Consistent Variance Curve Models

Hans Buehler*

Quantitative Products Analytics
Global Markets Equity
Deutsche Bank AG London
1 Great Winchester Street
London EC2N 2EQ, United Kingdom
e-mail: hans.buehler@db.com

Institut für Mathematik, MA 7-4
TU Berlin, Strasse des 17. Juni 136
10623 Berlin, Germany
e-mail: buehler@math.tu-berlin.de

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Abstract

We introduce a general approach to model a joint market of stock price and a term structure of variance swaps in an HJM-type framework. In such a model, strongly volatility-dependent contracts can be priced and risk-managed in terms of the observed stock and variance swap prices.

To this end, we introduce equity forward variance term-structure models and derive the respective HJM-type arbitrage conditions. We then discuss finite-dimensional Markovian representations of the fixed time-to-maturity forward variance swap curve and derive consistency results for both the standard case and for variance curves with values in a Hilbert space. For the latter, our representation also ensures non-negativity of the process.

We then give a few examples of such variance curve functionals and discuss briefly completeness and hedging in such models. As a further application, we show that the speed of mean-reversion in some standard stochastic volatility models should be kept constant when the model is recalibrated.

Keywords: Variance Swaps, Options on Variance, Market models, arbitrage free term-structure dynamics, Heath-Jarrow-Morton theory, Consistent parametrizations

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1 Introduction

Standard financial equity market models model exclusively the price process of the stock prices. The prices of liquid derivatives, like plain vanilla calls, are only used to calibrate the parameters of the model. A more natural approach would be to model the evolution of stock price and some liquid instruments simultaneously.

Various approaches, such as those by Brace et al. [BGKW01], Cont et al. [CFD02], Fengler et al. [FHM03], or Haffner [H04], take on the problem by modeling the stochastic evolution of implied

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volatility surfaces (and therefore European call prices) according to certain stylized facts or statistical observations. The price processes obtained from these so-called “stochastic implied volatility models”, however, are not guaranteed to remain martingales. Schönbucher [S99] and Schweizer/Wissel [SW05], on the other hand, focus on the term structure of implied volatility for a single fixed strike, which is too restrictive for the applications we have in mind.

In this article, we consider *variance swaps* as liquid derivatives and derive conditions such that the joint market of stock price and variance swap prices is free of arbitrage. Such models can then be used to price exotic options and allow the computation of hedges with respect to stock and variance swaps in a consistent way. This approach is therefore particularly suitable for options on realized variance and other strongly volatility depended products.

The closest model discussed in the literature to our knowledge is Dupire [D04], who discusses a stochastic volatility model which fits perfectly an initial term-structure of variance swaps. We will extend his approach to our setting in Section 2.2.2.

Variance Swap Markets

For the world's equity stock indices, a fairly liquid market of *variance swaps* has evolved in recent years. Given an index S, such a variance swap exchanges the payment of *realized variance* of the log-returns against a previously agreed strike price. The (zero mean) annualized realized variance for the period $[0,T]$ with business days $0 = t_0 < \ldots < t_n = T$ is usually defined as

$$\frac{c}{n} \sum_{i=1,\ldots,n} \left(\log \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2 ,$$

but contracts may vary. The constant c denotes the number of trading days per year. A standard result (e.g. Protter [P04], p. 66) gives that

$$\langle \log S \rangle_T = \lim_{n\uparrow\infty} \sum_{t_i \in \tau_n} \left(\log \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2$$

where the limit is taken over refining subdivisions $\tau_n = (0 = t_0^n < \cdots < t_n^n = T)$, that is $\lim_{n\uparrow\infty} \sup_{i=1,\ldots,n} |t_i - t_{i-1}| = 0$. We can focus without loss of generality on variance swaps with a zero strike price.

We will assume in this article that the realized variance paid by a variance swap is the realized quadratic variation of the logarithm of the index price, i.e. $\langle \log S\rangle_T$. We also assume that the stock price process is continuous.

Our starting point will be to assume that variance swaps are liquidly traded for all maturities.

The aim of this article is to *first* model the variance swap prices and then find an associated stock price process with compatible dynamics:

Problem (**P1**)
Given today's variance swap prices $V_0(T)$ for all maturities $T \in \mathbb{R}_{\geq 0}$, we want to model the price processes $V(T) = (V_t(T))_{t\geq 0}$ and the stock price S together,[1] *such that the joint market with all variance swaps and the index price itself is free of arbitrage.*

This is a term-structure problem closely related to the situation in Heath-Jarrow-Morton (HJM) interest rate models, where the aim is to construct arbitrage-free price processes of zero bonds (see [HJM92]). We carry this similarity further and introduce the *forward variance curve* $(v(T))_{T\geq 0}$ of the log-returns of S, defined as

$$v_t(T) := \partial_T V_t(T) \qquad t, T \geq 0$$

on some stochastic base $\mathbb{W} := (\Omega, \mathcal{F}_\infty, \mathbb{P}, \mathbb{F})$ which supports an extremal Brownian motion W.[2] The additional complication here is that the forward variance curve is required to remain non-negative and that we have to be able to define an associated stock price process with matching forward variances.

[1] For notational convenience, we set $V_t(T) := V_T(T)$ for all $t > T$.
[2] For details and a precise setup, please refer to Section 2.2.

Remark 1.1 *We want to emphasize an important difference between a forward discount factor and a forward variance swap: while a forward discount factor of zero amounts to the default of the central bank (an exogenous event which is not usually modelled within an interest rate model), a period of zero forward variance is a very natural state for equities. For example, stocks will not trade during holidays or if they are suspended from trading.*

The consequence is that we can not take the logarithm of the forward variance v to remove the difficulty of non-negativity.

Working with the forward variances v, we show an HJM-type result, namely that (under the assumptions of the next section) for each finite T, the process $v(T) = (v_t(T))_{t\geq 0}$ must be a non-negative martingale and therefore has no drift. This will be carried out in Section 2, where we will also introduce the "Musiela-parametrization" u of v in terms of a fixed *time-to-maturity* x,

$$u_t(x) := v_t(x+t) \qquad x, t \geq 0 \ .$$

It is then also shown in Theorem 2.2 that for all Brownian motions B on $\mathbb{W}$ the market of all variance swap price processes $(V(T))_{T\geq 0}$ and the B-"associated price process" S, defined by

$$\left.\begin{aligned} S_t &:= \mathcal{E}_t(X) \\ dX_t &:= \sqrt{u_t(0)}\, dB_t \end{aligned}\right\}$$

is free of arbitrage. In such a case we call the curve v a *variance curve model*, and B has the intuitive meaning of a "correlation structure". We want to emphasize that these no-arbitrage-conditions are very straightforward to enforce, in remarkable contrast to the severe difficulties in this respect with the aforementioned "stochastic implied volatility models".

Remark 1.2 *We want to stress that we do* not *attempt to develop a model in order to price variance swaps – on the contrary, we assume that their market prices are given. Instead, we want to make use of this information to construct a market model of variance; see also Section 2.1.3.*

Finite-Dimensional Realizations

The general framework above has two drawbacks: from a mathematical point of view, it is quite complicated to ensure that a process of real-valued curves such as variance swap curves remains non-negative. From a practical point of view, such a setting is also of limited use since it is difficult to handle for computational purposes. Hence, we are interested in forward curves which are given as a functional of a finite-dimensional Markov-processes: we aim to represent u as

$$u_t(x) = G(Z_t; x) \tag{1}$$

where $G : \mathcal{Z} \times \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ with $\mathcal{Z} \subset \mathbb{R}^m_{\geq 0}$ is a suitable positive function and where Z is an $\mathbb{R}^m$-valued *factor process* which is the strong solution to an SDE

$$dZ_t = \mu(Z_t)dt + \sum_{j=1}^{d} \sigma^j(Z_t) dW_t^j \ , \tag{2}$$

defined in terms of the d-dimensional standard Brownian motion W.

Clearly, the process Z itself depends on the initial value Z_0. Hence, we rather focus on the *factor model* $\mathbb{Z} = (\mu, \sigma)$ for which we assume that (2) has a unique solution for all $Z_0 \in \mathcal{Z}$.[3] We then call the pair $(G, \mathbb{Z})$ consistent if $u_t(x) := G(Z_t; x)$ is a variance curve model for all initial values $Z_0 \in \mathcal{Z}$ (we call it locally consistent if $G(Z_t; T-t)$ is only a local martingale). This leads to the natural question:

[3]Hence, $\mathbb{Z}$ represents the class of all processes which solve (2) for some $Z_0 \in \mathcal{Z}$.

Problem (**P2**)
When are a factor model $\mathbb{Z}$ and a functional G consistent?

This will be addressed in Section 2.3, and we will show in Theorem 2.3 that consistency essentially implies that the heat-equation

$$\partial_x G(z;x) = \mu(z)\,\partial_z G(z;x) + \frac{1}{2}\sigma^2(z)\,\partial_{zz} G(z;x)$$

holds.

These results are closely related to the concept of "finite-dimensional realizations" (FDR) for HJM interest rate models, which has been introduced by Björk/Christensen [BC99] and Björk/Svensson [BS01], and which has been discussed in a more general framework by Filipovic [F01] and Filipovic/Teichmann [FT04].

Following a similar approach, we understand $u = (u_t)_{t\geq 0}$ in Section 3 as a *function*-valued process, whose values lie in a suitable Hilbert-space $\mathcal{H}$. In general, it is very complicated to ensure non-negativity of the functions u_t in such a framework, since the cone of non-negative curves has no interior points. We address this problem by assuming that $G(\mathcal{Z}) \subset \mathcal{H}$ is a sub-manifold with boundary of non-negative curves of $\mathcal{H}$.[4] We then ask:

Problem (**P3**)
Given a process $u = (u_t)_{t\geq 0}$ which starts in $u_0 \in G(\mathcal{Z})$, when does u stay in $G(\mathcal{Z})$?

We will solve this problem locally in Section 3 by following closely ideas from Filipovic/Teichmann [FT04]: writing u as a solution to an $\mathcal{H}$-valued SDE

$$du_t = \partial_x u_t dt + \sum_{j=1}^{d} b_t^j(u_t)\,dW_t^j \ , \tag{3}$$

we show in Theorem 3.1 that u stays locally in $G(\mathcal{Z}) \subset \mathrm{dom}(\partial_x)$ if

$$b^j(u) \in T_u\mathcal{G}$$

for $j = 0, \ldots, d$ and $u \in \mathcal{G} \setminus \partial\mathcal{G}$. The first component b^0 is the Stratonovic-drift of u,

$$b_t^0(u) := \partial_x u_t - \frac{1}{2}\sum_{j=1}^{d} Db^j(u)\ b^j(u)$$

(we also show the relevant conditions on the boundary of $\mathcal{G}$).

Additionally, we prove that if u stays locally in $\mathcal{G}$ and if G is invertible, then it has a finite dimensional representation

$$u_t = G(Z_t)$$

in terms of a (locally) consistent factor model $\mathbb{Z}$ which is explicitly given in terms of b and G.

1.1 Examples and further Applications

In the second part of the article, we will discuss examples (Section 4): we will particularly focus on exponential-polynomial functionals G, a very well-known example of which is the forward variance curve

$$G(z;x) = z_2 + (z_1 - z_2)e^{-z_3 x} \qquad (z_1, z_2, z_3) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}^2 \ .$$

Similar to Filipovic [F01] for interest rates, we find that in general the coefficients in the exponentials (ie, z_3 above) must be constant if the curve is to produce arbitrage-free variance swap prices.

[4] We now understand $u_t(\cdot)$ and $G(Z_t;\cdot)$ as *functions* and therefore omit the x-argument.

We then apply these results to the standard market practise of recalibration of various models. Taking Heston's popular model [H93] as an example, we show that mean-reversion must theoretically be kept constant during the life of an exotic to ensure that its price process is a local martingale *in the real world of the institution*[5] where it is the result of frequent recalibration.

Section 4.4 is finally devoted to the question of market completeness. We explain how variance curve models can be used to obtain hedges of payoffs in terms of stock price and traded variance swaps, if a simple invertibility criterion on G is satisfied.

2 Variance Curve Models

We will assume a given probability space $\mathbb{W} = (\Omega, \mathcal{F}_\infty, \mathbb{P}, \mathbb{F})$ with an d-dimensional extremal Brownian motion $W = (W^j)_{j=1,\dots,d}$ on which we want to model a market of variance swaps and an index stock price process, which pays no dividends. We also assume that the prevailing interest rates are zero and set the initial stock price level to $S_0 = 1$.[6]

Our notations will mostly follow Revuz/Yor [RY99]: We denote by H^2_T the space of all square-integrable martingales up to T, and by L^2_T the space of all predicable square-integrable processes. The symbols H^{loc}_T and L^{loc}_T denote the respective local spaces, and we may drop the T in case the respective object is element of the relevant sets for all $T < \infty$.

We denote by $\mathcal{B}(U)$ the Borel-σ-algebra of a topological space U and by $\mathcal{P}$ the predictable σ-algebra on $\Omega \times \mathbb{R}_{\geq 0}$. Given a process $Z = (Z_t)_{t\geq 0}$, we define its image measure by $\mathbb{P}^Z[A] := \mathbb{P}[Z \in A]$ where $A \in \mathcal{B}[C[0,\infty)]$. The Lebesgue measure on $\mathbb{R}^n$ is denoted by λ^n and we set $\lambda := \lambda^1$. We also denote by $\mathcal{E}(X)$ the Doleans-Dade exponential of a process X.

We will use the symbols $x \vee y := \max(x,y)$, $x \wedge y := \min(x,y)$ and $x^+ := x \vee 0$. The set of positive numbers $x > 0$ is denoted by $\mathbb{R}_{>0}$, while $\mathbb{R}_{\geq 0}$ is the set of non-negative numbers.

2.1 Review of the Standard Case

For illustration, we assume in this subsection that we are *given* a continuous stock price process as a positive local martingale S on $\mathbb{W}$, such that its variance swap prices are finite. This is the reverse case to the rest of the article, where the variance curves will be given and S will be constructed accordingly.

This subsection is intended to build some understanding of the required properties of a variance swap model, but from a logical point of view it can be omitted and the reader may immediately proceed to Section 2.2.

Proposition 2.1 *If S is a positive local martingale, we can write it as*

$$S_t = \mathcal{E}_t(X) \tag{4}$$

where

$$X_t = \int_0^t \sqrt{\zeta_s}\, dB_s \tag{5}$$

for some $\sqrt{\zeta} \in L^{\mathrm{loc}}$ and a Brownian motion B. We also have that $X \in \mathcal{H}^2$.

Proof – Since S is a positive local martingale, we can write $S_t = \mathcal{E}_t(X)$ for some local martingale X (cf. [RY99] p. 328, prop. 1.6). Hence, there exist $z \in L^{\mathrm{loc}}(W)$ such that

$$X_t = \sum_{j=1}^d \int_0^t z^j_s \, dW^j_s \ .$$

[5] See Section 4.3 for details and terminology.

[6] These conditions are essentially equivalent to the assumption of deterministic interest rates and dividends.

Define $\zeta_t := \sum_{j=1}^d (z_t^j)^2$. Then, $\zeta_t^{-1/2} 1_{\zeta_t>0} \in L^2(X)$ since $\mathbb{E}[\int_0^t \zeta_s^{-1} 1_{\zeta_s>0}\, d\langle X\rangle_s] = \mathbb{E}[\int_0^t 1_{\zeta_s>0}\, ds] \le t < \infty$. Therefore, we can define

$$B_t := \int_0^t \frac{1}{\sqrt{\zeta_s}} 1_{\zeta_s>0}\, dX_t + \int_0^t 1_{\zeta_s=0}\, dW_s^1\ .$$

We have $\langle B\rangle_t = \int_0^t \left(\zeta_s^{-2}\zeta_s^2 1_{\zeta_s>0} + 1_{\zeta_s=0}\right) ds = t$, and since B is clearly adapted and continuous, it is a Brownian motion with the required property (5).

By assumption that the variance swap prices are finite, we also have that $\mathbb{E}[\int_0^T \zeta_s\, ds] < \infty$, ie $X \in \mathcal{H}_T^2$ for all finite T. □

Hence, any positive continuous stock price process can be written as a "stochastic volatility model" (such as the examples in Section 4)

$$\frac{dS_t}{S_t} = \sqrt{\zeta_t}\, dB_t\ .$$

Note, however, that the joint process (S,ζ) will generally not be Markov.

2.1.1 Variance Swaps

Since variance swaps are assumed to be tradable at any time t, the time-t price $V_t(T)$ is given as the expectation of the quadratic variation of $\log S$ under a pricing measure $\mathbb{P}$:

$$V_t(T) = \mathbb{E}_{\mathbb{P}}\left[\left\langle \log S\right\rangle_T \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}_{\mathbb{P}}\left[\int_0^T \zeta_s\, ds \,\middle|\, \mathcal{F}_t\right]\ . \tag{6}$$

It is clear from standard arbitrage-theory that the measure $\mathbb{P}$ is in general not unique, i.e. the price processes $V(T)$ of the variance swaps with maturities $T \ge 0$ are in general not determined by specifying the stock price process (S,ζ) alone.

It is therefore necessary to fix a pricing measure, which we will assume for the remainder of this subsection is $\mathbb{P}$. We want to stress that by constructing directly the variance swap prices (which is the subject of this article), the prices of variance swaps are given by the market and any pricing measure must have property (6).

Given that $V(T)$ is a martingale and due to the extremality of W, we find some $\gamma(T) \in L^{\mathrm{loc}}$ such that

$$V_t(T) = V_0(T) + \sum_{j=1}^d \int_0^t \gamma_s^j(T)\, dW_s^j\ .$$

2.1.2 Forward Variance

We also see that at any time t, the curve $V_t(\cdot)$ is absolutely continuous with respect to the Lebesgue measure λ, hence we can define the derivative along T,

$$v_t(T) := \partial_T V_t(T) = \mathbb{E}\left[\zeta_T \,\middle|\, \mathcal{F}_t\right] \qquad T,t \ge 0\ . \tag{7}$$

which is called the fixed maturity *T-forward variance* seen at time t ($v_t(T)$ is well defined for $T > t$). Note the conceptual similarity with the forward rate in interest rate modelling.

PROPOSITION 2.2 (HJM-Condition for Forward Variance) *For all $T \ge 0$, the process $v(T) = (v_t(T))_{t\ge0}$ defined by (7) is martingale and can be written as*

$$v_t(T) = v_0(T) + \int_0^t \beta_s(T)\, dW_s \tag{8}$$

with $\beta(T) := \partial_T\gamma(T) \in L^{\mathrm{loc}}$.[7]

[7] The fact that $\beta(T) = \partial_T\gamma(T)$ follows from the uniqueness of the representation of a local martingale on $\mathbb{F}$ with respect to W.

2.1.3 Pricing Variance Swaps using European Options

Neuberger [N92] has shown that the price of a variance swap in the present framework can be computed as

$$V_0(T) = -2\int_0^1 \frac{1}{K^2}\mathrm{Put(K,T)}\,dK - 2\int_1^\infty \frac{1}{K^2}\mathrm{Call(K,T)}\,dK \tag{9}$$

where Put(K,T) and Call(K,T) denote quoted market option prices, which need to be available for this approach (on this subtle point, see Buehler [B04] and the discussion of no-arbitrage conditions on Föllmer/Schied [FS04]). A good reference on variance swaps in practise is Demeterfi et al. [DDKZ99].

Here, we *do not* attempt to develop a pricing scheme for variance swaps (or forward variance swaps, for that matter). Rather, we can use the market prices (quoted or computed by the above formula) as *input* to develop a variance swap model which in turn can be used to price and hedge more exotic options such as, for example (but of course not exclusively), options on realized variance: e.g. a *call on realized variance* has payoff

$$\left(\int_0^T \zeta_s\,ds - K\right)^+ \tag{10}$$

and a *volatility swap* pays out

$$\sqrt{\int_0^T \zeta_s\,ds}\ .$$

It is natural to hedge such payoffs with the "forward" on the underlying quantity $\langle \log S\rangle_T$, i.e. with variance swaps. Note that this is even more convincing if we want to price options which pay out variance swaps, such as a call on a (forward started) variance swap.

2.2 Definitions and Basic Properties

We now introduce our variance curve models: we want to *specify* a family $v = (v(T))_{T\geq 0}$ of forward variance price processes $v(T) = (v_t(T))_{t\geq 0}$ under the martingale measure $\mathbb{P}$, which will describe the values of the expected future instantaneous variances, just as in the previous section: at each observation time t, the value $v_t(T)$ for $T \geq t$ will represent the expectation of the stock's instantaneous variance at time T conditional on $\mathcal{F}_t$. For past values $T < t$, we assume $v_t(T) := v_T(t)$, i.e. it is simply the observed variance.

Such a family v should allow us to construct a stock process such that the joint market of stock and variance swaps is arbitrage-free.

Definition 2.1 (Variance Curve Model) *We call a family $v = (v(T))_{T\geq 0}$ of processes $v(T) = (v_t(T))_{0\leq t\leq \tau}$ defined up to a strictly positive stopping time τ a* local variance curve model *on $\mathbb{W}$ if:*

(a) For all $T < \infty$, $v(T) = (v_t(T))_{0\leq t\leq\tau}$ is a non-negative *martingale with representation*

$$dv_t(T) = \sum_{j=1}^d \beta_t^j(T)\,dW_t^j \tag{11}$$

for some $\beta(T) \in L^{\mathrm{loc}}$ (this is the "HJM-condition" for variance curves).

(b) For all $T < \infty$, the initial variance swap prices are finite,

$$V_0(T) := \int_0^T v_0(s)\,ds < \infty\ . \tag{12}$$

(c) The process $v_t(t)$ is predictable (this is the case, for example, if $v_t(\cdot)$ is left-continuous).

We call v a global *variance curve model if $\tau = \infty$ can be chosen (in this case, the forward variance processes $v(T) = (v_t(T))_{0\leq t\leq T}$ are true martingales).*

By Proposition 2.2 it is clear that forward variance must be a local martingale, and finiteness of the variance swap prices is a very natural assumption, if we want to use them as liquid instruments.

NOTATION 1 *For notational convenience, we define* $v_t(T) := v_T(T)$ *for* $T < t \le \tau$*, hence the process* $v(T) = (v_t(T))_{0 \le t \le \tau}$ *is well-defined.*

One convenient consequence of this convention is that this allows us to write the family of *variance swap price processes* $V = (V(T))_{T \ge 0}$ as

$$V_t(T) := \int_0^T v_t(s)\, ds \tag{13}$$

whenever $t \le \tau$. This is insofar an important observation as in the variance swap framework here, the past information of the running realized variance,

$$V_t(t) = \int_0^t v_s(s)\, ds = \int_0^t \zeta_s\, ds$$

is still required to recover the price of a variance swap with maturity T. In interest rate theory, in contrast, the price of a zero bond at any time t depends solely on the future forward rates, and not on the value of the cash bond.

PROPOSITION 2.3 *If* v *is a global variance curve model, then the process* $V(T) = (V_t(T))_{t \ge 0}$ *is a martingale for each* $T < \infty$ *with dynamics*

$$dV_t(T) = \sum_{j=1}^d \gamma_t^j(T)\, dW_t^j$$

where $\gamma_t^j(T) = \int_0^T \beta_t^j(s)\, ds$.

Proof – It is clear that $V(T)$ defined by (13) is adapted and therefore is a martingale with $V_t(T) = \mathbb{E}\left[\, V_T(T) \,\middle|\, \mathcal{F}_t \,\right]$ and $V_0(T)$ given by (12). The representation of V via γ follows from (11) and the uniqueness of the representation of $V(T)$ with respect to W. □

Given a variance curve model, we call the positive process

$$\zeta_t := v_t(t) \tag{14}$$

the *short variance* of v. It is well defined by the requirements of Definition 2.1. Since also

$$\mathbb{E}\left[\int_0^T \zeta_s\, ds \right] = \int_0^T \mathbb{E}\left[\, v_s(s) \,\right] ds = V_0(T) < \infty \ ,$$

it follows that process $\sqrt{\zeta}$ is in L_T^2 for all $T < \infty$. This justifies the following definition:

DEFINITION 2.2 (Associated Stock Price Process) *For any variance curve model* v *and an arbitrary real-valued Brownian motion* B *on* $\mathbb{W}$*, the* B-associated stock price process *is defined as the local martingale*

$$S_t := \mathcal{E}_t(X) \qquad \textit{with} \qquad X_t := \int_0^t \sqrt{\zeta_s}\, dB_s \ . \tag{15}$$

The process X *is in* H_T^2 *for all* $T < \infty$ *and a variance swap on* S *has the price*

$$\mathbb{E}\left[\, \langle \log S \rangle_T \,\middle|\, \mathcal{F}_t \,\right] = \mathbb{E}\left[\, \langle X \rangle_T \,\middle|\, \mathcal{F}_t \,\right] = V_t(T) \ ,$$

where V *was defined in (13).*

It follows then directly by construction

Theorem 2.1 (Variance Swap Market Model) *Let v be a variance curve model, B a Brownian motion and S its associated stock price proces.*

Then, the joint market (S, V) is free of arbitrage.

We see a very convenient property of the current model approach: Once the variance curve model is fully specified by v_0 and β, an associated stock price process can easily be constructed to yield a full variance swap market model which is free of arbitrage.

Remark 2.1 (Interpretation of B) *Note that each B defined on the stochastic base $\mathbb{W}$ can be written as*

$$dB_t = \sum_{j=1}^{d} \rho_t^j \, dW_t^j$$

in terms of some predictable "correlation vector" $\rho \in [-1, +1]^d$ with $j = 1, \ldots, d$ and $||\rho||_2 = 1$.

Since the Brownian motion B (or ρ, for that matter) defines the "correlation structure" of S with its variance process, it has the intuitive meaning of a "skew parameter".

Note, however, while B can be chosen arbitrarily to yield a local martingale S, more care must be taken if S is required to be a true martingale.

2.2.1 Musiela-Parametrization

In the sprit of Musiela's parametrization [M93] of forward rates, we now introduce the respective process for variance curve models:

Definition 2.3 *We call*

$$u_t(x) := v_t(t + x)$$

the fixed time-to-maturity forward variance*, and $U_t(x) := \int_0^x u_t(s)\, ds$ the* fixed time-to-maturity variance swap.

Note that above definition is valid for each fixed t and almost all ω. To define a proper process u, we have to impose some additional regularity on v.

Proposition 2.4 *Let v be a local variance curve model. Assume that v_0 is differentiable in T, that β in (11) is $\mathcal{B}[\mathbb{R}^d] \times \mathcal{P}$-measurable and almost surely differentiable in T with*

$$\sqrt{\int_0^U \partial_T \beta_t^j(T)^2 \, dT} \in L_U^{\mathrm{loc}} \qquad \text{for all } j = 1, \ldots, d \text{ and all } U < \infty. . \tag{16}$$

Then, $\partial_T v_t(T)$ is well defined up to τ,[8] *and the fixed time-to-maturity forward variance process $u(x) = (u_t(x))_{0 \le t \le \tau}$ can be written as*

$$u_t(x) = u_0(x) + \int_0^t \partial_x u_s(x)\, ds + \sum_{j=1}^{d} \int_0^t b_s^j(x)\, dW_s^j \tag{17}$$

where $b_t^j(x) := \beta_t^j(t + x)$.

Proof – With the assumptions above, we have

$$\begin{aligned} u_t(x) &= v_t(t + x) \\ &= v_0(x) + \int_0^t \partial_T v_0(s + x)\, ds \end{aligned}$$

[8] We have $\partial_T v_t(T) = \partial_T v_0(T) + \sum_{j=1}^{d} \int_0^t \partial_T \beta_s^j(T)\, dW_s^j$.

$$
\begin{aligned}
&+\int_0^t \left\{\beta_r(r+x)+\int_r^t \partial_T\beta_r(s+x)\,ds\right\}\,dW_r \\
=\;& v_0(x)+\int_0^t \partial_T v_s(s+x)\,ds+\int_0^t \beta_r(r+x)\,dW_r \\
=\;& u_0(x)+\int_0^t \partial_T u_s(x)\,ds+\int_0^t b_r(x)\,dW_r\ ,
\end{aligned}
$$

as claimed. Note that (16) basically ensures that $\int_0^s \partial_T\beta_r(T)\,dW_r$ is a local martingale (see, for example, Protter [P04] p. 208). □

The reverse of the previous proposition constitutes the HJM-condition for the fixed time-to-maturity case: assume we start with a family u, when defines $v_t(T) := u_t(T-t)$ a (local) variance curve model?

Theorem 2.2 (HJM-condition for Variance Curve Models) *Let $u = (u(x))_{x\geq 0}$ be a family of non-negative adapted processes $u(x) = (u_t(x))_{0\leq t\leq\tau}$ where τ is a strictly positive stopping time. Assume that*

(a) The curve $v_t(\cdot)$ is almost surely in C^1.

(b) The process $u(x) = (u_t(x))_{0\leq t\leq\tau}$ has a representation

$$
du_t(x) = \partial_x u_t(x)\,dt + \sum_{j=1}^d b_t^j(x)\,dW_t^j\ . \tag{18}
$$

(c) The prices of variance swaps $U_0(x) := \int_0^x u_0(s)\,ds$ are finite for all $x<\infty$.

(d) The volatility coefficient b in (18) is C^1 in x and satisfies $\sqrt{\int_0^U \partial_x b_t^j(x)^2\,dx} \in L_U^{\mathrm{loc}}$ for all $U<\infty$.

Then, the family $v = (v(T))_{T\geq 0}$ given by

$$
v_t(T) := \begin{cases} u_t(T-t) & t\leq T\wedge\tau \\ u_T(0) & T<t\leq\tau \end{cases} \tag{19}
$$

defines a local variance curve model.

If, moreover, $\tau=\infty$, then v is a global variance curve model and for each T, and the processes $v(T) = (v_t(T))_{t\geq 0}$ are true martingales.

Proof – We have to satisfy the conditions of Definition 2.1. The finiteness of variance swap prices is satisfied by (b). Now assume v is defined by (19). As before,

$$
dv_t(T) = du_t(T-t) = \sum_{j=1}^d b_t^j(T-t)\,dW_t^j\ , \tag{20}
$$

i.e. $v(T)$ is a local martingale. The process $\zeta_t := u_t(0)$ is by construction well defined. □

This theorem allows us to specify u instead of v. We will therefore also refer to u as a "variance curve model" if it satisfies the conditions of Theorem 2.2.

Conclusion 2.1 *Theorems 2.1 and 2.2 answer* (**P1**) *from the introduction.*

Remark 2.2 *Despite the introduction of forward rates in terms of fixed-time-to-maturity by Musiela, it is more common in interest-rate theory to deal with fixed maturity objects because the maturities of underlying market instruments are typically fixed points in time (such as LIBOR rates and Swaps).*[9]

A variance curve, in contrast, is more naturally seen as a fixed time-to-maturity object, in particular given that the short end of the curve is the instantaneous variance of the log-price of the stock as seen in Definition 2.2.[10]

[9] In a typical LIBOR rate model, the short rate is not modelled.

[10] For example, an option on realized variance (such as a call (10) for example) is not an option on a variance swap.

2.2.2 Fitting the initial term structure

An advantage of the HJM-approach for interest-rates is that the current forward interest rate f_t (as a function of time-to-maturity x) is given as

$$f_t(x) = f_0(x) + \int_0^t (\partial_x f_s(x) + a_s(x))\, ds + \sum_{j=1}^d \int_0^t b_s^j(x)\, dW_s^j$$

with HJM-drift $a_t(x) := \sum_{j=1}^d b_t^j(x) \int_0^x b_t^j(y)\, dy$. This means that the initial curve f_0 can be estimated from market quotes without imposing additional constraints on the volatility structure b.

In contrast, our specification of u must remain non-negative, which renders specification of the volatility structure dependent on u_0.

In the main part of this article we will deal with finite-dimensional realizations of u, where this is not a concern (because we will write u in terms of a non-negative functional). However, if we were to work directly with u, we might consider parameterizing it as

$$u_t(x) = e^{y_t(x)} \ . \tag{21}$$

In this case, condition (17) translates into the condition that y can be written as

$$dy_t(x) = \left(\partial_x y_t(x) - \frac{1}{2} \sum_{j=1}^d \gamma_t^j(x)^2 \right) dt + \sum_{j=1}^d \gamma_t^j(x)\, dW_t^j \tag{22}$$

for $\gamma \in L^{\mathrm{loc}}$.

Remark 2.3 *In [D04], Dupire uses a model of the type (21) for the case of a one-dimensional Brownian motion driving the forward variance where the function γ is constant, i.e. $u(x)$ is a log-normal process. His article also contains details on hedging in such a framework.*

If we want to allow arbitrary initial curves and be able to choose the volatility structure independently from the chosen initial curve, this approach can be employed. Unfortunately, it does not allow u_t to be zero, hence models such as Heston's are not covered in this setting. It also forbids forward variances which are zero due to holidays or suspended trading.

However, we can extend this idea to fit an arbitrary variance curve model to observed market prices: assume that we are given an observed forward variance curve $w_0 \in C^1[\mathbb{R}_{\geq 0}]$ and a (global) variance curve model $\tilde{u} = (\tilde{u}(x))_{x \geq 0}$ in Musiela-parametrization with the property that if $\tilde{u}_0(x)$ is zero, then $w_0(x)$ is also zero.

Let

$$u_t(x) := \frac{w_0(t+x)}{\tilde{u}_0(t+x)} 1_{\tilde{u}_0(t+x)>0}\, \tilde{u}_t(x) \qquad t, x \geq 0 \ . \tag{23}$$

Proposition 2.5 *The family $u = (u(x))_{x \geq 0}$ is a global variance curve model and fits the initially observed forward variances, i.e.*

$$u_0 = w_0 \ .$$

Hence, we can turn any variance curve model (in particular the finite-dimensionally driven models of the next section) into a "fitting" model by simply rescaling the variance curve according to (23). A slightly different approach for linearly mean-reverting models is presented in Example 2 below.

2.3 Consistent Variance Curve Functionals

In the previous section, we have discussed variance curve models which were given in terms of general integrable processes. These have the aforementioned drawbacks: on one hand, it is very difficult to check whether a general model of the form (18) actually stays non-negative (this is particularly difficult for diffusions with values in Hilbert spaces, cf. equation (26) on page 13).

On the other hand, it is not clear how such models can be used in practise. Indeed, consider the situation in the reality of a trading floor: we do not actually see an infinite number of variance swap prices $(V_0(T))_{T\geq 0}$ in the market. Rather, a discrete set of swap prices will be interpolated by some functional which is parameterized by a finite-dimensional parameter vector.

Hence, we want to focus on variance curves which are given in terms of such finite-dimensionally parameterized *variance curve functionals.*

Definition 2.4 (Variance Curve Functional) *A* Variance Curve Functional *is a non-negative $C^{2,2}$-function $G : (z; x) \in \mathcal{Z} \times \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ such that $\int_0^T G(z; x)\, dx < \infty$ for all (z, T).*

The open subset $\mathcal{Z} \subset \mathbb{R}^m_{\geq 0}$ is called the parameter space *of G.*

Given a functional G and $Z_0 \in \mathcal{Z}$, we now have to find a factor process $Z = (Z_t)_{t\in\geq 0}$ such that

$$u_t(x) := G(Z_t; x) \ , \quad x \geq 0 \ ,$$

forms a variance curve model. To avoid arbitrage, we need to meet the conditions of Theorem 2.2.

We want to focus on diffusions Z which are strong solutions of an SDE

$$dZ_t = \mu(Z_t)\, dt + \sum_{j=1}^{d} \sigma^j(Z_t)\, dW_t^j \tag{24}$$

with locally Lipschitz coefficients $\mu : \mathcal{Z} \mapsto \mathbb{R}^m$ and $\sigma^j : \mathcal{Z} \mapsto \mathbb{R}^m$ for $j = 1, \ldots, d$ up to a strictly positive stopping time τ (which depends on Z_0). The set of coefficients (μ, σ) which admit a unique strong solution for all $Z_0 \in \mathcal{Z}$ will be denoted by Ξ. We call its elements *factor models.*

Note that (24) allows to define, say, the nth coordinate as “time”, i.e. $Z_t^n = t$: simply set $\mu_n(z) := 1$ and $\sigma_n^j(z) := 0$ for $j = 1, \ldots, d$. This way, a deterministic dependency of the coefficients μ and σ on time can be incorporated in the above formulation (see also Example 2 below).

Definition 2.5 (Consistent Factor Model) *A* locally consistent *factor model $\mathbb{Z} := (\mu, \sigma)$ for $(G, \mathcal{Z})$ is a factor model $(\mu, \sigma) \in \Xi$ such that for all $Z_0 \in \mathcal{Z}$, the solution $Z = (Z_t)_{0\leq t\leq\tau}$ to (24) stays in $\mathcal{Z}$ (i.e. $Z_t \in \mathcal{Z}$ for $t \leq \tau$) and*

$$u_t(x) := G(Z_t; x) \ , \quad x \geq 0 \ ,$$

is a local variance curve model. Each solution Z is called a locally consistent factor process*.*

We also call the pair $(G, \mathbb{Z})$ locally consistent*.*

Global *consistency is defined similarly.*

Naturally, we now have to ask whether a given curve functional G is consistent at all and if so, whether we can specify a particular consistent factor model. We can now prove the following theorem, which is closely related to proposition 3.1.1 in Filipovic [F01]. It paves the way to answer problem (**P2**):

Theorem 2.3 (HJM-condition for Variance Curve Functionals) *A factor model $\mathbb{Z} \in \Xi$ of processes is locally consistent with G if and only if for all $Z_0 \in \mathcal{Z}$, equation (24) has a solution Z which stays locally in $\mathcal{Z}$, and*

$$\partial_x G(z; x) = \mu(z)\, \partial_z G(z; x) + \frac{1}{2}\sigma^2(z)\, \partial_{zz} G(z; x) \tag{25}$$

holds on $\mathcal{Z} \times \mathbb{R}_{\geq 0}$.

Moreover, $\mathbb{Z}$ is globally consistent iff $\tau = \infty$ and $(G(Z_t; T - t))_{0\leq t\leq T}$ is a true martingale

The above equation (25) means that

$$\begin{aligned} \partial_x G(Z_t; x) \quad &= \quad \sum_{i=1}^{m} \mu^i(Z_t)\, \partial_{z^i} G(Z_t; x) \\ &+ \quad \frac{1}{2} \sum_{i,k=1}^{m} \left(\sum_{j=1}^{d} (\sigma^j)^i(Z_t)(\sigma^j)^k(Z_t) \right) \partial_{z^i z^k} G(Z_t; x) \end{aligned}$$

almost surely. We wrote $(\sigma^j)^i$ to denote the ith component of the m-dimensional vector σ^j.

Theorem 2.3 gives us the required condition for problem (**P2**) when a pair (G, Z) is consistent.

Remark 2.4 *Since equation (25) is the heat equation, consistency can be seen as equivalent to a solution of the heat-equation. For a given $\mathbb{Z} \in \Xi$ and a suitably integrable function $G(\cdot;0)$, a consistent functional G is therefore defined by*

$$G(z;x) := \mathbb{E}\left[\, G(Z_x;0) \,\middle|\, Z_0 = z \,\right] \ .$$

Here, we look at the heat equation from a different angle and ask: given an assumed parametrization G, can we find a factor model $\mathbb{Z} \in \Xi$ which is consistent with G, and therefore solves (25) ?

3 Variance Curve Models in Hilbert Spaces

We will now focus on problem (**P3**): Given u now as a solution of a general SDE of the form (18), and a curve functional G such that $G(\mathcal{Z})$ is a sub-manifold of a Hilbert-space $\mathcal{H}$, under which conditions on the coefficients of u can we find a consistent factor model $\mathbb{Z}$ such that

$$u_t = G(Z_t)?$$

(We now drop the x-argument since we understand u_t and $G(Z_t)$ in this section as elements of a function space.) Note that such a representation is also an efficient way to ensure non-negativity of the process u.

To be able to approach this question, we have to impose some regularity on the possible curves of u. Indeed, we will employ the theory of stochastic differential equations in Hilbert spaces, the standard reference on which is da Prato/Zabcyk [PZ92]; also see Teichmann [T05]. We will closely follow Björk/Svensson [BS01], Filipovic/Teichmann [FT04] and Teichmann [T05].

We remain on the space $\mathbb{W} = (\Omega, \mathcal{F}_\infty, \mathbb{P}, \mathbb{F})$ which supports an extremal d-dimensional Brownian motion W. Additionally we assume that we are also given a Hilbert-space $\mathcal{H}$, which will contain our forward variance curves.[11]

In $\mathcal{H}$, we assume u is given as a solution to a stochastic differential equation of the type

$$du_t = \partial_x u_t \, dt + \sum_{j=1}^d b^j(u_t)\, dW_t^j \tag{26}$$

with locally Lipschitz vector fields $\beta^1, \dots, \beta^d : U \subset \mathcal{H} \to \mathcal{H}$ where U is an open set. A (mild) solution[12] of such an equation typically only exists up to a strictly stopping explosion time τ, hence we focus on questions of local consistency. The operator $\partial_x : \mathrm{dom}(\partial_x) \subset \mathcal{H} \to \mathcal{H}$ is the generator of the strongly continuous semigroup $(T_t)_{t\geq 0}$ of shift operators $(T_t u)(x) := u(x+t)$; see [PZ92] for details.

Assumption 1 *The set $\mathcal{G} := G(\mathcal{Z}) \subset \mathcal{H}$ is a sub-manifold with boundary*[13] *(which is denoted by $\partial\mathcal{G}$).*

Definition 3.1 (Locally Consistency and FDR) *We say $u = (u_t)_{t\geq 0}$ is* locally consistent *with G if there exist a locally consistent $\mathbb{Z} \in \Xi$ for G (with stopping time η) such that*

$$u_t = G(Z_t)$$

for all $t \leq \tau \wedge \eta$. We call the pair $(G, \mathbb{Z})$ a finite dimensional representation *or* FDR *of the variance curve u.*

[11] For examples of suitable Hilbert-spaces, see Filipovic [F01].
[12] For concepts of solutions for equations in Hilbert-spaces, see da Prato/Zabcyk [PZ92] or Teichmann [T05].
[13] It is finite-dimensional by construction.

Let us define the Stratonovic drift of u,

$$b^0(u) := \partial_x u - \frac{1}{2}\sum_{j=1}^{d}(Db^j)(u)\ b^j(u)$$

where $(Db^j)(u)$ denotes the Frechet-derivative of b^j along u. Note that b^0 is only defined for $u \in \mathrm{dom}(\partial_x)$.

THEOREM 3.1 *The process $u = (u_t)_{t\geq 0}$ with $u_0 \in \mathcal{G}$ is locally consistent with G if and only if*

(a) $\mathcal{G} \subset dom(\partial_x)$.

(b) For all $u \in \mathcal{G} \setminus \partial\mathcal{G}$ and for $j = 0, \ldots, d$,

$$b^j(u) \in T_u\mathcal{G}\ . \tag{27}$$

(the tangent space $T_u\mathcal{G}$ in $u = G(z)$ is given by $\mathrm{Img}\ \partial_z G(z)$*).*

(c) For all $u \in \partial\mathcal{G}$,

$$b^0(u) \in (T_u\mathcal{G})_{\geq 0} \quad \textit{and} \quad b^j(u) \in T_u\partial\mathcal{G} \tag{28}$$

for $j = 1, \ldots, d$.[14]

For a proof, see Filipovic/Teichmann [FT04] or theorem 13 in [T05]. We also obtain:

COROLLARY 3.1 *If u is locally consistent with G and if G is invertible, then the factor model $\mathbb{Z} = (\mu, \sigma)$ specified by*

$$\sigma^j(z) = (\partial_z G)(z)^{-1}\ b^j(G(z))$$

for $j = 1, \ldots, d$ and

$$\mu(z) := (\partial_z G)(z)^{-1}\ b^0(G(z)) + \sum_{j=1}^{d}(\partial_z\sigma^j)(z)\,\sigma^j(z)\ .$$

is in Ξ and locally consistent with G.

CONCLUSION 3.1 (Local solution to problem (**P3**)) *At least locally, Theorem 3.1 solves problem (**P3**): A variance curve model admits an FDR (G, Z) if and only if (27) and (27) are satisfied. If G is invertible, the parameter process Z is given in Corollary 3.1.*

4 Applications

Having solved the abstract problems (**P1**) - (**P3**), let us finally present some examples. We mainly focus on exponential-polynomial variance curves, but also discuss a few other approaches. We link these results in Section 4.3 to the practice of daily recalibration of stochastic volatility models.

4.1 Exponential-Polynomial Variance Curve Models

DEFINITION 4.1 *The family of* Exponential-Polynomial Curve Funtional *is parameterized by a vector $z = (z_1, \ldots, z_n; z_{n+1}, \ldots, z_m) \in \mathbb{R}_{>0}{}^n \times \mathbb{R}^{m-n}$ and given as*

$$G(z; x) = \sum_{i=1}^{n} p_i(z; x)e^{-z_i x} \tag{29}$$

where p_i are polynomials of the form $p_i(z; x) = \sum_{j=0}^{N} a_{ij}(z)x^j$ with coefficents a_{ij} such that $p_i \geq 0$ λ^m-as. W.l.g. we can assume that $\deg(p^i) > \deg(p^{i+1})$.[15]

[14] We used $(T_u\mathcal{G})_{\geq 0}$ to denote the inward pointing tangent-space of $\mathcal{G}$ in the boundary point u.

[15] We denote by $\deg(p)$ the degree of a polynomial p.

(Also compare Bjoerk et al. [BS01] and Filipovic [F01].)

We assume that any factor process has $Z_t^i \neq Z_t^j$ for $i \neq j$, since otherwise we can just rewrite (29) accordingly. Also note that $\int_0^T G(z;x)\,dx < \infty$ for all $T < \infty$.

Lemma 4.1 *Let Z be a factor process consistent with an exponential-polynomial variance curve functional. Then, the first n coordinates $Z^1,\dots,Z^n$ are constant.*

Proof – We have

$$\begin{aligned}
\partial_x G &= -z_i \sum_{i=1}^n p_i(z;x)e^{-z_i x} + \sum_{i=1}^n \partial_x p_i(z;x)e^{-z_i x} \\
\partial_{z_j} G &= -p_j(z;x)xe^{-z_j x} 1_{j\le n} + \sum_{i=1}^n \partial_{z_j} p_i(z;x)e^{-z_i x} \\
\partial^2_{z_j z_j} G &= \left(p_j(z;x)x^2 e^{-z_j x} - 2\partial_{z_j} p_j(z;x)xe^{-z_j x}\right) 1_{j\le n} + \sum_{i=1}^n \partial^2_{z_j z_j} p_i(z;x)e^{-z_i x}
\end{aligned} \tag{30}$$

We ignore the mixed terms $\partial^2_{z_j z_k} G$, since we can already see that $\partial^2_{z_j z_j} G$ with $j \le n$ are the only terms in (25) which involve polynomials of degree $\deg(p_i)+2$ as factors in front of the exponentials $e^{-z_i x}$. Since we choose the z_i distinct, and because neither μ nor σ depends on x, this implies that $\sigma_i^2 = 0$ for $i \le n$. In other words, the states z_i for $i \le n$ cannot be random.

Next, we use (30) and find with the same reasoning (now applied to the polynomials of degree $\deg(p_i)+1$) that $\mu_i = 0$ for $i \le n$, so Z^i must be a constant. □

We will now present two particular exponential-polynomial curve functionals. In the light of Lemma 4.1, we will keep the exponentials constant but investigate the possible dynamics of the remaining parameters.

Example 1 (Linearly Mean-Reverting Variance Curve Models) *The Functional*

$$G(z;x) := z_2 + (z_1 - z_2)e^{-\kappa x} \ .$$

with $z \in \mathcal{Z} := \mathbb{R}_{\ge 0} \times \mathbb{R}_{>0}$ is consistent with a factor process Z if $\mu_1(z) = \kappa(z_2 - z_1)$ and $\mu_2(z) = 0$ (that is, Z^2 must be a martingale). The volatility parameters can be freely specified, as long as Z_t remains within $\mathcal{Z}$.

We call such a model a linearly mean-reverting variance curve model.

Proof – Theorem 2.3 (25) implies that we have to match

$$-\kappa(z_1 - z_2)e^{-\kappa x} = \mu_1(z)e^{-\kappa x} + \mu_2(z)(1 - e^{-\kappa x})$$

Since the left hand side has no term constant in x, we must have $\mu_2(z) = 0$ (i.e. Z^2 is martingale), and then $\mu_1(z) = \kappa(z_2 - z_1)$. □

A popular parametrization is $\sigma_2 = 0$ and $\sigma_1(z_1) = \nu\sqrt{z_1}$ for some $\nu > 0$, which has been introduced by Heston [H93] (Z^1 is then the square of the short-volatility of the associated stock price process). Another possible choice is

$$\mu(z) = \begin{pmatrix} \kappa(z_2 - z_1) \\ 0 \end{pmatrix} \qquad \sigma(z) = \begin{pmatrix} \nu z_1^\alpha & 0 \\ \eta\rho z_2 & \eta\hat\rho z_2 \end{pmatrix}$$

with constants $\alpha \in [\frac{1}{2}, 2]$, $\nu, \eta \in \mathbb{R}_{>0}$, $\rho \in (-1, 0]$ and $\hat\rho = \sqrt{1-\rho^2}$. Such a process has a stochastic mean-reversion level.

Example 2 (Fitting Heston to the market) *In the light of the discussion in Section 2.2.2, let us to show an approach here to fit a Heston-model to an observed variance swap curve while retaining computational tractability.*

To this end, consider Heston's model [H93] with a time-dependent mean-reversion level,

$$\begin{aligned} dZ^1_t &= \kappa(\theta(Z^2_t) - Z^1_t)\,dt + \nu\sqrt{Z^1_t}\,dW^1_t \\ dZ^2_t &= dt \end{aligned}$$

with the associated stock price process given by a constant correlation ρ.[16]

Assume now that we observe a market variance curve $\hat{u}_0 \in C^1[\mathbb{R}_{\geq 0}]$ *and let* $\theta(x) := \kappa\hat{u}_0(x) + \partial_x\hat{u}_0(x)$. *If* $\theta(x) \geq 0$ *(such that* $Z^1_t \geq 0$*), then* Z *is fits the market in the sense* $\mathbb{E}\left[\, Z^1_x \,\middle|\, Z_0 = (u_0(0), 0) \,\right] = \hat{u}_0(x)$.

The characteristic function of the logarithm of the stock price in this model can be computed using standard methods; see Overhaus et al [OBBFJL06] for details.

The next functional is a generalization of the linearly mean-reverting case above:

Example 3 (Double Mean-Reverting Variance Curve Models) *Let* $c, \kappa > 0$ *constant and let* $z = (z_1, z_2, z_3) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^2_{>0}$. *The Curve Functional*

$$G(z;x) := z_3 + (z_1 - z_3)e^{-\kappa x} + (z_2 - z_3)\begin{cases} \frac{\kappa}{\kappa - c}\left(e^{-cx} - e^{-\kappa x}\right) & (\kappa \neq c) \\ \kappa x\, e^{-\kappa x} & (\kappa = c) \end{cases} \tag{31}$$

is consistent with any factor model $\mathbb{Z} = (\mu, \sigma)$ *such that*

$$\begin{aligned} dZ^1_t &= \kappa(Z^2_t - Z^1_t)\,dt + \sigma_1(Z_t)\,dW_t \\ dZ^2_t &= c(Z^3_t - Z^2_t)\,dt + \sigma_2(Z_t)\,dW_t \\ dZ^3_t &= \sigma_3(Z_t)\,dW_t \end{aligned}$$

and is called a double mean-reverting variance curve model.

It turns out to be a flexible and applicable model: at the time of writing, the variance functional (31) fits the variance swap market of major indices well, so this kind of double mean-reverting model is a good candidate for a variance curve model. The practical implementation of such a model is discussed in [B05] and, from a practical point of view, in Overhaus et al. [OBBFJL06] where the reader also finds example calibrations. A similar model has also been proposed by Duffie et al. [DPS00] who use $\sigma_1(z) = \sqrt{z_1}$ and $\sigma_2(z) = \sqrt{z_2}$ and a particular sparse correlation structure.

4.1.1 Exponential Curves

As in (29), let $(p_i)_{i=1,\dots,D}$ be polynomials and let

$$g(z;x) = \sum_{i=1}^{d} p_i(z;x)e^{-z_i x}$$

with $z = (z_1, \dots, z_d; z_{d+1}, \dots, z_m) \in {\mathbb{R}_{>0}}^d \times \mathbb{R}^{m-d}$. Set

$$G(z;x) := \exp(g(z;x)) \ . \tag{32}$$

Using Theorem 2.3, a necessary condition for a consistent pair is

$$\partial_x g(z;x) = \mu(z)\,\partial_z g(z;x) + \frac{1}{2}\sigma^2(z)\left\{(\partial_z g(z;x))^2 + \partial_{zz} g(z;x)\right\} \ . \tag{33}$$

where $(\partial_z g(z;x))^2 = \sum_{i,j=1}^m \partial_{z^i} g(z;x)\partial_{z^j} g(z;x)$. As a result, we obtain the following lemma,[17] whose proof is very similar to the proof of Lemma 4.1.

[16] Note that $Z^2_t = Z^2_0 + t$.

[17] Compare theorems 3.6.1 and 3.6.2 from Filipovic [F01] p.52ff.

LEMMA 4.2 *If Z is a consistent factor process for the functional G, then the coordinates $Z^1, \ldots, Z^m$ are constant. Moreover, there must be at least one pair $i \neq j$ such that $Z^i = 2Z^j$, otherwise Z is entirely constant.*

EXAMPLE 4 (Exponential Mean-Reverting Models) *Let*

$$g(z;x) = z_2 + (z_1 - z_2)e^{-\kappa x} + \frac{z_3}{4\kappa}(1 - e^{-2\kappa x})$$

with $(z_1, z_2, z_3) \in \mathcal{Z} := \mathbb{R} \times \mathbb{R} \times \mathbb{R}_{>0}$.

Then, $\mu_1(z) = \kappa(z_2 - z_1)$, $\sigma_1(z) = \sqrt{z_3}$ and $\mu_2 = \mu_3 = \sigma_2 = \sigma_3 = 0$, i.e. the only consistent factor model is the exponential Ornstein-Uhlenbeck stochastic volatility model discussed in depth by Fouque et al. in [FPS00].

Proof – The result is a relatively straight-forward consequence of (33) given that μ and σ must be defined for all $z \in \mathcal{Z}$ and that $\sigma_2 \geq 0$. $\square$

4.2 Variance Swap Curves given in terms of Volatility

In this section, we briefly want to discuss a few variance curve term-structure interpolation schemes in terms of a *volatility* function, i.e. schemes where the variance swap price is given as a functional

$$U_t(x) = \Sigma^2(Z_t;x)x \quad \text{where} \quad \Sigma(z;x) = z_1 + z_2 w(x) \tag{34}$$

for some volatility "term-structure" function w.

Such curves arise if standard forms of *implied volatility* term-structure functionals are employed to model the variance swap curve. This approach might be appealing if the implied volatility term-structure is well captured by a particular choice of w.[18] A few choices which have been considered for implied volatility interpolation are

$$\begin{aligned} w_0(x) &:= 0 && (35)\\ w_1(x) &:= \ln(1+x) && (36)\\ w_2(x) &:= \sqrt{\epsilon + x} && (37)\\ w_3(x) &:= 1/\sqrt{\epsilon + x} && (38) \end{aligned}$$

We call w_0 the "Black&Scholes" case (see also Haffner p. 87 in [H04] for a discussion of implied volatility term-structure interpolation). In our previous notation,

$$G(z;x) := \partial_x \left(\Sigma(z;x)^2 x\right) = \Sigma(z;x)^2 + 2\Sigma(z;x)\Sigma'(z;x)x \ . \tag{39}$$

Using Theorem 2.3 we obtain:

COROLLARY 4.1

(a) Case (35) is consistent iff $\mu_1(z_1) = 0$, i.e. Z^1 is a non-negative martingale.

(b) Case (35) is the only case of (35)-(38) which is consistent.

[18] Gatheral [G04] notes that in some stock price models, the shape of the term-structure of implied volatility is similar to the shape of the variance swap curve

4.3 Conclusions for the Recalibration of Stochastic Volatility Models

The results of the previous section can be put to use when it comes to the issue of recalibration of stochastic volatility models.

In most financial institutions, pricing models will be recalibrated on a daily basis to the then observed market data. For example, if an institution wishes to use Heston's model [H93],

$$
\begin{aligned}
dv_\tau &:= \kappa(m - v_\tau)\,dt + \nu\sqrt{v_\tau}\,dW^1_\tau \qquad v_0 = v_0 && (40)\\
dX_\tau &:= \sqrt{\tilde{v}_\tau}\,d(\rho\tilde{W}^1_\tau + \sqrt{1-\rho^2}W^2_\tau) \qquad \tilde{X}_0 = 0 && (41)
\end{aligned}
$$

with parameters $z = (v_0; \kappa, m, \nu, \rho) \in \mathcal{Z} := \mathbb{R}_{\geq 0} \times \mathbb{R}^3_{>0} \times (-1, 0]$, "calibration" is the process of determining the parameter vector z from observed liquid option prices.[19]

This procedure can face two problems: on one hand, the calibration routine itself can be imperfect in the sense that it does not produce robust parameters from given input data.[20] On the other hand, the model itself can be a flawed picture of reality.

Consistency conditions of the sort derived above (such as the requirement of constant speeds of mean-reversion for the models[21] discussed in Section 4 to avoid arbitrage) can now be used in both cases to improve the situation: if we trust in the calibration routine itself, severe violations of such consistency conditions as a result of daily recalibration serve as an indication that the model is indeed a flawed.

If this is not the case, then the consistency conditions can be used to improve the stability of the calibration itself. Indeed, the calibration of Heston's model becomes more robust if we fix the speed of mean-reversion, or if we penalize a divergence from the previously calibrated value.

4.4 Hedging with Variance Curve Models

In this final subsection, we will discuss the completeness of variance curve models.

Assume that we are given a globally consistent pair $(G, \mathbb{Z})$ and a correlation vector ρ such that the associated stock price S is the unique solution to the SDE[22]

$$
dS_t = S_t\sqrt{G(Z_t;0)}\sum_{j=1}^d \rho^j(S_t;Z_t)\,dW^j_t
$$

and a true martingale. Then, the vector $Y := (S, Z)$ is Markov.

One key point is that we only want to price and hedge economically relevant contingent claims, i.e. those whose payoffs H_T depend solely on the past values of the traded instruments stock S and a (finite) number of variance swaps. Mathematically, this means that we restrict our attention to payoffs H_T which are measurable with respect to the filtration $\mathbb{F}^{S,V}$ generated by $(S, (V(T))_{T\geq 0})$, which is usually a strict sub-filtration of $\mathbb{F}$.

The idea is to use observed variance swaps prices and then to invert G in some sense to obtain Z. Since we observe variance swaps and not forward variances, we formulate our invertibility assumption 2 in terms of the former: let

$$
\hat{G}(z;x) := \int_0^x G(z;y)\,dy
$$

and define for $0 \leq x_1 < \cdots < x_m$ the projection

$$
\hat{G}_{x_1,\ldots,x_m}(z) := \left(\hat{G}(z;x_1), \ldots, \hat{G}(z;x_m)\right) .
$$

[19]Strictly speaking, v_0 as the short-variance of the stock price is observable if the path of S is monitored continuously. In practise, however, this will not be possible and v_0 as often regarded as just another parameter of the model.

[20]At the very least, the routine should be deterministic and "continuous" in the sense that small disturbances in the input only yield small changes in the output.

[21]It can also be shown that in Heston's model, the product $\nu\rho$ must be constant, too. This is shown in [B05] using a similar approach as discussed here applied to "entropy swaps".

[22]Note that $s\sqrt{G(z;0)}\rho^j(s,z)$ is locally Lipschitz since $G(\cdot;0)$ is C^2 and because ρ is bounded.

Assumption 2 (Invertibility) *There exists an $\tau^* > 0$ and $\tau_m > \cdots > \tau_1 > \tau^*$ such that*

$$\hat{G}_{x_1,\ldots,x_m} : \mathcal{Z} \longrightarrow \mathbb{R}^m_{\geq 0}$$

with $x_k := \tau_k - s$ $(k = 1, \ldots, m)$ is invertible for all $0 \leq s \leq \tau^*$.

The motivation behind the previous assumption is that on each interval $\mathcal{I}^\ell := [t_\ell, t_{\ell+1})$ with $t_\ell := \ell\tau^*$ $(\ell = 0, 1, \ldots)$, we can use the variance swaps with maturities $T_1^\ell := t_\ell + \tau_1, \ldots, T_m^\ell := t_\ell + \tau_m$ to retrieve Z_t by way of

$$Z_t = \hat{G}^{-1}_{T_1^\ell - t, \ldots, T_m^\ell - t}\Big(V_t(T_1^\ell) - V_t(t), \ldots, V_t(T_m^\ell) - V_t(t)\Big) \ .$$

Let L such that $T \in \mathcal{I}^L$. This gives us the representation of Z over the entire interval $[0, T]$ as

$$Z_t = F_t\Big(V_t(t); \bar{V}_t\Big)$$

with $\bar{V}_t := \big(V_t(T_1^0), \ldots, V_t(T_m^L)\big)$ and

$$F_t(v; v_1^0, \ldots, v_m^L) := \sum_{\ell=0}^{L} 1_{t \in \mathcal{I}^j} \, \hat{G}^{-1}_{T_1^\ell - t, \ldots, T_m^\ell - t}\Big(v_1^\ell - v, \ldots, v_m^\ell - v\Big) \ .$$

This function is by construction C^2 in v and $\bar{v} \in \mathbb{R}^M$ with $M := m(L+1)$. It is also piece-wise C^1 in t. This implies that

$$dZ_t - \mu(Z_t)\, dt = \sum_{k=1}^{M} \partial_{V^k} F_t\big(V_t(t); V_t(T_1), \ldots, V_t(T_M)\big)\, dV_t(T_k) \tag{42}$$

holds on $[0, T]$ in terms of the variance swaps with maturities $(T_1, \cdots, T_M) \equiv (T_1^0, \ldots, T_m^L)$.

Market Completeness

Let us now consider payoffs of the form

$$H_T := \bar{h}\Big(S_T, V_T(T); V_T(T_1^*), \ldots, V_T(T_N^*)\Big) \tag{43}$$

where $T < T_1^* < \ldots < T_N^*$ and where $\bar{h}$ is a bounded non-negative C^2-function. Also recall that $V_T(T) = \int_0^T \zeta_s \, ds$ denotes the realized variance up to T. We can rewrite this as

$$H_T := \bar{g}\Big(S_T; V_T(T), Z_T\Big)$$

in terms of the bounded C^2 function $\bar{g}(s, v, z) := \bar{h}(s, v; \hat{G}(z; T_1^* - T) + v, \ldots, \hat{G}(z; T_N^* - T) + v)$. Because the vector $(S_t, V_t(t), Z_t)_{t \geq 0}$ is Markov, there exists a measurable function g such that

$$g_t(S_t, V_t(t); Z_t) := \mathbb{E}\left[\, H_T \,\middle|\, \mathcal{F}_t \,\right] \ .$$

Assumption 3 *Assume that*

$$(P_t\bar{g})(s_0, z_0) := \mathbb{E}\left[\, \bar{g}(S_t, Z_t) \,\middle|\, S_0 = s_0, Z_0 = z_0 \,\right]$$

is C^2 in (s_0, z_0) for all C^2-functions $\bar{g}$.

Using equation (42), it follows that

$$h_t\big(s, v, \bar{v}\big) := g_t\big(s, v; F_t(v; \bar{v})\big)$$

is C^1 in $(S, \bar{V})$. By the martingale property of $\mathbb{E}[H_T|\mathcal{F}_t]$ (recall that $\bar{g}$ is bounded), the above assumption implies via Ito (and, if necessary, an approximation of h by C^2 functions) that

$$dh_t(S_t, V_t(t), \bar{V}_t) \quad = \quad \partial_s h_t(\cdots)\, dS_t + \sum_{k=1}^{M} \partial_{V_k} h_t(\cdots)\, dV_t(T_k) \ .$$

Standard extensions from the European case (43) to more general measurable payoffs yield

Theorem 4.1 (Completeness) *Under assumptions 2 and 3, the market of payoffs on (S, V) is complete.*

Corollary 4.2 (Delta-hedging with stock and variance swaps) *Let $H_T \in L^{\mathrm{loc}}_T(\mathcal{F}^{S,V})$ such that*

$$h_t(s, v, \bar{v}) := \mathbb{E}\left[\, H_T \,\middle|\, S_t = s, V_t(t) = v, \bar{V} = \bar{v}\,\right]$$

is C^1 in (s, z). Then, "delta-hedging works", i.e.

$$dH_t = \partial_s h_t(S_t, V_t(t), \bar{V}_t)\, dS_t + \sum_{k=1}^{M} \partial_{\bar{v}^k} h_t(S_t, V_t(t), \bar{V}_t)\, dV_t(T_k) \ .$$

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