Robust Statistical Arbitrage Strategies

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Abstract. We investigate statistical arbitrage strategies when there is ambiguity about the underlying time-discrete financial model. Pricing measures are assumed to be martingale measures calibrated to prices of liquidly traded options, whereas the set of admissible physical measures is not necessarily implied from market data. Our investigations rely on the mathematical characterization of statistical arbitrage, which was originally introduced by [Bon03]. In contrast to pure arbitrage strategies, statistical arbitrage strategies are not entirely risk-free, but the notion allows to identify strategies which are profitable on average, given the outcome of a specific σ-algebra. Besides a characterization of robust statistical arbitrage, we also provide a super-/sub-replication theorem for the construction of statistical arbitrage strategies based on path-dependent options. In particular, we show that the range of statistical arbitrage-free prices is, in general, much tighter than the range of arbitrage-free prices.

Keywords: Statistical Arbitrage, Robust Valuation, Trading Strategies, Super-Replication Duality

1. Introduction

Model-independent and robust finance aims, in particular, at calculating arbitrage-free price bounds for exotic derivatives, such that these bounds do not depend on any specific model assumptions. This approach, however, lacks applicability, since the price bounds turn out to be rather broad. In this paper, we instead consider a weaker notion of arbitrage, the so called $\mathcal{G}$-arbitrage, referring to strategies which are profitable on average, given information contained in the σ-algebra $\mathcal{G}$. Such strategies are not risk-free in general. In particular, there may be outcomes which lead to severe losses with positive probability. Nevertheless, such strategies can be of large interest in practice, especially in markets which do not allow for arbitrage. We show that through substituting the notion of arbitrage by the weaker notion of $\mathcal{G}$-arbitrage, the range of prices of a derivative which do not allow for $\mathcal{G}$-arbitrage becomes significantly tighter than the range of prices which do not allow for arbitrage. Further, we show how $\mathcal{G}$-arbitrage strategies can be explicitly constructed based on path-dependent options.

Our methodology builds on the theory of model-independent pricing. Therein, one usually only assumes that all pricing measures satisfy the martingale property and are appropriately calibrated to market prices of liquidly traded options. Hence, each potential model is associated to a martingale measure $\mathbb{Q}$. The expected value $\mathbb{E}_{\mathbb{Q}}[c]$ then corresponds to the price of a derivative with payoff function c under the measure $\mathbb{Q}$. Therefore, minimizing and maximizing this value over all calibrated martingale models and corresponding pricing measures $\mathbb{Q}$ yields model-independent price bounds for the payoff c. Computing prices under martingale measures excludes model-free arbitrage, a result that was recently proved in [ABPS16] under some technical conditions. In the special case, where models are calibrated to a continuum of call- and put prices, the marginal distributions of $\mathbb{Q}$ are fixed (see [BL78]) and a popular technique for solving this valuation problem is called martingale optimal transport (compare e.g. [BHLP13], [BJ16], [DS14], [BNT17]). According to the duality theorem of martingale optimal transportation theory [BHLP13, Theorem 1.1.], upper and lower robust price bounds for exotic derivatives, calculated via the martingale transport approach, can be interpreted as extremal prices of super- and sub-replication strategies. These are semi-static trading strategies consisting of static positions in liquid options as well as self-financing, dynamic positions in the

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underlying security. Prices outside of these bounds lead to model-independent arbitrage opportunities. [LS19] and [Ses20] show how arbitrage-free price bounds can be further tightened by introducing additional constraints on the set of possible measures.

We build on the above outlined theory but replace the definition of arbitrage by the weaker notion of $\mathcal{G}$-arbitrage. The definition of $\mathcal{G}$-arbitrage for self-financing trading strategies which was defined in [RRS19] assumes a fixed physical measure $\mathbb{P}$ as well as a specific risk-neutral measure $\mathbb{Q}$ used for pricing. We consider a model-independent and robust version of $\mathcal{G}$-arbitrage in the following sense. First, and in line with the model-independent setting, all martingale measures calibrated to option prices are considered as potential pricing measures. Second, instead of assuming a fixed real-world measure, we allow for an arbitrary set $\mathcal{P}$ of admissible physical measures. In this way, we extend the notion of $\mathcal{G}$-arbitrage to a setting which is model-independent w.r.t. $\mathbb{Q}$ and robust w.r.t. $\mathbb{P}$. We refer to corresponding strategies consisting of self-financing positions in the underlying security and static positions in liquid options as $\mathcal{P}$-robust $\mathcal{G}$-arbitrage strategies. A frequently considered special case of $\mathcal{G}$-arbitrage is *statistical arbitrage*. Corresponding strategies are especially important when it comes to applications as they induce an average gain regardless of the outcome of the underlying security at terminal time. In practice, the term *statistical arbitrage* is often associated with pairs trading strategies which build on cointegrated pairs of assets (compare e.g. [AL10] and [MPZ16]). These strategies, however, do not fall within the rigorous definition of statistical arbitrage as introduced in [Bon03], which is also in line with our framework.

Our paper contributes to the existing literature in the following way. First, under some technical conditions, we show that a market is free of $\mathcal{P}$-robust $\mathcal{G}$-arbitrage if for all $\mathbb{P} \in \mathcal{P}$ there is some calibrated martingale measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that the density processes $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \in \mathcal{G}$. Second, we provide a duality theorem which characterizes $\mathcal{P}$-robust $\mathcal{G}$-arbitrage free prices by super-, resp. sub-replication strategies. This result has important practical implications. In particular, it implies price bounds for path-dependent derivatives which are much tighter than the classical no-arbitrage bounds. In addition, it yields a method for the explicit construction of corresponding $\mathcal{P}$-robust $\mathcal{G}$-arbitrage strategies, if derivative prices should lie outside of the associated bounds. In this way, our result allows to detect market prices for derivatives lying outside of these price bounds and to set up trading strategies that profit from this price deviation. Finally, within various empirical examples, we investigate the performance of different trading strategies when the market allows for robust statistical arbitrage. In particular, we also compare the performance of the statistical arbitrage strategies as implied by our duality result with the classical pairs trading strategies.

The paper is structured as follows. Section 2 introduces the general setting and the different notions of arbitrage. Section 3 deals with $\mathcal{G}$-arbitrage and its characterization in the model-dependent case, i.e., when considering only a single-reference measure $\mathbb{P}$. Section 4 extends the setting to an arbitrary set $\mathcal{P}$ of possible probability measures and contains the main results including a characterization of $\mathcal{P}$-robust $\mathcal{G}$-arbitrage as well as a super-, resp. sub-replication result. In Section 5 we apply the resulting statistical arbitrage strategies in different settings and compare the performance in simulated environments as well as based on market data.

2. Notions of Arbitrage

In this section, we discuss different notions of arbitrage in the model-dependent, model-independent, and robust settings. Therefore, let us first introduce some basic notation used throughout the article. We consider a discrete set of $n \in \mathbb{N}$ times $t_1, \ldots, t_n$ with $t_1 < \cdots < t_n$ and a real-valued Borel-measurable process denoted by $S = (S_{t_i})_{i \in \{1,\ldots,n\}}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with initial value S_{t_0}. In most applications, the process S can be considered as the canonical process on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, i.e.,

$$S_{t_i} : (x_1, \ldots, x_n) \mapsto x_i$$

for all $i = 1, \ldots, n$. The framework involving the canonical process can easily be extended to an arbitrary Borel-measurable process and poses no severe restrictions, in particular, our main results do not rely on this assumption and can easily be transported to other, more specific frameworks. If S is defined on a different specific probability space, then we will describe the setting explicitly. In financial applications, S usually describes the evolution of the price of an underlying security over time. If not mentioned otherwise, we denote by $\mathbb{F}$ the natural filtration of S. To simplify the exposition, we normalize interest rates to zero and we do not consider dividends. Further, we assume that the underlying security S as well as derivatives on S can be bought and sold liquidly at times $t_0 < t_1 < \cdots < t_n$ without transaction costs. In such a setting arbitrage opportunities can be specified dependent on a particular model characterized by a real-world measure $\mathbb{P}$. Alternatively, when there is ambiguity about $\mathbb{P}$, arbitrage can be defined also in a robust way, such that a whole set of real-world measures $\mathcal{P}$ is admitted. Finally, arbitrage can be considered model-independent, i.e., regardless of any specification of $\mathbb{P}$ or $\mathcal{P}$. In the following, we formalize these definitions and characterize the corresponding weaker forms of $\mathcal{G}$-arbitrage and statistical arbitrage opportunities.

2.1. **Model-Dependent Arbitrage.**

Let us consider a financial market model characterized by the stochastic process $S = (S_{t_i})_{i \in \{1,\ldots,n\}}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Note that the real-world measure $\mathbb{P}$ is fixed in this setting. Let $\Delta = (\Delta_j)_{j=1,\ldots,n-1}$ denote a zero-cost self-financing trading strategy, where Δ_j are bounded and measurable functions. For some $i \in \mathbb{N}$, the set of all bounded and measurable functions $f : \mathbb{R}^i \to \mathbb{R}$ is in the following denoted by $B(\mathbb{R}^i)$. The payoff of the strategy is defined via the time-discrete stochastic integral as

$$(\Delta \cdot S)_n := \sum_{i=0}^{n-1} \Delta_i(S_{t_0}, \ldots, S_{t_i})(S_{t_{i+1}} - S_{t_i}).$$

We call Δ a $\mathbb{P}$-arbitrage or simply an arbitrage opportunity, if Δ has non-negative payoff and with positive probability leads to some gains.

Definition 2.1 (Arbitrage). *a) A self-financing trading strategy Δ is called a $\mathbb{P}$-arbitrage (or an* arbitrage strategy*) if*

(i) $(\Delta \cdot S)_n \geq 0$ $\mathbb{P}$-a.s., and

(ii) $\mathbb{E}_{\mathbb{P}}[(\Delta \cdot S)_n] > 0$.

b) The market model satisfies the no-arbitrage condition if there exists no arbitrage strategy in the market, i.e., if $(\Delta \cdot S)_n \geq 0$ $\mathbb{P}$-a.s. implies $(\Delta \cdot S)_n = 0$ $\mathbb{P}$-a.s..

Next, we relax the definition of an arbitrage by considering a broader class of profitable strategies, which are called $\mathcal{G}$-arbitrage strategies (see also [RRS19] and [Bon03]). These are strategies which have a non-negative expected payoff given the information on a σ-algebra $\mathcal{G} \subset \sigma(S_{t_1}, \ldots, S_{t_n})$ and which yield a positive payoff on average.

Definition 2.2 ($\mathcal{G}$-Arbitrage). *Let $\mathcal{G} \subset \sigma(S_{t_1}, \ldots, S_{t_n})$ be a σ-algebra.*

*a) A self-financing trading strategy Δ is called a $\mathcal{G}$-*arbitrage strategy *(or profitable $\mathcal{G}$-strategy) if*

(i) $\mathbb{E}_{\mathbb{P}}[(\Delta \cdot S)_n \mid \mathcal{G}] \geq 0$ $\mathbb{P}$-a.s., and

(ii) $\mathbb{E}_{\mathbb{P}}[(\Delta \cdot S)_n] > 0$.

b) The market model satisfies the no $\mathcal{G}$-arbitrage condition if there exists no $\mathcal{G}$-arbitrage strategy, i.e., if $\mathbb{E}_{\mathbb{P}}[(\Delta \cdot S)_n \mid \mathcal{G}] \geq 0$ $\mathbb{P}$-a.s. implies $\mathbb{E}_{\mathbb{P}}[(\Delta \cdot S)_n] = 0$.

As an important special case of a $\mathcal{G}$-arbitrage, we define a statistical arbitrage opportunity.

Definition 2.3 (Statistical Arbitrage). *A self-financing trading strategy Δ is called a* statistical-arbitrage *if Definition 2.2 holds for the particular choice $\mathcal{G} = \sigma(S_{t_n})$.*

Hence, a statistical arbitrage is a trading strategy which leads to an average profit for every possible outcome of S_{t_n}. Nevertheless, in contrast to a pure arbitrage strategy, such a strategy is not risk-free, since (potentially high) losses are only excluded on average given $\sigma(S_{t_n})$. Thus, losses are still possible for events which occur with positive probability. For an investigation on the associated risks of statistical arbitrage also compare [RRS19, Section 3.4]. In most applications and examples, we study exactly this specialization of $\mathcal{G}$-arbitrage. However, we establish all of the theoretical results for the more general notion of $\mathcal{G}$-arbitrage with an arbitrary choice of $\mathcal{G} \subset \sigma(S_{t_1}, \dots, S_{t_n})$. For an overview of possible choices for $\mathcal{G}$ we refer to [RRS19, Section 2].

2.2. **Robust Arbitrage.** Next, we define different types of arbitrage opportunities in both a robust and a model-independent setting when there is ambiguity about the choice of the correct real-world measure. We denote by $\mathcal{P}$ an arbitrary set of real-valued n-dimensional probability measures and denote by $\mathbb{P} \in \mathcal{P}$ some reference measure in the set $\mathcal{P}$.

First, we specify all pricing measures which are consistent with market data. Therefore, suppose we can observe $N \in \mathbb{N} \cup \{\infty\}$ different financial instruments with payoff functions $(\varphi_i(S_{t_1}, \dots, S_{t_n}))_{i=1,\dots,N}$ traded at market prices p_i for $i = 1, \dots, N$. W.l.o.g. we assume that they can be bought at zero prices[1], i.e., we set $p_i = 0$ for all $i = 1, \dots, N$. In particular, we observe that strategies of the form

$$f(x_1, \dots, x_n) = \sum_{i=1}^{N} a_i \varphi_i(x_1, \dots, x_n) + \sum_{i=0}^{n-1} \Delta_i(x_1, \dots, x_i)(x_{i+1} - x_i) \tag{2.1}$$

can be bought and sold for price zero. This leads to the following definition of robust arbitrage (compare also [BN15, Definition 1.1], where exactly the same notion of robust arbitrage is used). In the following we abbreviate $\Phi = \{\varphi_i,\ i = 1, \dots, N\}$.

Definition 2.4 ($\mathcal{P}$-Robust Arbitrage)**.** *A $\mathcal{P}$-robust arbitrage strategy is a strategy with payoff f of the form (2.1) for some $a_i \in \mathbb{R}$ and some $\Delta_i \in B(\mathbb{R}^i)$ such that $f \geq 0$ $\mathbb{P}$-almost surely for all $\mathbb{P} \in \mathcal{P}$ and $\mathbb{E}_{\mathbb{P}}[f] > 0$ for some $\mathbb{P} \in \mathcal{P}$.*

We stress that the latter definition generalizes Definition 2.2 in two ways: First, it involves an ambiguity class of probability measures instead of a fixed a-priori measure. Second, it includes static trading in liquidly traded options. Next, we define a robust notion of $\mathcal{G}$-arbitrage.

Definition 2.5 ($\mathcal{P}$-Robust $\mathcal{G}$-Arbitrage)**.** *A $\mathcal{P}$-robust $\mathcal{G}$-arbitrage strategy is a strategy with payoff f of the form (2.1) for some $a_i \in \mathbb{R}$ and some $\Delta_i \in B(\mathbb{R}^i)$ such that for all $\mathbb{P} \in \mathcal{P}$ it holds $\mathbb{E}_{\mathbb{P}}[f \mid \mathcal{G}] \geq 0$ $\mathbb{P}$-almost surely. Further, there exists some $\tilde{\mathbb{P}} \in \mathcal{P}$ such that $\mathbb{E}_{\tilde{\mathbb{P}}}[f] > 0$.*

The notion of $\mathcal{P}$-robust arbitrage is related to model-independent arbitrage which refers to strategies with payoff f of the form (2.1) for some $a_i \in \mathbb{R}$ and some $\Delta_i \in B(\mathbb{R}^i)$ such that $f > 0$, see also [ABPS16, Definition 1.2]. Hence, a model-independent arbitrage strategy is a strategy which allows to profit regardless of any model assumptions. In particular, each model-independent arbitrage strategy is a $\mathcal{P}$-robust arbitrage strategy and each $\mathcal{P}$-robust arbitrage strategy is a $\mathcal{P}$-robust $\mathcal{G}$-arbitrage strategy. According to Definition 2.5 a market is free of $\mathcal{P}$-robust $\mathcal{G}$-arbitrage if and only if $\mathbb{E}_{\mathbb{P}}[f \mid \mathcal{G}] \geq 0$ $\mathbb{P}$-almost surely for all $\mathbb{P} \in \mathcal{P}$ implies $\mathbb{E}_{\mathbb{P}}[f \mid \mathcal{G}] = 0$ $\mathbb{P}$-almost surely for all $\mathbb{P} \in \mathcal{P}$.

Remark 2.6. The set of $\mathcal{P}$-robust arbitrage strategies and $\mathcal{P}$-robust $\mathcal{G}$-arbitrage strategies depends crucially on the set of considered statically traded options Φ. Compare also Example 3.6 where we show the influence of available static instruments on the set of $\mathcal{G}$-arbitrage strategies.

2.3. **Fundamental Theorems.** Absence of arbitrage is strongly connected with the existence of martingale measures. If there are no financial instruments for which prices can be observed, i.e., if $\Phi = \emptyset$, we define

$$\mathcal{Q}_{\emptyset} = \{\mathbb{Q} \in \mathcal{M}(\mathbb{R}^n) \mid \mathbb{Q} \text{ is a martingale measure }\},$$

where $\mathcal{M}(\mathbb{R}^n)$ denotes the set of n-dimensional real-valued probability measures. The condition $\mathbb{Q}$ to be a martingale measure w.r.t. the natural filtration can be expressed as

$$\mathbb{E}_{\mathbb{Q}}\left[\Delta(S_{t_1}, \dots, S_{t_j})(S_{t_{j+1}} - S_{t_j})\right] = 0$$

[1] This can always be achieved by considering $\varphi_i - p_i$ instead of φ_i.

for all $j = 1, \dots, n-1$ and all $\Delta \in C_b(\mathbb{R}^j)$(compare e.g. (Beiglböck et al. 2013, Lemma 2.3).) If $\Phi \neq \emptyset$, then we specify the set of admissible martingale measures which are consistent with given market prices as

$$\mathcal{Q}_\Phi = \{\mathbb{Q} \in \mathcal{M}(\mathbb{R}^n) \mid \mathbb{E}_\mathbb{Q}[\varphi_i(S_{t_1}, \dots, S_{t_n})] = 0 \text{ for } i = 1, \dots, N \text{ and } \mathbb{Q} \text{ is a martingale measure}\}.$$

In particular, strategies of the form (2.1) have zero expected values under measures $\mathbb{Q} \in \mathcal{Q}_\Phi$.

Remark 2.7. If Φ consists of payoffs of call-options maturing at times $t_1, \dots, t_n$ for a continuous range of strikes, then $\mathcal{Q}_\Phi$ describes the set of all martingale measures with fixed marginal distributions $\mu_1, \dots, \mu_n$ which are implied by option prices via the Breeden-Litzenberger result (see [BL78]). In this situation martingale optimal transport approaches can be applied to compute model-free price bounds for derivatives, compare e.g. [BHLP13], [ABPS16], [DS14], [BJ16], [HK15].

Note that the pricing measures in $\mathcal{Q}_\Phi$ are not necessarily equivalent to any measure in the set $\mathcal{P}$ of real-world measures which we consider in the robust setting. Similarly to the approach in [BN15] we now consider all pricing measures which are absolutely continuous with respect to some reference measure $\mathbb{P} \in \mathcal{P}$, i.e., the set

$$\mathcal{Q}_\Phi^\mathcal{P} = \{\mathbb{Q} \in \mathcal{Q}_\Phi : \mathbb{Q} \ll \mathbb{P} \text{ for some } \mathbb{P} \in \mathcal{P}\}.$$

Hence, $\mathcal{Q}_\Phi^\mathcal{P}$ characterizes all pricing measures which are consistent with market prices and which are absolutely continuous w.r.t. some real-world measure $\mathbb{P}$ in the ambiguity set $\mathcal{P}$.[2]

The latter defined sets of martingale measures give rise to alternative characterizations of the previously introduced no-arbitrage conditions through the following theorems.

Theorem 2.8 (Robust Fundamental Theorem of Asset Pricing, [BN15])**.** *The following are equivalent.*

(i) There exists no $\mathcal{P}$-robust arbitrage.
(ii) For all $\mathbb{P} \in \mathcal{P}$ there exists some $\mathbb{Q} \in \mathcal{Q}_\Phi^\mathcal{P}$ such that $\mathbb{P} \ll \mathbb{Q}$.

This result is due to [BN15, Theorem 5.1][3] and represents an adaption of the theorem in [DS94] to the robust situation. See [ABPS16, Theorem 1.3] for a model-independent version of this result. In the recent literature, results have been established which characterize $\mathcal{G}$-arbitrage by an assertion similar to the preceding fundamental theorems. First, in [Bon03] a fundamental characterization was stated for self-financing trading strategies and the particular case $\mathcal{G} = \sigma(S_{t_n})$. Then, [KL17] extended the result to general sigma-algebras $\mathcal{G}$ and general (not necessarily tradeable) payoffs with zero price. The most recent contribution related to this fundamental theorem stems from [RRS19]. Therein the authors argue that a completeness assumption ensures validity of the characterization of no $\mathcal{G}$-arbitrage when considering a fixed underlying physical probability measure. A combination of all these contributions leads to the following result.

Theorem 2.9 ([Bon03], [KL17], [RRS19])**.** *In a complete arbitrage-free market without traded options, i.e., $\Phi = \emptyset$, the following are equivalent.*

(i) There is no $\mathcal{G}$-arbitrage.
(ii) It holds $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \in \mathcal{G}$ for the unique $\mathbb{Q} \in \mathcal{Q}_\emptyset$ such that $\mathbb{Q} \sim \mathbb{P}$.

In the latter theorem and in the following, by abuse of notation, we write $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \in \mathcal{G}$ to express that $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}$ is $\mathcal{G}$-measurable.

A major contribution of the present paper is to generalize Theorem 2.9 to the situation in which the physical measure $\mathbb{P}$ is not fixed but contained in a class of physical measures and additional trading in static options is allowed. Furthermore, we replace the completeness assumption by a more general assumption on the set of available static options. The set of

[2]In the case $\Phi = \emptyset$, we have $\mathcal{Q}_\emptyset^\mathcal{P} = \{\mathbb{Q} \in \mathcal{Q}_\emptyset : \mathbb{Q} \ll \mathbb{P} \text{ for some } \mathbb{P} \in \mathcal{P}\}$.

[3]For this result, the authors assume additionally a specific product structure of $\mathbb{P} \in \mathcal{P}$ as well as convexity of $\mathcal{P}$. Moreover, the considered trading strategies are assumed to be universally measurable. The corresponding framework is described in further detail within [BN15, Section 1.2.]. In particular, the extreme cases $\mathcal{P} = \{\mathbb{P}\}$ and $\mathcal{P} = \mathcal{M}(\mathbb{R}^n)$ are covered in this framework.

available options is then required to be rich enough to enable the costless super-replication of particular payoffs of interest.

Our results do not rely on the quasi-sure setting of [BN15] but instead are based on a combination of arguments applied to each physical measure separately.

3. Characterization of Model-Dependent $\mathcal{G}$-Arbitrage

In this section, we establish conditions for the absence of $\mathcal{G}$-arbitrage and we study their implications for the valuation of derivatives. While we still assume a single unambigious physical measure $\mathbb{P}$, from now on we assume, in addition to dynamic trading in the underlying security that we are able to trade statically in zero-price instruments $(\varphi_i)_{i=1,\dots,N} \in \Phi$ with $N \in \mathbb{N} \cup \{\infty\}$. Moreover, we substitute the completeness assumption in [RRS19] by the more general assumption stated in Assumption 3.1. We generalize to $\mathbb{P} \in \mathcal{P}$ in Section 4.

3.1. Absence of Model-Dependent $\mathcal{G}$-Arbitrage.

To derive suitable conditions for the absence of $\mathcal{G}$-arbitrage in case of a single reference measure $\mathbb{P}$, we require an additional assumption.

Assumption 3.1. *Let one of the following assumptions hold.*

(i) Assume

$$\mathbb{E}_{\mathbb{P}}\left[\sum_{i=1}^{N} a_i\varphi_i + (\Delta \cdot S)_n \;\middle|\; \mathcal{G}\right] \geq \mathbb{E}_{\mathbb{P}}\left[\frac{\mathbb{E}_{\mathbb{P}}[\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \mid \mathcal{G}]}{\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}} - 1 \;\middle|\; \mathcal{G}\right] \;\mathbb{P}\text{-a.s.} \tag{3.1}$$

for some $a_i \in \mathbb{R}$, $\Delta_j \in B(\mathbb{R}^j)$ and some $\mathbb{Q} \in \mathcal{Q}_{\Phi}^{\{\mathbb{P}\}}$ such that $\mathbb{Q} \sim \mathbb{P}$.

(ii) Assume $\mathcal{G} = \sigma(S_{t_n})$, $\mathbb{E}_{\mathbb{P}}[S_{t_i}^2] < \infty$ for all $i = 1, \dots, n$, $\{(\varphi_i)_{i=1,\dots,N}\}$ comprises call- and put options maturing at time t_n for all strikes from a compact interval K and that the t_n marginal of $\mathbb{P}$ is supported on K and equivalent with the Lebesgue-measure on K.

Under this assumption, we obtain the following characterization of model-dependent $\mathcal{G}$-arbitrage.

Proposition 3.2 (Characterization of $\mathcal{G}$-Arbitrage)**.** *Let Assumption 3.1 be valid. Then the following are equivalent.*

(i) There is no $\mathcal{G}$-arbitrage.

(ii) There exists some $\mathbb{Q} \in \mathcal{Q}_{(\varphi_i)}^{\{\mathbb{P}\}}$ with $\mathbb{P} \sim \mathbb{Q}$ such that the density $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}$ is $\mathcal{G}$-measurable.

To avoid confusion we stress that here the notion $\mathcal{G}$-arbitrage relies rather on Definition 2.5 with the specific choice $\mathcal{P} = \{\mathbb{P}\}$ than on Definition 2.2 where no trading in options is considered, i.e., in accordance with our notation we consider absence of $\{\mathbb{P}\}$-robust $\mathcal{G}$-arbitrage.

Proof. Follows by the more general Proposition 4.3 with the particular choice $\mathcal{P} = \{\mathbb{P}\}$. If Assumption 3.1 (ii) is valid, then we can derive the result by Proposition 4.6. □

Before moving on to the implications of absence of $\mathcal{G}$-arbitrage for the valuation of derivatives, we discuss Assumption 3.1 (i) and its ramifications in different situations. Assumption 3.1 (ii) will be discussed in Section 4.3.

Remark 3.3 (Validity of Assumption 3.1).

(i) Assumption 3.1 (i) is obviously fulfilled if we can find some $\mathbb{P}$-equivalent probability measure $\mathbb{Q} \in \mathcal{Q}_{\Phi}$ s.t. $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \in \mathcal{G}$. Then,

$$\mathbb{E}_{\mathbb{P}}\left[\frac{\mathbb{E}_{\mathbb{P}}[\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \mid \mathcal{G}]}{\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}} - 1 \;\middle|\; \mathcal{G}\right] = 0 \;\mathbb{P}\text{-a.s.}$$

which can be trivially replicated by $\Delta_j \equiv 0,\; a_i = 0$ for all i, j.

(ii) In a market free of $\mathcal{G}$-arbitrage there are two possibilities how Assumption 3.1 (i) can be fulfilled: Either there exists some equivalent martingale measure $\mathbb{Q} \in \mathcal{Q}_{\Phi}^{\{\mathbb{P}\}}$ such that the right-hand side of (3.1) vanishes or if this is not the case, then we add options φ_i to the market which admit a $\mathcal{G}$-arbitrage opportunity and super-replicate $\frac{\mathbb{E}_{\mathbb{P}}[\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \mid \mathcal{G}]}{\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}} - 1$ conditional on $\mathcal{G}$. Such an approach is also shown within Example 3.6 below.

(iii) The assumption is fulfilled if the market is weakly complete, i.e., if each integrable random variable can be replicated.

(iv) Assume $\mathbb{P} \sim \mathbb{Q}$ for some $\mathbb{Q} \in \operatorname{ext} \mathcal{Q}_\Phi^{\{\mathbb{P}\}}$, where $\operatorname{ext} \mathcal{Q}_\Phi^{\{\mathbb{P}\}}$ denotes the set of extremal measures $\tilde{\mathbb{Q}} \in \mathcal{Q}_\Phi^{\{\mathbb{P}\}}$, i.e., the set of all measures $\tilde{\mathbb{Q}}$ for which $\tilde{\mathbb{Q}} = \lambda\tilde{\mathbb{Q}}_1 + (1-\lambda)\tilde{\mathbb{Q}}_2$ for some $\lambda \in (0,1)$ implies $\tilde{\mathbb{Q}} = \tilde{\mathbb{Q}}_1 = \tilde{\mathbb{Q}}_2$. Then, we obtain the representation

$$Z := \frac{\mathbb{E}_{\mathbb{P}}[\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \mid \mathcal{G}]}{\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}} - 1 = a_0 + \sum_{i=1}^N a_i \varphi_i + (\Delta \cdot S)_n \quad \mathbb{Q}\text{-a.s.} \tag{3.2}$$

for some $a_i \in \mathbb{R}$, $\Delta_j \in B(\mathbb{R}^j)$. The semi-static replication is due to [AL17, Theorem 3.1.]. Note for this step that $\mathbb{E}_{\mathbb{Q}}[|Z|] \leq 2 < \infty$ and therefore $Z \in L^1(\mathbb{Q})$ which implies together with [AL17, Remark 3.3.] the semi-static representation. Compare also [CM19] for a representation result similar to (3.2) in the two-dimensional case with tradeable vanilla options (and thus by Remark 2.7 fixed marginal distributions of $\mathbb{Q}$). Since $\mathbb{Q} \sim \mathbb{P}$ the representation (3.2) also holds $\mathbb{P}$-almost surely. We observe $\mathbb{E}_{\mathbb{Q}}[Z] = 0$ which by (3.2) yields $a_0 = 0$.

If we instead assume that $\Phi = \emptyset$ and $\mathcal{Q}_\emptyset^{\{\mathbb{P}\}} \neq \emptyset$ and that there exists more than one measure in $\mathcal{Q}_\emptyset^{\{\mathbb{P}\}}$ which is equivalent to $\mathbb{P}$, then we cannot have $\mathbb{Q} \in \operatorname{ext} \mathcal{Q}_\emptyset^{\{\mathbb{P}\}}$, since all $\mathbb{P}$-equivalent measures are not extreme points of $\mathcal{Q}_\emptyset^{\{\mathbb{P}\}}$, see e.g. [FS16, Theorem 5.39] in a discrete-time setting or [Del92, Theorem 6.5] in a continuous-time setting. Hence, in absence of the possibility for static-trading in options, we are only able to apply Proposition 3.2 if the set of equivalent martingale measures is a singleton. In this case we recover Theorem 2.9.

3.2. **$\mathcal{G}$-Arbitrage-Free Valuation of Derivatives.** In a market which is free of $\mathcal{G}$-arbitrage, we aim at determining a $\mathcal{G}$-arbitrage-free price for some derivative $c(S_{t_1}, \dots, S_{t_n})$. This leads to the super-replication problem

$$\begin{aligned} P(\mathcal{G}, c) := \inf \Bigg\{ d \in \mathbb{R} : \ \exists (a_i)_i \in \mathbb{R},\ (\Delta_j)_j \in B(\mathbb{R}^j) \text{ s.t. } d + \mathbb{E}_{\mathbb{P}}\left[\sum_{i=1}^N a_i\varphi_i + (\Delta \cdot S)_n\right] > \mathbb{E}_{\mathbb{P}}[c] \\ \text{and} \quad d + \mathbb{E}_{\mathbb{P}}\left[\sum_{i=1}^N a_i\varphi_i + (\Delta \cdot S)_n \;\middle|\; \mathcal{G}\right] \geq \mathbb{E}_{\mathbb{P}}[c \mid \mathcal{G}] \ \mathbb{P}\text{-a.s.} \Bigg\}. \end{aligned}$$

Indeed, the value $P(\mathcal{G}, c)$ defines the highest possible price for c which excludes $\mathcal{G}$-arbitrage. More specifically, suppose $P(\mathcal{G}, c)$ is attained and c is traded for a price $p(c) > P(\mathcal{G}, c)$, then one can lock in a $\mathcal{G}$- arbitrage profit by initiating a short position in c as well as a long position in the optimal strategy $f = d + \sum_{i=1}^N a_i\varphi_i + (\Delta \cdot S)_n$ at price $P(\mathcal{G}, c)$ and investing the profit $p(c) - P(\mathcal{G}, c)$ at the risk-free rate.

As in the classical no arbitrage theory we derive a super-replication theorem which arises basically as a consequence of the characterization of $\mathcal{G}$-arbitrage.

Corollary 3.4 (Super-Replication). *Assume $\mathcal{Q}_\Phi^{\{\mathbb{P}\}} \neq \emptyset$ and assume Assumption 3.1 to be valid for the market with securities Φ as well as for traded securities $\Phi \cup \{c - p(c)\}$ for any price $p(c) \in [D(\mathcal{G}, c), P(\mathcal{G}, c)]$ of payoff c. If the market is $\mathcal{G}$-arbitrage free, then we have*

$$P(\mathcal{G}, c) = \sup_{\mathbb{Q} \in \mathcal{Q}_\Phi^{\{\mathbb{P}\}}} \{\mathbb{E}_{\mathbb{Q}}[c] : \mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \in \mathcal{G} \text{ and } \mathbb{P} \sim \mathbb{Q}\} =: D(\mathcal{G}, c)$$

Proof. Follows by Corollary 4.4 with the specific choice $\mathcal{P} = \{\mathbb{P}\}$. $\square$

Recall that we want to compute the maximal price $p_{max}(c)$ and minimal price $p_{min}(c)$ for a derivative $c(S_{t_1}, \dots, S_{t_n})$ such that the market complemented by this derivative still does not allow for $\mathcal{G}$-arbitrage. This means by the risk-neutral pricing approach, we face the problem

$$p_{max}(c) = \sup_{\mathbb{Q} \in \mathcal{Q}_\Phi^{\{\mathbb{P}\}}} \{\mathbb{E}_{\mathbb{Q}}[c] \ : \ \nexists \mathcal{G}\text{-arbitrage in the market } \Phi \cup \{c - \mathbb{E}_{\mathbb{Q}}[c]\}\} \tag{3.3}$$

as well as the corresponding minimization problem. Applying Proposition 3.2, we can reformulate this problem as

$$\sup_{\mathbb{Q}\in\mathcal{Q}_\Phi^{\{\mathbb{P}\}}} \{\mathbb{E}_\mathbb{Q}[c] : \mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \in \mathcal{G} \text{ and } \mathbb{P}\sim\mathbb{Q}\} = p_{max}(c). \tag{3.4}$$

We show that this problem, which we call the dual problem, can be solved by the consideration of the primal problem $P(\mathcal{G}, c)$, i.e., both valuation approaches, super-replication approach and the risk-neutral pricing approach yield the same value.

Similarly, we can derive a representation for the minimal price $p_{min}(c)$ through a sub-replication strategy.

Remark 3.5 (Sub-Replication). Under the same assumptions as imposed in Corollary 3.4, we can derive a sub-replication theorem. This follows from the identity $\inf_\mathbb{Q} \mathbb{E}_\mathbb{Q}[c] = -\sup_\mathbb{Q} \mathbb{E}_\mathbb{Q}[-c]$. The duality relation can then be expressed as

$$\begin{aligned}
\sup\Bigg\{ d\in\mathbb{R} : \exists (a_i)_i \in \mathbb{R},\ (\Delta_j)_j \in B(\mathbb{R}^j) \text{ s.t. } d + \mathbb{E}_\mathbb{P}\left[\sum_{i=1}^N a_i\varphi_i + (\Delta\cdot S)_n\right] < \mathbb{E}_\mathbb{P}[c] \\
\text{and}\quad d + \mathbb{E}_\mathbb{P}\left[\sum_{i=1}^N a_i\varphi_i + (\Delta\cdot S)_n \,\middle|\, \mathcal{G}\right] \le \mathbb{E}_\mathbb{P}[c\mid\mathcal{G}]\ \mathbb{P}-\text{a.s.}\Bigg\} \\
= \inf_{\mathbb{Q}\in\mathcal{Q}_\Phi^{\{\mathbb{P}\}}} \{\mathbb{E}_\mathbb{Q}[c] : \mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}\in\mathcal{G} : \mathbb{P}\sim\mathbb{Q}\}.
\end{aligned}$$

These results allow, in particular, to construct $\mathcal{G}$-arbitrage strategies, which involve trading in derivatives with prices that do not lie within the range of $\mathcal{G}$-arbitrage-free prices.

3.3. **Examples.** In this subsection, we provide several examples illustrating the influence of the no $\mathcal{G}$-arbitrage condition on the valuation of financial derivatives.

Example 3.6 (Statistical arbitrage, counter example from [RRS19], revisited). In [RRS19, Section 3.1. and Section 3.2.], the authors study an example for a financial market without traded options, i.e., $\Phi = \emptyset$, in which there exists no possibility for statistical arbitrage. This constitutes a counter example to the main assertion in [Bon03] since in this example, there does not exist some $\mathbb{Q}\in\mathcal{Q}_\emptyset^{\{\mathbb{P}\}}$ with $\mathbb{Q}\sim\mathbb{P}$ possessing a density $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}\in\sigma(S_{t_n})$, although the market is free of statistical arbitrage. We remark that this observation is not contradicting our Proposition 3.2, since the assumptions of the proposition are not entirely fulfilled in this example.
More precisely, in [RRS19], the authors consider a two time-step trinomial model with 6 possible scenarios, i.e., $\Omega = (\omega_1,\dots,\omega_6)$. Compare Figure 1 for an illustration of the possible developments of the underlying process.

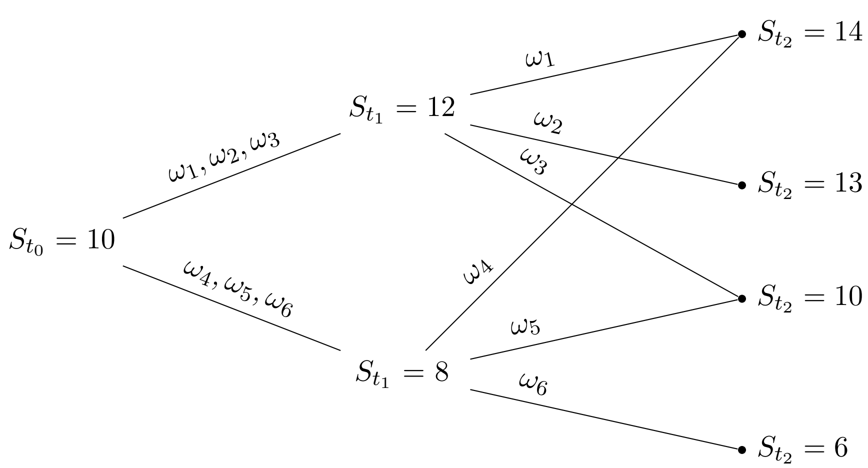

FIGURE 1. Illustration of the possible developments of $(S_{t_i})_{i\in 0,1,2}$ in the financial market considered in an example in [RRS19] and revisited in Example 3.6.

The physical measure $\mathbb{P}$ is then chosen such that

$$\mathbb{P}(\omega_1)=0.15,\ \mathbb{P}(\omega_2)=0.2,\ \mathbb{P}(\omega_3)=0.3,\ \mathbb{P}(\omega_4)=0.05,\ \mathbb{P}(\omega_5)=0.1,\ \mathbb{P}(\omega_6)=0.2.$$

In this situation, the set of possible, not necessarily equivalent, martingale measures is given by

$$\mathcal{Q}_{\emptyset}^{\{\mathbb{P}\}}=\Bigg\{\mathbb{Q}\in\mathcal{M}(\mathbb{R}^6)\ \Bigg|\ \mathbb{Q}(\omega_1)=-\frac{3}{4}\mathbb{Q}(\omega_2)+\frac{1}{4},\ \mathbb{Q}(\omega_3)=-\frac{1}{4}\mathbb{Q}(\omega_2)+\frac{1}{4},\ \mathbb{Q}(\omega_4)=\mathbb{Q}(\omega_6)-\frac{1}{4},$$
$$\mathbb{Q}(\omega_5)=-2\mathbb{Q}(\omega_6)+\frac{3}{4},\ \text{with } \mathbb{Q}(\omega_2)\in\left[0,\frac{1}{3}\right],\ \mathbb{Q}(\omega_6)\in\left[1/4,\frac{3}{8}\right]\Bigg\}.$$

[RRS19] show that there does not exist a $\mathbb{Q}\in\mathcal{Q}_{\emptyset}^{\{\mathbb{P}\}}$ such that $\mathbb{Q}\sim\mathbb{P}$ and $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}\in\sigma(S_{t_2})$. We show that it is, however, possible to incorporate options φ_i in a way such that statistical arbitrage opportunities emerge. In particular, we show that in such a setting the assumptions of Proposition 3.2 are met.

(i) We consider a path-dependent option $\varphi_1(S_{t_1},S_{t_2})$ with payoff net of its price given by

$$\varphi_1(12,14)=1,\ \varphi_1(12,13)=0,\ \varphi_1(12,10)=1,\ \varphi_1(8,14)=0,\ \varphi_1(8,10)=-2,\ \varphi_1(8,6)=0$$

and we calculate

$$\mathbb{E}_{\mathbb{P}}[\varphi_1\mid S_{t_2}]=\frac{3}{4}\mathbb{1}_{\{S_{t_2}=14\}}+\frac{1}{4}\mathbb{1}_{\{S_{t_2}=10\}}\geq 0\ \mathbb{P}-\text{a.s.},\quad\text{and}\quad \mathbb{E}_{\mathbb{P}}[\varphi_1]>0. \tag{3.5}$$

This means φ_1 admits a statistical arbitrage opportunity. Further, the constraint $\mathbb{E}_{\mathbb{Q}}[\varphi_1]=0$ shrinks the set of admissible martingale measures to

$$\mathcal{Q}_{\{\varphi_1\}}^{\{\mathbb{P}\}}=\Bigg\{\mathbb{Q}\in\mathcal{M}(\mathbb{R}^6)\ \Bigg|\ \mathbb{Q}(\omega_2)=\frac{1}{3}\left(1-4\mathbb{Q}(\omega_1)\right),\ \mathbb{Q}(\omega_3)=\frac{1}{6}\left(2\mathbb{Q}(\omega_1)+1\right),$$
$$\mathbb{Q}(\omega_4)=\frac{1}{12}\left(1-4\mathbb{Q}(\omega_1)\right),\ \mathbb{Q}(\omega_5)=\frac{1}{12}\left(8\mathbb{Q}(\omega_1)+1\right),$$
$$\mathbb{Q}(\omega_6)=\frac{1}{3}\left(1-\mathbb{Q}(\omega_1)\right),\ \mathbb{Q}(\omega_1)\in[0,1/4]\Bigg\}.$$

Moreover, it holds

$$\mathbb{E}_{\mathbb{P}}\left[\frac{\mathbb{E}_{\mathbb{P}}[\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}\mid S_{t_2}]}{\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}}-1\ \middle|\ S_{t_2}\right]=\left(\frac{9}{16}\frac{\mathbb{Q}(\omega_4)}{\mathbb{Q}(\omega_1)}+\frac{1}{16}\frac{\mathbb{Q}(\omega_1)}{\mathbb{Q}(\omega_4)}-\frac{3}{8}\right)\mathbb{1}_{\{S_{t_2}=14\}}$$
$$+\left(\frac{9}{16}\frac{\mathbb{Q}(\omega_5)}{\mathbb{Q}(\omega_3)}+\frac{1}{16}\frac{\mathbb{Q}(\omega_3)}{\mathbb{Q}(\omega_5)}-\frac{3}{8}\right)\mathbb{1}_{\{S_{t_2}=10\}}\ \mathbb{P}\text{-a.s.}$$

This means that for all $\mathbb{Q}\in\mathcal{Q}_{\{\varphi_1\}}^{\{\mathbb{P}\}}$ with $\mathbb{Q}\sim\mathbb{P}$ we can find some $a\in\mathbb{R}$ such that

$$\mathbb{E}_{\mathbb{P}}[a\varphi_1\mid S_{t_2}]\geq\mathbb{E}_{\mathbb{P}}\left[\frac{\mathbb{E}_{\mathbb{P}}[\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}\mid\mathcal{G}]}{\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}}-1\ \middle|\ S_{t_2}\right]\ \mathbb{P}\text{-a.s.}$$

and Assumption 3.1 is fulfilled. We observe that the assertion of Proposition 3.2 is valid: Through adding the additional option φ_1, there exists statistical arbitrage. Indeed, we still cannot find a S_{t_2}-measurable density $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}$ for some equivalent martingale measure $\mathbb{Q}$ since the set of admissible measures has become smaller through the incorporation of φ_1.

(ii) We incorporate a call option with maturity t_2 and strike 13 traded for price $1/8$ as well as a call option with strike 10 which is traded for price $5/4$, i.e., we additionally consider

$$\varphi_1(S_{t_1},S_{t_2})=(S_{t_2}-13)^+-1/8\ \text{ and } \varphi_2(S_{t_1},S_{t_2})=(S_{t_2}-10)^+-5/4.$$

This shrinks $\mathcal{Q}_{\{\varphi_1\}}^{\{\mathbb{P}\}}$ to a singleton. The only element $\mathbb{Q}$ is given by

$$\mathbb{Q}(\omega_1)=1/16,\ \mathbb{Q}(\omega_2)=1/4,\ \mathbb{Q}(\omega_3)=3/16,$$
$$\mathbb{Q}(\omega_4)=1/16,\ \mathbb{Q}(\omega_5)=1/8,\ \mathbb{Q}(\omega_6)=5/16.$$

and it holds obviously $\mathbb{P}\ll\mathbb{Q}$ for the only extreme point $\mathbb{Q}\in\operatorname{ext}\mathcal{Q}_{\{\varphi_1\}}^{\{\mathbb{P}\}}$. Nevertheless, we have that $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\notin\sigma(S_{t_2})$ as $\mathbb{Q}(\omega_1)/\mathbb{Q}(\omega_4)\neq\mathbb{P}(\omega_1)/\mathbb{P}(\omega_4)=3$ and $\mathbb{Q}(\omega_3)/\mathbb{Q}(\omega_5)\neq$

$\mathbb{P}(\omega_3)/\mathbb{P}(\omega_5) = 3$. Thus, since the assumptions of Proposition 3.2 are fulfilled, there exist statistical arbitrage opportunities. Indeed, we can identify a statistical arbitrage strategy through

$$f = a_1\varphi_1(S_{t_1}, S_{t_2}) + a_2\varphi_2(S_{t_1}, S_{t_2}) + \Delta_0(S_{t_1} - S_{t_0}) + \Delta_1(S_{t_1})(S_{t_2} - S_{t_1})$$

by setting $a_1 = 1,\ a_2 = 2, \Delta_0 = 1,\ \Delta_1(S_{t_1}) = -2\mathbb{1}_{\{S_{t_1}=12\}} - 5/2\mathbb{1}_{\{S_{t_1}=8\}}$.

In Appendix B we derive conditions for statistical arbitrage strategies and verify that the latter choice of parameters defines an arbitrage strategy. We remark that this choice is not uniquely defined and there is an infinite amount of different statistical arbitrage strategies.

Example 3.7 (Statistical-arbitrage-free price bounds). We consider a two time-step stochastic process with initial value $S_{t_0} = 15/8$. Further, we consider the probability measure

$$\mathbb{P} = 1/4 \cdot \delta_{(1,1)} + 1/2 \cdot \delta_{(2,1)} + 1/8 \cdot \delta_{(2,2)} + 1/8 \cdot \delta_{(2,3)}.$$

Compare Figure 2 for an illustration of all possible paths. We further assume that we have no

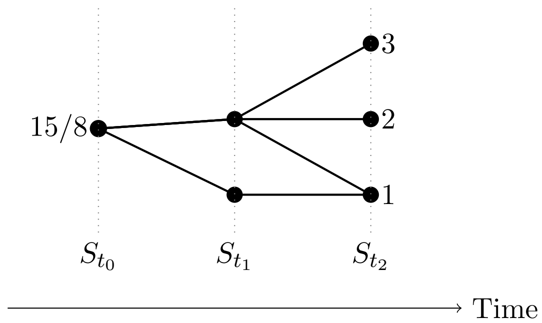

FIGURE 2. The paths with positive probability in Example 3.7.

price information on other traded options, i.e., $\{(\varphi_i)_{i=1,\dots,N}\} = \emptyset$. Let $\mathbb{Q} \in \mathcal{Q}_\emptyset^{\{\mathbb{P}\}}$. Then, due to the martingale property, it has to hold that

$$\mathbb{E}_\mathbb{Q}[S_{t_2} \mid S_{t_1} = 1] = 1 \text{ as well as } \mathbb{E}_\mathbb{Q}[S_{t_2} \mid S_{t_1} = 2] = 2 \text{ and } \mathbb{E}_\mathbb{Q}[S_{t_1}] = S_{t_0}.$$

If $\mathbb{Q} \ll \mathbb{P}$, then we obtain the particular form

$$\mathbb{Q} = q_{11} \cdot \delta_{(1,1)} + q_{21} \cdot \delta_{(2,1)} + q_{22} \cdot \delta_{(2,2)} + q_{23} \cdot \delta_{(2,3)} \tag{3.6}$$

with coefficients which fulfil $\sum_{i,j} q_{ij} = 1,\ q_{ij} \geq 0$. Thus, the martingale conditions write as $q_{21} = q_{23}$ and $q_{11} + 2\,(q_{21} + q_{22} + q_{23}) = 15/8$. The set of admissible martingale measures is given by

$$\mathcal{Q}_\emptyset^{\{\mathbb{P}\}} = \left\{ \mathbb{Q} \text{ as in (3.6)} \;\middle|\; q_{11} = 1/8,\ q_{21} = q_{23},\ q_{22} = 7/8 - 2q_{21}, \text{ and } \ q_{21} \in [0, 7/16] \right\}.$$

In this situation, we compute for $\mathbb{Q} \in \mathcal{Q}_\emptyset^{\{\mathbb{P}\}}$ with $\mathbb{Q} \sim \mathbb{P}$, i.e., for $q_{21} \in (0, 7/16)$ the following identity

$$\mathbb{E}_\mathbb{P}\left[\frac{\mathbb{E}_\mathbb{P}[\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \mid S_{t_2}]}{\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}} - 1 \;\middle|\; S_{t_2} \right] = \frac{1}{9}\left(8q_{21} + \frac{1}{2q_{21}} - 4\right)\mathbb{1}_{\{S_{t_2}=1\}} \geq 0 \ \mathbb{P}\text{-a.s.}.$$

The latter line vanishes for the specific choice $q_{21} = 1/4$ which shows that Assumption 3.1 is fulfilled. Next, we want to consider price bounds for derivatives which do not allow for statistical arbitrage. We observe that

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \in \sigma(S_{t_2}) \Leftrightarrow \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(S_{t_1} = 1, S_{t_2} = 1) = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(S_{t_1} = 2, S_{t_2} = 1) \Leftrightarrow \frac{q_{21}}{q_{11}} = 2.$$

We account for this additional linear constraint, when optimizing w.r.t. all $\sigma(S_{t_2})$-measurable densities. Consider some derivative with payoff function $c(x_1, x_2) = |x_2 - x_1|$ for which we want to compute its statistical arbitrage-free price. We obtain as minimal and maximal solution of the linear systems

$$\inf_{\mathbb{Q}\in\mathcal{Q}_\emptyset^{\{\mathbb{P}\}}} \{\mathbb{E}_\mathbb{Q}[c] : \mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \in \sigma(S_{t_2}) : \mathbb{Q} \sim \mathbb{P}\} = 0.5$$

and the same maximal price

$$\sup_{\mathbb{Q}\in\mathcal{Q}_\emptyset^{\{\mathbb{P}\}}} \{\mathbb{E}_\mathbb{Q}[c] : \mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \in \sigma(S_{t_2}) : \mathbb{Q} \sim \mathbb{P}\} = 0.5,$$

since the single possible measure fulfilling all constraints is given by

$$\mathbb{Q} = \frac{1}{8}\delta_{(1,1)} + \frac{1}{4}\delta_{(2,1)} + \frac{3}{8}\delta_{(2,2)} + \frac{1}{4}\delta_{(2,3)}.$$

In contrast, the no-arbitrage bounds (i.e. the extremal solutions of the linear system without the constraint $\frac{q_{21}}{q_{11}} = 2$) are significantly wider

$$\inf_{\mathbb{Q}\in\mathcal{Q}_\emptyset^{\{\mathbb{P}\}}} \mathbb{E}_\mathbb{Q}[c] = 0 \quad \text{and} \quad \sup_{\mathbb{Q}\in\mathcal{Q}_\emptyset^{\{\mathbb{P}\}}} \mathbb{E}_\mathbb{Q}[c] = 0.875.$$

Next, we show how to exploit statistical arbitrage if the derivative price differs from the only possible statistical arbitrage-free price 0.5. For this, assume c is traded at price 0.25, then borrow 0.25, buy c and invest in a self-financing strategy $(\Delta_0, \Delta_1) = (2, 0)$. The payoff of this strategy is given by $f(S_{t_1}, S_{t_2}) = |S_{t_2} - S_{t_1}| - 0.25 + 2(S_{t_1} - S_{t_0})$. One obtains

$$\mathbb{E}_\mathbb{P}\left[f(S_{t_1}, S_{t_2}) \mid S_{t_2}\right] = \mathbb{1}_{\{S_{t_2}=3\}} \geq 0 \text{ } \mathbb{P}\text{-a.s.}, \quad \text{and} \quad \mathbb{E}_\mathbb{P}\left[f(S_{t_1}, S_{t_2})\right] = 1/8 > 0,$$

and thus a statistical arbitrage opportunity. However, since the price for c lies within the arbitrage-free bounds, the strategy is not a $\mathbb{P}$-arbitrage strategy. Indeed, it holds conditional on the outcome $\{(S_{t_1}, S_{t_2}) = (1, 1)\}$ that

$$f(1, 1) = -2 < 0.$$

We extend this setting in Example 4.8 by allowing for a broader class of physical measures.

Example 3.8 (Complete models). We consider some time-discrete model with price process $(S_t)_t$ under $\mathbb{P}$. Further we have no price information on traded options, i.e., $\Phi = \emptyset$. Assume the model is complete and arbitrage-free, then there exists a unique $\mathbb{Q} \in \mathcal{Q}_\emptyset^{\{\mathbb{P}\}}$ such that $\mathbb{Q} \sim \mathbb{P}$. If it holds that $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \in \mathcal{G}$, then there exists a unique $\mathcal{G}$-arbitrage-free price for each claim which necessarily coincides with the arbitrage-free price. If $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \notin \mathcal{G}$, then Corollary 3.4 is not applicable and we cannot find a $\mathcal{G}$-arbitrage-free price.

4. $\mathcal{G}$-Arbitrage under Uncertainty w.r.t. $\mathbb{P}$

In this section, we generalize the previous results to the situation when there is uncertainty about the physical measure $\mathbb{P}$. Therefore, we introduce a set of probability measures $\mathcal{P}$ which serve as potential physical measures, and we refer to this as the ambiguity set. First, we face the most general case and assume the measures contained in $\mathcal{P}$ are not necessarily required to be dominated nor parametrized. Moreover, as in the previous section, we fix a set of statically tradable options Φ which can be bought and sold at price zero.

4.1. **Absence of $\mathcal{P}$-Robust $\mathcal{G}$-Arbitrage.** Similar to the previous section we pose the following assumption.

Assumption 4.1. *There exist $a_i \in \mathbb{R}$, $\Delta_j \in B(\mathbb{R}^j)$ such that for all $\mathbb{P} \in \mathcal{P}$ and some $\mathbb{Q} \in \mathcal{Q}_\Phi^\mathcal{P}$ with $\mathbb{Q} \sim \mathbb{P}$ it holds*

$$\mathbb{E}_\mathbb{P}\left[\sum_{i=1}^N a_i\varphi_i + (\Delta\cdot S)_n \;\middle|\; \mathcal{G}\right] \geq \mathbb{E}_\mathbb{P}\left[\frac{\mathbb{E}_\mathbb{P}[\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \mid \mathcal{G}]}{\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}} - 1 \;\middle|\; \mathcal{G}\right] \text{ } \mathbb{P}\text{-a.s.} \tag{4.1}$$

The latter assumption imposes two major requirements. First, the existence of an equivalent consistent martingale measure for all $\mathbb{P} \in \mathcal{P}$ and second the conditional super-replication expressed by (4.1). The assumption on the existence of some $\mathbb{P}$-equivalent martingale measure $\mathbb{Q} \in \mathcal{Q}_{(\varphi_i)}^{\mathcal{P}}$ is tantamount to the absence of $\{\mathbb{P}\}$-robust arbitrage. Thus, the assumption can equivalently be understood as non-existence of $\{\mathbb{P}\}$-robust arbitrage opportunities for all $\mathbb{P} \in \mathcal{P}$.

Remark 4.2. Absence of $\{\mathbb{P}\}$-robust arbitrage for all $\mathbb{P} \in \mathcal{P}$ directly implies absence of $\mathcal{P}$-robust arbitrage. If all $\mathbb{P} \in \mathcal{P}$ are even equivalent, then absence of $\{\mathbb{P}\}$-robust arbitrage for all $\mathbb{P} \in \mathcal{P}$ even coincides with absence of $\mathcal{P}$-robust arbitrage:
To see this, assume absence of $\mathcal{P}$-robust arbitrage and assume f to be some $\{\mathbb{P}\}$-robust arbitrage strategy. Then, since we exclude $\mathcal{P}$-robust arbitrage, there exists some $\mathbb{P}' \in \mathcal{P}$ s.t. $\mathbb{P}'(f < 0) > 0$. By the equivalence of the measures this however leads to the contradiction $\mathbb{P}(f < 0) > 0$.

The situation that all physical measures in $\mathcal{P}$ are equivalent appears often in practice when $\mathcal{P}$ is implied by some parametric financial model.

Absence of $\{\mathbb{P}\}$-robust arbitrage for all $\mathbb{P} \in \mathcal{P}$ is, by definition, a weaker notion than absence of $\mathcal{P}$-robust $\mathcal{G}$-arbitrage. Thus, it appears sensible to assume absence of $\{\mathbb{P}\}$-robust arbitrage for all $\mathbb{P} \in \mathcal{P}$ when characterizing absence of $\mathcal{P}$-robust $\mathcal{G}$-arbitrage as it is done in Proposition 4.3.

Further, inequality (4.1) means that, given information on $\mathcal{G}$, the market with its traded options is rich enough to super-replicate the right-hand side for price zero. Obviously this assumption is weaker than a completeness assumption which is required for the main result from Rein et. al. (2019). However, we stress that this requirement may shrink the set of possible ambiguity sets $\mathcal{P}$ considerably.

In addition to the discussion in Remark 3.3 which mainly can be extended to the situation at hand, we show within Example 4.7 how even for an undominated set of physical measures $\mathcal{P}$ Assumption 4.1 can be met by allowing for common options to be considered as part of the trading strategies used for the super-replication. Assumption 4.1 now allows to extend Proposition 3.2.

Proposition 4.3 (Characterization of $\mathcal{P}$-Robust $\mathcal{G}$-Arbitrage)**.** *Let Assumption 4.1 be valid. Then the following are equivalent.*

(i) There is no $\mathcal{P}$-robust $\mathcal{G}$-arbitrage.

(ii) For all $\mathbb{P} \in \mathcal{P}$ there exists some $\mathbb{Q} \in \mathcal{Q}_{\Phi}^{\mathcal{P}}$ with $\mathbb{P} \sim \mathbb{Q}$ such that the density $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}$ is $\mathcal{G}$-measurable.

Proof. First assume absence of $\mathcal{P}$-robust $\mathcal{G}$-arbitrage and pick some $\mathbb{P} \in \mathcal{P}$.
Under Assumption 4.1 there exists some $\mathbb{Q} \in \mathcal{Q}_{\Phi}^{\mathcal{P}}$ such that $\mathbb{Q} \sim \mathbb{P}$ and $Z := \frac{\mathbb{E}_{\mathbb{P}}[\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \mid \mathcal{G}]}{\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}} - 1$ can be super-replicated conditional on $\mathcal{G}$. We obtain

$$\mathbb{E}_{\mathbb{P}}[Z \mid \mathcal{G}] \leq \mathbb{E}_{\mathbb{P}}\left[\sum_{i=1}^{N} a_i\varphi_i + (\Delta \cdot S)_n \;\middle|\; \mathcal{G}\right] \ \mathbb{P}\text{-a.s.} \tag{4.2}$$

for some $a_i \in \mathbb{R}$, $\Delta_j \in B(\mathbb{R}^j)$. Similar to the argumentation in the proof of [KL17, Proposition 6], we get by applying Jensen's inequality for conditional expectations to $x \mapsto 1/x$ that

$$\mathbb{E}_{\mathbb{P}}\left[\sum_{i=1}^{N} a_i\varphi_i + (\Delta \cdot S)_n \;\middle|\; \mathcal{G}\right] \geq \mathbb{E}_{\mathbb{P}}\left[\frac{\mathbb{E}_{\mathbb{P}}[\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \mid \mathcal{G}]}{\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}} - 1 \;\middle|\; \mathcal{G}\right] \geq \frac{\mathbb{E}_{\mathbb{P}}[\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \mid \mathcal{G}]}{\mathbb{E}_{\mathbb{P}}[\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \mid \mathcal{G}]} - 1 = 0 \tag{4.3}$$

with equality iff $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \in \mathcal{G}$. By assumption, (4.3) holds $\mathbb{P}$-almost surely for all $\mathbb{P} \in \mathcal{P}$. But absence of $\mathcal{P}$-robust $\mathcal{G}$-arbitrage implies that left-hand side and right-hand side vanish and thus we indeed have $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \in \mathcal{G}$.

Conversely, pick some $f = \sum_{i=1}^{N} a_i\varphi_i(S_{t_1}, \dots, S_{t_n}) + (\Delta \cdot S)_n$ such that $\mathbb{E}_{\tilde{\mathbb{P}}}[f \mid \mathcal{G}] \geq 0$ $\tilde{\mathbb{P}}$-a.s. for all $\tilde{\mathbb{P}} \in \mathcal{P}$. Pick some $\mathbb{P} \in \mathcal{P}$, then there exists by assumption $\mathbb{Q} \in \mathcal{Q}_{\Phi}^{\mathcal{P}}$ with $\mathbb{Q} \sim \mathbb{P}$ and a $\mathcal{G}$-measurable density $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}$. By the $\mathcal{G}$-measurability of the density $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}$, we have for all

$\mathbb{P}$- and $\mathbb{Q}$-integrable random variables X the following identity $\mathbb{E}_{\mathbb{P}}[X \mid \mathcal{G}] = \mathbb{E}_{\mathbb{Q}}[X \mid \mathcal{G}]$ $\mathbb{Q}$-a.s. (compare [KL17, Proposition 2]). This leads to

$$0 = \mathbb{E}_{\mathbb{Q}}[f] = \mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}[f \mid \mathcal{G}]] = \mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{P}}[f \mid \mathcal{G}]]. \tag{4.4}$$

The latter line (4.4) implies $\mathbb{E}_{\mathbb{P}}[f \mid \mathcal{G}] = 0$ $\mathbb{Q}$-a.s. and therefore also $\mathbb{P}$-almost surely. Since $\mathbb{P} \in \mathcal{P}$ was chosen arbitrarily, there exists no $\mathcal{P}$-robust $\mathcal{G}$-arbitrage and the assertion is shown. ☐

4.2. **$\mathcal{P}$-Robust $\mathcal{G}$-Arbitrage-Free Valuation of Derivatives.** As in the model-dependent case of Section 3, the primal problem $P(\mathcal{G}, c)$ defines the largest price for c which does not allow for $\mathcal{G}$-arbitrage in the augmented market.

$$\begin{aligned} P(\mathcal{G}, c) : = \inf \Bigg\{ & d \in \mathbb{R} : \ \exists (a_i)_i \in \mathbb{R}, \ (\Delta_j)_j \in B(\mathbb{R}^j) \text{ such that} \\ & d + \mathbb{E}_{\mathbb{P}} \left[\sum_{i=1}^N a_i \varphi_i + (\Delta \cdot S)_n \;\middle|\; \mathcal{G} \right] \geq \mathbb{E}_{\mathbb{P}}[c \mid \mathcal{G}] \ \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}, \\ & d + \mathbb{E}_{\mathbb{P}} \left[\sum_{i=1}^N a_i \varphi_i + (\Delta \cdot S)_n \right] > \mathbb{E}_{\mathbb{P}}[c] \text{ for some } \mathbb{P} \in \mathcal{P} \Bigg\}. \end{aligned} \tag{4.5}$$

Analogously to the previous derivations, we show that this problem can be solved by the consideration of a dual problem. We consider the dual problem for the derivation of the maximal price $p_{max}(c)$ of a derivative $c(S_{t_1}, \dots, S_{t_n})$ such that the market complemented by this derivative does not allow for $\mathcal{P}$-robust $\mathcal{G}$-arbitrage, i.e, we consider the problem

$$D(\mathcal{G}, c) := \sup_{\mathbb{Q} \in \mathcal{Q}_\Phi^{\mathcal{P}}} \{\mathbb{E}_{\mathbb{Q}}[c] : \mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \in \mathcal{G} \text{ and } \mathbb{P} \sim \mathbb{Q} \text{ for some } \mathbb{P} \in \mathcal{P}\} = p_{max}(c). \tag{4.6}$$

The characterization of $\mathcal{P}$-robust $\mathcal{G}$-arbitrage in Proposition 4.3 then directly implies the following super-replication statement.

Corollary 4.4 ($\mathcal{P}$-Robust Super-Replication)**.** *Assume $\mathcal{Q}_\Phi^{\mathcal{P}} \neq \emptyset$ and let Assumption 4.1 be valid for the market with securities Φ as well as for traded securities $\Phi \cup \{c - p(c)\}$ for any $p(c) \in [D(\mathcal{G}, c), P(\mathcal{G}, c)]$ of payoff c. If the market is $\mathcal{G}$-arbitrage free, then we have*

$$D(\mathcal{G}, c) = P(\mathcal{G}, c).$$

Proof. The idea of the proof is similar to the proof of the super-replication theorem in model-independent finance stated in [ABPS16, Theorem 1.4]. We first show $P(\mathcal{G}, c) \geq D(\mathcal{G}, c)$. Consider some payoff

$$f(x_1, \dots, x_n) = d + \sum_{i=1}^N a_i \varphi_i(x_1, \dots, x_n) + \sum_{i=0}^{n-1} \Delta_i(x_1, \dots, x_i)(x_{i+1} - x_i)$$

such that $\mathbb{E}_{\mathbb{P}}[f \mid \mathcal{G}] \geq \mathbb{E}_{\mathbb{P}}[c \mid \mathcal{G}]$ holds $\mathbb{P}$-almost surely for all $\mathbb{P} \in \mathcal{P}$ as well as $\mathbb{E}_{\mathbb{P}}[f] > \mathbb{E}_{\mathbb{P}}[c]$ for some $\mathbb{P} \in \mathcal{P}$. Pick some $\mathbb{Q} \in \mathcal{Q}_\Phi^{\mathcal{P}}$ with $\mathbb{Q} \sim \tilde{\mathbb{P}}$ for some $\tilde{\mathbb{P}} \in \mathcal{P}$ such that $\mathrm{d}\mathbb{Q}/\mathrm{d}\tilde{\mathbb{P}} \in \mathcal{G}$. We obtain

$$\begin{aligned} d = \mathbb{E}_{\mathbb{Q}}[f] &= \mathbb{E}_{\tilde{\mathbb{P}}} \left[\mathbb{E}_{\tilde{\mathbb{P}}}[\mathrm{d}\mathbb{Q}/\mathrm{d}\tilde{\mathbb{P}} \cdot f \mid \mathcal{G}] \right] = \mathbb{E}_{\tilde{\mathbb{P}}} \left[\mathrm{d}\mathbb{Q}/\mathrm{d}\tilde{\mathbb{P}} \cdot \mathbb{E}_{\tilde{\mathbb{P}}}[f \mid \mathcal{G}] \right] \\ &\geq \mathbb{E}_{\tilde{\mathbb{P}}} \left[\mathrm{d}\mathbb{Q}/\mathrm{d}\tilde{\mathbb{P}} \cdot \mathbb{E}_{\tilde{\mathbb{P}}}[c \mid \mathcal{G}] \right] = \mathbb{E}_{\mathbb{Q}}[c]. \end{aligned}$$

Thus, $P(\mathcal{G}, c) \geq D(\mathcal{G}, c)$. Next, we want to show that indeed equality holds. Striving for a contradiction, we assume the inequality is strict, i.e., we assume there exists some price $p(c) \in \mathbb{R}$ of payoff c such that $P(\mathcal{G}, c) > p(c) > D(\mathcal{G}, c)$. Define an additional financial instrument by $\varphi = p(c) - c$ and add it to the market. Then, we want to apply Proposition 4.3 to the extended market. By assumption, the extended market fulfils Assumption 4.1. Therefore, we conclude that the following statements are equivalent.

(i) There exists no $\mathcal{P}$-robust $\mathcal{G}$-arbitrage, i.e., there is no

$$f(x_1, \dots, x_n) = \sum_{i=1}^N a_i \varphi_i(x_1, \dots, x_n) + a_0 \varphi(x_1, \dots, x_n) + \sum_{i=0}^{n-1} \Delta_i(x_1, \dots, x_i)(x_{i+1} - x_i) \tag{4.7}$$

for some $a_i, a_0 \in \mathbb{R},\ \Delta_i \in B(\mathbb{R}^i)$ for $i = 1, \ldots, N$, such that $\mathbb{E}_{\mathbb{P}}[f \mid \mathcal{G}] \geq 0$ $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}$ and $\mathbb{E}_{\mathbb{P}}[f] > 0$ for some $\mathbb{P} \in \mathcal{P}$.

(ii) For all $\mathbb{P} \in \mathcal{P}$ there exists some $\mathbb{Q} \in \mathcal{Q}^{\mathcal{P}}_{\Phi \cup \{\varphi\}}$ s.t. $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \in \mathcal{G}$ and $\mathbb{Q} \sim \mathbb{P}$.

If there exists no $\mathcal{P}$-robust $\mathcal{G}$-arbitrage, then by (ii) for all $\mathbb{P} \in \mathcal{P}$ there exists some $\mathbb{Q} \in \mathcal{Q}^{\mathcal{P}}_{\Phi}$ with $\mathbb{Q} \sim \mathbb{P}$ such that $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \in \mathcal{G}$ and $\mathbb{E}_{\mathbb{Q}}[\varphi] = 0$. This means $\mathbb{E}_{\mathbb{Q}}[c] = p(c)$ which implies, in particular, that $p(c) \leq P(\mathcal{G}, c)$ contradicting the assumption.

In the other case, if there does exist $\mathcal{P}$-robust $\mathcal{G}$-arbitrage, then there exists some f as in (4.7) such that $\mathbb{E}_{\mathbb{P}}[f \mid \mathcal{G}] \geq 0$ $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}$ as well as $\mathbb{E}_{\mathbb{P}}[f] > 0$ for some $\mathbb{P} \in \mathcal{P}$. This implies for $a_0 > 0$ that

$$\mathbb{E}_{\mathbb{P}}[f/a_0 + c|\mathcal{G}] = p(c) + \mathbb{E}_{\mathbb{P}}\left[\sum_{i=1}^N (a_i/a_0)\varphi_i + (\Delta/a_0 \cdot S)_n \,\middle|\, \mathcal{G}\right] \geq \mathbb{E}_{\mathbb{P}}[c \mid \mathcal{G}] \text{ } \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P},$$

and

$$\mathbb{E}_{\mathbb{P}}[f/a_0 + c] = p(c) + \mathbb{E}_{\mathbb{P}}\left[\sum_{i=1}^N (a_i/a_0)\varphi_i + (\Delta/a_0 \cdot S)_n\right] > \mathbb{E}_{\mathbb{P}}[c] \text{ for some } \mathbb{P} \in \mathcal{P}.$$

Therefore, $p(c) \geq P(\mathcal{G}, c)$ which contradicts the assumption.
If $a_0 < 0$, then we similarly may derive

$$-p(c) \geq P(\mathcal{G}, -c),$$

or equivalently,

(4.8)

$$\begin{aligned}
&p(c) \leq -P(\mathcal{G}, -c) \\
&\quad = -\inf \Bigg\{ d \in \mathbb{R} :\ \exists (a_i)_i \in \mathbb{R},\ (\Delta_j)_j \in B(\mathbb{R}^j) \text{ s.t. for } f = d + \sum_{i=1}^N a_i\varphi_i + (\Delta \cdot S)_n \text{ we have} \\
&\qquad\qquad \mathbb{E}_{\mathbb{P}}\left[f \mid \mathcal{G}\right] \geq \mathbb{E}_{\mathbb{P}}[-c \mid \mathcal{G}] \text{ } \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}, \quad \mathbb{E}_{\mathbb{P}}\left[f\right] > \mathbb{E}_{\mathbb{P}}[-c] \text{ for some } \mathbb{P} \in \mathcal{P} \Bigg\} \\
&\quad = -\inf \Bigg\{ -d \in \mathbb{R} :\ \exists (a_i)_i \in \mathbb{R},\ (\Delta_j)_j \in B(\mathbb{R}^j) \text{ s.t. for } f = d + \sum_{i=1}^N a_i\varphi_i + (\Delta \cdot S)_n \text{ we have} \\
&\qquad\qquad \mathbb{E}_{\mathbb{P}}\left[f \mid \mathcal{G}\right] \leq \mathbb{E}_{\mathbb{P}}[c \mid \mathcal{G}] \text{ } \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}, \quad \mathbb{E}_{\mathbb{P}}\left[f\right] < \mathbb{E}_{\mathbb{P}}[c] \text{ for some } \mathbb{P} \in \mathcal{P} \Bigg\} \\
&\quad = \sup \Bigg\{ d \in \mathbb{R} :\ \exists (a_i)_i \in \mathbb{R},\ (\Delta_j)_j \in B(\mathbb{R}^j) \text{ s.t. for } f = d + \sum_{i=1}^N a_i\varphi_i + (\Delta \cdot S)_n \text{ we have} \\
&\qquad\qquad \mathbb{E}_{\mathbb{P}}\left[f \mid \mathcal{G}\right] \leq \mathbb{E}_{\mathbb{P}}[c \mid \mathcal{G}] \text{ } \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}, \quad \mathbb{E}_{\mathbb{P}}\left[f\right] < \mathbb{E}_{\mathbb{P}}[c] \text{ for some } \mathbb{P} \in \mathcal{P} \Bigg\}
\end{aligned}$$

(4.9)

$$\begin{aligned}
&\quad \leq \inf_{\mathbb{Q} \in \mathcal{Q}^{\mathcal{P}}_{\Phi}} \{\mathbb{E}_{\mathbb{Q}}[c] : \mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \in \mathcal{G} : \mathbb{P} \sim \mathbb{Q} \text{ for some } \mathbb{P} \in \mathcal{P}\} \\
&\quad \leq \sup_{\mathbb{Q} \in \mathcal{Q}^{\mathcal{P}}_{\Phi}} \{\mathbb{E}_{\mathbb{Q}}[c] : \mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \in \mathcal{G} : \mathbb{P} \sim \mathbb{Q} \text{ for some } \mathbb{P} \in \mathcal{P}\} = D(\mathcal{G}, c)
\end{aligned}$$

in contrast to the assumption $p(c) > P(\mathcal{G}, c)$. Inequality (4.9) can be derived completely analogue to $D(\mathcal{G}, c) \geq P(\mathcal{G}, c)$ which was already shown in the beginning of the proof.
If $a_0 = 0$, then the market with the instruments Φ but without φ is not $\mathcal{G}$-arbitrage-free, which is contradicting the assumptions of the theorem. $\square$

We obtain an alternative representation of the latter result which is especially useful for numerical implementation. It states that we may equivalently solve the single reference measure problem from Corollary 3.4 for all $\mathbb{P} \in \mathcal{P}$ and then take the maximum over all measures $\mathbb{P} \in \mathcal{P}$.

Remark 4.5 (Alternative Representation). Under the same assumptions as in Corollary 4.4, it holds that

$$
\begin{aligned}
D(\mathcal{G},c) &= \sup_{\mathbb{P}\in\mathcal{P}} \sup_{\mathbb{Q}\in\mathcal{Q}_\Phi^{\{\mathbb{P}\}}} \{\mathbb{E}_\mathbb{Q}[c] : \mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}\in\mathcal{G} \text{ and } \mathbb{Q}\sim\mathbb{P}\} \\
&= \sup_{\mathbb{P}\in\mathcal{P}} \Bigg\{ d\in\mathbb{R} :\ \exists (a_i)_i \in \mathbb{R},\ (\Delta_j)_j \in B(\mathbb{R}^j) \text{ s.t.} \\
&\qquad d + \mathbb{E}_\mathbb{P}\left[\sum_{i=1}^N a_i\varphi_i + (\Delta\cdot S)_n \,\middle|\, \mathcal{G}\right] \geq \mathbb{E}_\mathbb{P}[c \mid \mathcal{G}] \ \mathbb{P}\text{-a.s.}, \\
&\qquad d + \mathbb{E}_\mathbb{P}\left[\sum_{i=1}^N a_i\varphi_i + (\Delta\cdot S)_n\right] > \mathbb{E}_\mathbb{P}[c] \Bigg\} = P(\mathcal{G},c).
\end{aligned} \tag{4.10}
$$

The equality in the first line of (4.10) follows by definition and because we have

$$\left\{\mathbb{Q}\in\mathcal{Q}_\Phi^{\{\mathbb{P}\}}\ :\ \mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}\in\mathcal{G} \text{ and } \mathbb{Q}\sim\mathbb{P}\right\} \neq \emptyset$$

by absence of $\mathcal{G}$-arbitrage. The second equality follows by applying Corollary 3.4 and taking the supremum over all $\mathbb{P}\in\mathcal{P}$. We stress that the identity in the second line of (4.10) only allows to compute robust price bounds, but not to compute optimal robust strategies.

4.3. **$\mathcal{P}$-Robust Statistical Arbitrage with Trading in Vanilla Options.** Next, we consider the following important specification of the previous setting. Assume $\mathcal{G}=\sigma(S_{t_n})$ and that Φ comprises call- and put options maturing at t_n for all strikes within a compact interval K. In this setting we require that the marginal distribution of some $\mathbb{P}\in\mathcal{P}$ at time t_n is supported on K and is equivalent w.r.t. the Lebesgue-measure. The condition on the compact support and the equivalence of the time t_n marginal imposes a strong technical restriction. However, in practice, this assumption seems natural since one obviously can only consider prices for a finite range of strikes. Thus, the risk-neutral pricing measures used for the valuation of the vanilla options that are supposed to be equivalent to $\mathbb{P}$ are also only supported on K. In the same way the assumption on the ability to observe a continuum of prices of call and put options is only a technical assumption, but no severe constraint in practice, since we can usually approximate missing prices in an adequate way, e.g. by spline interpolation.

We are able to derive a modification of Proposition 4.3 without having to impose additional conditions on the replicability as in Assumption 4.1.

Proposition 4.6 (Characterization of Statistical-Arbitrage). *We assume $\mathcal{Q}_\Phi\neq\emptyset$, $\mathbb{E}_\mathbb{P}[S_{t_i}^2]<\infty$ for all $i=1,\dots,n$ and all $\mathbb{P}\in\mathcal{P}$. Then the following are equivalent.*

(i) For all $\mathbb{P}\in\mathcal{P}$ there is no $\{\mathbb{P}\}$-robust statistical arbitrage.
(ii) For all $\mathbb{P}\in\mathcal{P}$ there exists some $\mathbb{Q}\in\mathcal{Q}_\Phi^\mathcal{P}$ with $\mathbb{P}\sim\mathbb{Q}$ such that the density $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}$ is $\sigma(S_{t_n})$-measurable.

Proof. First, we assume $\mathcal{P}=\{\mathbb{P}\}$.
Due to the density of the rational numbers in $\mathbb{R}$ it is possible to enumerate an increasing sequence of traded put and call options (and their corresponding rational strikes contained in K) denoted by $(\Phi_m)_{m\in\mathbb{N}}$ such that Φ_m contains m options and such that it holds $\overline{\bigcup_{m\in\mathbb{N}}\Phi_m}=\Phi$. Further, by the result from [BL78], measures which are consistent with the prices from Φ have a common law at time t_n. This means, we have for all integrable functions g

$$\sup_{\mathbb{Q}\in\mathcal{Q}_\Phi} \mathbb{E}_\mathbb{Q}[g(S_{t_n})] = \inf_{\mathbb{Q}\in\mathcal{Q}_\Phi} \mathbb{E}_\mathbb{Q}[g(S_{t_n})].$$

This implies in particular

$$0 = \inf_{m\in\mathbb{N}} \left(\sup_{\mathbb{Q}\in\mathcal{Q}_{\Phi_m}} \mathbb{E}_\mathbb{Q}[g(S_{t_n})] - \inf_{\mathbb{Q}\in\mathcal{Q}_{\Phi_m}} \mathbb{E}_\mathbb{Q}[g(S_{t_n})] \right). \tag{4.11}$$

Thus, for $f \in L^1(\mathbb{P})$ the sequence $\left(x_m^{(f)}\right)_{m\in\mathbb{N}}$ defined through

$$x_m^{(f)} := \sup_{\mathbb{Q}\in\mathcal{Q}_{\Phi_m}} \mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{P}}[f(S_{t_1},\dots,S_{t_n})|S_{t_n}]] - \inf_{\mathbb{Q}\in\mathcal{Q}_{\Phi_m}} \mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{P}}[f(S_{t_1},\dots,S_{t_n})|S_{t_n}]]$$

is decreasing and according to (4.11) bounded from below by $\inf_{m\in\mathbb{N}} x_m^{(f)} = 0$ which together yields

$$\lim_{m\to\infty} x_m^{(f)} = 0. \tag{4.12}$$

Let $\varepsilon > 0$ and pick some $\Delta_j \in C_b(\mathbb{R}^j)$ for $j = 1,\dots,n-1$. Then, according to (4.12), there exists some $m_\varepsilon \in \mathbb{N}$ such that we have

$$\begin{aligned}\sup_{j\in\{1,\dots,n-1\}} \Bigg(&\sup_{\mathbb{Q}\in\mathcal{Q}_{\Phi_{m_\varepsilon}}} \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{P}}[\Delta_j(S_{t_1},\dots,S_{t_j})(S_{t_{j+1}}-S_{t_j}) \mid S_{t_n}]\right] \\ &- \inf_{\mathbb{Q}\in\mathcal{Q}_{\Phi_{m_\varepsilon}}} \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{P}}[\Delta_j(S_{t_1},\dots,S_{t_j})(S_{t_{j+1}}-S_{t_j}) \mid S_{t_n}]\right]\Bigg) \leq \varepsilon.\end{aligned} \tag{4.13}$$

Absence of $\{\mathbb{P}\}$-robust statistical arbitrage in the multi-period model with traded instruments Φ_{m_ε} implies absence of $\{\mathbb{P}\}$-robust arbitrage (i.e. $\mathbb{P}$-arbitrage) in the one-period model (S_{t_0}, S_{t_n}) with traded instruments Φ_{m_ε}. To see this, assume there exists some strategy

$$g(S_{t_n}) = \sum_{i=1}^{m_\varepsilon} a_i\varphi_i(S_{t_n}) + \Delta_0(S_{t_n} - S_{t_0}) \geq 0 \ \mathbb{P}\text{-almost-surely},$$

with strict inequality for at least one event which possesses a positive probability w.r.t $\mathbb{P}$. But then, g is S_{t_n}-measurable and would also be a statistical arbitrage strategy. Hence, the fundamental theorem of asset pricing [FS16, Theorem 1.7.][4] implies the existence of some $\mathbb{P}$-equivalent martingale measure $\tilde{\mathbb{Q}}_\varepsilon$ such that $\mathbb{E}_{\tilde{\mathbb{Q}}_\varepsilon}[\varphi_i] = 0$ for $i = 1,\dots,m_\varepsilon$ and $\mathbb{E}_{\tilde{\mathbb{Q}}_\varepsilon}[\Delta_0(S_{t_n} - S_{t_0})] = 0$ for all $\Delta_0 \in \mathbb{R}$. Further, the theorem states that the density $\mathrm{d}\tilde{\mathbb{Q}}_\varepsilon/\mathrm{d}\mathbb{P}$ is bounded by some constant. We define a new $\mathbb{P}$-equivalent measure $\mathbb{Q}_\varepsilon$ through the bounded Radon-Nikodym derivative

$$\frac{\mathrm{d}\mathbb{Q}_\varepsilon}{\mathrm{d}\mathbb{P}} = \mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\tilde{\mathbb{Q}}_\varepsilon}{\mathrm{d}\mathbb{P}} \,\middle|\, S_{t_n}\right]$$

and obtain $\mathrm{d}\mathbb{Q}_\varepsilon/\mathrm{d}\mathbb{P} \in \sigma(S_{t_n})$ as well as $\mathbb{E}_{\mathbb{Q}_\varepsilon}[\varphi_i] = \mathbb{E}_{\mathbb{Q}_\varepsilon}[\Delta_0(S_{t_n} - S_{t_0})] = 0$. In a next step, we apply the one-period super-hedging theorem (see e.g. [FS16, Corollary 7.15.]) and obtain for all $j = 1,\dots,n-1$ by applying (4.13) the existence of some $\overline{d}, \overline{\Delta_0}, \overline{a_i}, \underline{d}, \underline{\Delta_0}, \underline{a_i} \in \mathbb{R}$ with $\overline{d} - \underline{d} \leq \varepsilon$ such that

$$\begin{aligned}\overline{d} + \overline{\Delta_0}(S_{t_n} - S_{t_0}) &+ \sum_{i=1}^{m_\varepsilon} \overline{a_i}\varphi_i(S_{t_n}) \\ &\geq \mathbb{E}_{\mathbb{P}}[\Delta_j(S_{t_1},\dots,S_{t_j})(S_{t_{j+1}} - S_{t_j}) \mid S_{t_n}] \\ &\geq \underline{d} + \underline{\Delta_0}(S_{t_n} - S_{t_0}) + \sum_{i=1}^{m_\varepsilon} \underline{a_i}\varphi_i(S_{t_n}).\end{aligned}$$

The latter inequalities hold $\mathbb{P}$-almost surely. In combination with absence of statistical arbitrage the above inequalities yield $\overline{d} \geq 0 \geq \underline{d}$. This leads to

$$\begin{aligned}-\varepsilon \leq \underline{d} &\leq \mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}_\varepsilon}{\mathrm{d}\mathbb{P}}\left(\mathbb{E}_{\mathbb{P}}[\Delta_j(S_{t_1},\dots,S_{t_j})(S_{t_{j+1}} - S_{t_j}) \mid S_{t_n}] - \underline{\Delta_0}(S_{t_n} - S_{t_0}) - \sum_{i=1}^{m_\varepsilon} \underline{a_i}\varphi_i(S_{t_n})\right)\right] \\ &= \mathbb{E}_{\mathbb{Q}_\varepsilon}[\Delta_j(S_{t_1},\dots,S_{t_j})(S_{t_{j+1}} - S_{t_j})].\end{aligned}$$

[4]The result does not include traded options. But in the one-period model, we may simply consider a finite amount of traded options φ_i as additional tradable assets. The martingale property then writes for these options as $\mathbb{E}_{\tilde{\mathbb{Q}}_\varepsilon}[\varphi_i] = 0$.

In the same manner we get $\mathbb{E}_{\mathbb{Q}_\varepsilon}[\Delta_j(S_{t_1},\dots,S_{t_j})(S_{t_{j+1}}-S_{t_j})] \le \varepsilon$. Now consider a sequence of positive real numbers $(\varepsilon_k)_k$ with $\varepsilon_k \downarrow 0$ for $k\to\infty$. By repeating the previous steps we obtain a sequence of measures $\mathbb{Q}_{\varepsilon_k} \in \mathcal{Q}_{\Phi_{m_{\varepsilon_k}}}$ with $\mathrm{d}\mathbb{Q}_{\varepsilon_k}/\mathrm{d}\mathbb{P} \in \sigma(S_{t_n})$, $\mathbb{E}_{\mathbb{Q}_{\varepsilon_k}}[\varphi_i]=0$ for $i=1,\dots,m_{\varepsilon_k}$ and $|\mathbb{E}_{\mathbb{Q}_{\varepsilon_k}}[\Delta_j(S_{t_1},\dots,S_{t_j})(S_{t_{j+1}}-S_{t_j})]| \le \varepsilon_k$ for all $j \le n-1$.
Next, we show the tightness of $(\mathbb{Q}_{\varepsilon_k})_k$. For this, we observe that $(\mathrm{d}\mathbb{Q}_{\varepsilon_k}/\mathrm{d}\mathbb{P})_k$ is uniformly bounded from below by some $c>0$ and from above by some $C<\infty$.

Otherwise, there exists some $x\in K$ such that $\sup_k \mathrm{d}\mathbb{Q}_{\varepsilon_k}/\mathrm{d}\mathbb{P}(x)=\infty$. But, by assumption there exists $k\in\mathbb{N}$ such that $\mathrm{d}\mathbb{Q}_{\varepsilon_{k'}}/\mathrm{d}\mathbb{P}(x)=\mathrm{d}\mathbb{Q}_{\varepsilon_k}/\mathrm{d}\mathbb{P}(x)<\infty$ for all $k'>k$. To see this, we observe that, if consistency of $\mathbb{Q}_{\varepsilon_k}$ with call (or put) options with strike x with all rational strikes in a sufficient small environment of x is required, then we have

$$\frac{\mathrm{d}\mathbb{Q}_{\varepsilon_k}}{\mathrm{d}\lambda}(x)=\frac{\mathrm{d}^2}{\mathrm{d}y^2}\left(\mathbb{E}_{\mathbb{Q}_{\varepsilon_k}}\left[(S_{t_n}-y)^+\right]\right)\Big|_{y=x},$$

where λ denotes the Lebesgue-measure on $\mathbb{R}$ and k is an index such that $\Phi_{m_{\varepsilon_k}}$ contains all the options in a small environment of k. Then, for $k'\ge k$, $\Phi_{m_{\varepsilon_{k'}}}$ contains at least the same options, i.e., $\Phi_{m_{\varepsilon_k}} \subset \Phi_{m_{\varepsilon_{k'}}}$. This fixes

$$\frac{\mathrm{d}\mathbb{Q}_{\varepsilon'_k}}{\mathrm{d}\mathbb{P}}(x)=\frac{\mathrm{d}\mathbb{Q}_{\varepsilon_{k'}}}{\mathrm{d}\lambda}(x)\cdot\frac{\mathrm{d}\lambda}{\mathrm{d}\mathbb{P}}(x)$$

for all $k'>k$. In the same way, we obtain existence of a lower bound greater than zero.

Certainly, for all $\delta>0$ there exists some compact set $\tilde{K}\subset\mathbb{R}^n$ such that $\mathbb{P}(\mathbb{R}^n\backslash\tilde{K})<\delta$. Then we obtain

$$\mathbb{Q}_{\varepsilon_k}(\mathbb{R}^n\backslash\tilde{K})=\int_{\mathbb{R}^n\backslash\tilde{K}}\frac{\mathrm{d}\mathbb{Q}_{\varepsilon_k}}{\mathrm{d}\mathbb{P}}\mathrm{d}\mathbb{P}\le C\delta.$$

This means in particular $(\mathbb{Q}_{\varepsilon_k})_k$ is tight. Then, Prokhorov's theorem implies the existence of a subsequence (which we denote with the same notation as the original sequence) and some probability measure $\mathbb{Q}$ such that $\mathbb{Q}_{\varepsilon_k}\to\mathbb{Q}$ weakly.

Now, we want to show that $\mathbb{Q}$ is the desired equivalent martingale measure with $\sigma(S_{t_n})$-measurable density.

Let $j\in\mathbb{N}$, $j\le n-1$. Then, we observe that by assumption and since $\mathrm{d}\mathbb{Q}_{\varepsilon_k}/\mathrm{d}\mathbb{P}$ is uniformly bounded, it holds $\mathbb{E}_{\mathbb{Q}_{\varepsilon_k}}\left[\left(\Delta_j(S_{t_1},\dots,S_{t_j})(S_{t_{j+1}}-S_{t_j})\right)^2\right]<\infty$ for all $k\in\mathbb{N}$. We apply [FS16, Lemma 5.61] and get

$$\left|\mathbb{E}_{\mathbb{Q}}[\Delta_j(S_{t_1},\dots,S_{t_j})(S_{t_{j+1}}-S_{t_j})]\right|=\lim_{k\to\infty}\left|\mathbb{E}_{\mathbb{Q}_{\varepsilon_k}}[\Delta_j(S_{t_1},\dots,S_{t_j})(S_{t_{j+1}}-S_{t_j})]\right|\le\lim_{k\to\infty}\varepsilon_k=0.$$

Completely analogue we get with $\mathbb{E}_{\mathbb{Q}_{\varepsilon_k}}[\varphi_i^2]<\infty$ for all $i=1,\dots,N$ that

$$|\mathbb{E}_{\mathbb{Q}}[\varphi_i]|=\lim_{k\to\infty}|\mathbb{E}_{\mathbb{Q}_{\varepsilon_k}}[\varphi_i]|=0.$$

The bounded densities imply that weak convergence preserves absolute continuity w.r.t. $\mathbb{P}$. Indeed: We define for $g\in C_b(\mathbb{R}^n)$ the functional $F(g)=\int g\mathrm{d}\mathbb{Q}$. The Cauchy-Schwarz inequality then leads to

$$|F(g)|=\lim_{k\to\infty}\left|\int_{\mathbb{R}^n}g\frac{\mathrm{d}\mathbb{Q}_{\varepsilon_k}}{\mathrm{d}\mathbb{P}}\mathrm{d}\mathbb{P}\right|\le C\|g\|_{L^2(\mathbb{P})}.$$

Thus F is bounded and, according to the Hahn-Banach theorem, can be extended to the Hilbert space $L^2(\mathbb{P})$. The Riesz' representation theorem implies the existence of some $f\in L^2(\mathbb{P})$ such that for all $g\in L^2(\mathbb{P})$ (and thus for all $g\in C_b(\mathbb{R}^n)$) it holds $F(g)=\langle f,g\rangle_{L^2(\mathbb{P})}=\int fg\mathrm{d}\mathbb{P}$, i.e., $f=\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}$ $\mathbb{P}$-a.s., i.e., $\mathbb{Q}\ll\mathbb{P}$. In the same way we get $\mathbb{P}\ll\mathbb{Q}$ by considering the functional $\tilde{F}(g)=\int g\mathrm{d}\mathbb{P}$ and observing that

$$\left|\tilde{F}(g)\right|=\left|\int_{\mathbb{R}^n}g\mathrm{d}\mathbb{P}\right|=\lim_{k\to\infty}\left|\int_{\mathbb{R}^n}g\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}_{\varepsilon_k}}\mathrm{d}\mathbb{Q}_{\varepsilon_k}\right|\le\frac{1}{c}\lim_{k\to\infty}\int_{\mathbb{R}^n}g^2\mathrm{d}\mathbb{Q}_{\varepsilon_k}=\frac{1}{c}\|g\|_{L^2(\mathbb{Q})}$$

for $g \in C_b(\mathbb{R}^n)$. It remains to show that $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \in \sigma(S_{t_n})$. Pick some $f \in C_b(\mathbb{R}^n)$. Then, we observe that by (4.12) and (4.13) there exists some $k \in \mathbb{N}$ and corresponding $\varepsilon_k > 0$ such that

$$\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{P}}[f(S_{t_1},\dots,S_{t_n}) \mid S_{t_n}]\right] - \mathbb{E}_{\mathbb{Q}_{\varepsilon_k}}\left[\mathbb{E}_{\mathbb{P}}[f(S_{t_1},\dots,S_{t_n}) \mid S_{t_n}]\right] \leq \varepsilon_k.$$

If necessary we scale ε_k appropriately to ensure that the latter inequality also holds for all indices greater than k. The $\sigma(S_{t_n})$-measurability of $\mathrm{d}\mathbb{Q}_{\varepsilon_k}/\mathrm{d}\mathbb{P}$, applied to [KL17, Proposition 2] leads then to

$$\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{P}}[f(S_{t_1},\dots,S_{t_n}) \mid S_{t_n}]\right] - \mathbb{E}_{\mathbb{Q}_{\varepsilon_k}}\left[f(S_{t_1},\dots,S_{t_n})\right] \leq \varepsilon_k.$$

We let $k \to \infty$ and obtain by using the weak convergence of $\mathbb{Q}_{\varepsilon_k}$ that

$$\mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{P}}[f(S_{t_1},\dots,S_{t_n}) \mid S_{t_n}]] = \mathbb{E}_{\mathbb{Q}}[f(S_{t_1},\dots,S_{t_n})]. \tag{4.14}$$

Next, we borrow some arguments from the proof of [KL17, Proposition 2]. Let $h \in C_b(\mathbb{R})$ be arbitrary. Then we get through an application of (4.14)

$$\mathbb{E}_{\mathbb{Q}}[f(S_{t_1},\dots,S_{t_n}) \cdot h(S_{t_n})] = \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{P}}[f(S_{t_1},\dots,S_{t_n}) \mid S_{t_n}] \cdot h(S_{t_n})\right].$$

This means $\mathbb{E}_{\mathbb{Q}}[f \mid S_{t_n}] = \mathbb{E}_{\mathbb{P}}[f \mid S_{t_n}]$. We get

$$\begin{aligned}
0 &= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}[f(S_{t_1},\dots,S_{t_n}) \mid S_{t_n}] - \mathbb{E}_{\mathbb{P}}[f(S_{t_1},\dots,S_{t_n}) \mid S_{t_n}]\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[f(S_{t_1},\dots,S_{t_n})\right] - \mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}[f(S_{t_1},\dots,S_{t_n}) \mid S_{t_n}]\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right] &(4.15)\\
&= \mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} f(S_{t_1},\dots,S_{t_n})\right] - \mathbb{E}_{\mathbb{P}}\left[f(S_{t_1},\dots,S_{t_n})\mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \,\middle|\, S_{t_n}\right]\right] &(4.16)\\
&= \mathbb{E}_{\mathbb{P}}\left[\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} - \mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \,\middle|\, S_{t_n}\right]\right) f(S_{t_1},\dots,S_{t_n})\right]
\end{aligned}$$

where the equality between (4.15) and (4.16) follows with the tower-property of the conditional expectation. As f was chosen arbitrarily, this implies $\mathbb{P}$-almost surely that

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \,\middle|\, S_{t_n}\right] \in \sigma(S_{t_n}).$$

We still need to show that $\mathbb{Q}$ fulfils the martingale property. For this we pick some arbitrary $\tilde{\Delta}_j \in C_b(\mathbb{R}^j)$ for $j = 1,\dots,n-1$. Then by a repeated application of the preceding steps we obtain some measure $\tilde{\mathbb{Q}}$ with the same marginal as $\mathbb{Q}$ at time t_n, with $\tilde{\mathbb{Q}} \sim \mathbb{P}$, $\mathrm{d}\tilde{\mathbb{Q}}/\mathrm{d}\mathbb{P} \in \sigma(S_{t_n})$ and which fulfils

$$\mathbb{E}_{\tilde{\mathbb{Q}}}[\tilde{\Delta}_j(S_{t_1},\dots,S_{t_j})(S_{t_{j+1}} - S_{t_j})] = 0 \text{ for } j = 1,\dots,n-1.$$

We observe that $\mathrm{d}\tilde{\mathbb{Q}}/\mathrm{d}\mathbb{P} \in \sigma(S_{t_n})$ combined with $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \in \sigma(S_{t_n})$ implies $\mathrm{d}\tilde{\mathbb{Q}}/\mathrm{d}\mathbb{P} \cdot \mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q} = \mathrm{d}\tilde{\mathbb{Q}}/\mathrm{d}\mathbb{Q} \in \sigma(S_{t_n})$. But this means

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}[\tilde{\Delta}_j(S_{t_1},\dots,S_{t_j})(S_{t_{j+1}} - S_{t_j})] \\
&\quad = \mathbb{E}_{\tilde{\mathbb{Q}}}\left[\mathbb{E}_{\mathbb{Q}}[\tilde{\Delta}_j(S_{t_1},\dots,S_{t_j})(S_{t_{j+1}} - S_{t_j}) \mid S_{t_n}]\right] &(4.17)\\
&\quad = \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}}\tilde{\Delta}_j(S_{t_1},\dots,S_{t_j})(S_{t_{j+1}} - S_{t_j}) \,\middle|\, S_{t_n}\right]\right] \\
&\quad = \mathbb{E}_{\tilde{\mathbb{Q}}}\left[\tilde{\Delta}_j(S_{t_1},\dots,S_{t_j})(S_{t_{j+1}} - S_{t_j})\right] = 0.
\end{aligned}$$

Equality (4.17) follows since $\mathbb{Q}$ and $\tilde{\mathbb{Q}}$ possess the same t_n-marginals. Taken as a whole, this shows the martingale property of $\mathbb{Q}$. Now assume $\mathcal{P}$ is not a singleton, then the assertion follows straightforward.

The proof of $(ii) \Rightarrow (i)$ is similar to the proof of this implication in Proposition 4.3. $\square$

4.4. **Examples.** In this section, we first consider important examples illustrating that the requirements of Assumption 4.1 can be fulfilled in relevant situations. We investigate therein important special cases of $\mathcal{G}$ that are also mentioned in the related literature, compare e.g. [RRS19, Section 2]. Further, we extend Example 3.7 to the setting with uncertainty about the physical measure $\mathbb{P}$ and we apply Corollary 4.4 in this setting. Afterwards, we discuss an example where the set $\mathcal{P}$ of physical measures is parameterized. This situation is highly relevant for practical applications and shows how our results can be applied to exploit statistical arbitrage strategies.

Example 4.7. We consider an arbitrary set of traded options Φ as well as subsets of the largest set of physical measures such that Assumption 4.1 can be met:

$$\mathcal{P} \subset \{\mathbb{P} \in \mathcal{M}(\mathbb{R}^n) \mid \text{ for all } \mathbb{P} \text{ there exists } \mathbb{Q} \in \mathcal{Q}_\Phi^{\mathcal{P}} : \ \mathbb{Q} \sim \mathbb{P}\}.$$

This allows us to investigate special situations in which the replication assumption (4.1) is fulfilled.

(a) We face $\mathcal{G} = \sigma\left(\{S_{t_j} > K\}\right)$ for some $K \in \mathbb{R}$ and some $j \in \{1, \dots, n\}$. In this situation we consider call options with strike K and traded price p, i.e., these options generate a net payoff $\varphi(S) = (S_{t_j} - K)^+ - p$. Then, it holds for $\mathbb{P} \in \mathcal{P}$ on $\{S_{t_j} > K\}$:

$$\mathbb{E}_{\mathbb{P}}[\varphi \mid \mathcal{G}] = \frac{\mathbb{E}_{\mathbb{P}}[(S_{t_j} - K)_+]}{\mathbb{P}(S_{t_j} > K)} - p.$$

Thus, if

$$\begin{aligned} & p \notin \left[\inf_{\mathbb{P} \in \mathcal{P}} \frac{\mathbb{E}_{\mathbb{P}}[(S_{t_j} - K)^+]}{\mathbb{P}(S_{t_j} > K)}, \ \sup_{\mathbb{P} \in \mathcal{P}} \frac{\mathbb{E}_{\mathbb{P}}[(S_{t_j} - K)^+]}{\mathbb{P}(S_{t_j} > K)}\right], \\ & \text{and } p \in \left[\inf_{\mathbb{Q} \in \mathcal{Q}_\Phi : \mathbb{Q} \sim \mathbb{P}} \mathbb{E}_{\mathbb{Q}}[(S_{t_j} - K)^+], \ \sup_{\mathbb{Q} \in \mathcal{Q}_\Phi : \mathbb{Q} \sim \mathbb{P}} \mathbb{E}_{\mathbb{Q}}[(S_{t_j} - K)_+]\right] \text{ for all } \mathbb{P} \in \mathcal{P}, \end{aligned} \tag{4.18}$$

then we can always find some $a \in \mathbb{R}$ such that on $\{S_{t_j} > K\}$ it holds for all $\mathbb{P} \in \mathcal{P}$ and some $\mathbb{Q} \in \mathcal{Q}_{\Phi \cup \{\varphi\}}$ with $\mathbb{Q} \sim \mathbb{P}$ the following inequality

$$\mathbb{E}_{\mathbb{P}}[a\varphi \mid \mathcal{G}] \geq \mathbb{E}_{\mathbb{P}}\left[\frac{\mathbb{E}_{\mathbb{P}}[\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \mid \mathcal{G}]}{\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}} - 1 \,\middle|\, \mathcal{G}\right],$$

since the left-hand-side of the latter equation imposes a $\mathcal{G}$-arbitrage that can be scaled arbitrarily large. To obtain a conditional super-replication on $\{S_{t_j} < K\}$ we may likewise consider put options with strike K and some price $\tilde{p}$ such that

$$\begin{aligned} & \tilde{p} \notin \left[\inf_{\mathbb{P} \in \mathcal{P}} \frac{\mathbb{E}_{\mathbb{P}}[(K - S_{t_j})^+]}{\mathbb{P}(S_{t_j} < K)}, \ \sup_{\mathbb{P} \in \mathcal{P}} \frac{\mathbb{E}_{\mathbb{P}}[(K - S_{t_j})^+]}{\mathbb{P}(S_{t_j} < K)}\right], \\ & \text{and } \tilde{p} \in \left[\inf_{\mathbb{Q} \in \mathcal{Q}_{\Phi \cup \{\varphi\}} : \mathbb{Q} \sim \mathbb{P}} \mathbb{E}_{\mathbb{Q}}[(K - S_{t_j})^+], \ \sup_{\mathbb{Q} \in \mathcal{Q}_{\Phi \cup \{\varphi\}} : \mathbb{Q} \sim \mathbb{P}} \mathbb{E}_{\mathbb{Q}}[(K - S_{t_j})^+]\right] \text{ for all } \mathbb{P} \in \mathcal{P}. \end{aligned} \tag{4.19}$$

In the special case that call options for all strikes with maturity t_j are available for trading (the martingale optimal transport case), then the marginal distribution μ_j at time t_j is fixed for all equivalent martingale measures. Hence, condition (4.18) reduces to

$$\mathbb{E}_{\mu_j}[(S_{t_j} - K)^+] \notin \left[\inf_{\mathbb{P} \in \mathcal{P}} \frac{\mathbb{E}_{\mathbb{P}}[(S_{t_j} - K)^+]}{\mathbb{P}(S_{t_j} > K)}, \ \sup_{\mathbb{P} \in \mathcal{P}} \frac{\mathbb{E}_{\mathbb{P}}[(S_{t_j} - K)^+]}{\mathbb{P}(S_{t_j} > K)}\right].$$

In the same way (4.19) can be simplified.

(b) We study for some $j \in \{1, \dots, n\}$ the case $\mathcal{G} = \sigma(\{S_{t_j} \in K_i\})$ where $(K_i)_{i=1,\dots,n}$ is a partition of $\mathbb{R}$. We assume the structure $K_i = \left(\underline{K_i}, \overline{K_i}\right]$ with $\underline{K_1} = -\infty$, $\overline{K_n} = \infty$. Then, we consider the case $1 < i < n$, otherwise the super-replication can be obtained as in a). Similar to the exposition from a) we consider the following three vanilla options $\varphi_1, \varphi_2, \varphi_3$ with prices p_1, p_2 and p_3:

$$\varphi_1(S) = (S_{t_j} - \underline{K_i})^+ - p_1,\ \varphi_2(S) = \left(S_{t_j} - \left(\frac{\underline{K_i} + \overline{K_i}}{2}\right)\right)^+ - p_2,\ \varphi_3(S) = (\overline{K_i} - S_{t_j})^+ - p_3,$$

and construct a butterfly payoff:

$$\varphi(S) := \varphi_1(S) - 2\varphi_2(S) + \varphi_3(S).$$

We set $p = p_1 + 2p_2 - p_3$ and $\tilde{\varphi}(S) := \varphi(S) + p$. Then, we have $\tilde{\varphi}(x_1, \dots, x_n) > 0 \Leftrightarrow x_j \in (\underline{K_i}, \overline{K_i})$ and, analogue to a), we get fulfilment of Assumption 4.1 if

$$\begin{aligned} &p \notin \left[\inf_{\mathbb{P}\in\mathcal{P}} \frac{\mathbb{E}_{\mathbb{P}}\left[\tilde{\varphi}(S)\right]}{\mathbb{P}(S_{t_j} \in K_i)},\ \sup_{\mathbb{P}\in\mathcal{P}} \frac{\mathbb{E}_{\mathbb{P}}\left[\tilde{\varphi}(S)\right]}{\mathbb{P}(S_{t_j} \in K_i)}\right], \\ &\text{and } p \in \left[\inf_{\mathbb{Q}\in\mathcal{Q}_\Phi:\mathbb{Q}\sim\mathbb{P}} \mathbb{E}_{\mathbb{Q}}\left[\tilde{\varphi}(S)\right],\ \sup_{\mathbb{Q}\in\mathcal{Q}_\Phi:\mathbb{Q}\sim\mathbb{P}} \mathbb{E}_{\mathbb{Q}}\left[\tilde{\varphi}(S)\right]\right] \text{ for all } \mathbb{P} \in \mathcal{P}. \end{aligned} \tag{4.20}$$

Again, in the martingale optimal transport case, the above equations simplify.

(c) In the same manner as in a) and b) we can also meet Assumption 4.1 by the inclusion of traded options in the case where $\mathcal{G}$ is generated by path-dependent events. If $\mathcal{G} = \sigma\left(\{\max_{i\in\{1,\dots,n\}} S_{t_i} > K\}\right)$ or $\mathcal{G} = \sigma\left(\{\max_{i\in\{1,\dots,n\}} S_{t_i} \in K_i\}\right)$ for some partition $(K_i)_i$ as in b), then we may consider lookback options with payoff

$$\left(\max_{i\in\{1,\dots,n\}} S_{t_i} - K\right)^+ \quad \text{and} \quad \left(K - \max_{i\in\{1,\dots,n\}} S_{t_i}\right)^+$$

for associated strikes K. If the price of these options fulfils analogue conditions to (4.18), (4.19) and (4.20) respectively, then the inclusion of these options ensures validity of Assumption 4.1.

Example 4.8. We reconsider the setting from Example 3.7, but instead of fixing a specific probability measure $\mathbb{P}$, we allow for all physical measures within an ambiguity set of possible physical measures given by

$$\tilde{\mathcal{P}} = \left\{\mathbb{P}\ \middle|\ \mathbb{P} = p_{11}\delta_{(1,1)} + p_{21}\delta_{(2,1)} + p_{22}\delta_{(2,2)} + p_{23}\delta_{(2,3)},\ p_{ij} > 0,\ \sum_{i,j} p_{ij} = 1\right\}.$$

The set of martingale measures then remains unchanged

$$\mathcal{Q}_\emptyset^{\tilde{\mathcal{P}}} = \left\{\mathbb{Q} \text{ as in (3.6)} \mid q_{11} = 1/8,\ q_{21} = q_{23}, q_{22} = 7/8 - 2q_{21},\ \text{with }\ q_{21} \in [0, 7/16]\right\}.$$

For an arbitrary $\mathbb{P} \in \tilde{\mathcal{P}}$, we compute for $\mathbb{Q} \in \mathcal{Q}_\emptyset^{\tilde{\mathcal{P}}}$ with $\mathbb{Q} \sim \mathbb{P}$ the following conditional expectation

$$\mathbb{E}_{\mathbb{P}}\left[\frac{\mathbb{E}_{\mathbb{P}}[\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \mid S_{t_2}]}{\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}} - 1\ \middle|\ S_{t_2}\right] = \frac{1}{(p_{11} + p_{21})^2}\left(8q_{21}p_{11}^2 + \frac{p_{21}^2}{8q_{21}} - 2p_{11}p_{21}\right)\mathbb{1}_{\{S_{t_2}=1\}} \geq 0 \ \mathbb{P}\text{-a.s.}$$

which vanishes if and only if $q_{21} = p_{21}/(8p_{11})$, i.e., if and only if $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \in \sigma(S_{t_2})$. In that case, the market is $\tilde{\mathcal{P}}$-robust statistical arbitrage free by Proposition 4.6. Hence, in order for the market to be free of $\tilde{\mathcal{P}}$-robust statistical arbitrage, each element $\mathbb{Q} \in \mathcal{Q}_\emptyset^{\tilde{\mathcal{P}}}$ has to fulfil the constraint $q_{21} \in [0, 7/16]$. Thus, for all physical measures $\mathbb{P} \in \tilde{\mathcal{P}}$, a measure $\mathbb{Q}$ can only be chosen such that $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \in \sigma(S_{t_2})$, if we shrink the set of admissible physical measures and consider only subsets of

$$\mathcal{P} = \{\mathbb{P} \in \tilde{\mathcal{P}} \mid p_{21}/p_{11} < 7/2\}$$

as ambiguity sets. For example, assume we want to assign every likely path a certain probability which is neither too high nor too low, then one could consider the set

$$\mathcal{P}_1 = \{\mathbb{P} \in \tilde{\mathcal{P}} \mid p_{ij} \in [0.2, 0.4]\} \subset \mathcal{P}.$$

This choice implies, in particular, that for all $\mathbb{P} \in \mathcal{P}_1$, there exists some $\mathbb{Q} \in \mathcal{Q}_\emptyset^{\mathcal{P}_1}$ such that $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \in \sigma(S_{t_2})$. Thus, Assumption 4.1 is fulfilled and the market is free of $\mathcal{P}_1$-robust statistical arbitrage. Next, consider for some $\mathbb{P} \in \mathcal{P}_1$ a measure $\mathbb{Q} \in \mathcal{Q}_\emptyset^{\mathcal{P}_1}$ with $\mathbb{Q} \sim \mathbb{P}$ such that $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \in \sigma(S_{t_2})$. Then, the $\mathbb{Q}$-expectation of some payoff $c(S_{t_1}, S_{t_2})$ computes as

$$\mathbb{E}_\mathbb{Q}[c] = \frac{1}{8}\left(c(1,1) + \frac{p_{21}}{p_{11}}c(2,1) + \left(7 - 2\frac{p_{21}}{p_{11}}\right)c(2,2) + \frac{p_{21}}{p_{11}}c(2,3)\right).$$

For the specific payoff $c(S_{t_1}, S_{t_2}) = |S_{t_2} - S_{t_1}|$, we obtain

$$\mathbb{E}_\mathbb{Q}[|S_{t_2} - S_{t_1}|] = \frac{1}{4}\frac{p_{21}}{p_{11}}.$$

Applying Remark 4.5 and the constraint $p_{ij} \in [0.2, 0.4]$ implies the following $\mathcal{P}_1$-robust statistical arbitrage free price bounds

$$\sup_{\mathbb{P}\in\mathcal{P}_1} \sup_{\mathbb{Q}\in\mathcal{Q}_\emptyset^{\{\mathbb{P}\}}} \{\mathbb{E}_\mathbb{Q}[|S_{t_2} - S_{t_1}|] : \mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \in \sigma(S_{t_2}) : \mathbb{Q} \sim \mathbb{P}\} = 0.5$$

and

$$\inf_{\mathbb{P}\in\mathcal{P}_1} \inf_{\mathbb{Q}\in\mathcal{Q}_\emptyset^{\{\mathbb{P}\}}} \{\mathbb{E}_\mathbb{Q}[|S_{t_2} - S_{t_1}|] : \mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \in \sigma(S_{t_2}) : \mathbb{Q} \sim \mathbb{P}\} = 0.125.$$

These bounds are significantly tighter than the $\mathcal{P}_1$-robust arbitrage free price bounds given by 0 and 0.875. The latter can be computed similarly to the single reference measure approach presented in Example 3.7. By Corollary 4.4, we may derive self-financing strategies which allow for $\mathcal{P}_1$-robust statistical arbitrage if the price for $|S_{t_2} - S_{t_1}|$ does not lie within the interval $[0.125, 0.5]$. The strategies f_{up} and f_{down} which attain the price bounds are given by

$$\begin{aligned} f_{\text{down}}(S_{t_1}, S_{t_2}) &= 0.125 - (S_{t_1} - S_{t_0}) + \mathbb{1}_{\{S_{t_1}=2\}}(S_{t_2} - S_{t_1}), \\ f_{\text{up}}(S_{t_1}, S_{t_2}) &= 0.5 - 4(S_{t_1} - S_{t_0}) + \mathbb{1}_{\{S_{t_1}=2\}}(S_{t_2} - S_{t_1}). \end{aligned}$$

Compare Appendix B for details of the calculation.

Next, we specialize the setting from the previous section by considering the parametric case

$$\mathcal{P} = \{(\mathbb{P}_\theta)_{\theta\in\Theta}\} \text{ for some } \Theta \subset \mathbb{R}^m.$$

The setting under consideration is well motivated by the situation faced in plenty practical applications. A commonly used approach by practitioners is to estimate a real-world measure $\mathbb{P}$ according to historical data of stock returns. The measure $\mathbb{P}$ is then usually based on some modeling assumption of the underlying asset and the related model parameters are estimated from historical price data. This estimation involves a range of sources of uncertainty. Even if the class of financial models is chosen perfectly appropriate and can be matched very well to historical data, then the future development of the stock and its returns may still significantly differ from the past evolution. Assume we consider as *true* model a financial model which comprises a parameter $\theta \in \Theta$ that can be calibrated to historical data. In other words, we consider an underlying asset price process $(S_{t_i}^\theta)_{i=1,\dots,n}$ specified under the real-world measure $\mathbb{P}_\theta$ which in turn depends on the choice of the parameter θ. Then, the class of possible measures is given by $\mathcal{P} = (\mathbb{P}_\theta)_{\theta\in\Theta}$ for some set Θ of admissible parameters. By considering such a class of parametric probability measures $\mathcal{P}$, we take into account the uncertainty inherent in potential estimation errors as well as the possibility of future changes in the model parameters. In the sequel, we provide an example illustrating the practical applicability of the latter results for such a parametric case.

Example 4.9 (CRR-Model with parameter uncertainty). We consider the binomial Cox-Ross-Rubinstein(CRR) model from [CRR79] with $n = 10$ time steps. The underlying process, which starts at $S_{t_0} = 100$, moves in each step either upwards or downwards, i.e., we investigate for $i = 1, \dots, 10$, the following probabilities

$$p := \mathbb{P}(S_{t_i} = uS_{t_{i-1}}) = 1 - \mathbb{P}(S_{t_i} = dS_{t_{i-1}})$$

with specific values $u = 1.1$ and $d = 1/u$. We account for uncertainty w.r.t. the probability p, since we do not assign a concrete value to p, but instead suppose that the probability p lies in the interval $[0.4, 0.6]$. For sake of simplicity, we allow trading only at times t_5 and t_{10}. We check numerically that this market with two trading times does neither allow for statistical arbitrage nor for arbitrage. Moreover, for each $p \in [0.4, 0.6]$, we can find some martingale measure $\mathbb{Q} \sim \mathbb{P}$ s.t. $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \in \sigma(S_{t_{10}})$. We consider a derivative with payoff function

$$c(S_{t_5}, S_{t_{10}}) = \left(\frac{1}{2}(S_{t_5} + S_{t_{10}}) - S_{t_0}\right)^+.$$

With the numerical routine described in the Appendix A, we obtain the robust no-statistical arbitrage price bounds

$$[9.35,\ 9.88],$$

which are significantly tighter than the robust no arbitrage price bounds

$$[4.76,\ 27.41].$$

Now assume c is traded for a price $p(c) = 5$. This price does not allow for arbitrage, but for statistical arbitrage. To exploit statistical arbitrage, we need to sell the maximal sub-replicating strategy (Δ_0, Δ_1) and buy c for the price 5. This means we obtain a payoff

$$f = -\Delta_0 \,(S_{t_5} - S_{t_0}) - \Delta_1(S_{t_5})\,(S_{t_{10}} - S_{t_5}) + c(S_{t_5}, S_{t_{10}}) - 5.$$

We run $1,000,000$ simulations for the price process $(S_{t_i})_{i=1,\dots,10}$ and choose in each simulation $p \in [0.4, 0.6]$ randomly. The results of this simulation and the success of the trading strategy f are displayed in the first row of Table 1. We further investigate how the trading price $p(c)$ influences the profitability of the corresponding statistical arbitrage strategies. For prices which are above the no-statistical arbitrage interval we use the minimal super-replication strategy instead of the maximal sub-replication strategy which is used for prices below. The results indicate that all strategies are profitable on average. Nevertheless, the strategies are also very risky, often lead to losses and are especially prone to large-scale losses. The more the trading price $p(c)$ deviates from the no-statistical arbitrage price interval, the more likely are gains.

Table 1. The table shows the profits from strategies involving a *mispriced* derivative such that the sub-replication and super-replication strategies lead to average profits.

Traded Price	Average Profit	Best Gain	Worst Loss	% Loss Scenarios	% Gain Scenarios
5	4.68	88.15	−45.97	55.69%	44.31%
9	0.67	84.15	−49.97	61.44%	38.55%
10	0.41	52.53	−98.92	38.21%	61.78%
20	10.37	62.53	−88.92	20.90%	79.09%

5. Empirical Investigation and Applications

In the sequel, we compute several trading strategies relying on statistical arbitrage in various different settings and we investigate their behaviour as well as their performance in comparison with other commonly used trading strategies.

5.1. **Statistical Arbitrage versus Pairs-Trading.** We assume that the process S models the evolution of the spread between two stocks S^1 and S^2, i.e., we consider

$$(S_{t_i})_{i=1,\dots,n} = (a\,S^1_{t_i} + b\,S^2_{t_i})_{i=1,\dots,n}, \quad \text{for some } a, b \in \mathbb{R}.$$

Suppose that under all admissible physical measures $\mathbb{P} \in \mathcal{P}$, the spread process S is stationary. In that situation, pairs-trading strategies are commonly applied in practice. These make use of the mean reversion property of stationary processes by trading in the direction opposite to the deviation of the process from its long-term mean, provided that the deviation is large enough.

We refer to [AL10] and [MPZ16] for an elaborate investigation of pairs-trading strategies. In the spirit of [EVDHM05], we define a general pairs trading strategy $\Delta^{\varepsilon,\delta}$ by the trading action at time t_i,

$$\Delta_i^{\varepsilon,\delta} - \Delta_{i-1}^{\varepsilon,\delta} = \begin{cases} -1 & \text{if } S_{t_i} \geq \mu + \varepsilon \text{ and } \Delta_{i-1}^{\varepsilon,\delta} = 0, \\ -1 & \text{if } S_{t_i} \geq \mu - \delta \text{ and } \Delta_{i-1}^{\varepsilon,\delta} = 1, \\ 1 & \text{if } S_{t_i} \leq \mu - \varepsilon \text{ and } \Delta_{i-1}^{\varepsilon,\delta} = 0, \\ 1 & \text{if } S_{t_i} \leq \mu + \delta \text{ and } \Delta_{i-1}^{\varepsilon,\delta} = -1, \\ 0 & \text{else,} \end{cases}$$

for some $\varepsilon > \delta > 0$. This means, the strategy triggers trading, if the deviation of the spread from its mean is larger than ϵ. A long position is entered, if the deviation is negative, and a short position, if the deviation is positive. We liquidate the position, if the process returns to its long-term mean $\mu \in \mathbb{R}$ up to (some small) δ. Thus, $\Delta_i^{\epsilon,\delta}$ indicates whether in period $[t_i, t_{i+1})$ a short position ($\Delta_i^{\epsilon,\delta} = -1$) or a long position ($\Delta_i^{\epsilon,\delta} = +1$) is held. Figure 3 illustrates such a strategy. In practice, a frequent choice of parameters is $\varepsilon = 2\sigma$.

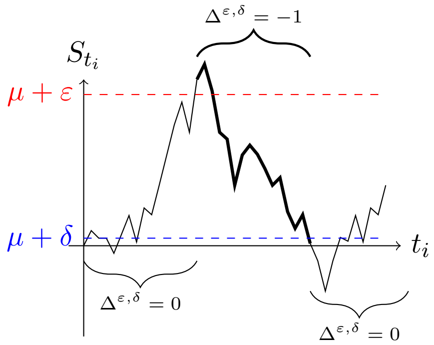

FIGURE 3. Illustration of the trading strategy $\Delta^{\varepsilon,\delta}$. The trading is triggered when S exceeds the threshold $\mu + \varepsilon$. This position is cleaned as soon as S falls below $\mu + \delta$.

In the following, we compare the statistical arbitrage approach involving exotic options with the pairs-trading approach outlined above in different settings.

5.1.1. *AR(1)-Model.* In a continuous time setting, the dynamics of the spread may be modelled by an Ornstein-Uhlenbeck process, which possesses the mean-reversion property. For a discrete and finite time horizon, $t_0 \leq t_1 \leq \cdots \leq t_n$ with constant time lag $t_i - t_{i-1} = \bar{t}$ for all i, we use an AR(1)-process, which respresents a discretized version of the Ornstein-Uhlenbeck process. For some starting value S_{t_0}, we define the process S recursively by setting

$$S_{t_{i+1}} = S_{t_i} + \kappa\,\bar{t}(\mu - S_{t_i}) + \sigma\,\eta_i \text{ for } i = 1, \dots, n,$$

with $\eta_i \sim \mathcal{N}(0, \bar{t})$ i.i.d., mean level $\mu \in \mathbb{R}$, as well as volatility parameter $\sigma > 0$ and speed of mean-reversion $\kappa > 0$. A closed-form solution may be derived iteratively (compare e.g. [SS17]) as

$$\begin{aligned} S_{t_i} &= \kappa\,\mu\,\bar{t} \sum_{j=0}^{i-1} (1 - \kappa\,\bar{t})^j + (1 - \kappa\,\bar{t})^i S_{t_0} + \sum_{j=0}^{i-1} (1 - \kappa\,\bar{t})^j \sigma\,\eta_{i-j} \\ &= \mu(1 - (1 - \kappa\,\bar{t})^i) + (1 - \kappa\,\bar{t})^i S_{t_0} + \sum_{j=0}^{i-1} (1 - \kappa\,\bar{t})^j \sigma\,\eta_{i-j}. \end{aligned}$$

In particular, we have $S_{t_i} \sim \mathcal{N}\left(\mu(1 - (1 - \kappa\bar{t})^i) + (1 - \kappa\bar{t})^i S_{t_0}, \sqrt{\left(\sum_{j=0}^{i-1} (1 - \kappa\bar{t})^j \sigma\bar{t} \right)^2} \right)$.

Remark 5.1. When assuming an $AR(1)$-model, a pairs-trading strategy $\Delta^{\varepsilon,\delta}$ cannot be considered a statistical arbitrage strategy in the sense defined in Section 2. The reason is that for all $N \in \mathbb{N}$ the outcome $\{S_{t_n} \geq N\}$ has positive probability. For N large enough, however, it holds $\mathbb{E}\left[\Delta^{\varepsilon,\delta} \mid S_{t_n} \geq N\right](\omega) < 0$ for some ω with positive probability, since the pairs-trading strategy enters a short-position in S at a previous trading time if the deviation from the mean was large enough. Then the loss of $\Delta^{\varepsilon,\delta}$ at t_n outreaches previous gains.

Next, we compare a classical pairs trading strategy with statistical arbitrage strategies.

Example 5.2 (AR(1)-Process with uncertainty about the mean level)*.* We consider a call option on the spread S with payoff $c(S_{t_{100}}) = (S_{t_{100}} - S_{t_0})_+$. Suppose an investor observes the current value $S_{t_0} = 0$ and assumes the spread to follow an AR(1) process with specific parameters $\bar{t} = 1$, $\sigma = 1$, $\kappa = 0.001$, but she is uncertain about the long term mean level μ. Therefore, she considers $\mu \in [-1, 1]$. In the replication strategy for the call option, we allow trading at t_0, t_{50} and t_{100} and we test numerically for the existence of robust statistical arbitrage strategies in the market. For this step, we discretize the joint distribution $\mathbb{P}(S_{t_{50}} = x_i, S_{t_{100}} = y_j)$ on a 50×50 grid $(x_i, y_j)_{(i,j)\in\{1,\dots,50\}^2}$. Numerical computations yield a tight robust no-statistical arbitrage price interval for c which is given by $[4.005, 4.031]$. The bounds are attained by robust sub- and super-replication strategies, which allow to construct robust statistical arbitrage strategies for derivatives outside these price bounds. Assume c is traded at the market at price $p(c) = 6$. We use the super-replication result from Corollary 3.4 to find some strategy

$$f = 4.005 + \Delta_0(S_{t_{50}} - S_{t_0}) + \Delta_1(S_{t_{50}})(S_{t_{100}} - S_{t_{50}})$$

such that $\mathbb{E}_{\mathbb{P}}[f \mid S_{t_{100}}] \geq \mathbb{E}_{\mathbb{P}}[c \mid S_{t_{100}}] = (S_{t_{100}} - S_{t_0})_+$.
We take a short position in c and a long position in f which leads to a payoff

$$\tilde{f} := 6 + \Delta_0(S_{t_{50}} - S_{t_0}) + \Delta_1(S_{t_{50}})(S_{t_{100}} - S_{t_{50}}) - c(S_{t_{100}}) \quad \text{for all choices of } \mu \in [-1, 1].$$

Next, we run $1,000,000$ simulations of the AR(1)-process, where in each simulation we choose $\mu \in [-1, 1]$ randomly. We compare the success of the strategy $\tilde{f}$ with a classical pairs-trading strategy $\Delta^{\varepsilon,\delta}$ for different choices of ε and for fixed $\delta = 0.001$. For the pairs-trading strategy, we allow trading at all intermediate times $t_1, \dots, t_{99}$. The results are displayed in Table 2. They indicate that $\tilde{f}$ is more profitable on average, but also more risky since it allows for both higher extreme gains and losses. We also remark that the pairs trading strategy becomes more profitable by increasing the mean reversion parameter κ. A similar pattern can be observed also in Example 5.3. Figure 4 depicts a typical evolution of the underlying process as well as the gain process of the strategy $\Delta^{2\sigma,\delta}$ for $\mu = 0$.

TABLE 2. This table compares the profits of the strategy involving a spread option with the profits of a classical pairs trading strategy. We account for uncertainty w.r.t the long-term mean by allowing μ to vary in $[-1, 1]$.

Strategy	Average Profit	Best Gain	Worst Loss	% Loss Scenarios	% Gain Scenarios
$\tilde{f}$	2.05	498.74	-536.17	29.92%	70.07%
$\Delta^{\sigma,\delta}$	0.50	42.81	-45.09	47.01%	52.98%
$\Delta^{2\sigma,\delta}$	0.51	33.45	-45.53	45.98%	54.01%
$\Delta^{3\sigma,\delta}$	0.48	32.15	-43.23	45.01%	54.97%

The next example illustrates a profitable trading strategy in a market which allows for statistical arbitrage.

Example 5.3 (AR(1)-Model allowing for statistical arbitrage)*.* We consider the AR(1)-model from Example 5.2, but without uncertainty about the mean μ and we increase the mean reversion parameter by the factor 100. More specifically, we have $\bar{t} = 1$, $\sigma = 1$, $\mu = S_{t_0} = 0$, $\kappa = 0.1$. Due to the strong mean reversion property, such a market seems to be perfectly suited for the

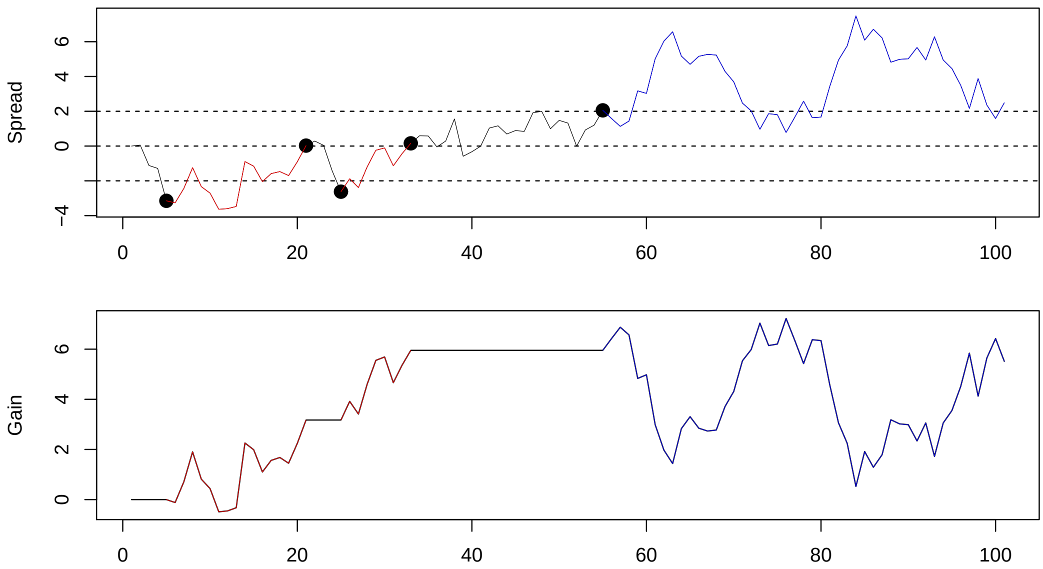

Figure 4. Top: Typical evolution of the spread when modelled by an AR(1) process with parameters $\bar{t} = 1, \mu = 0, \kappa = 0.001, \sigma = 1$. The entry points for a corresponding strategy $\Delta^{\varepsilon,\delta}$ with $\varepsilon = 2 \cdot \sigma$ and $\delta = 0.001$ are indicated by dashed lines. Each trading event is indicated by a black circle. The red color indicates periods in which the trader holds a long position whereas the blue line indicates short positions in S. Bottom: The corresponding gain process $\Delta^{\varepsilon,\delta} \cdot S$ of the pairs trading strategy $\Delta^{\varepsilon,\delta}$.

implementation of a pairs trading strategy. We compute numerically (compare the Appendix A) the quantity

(5.1)
$$P(\sigma(S_{t_n}),0)^K := \inf \Bigg\{ d \in \mathbb{R} : \ \exists (a_i)_i \in \mathbb{R}, \ (\Delta_j)_j \in B(\mathbb{R}^j), \ |(\Delta_j)_j| < K, |(a_i)_i| < K \text{ s.t.} \\ d + \mathbb{E}_{\mathbb{P}}\left[\sum_{i=1}^N a_i \varphi_i + (\Delta \cdot S)_n \ \middle| \ \sigma(S_{t_n}) \right] \geq 0, d + \mathbb{E}_{\mathbb{P}}\left[\sum_{i=1}^N a_i \varphi_i + (\Delta \cdot S)_n \right] > 0 \Bigg\}$$

for some real constant K which can be interpreted as the maximal amount a trader is willing to invest in each position. If we choose $K = \infty$, then we get $D(\sigma(S_{t_n}),0)^K = -\infty$. This can be explained because the market allows for statistical arbitrage and statistical arbitrage strategies then can be scaled arbitrarily large. We run $1,000,000$ simulations of this market model and we compare the success of the strategy from (5.1) for different values of K with the success of the pairs trading strategy $\Delta^{2\sigma,\delta}$. The results provided in Table 3 show that the average profit, the best gain and the worst gain scale (up to effects which rely on the randomness of the simulations) linearly with K. In this setting, which seems perfectly suited for pairs trading, we observe that those strategies which occur as solutions to problem (5.1) represent alternative profitable strategies, but are more risky. While the pairs trading strategy yields scenarios associated with gains in 99.93% of the simulations, the alternative strategies only lead to gains in approximately 73% of the cases.

Let us now introduce uncertainty w.r.t the mean reversion parameter κ. More specifically, suppose $\kappa \in [0.005, 0.1]$, i.e., $\mathcal{P}$ comprises all measures $\mathbb{P}$ with some mean reversion parameter from $[0.005, 0.1]$. In particular, this means we consider scenarios with a low mean reversion rate as well as scenarios with strong mean reversion speed. Note that each market allows for statistical arbitrage. Given K, we then compute a robust statistical arbitrage strategy by

TABLE 3. This table compares the profit of a statistical arbitrage strategy induced by (5.1) with the profit of a pairs trading strategy. The mean reversion parameter is rather high at $\kappa = 0.1$.

Strategy	Average Profit	Best Gain	Worst Loss	% Loss Scenarios	% Gain Scenarios
$K = 10$	16.25	152.86	−187.61	26.96%	73.04%
$K = 100$	162.39	1632.92	−1784.34	26.99%	73.01%
$K = 1000$	1623.28	15925.15	−19775.06	26.97%	73.03%
$\Delta^{2\sigma,\delta}$	14.52	43.01	−5.155	0.06%	99.93%

solving

$$\begin{aligned}
&\inf\Bigg\{d \in \mathbb{R} :\ \exists (a_i)_i \in \mathbb{R},\ (\Delta_j)_j \in B(\mathbb{R}^j),\ |(\Delta_j)_j| < K, |(a_i)_i| < K \text{ s.t.} \\
&\qquad d + \mathbb{E}_{\mathbb{P}}\left[\sum_{i=1}^N a_i\varphi_i + (\Delta\cdot S)_n \,\middle|\, \sigma(S_{t_n})\right] \geq 0 \text{ for all } \mathbb{P} \in \mathcal{P} \\
&\qquad d + \mathbb{E}_{\mathbb{P}}\left[\sum_{i=1}^N a_i\varphi_i + (\Delta\cdot S)_n\right] > 0 \text{ for some } \mathbb{P} \in \mathcal{P}\Bigg\}
\end{aligned} \tag{5.2}$$

We compare the robust statistical arbitrage strategy for $K = 10$ with the pairs trading strategy $\Delta^{2\sigma,\delta}$. The results are displayed in Table 4.

TABLE 4. This table compares the profit of a statistical arbitrage strategy induced by (5.2) with the profit of a pairs trading strategy. The speed of mean reversion parameter is uncertain and randomly chosen in each simulation for $\kappa \in [0.005, 0.1]$.

Strategy	Average Profit	Best Gain	Worst Loss	% Loss Scenarios	% Gain Scenarios
$K = 10$	0.19	164.07	−203.28	49.96%	50.04%
$\Delta^{2\sigma,\delta}$	9.94	38.24	−35.69	7.96%	92.04%

Our calculations reveal that the classical pairs trading strategy clearly outperforms the statistical arbitrage strategy. The latter is more risky with approximately 50% of the scenarios associated with losses while the pairs trading strategy produces gains in approximately 92% of all simulations. An explanation for this is that we trade the statistical arbitrage strategy only at initial time and at one intermediate time t_{50} whereas the pairs trading strategy can be adjusted at every time step. To test this, we successively increase the number of possible trading dates for the statistical arbitrage strategy to 4, 9 and 19 intermediate times, respectively. More precisely, we initiate new statistical arbitrage strategies with shorter maturity $\frac{100}{\#\text{intermediate times}+1}$ at all equidistant intermediate times by taking into account information on the development of the underlying process. We then obtain after 10,000 simulations the results provided in Table 5.3. These are further depicted within histograms which are collected in Figure 5 and indicate a significant improvement of the performance for increasing number of intermediate trading points. The average profit increases and also the percentage of scenarios leading to gains rises. We remark, however, that the results are derived without considering transaction costs. Obviously an increase of intermediate trading activities also incurs additional costs caused by these transactions. Moreover, the average profit, maximal gains and losses can be scaled by allowing for different maximal investment amounts K.

To obtain a fair comparison with the pairs trading strategy we also investigate the pairs trading strategy within the AR(1)-model when only allowing trading at a reduced amount of intermediate trading days. The results are displayed in Table 6.

TABLE 5. This table shows the profit of statistical arbitrage strategies when the amount of trading points increases. We account for uncertainty w.r.t. the speed of mean reversion $\kappa \in [0.005, 0.1]$ and we allow for a maximal investment of $K = 10$ at each trading time.

Strategy	Average Profit	Best Gain	Worst Loss	% Loss Scenarios	% Gain Scenarios
1 trade	0.19	164.07	−203.28	49.96%	50.04%
4 trades	3.61	481.18	−311.91	49.63%	50.37%
9 trades	17.61	428.85	−356.52	43.94%	56.06%
19 trades	23.07	461.53	−373.05	40.44%	59.56%

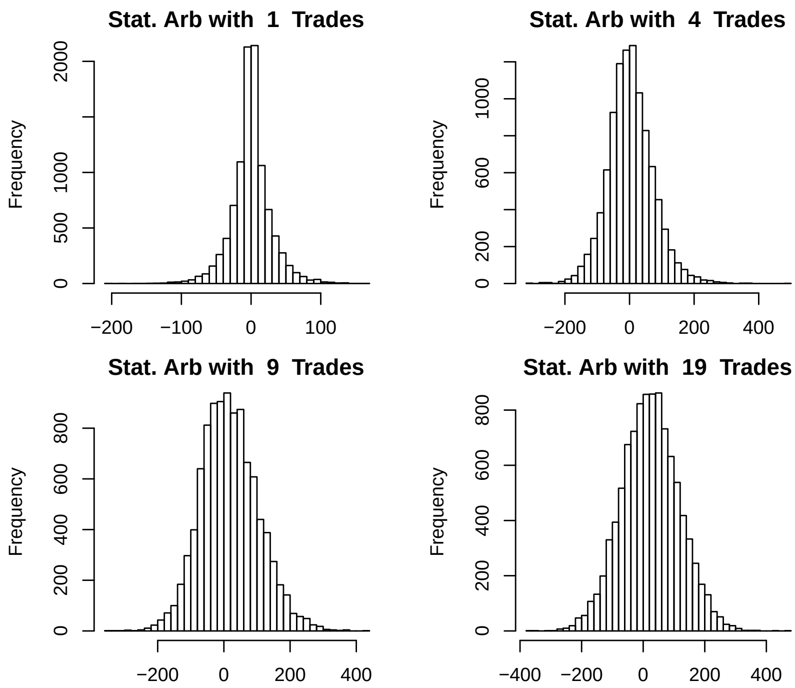

FIGURE 5. These histograms illustrate the distributions of the profits which are displayed in Table 5.3. For this, we take into account 10,000 simulations. The figures particularly indicate that allowing for more intermediate trading days shifts the mean of the distribution of the profits to the right, i.e. positive gains are more likely to happen.

TABLE 6. This table shows the profit of pairs trading strategies when the amount of trading points is reduced to a certain number of intermediate trading points.

Strategy	Average Profit	Best Gain	Worst Loss	% Loss Scenarios	% Gain Scenarios
1 trade	1.51	21.93	−16.34	12.23%	87.77%
4 trades	3.97	22.26	−23.58	13.75%	86.25%
9 trades	6.04	30.45	−24.61	12.09%	87.91%
19 trades	7.71	31.74	−30.32	10.71%	89.29%

A major problem in practice is that pairs trading only leads to a stable average profit as long as the cointegration relationship persists. However, if the cointegration relation breaks down, the pairs trading strategy will on average lead to losses as the already initiated positions will not be liquidated. In contrast, the statistical arbitrage approach does not rely on a cointegration assumption and therefore also works in scenarios in which the deviation from the mean at terminal time is rather large. For illustration we consider $1,000$ scenarios in which the deviation from the mean $\mu = 0$ at time t_{100} is larger than 12σ. These are scenarios which can be interpreted as scenarios in which the cointegration relation did break down. In Table 7 we compare the success of pairs trading and statistical arbitrage strategies in these scenarios. We allow for three intermediate trading days in both strategies and set $K = 10$ for the statistical arbitrage strategy. Indeed, the statistical arbitrage strategy clearly outperforms the pairs trading strategy given these adverse scenarios. This is also observable in Figure 6 where we depict the empirical distribution of the profits. The pairs trading strategy leads remarkably often to zero gains because no trading is initiated. If trading is initiated, then, given these scenarios, the histogram reveals that the strategy leads with high probability to losses. In contrast, such a behaviour is not observable for the statistical arbitrage strategy which leads to average profits.

TABLE 7. We compare the performance of the pairs trading with the performance of the statistical arbitrage strategy when considering only scenarios which have a deviation of at least 12σ from the mean at terminal time.

Strategy	Average Profit	Best Gain	Worst Loss	% Loss Scenarios	% Gain Scenarios
Pairs Trading	−7.85	21.19	−22.72	91.4%	8.6%
Statistical Arbitrage	1.26	294.33	−413.55	46.7%	53.3%

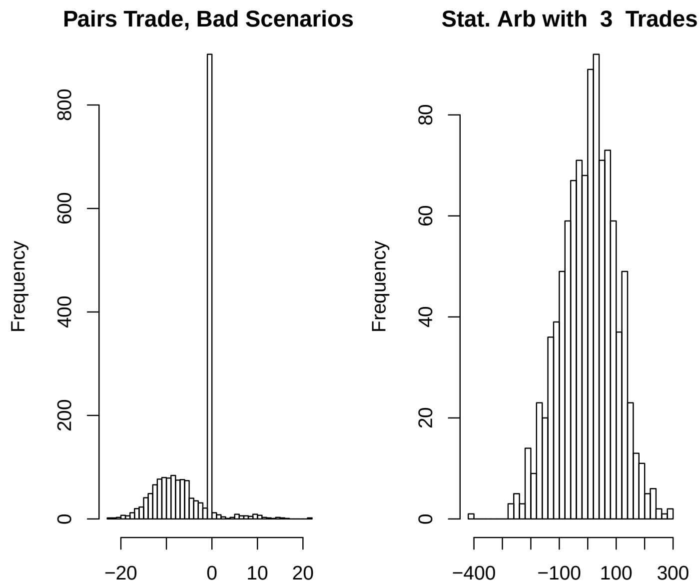

FIGURE 6. These two histograms illustrate the empirical distributions of profits and losses of the statistical arbitrage strategy as well as of the pairs trading stratey in scenarios with a large mean deviation at terminal time.

5.2. **Application to EURO STOXX 50 Data and Backtesting.** Eventually, we investigate the performance of the previously introduced profitable market strategies when applied to real market data. We decide to consider daily prices of the Eurozone stock index *EURO STOXX 50* over a period of five years from January 2014 until December 2018. This means we take into account 1244 consecutive trading days and 60 months of trading activities.

We want to invest in a renewed statistical arbitrage strategy every month. If today is denoted by t_0, then since an average month possesses 21 trading days, this strategy involves the future dates in the middle of next month $t_0 + 11$ and at the end of next month $t_0 + 21$. In order to estimate the physical probabilities $\mathbb{P}(S_{t_0+11}, S_{t_0+21})$ we use a two-dimensional kernel density estimation with an axis-aligned bivariate normal kernel (compare e.g. [Rip02]). To perform the estimation we observe an amount of $d \in \mathbb{N}$ pairs $((S_{t_i+11} - S_{t_i})/S_{t_i}, (S_{t_i+21} - S_{t_i})/S_{t_i})$ where t_i is some day in the past, lying at least 21 days back. Based on these observations we estimate the kernel to obtain joint distributions $\mathbb{P}\left((S_{t_0+11} - S_{t_0})/S_{t_0}, (S_{t_0+21} - S_{t_0})/S_{t_0}\right)$. To be able to start trading in January 2014, we use earlier historic price data to obtain estimations for the joint probabilities at the start of the trading period. Additionally, the kernel uses a bandwidth $h \in \mathbb{R}$ which has to be pre-specified and the density is evaluated on a 50×50 grid and normed such that the sum of all probabilities is 1, i.e., we estimate

$$\mathbb{P}\bigg(S_{t_0+11} = S_{t_0}x_k + S_{t_0},\ S_{t_0+21} = S_{t_0}y_l + S_{t_0}\bigg) = \mathbb{P}\bigg(\frac{S_{t_0+11} - S_{t_0}}{S_{t_0}} = x_k,\ \frac{S_{t_0+21} - S_{t_0}}{S_{t_0}} = y_l\bigg)$$

for $x_k, y_l \in \mathbb{R}$ with $k, l = 1, \dots, 50$.

We then compute $D(\sigma(S_{t_j}), 0)^K$ for $K = 1$, as defined in (5.1). In a first simulation, we use every month the last $d = 21$ days for the kernel estimation and realize the corresponding trading strategy $\Delta_0,\ \Delta_1$ for the upcoming month and compute its profit. Further, we account for uncertainty w.r.t. $\mathbb{P}$ by allowing for a range of bandwidths

$$h \in [0.95 \cdot \tilde{h},\ 1.05 \cdot \tilde{h}]. \tag{5.3}$$

where we use the rule of thumb $\tilde{h} = 1.06 \min(\tilde{\sigma}, R/1.34) d^{-1/5}$ for $\tilde{\sigma}$ the empirical standard deviation and R the interquartile range of the data. See also [Rip02, Chapter 5.6.]. We repeat the estimation procedure and the computation of (5.2) to determine the corresponding trading strategies on a monthly basis and obtain the results displayed in the first row of Table 8. The results in the second to third row of Table 8 correspond to different numbers of considered observations used for the kernel estimations.

Table 8. This table compares the performance of statistical arbitrage strategies backtested on EURSOTOXX 50 data for a consecutive period of 60 months. The first row depicts results using the estimated probabilities, where the probabilities are estimated based on the observations of the past month. The other rows show the results when accounting for different number of observations d.

	Profit/Month	Best Month	Worst Month	% Months w. Losses	% Months w. Gains
$d = 21$	3.3	352.47	−275.63	50%	50%
$d = 63$	23.83	274.14	−343.62	43.1%	56.9%
$d = 126$	9.46	463.46	−355.06	43.1%	56.9%

In Figure 7 we depict the average profit per month in dependence of the used number of observations for the estimated density and compare it with statistical arbitrage strategies using fixed bandwidths $h \in \left\{0.95\tilde{h},\ \tilde{h},\ 1.05\tilde{h}\right\}$. The average profit per month of the robust strategy obtained via (5.2) is, with a small amount of exceptions, strictly positive.

6. Summary

In this paper, we discussed statistical arbitrage strategies in a robust sense. Our theoretical results comprise conditions for the absense of $\mathcal{P}$-robust $\mathcal{G}$-arbitrage strategies in financial markets with instruments $(\varphi_i)_{i=1,\dots,N}$ and underlying security $(S_{t_i})_{i=1,\dots,n}$. Moreover, we provide super- (resp. sub-)replication strategies for the construction of $\mathcal{P}$-robust $\mathcal{G}$-arbitrage strategies in such markets when an additional path-dependent derivative $c(S_{t_1}, \dots, S_{t_n})$ is included. These strategies involve static positions in $\{(\varphi_i)_{i=1,\dots,N}, c\}$ and investments in the underlying security according to a self-financing trading strategy $\Delta(S_{t_1}, \dots, S_{t_n})$. Thus, our results allow to detect *mispriced* options and to develop profitable trading strategies based on these. We provide several examples illustrating the implementation and performance of these strategies in practice. In the empirical part, we then compare the (robust) statistical arbitrage approach with classical

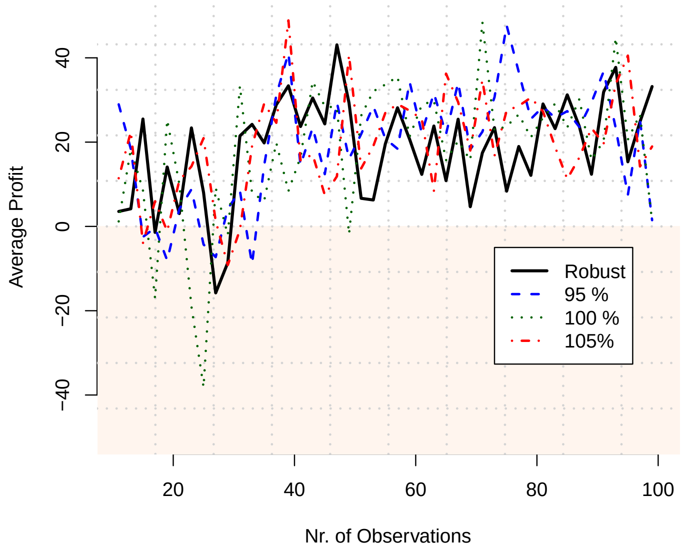

FIGURE 7. This figure depicts the average monthly profit of a robust statistical arbitrage strategy in dependence of the number of trading days which were used to estimate the future joint distributions. We also compare the profit with the profit of statistical arbitrage strategies where the bandwidth is fixed.

pairs trading strategies in a simulated AR(1)-model. Moreover, we perform a backtesting study based on the EUROSTOXX 50 index which demonstrates the applicability of the introduced method in a market allowing for robust statistical arbitrage.

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References

[ABPS16] Acciaio, Beatrice ; Beiglböck, Mathias ; Penkner, Friedrich ; Schachermayer, Walter: A model-free version of the fundamental theorem of asset pricing and the super-replication theorem. In: *Mathematical Finance* 26 (2016), Nr. 2, S. 233–251

[ACJ19] Alfonsi, Aurélien ; Corbetta, Jacopo ; Jourdain, Benjamin: Sampling of one-dimensional probability measures in the convex order and computation of robust option price bounds. In: *International Journal of Theoretical and Applied Finance* 22 (2019), Nr. 03, S. 1–41

[AL10] Avellaneda, Marco ; Lee, Jeong-Hyun: Statistical arbitrage in the US equities market. In: *Quantitative Finance* 10 (2010), Nr. 7, S. 761–782

[AL17] Acciaio, Beatrice ; Larsson, Martin: Semi-static completeness and robust pricing by informed investors. In: *The Annals of Applied Probability* 27 (2017), Nr. 4, S. 2270–2304

[BHLP13] Beiglböck, Mathias ; Henry-Labordère, Pierre ; Penkner, Friedrich: Model-independent bounds for option prices: a mass transport approach. In: *Finance and Stochastics* 17 (2013), Nr. 3, S. 477–501

[BJ16] Beiglböck, Mathias ; Juillet, Nicolas: On a problem of optimal transport under marginal martingale constraints. In: *The Annals of Probability* 44 (2016), Nr. 1, S. 42–106

[BL78] Breeden, Douglas T. ; Litzenberger, Robert H.: Prices of state-contingent claims implicit in option prices. In: *Journal of Business* 54 (1978), Nr. 4, S. 621–651

[BN15] Bouchard, Bruno ; Nutz, Marcel: Arbitrage and duality in nondominated discrete-time models. In: *The Annals of Applied Probability* 25 (2015), Nr. 2, S. 823–859

[BNT17] Beiglböck, Mathias ; Nutz, Marcel ; Touzi, Nizar: Complete duality for martingale optimal transport on the line. In: *The Annals of Probability* 45 (2017), Nr. 5, S. 3038–3074

[Bon03] Bondarenko, Oleg: Statistical arbitrage and securities prices. In: *The Review of Financial Studies* 16 (2003), Nr. 3, S. 875–919

[CM19] Campi, Luciano ; Martini, Claude: On the support of extremal martingale measures with given marginals: the countable case. In: *Advances in Applied Probability* 51 (2019), Nr. 2, S. 570–605

[CRR79] Cox, John C. ; Ross, Stephen A. ; Rubinstein, Mark: Option pricing: A simplified approach. In: *Journal of Financial Economics* 7 (1979), Nr. 3, S. 229–263

[Del92] Delbaen, Freddy: Representing martingale measures when asset prices are continuous and bounded. In: *Mathematical Finance* 2 (1992), Nr. 2, S. 107–130

[DS94] Delbaen, Freddy ; Schachermayer, Walter: A general version of the fundamental theorem of asset pricing. In: *Mathematische Annalen* 300 (1994), Nr. 1, S. 463–520

[DS14] Dolinsky, Yan ; Soner, H M.: Martingale optimal transport and robust hedging in continuous time. In: *Probability Theory and Related Fields* 160 (2014), Nr. 1-2, S. 391–427

[EGLO19] Eckstein, Stephan ; Guo, Gaoyue ; Lim, Tongseok ; Obloj, Jan: Robust pricing and hedging of options on multiple assets and its numerics. In: *arXiv preprint arXiv:1909.03870* (2019)

[EK19] Eckstein, Stephan ; Kupper, Michael: Computation of optimal transport and related hedging problems via penalization and neural networks. In: *Applied Mathematics & Optimization* (2019), S. 1–29

[EVDHM05] Elliott, Robert J. ; Van Der Hoek, John ; Malcolm, William P.: Pairs trading. In: *Quantitative Finance* 5 (2005), Nr. 3, S. 271–276

[FS16] Föllmer, Hans ; Schied, Alexander: *Stochastic Finance: An Introduction in Discrete Time.* Walter de Gruyter, 2016

[GO19] Guo, G ; Obloj, J: Computational methods for martingale optimal transport problems. In: *Annals of Applied Probability* (2019)

[HK15] Hobson, David ; Klimmek, Martin: Robust price bounds for the forward starting straddle. In: *Finance and Stochastics* 19 (2015), Nr. 1, S. 189–214

[HL13] Henry-Labordère, Pierre: Automated option pricing: Numerical methods. In: *International Journal of Theoretical and Applied Finance* 16 (2013), Nr. 08, S. 1–27

[KL17] Kassberger, Stefan ; Liebmann, Thomas: Additive portfolio improvement and utility-efficient payoffs. In: *Mathematics and Financial Economics* 11 (2017), Nr. 2, S. 241–262

[LS19] Lütkebohmert, Eva ; Sester, Julian: Tightening robust price bounds for exotic derivatives. In: *Quantitative Finance* 19 (2019), Nr. 11, S. 1797–1815

[MPZ16] Moura, Carlos E. ; Pizzinga, Adrian ; Zubelli, Jorge: A pairs trading strategy based on linear state space models and the Kalman filter. In: *Quantitative Finance* 16 (2016), Nr. 10, S. 1559–1573

[Rip02] Ripley, BD: *Modern Applied Statistics with S.* Springer, New York, 2002

[RRS19] Rein, Christian ; Rüschendorf, Ludger ; Schmidt, Thorsten: Generalized statistical arbitrage concepts and related gain strategies. In: *Available at SSRN 3423944* (2019)

[Ses20] Sester, Julian: Robust price bounds for derivative prices in Markovian models. In: *International Journal of Theoretical and Applied Finance, Forthcoming* (2020). http://dx.doi.org/10.1142/S0219024920500156. – DOI 10.1142/S0219024920500156

[SS17] Shumway, Robert H. ; Stoffer, David S.: *Time Series Analysis and its Applications.* Springer, 2017

Appendix A. Numerics

In this appendix, we explain how the primal and the dual problem from the super-replication theorem stated in Corollary 4.4 can be implemented and solved numerically in the case of $\mathcal{G} = \sigma(S_{t_n})$. For better readability we consider only two times $t_1 < t_2$. The approach can, however, easily be extended to an arbitrary number of time points. Both methods are based on discrete probability distributions. This means we either discretize the support of S_{t_1} and S_{t_2}, if the random variables are continuous, or we use a discrete model. Thus, we assume from now on that $\operatorname{supp}(S_{t_1}) = \{x_1, \dots, x_{n_1}\}$ and $\operatorname{supp}(S_{t_2}) = \{y_1, \dots, y_{n_2}\}$ for some integers $n_1, n_2 \in \mathbb{N}$. The presented numerical routine relates to several recent works investigating martingale optimal transport from a numerical point of view. In particular, these are [HL13], [EK19], [EGLO19], [ACJ19] and [GO19]. These articles become especially relevant in the case when Φ consists of a continuum of prices of call options and the marginal distributions are therefore fixed. Then we may apply the procedure from [ACJ19] to ensure the increasing convex order of the discretized marginals which are then involved. However, in the examples within this paper the case with fixed marginals is not covered and we therefore do not go into further detail concerning this procedure. The discretization of $\operatorname{supp}(S_{t_1}) = \{x_1, \dots, x_{n_1}\}$ and $\operatorname{supp}(S_{t_2})$ in Section 5.2 was performed equidistantly with 50 supporting values. Certainly, this is not the unique way to perform this step, but turns out to be sufficient within our considerations. The used code can be found at `https://github.com/juliansester/statistical-arbitrage`.

A.1. **The Primal Problem.** For the primal problem, the condition

$$d + \mathbb{E}_{\mathbb{P}}\left[\sum_{i=1}^{N} a_i \varphi_i + (\Delta \cdot S)_n \;\middle|\; S_{t_n}\right] \geq \mathbb{E}_{\mathbb{P}}[c \mid S_{t_n}]$$

from the super-replication theorem in Corollary 3.4 can be written as

$$\begin{aligned} &d + \sum_{i=1}^{N} a_i \sum_{k=1}^{n_1} \varphi_i(x_k, y_j) \frac{p_{kj}}{\sum_{l=1}^{n_1} p_{lj}} \\ &+ \Delta_0 \cdot \left(\sum_{k=1}^{n_1} x_k \frac{p_{kj}}{\sum_{l=1}^{n_1} p_{lj}} - S_{t_0}\right) + \sum_{i=1}^{n_1} \Delta_1(x_i) \cdot \frac{p_{ij}}{\sum_{k=1}^{n_1} p_{kj}} (y_j - x_i) \\ &\geq \sum_{i=1}^{n_1} c(x_i, y_j) \frac{p_{ij}}{\sum_{k=1}^{n_1} p_{kj}}, \end{aligned} \tag{A.1}$$

for all $j = 1, \dots, n_2$. In the above equations we used the notation $\mathbb{P}(x_i, y_j) = p_{ij}$. We solve this linear programming problem for the minimal $d \in \mathbb{R}$ and some real values $a_i, \Delta_0,\ \Delta_1(x_i)$. We observe that this problem only respects n_2 inequality constraints. Thus, it is numerically significantly faster to solve than the dual problem with its $3n_1 + n_2 + N + 2$ equality constraints. Moreover, the primal approach directly yields some super-replication strategy allowing to exploit statistical arbitrage. If necessary we use spline-interpolation to obtain a function Δ_1 which is defined not only on the discrete values $(x_1, \dots, x_{n_1})$.

A.2. **The Dual Problem.** To solve the dual maximization problem

$$D(\sigma(S_{t_n}), c) = \sup_{\mathbb{Q} \in \mathcal{Q}_\Phi^{\{\mathbb{P}\}}} \{\mathbb{E}_{\mathbb{Q}}[c(S_{t_1}, S_{t_2})] : \mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \in \sigma(S_{t_2}) : \mathbb{Q} \sim \mathbb{P}\}$$

for a fixed measure $\mathbb{P}$, we look for some measure $\mathbb{Q}$ defined on $(x_i, y_j)_{i,j}$, denoted by $q_{ij} = \mathbb{Q}(x_i, y_j)$, which solves the martingale constraints $\mathbb{E}_{\mathbb{Q}}[S_{t_2} \mid S_{t_1}] = S_{t_1}$ as well as $\mathbb{E}_{\mathbb{Q}}[S_{t_1}] = S_{t_0}$, or equivalently,

$$\sum_{j=1}^{n_2} q_{ij}(y_j - x_i) = 0 \quad \text{for } i = 1, \dots, n_1, \tag{A.2}$$

and

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} q_{ij} x_i = S_{t_0}. \tag{A.3}$$

Further, we account for the non-negativity constraint, $q_{ij} \geq 0$, the probability measure property

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} q_{ij} = 1, \tag{A.4}$$

and the measurability condition $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \in \sigma(S_{t_2})$, which is equivalent to

$$\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}(x_i, y_j) = \mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}(x_k, y_j) \quad \text{for all } i, k \in \{1, \dots, n_1\},\ j \in \{1, \dots, n_2\}.$$

We can rewrite this constraint as

$$q_{ij} p_{kj} - q_{kj} p_{ij} = 0 \quad \text{for all } i, k \in \{1, \dots, n_1\},\ j \in \{1, \dots, n_2\}. \tag{A.5}$$

By considering traded options $(\varphi_k)_{k=1,\dots,N}$ we obtain the additional restrictions

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} q_{ij} \varphi_k(x_i, y_j) = 0 \quad \text{for all } k = 1, \dots, N. \tag{A.6}$$

We then maximize $\mathbb{E}_{\mathbb{Q}}[c] = \sum_{i,j} q_{ij} c(x_i, y_j)$ subject to the constraints (A.2), (A.3), (A.4), (A.5), (A.6) above. This optimization problem turns out to be a linear programming problem with a single inequality constraint as well as $3n_1 + n_2 + N + 2$ equality constraints.

A.3. **Addressing Uncertainty w.r.t the Physical Measure.** When considering a set of physical measures $\mathcal{P}$ consisting of more than a single element, then we use Remark 4.5 to obtain the maximal/minimal solution of the robust super-/sub-hedging problems and primal problems, respectively. More explicitely, we compute the problems from Section A.1 and A.2 for a large amount of (randomly chosen) elements $\mathbb{P} \in \mathcal{P}$. The elements which lead to the most extremal values then imply the robust price bounds. If we are not only interested in the robust price interval, but want to obtain a robust strategy, then we add conditions of the form (A.1) for all $\mathbb{P}$ used for the numerical computation, before solving the system of linear inequalities.

Appendix B. Detailed Calculations within the Examples

In this appendix we provide detailed derivations of the results which were - for sake of better readability - omitted within the mentioned examples.

B.1. **Calculations from Example 3.6.** Consider some strategy

$$\begin{aligned} f =& a_1\left((S_{t_2}-13)^+-1/8\right)+a_2\left((S_{t_2}-10)^+-5/4\right)+\Delta_0(S_{t_1}-10)\\ &+\Delta_{11}\mathbb{1}_{\{S_{t_1}=12\}}(S_{t_2}-12)+\Delta_{12}\mathbb{1}_{\{S_{t_1}=8\}}(S_{t_2}-8) \end{aligned}$$

for parameters $a_1, a_2, \Delta_0,\ \Delta_{11},\ \Delta_{12} \in \mathbb{R}$, where we abbreviated $\Delta_{11} = \Delta_1(12)$ and $\Delta_{12} = \Delta_1(8)$. We compute

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}[f \mid S_{t_2}] =& a_1\left((S_{t_2}-13)^+-1/8\right)+a_2\left((S_{t_2}-10)^+-5/4\right)+\Delta_0(\mathbb{E}_{\mathbb{P}}[S_{t_1}\mid S_{t_2}]-10)\\ &+\Delta_{11}\mathbb{P}(S_{t_1}=12\mid S_{t_2})(S_{t_2}-12)+\Delta_{12}\mathbb{P}(S_{t_1}=8\mid S_{t_2})(S_{t_2}-8)\ \mathbb{P}\text{-a.s..} \end{aligned} \tag{B.1}$$

We calculate the following expressions

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}[S_{t_1}\mid S_{t_2}] &= 11\cdot \mathbb{1}_{\{S_{t_2}=14\}}+12\cdot \mathbb{1}_{\{S_{t_2}=13\}}+11\cdot \mathbb{1}_{\{S_{t_2}=10\}}+8\cdot \mathbb{1}_{\{S_{t_2}=6\}}\ \mathbb{P}\text{-a.s.},\\ \mathbb{P}(S_{t_1}=12\mid S_{t_2}) &= \frac{3}{4}\cdot \mathbb{1}_{\{S_{t_2}=14\}}+\mathbb{1}_{\{S_{t_2}=13\}}+\frac{3}{4}\cdot \mathbb{1}_{\{S_{t_2}=10\}}\ \mathbb{P}\text{-a.s.},\\ \mathbb{P}(S_{t_1}=8\mid S_{t_2}) &= \frac{1}{4}\mathbb{1}_{\{S_{t_2}=14\}}+\frac{1}{4}\mathbb{1}_{\{S_{t_2}=10\}}+\mathbb{1}_{\{S_{t_2}=6\}}\ \mathbb{P}\text{-a.s.} \end{aligned}$$

Substituting the latter quantities in (B.1) yields

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}[f\mid S_{t_2}] =& \mathbb{1}_{\{S_{t_2}=6\}}\left(-\frac{1}{8}a_1-\frac{5}{4}a_2-2\Delta_0-2\Delta_{12}\right)\\ &+\mathbb{1}_{\{S_{t_2}=10\}}\left(-\frac{1}{8}a_1-\frac{5}{4}a_2+\Delta_0-\frac{3}{2}\Delta_{11}+\frac{1}{2}\Delta_{12}\right)\\ &+\mathbb{1}_{\{S_{t_2}=13\}}\left(-\frac{1}{8}a_1+\frac{7}{4}a_2+2\Delta_0+\Delta_{11}\right)\\ &+\mathbb{1}_{\{S_{t_2}=14\}}\left(\frac{7}{8}a_1+\frac{11}{4}a_2+\Delta_0+\frac{3}{2}\Delta_{11}+\frac{3}{2}\Delta_{12}\right)\ \mathbb{P}\text{-a.s.} \end{aligned}$$

For f being a statistical arbitrage strategy, each of the latter brackets should be non-negative with at least one bracket strictly positive. It is straight forward to verify that this is fulfilled for the specific choice $a_1 = 1,\ a_2 = 2,\ \Delta_0 = 1,\ \Delta_{11} = -2$ and $\Delta_{12} = -5/2$.

B.2. **Calculations from Example 4.8.** We verify that the provided strategies f_{up} and f_{down} are indeed statistical arbitrage strategies attaining the robust price bounds.
We have

$$\mathbb{E}_{\mathbb{P}}[c(S_{t_1},S_{t_2})\mid S_{t_2}] = \mathbb{E}_{\mathbb{P}}[|S_{t_2}-S_{t_1}|\mid S_{t_2}] = \mathbb{1}_{\{S_{t_2}=3\}}+\frac{p_{21}}{p_{11}+p_{21}}\mathbb{1}_{\{S_{t_2}=1\}}\ \mathbb{P}\text{-a.s.},$$

and compute further

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}[f_{\text{down}} \mid S_{t_2}] =& \frac{1}{8} - \mathbb{E}_{\mathbb{P}}[S_{t_1} \mid S_{t_2}] + \frac{15}{8} + \mathbb{E}_{\mathbb{P}}\left[\mathbb{1}_{\{S_{t_1}=2\}} S_{t_2} \mid S_{t_2}\right] - \mathbb{E}_{\mathbb{P}}\left[\mathbb{1}_{\{S_{t_1}=2\}} S_{t_1} \mid S_{t_2}\right] \\
=& 2 - \left(2\mathbb{1}_{\{S_{t_2}=3\}} + 2\mathbb{1}_{\{S_{t_2}=2\}} + \left(2\frac{p_{21}}{p_{11}+p_{21}} + \frac{p_{11}}{p_{11}+p_{21}}\right)\mathbb{1}_{\{S_{t_2}=1\}}\right) \\
&+ 3\mathbb{P}(S_{t_2}=2 \mid S_{t_2}=3)\mathbb{1}_{\{S_{t_2}=3\}} + 2\mathbb{P}(S_{t_2}=2 \mid S_{t_2}=2)\mathbb{1}_{\{S_{t_2}=2\}} \\
&+ \mathbb{P}(S_{t_2}=2 \mid S_{t_2}=1)\mathbb{1}_{\{S_{t_2}=1\}} - 2\mathbb{P}(S_{t_2}=2 \mid S_{t_2}) \ \mathbb{P}\text{-a.s.}
\end{aligned}$$

Inserting the corresponding conditional probabilities then yields

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}[f_{\text{down}} \mid S_{t_2}] =& 2 - 2\mathbb{1}_{\{S_{t_2}=3\}} - 2\mathbb{1}_{\{S_{t_2}=2\}} - \frac{2p_{21}+p_{11}}{p_{11}+p_{21}}\mathbb{1}_{\{S_{t_2}=1\}} + 3\mathbb{1}_{\{S_{t_2}=3\}} + 2\mathbb{1}_{\{S_{t_2}=2\}} \\
&+ \frac{p_{21}}{p_{11}+p_{21}}\mathbb{1}_{\{S_{t_2}=1\}} - 2\mathbb{1}_{\{S_{t_2}=3\}} - 2\mathbb{1}_{\{S_{t_2}=2\}} - 2\frac{p_{21}}{p_{11}+p_{21}}\mathbb{1}_{\{S_{t_2}=1\}} \\
=& 2 - \left(\frac{3p_{21}+p_{11}}{p_{11}+p_{21}}\right)\mathbb{1}_{\{S_{t_2}=1\}} - 2\mathbb{1}_{\{S_{t_2}=2\}} - \mathbb{1}_{\{S_{t_2}=3\}} \\
=& 2\left(1 - \mathbb{1}_{\{S_{t_2}=1\}} - \mathbb{1}_{\{S_{t_2}=2\}} - \mathbb{1}_{\{S_{t_2}=3\}}\right) + \frac{p_{11}-p_{21}}{p_{11}+p_{21}}\mathbb{1}_{\{S_{t_2}=1\}} + \mathbb{1}_{\{S_{t_2}=3\}} \\
=& \frac{p_{11}-p_{21}}{p_{11}+p_{21}}\mathbb{1}_{\{S_{t_2}=1\}} + \mathbb{1}_{\{S_{t_2}=3\}} \ \mathbb{P}\text{-a.s.}
\end{aligned}$$

Since $p_{ij} \in [0.2, 0.4]$ we have $p_{11} - p_{21} \leq 0.2 \leq p_{21}$. This means

$$\mathbb{E}_{\mathbb{P}}[f_{\text{down}} \mid S_{t_2}] = \frac{p_{11}-p_{21}}{p_{11}+p_{21}}\mathbb{1}_{\{S_{t_2}=1\}} + \mathbb{1}_{\{S_{t_2}=3\}} \leq \frac{p_{21}}{p_{11}+p_{21}}\mathbb{1}_{\{S_{t_2}=1\}} + \mathbb{1}_{\{S_{t_2}=3\}} = \mathbb{E}_{\mathbb{P}}[c \mid S_{t_2}] \ \mathbb{P}\text{-a.s.}$$

Equality holds if and only if $2p_{21} = p_{11}$, i.e. if $p_{21} = 0.2$ and $p_{11} = 0.4$. An analogue computation leads to

$$\mathbb{E}_{\mathbb{P}}[f_{\text{up}} \mid S_{t_2}] = \mathbb{1}_{\{S_{t_2}=3\}} + \frac{4p_{11}-p_{21}}{p_{21}+p_{11}}\mathbb{1}_{\{S_{t_2}=1\}} \ \mathbb{P}\text{-a.s.}$$

We have $4p_{11} - p_{21} \geq 0.4 \geq p_{21}$ and thus

$$\mathbb{E}_{\mathbb{P}}[f_{\text{up}} \mid S_{t_2}] \geq \mathbb{E}_{\mathbb{P}}[c \mid S_{t_2}] \ \mathbb{P}\text{-a.s.}$$

Equality holds if and only if $2p_{11} = p_{21}$, i.e. if $p_{11} = 0.2$ and $p_{21} = 0.4$.

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