On the origin of power law tails in price fluctuations

J. Doyne Farmer[1] and Fabrizio Lillo[1,2]

[1]*Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, NM 87501*

[2]*Istituto Nazionale per la Fisica della Materia, Unità di Palermo, Italy*

(Dated: February 2, 2008)

In a recent Nature paper, Gabaix et al. [1] presented a testable theory to explain the power law tail of price fluctuations. The main points of their theory are that volume fluctuations, which have a power law tail with exponent roughly -1.5, are modulated by the average market impact function, which describes the response of prices to transactions. They argue that the average market impact function follows a square root law, which gives power law tails for prices with exponent roughly -3. We demonstrate that the long-memory nature of order flow invalidates their statistical analysis of market impact, and present a more careful analysis that properly takes this into account. This makes it clear that the functional form of the average market impact function varies from market to market, and in some cases from stock to stock. In fact, for both the London Stock Exchange and the New York Stock Exchange the average market impact function grows much slower than a square root law; this implies that the exponent for price fluctuations predicted by modulations of volume fluctuations is much too big. We find that for LSE stocks traded in the electronic market the distribution of transaction volumes does not even have a power law tail. This makes it clear that volume fluctuations do not determine the power law tail of price returns.

Gabaix *et al.* [1] have recently proposed a testable theory for the origin of power law tails in price fluctuations. In essence, their proposal is that they are driven by fluctuations in the volume of transactions, modulated by a deterministic market impact function. More specifically, they argue that the distribution of large trade sizes scales as $P(V > x) \sim x^{-\gamma}$, where V is the volume of the trade and $\gamma \approx 3/2$. Based on the assumption that agents are profit optimizers, they argue that the average market impact function[1] is a deterministic function of the form $r = kV^{\beta}$, where r is the the change in the logarithm of price resulting from a transaction of volume V, k is a constant, and $\beta = 1/2$. This implies that large price returns r have a power law distribution with exponent $\alpha = \gamma/\beta \approx 3$. They argue that their theory is consistent with the data, even though their hypothesis about market impact appears to contradict several other previous studies [2, 3, 4] in the same markets they study (the New York and Paris Stock Exchanges).

I. PROBLEMS WITH THE TEST OF GABAIX ET AL.

Gabaix et al. [1] present statistical evidence that appears to show that the NYSE and Paris data are consistent with the hypothesis that the average market impact follows a square root law. In this section we show that their test may have problems in circumstances (such as those of the real data) in which orders have long-memory properties. This weakens their test, so that it lacks the power to reject reasonable alternative hypotheses and may give misleading results.

Their method to test the hypothesis of square root price impact is to investigate $E[r^2|V]$ over a given time interval, e.g. 15 minutes, where r is the price shift and $V = \sum_{i=1}^{M} V_i$ is the sum of the volumes of the M transactions occurring in that time interval. They have chosen to analyze r^2 rather than r because of its properties under time aggregation. To see why this might be useful, assume the return due to each transaction i is of the form $r_i = k\epsilon_i V_i^{\beta} + u_i$, where u is an IID noise process that is uncorrelated with V_i, and ϵ_i is the sign of the transaction. The squared return for the interval is then of the form

$$r^2 = \sum_{i=1,j=1}^{M} (k\epsilon_i V_i^{\beta} + u_i)(k\epsilon_j V_j^{\beta} + u_j) \qquad (1)$$

Under the assumption that V_i, V_j, ϵ_i, and ϵ_j are all uncorrelated, when $\beta = 1/2$ it is easy to show that $E[r^2|V] = a + b\,V$, where a and b are constants.

The problem is that for the real data V_i, V_j, ϵ_i, and ϵ_j are strongly correlated, and indeed, the sequence of signs ϵ_i is a long-memory process [5, 6]. To demonstrate the gravity of this problem, we use real transactions V_i, but introduce an artificial and deterministic market impact function of the form $r_i = kV_i^{\beta}$ with $\beta \neq 0.5$. We first fix the number of transactions, and then repeat the same procedure using a fixed time period. We examine blocks of trades with M transactions, $\{\epsilon_i, V_i\}$, $i = 1, ..., M$, where $\epsilon_i = +1$ (-1) for buyer (seller) initiated trades and V_i is the volume of the trade in number of shares. For each trade we create an artificial price return $r_i = k\epsilon_i V_i^{\beta}$, where k is a constant. Then for each block of M trades we compute $r = \sum_{i=1}^{M} r_i = k \sum_{i=1}^{M} \epsilon_i V_i^{\beta}$ and $V = \sum_{i=1}^{M} V_i$. Since we are using the real order

[1] One should more properly think of the market impact as a response to the order initiating the trade. That is, in every transaction there is a just-arrived order that causes the trade to happen, and this order tends to alter the best quoted price in the direction of the trade, e.g. a buy order tends to drive the price up, and a sell order tends to drive it down.

flow we are incorporating the correct autocorrelation of the signs ϵ_i and transaction sizes V_i. Figure 1(a) shows $E[r^2|V]$ for different values of M and $\beta = 0.3$ for the British stock Vodafone in the period from May 2000 to December 2002, a series which contains approximately 10^6 trades. We see that for small values of M the quantity $E[r^2|V]$ follows the artificial market impact functional form $E[r^2|V] \sim V^{2\beta} = V^{0.6}$, but when M is large the relation between $E[r^2|V]$ and V becomes linear. The value $M = 40$ is roughly the average number of trades in a 15 minute interval. We also show error bars computed as specified by Gabaix et al. We cannot reject the null hypothesis of a linear relation between $E[r^2|V]$ and V with 95% confidence, even though we have a large amount of data, and we know by construction that β is quite different from 1/2. We have also performed tests on other stocks, which give similar results.

One can ask whether it makes a difference that we used a fixed number of transactions rather than a fixed time interval. To test this we repeat the procedure using a fixed time interval of 15 minutes. Figure 1(b) shows the result. We see an even clearer linear relation between $E[r^2|V]$ and V than before, so that the test once again fails.

Why doesn't this test work? To gain some understanding of this, we repeat the same test but shuffle the order of the data, which breaks the correlation structure. As shown in Figure 1(c), the result in this case is far from linear even when $M = 40$, and the test easily shows that the market impact does not follow a square root law. Thus, we see that the problem lies in the autocorrelation structure of the real data.

In conclusion our numerical simulations show that the linearity test of $E[r^2|V]$ lacks power to test for a square root market impact with data containing the correlation structure of real data. In fact, even a deterministic market impact like $r \sim V^{0.3}$ is consistent with the relation $E[r^2|V] = a + b\ V$ for a sufficiently large number of trades. Doing this for a fixed time interval rather than a fixed number of trades time makes this even more evident. Thus the test of Gabaix et al. provides no evidence that the average market impact follows a square root law.

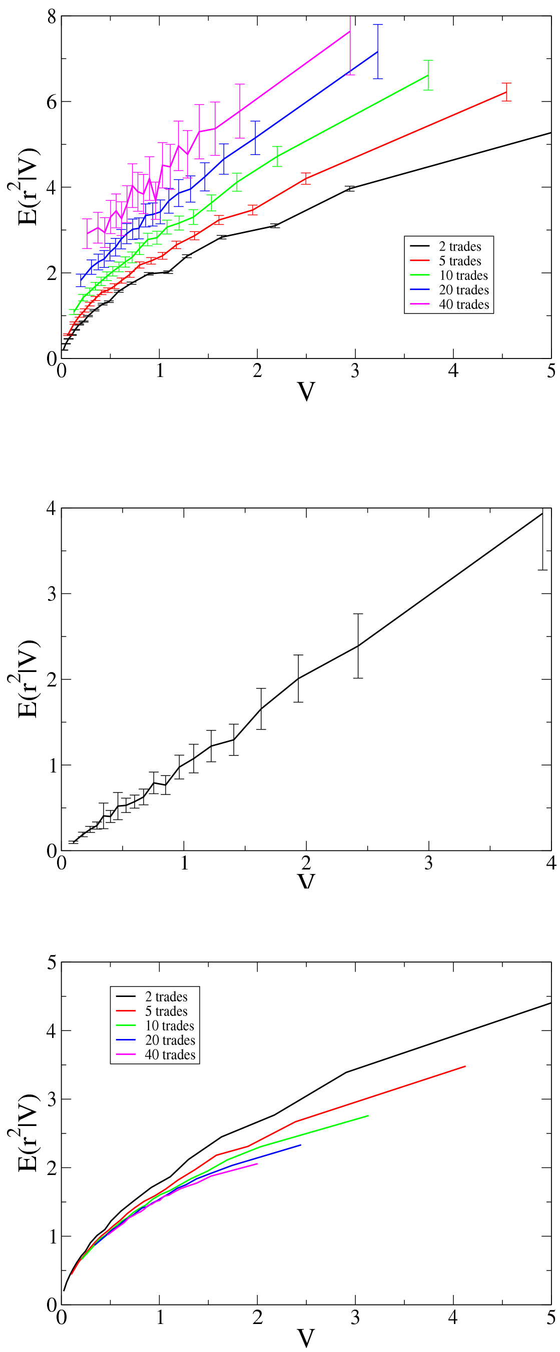

FIG. 1: A demonstration that the statistical test of Gabaix et al. [1] fails due to the strong autocorrelations in real data. The expected value of the squared price return, $E[r^2|V]$, is plotted as a function of total transaction size $V = \sum_{i=1}^{M} V_i$, where V_i is the size of transaction i. Each transaction causes a simulated market impact of the form $r_i = k\epsilon_i V_i^{\beta}$, to generate total return $r = \sum_{i=1}^{M} r_i$. The transaction series V_i and ϵ_i are from the real data from the electronic market for the British stock Vodafone, and contain roughly 10^6 events. The error bars are the 95% confidence intervals computed following the procedure specified by Gabaix et al. (a) shows the results for a fixed number of transactions, with M varying from 2 to 40; the curves are in ascending order of M; (b) is the same using a fixed time interval of 15 minutes, with variable M; and (c) is the same as (a) with the order of the transactions randomly shuffled. For (a) and (b) we see straight lines for large M, indicating that the test is passed, even though by construction the market impact does not follow the $r \sim V^{0.5}$ hypothesis, whereas for the shuffled data the test quite clearly shows us that the hypothesis is false.

II. PLACING ERROR BARS ON THE AVERAGE MARKET IMPACT

While there have been many previous studies of average market impact, they have not included the statistical analysis needed to assign good error bars. In this section we present results about average market impact at the level of individual ticks. We show that it does not generally follow a square root law, and that it varies from market to market and in some cases from stock to stock in a substantial and statistically significant way.

Realistic error bars for the average market impact are difficult to assess due to the fact that volatility is a long-memory process [7, 8]. That is, its time series has a slowly decaying power law autocorrelation function that is asymptotically of the form $\tau^{-\kappa}$, with $\kappa < 1$ so that the integral is unbounded. This makes error analysis complicated, since data from the distant past have a strong effect on data in the present. Because volatility is long-memory, the price returns that fall in a given volume bin V_a, which are by definition all of the same sign, are also long-memory. This means that the errors in measuring market impact are much larger than one would expect from intuition based on an IID hypothesis.

We analyze the market impact only for orders (or portions of orders) that result in immediate transactions. We call the portion of an order that results in an immediate transaction an effective market order, and for the remainder of the paper V_i represents effective market order size rather than transaction size. Each order of size V_i generates a price return $r_i = \log p_a - \log p_b$, where p_b is the midpoint price quote just before the order is placed and p_a is the midpoint price quote just after. We analyze buy and sell orders separately. The electronic (SETS) data for the LSE has the advantage that the data set contains a record of orders, and so we can distinguish buy and sell orders unambiguously, but has the disadvantage that it omits trades made in the upstairs market[2]. For the NYSE data we use the trades and quotes (TAQ) data to infer orders and their signs using the Lee and Ready algorithm [9]; to identify orders we lump together all trades with the same timestamp and order code. To estimate the average market impact we sort the events $(V_i,\ r_i)$ with the same sign ϵ_i into bins based on V_i and plot the average value of V_i for each bin against the average value of r_i, as shown in Figure 2. We choose the bins so that each bin has roughly the same number of points in it.

To assign error bars for each bin we use the variance plot method [7]. For each bin we split the events into m subsamples with $n = K/m$ points, where K is the number of records in the bin. The subsamples are chosen to be blocks of values adjacent in time. For each subsample i we compute the mean $\mu_i^{(n)}$, $i = 1, ..., m$. Then we compute the standard deviation of the $\mu_i^{(n)}$ which we indicate as $\sigma^{(n)}$. By plotting $\sigma^{(n)}$ versus n in a log-log plot we compute the Hurst exponent H by fitting the data with a power-law function $\sigma^{(n)} = An^{H-1}$. We compute the error in the mean of the entire sample of K points by extrapolating the fitted function to the value $m = K$, i.e. $\sigma = \hat{A}\ K^{\hat{H}-1}$ where $\hat{A}$ and $\hat{H}$ are the ordinary least square estimate of the parameters A and H. Interestingly, for smaller values of V_i we find Hurst exponents substantially larger than 1/2, whereas for large values of V_i the Hurst exponents are much closer to 1/2. When $H > 1/2$ the error bars are typically much larger than standard errors[3].

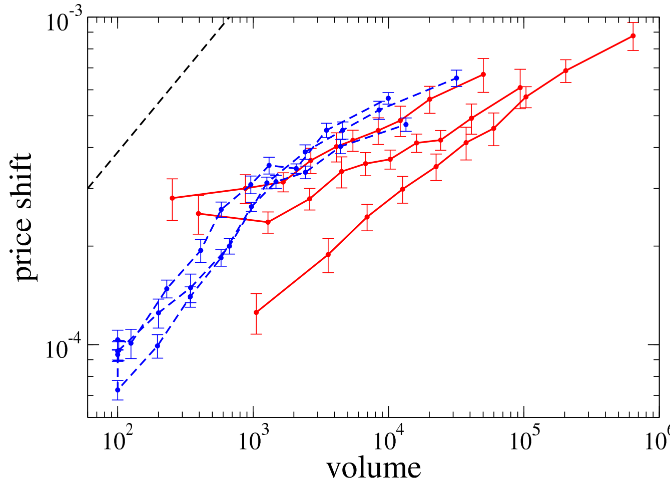

FIG. 2: Market impact function for buy orders of three stocks traded in the New York Stock Exchange (blue, dashed) and three stocks traded in the London Stock Exchange (red, solid). Orders of similar size V_i are binned together; on the horizontal axis we show the average volume of the orders in each bin, and on the vertical axis the average size of the logarithmic price change for the orders in that bin. In both cases comparison to the dashed black line in the corner, which has slope 1/2, makes it clear that the behavior for large volume does not follow a law of the form $r_i \sim V_i^{1/2}$. Error bars are computed using the variance plot method [7] as described in the text.

In Figure 2 we show empirical measurements of the average market impact for the New York Stock Exchange and for the London Stock Exchange. We consider three highly capitalized stocks for each exchange, Lloyds (LLOY), Shell (SHEL) and Vodafone (VOD) for the LSE, and General Electric (GE), Procter & Gamble (PG) and AT&T (T) for the NYSE. For LSE stocks we consider the period May 2000- December 2002, while for NYSE stocks we consider the time period 1995-1996. The data for the NYSE are consistent with results reported earlier without error bars [3], while the LSE market impact data is new. The NYSE data clearly do not follow a power law across the whole range, consistent with earlier results in references [2, 3]. While $\beta(V_i) \approx 0.5$ for small V_i, for larger V_i it appears that $\beta(V_i) < 0.2$. As shown in reference [3], this transition occurs for smaller values

[2] The relative impact on price formation of the upstairs and downstairs markets is not clear. On one hand, the upstairs market contains the largest trades. On the other hand, because these trades are arranged privately and then printed in the transaction record later, they may not have as large an effect on price formation.

[3] Since we choose the bins to have roughly the same number of points, the difference in Hurst exponent between bins with large and small V cannot be due to a difference in the mean interval between samples.

of V_i for stocks with lower capitalization. Thus, the assumption that $\beta = 0.5$ breaks down for high volumes, precisely where it is necessary in order for the theory of Gabaix et al. to hold. For the London data the power law assumption seems more justified across the whole range, but the exponent is too low; a least squares fit gives $\beta \approx 0.26$. While we have not attempted to compute error bars for the regression, a visual comparison with the error bars of the individual bins makes it quite clear that $\beta = 1/2$ is inconsistent with either the London or the NYSE data. A separate study of eleven LSE stocks gives $\beta = 0.26 \pm 0.02$ for buy orders and 0.23 ± 0.02 for sell orders [14]; in as yet unpublished work this has been extended to 50 stocks, with similar results. Our earlier study for the NYSE was based on 1000 stocks [3]. It is clear that the average market impact functions are qualitatively different for LSE and NYSE stocks, and that for NYSE stocks the functional form varies with market capitalization [3].

Even if we abandon the prediction that the average market impact is a square root law, one might imagine that we could explain fluctuations in prices in terms of fluctuations in volume modulated by average market impact of the form $r_i = kV_i^{\beta}$. However, if this were true, for the NYSE the predicted exponent for price fluctuations would be $\alpha = \gamma/\beta \approx 1.5/0.25 = 6$, which is much too large to agree with the data. (A typical value [10] is $\alpha \approx 3$). To make matters even worse, the power law hypothesis for volume or market impact appears to fail in some other markets. In the Paris Stock Exchange Bouchaud et al. [4] have suggested that the average market impact function[4] is of the form $\log V_i$, yielding $\beta \to 0$ in the limit as $V_i \to \infty$. For the London Stock Exchange the power law hypothesis for average market impact seems reasonable, but with an exponent significantly smaller than 1/2. Moreover, the volume for the electronic market is not power law distributed, as discussed in the next section.

Note that we are making all the above statements for individual orders, whereas many studies have been done based on aggregated data over a fixed time interval. Aggregating the data in time complicates the discussion, since the functional form of the market impact generally depends on the length of the time interval. Hence it is more meaningful to do the analysis based on individual transactions.

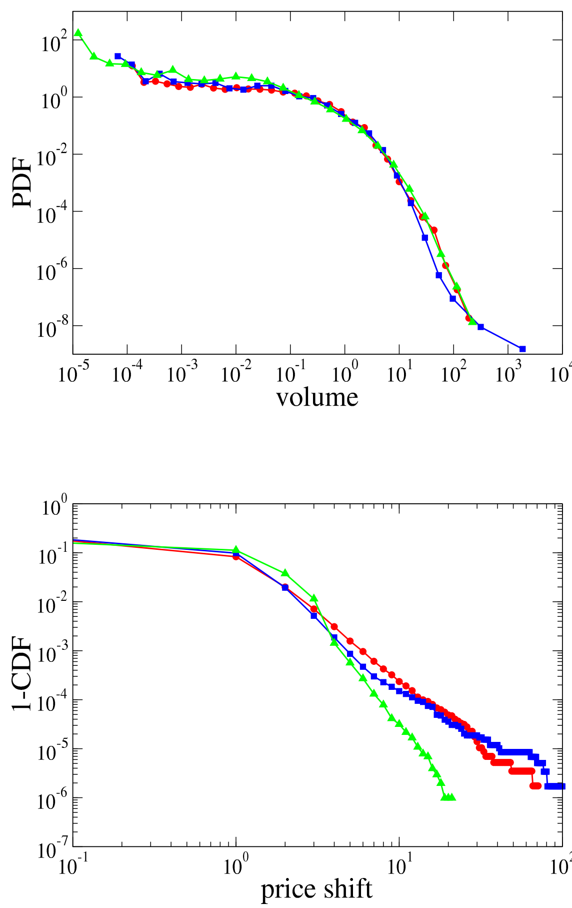

FIG. 3: (a) The probability density of normalized volume for three typical high volume stocks in the LSE, LLOY (red, circles), SHEL (blue, squares), and VOD (green, triangles) in the period May 2000- December 2002, based on data from the electronic exchange. There are approximately 10^6 data points for each stock. (b) $1-P(r)$, where $P(r)$ is the cumulative density function of returns induced by the same transactions in (a). For the normalized volume there is no clear evidence for power law tails; in contrast for returns this is quite plausible. Furthermore, the volume distributions are essentially identical, whereas the return distribution for VOD decays more steeply than the others.

III. VOLUME DISTRIBUTION

The theory of Gabaix et al. explains the power law of returns in terms of the power law of volume, so if volume doesn't have a power law, then returns shouldn't either. The existence of a power law tail for volume seems to vary from market to market. For the NYSE we confirm the observation of power law tails for volume reported earlier [11]. However in Figure 3 we show the distribution of volumes for three stocks in the electronic market of the LSE. In order to compare different stocks we normalize the data by dividing by the sample mean for each stock. All three stocks have strikingly similar volume distributions; this is true for the roughly twenty stocks that we have studied. There is no clear evidence

[4] For the NYSE the logarithmic form for average market impact is a reasonable approximation for small V_i, but breaks down for higher V_i

for power law scaling, even though the power law scaling of the corresponding return distributions shown in Figure 3(b) is rather clear. If one attempts to fit lines to the larger volume range of the curve (roughly $10^1 - 10^2$), the exponent of the cumulative distribution corresponding to Figure 3(a) is highly uncertain but it is at least 3, which together with the measured values of β would imply $\alpha \approx 3/0.3 \approx 10$. In contrast, the measured exponents for Figure 3(b) are roughly 2.2, 2.5, and 4.3 for SHEL, LLOY, and VOD respectively. It is noteworthy that VOD has a much larger α than the other stocks, even though it has essentially the same volume distribution and a similar volume distribution; if anything from Figure 2 it's β is larger than that of the other stocks, which according to $\alpha = \gamma/\beta$ would imply a smaller α. This provides yet more evidence that the power law tails of returns are not driven by those of volume.

Note that one of the differences between the NYSE and the LSE data examined here, which may be the underlying cause of the difference in their distributions, is that the data from the NYSE includes upstairs market trades, whereas the LSE data does not.

IV. CONCLUSION

We have shown that the conclusions of Gabaix et al. [1] are suspect for three different reasons: First, their statistical analysis in claiming the existence of a square root law for average market impact lacks power to reject alternative hypotheses in the presence of the strong autocorrelations that are present in real data; Second, new measurements of the average market impact with proper error bars show that it does not follow a square root law; Third, for electronic data the London Stock Exchange the distribution of volumes does not have a power law tail, and there are substantial variations between the return distributions that are not reflected in variations in volume or average market impact. Thus, it seems that the distribution of large price fluctuations cannot be explained as a simple transformation of volume fluctuations.

This leaves open the question of what really causes the power law tails of prices. We believe that the correct explanation lies in the extension of theories based on the stochastic properties of order placement and price formation [12, 13, 14], which naturally give rise to fluctuations in the response of prices to orders. Further work is clearly needed.

Note added in press: In a recent study it has been shown that large price fluctuations in the NYSE and the electronic portion of the LSE are driven by fluctuations in liquidity [15]. That is, if one matches up returns with the orders that generate them, the conditional distribution of large returns is essentially independent of order size. This has been confirmed for the NYSE and Island by Weber and Rosenow [16]. The idea that the tail of prices is driven by fluctuations in liquidity rather than fluctuations in the number of trades was implicitly suggested earlier by results of Plerou et al. [17].

Acknowledgments

We would like to thank the McKinsey Corporation, Credit Suisse First Boston, The McDonnel Foundation, Bob Maxfield, and Bill Miller for supporting this research, and Janos Kertesz, Moshe Levy, Rosario Mantegna, and Ilija Zovko for valuable conversations.

[1] Gabaix, X., Gopikrishnan, P., Plerou, V. and Stanley, H.E. A theory of power-law distributions in financial market fluctuations, *Nature* **423**, 267-270 (2003).

[2] Plerou, V., Parameswaran, G., Gabaix, X., and Stanley, H.E., Qauntifying stock price response to demand fluctuations, *Phys. Rev. E* **66** 027104 (2002).

[3] Lillo,F. Farmer. J.D. & Mantegna, R.N., Master curve for price-impact Function, *Nature* **421**, 129-130 (2003).

[4] Potters, M. & Bouchaud, J.-P., More statistical properties of order books and price impact, *Physica A* **324**, 133-140 (2003)

[5] Bouchaud, J-P., Gefen, Y., Potters, M., and Wyart, M., Fluctuations and response in financial markets: the subtle nature of 'random' price changes, xxx.lanl.gov/cond-mat/0307332, 2003.

[6] Lillo, F. and Farmer, J.D., The long memory of the efficient market, in preparation.

[7] Beran, J. *Statistics for Long-Memory Processes*, Chapman & Hall (1994).

[8] S.-H. Poon and C.W.J. Granger, Forecasting volatility in financial markets: A review, *J. of Economic Literature* 41, 478-539 (2003).

[9] C. M. C. Lee and M. J. Ready, *Journal of Finance*, **46** 733-746 (1991)

[10] V. Plerou, P. Gopikrishnan, L.A.N. Amaral, M. Meyer and H.E. Stanley, Scaling of the Distribution of Price Fluctuations of Individual Companies, Phys. Rev. E **60** 6519-2529 (1999).

[11] P. Gopikrishnan, V. Plerou, X. Gabaix and H.E. Stanley, Statistical properties of share volume traded in financial markets, Phys. Rev. E **62** R4493-R4496 (2000).

[12] Daniels, M, Farmer, J.D., Gillemot, L., Iori, G., and Smith, D.E. Quantitative model of price diffusion and market friction based on trading as a mechanistic random process, *Physical Review Letters* **90**, 108102 (2003).

[13] Smith, E., Farmer, J.D., Gillemot, L., and Krishnamurthy, S., Statistical theory of the continuous double auction. To appear in *Quantitative Finance*, 2003.

[14] J.D. Farmer, P. Patelli, I.I. Zovko, The predictive power of zero intelligence in financial markets, http://xxx.lanl.gov/cond-mat/0309233.

[15] J.D. Farmer, L. Gillemot, F. Lillo, S. Mike, and

A. Sen, What really causes large price changes?, http://xxx.lanl.gov/cond-mat/0312703 (2003).
[16] P. Weber and B. Rosenow, Large stock price changes: volume or liquidity?, http://xxx.lanl.gov/cond-mat/0401123 (2004).
[17] V. Plerou, P. Gopikrishnan, L.A.N. Amaral, X. Gabaix, and H.E. Stanley, Economic fluctuations and anomalous diffusion, *Phys. Rev. E.* **62** R3023 (2000).