||^2]ds < \infty$. Moreover, applying Ito's formula to the process $e^{-rt}W_t$ yields

$$d(e^{-rt}W_t) = -re^{rt}W_tdt + e^{-rt}dW_t = \pi_t \cdot e^{-rt}dX_t - \pi_t \cdot e^{-rt}prdt$$

and, therefore,

$$W_t = w + e^{rt} \left(\int_0^t \pi_s \cdot e^{-rs} dX_s - \int_0^t \pi_s \cdot e^{-rs} pr ds \right)$$
(A.3)

for any $t \ge 0$ and any admissible control π . Thus, the SDE for W has a unique strong solution W^* for the particular case $\pi = \pi^*$ for any initial data, given by the above integral (note that the stochastic integral is well defined, since $dX_s = A(\mu - X_s)ds + \sigma dB_s$, π^* and X are continuous, and again $\int_0^T e^{-2rs} \mathbb{E}[||\pi_s'^*\sigma||^2] ds < \infty$).

As for the regularity conditions, we simply adapt the proof of Theorem 2.1. of Lintilhac and Tourin (2016) for a related model, which guarantee the uniform \mathbb{P} -integrability of the family of random variables $(H(\tau, X_{\tau}, W_{\tau}^*))_{\tau \in [0,T]}$ where τ is a \mathcal{F} -stopping time, and which we simply adapt to the parameters of the present model obtaining the sufficient conditions stated in Proposition 3.2.

The key observation to adapt their proof is that in our case we also have that the hypothesized value function is of the form $H(t, x, w) = -\exp(-\gamma w e^{r(T-t)} - \frac{1}{2}x'A_2(t)x - A_1(t)x - A_0(t))$ for some explicit smooth functions $A_i(t)$ that we computed in the proof of Proposition 3.1, and our X is also a matrix Ornstein-Uhlenbeck process under \mathbb{P} with SDE $dX_t = A(\mu - X_t)dt + \sigma dB_t$, and

$$\gamma W_{\tau}^* e^{r(T-\tau)} = \gamma w e^{r(T-\tau)} + \gamma \int_0^{\tau} \pi_s^* \cdot e^{r(T-s)} (A(\mu - X_s) - pr) ds + \gamma \int_0^{\tau} \pi_s^* \cdot e^{r(T-s)} \sigma dB_s$$

as we showed in (A.3). Thus, using the Cauchy-Schwarz inequality as in their proof, the part corresponding to $-\frac{1}{2}X'_{\tau}A_2(\tau)X_{\tau} - A_1(\tau)X_{\tau} - A_0(\tau)$ in the above expression for $H(\tau, X_{\tau}, W_{\tau}^*)$ may be bounded exactly as they do. As for the part corresponding to $-\gamma W_{\tau}^* e^{r(T-\tau)}$, we can again repeat their exact reasoning, but noting that the quadratic term in X_s in the first integral above is now $X'_s C_0(s) X_s$ for the matrix $C_0(s)$ that we defined before, and likewise the term in X_s in the second integral is $X'_s C_1(s)$, which following their proof gives respectively the two explicit sufficient conditions that we stated in Proposition 3.2.

A.2 Proof of Theorem 3.2

The proof of this follows the same lines as the previous one and is actually much simpler, so we just indicate the relevant changes. The HJB equation is now

$$0 = \partial_t H + (\mu - x)' A' \nabla_x H + \frac{1}{2} \text{Tr}(\sigma \sigma' \nabla_{xx} H) + \sup_{\pi} \left(\left(\pi' A(\mu - x) + (w - \pi' p)r \right) \partial_w H + \frac{1}{2} \pi' \sigma \sigma' \pi \partial_{ww} H + \pi' \sigma \sigma' \nabla_{xw} H - \frac{\gamma(t)}{2} \pi' \sigma \sigma' \pi \right)$$

with terminal condition H(T, x, w) = w.

Guessing that the value function will now be of the form $H(t, x, w) = we^{r(T-t)} + a(t) + b(t)'x + \frac{1}{2}x'c(t)x$ for a scalar a(t), an N-dimensional vector b(t), and a symmetric $N \times N$ matrix c(t), and plugging this into the above equation, we obtain the hypothesized optimal control given in the statement of the theorem and the above PDE gets reduced to the following system of three first-order linear matrix ODEs

$$0 = \partial_t c - A'c - cA + e^{2r(T-t)}A'(\gamma(t)\sigma\sigma')^{-1}A$$

$$0 = \partial_t b - A'b + cA\mu - e^{2r(T-t)}A'(\gamma(t)\sigma\sigma')^{-1}(A\mu - pr)$$

$$0 = \partial_t a + \frac{1}{2}(\mu'A'b + b'A\mu) + \frac{1}{2}\text{Tr}(\sigma\sigma'c) + \frac{e^{2r(T-t)}}{2}(A\mu - pr)'(\gamma(t)\sigma\sigma')^{-1}(A\mu - pr)$$

with terminal conditions a(T) = b(T) = c(T) = 0.

This system has an explicit bounded solution in [0, T], since the classical solution of the general first-order linear matrix ODE $\partial_t y + uy + v(t) = 0$ with y(T) = 0 is given by

$$y(t) = \int_{t}^{T} \exp((s-t)u) v(s) ds,$$

if v(s) is continuous on [0, T], in which case it is automatically bounded on [0, T] as well; and similarly the classical solution of $\partial_t y + uy + yu' + v(t) = 0$ with y(T) = 0 for a symmetric v is given by

$$y(t) = \int_{t}^{T} \exp((s-t)u) v(s) \exp((s-t)u') ds.$$

Thus, the HJB equation has an explicit classical solution which has quadratic growth in the state variables uniformly in t. A classical verification result (cf. for example Theorem 4.3 of Guyon & Labordère (2013)) then yields that our hypothesized optimal control is indeed optimal provided that it is admissible, which may be checked exactly as in the proof of Theorem 3.1. \square

A.3 Proof of Theorem 4.1

Proof of Proposition 4.1:

The corresponding dynamic programming equation is in this case

$$0 = (\partial_t + \mathcal{L}_X)H + \left(\pi'A(\mu - x) + (w - \pi'p)r\right)\partial_w H + \frac{1}{2}\pi'\sigma\sigma'\pi\partial_{ww}H + \pi'\sigma\sigma'\nabla_{xw}H - \frac{\gamma(t)}{2}\pi'\sigma\sigma'\pi + \sup_I\left(I'\nabla_\pi H - \frac{1}{2}I'CI\right)$$

with terminal condition $H(T, x, w, \pi) = w$ and where the supremum is obviously attained at $I^* = C^{-1}\nabla_{\pi}H$.

Substituting this back in the above equation and plugging the stated ansatz we obtain that

$$0 = \frac{1}{2}\pi'\partial_{t}a\pi + \pi'(\partial_{t} + \mathcal{L}_{X})b + (\partial_{t} + \mathcal{L}_{X})d + \pi'(A(\mu - x) - pr)e^{r(T-t)} - \frac{\gamma(t)}{2}\pi'\sigma\sigma'\pi + \frac{1}{2}(a\pi + b)'C^{-1}(a\pi + b).$$

Matching the coefficients for $\pi'(\cdot)\pi$, $\pi'(\cdot)$, and the constant yields the above differential equations.

Before we prove the next proposition, we state here the following result for comparison and existence of solutions of matrix Riccati ODEs (cf. Theorem 2.2.2 in Kratz (2011)), which we will use in our proof.

Theorem A.1. Let $L_1(t), L_2(t), L(t), N_1(t), N_2(t) \in \mathbb{R}^{d \times d}$ be piecewise continuous on \mathbb{R} . Moreover, suppose $L_1(t), L_2(t), N_1(t), N_2(t)$ and $S_1, S_2 \in \mathbb{R}^{d \times d}$ are symmetric. Let T > 0 and

$$S_1 > S_2, L_1 > L_2 > 0, N_1 > N_2,$$

on [0,T]. Assume that the terminal value problem

$$\partial_t H_1 + H_1 L_1 H_1 + M H_1 + H_1 M + N_1 = 0, \quad H_1(T) = S_1,$$

has a (symmetric) solution H_1 on [0,T]. Then the terminal value problem

$$\partial_t H_2 + H_2 L_2 H_2 + M H_2 + H_2 M + N_2 = 0, \quad H_2(T) = S_2,$$

has a (symmetric) solution H_2 on [0,T] and $H_1(t) \ge H_2(t)$ for all $t \in [0,T]$.

We are now in a position to give the following:

Proof of Proposition 4.2:

(1) The first statement follows directly from the comparison Theorem A.1 stated before, since the matrix Riccati ODE

$$\partial_t a + aC^{-1}a = 0$$

with terminal condition a(T) = 0 has the obvious symmetric solution a(t) = 0 defined on all [0, T]. Thus, (4.2) has a symmetric classical solution a(t) defined on all [0, T] with $a \le 0$, which is bounded because [0, T] is compact and a is differentiable hence continuous.

As for the particular solution when $\gamma(t) = \gamma$, simply note that pre- and post-multiplying (4.2) by $C^{-1/2}$ and defining $\tilde{a} := C^{-1/2} a C^{-1/2}$ gives the new Riccati

$$\partial_t \tilde{a} - \gamma C^{-1/2} \sigma \sigma' C^{-1/2} + \tilde{a}^2 = 0,$$

whose solution is $\tilde{a}(t) = D \tanh(D(t-T))$.

(2) The existence of solutions with polynomial growth and their probabilistic representation in the above form follow from a vector-valued version of the Feynman-Kac theorem (see Appendix A.3 of Cartea et al. (2018) for a proof of how to adapt the one-dimensional case) provided that the appropriate regularity conditions hold. Using, for example, Condition 2 of Appendix E in Duffie (2010), it is sufficient that all the functions of (t, x) $A(\mu - x)$, σ , $a(t)'C^{-1}$, $e^{r(T-t)}(A(\mu - x) - rp)$ (and $b(t, x)'C^{-1}b(t, x)$ for the existence of d) are uniformly Lipschitz in x, they and their first and second derivatives in x are continuous with polynomial growth in x uniformly in t, and $a(t) \leq 0$. All of these properties are straightforward to check in this case because all the corresponding functions are given explicitly and are simple, and the required properties for a follow from (1).

The fact that b has linear growth in x uniformly in t is then a consequence of the probabilistic representation (4.5). Indeed, Fubini's theorem implies that

$$b(t,x) = \int_{t}^{T} : \exp\left(\int_{t}^{s} a'(u)C^{-1}du\right) : e^{r(T-s)}(A\mathbb{E}_{t,x}[\mu - X_{s}] - rp)ds$$

whereas the fact that

$$X_{t+\Delta t} = e^{-A\Delta t}X_t + (I - e^{-A\Delta t})\mu + \int_t^{t+\Delta t} e^{-A(\Delta t + t - s)}\sigma dB_s$$

yields

$$\mathbb{E}_{t,x} [\mu - X_s] = e^{-A(s-t)} (\mu - x).$$

Combining the two pieces and using the boundedness of a and the compactness of [0, T] gives the desired uniform bound in t.

The quadratic growth of d in x uniformly in t is then obvious looking at its probabilistic representation and using the linear growth of b.

Proof of Proposition 4.3:

Combining the two previous propositions, we have already found an explicit classical solution of the associated HJB equation with quadratic growth in the state variables uniformly in t, so using again Theorem 4.3 in Guyon and Labordère (2013), we just have to verify that the candidate intensity given in Proposition 4.1 is admissible.

For this, first of all note that the corresponding SDEs controlled by the above intensity have an unique (strong) solution for any initial data. Indeed, given I^* and the definition of I as $d\pi = Idt$, we can solve explicitly the corresponding first-order linear matrix ODE for π^* yielding, for $s \ge t$,

$$\pi_s^* = \pi_t + \int_t^s : \exp\left(\int_u^s Rate(v)dv\right) : Aim(u, X_u)du,$$

and this π^* in turn defines W^* like in the proof of Theorem 3.1.

Finally, from the above construction it is obvious that both π_t^* and I_t^* are \mathcal{F}_t -adapted and predictable (in fact, they have continuous paths), and the property that π^* is in $L^2([0,T]\times\Omega)$ (i.e., that $\int_0^T \mathbb{E}[||\pi_s^*||^2]ds < \infty$) stems from the observation that Rate(u) is deterministic and bounded (because of Proposition 4.3.1), Aim(t,x) has linear growth in x uniformly in t (by Proposition 4.3.2), and X is a Gaussian process (so it is in $L^2([0,T]\times\Omega)$).

 I_t^* is likewise in $L^2([0,T]\times\Omega)$ since, as we saw in Proposition A.2, $I_t^*=C^{-1}(a(t)\pi_t^*+b(t,X_t))$ and we can therefore use the triangular inequality, the just shown fact that π_t^* is in $L^2([0,T]\times\Omega)$, and the same arguments as above that a(t) is bounded (because of Proposition 4.3.1), that b(t,x) has linear growth in x uniformly in t (by Proposition 4.3.2), and that X is a Gaussian process, to conclude.

References

- [1] Almgren, R. and Chriss, N. (2001). Optimal execution of portfolio transactions. *Journal of Risk*, 3, 5-40.
- [2] Avellaneda, M., and Lee, J. H. (2010). Statistical arbitrage in the US equities market. *Quantitative Finance*, 10(7), 761-782.
- [3] Benth, F. E and Karlsen, K. H. (2005). A note on Merton's portfolio selection problem for the Schwartz mean-reversion model. *Stochastic analysis and applications*, 23(4), 687-704.
- [4] Bertsimas, D. and Lo, A. (1998). Optimal control of execution costs. *Journal of Financial Markets*, 1, 1-50.
- [5] Bismuth, A., Guéant, O. and Pu, J. (2019). Portfolio choice, portfolio liquidation, and portfolio transition under drift uncertainty. *Mathematics and Financial Economics*, 13(4), 661-719.
- [6] Boyd, S., Busseti, E., Diamond, S., Kahn, R., Koh, K., Nystrup, P., and Speth, J. (2017), Multiperiod trading via convex optimization. *Foundations and Trends in Optimization*, 3(1), 1-76.
- [7] Brendle, S. (2006). Portfolio selection under incomplete information. *Stochastic processes* and their applications, 116(5), 701-723.
- [8] Cartea, Á., Jaimungal, S., and Peñalva, J. (2015). Algorithmic and high frequency trading. Cambridge University Press, Cambridge.
- [9] Cartea, Á. and Jaimungal, S. (2016). Algorithmic trading of co-integrated assets. *International Journal of Theoretical and Applied Finance*, 19(06), 165038.
- [10] Cartea, Á., Gan, L., and Jaimungal, S. (2018). Trading cointegrated assets with price impact. To appear in *Mathematical Finance*.
- [11] Chiu, M. C. and Wong, H. Y. (2011). Mean-variance portfolio selection of cointegrated assets. J. Econ. Dyn. Control, 35, 1369-1385.
- [12] DeMiguel, V., Mei, X. and Nogales, F. J. (2016). Multiperiod portfolio optimization with multiple risky assets and general transaction costs. *Journal of Banking and Finance*, 69, p. 108-120.
- [13] Duffie, D. (2010). Dynamic asset pricing theory. Princeton University Press, Princeton, 3rd edition.
- [14] Duffie, D., Filipovic, D., and Schachermayer, W. (2003). Affine processes and applications in finance. *The Annals of Applied Probability*, 13(3), 984-1053.
- [15] Fouque, J.-P., Papanicolaou, A., and Sircar, R. (2015). Filtering and portfolio optimization with stochastic unobserved drift in asset returns. *Communications in Mathematical Sciences*, 13(4), 935-953.
- [16] Garleanu, N. and Pedersen, L. H. (2013). Dynamic trading with predictable returns and transaction costs. *J. Finance*, 68(6), 23092340.
- [17] Garleanu, N. and Pedersen, L. H. (2016). Dynamic portfolio choice with frictions. J. Econ. Theory, 164, 487-516.

- [18] Guo, X., Lai, T. L., Shek, H., and Wong, S (2016). Quantitative trading: algorithms, analytics, data, models, optimization. Chapman and Hall/CRC, Boca Raton.
- [19] Guyon, J., and Labordère, P. H. (2013). Nonlinear option pricing. Chapman and Hall/CRC, Boca Raton.
- [20] Jurek, J. W. and Yang, H. (2007) Dynamic portfolio selection in arbitrage. Working Paper, Harvard University.
- [21] Kitapbayev, Y. and Leung, T. (2018). Mean reversion trading with sequential deadlines and transaction costs. *International Journal of Theoretical and Applied Finance*, 21(1), 1850004.
- [22] Kratz, D.-M. P. (2011). Optimal liquidation in dark pools in discrete and continuous time. Ph. D. thesis, Humboldt-Universitat zu Berlin.
- [23] Lei, Y. and Xu, J. (2015), Costly arbitrage through pairs trading. J. Econ. Dyn. Control, 56, 119.
- [24] Leung, T., and Li, X. (2015). Optimal mean reversion trading with transaction costs and stop-loss exit. *Int. J. Theor. Appl. Finance*, 18(3), 1-31.
- [25] Lettau, M, and Pelger, M. (2018). Estimating latent asset-pricing factors, Working Paper, Stanford University.
- [26] Liang, Z., Yuen, K. C., and Guo, J. (2011). Optimal proportional reinsurance and investment in a stock market with OrnsteinUhlenbeck process. *Insurance: Mathematics and Economics*, 49(2), 207215.
- [27] Lintilhac, P., and Tourin, A. (2016). Model-based pairs trading in the bitcoin markets. *Quantitative Finance*, 17(5), 703-716.
- [28] Liu, J. and Timmermann, A. (2013) Optimal convergence trade strategies. Review of Financial Studies, 26(4), 10481086.
- [29] Ludwig, S. (2012). Optimal portfolio allocation of commodity related assets using a controlled forward-backward stochastic algorithm. Ph. D. thesis, Rubrecht-Karls Universität Heidelberg.
- [30] Merton, R. (1971). Optimum consumption and portfolio rules in a continuous time model. Journal of Economic Theory, 3, 373-413.
- [31] Moreau, L., Muhle-Karbe, J., and Soner, H.M. (2017). Trading with small price impact. Mathematical Finance, 27(2), 350-400.
- [32] Moutari, S., Sandjo, A. N., and Colin, F. (2017). An Explicit Solution for a Portfolio Selection Problem with Stochastic Volatility. *Journal of Mathematical Finance*, 7, 199-218.
- [33] Mudchanatongsuk, S., Primbs, J. A., and Wong, W. (2008). Optimal pairs trading: A stochastic control approach. 2008 American Control Conference, 1035-1039.
- [34] Muhle-Karbe, J., Liu, R., Weber, M. (2017). Rebalancing with linear and quadratic costs. SIAM Journal on Control and Optimization, 55(6), p. 3533-3563.
- [35] Ngo, M.- M. and Pham, H. (2016), Optimal switching for pairs trading rule: A viscosity solutions approach, *J. Math. Anal. Appl.*, 441(1), 403425.

- [36] Obizhaeva, A. and Wang, J. (2013). Optimal trading strategy and supply/demand dynamics. *Journal of Financial Markets*, 16 (1), 1-32.
- [37] Papanicolaou, G. and Yeo, J. (2017). Risk control of mean-reversion time in statistical arbitrage. Risk and Decision Analysis, 6, 263-290.
- [38] Pelger, M. and Xiong, R. (2018). Interpretable proximate factors for large dimensions, Working Paper, Stanford University.
- [39] Rogers, L. C. G. and Singh, S. (2010). The cost of illiquidity and its effects on hedging. *Mathematical Finance*, 20(4), 597-615.
- [40] Tourin, A. and Yan, R. (2013). Dynamic pairs trading using the stochastic control approach. Journal of Economic Dynamics and Control, 37(10), 1972 - 1981.