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Trend estimation of financial time series

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SUMMARY

We propose to decompose a financial time series into trend plus noise by means of the exponential smoothing filter. This filter produces statistically efficient estimates of the trend that can be calculated by a straightforward application of the Kalman filter. It can also be interpreted in the context of penalized least squares as a function of a smoothing constant has to be minimized by trading off fitness against smoothness of the trend. The smoothing constant is crucial to decide the degree of smoothness and the problem is how to choose it objectively. We suggest a procedure that allows the user to decide at the outset the desired percentage of smoothness and derive from it the corresponding value of that constant. A definition of smoothness is first proposed as well as an index of relative precision attributable to the smoothing element of the time series. The procedure is extended to series with different frequencies of observation, so that comparable trends can be obtained for say, daily, weekly or intraday observations of the same variable. The theoretical results are derived from an integrated moving average model of order $(1,1)$ underlying the statistical interpretation of the filter. Expressions of equivalent smoothing constants are derived for series generated by temporal aggregation or systematic sampling of another series. Hence, comparable trend estimates can be obtained for the same time series with different lengths, for different time series of the same length and for series with different frequencies of observation of the same variable.

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1. INTRODUCTION

This article is concerned with the problem of decomposing an observed financial time series into two unobserved components, trend and noise. The basic goal of such type of analysis is to enable

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the analyst to study the behavior of each component separately, on the assumption that they have different causal forces.

In an economic context, there is a tradition for decomposing time series into components such as trend, seasonal and irregular by means of statistical packages like X-12-ARIMA (see Findley *et al.* [1]) or TRAMO-SEATS (Gómez and Maravall [2]) which are easy to use and enable the analyst to carry out a very thorough time series analysis. Another method commonly employed to decompose an economic time series consists of an application of the Hodrick–Prescott (HP) filter (see Hodrick and Prescott [3]). In most cases the observed series is assumed to be $I(2)$ (Integrated of order 2) as indicated by Maravall [4]. Nevertheless, as a referee pointed out 'There is no reason why the methodology of SEATS cannot handle an $I(1)$ specification, but the software tends to favor the $I(2)$ specification.' Something similar occurs with the X-12-ARIMA methodology.

On the other hand, when working with financial time series, a usual assumption is that the series behaves as a random walk, that is, as an $I(1)$ process. For instance, Baillie and Bollerslev [5] found that the exchange rates of several currencies against the U.S. Dollar behave as random walks. Similarly, Narayan and Smyth [6] showed that the stock prices of the Organization for Economic Co-operation and Development countries should be considered $I(1)$ processes. Moreover, Tsay [7] uses a random walk with drift as a conventional model for prices. Thus, the random walk model plays a central role in the financial literature. To be consistent with this idea, since we are interested in decomposing a financial time series into trend plus noise, we should use the exponential smoothing (ES) filter rather than the HP filter. The distinction between these filters is thoroughly exposed by King and Rebelo [8]. For our purposes it suffices to say that the ES filter employs an $I(1)$ representation for the trend, while the HP filter uses an $I(2)$ representation.

This paper is structured as follows. The following section describes the statistical basis of the ES filter along with its interpretation. There, we emphasize the role played by the smoothing constant required by the filter and show how to choose that constant, based on a measure of smoothness proposed by Guerrero [9]. In Section 3 we show how to extend the applicability of the method to series with different frequencies of observation. Section 4 focuses on the empirical basis of the method and argues in favor of using daily time series data as the standard of reference. Section 5 provides some illustrative numerical examples. In Section 6 we conclude with some final remarks.

2. STATISTICAL DESCRIPTION OF THE ES FILTER

This section presents results that are similar in nature to those appearing in Guerrero [10] for the HP filter. However, the results shown here pertain specifically to the ES filter and we consider them of such importance that they must be shown here for completeness and to make this paper self-contained. The remaining sections of this work are completely new, in comparison with the material appearing in papers previously published by one of us (i.e. Guerrero [9, 10]).

Let us assume that an observed time series $\{Y_1, \ldots, Y_N\}$ can be represented by the following unobserved component model:

$$Y_t = \tau_t + \eta_t \quad \text{for } t = 1, \ldots, N \tag{1}$$

where τ_t denotes the (unobserved) trend and η_t the (unobserved) noise at time t. We will assume that the time series $\{Y_t\}$ is $I(1)$ in such a way that its trend will also be $I(1)$ and the noise process will be stationary. By using this model we do not pretend to say that it represents the true data generating process. It is only an easy-to-use representation that captures some stylized facts of the time series under study. There are other representations equally useful in practice to capture some other features of the data. For instance, conditional heteroscedastic models play a central role in the literature on financial volatility, as indicated by Tsay [7, Chapter 3]. Therefore, we can choose different models for the same series, depending on the problem we want to attack. In many cases we can use a time series model for the mean together with a volatility model for the variance. We shall use the term filter very frequently, with a filter defined here as any operation on the observed series that yields a new series, which in the present case will be the estimated trend. Gómez [11] proved that the HP filter produces results equivalent to those obtained with any of the following three methods: (i) smoothing by penalized least squares (PLS), (ii) Kalman filtering plus smoothing and (iii) Wiener–Kolmogorov (WK) filtering. Such a result can be extended in a straightforward manner to the ES filter, since the ES and the HP filters share the same statistical basis.

The PLS smoothing approach that leads to the ES filter establishes that the trend must minimize the function $M(\lambda)$ defined as

$$M(\lambda)=\sum_{t=1}^{N}(Y_t-\tau_t)^2+\lambda\sum_{t=2}^{N}(\nabla\tau_t)^2 \tag{2}$$

where ∇ denotes the difference operator given by $\nabla Z_t = Z_t - Z_{t-1}$ for every variable Z and index t, and the constant $\lambda>0$ is a smoothing parameter. By writing

$$F=\sum_{t=1}^{N}(Y_t-\tau_t)^2 \quad \text{and} \quad S=\sum_{t=2}^{N}(\nabla\tau_t)^2 \tag{3}$$

we see that F measures the goodness of fit of the trend to the data, while S is related to trend smoothness. The constant λ controls the emphasis given to F and S in the minimization of $M(\lambda)$, hence, it plays an important role in expression (2). On the one hand, when $\lambda\to 0$ the fit of the trend to the observed series is emphasized over its smoothness, consequently $\tau_t \to Y_t$ for all t. On the other hand, if $\lambda\to\infty$ the smoothness constraint dominates the solution and we obtain the constant trend model $\nabla\tau_t = 0$.

The minimization problem can be written in matrix notation as

$$\min_{\boldsymbol{\tau}} M(\lambda)=(\mathbf{Y}-\boldsymbol{\tau})'(\mathbf{Y}-\boldsymbol{\tau})+\lambda(K_{1N}\boldsymbol{\tau})'(K_{1N}\boldsymbol{\tau}) \tag{4}$$

with $\mathbf{Y}=(Y_1,\ldots,Y_N)'$, $\boldsymbol{\tau}=(\tau_1,\ldots,\tau_N)'$ and K_{1N} the $(N-1)\times N$ matrix given by

$$K_{1N}=\begin{pmatrix} -1 & 1 & 0 & 0 & \ldots & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & \ldots & 0 & 0 & 0\\ & & & & \ldots & & & \\ 0 & 0 & 0 & 0 & \ldots & -1 & 1 & 0\\ 0 & 0 & 0 & 0 & \ldots & 0 & -1 & 1 \end{pmatrix} \tag{5}$$

Therefore, the minimum of $M(\lambda)$ is obtained by calculating the derivative of $M(\lambda)$ with respect to $\boldsymbol{\tau}$, equating it to zero (evaluated at $\boldsymbol{\tau}=\hat{\boldsymbol{\tau}}$) and solving the resulting equation. By doing so, we get

$$\hat{\boldsymbol{\tau}}=(I_N+\lambda K_{1N}'K_{1N})^{-1}\mathbf{Y} \tag{6}$$

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where $(I_N + \lambda K'_{1N} K_{1N})^{-1}$ is a symmetric positive definite matrix. As the second derivative of $M(\lambda)$ evaluated at $\boldsymbol{\tau} = \hat{\boldsymbol{\tau}}$ can be shown to be a symmetric and positive definite matrix we know this estimator minimizes $M(\lambda)$. It should be noticed that expression (6) is well known in the filtering and data graduation fields (e.g. Hodrick and Prescott [3]) as well as in the penalized splines literature (see Ruppert *et al.* [12]). The PLS approach has the advantage of showing explicitly the roles played by λ, F and S. However, to obtain $\hat{\boldsymbol{\tau}}$ with (6) it is necessary to invert an $N \times N$ matrix, which may cause instability and lack of precision of the numerical solution when N is large. To avoid this problem it is preferable to use the Kalman filter with smoothing.

Kalman filtering requires the formulation of a state-space model that uses a state equation and a measurement equation, given, respectively, by

$$\mathbf{X}_t = A_t \mathbf{X}_{t-1} + \mathbf{W}_t \quad \text{and} \quad Y_t = \mathbf{c}'_t \mathbf{X}_t + \eta_t \quad \text{for } t = 1, \ldots, N \tag{7}$$

with $\{\mathbf{W}_t\}$ and $\{\eta_t\}$ two independent zero-mean white noise processes, that is, they are sequences of random errors serially uncorrelated and identically distributed. For the ES filter we have

$$\mathbf{X}_t = \tau_t, \quad A_t = 1, \quad \mathbf{W}_t = \varepsilon_t \quad \text{and} \quad \mathbf{c}'_t = 1 \quad \text{for } t = 1, \ldots, N \tag{8}$$

with $\mathrm{Var}(\varepsilon_t) = \sigma_\varepsilon^2$ and $\mathrm{Var}(\eta_t) = \sigma_\eta^2$. Thus, the state and measurement equations for the ES filter are

$$\tau_t = \tau_{t-1} + \varepsilon_t \quad \text{and} \quad Y_t = \tau_t + \eta_t \tag{9}$$

Besides, λ is given by the variance ratio $\sigma_\eta^2 / \sigma_\varepsilon^2$. Thus, to equate the results of the Kalman filter to those obtained with (6) we assume $\sigma_\varepsilon^2 = 1$ and $\sigma_\eta^2 = \lambda$.

Another equivalent method is the WK filter, which arises by assuming that expressions (9) are true. As shown by King and Rebelo [8] the formula for this filter can be obtained from the first-order condition of the minimization problem, that is,

$$0 = -2(Y_t - \hat{\tau}_t) + 2\lambda(\hat{\tau}_t - \hat{\tau}_{t-1}) - 2\lambda(\hat{\tau}_{t+1} - \hat{\tau}_t) \quad \text{for } t = 2, \ldots, N-1 \tag{10}$$

By solving this equation we obtain the symmetric WK filter

$$\hat{\tau}_t = \left(\frac{1/\lambda}{(1-B)(1-B^{-1}) + 1/\lambda} \right) Y_t \tag{11}$$

This formula is best justified by the result that follows from Assumption A in Bell [13], which states that the initial values of the process are independent of differenced signal and noise. As with the Kalman filter, the estimated trend for the WK filter is obtained by assuming $\sigma_\varepsilon^2 = 1$ and $\sigma_\eta^2 = \lambda$, so that $\hat{\tau}_t = (\sigma_\varepsilon^2 / \sigma_\eta^2)[(1-B)(1-B^{-1}) + \sigma_\varepsilon^2 / \sigma_\eta^2]^{-1} Y_t$, where B^{-1} is such that $B^{-1} X_t = X_{t+1}$ for every variable X and index t. This filter produces an estimator of $\{\tau_t\}$ with minimum mean-square error (MSE) if a complete realization (from $t = -\infty$ to $t = \infty$) of the series $\{Y_t\}$ is available. Otherwise, we could extend the observed series with a few backcasts and forecasts as did Kaiser and Maravall [14]. For the HP filter, they found that only four backcasts and forecasts are required to reproduce the infinite filter effect. It is possible to combine the forecasting formula with the WK formula to obtain an exact finite-sample filter and another approach is to use the matrix formulas provided by McElroy [15].

There is a strong connection between the ES filter and the so-called scatterplot smoothing technique employed within the context of semiparametric models (see Ruppert *et al.* [12] for a thorough discussion of this subject). One of such techniques is penalized splines, which may

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be deemed as an extension of linear regression modeling and produces smooth curves by way of a smoothing parameter λ, as with the ES filter. In that context, we can find a mixed model representation of the penalized spline that could lead us to use Kalman filtering for computations and to select λ via Maximum Likelihood (or else use some other approaches, as generalized cross-validation or Akaike's information criterion). In what follows, we propose to select λ with a different approach that comes out by defining first an index of relative precision attributable to smoothness. We prefer this approach over the former ones because this way we have control on the amount of smoothness to be achieved by the trend and this fact allows us to establish more valid comparisons among trends.

To use the ES filter in practice it is essential to choose the value of the smoothing constant (λ) that determines the behavior of the trend completely. The selection of that constant cannot be taken lightly because both smoothness and fit of the trend will depend on its value. In fact, selection of λ must take into consideration the number of data points available and the frequency of observations. Thus, let us suppose that Equations (7) are true, with $\{\eta_t\}$ and $\{\varepsilon_t\}$ two uncorrelated zero-mean white noise processes, with variances σ_η^2 and σ_ε^2, respectively. In matrix terms we have

$$\mathbf{Y} = \boldsymbol{\tau} + \boldsymbol{\eta} \quad \text{with } E(\boldsymbol{\eta}) = \mathbf{0}, \quad \text{Var}(\boldsymbol{\eta}) = \sigma_\eta^2 I_N \tag{12}$$

and

$$K_{1N}\boldsymbol{\tau} = \boldsymbol{\varepsilon} \quad \text{with } E(\boldsymbol{\varepsilon}) = \mathbf{0}, \quad \text{Var}(\boldsymbol{\varepsilon}) = \sigma_\varepsilon^2 I_{N-1} \tag{13}$$

Now, since $E(\boldsymbol{\varepsilon}\boldsymbol{\eta}') = 0$, we get

$$\begin{pmatrix} \mathbf{Y} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} I_N \\ K_{1N} \end{pmatrix} \boldsymbol{\tau} + \begin{pmatrix} \boldsymbol{\eta} \\ -\boldsymbol{\varepsilon} \end{pmatrix} \quad \text{with } E\begin{pmatrix} \boldsymbol{\eta} \\ -\boldsymbol{\varepsilon} \end{pmatrix} = \mathbf{0}, \quad \text{Var}\begin{pmatrix} \boldsymbol{\eta} \\ -\boldsymbol{\varepsilon} \end{pmatrix} = \begin{pmatrix} \sigma_\eta^2 I_N & 0 \\ 0 & \sigma_\varepsilon^2 I_{N-1} \end{pmatrix} \tag{14}$$

Therefore, as did Guerrero [10] with the HP filter, we use generalized least squares to produce the linear estimator with minimum MSE, given by

$$\hat{\boldsymbol{\tau}} = (\sigma_\eta^{-2} I_N + \sigma_\varepsilon^{-2} K_{1N}' K_{1N})^{-1} \sigma_\eta^{-2} \mathbf{Y} \tag{15}$$

whose MSE matrix becomes

$$\Gamma = \text{Var}(\hat{\boldsymbol{\tau}}) = (\sigma_\eta^{-2} I_N + \sigma_\varepsilon^{-2} K_{1N}' K_{1N})^{-1} \tag{16}$$

For this result to be completely equivalent to that provided by the WK filter, we also assume that the initial value of $\{Y_t\}$ is independent of $\boldsymbol{\eta}$ and $\boldsymbol{\varepsilon}$. By looking at the precision matrix Γ^{-1}, we see that it is the sum of two precision matrices, $\sigma_\eta^{-2} I_N$ corresponding to model (12) and $\sigma_\varepsilon^{-2} K_{1N}' K_{1N}$ associated with (13). To measure the proportion of P in $(P+Q)^{-1}$, where P and Q are $N \times N$ positive definite matrices, we will use the index proposed in Guerrero [9, 10]. Such an index is

$$\Lambda(P; P+Q) = \text{tr}[P(P+Q)^{-1}]/N \tag{17}$$

where tr(.) denotes trace of a matrix. This index has the following properties: (i) it takes on values between zero and one, (ii) it is invariant under linear nonsingular transformations of the variable involved, (iii) it behaves linearly and (iv) it is symmetric, in the sense that $\Lambda(P; P+Q) + \Lambda(Q; P+Q) = 1$.

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Thus, we use (17) to quantify the proportion of precision attributable to trend smoothness induced by (13). It is considered as an index of relative precision contributed by the smoothing model and its expression becomes

$$S(\lambda; N) = \Lambda(\sigma_\varepsilon^{-2} K'_{1N} K_{1N}; \Gamma)$$
$$= 1 - \mathrm{tr}[(I_N + \lambda K'_{1N} K_{1N})^{-1}]/N \tag{18}$$

with $\lambda = \sigma_\eta^2/\sigma_\varepsilon^2$. This index depends only on the values of λ and N because the matrix K_{1N} is fixed. It is clear that $S(\lambda; N) \to 0$ when $\lambda \to 0$ and $S(\lambda; N) \to 1$ when $\lambda \to \infty$. For convenience, we will use the notation $100S(\lambda; N)\%$ or simply $S\%$, to interpret it as a percentage of smoothness achieved by the ES filter.

In Figure 1 we can appreciate the percentage of smoothness achieved by different values of N and λ. Figure 1(a) shows that the smoothness grows quickly as the sample size grows. When $N = 50$, the percentage of smoothness is higher than 90% for λ as small as 50, and the smoothness stays basically constant when the sample size is larger than 50, in spite of the value of λ. Similarly, Figure 1(b) allows us to appreciate the effect of the sample size for fixed λ values. For $\lambda > 150$, the percentage of smoothness is greater than 95% in the three cases shown in that figure.

Unfortunately, from (18) it is not possible to obtain an analytic expression for λ as a function of N and $S\%$. Therefore, in Table I we present the values of λ corresponding to some percentages of smoothness, for different lengths (sample sizes) of a daily series. These values were obtained numerically, for given values of N and $S\%$, by solving Equation (18) for λ. To simplify the selection of λ in practical applications, we searched for an approximating function of N and $S\%$ that would provide a good fit to the values in Table I. Thus, we fitted several regression models for each $S\%$ and the estimation results of the best generic fitting model are shown in Table II. We should bear in mind that the values of λ produced by the regression model are just approximations to the true values in Table I. We should also notice that it is not possible to get λ for large values of $S\%$ when the sample size is small. Further, in Table II we require that $b_1 + b_0 N$ be positive in order for λ to be positive. It is worth stressing that the models shown in Table II are useful to interpolate and, more importantly, extrapolate the λ values for larger sample sizes than those shown in Table I.

Table III shows the minimum sample sizes required to produce a percentage of smoothness at most equal to $S\%$ by using Equation (18), as they appear in Table I, and compares them with those

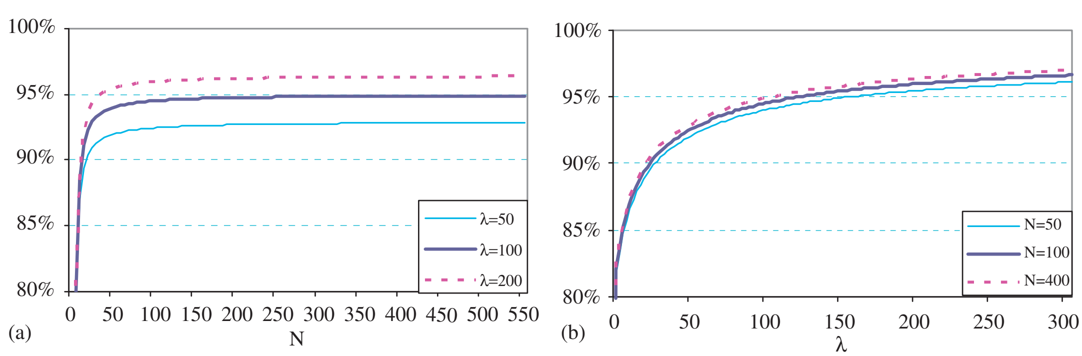

Figure 1. Behavior of $S\%$ for different values of λ and N.

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Table I. Values of λ as a function of sample size N and $S\%$ (for a daily series).

	Percentage of smoothness $S\%$										
N	50%	55%	60%	65%	70%	75%	80%	85%	90%	92.5%	95%
4	1.366	1.955	2.962	5.012	11.230	NA	NA	NA	NA	NA	NA
8	0.984	1.333	1.851	2.666	4.056	6.756	13.446	48.14	NA	NA	NA
12	0.894	1.198	1.638	2.312	3.415	5.410	9.608	21.34	109.66	NA	NA
16	0.854	1.138	1.546	2.161	3.153	4.903	8.437	17.38	55.37	195.79	NA
20	0.832	1.105	1.494	2.078	3.010	4.633	7.842	15.65	44.47	108.49	NA
28	0.807	1.068	1.439	1.989	2.859	4.351	7.236	14.00	36.71	76.67	277.48
36	0.794	1.048	1.409	1.942	2.780	4.205	6.930	13.19	33.38	66.62	195.80
44	0.785	1.036	1.391	1.913	2.731	4.116	6.746	12.72	31.51	61.40	167.83
52	0.780	1.028	1.378	1.894	2.698	4.056	6.622	12.40	30.30	58.15	153.06
60	0.776	1.022	1.369	1.879	2.674	4.013	6.534	12.18	29.46	55.94	143.68
72	0.771	1.016	1.359	1.864	2.649	3.968	6.440	11.94	28.58	53.68	134.53
84	0.768	1.011	1.352	1.853	2.631	3.935	6.374	11.78	27.98	52.14	128.54
96	0.766	1.008	1.347	1.845	2.618	3.911	6.325	11.66	27.55	51.04	124.30
108	0.764	1.005	1.343	1.839	2.608	3.893	6.288	11.56	27.21	50.20	121.15
120	0.763	1.003	1.340	1.834	2.600	3.878	6.258	11.49	26.95	49.55	118.71
136	0.761	1.001	1.337	1.829	2.591	3.863	6.227	11.41	26.68	48.87	116.21
152	0.760	0.999	1.334	1.825	2.584	3.851	6.202	11.35	26.46	48.34	114.30
168	0.759	0.998	1.332	1.822	2.579	3.841	6.183	11.31	26.29	47.92	112.78
184	0.758	0.997	1.331	1.819	2.574	3.833	6.166	11.27	26.15	47.58	111.55
200	0.758	0.996	1.329	1.817	2.570	3.826	6.153	11.23	26.04	47.30	110.53
220	0.757	0.995	1.328	1.814	2.567	3.819	6.139	11.20	25.91	47.00	109.48
240	0.756	0.994	1.326	1.812	2.563	3.813	6.127	11.17	25.81	46.76	108.61
260	0.756	0.993	1.325	1.811	2.560	3.808	6.117	11.15	25.73	46.55	107.89
280	0.755	0.992	1.324	1.809	2.558	3.804	6.109	11.12	25.66	46.38	107.28
300	0.755	0.992	1.324	1.808	2.556	3.801	6.101	11.11	25.60	46.23	106.75
324	0.755	0.991	1.323	1.807	2.554	3.797	6.094	11.09	25.53	46.07	106.20
348	0.754	0.991	1.322	1.806	2.552	3.793	6.087	11.07	25.48	45.94	105.74
372	0.754	0.990	1.321	1.805	2.551	3.791	6.081	11.06	25.43	45.82	105.34
396	0.754	0.990	1.321	1.804	2.549	3.788	6.077	11.05	25.39	45.72	104.99
420	0.754	0.990	1.320	1.803	2.548	3.786	6.072	11.04	25.35	45.63	104.68
448	0.753	0.989	1.320	1.802	2.547	3.784	6.068	11.02	25.31	45.54	104.36
476	0.753	0.989	1.319	1.802	2.546	3.782	6.064	11.01	25.28	45.46	104.08
504	0.753	0.989	1.319	1.801	2.545	3.780	6.060	11.01	25.25	45.39	103.83
532	0.753	0.989	1.319	1.800	2.544	3.778	6.057	11.00	25.22	45.32	103.61
560	0.753	0.988	1.318	1.800	2.543	3.777	6.054	10.99	25.20	45.27	103.41

of Table II. These values are similar, but the minimum values in Table I are generally greater than or equal to the values of N obtained with the regression model. Thus, Table III allows us to see the distorting effect of using the fitting model rather than Equation (18).

It is important to say that another way of looking at the smoothness property of a trend produced by a filter is by means of spectral analysis. To assess smoothing in the frequency domain we should look at squared gain plots, as did Kaiser and Maravall [14] when analyzing the HP filter. That kind of approach was also adopted by King and Rebelo [8] to compare the properties of the ES and HP filters within the context of cyclical analysis. Moreover, several tools have been developed in the frequency domain for performing cyclical analysis. For instance, Kaiser and Maravall [14] suggests to choose the λ value for the HP filter by fixing the length of the period over which the

Table II. Estimation results of fitting models that relate λ with N and $S\%$ (daily series).

	Model form: $\lambda = N/(b_1 + b_0 N)$		
$S\%$	b_0	b_1	R^2
50	1.330926	−2.441986	0.9989
55	1.013488	−2.048590	0.9988
60	0.760049	−1.720560	0.9990
65	0.557036	−1.446806	0.9994
70	0.394926	−1.220657	0.9997
75	0.265943	−0.966744	0.9989
80	0.166080	−0.746887	0.9991
85	0.091809	−0.559849	0.9992
90	0.040247	−0.366094	0.9991
92.5	0.022526	−0.273268	0.9992
95	0.009950	−0.177600	0.9994

Table III. Minimum values of N required to produce at most $S\%$. From Tables I and II.

$S\%$	50%	55%	60%	65%	70%	75%	80%	85%	90%	92.5%	95%
N	3	3	3	3	4	5	6	7	11	14	21
$N_{\text{regression}}$	2	3	3	3	4	4	5	7	10	13	18

analyst wishes to measure cyclical activity. The same reasoning could be applied with the ES filter, but we did not follow that approach in this paper, because we are not interested in performing cyclical analysis in particular, but in obtaining a trend that provides a nice visual representation of the data. That is, our aim is to obtain an appropriate estimate of the trend (in terms of the percentage of smoothness attained) not in detrending the series in order to study the underlying cyclical activity.

3. EXTENSION TO OTHER THAN DAILY FREQUENCIES

To estimate the trend of a series with frequency of observation different than daily, it is not adequate to use the same λ obtained for a daily series. To see why let us assume that the observation period spans years 2000–2002. Thus, we either have 1095 daily data, or 781 daily data by considering 5-day weeks, or only 156 weekly data, so that the sample size changes for each type of periodicity under consideration. Therefore, for each of those series and the same $S\%$ Table I would lead to different λ values. However, we should bear in mind that the long term behavior of the series must be essentially the same, no matter what the periodicity of the series is. Moreover, it is important to realize that a series with lower frequency of observation is related to that with higher frequency by means of some type of aggregation mechanism. This fact has been recognized by Maravall and del Río [16], who proposed different solutions to find λ values that produce equivalent results on series with different periodicities, from a frequency domain perspective.

In the present case, the selection of the smoothing constant for a nondaily series will be based on a time domain methodology that produces an equivalent constant, in the sense of producing the same percentage of smoothness as that obtained with a daily series. We start by considering the type of aggregation that links a lower-frequency series $\{Y^*_T\}$ with a higher-frequency series $\{Y_t\}$. The aggregation is assumed to be linear, that is

$$Y^*_T = \sum_{j=1}^{k} c_j Y_{k(T-1)+j} \quad \text{for } T=1,\ldots,n \tag{19}$$

with $n=\lfloor N/k \rfloor$, where $\lfloor x \rfloor$ denotes the integer part of a real number x, and k is the number of observations Y_t between two successive observations Y^*_T. The c_js are constants that define the type of aggregation, e.g. $c_1=\cdots=c_k=1$ are used to aggregate a flow series and $c_1=\cdots=c_k=1/k$ are used when working with an index or an annualized flow series (which is also considered a flow series). When working with a series of stocks, the aggregated series is generated by systematic sampling. In that case the usual values are $c_1=1$, $c_2=\cdots=c_k=0$ or $c_1=\cdots=c_{k-1}=0$, $c_k=1$. Without loss of generality, in what follows we shall assume that $c_1=\cdots=c_k=1$ for a flow series and $c_1=\cdots=c_{k-1}=0$, $c_k=1$ for a series of stocks. Thus, let T and t be the time sub-index for the aggregated and disaggregated series, respectively, then we have

$$Y^*_{T-j} = \begin{cases} Y_{t-jk}+Y_{t-jk-1}+\cdots+Y_{t-(j+1)k+1} & \text{for flows} \\ Y_{t-jk} & \text{for stocks} \end{cases}, \quad j=0,1,\ldots,T-1 \tag{20}$$

The model to be applied to the aggregated data preserves the form (12)–(13), in particular the order of integration (as shown by Brewer [17]), that is,

$$\mathbf{Y}^* = \boldsymbol{\tau}^* + \boldsymbol{\eta}^* \quad \text{with } E(\boldsymbol{\eta}^*)=\mathbf{0}, \quad \text{Var}(\boldsymbol{\eta}^*)=\sigma^{*2}_{\eta} I_n \tag{21}$$

$$K_{1n}\boldsymbol{\tau}^* = \boldsymbol{\varepsilon}^* \quad \text{with } E(\boldsymbol{\varepsilon}^*)=\mathbf{0}, \quad \text{Var}(\boldsymbol{\varepsilon}^*)=\sigma^{*2}_{\varepsilon} I_{n-1} \tag{22}$$

and $E(\boldsymbol{\varepsilon}^*\boldsymbol{\eta}^{*'})=0$, where $*$ is used to denote aggregated variables. As with (15), the trend estimator is given by

$$\hat{\boldsymbol{\tau}}^* = (\sigma^{*-2}_{\eta} I_n + \sigma^{*-2}_{\varepsilon} K'_{1n}K_{1n})^{-1}\sigma^{*-2}_{\eta}\mathbf{Y}^* \tag{23}$$

Even though expressions (15) and (23) are of the same form, the resulting trends are different. In fact, aggregating the trend $\{\hat{\tau}_t\}$ estimated from the disaggregated series will produce different values than those of the estimated trend $\{\hat{\tau}^*_T\}$ obtained directly from the aggregated series. Nevertheless, we show below that it is possible to find a smoothing constant λ for a disaggregated series that is equivalent to the λ^*_k value for the aggregated data, in the sense of producing the same percentage of smoothness.

From Equations (21)–(22) it follows that

$$\nabla^* Y^*_T = \varepsilon^*_T + \nabla^* \eta^*_T \tag{24}$$

where ∇^* is the difference operator for the aggregated series, so that $\{Y^*_T\}$ is represented by an IMA$(1,1)$, i.e. an Integrated Moving Average model of order $(1,1)$ whose variance γ^*_0 and autocovariance γ^*_1 are given by

$$\gamma^*_0 = \sigma^{*2}_{\varepsilon} + 2\sigma^{*2}_{\eta} \quad \text{and} \quad \gamma^*_1 = -\sigma^{*2}_{\eta} \tag{25}$$

Similarly, from Equations (12)–(13) we know that the disaggregated series follows the IMA$(1,1)$ model $\nabla Y_t = \varepsilon_t + \nabla\eta_t$ with variance $\gamma_0 = \sigma^2_{\varepsilon} + 2\sigma^2_{\eta}$ and covariance $\gamma_1 = -\sigma^2_{\eta}$. To see how the two

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IMA(1, 1) models relate to each other, let us first consider a flow series, i.e. let $Y^*_{T-j} = S_k Y_{t-jk}$ with $S_k = 1 + B + \cdots + B^{k-1}$. Since $\nabla_k = 1 - B^k = S_k \nabla$ we have $\nabla_k Y_t = S_k(\varepsilon_t + \nabla \eta_t)$ so that $\nabla_k S_k Y_t = S_k^2 \varepsilon_t + S_k \nabla_k \eta_t$. Then, as $\nabla^* Y^*_T = \nabla_k S_k Y_t$ we get

$$\nabla Y^*_T = S_k^2 \varepsilon_t + S_k \nabla_k \eta_t \quad \text{for flows} \tag{26}$$

Now, for a stock series we know that $Y^*_{T-j} = Y_{t-jk}$ and $\nabla^* Y^*_T = \nabla_k Y_t$ so that the previous derivation leads us to

$$\nabla Y^*_T = S_k \varepsilon_t + \nabla_k \eta_t \quad \text{for stocks} \tag{27}$$

Therefore, the autocovariance generating function (AGF) of the disaggregated series, $\gamma(B) = \sum_{j=-\infty}^{\infty} \gamma_j B_j$, is given by

$$\gamma(B) = \begin{cases} S_k^2 \bar{S}_k^2 \sigma_\varepsilon^2 + S_k \nabla_k \bar{S}_k \bar{\nabla}_k \sigma_\eta^2 & \text{for flows} \\ S_k \bar{S}_k \sigma_\varepsilon^2 + \nabla_k \bar{\nabla}_k \sigma_\eta^2 & \text{for stocks} \end{cases} \tag{28}$$

where an upper bar denotes the corresponding polynomial with B replaced by B^{-1} (i.e. $\bar{\nabla}_k = 1 - B^{-k}$).

The values σ_ε^2, σ_η^2, σ_ε^{*2} and σ_η^{*2} that make the results of both ES filters become equivalent are obtained by equating γ_0^* and γ_1^* to γ_0 and γ_{1k}. That is, by asking the following system of equations to hold true:

$$\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \sigma_\varepsilon^{*2} \\ \sigma_\eta^{*2} \end{pmatrix} = \begin{pmatrix} a_{11,k} & a_{12,k} \\ a_{21,k} & a_{22,k} \end{pmatrix} \begin{pmatrix} \sigma_\varepsilon^2 \\ \sigma_\eta^2 \end{pmatrix} \tag{29}$$

where $a_{11,k}$ and $a_{21,k}$ are the coefficients of B^0 and B^k in the polynomial $S_k^2 \bar{S}_k^2$ for a flow series or in the polynomial $S_k \bar{S}_k$ for a stock series. In the same way, $a_{12,k}$ and $a_{22,k}$ are the coefficients of B^0 and B^k in the polynomial $S_k \nabla_k \bar{S}_k \bar{\nabla}_k$ for a flow series or in the polynomial $\nabla_k \bar{\nabla}_k$ if the series is of stocks. This system has a unique solution for σ_ε^{*2} and σ_η^{*2}, or for σ_ε^2 and σ_η^2, depending on which pair of variances is known. If the pair σ_ε^{*2} and σ_η^{*2} is known, so that $\lambda_k^* = \sigma_\eta^{*2} / \sigma_\varepsilon^{*2}$ is known, we solve the system to get

$$\sigma_\varepsilon^2 = \frac{a_{22,k} + (a_{12,k} + 2a_{22,k})\lambda_k^*}{a_{11,k} a_{22,k} - a_{12,k} a_{21,k}} \quad \text{and} \quad \sigma_\eta^2 = \frac{-a_{21,k} - (a_{11,k} + 2a_{21,k})\lambda_k^*}{a_{11,k} a_{22,k} - a_{12,k} a_{21,k}} \tag{30}$$

from which it follows that $\lambda = \sigma_\eta^2 / \sigma_\varepsilon^2$ is given by

$$\lambda = \frac{-a_{21,k} - (a_{11,k} + 2a_{21,k})\lambda_k^*}{a_{22,k} + (a_{12,k} + 2a_{22,k})\lambda_k^*} \tag{31}$$

Now, since $\nabla_k \bar{\nabla}_k = 2 - (B^k + B^{-k})$, $S_k \nabla_k \bar{S}_k \bar{\nabla}_k = 2k - k(B^k + B^{-k}) + P_{1,k}(B, B^{-1})$, $S_k \bar{S}_k = k + P_{2,k}(B, B^{-1})$ and $S_k^2 \bar{S}_k^2 = 2\sum_{i=1}^{k-1} i^2 + k^2 + (k^3 - k)(B^k + B^{-k})/6 + P_{3,k}(B, B^{-1})$, where $P_{1,k}(B, B^{-1})$, $P_{2,k}(B, B^{-1})$ and $P_{3,k}(B, B^{-1})$ are polynomials in B and B^{-1} that have no powers of type B^{ik}, for $i = 0, 1$. By inspection of the polynomials involved, we found for flows

$a_{11,k}=(2k^3+k)/3$, $a_{21,k}=(k^3-k)/6$, $a_{12,k}=2k$ and $a_{22,k}=-k$, whereas for stocks, $a_{11,k}=k$, $a_{21,k}=0$, $a_{12,k}=2$ and $a_{22,k}=-1$. Therefore, we finally get

$$\lambda=\begin{cases}(k^2-1)/6+k^2\lambda_k^* & \text{for flows}\\ k\lambda_k^* & \text{for stocks}\end{cases} \tag{32}$$

When the smoothing constant λ for the disaggregated series is known, it should be clear that we can also use expression (32) to solve for the corresponding value of λ_k^*.

To see how the previous result works with an observed time series, let us consider the Daily Exchange Rate (Pesos/U.S. Dollar) series shown in Figure 2 together with its trend. The sample period covers data from January, 2004 to March, 2006 and considers 5-day weeks, thus the sample consists of $N=580$ daily observations. Table II indicates using $\lambda=11.008$ and $\lambda=103.694$ for 85% and 95% smoothness, respectively. As the percentage of smoothness grows, the trend emphasizes the smoothness component and reveals the long-term component of the series, which is what we want to see.

In Figure 3 we show the weekly Exchange Rate (Pesos/U.S. Dollar) series and its trend for the same period considered before. We should stress that this is a series of flows (it is the average rate for the period under consideration). The sample now consists of $n=116$ weekly observations, considering 5-day weeks. The values of λ are obtained with the formula $\lambda=4+25\lambda_5^*$. Therefore, the smoothing constants for the weekly series are $\lambda_5^*=0.280$ for $S\%=85\%$ and $\lambda_5^*=3.988$ for $S\%=95\%$. By comparing the resulting trends with those in Figure 2, we see that trends with the same percentage of smoothness show essentially the same dynamic behavior, no matter what the frequency of observation of the data is.

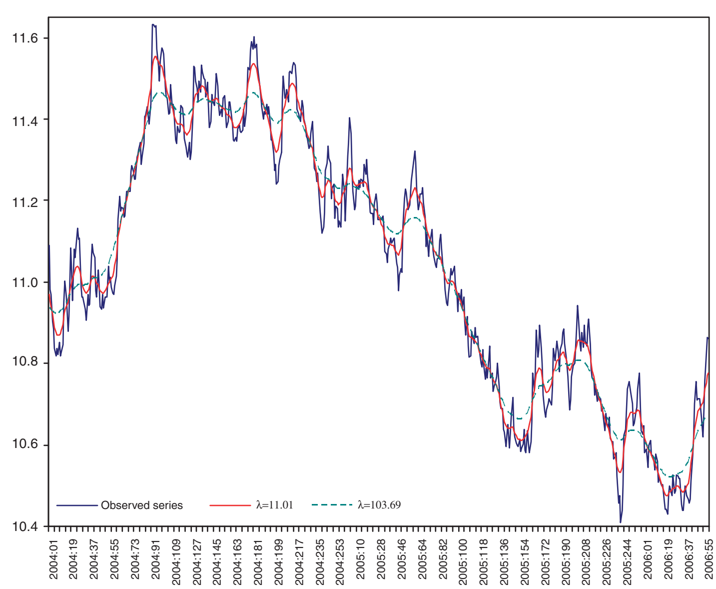

Figure 2. Daily exchange rate (Pesos/U.S. Dollar) and trends with $S\%=85\%$ ($\lambda=11.01$) and $S\%=95\%$ ($\lambda=103.69$).

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Figure 3. Weekly exchange rate (Pesos/U.S. Dollar) and trends with $S\%=85\%$ ($\lambda=0.28$) and 95% ($\lambda=3.99$).

4. WHY USE DAILY DATA AS THE STANDARD FREQUENCY?

In this section we argue that the λ values in Table II, as well as the values obtained from the fitting models in Table III, are best suited for time series of daily observations. This argument relies on the fact that, for a given standard frequency of observation, the smoothness achieved must reflect what is expected from a trend with a given percentage of smoothness. That is, if the selected $S\%$ is high the resulting trend should emphasize smoothness over fit; while low values of $S\%$ should work in the opposite direction. Thus, we present here a few numerical examples that lend empirical support to the use of a daily series over series with other frequencies. In fact, we carried out several other analyses before reaching our conclusion of using a daily frequency of observation as standard of reference. That is, we assumed first that the λ value produced by the fitting model of Table II was valid for a given series, say a quarterly series, and produced the trend with such a value. Next, on the assumption that the values in Table II were valid for a monthly series and using the results in the previous section, we derived the equivalent smoothing constant for the corresponding quarterly series and produced another trend for the same series.

We repeated this procedure by changing the assumption that the values in Table II were valid for series with another frequency of observation (biweekly, weekly, daily or intraday). Afterwards, we considered another series, say a monthly series, and repeated the whole process anew. Finally, we studied the resulting trends with different percentages of smoothness, particularly with 65% and 85% smoothness, by visual inspection. It should be stressed that this kind of analysis was required for the ES filter and not for the HP filter because the latter arose naturally when analyzing quarterly economic time series.

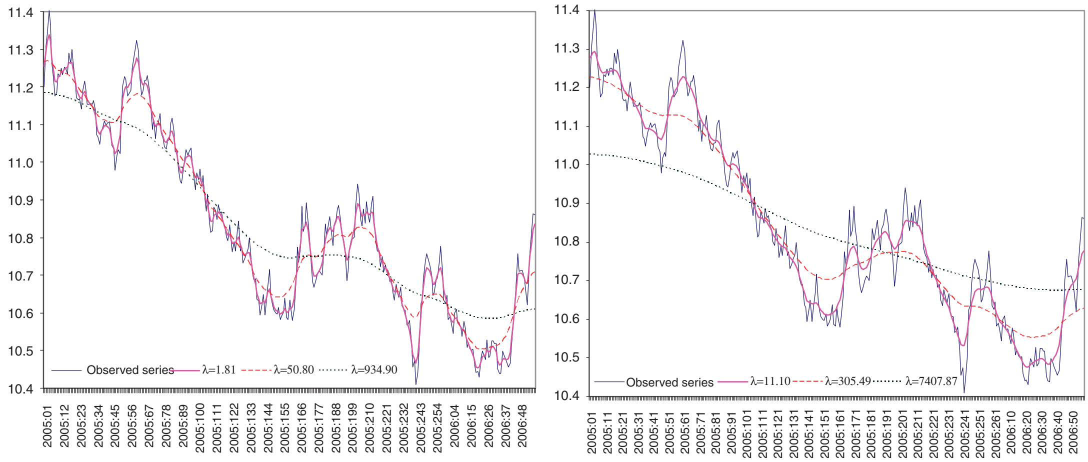

Figure 4. Daily exchange rate (Pesos/U.S. Dollar). Observed series and trends with: $S\%=65\%$ (left): daily ($\lambda=1.81$), weekly ($\lambda=50.80$) and monthly ($\lambda=934.90$); $S\%=85\%$ (right): daily ($\lambda=11.10$), weekly ($\lambda=305.49$) and monthly ($\lambda=7407.87$).

4.1. Daily exchange rate

In Figure 4 we present the daily Exchange Rate (Mexican Pesos/U.S. Dollar) series and two trends, with 65 and 85% smoothness respectively, for different frequencies of observation (daily, weekly and monthly). The sample period runs from January, 2005 to March, 2006. We obtained the trends for the different periodicities by assuming that the value of λ for each periodicity was given as in Table II and applied the results of the previous section to obtain the equivalent value of the smoothing constant for the daily series.

Using a daily basis we get $N=319$ and $\lambda=1.810$. The graph in the left of Figure 4 allows us to see that the trend neither emphasizes the fit to the data nor its smoothness, which is what we expect to see for 65% of smoothness. Now, in the weekly case with $N=63$ we get $\lambda=1.872$ from Table II, then we assume 5-day weeks (i.e. $n=20$, $k=5$ and $\lambda_5^*=1.872$) to obtain $\lambda=50.80$ from expression (32). In this case, the figure shows an apparent undue emphasis of smoothness over fit of the trend. This fact is even more pronounced in the monthly case, where we used $N=15$, $n=5$, $k=20$, $\lambda_{20}^*=2.171$ and $\lambda=934.90$. Moreover, we could not assume a quarterly series because we would have $N=1$ and the method requires at least three quarters to produce a trend with 65% smoothness. The graph in the right of Figure 4 shows results similar to those of the previous exercise, but now with 85% smoothness. Using a daily basis we have $N=319$, $\lambda=11.104$ and the resulting trend shows a smooth long-term behavior, while keeping a reasonable fit to the data. With weekly ($N=63$, $\lambda_5^*=12.059$, $k=5$ and $\lambda=305.486$) and monthly ($N=15$, $\lambda_{20}^*=18.353$, $k=20$ and $\lambda=7407.871$) bases the trend overemphasizes smoothness and tends to behave as a constant mean.

4.2. Mexican federal funds—CETES28

The following exercise considers the weekly yield rate of the Mexican federal funds called CETES28. This series is shown in Figure 5 together with its trend with 85% of smoothness. Since

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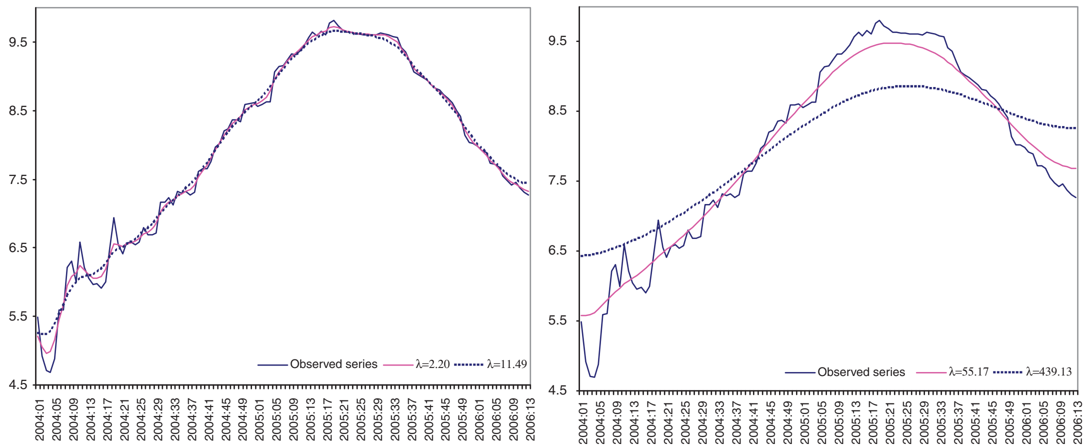

Figure 5. Weekly yield rate of CETES28 and its trend with 85% smoothness. Left: daily ($\lambda = 2.20$) and weekly ($\lambda = 11.49$). Right: monthly ($\lambda = 55.17$) and quarterly ($\lambda = 439.13$).

the sample covers data from January, 2004 to March, 2006 ($n = 117$ weeks) the trend for daily data is obtained by assuming $N = 585$ days, so that $\lambda_5^* = 11.007$. The series is of stocks, thus for the weekly case we get from expression (32) $k = 5$ and $\lambda = 2.201$. Now, by considering the weekly data directly we know that $N = 117$, thus we get $\lambda = 11.491$ from Table II.

In the right of that figure we also show the corresponding exercise with monthly and quarterly periodicity. For the monthly data we assumed $N = 29$, so that $\lambda = 13.792$. As a result, for monthly data we have $k = 4$ and $\lambda_4^* = 55.168$. For quarterly data (with $N = 9$ quarters) we obtained $\lambda = 33.780$, consequently we used $k = 13$ to produce $\lambda_{13}^* = 439.133$. It should be noticed that as the basis period increases, the trend becomes smoother. When the periodicity is daily or weekly, the trend shows a balance between smoothness and fit. On the other hand, if a monthly or quarterly basis is used, the smoothness component is over-emphasized. Thus, in this example neither the monthly nor the quarterly periodicities seem suitable as the standard basis of reference for trend smoothness.

4.3. Mexican stock exchange—IPC

Data with intraday frequency of observation have become common use in finance due to the advances in data acquisition and storage. Therefore, one objective of this work is to produce a method for trend estimation of time series with intraday periodicity. In Figure 6 we plot the trend of the intraday Mexican Stock Exchange index, named IPC (which is Mexico's most important financial index) with 55 and 65% smoothness and daily and weekly base periodicities. The sample period runs from April 17, 2006 to May 12, 2006, and covers $N = 19$ days, so that $\lambda = 1.104$ for $S\% = 55\%$ and $\lambda = 2.079$ for $S\% = 65\%$. There are 78 observations per day, so for daily data we get $k = 78$, $\lambda_{78}^* = 86.112$ for $S\% = 55\%$ and $\lambda_{78}^* = 162.162$ for $S\% = 65\%$. For this sample period there are $N = 3$ complete 5-day weeks, thus Table II indicates $\lambda = 3.025$ for $S\% = 55\%$ and $\lambda = 13.375$ for $S\% = 65\%$. Therefore, for a weekly basis we have $k = 390$ and $\lambda_{390}^* = 1179.75$ for $S\% = 55\%$, and $\lambda_{390}^* = 5216.25$ for $S\% = 65\%$. We did not use monthly or quarterly periodicities because the resulting sample size became smaller than $N = 1$. It should be clear from this example that

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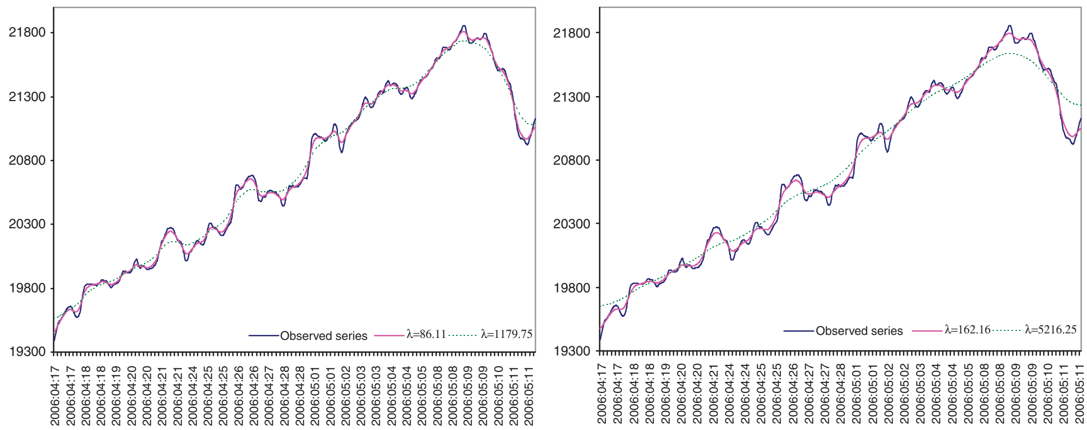

Figure 6. Intraday IPC series and trends with: $S\%=55\%$, left: daily ($\lambda=86.11$) and weekly ($\lambda=1179.75$); $S\%=65\%$, right: daily ($\lambda=162.16$) and weekly ($\lambda=5216.25$).

estimated trends for series with weekly periodicity become too smooth for low percentages of smoothness. Moreover, it is not possible to obtain trends with a weekly basis and relatively high percentages of smoothness. For the aforementioned reasons we concluded that the most suitable basis periodicity is the daily one.

5. FURTHER ILLUSTRATIVE EXAMPLES

The proposed method is now applied to some of Mexico's financial time series to illustrate different situations that may arise in practice.

5.1. Same time series with different sample periods

In this illustration we estimate the trend of the weekly yield rate of CETES28 employed previously (see Figure 5), but with different sample periods. In fact, the sample period covers the third week of November, 2005 up to the last week of March, 2006. We have to get first the λ^* value of the weekly series equivalent to the constant λ used with the daily series. Since we are dealing with a stock series, the equivalent smoothing constants are related by $\lambda=5\lambda_5^*$. With $N=100$ daily data the λ values that produce 65 and 85% smoothness of the trend are $\lambda=1.843$ and $\lambda=11.600$, respectively. Therefore, for the weekly series with $n=20$ we get $\lambda_5^*=0.369$ when $S\%=65\%$ and $\lambda_5^*=2.320$ when $S\%=85\%$. The resulting trends are shown in Figure 7. In particular, the trend with 85% smoothness shown in this figure resembles closely that in Figure 5. This fact can be appreciated numerically in Table IV. There we see that the trend figures are very close to each other, mainly at the middle and the end of the shorter sample period.

5.2. Time series with missing data

We now use the method to obtain the trend for the daily 5-day week IPC index. This is a stock series whose period of observation runs from November 14, 2005 to March 31, 2006. In this case

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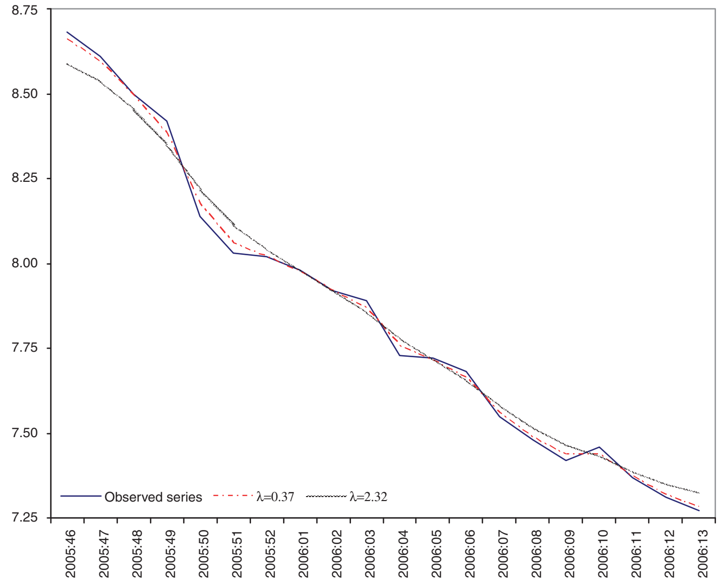

Figure 7. Weekly yield rate of CETES28 and trends with $S\%=65\%$ ($\lambda=0.37$) and $S\%=85\%$ ($\lambda=2.32$).

Table IV. Weekly yield rate of CETES28. Selected trend values with $S\%=85\%$.

		Sample used for trend estimation	
Week	Observed data	2004:01–2006:13	2005:46–2006:13
2005:46	8.68	8.650	8.585
2005:47	8.61	8.569	8.534
2005:48	8.50	8.470	8.452
⋮	⋮	⋮	⋮
2006:01	7.98	7.979	7.978
2006:02	7.92	7.917	7.917
2006:03	7.89	7.854	7.853
⋮	⋮	⋮	⋮
2006:11	7.37	7.384	7.385
2006:12	7.31	7.345	7.346
2006:13	7.27	7.321	7.323

we want to emphasize the fact that holidays produce missing values in the series, but that does not prevent us from obtaining the trend with the data available. This is so because we use the Kalman filter to perform the calculations. In fact, all we need to do to get the estimated trend values for missing data is to skip the filter step in the Kalman filter recursions. The period of observation is the same as in the previous CETES28 example; hence, the λ values are also the same as in that exercise. We present the resulting estimated trend in Figure 8.

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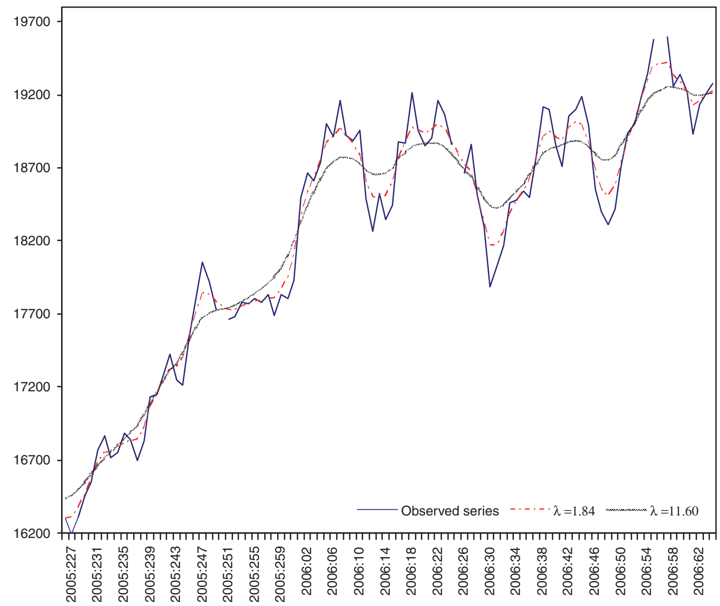

Figure 8. Daily IPC index and trends with 65% ($\lambda = 1.84$) and 85% ($\lambda = 11.60$) smoothness.

5.3. Intraday time series

The following application makes use of intraday data on the IPC. The frequency of observation of the series is every 5 min, from 8:30 a.m. to 3 p.m. while the Mexican Stock Exchange is open. The sample period runs from April 17, 2006 to May 12, 2006 (19 working days), hence, the daily number of observations is 78, for a total number of 1482 5-minute observations. To obtain the smoothing constants for the trend we first obtained the λ values for $N = 19$ days from Table II. These values are $\lambda = 2.079$ and 16.040 for percentages of smoothness of 65 and 85%, respectively. The equivalent smoothing constants for the series of 5-minute observations are obtained from the relation $\lambda = k\lambda_k^*$ with $k = 78$ and λ_{78}^* the smoothing constant previously obtained. The resulting values are $\lambda = 162.162$ and 1251.12 for 65 and 85%, respectively. In Figure 9 we appreciate the smooth curve representing the trend, which behaves as the dynamic mean of the series and provides a nice visual representation of the data.

6. FINAL REMARKS

In this paper we propose to choose the smoothing constant of the ES filter with a method that arises by defining an index of smoothness that measures the relative precision of the smoothness component in the statistical model underlying the filter. The smoothing constant is shown to be a function of the desired percentage of smoothness for the trend, the sample size and the periodicity of the series. We could not get a closed expression of the smoothing constant, but we were able to provide an approximating formula with empirical support. Once the smoothing constant is chosen,

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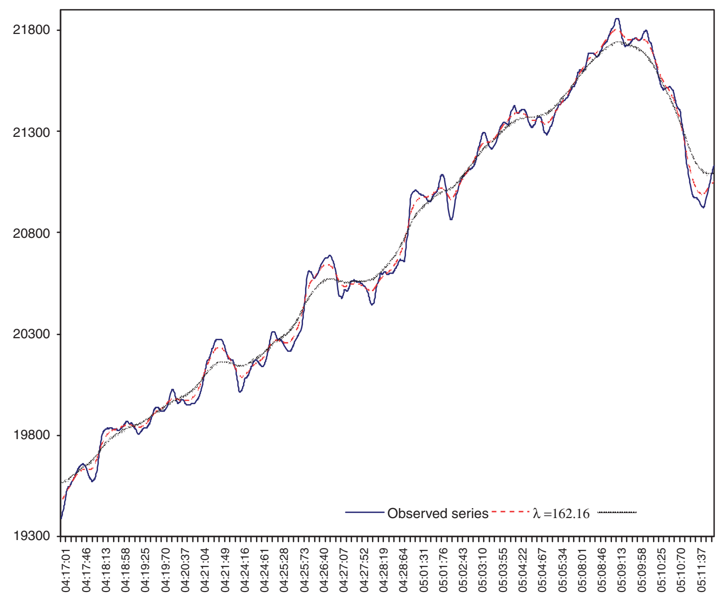

Figure 9. Intraday IPC index and trends with 65% ($\lambda = 162.16$) and 85% ($\lambda = 1251.12$) smoothness.

we suggest employing the Kalman filter to perform the required calculations for estimating the trend; thus, avoiding numerical problems that may arise from inverting large dimensional matrices.

We show that the procedure for choosing the smoothing constant of a daily time series can be generalized to time series with other than daily periodicities. The expression that relates equivalent smoothing constants is very simple for both series of flows or stocks. We also provide some evidence on the empirical benefits of using the daily frequency of observation as standard basis of reference. Thus, we consider the problem of obtaining the trend for the same time series observed with different periodicities. In that case, we show with some empirical examples that trends with equal degrees of smoothness, for different periodicities, show essentially the same dynamic behavior.

We illustrate the kind of results that can be obtained in practical applications by way of numerical examples. In particular, we consider situations in which the trend is needed for time series with lower than daily frequency or with intraday data. In those cases we show explicitly how to calculate the corresponding smoothing constant, starting from the equivalent daily one. Another situation of practical importance occurs when a time series has missing data, and we illustrate how that fact does not pose any problem for obtaining the trend. Another illustration allows us to see that obtaining trends for the same time series, but with different sample sizes, produces very similar numerical results.

Finally, we emphasize that the degree of smoothness for trend estimation should be chosen by the user at the outset. In fact, we recommend to choose the percentage of smoothness on the basis of expert judgment. As the method is mainly designed to enhance comparability of trends we suggest carrying out a pilot study of the series to be smoothed (e.g. a group of interest rates). Thus, by visual inspection of the trends in the study group, we can fix an appropriate percentage and apply that value to all the series in that category. Even for just one series, say an intraday series, the idea is to calculate the trend for that series each and every day, with the same percentage of smoothness, no matter what that percentage is. Once the percentage of smoothness has been decided, the resulting trends for different time series of the same length and frequency of observation or for the same series with different lengths can be compared appropriately.

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