Volatility-of-Volatility Risk

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Abstract

We show that time-varying volatility of volatility is a significant risk factor which affects the cross-section and the time-series of index and VIX option returns, beyond volatility risk itself. Volatility and volatility-of-volatility measures, identified model-free from options data as the VIX and VVIX indices, respectively, are only weakly related to each other. Delta-hedged index and VIX option returns are negative on average, and more negative for strategies more exposed to volatility and volatility-of-volatility risks. Volatility and volatility of volatility significantly and negatively predict future delta-hedged option payoffs. The evidence is consistent with a no-arbitrage model featuring time-varying volatility and volatility-of-volatility factors, which are negatively priced by investors.

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1 Introduction

Recent studies show that volatility risks significantly affect asset prices and the macroeconomy.[1] In the data, asset market volatility is often captured by the volatility index (VIX). Calculated in real time from the cross-section of S&P500 option prices, the VIX index provides a risk-neutral forecast of the index volatility over the next 30 days. The VIX index exhibits substantial fluctuations, which in the data and in many economic models drive the movements in asset prices and risk premia. Interestingly, the volatility of the VIX index itself varies over time. Computed from VIX options in an analogous way to the VIX, the volatility-of-volatility index (VVIX) directly measures the risk-neutral expectations of the volatility of volatility in the financial markets. In the data, we find that the VVIX has separate dynamics from the VIX, so that fluctuations in volatility of volatility are not directly tied to movements in market volatility. We further show that volatility-of-volatility risks are a significant risk factor which affects the time-series and the cross-section of index and VIX option returns, above and beyond volatility risks. The evidence is consistent with a no-arbitrage model which features time-varying market volatility and volatility-of-volatility factors which are priced by the investors. In particular, volatility and volatility of volatility have negative market prices of risk, so that investors dislike increases in volatility and volatility of volatility, and demand a risk compensation for the exposure to these risks.

Our no-arbitrage model extends the one-factor stochastic volatility specification of equity returns in Bakshi and Kapadia (2003). Specifically, we introduce a separate time-varying volatility-of-volatility risk factor which drives the conditional variance of the variance of market returns.[2] We use the model to characterize the payoffs to delta-hedged equity and VIX options. The zero-cost, delta-hedged positions represent the gains on a long position in

[1] See e.g. Bansal and Yaron (2004), Bloom (2009), Bansal, Kiku, Shaliastovich, and Yaron (2014), Fernandez-Villaverde and Rubio-Ramírez (2013) for the discussion of macroeconomic volatility risks, and Coval and Shumway (2001), Bakshi and Kapadia (2003), Campbell, Giglio, Polk, and Turley (2012) for market volatility risks.

[2] We use the terms “variance” and ”volatility” interchangeably unless specified otherwise.

the option, continuously hedged by an offsetting short position in the underlying asset. As argued in Bakshi and Kapadia (2003), delta-hedged option payoffs most cleanly isolate the exposures to volatility risks, and thus are very useful to study the pricing of volatility-related risks.[3] Indeed, we show that the expected payoff on the delta-hedged index option positions capture risk compensations for both volatility and volatility-of-volatility risks, while for VIX options, the expected gains primarily reflect the compensation for volatility-of-volatility risk. Imposing a standard linear risk premium assumption, we can further decompose the risk premia components of the expected gains on the index options into the product of the market price of risk, the risk exposure, and the time-varying quantity of each source of risk. For VIX options, the expected gains do not generally admit a linear factor structure, so we resort to numerical calibration techniques to support our theoretical arguments.

The model delivers clear, testable predictions for expected option returns and their relation to volatility and volatility-of-volatility risks. In the model, if investors dislike volatility and volatility of volatility, so that the market prices of these risks are negative, delta-hedged equity and VIX option gains are negative on average. In the cross-section, the average returns are more negative for option strategies which have higher exposure to the volatility and volatility-of-volatility risks. Finally, in the time series, higher volatility and volatility of volatility predict more negative delta-hedged option gains in the future.

We find that the model implications for volatility and volatility-of-volatility risks are strongly supported in the cross-section of index and VIX option returns. First, we use the VIX and VVIX indices to validate our volatility measures, and compare and contrast their dynamics. In our monthly 2006 - 2016 sample, the option-implied volatility measures capture meaningful information about the uncertainty in future market returns and market volatility in the data. Using predictive regressions, we show that the VIX is a significant predictor of the future realized variance of market returns, while the VVIX significantly

[3]For example, unlike delta-hedged positions, zero-beta straddles analyzed in Coval and Shumway (2001) are not dynamically rebalanced and may contain a significant time-decay option premium component.

forecasts future realized variation in the VIX index itself. Including both volatility measures at the same time, we find that the predictive power is concentrated with the corresponding factor (i.e., the VIX for market return volatility, the VVIX for VIX volatility). This evidence confirms that the measured VIX and VVIX indices can indeed separately capture volatility and volatility-of-volatility movements in the asset markets.

In the time-series, the VVIX behaves quite differently from the VIX, consistent with a setup of our model which separates market volatility from volatility of volatility. The VVIX is much more volatile, and is less persistent than the VIX. The correlation between the two series is 0.27. While both volatility measures share several common peaks, most notably during the financial crisis, other times of economic distress and economic uncertainty, such as the Eurozone debt crisis and flash crash in May 2010 as well as the U.S. debt ceiling crisis in August 2011, are characterized by large increases in the VVIX with relatively little action in the VIX.

On average, the risk-neutral volatilities of the market return and market volatility captured by the VIX and VVIX exceed the realized volatilities of returns and the VIX. The difference between the risk-neutral and physical volatilities of market returns is known as the variance premium (variance-of-variance premium for the VIX), and the findings of positive variance and variance-of-variance premium suggest that investors dislike variance and variance-of-variance risks, and demand a premium for being exposed to these risks.

We next turn to the asset-price evidence from the equity index and VIX option markets. In line with our model, we consider discrete-time counterparts to the continuously-rebalanced delta-hedged gains; this approach is similar to Bakshi and Kapadia (2003)[4]. Consistent with the evidence in previous studies, the average delta-hedged returns on out-of-the-money equity index calls and puts are significantly negative in our sample. The novel evidence in our paper is that the average delta-hedged returns on VIX options are also negative and statistically

[4] We perform a numerical analysis of the model to verify the adequacy of the approximation of the option delta by the delta computed from the Black-Scholes formula.

significant at all strikes, except for out-of-the-money puts and in-the-money calls which are marginally significant. Estimates of the loss for call options range from -0.25% of the index value for in-the-money VIX calls to -1.20% for out-of-the-money calls. The negative average returns on index and VIX options directly suggest that the market prices of volatility and volatility-of-volatility risks are negative.

We then show that the cross-sectional spreads in average option returns are significantly related to the volatility and volatility-of-volatility risks. In lieu of calculating exact model betas, we compute proxies for the option exposures to the underlying risks using the Black and Scholes (1973) vega and volga. Vega represents an increase in the Black-Scholes value of the option as the implied volatility increases by one percentage point, and thus provides an estimate for the exposure of equity options to volatility risks, and of VIX options to volatility-of-volatility risks. Volga is the second partial derivative of the option price with respect to the volatility, which we use to measure the sensitivity of the index option price to the volatility-of-volatility risks. Vega and volga vary intuitively with the moneyness of the option in the cross-section, and help us proxy for the betas of the options to the underlying risks. Empirically, we document that average option returns are significantly and negatively related to our proxies for volatility and volatility-of-volatility risks. Hence, using the cross-section of equity index options and VIX options, we find strong evidence for a negative market price of volatility and volatility-of-volatility risks.

Finally, we consider a predictive role of our volatility measures for future option returns. In the model, expected delta-hedged gains are time-varying and are driven by the volatility and volatility of volatility (by volatility of volatility for VIX options). In particular, as option betas are all positive, when the market prices of volatility-related risks are negative, both volatility measures should forecast future returns with a negative sign. This model prediction is consistent with the data. The VIX and VVIX significantly negatively predict future index option returns, and the VVIX is a significant negative predictor of option returns on the

VIX. Hence, using the cross-sectional and time-series evidence from the option markets, we find strong support that both volatility and volatility-of-volatility risks are separate priced sources of in the option markets, and have negative market prices of risks.

Related Literature. Our paper is most closely related to Bakshi and Kapadia (2003) who consider the implications of volatility risk for equity index option markets. We extend their approach to include volatility-of-volatility risk, and bring evidence from VIX options. To help us focus on the volatility-related risks, we consider dynamic delta-hedging strategies where a long position in an option is dynamically hedged by taking an offsetting position in the underlying. Delta-hedged strategies are also used in Bertsimas, Kogan, and Lo (2000), Cao and Han (2013) and Frazzini and Pedersen (2012), and are a standard risk management technique of option traders in the financial industry (Hull (2011)). In an earlier study, Coval and Shumway (2001) consider the returns on zero-beta straddles to identify volatility risk sensitive assets. Zhang and Zhu (2006) and Lu and Zhu (2010) highlight the nature and importance of volatility risks by analyzing the pricing of VIX futures. Also notably our analysis suggests that variance dynamics are richer than that of the square-root process typically considered in the literature — these findings are consistent with the results of Christoffersen, Jacobs, and Mimouni (2010) and Branger, Kraftschik, and Völkert (2016).

In a structural approach, Bollerslev, Tauchen, and Zhou (2009) consider a version of the Bansal and Yaron (2004) long-run risks model which features recursive utility and fluctuations in the volatility and volatility of volatility of the aggregate consumption process. They show that in equilibrium, investors require compensation for the exposure to volatility and volatility-of-volatility risks. With preference for early resolution of uncertainty, the market prices of the two risks are negative. As a result, the variance risk premium is positive on average, and can predict future equity returns. Bollerslev et al. (2009) and Drechsler and Yaron (2011) show that the calibrated version of such a model can account for the key features of equity markets and the variance premium in the data. Our empirical results are consistent with the economic intuition in these models and complement the empirical evidence in these studies.

Finally, it is worth noting that in our paper we abstract from jumps in equity returns, and focus on diffusive volatilities as the main drivers of asset prices and risk premia. For robustness, we confirm that our predictability results are robust to controlling for jump risk measures such as the slope of the implied volatility curve, realized jump intensity (Barndorff-Nielsen and Shephard (2006) and Wright and Zhou (2009)), and risk-neutral skewness (Bakshi, Kapadia, and Madan (2003)). Hence, we argue that the VIX and VVIX have a significant impact on option returns even in the presence of stock market and volatility jumps; we leave a formal treatment of jumps for future research. Reduced-form models which highlight the role of jumps include Bates (2000), Pan (2002), and Duffie, Pan, and Singleton (2000), among others. Muravyev (2016) documents that inventory risk faced by market-makers also affects option prices.

Our paper proceeds as follows. In Section 2 we discuss our model which links expected delta-hedged equity and volatility option gains to risk compensations for volatility and volatility-of-volatility risk. In Section 3, we consider an illustrative numerical example to show the relation between VIX option gains and the volatility-related risks. In Section 4, we describe the construction of both the model-free implied variance measures and high-frequency realized variance measures, and summarize their dynamics in the time-series. We show that the implied variances have a strong ability to forecast future realized variance. Section 5 provides the empirical evidence from option prices by empirically implementing the delta-hedged option strategies in our model. Section 6 presents robustness tests for alternative measures of variance, as well as robustness of the results in the presence of jump risks. Section 7 concludes the paper.

2 Model

In this section we describe our model for stock index returns, as well as for equity and volatility option prices. Our model is an extension of Bakshi and Kapadia (2003) and features separate time-variation in the market volatility and volatility of volatility. Both volatility risks are priced, and affect the level and time-variation of the expected asset payoffs.

2.1 Dynamics of Equity and Equity Option Prices

Under the physical measure ($\mathbb{P}$) the stock index price S_t evolves according to:

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu(S_t, V_t, \eta_t)dt + \sqrt{V_t}dW_t^1, \\ dV_t &= \theta(V_t)dt + \sqrt{\eta_t}dW_t^2, \\ d\eta_t &= \gamma(\eta_t)dt + \phi\sqrt{\eta_t}dW_t^3, \end{aligned} \tag{2.1}$$

where W^i $(i = 1, 2, 3)$ are the Brownian motions which drive stock returns, the stock return variance, and the variance of variance, respectively. The Brownian components can be correlated, i.e., we assume $dW_t^i\, dW_t^j = \rho_{i,j}\, dt$ for all $i \neq j$. V_t is the variance of instantaneous returns, and η_t is the variance of innovations in V_t. Note that the drift of the variance V_t only depends on itself, and not on the stock price S_t or the volatility of volatility η_t. Similarly, the drift of the volatility of volatility η_t is a function of the volatility of volatility η_t.

Under the risk-neutral measure ($\mathbb{Q}$) the stock price S_t follows a similar process, but with the drifts adjusted by the risk compensations for the corresponding risks:

$$\begin{aligned} \frac{dS_t}{S_t} &= r_f dt + \sqrt{V_t}d\widetilde{W}_t^1, \\ dV_t &= (\theta(V_t) - \lambda_t^V)dt + \sqrt{\eta_t}d\widetilde{W}_t^2, \\ d\eta_t &= (\gamma(\eta_t) - \lambda_t^\eta)dt + \phi\sqrt{\eta_t}d\widetilde{W}_t^3. \end{aligned} \tag{2.2}$$

In this representation, the $\widetilde{W}^i$ represent Brownian motions under the risk-neutral measure $\mathbb{Q}$. λ_t^V captures the compensation for variance risk, and λ_t^η reflects the compensation for innovations in the stochastic variance of variance. If investors dislike variance and variance-of-variance risks, the two risk compensations are negative. In this case, the variances have higher drifts under the risk-neutral than under the physical measure.

Let $C_t(K,\tau)$ denote the time t price of a call option on the stock with strike price K and time to maturity τ. Assume the risk-free rate r_f is constant. To simplify the presentation, we further abstract from dividends. While we focus our discussion on call options, the case of put options follows analogously. Given the specified dynamics of the stock price under the two probability measures, the option price is given by a twice-differentiable function C of the state variables: $C_t(K,\tau) = C(S_t, V_t, \eta_t, t)$. By Itô's Lemma,

$$dC_t = \frac{\partial C}{\partial S} dS_t + \frac{\partial C}{\partial V} dV_t + \frac{\partial C}{\partial \eta} d\eta_t + b_t dt, \tag{2.3}$$

for a certain drift component b_t.

The discounted option price $e^{-r_f t} C_t$ is a martingale under $\mathbb{Q}$ and thus has zero drift. We use Itô's Lemma again to obtain that:

$$\frac{\partial C}{\partial S} S_t r_f + \frac{\partial C}{\partial V}\left(\theta(V_t) - \lambda_t^V\right) + \frac{\partial C}{\partial \eta}\left(\gamma(\eta_t) - \lambda_t^\eta\right) + b_t - r_f C_t = 0. \tag{2.4}$$

This implies that:

$$b_t = r_f\left(C_t - \frac{\partial C}{\partial S} S_t\right) - \frac{\partial C}{\partial V}(\theta(V_t) - \lambda_t^V) - \frac{\partial C}{\partial \eta}(\gamma(\eta_t) - \lambda_t^\eta). \tag{2.5}$$

Let $\Pi_{t,t+\tau}$ stand for the delta-hedged option gain for call options:

$$\Pi_{t,t+\tau} \equiv C_{t+\tau} - C_t - \int_t^{t+\tau} \frac{\partial C}{\partial S} dS_u - \int_t^{t+\tau} r_f\left(C_u - \frac{\partial C}{\partial S} S_u\right) du. \tag{2.6}$$

The delta-hedged option gain represents the gain on a long position in the option, continuously hedged by an offsetting short position in the stock, with the net balance earning the risk-free rate.

Combining equations (2.3) and(2.5) and noting that $C_{t+\tau} = C_t + \int_t^{t+\tau} dC_u$, we see that the delta-hedged gain for call options is given as

$$\begin{aligned}\Pi_{t,t+\tau} &\equiv C_{t+\tau} - C_t - \int_t^{t+\tau} \frac{\partial C}{\partial S} dS_u - \int_t^{t+\tau} r_f \left(C_u - \frac{\partial C}{\partial S} S_u \right) du \\ &= \int_t^{t+\tau} \lambda_u^V \frac{\partial C}{\partial V} du + \int_t^{t+\tau} \lambda_u^\eta \frac{\partial C}{\partial \eta} du + \int_t^{t+\tau} \frac{\partial C}{\partial V} \sqrt{\eta_u} dW_u^2 + \int_t^{t+\tau} \frac{\partial C}{\partial \eta} \phi \sqrt{\eta_u} dW_u^3.\end{aligned} \tag{2.7}$$

Since the expectation of Itô integrals is zero, the expected delta-hedged equity option gains are given by

$$\mathbb{E}_t \left[\Pi_{t,t+\tau} \right] = \mathbb{E}_t \left[\int_t^{t+\tau} \lambda_u^V \frac{\partial C}{\partial V} du \right] + \mathbb{E}_t \left[\int_t^{t+\tau} \lambda_u^\eta \frac{\partial C}{\partial \eta} du \right]. \tag{2.8}$$

The expected option gains depend on the risk compensation components for the volatility and volatility-of-volatility risks (λ_t^V and λ_t^η), and the option price exposures to these two sources of risks ($\frac{\partial C}{\partial V}$ and $\frac{\partial C}{\partial \eta}$).

For tractability, following the literature let us further assume that the risk premium structure is linear:

$$\lambda_t^V = \lambda^V V_t, \qquad \lambda_t^\eta = \lambda^\eta \eta_t, \tag{2.9}$$

where λ^V is the market price of the variance risk and λ^η is the market price of the variance-of-variance risk. We can further operationalize (2.8) by applying Itô-Taylor expansions (Milstein (1995)). This gives us a linear factor structure (see details in Appendix A):

$$\frac{\mathbb{E}_t \left[\Pi_{t,t+\tau} \right]}{S_t} = \lambda^V \beta_t^V V_t + \lambda^\eta \beta_t^\eta \eta_t. \tag{2.10}$$

The sensitivities to the risk factors are given by:

$$\beta_t^V = \sum_{n=0}^{\infty} \frac{\tau^{1+n}}{(1+n)!} \Phi_{t,n}^V > 0, \qquad \beta_t^\eta = \sum_{n=0}^{\infty} \frac{\tau^{1+n}}{(1+n)!} \Phi_{t,n}^\eta > 0, \tag{2.11}$$

where $\Phi_{t,n}^V$ and $\Phi_{t,n}^\eta$ are positive functions which depend on the moneyness of the option and on the sensitivities $\frac{\partial C}{\partial V}$ and $\frac{\partial C}{\partial \eta}$, respectively. Hence, the expected payoff on the delta-hedged option position combines the risk compensations for the volatility and volatility-of-volatility risks. The two risk compensations are given by the product of the market price of risk, the exposure of the asset to the corresponding risk, and the quantity of risk. In particular, options are positive-beta assets with respect to both volatility and volatility-of-volatility risks. Hence, if investors dislike volatility and volatility-of-volatility risks so that their market prices of risks are negative, the expected option payoffs are negative as well.

2.2 Dynamics of VIX Option Prices

The squared VIX index for a time to maturity τ is the annualized risk-neutral expectation of the quadratic variation of returns from time t to $t + \tau$, given by:

$$VIX_t^2 = \frac{1}{\tau} \mathbb{E}_t^{\mathbb{Q}} \left[\int_t^{t+\tau} V_s \, ds \right]. \tag{2.12}$$

Given our model assumptions, the VIX index is a function of the stock market variance: $VIX_t = VIX(V_t)$. For example, in a linear model where the variance drift $\theta(V_t)$ is linear in V_t, the squared VIX is a linear function of the stock market variance V_t.

Let F_t be the time t price of a VIX futures contract expiring at $t + \tau$. Under the assumptions of no arbitrage and continuous mark-to-market, F_t is a martingale under the risk-neutral measure $\mathbb{Q}$:

$$F_t = \mathbb{E}_t^{\mathbb{Q}}\left[VIX_{t+\tau}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[VIX(V_{t+\tau})\right]. \tag{2.13}$$

Under our model structure, the futures price F is a function of the market variance V_t and volatility of volatility η. Under economically plausible scenarios, the futures price is monotone in the two volatility processes.[5] Knowing F_t and η_t is sufficient for V_t, so we can re-write the economic states $\begin{bmatrix} V_t & \eta_t \end{bmatrix}$ in terms of $\begin{bmatrix} F_t & \eta_t \end{bmatrix}$.

Let C_t^* be the time t price of a VIX call option, whose underlying is a VIX forward contract. The option price is given by a twice differentiable function C^* of the state variables, so that $C_t^*(K,\tau) = C^*(F_t, \eta_t, t)$. By Itô's Lemma:

$$dC_t^* = \frac{\partial C^*}{\partial F} dF_t + \frac{\partial C^*}{\partial \eta} d\eta_t + b_t^* dt, \tag{2.14}$$

for a drift component b_t^*.

Under the risk-neutral measure $\mathbb{Q}$, the discounted VIX option price process $e^{-r_f t} C_t^*$ is a martingale, so it must have zero drift:

$$\frac{\partial C^*}{\partial F} \mathcal{D}^{\mathbb{Q}}[F_t] + \frac{\partial C^*}{\partial \eta} (\gamma(\eta_t) - \lambda_t^{\eta}) - r_f C_t^* + b_t^* dt = 0, \tag{2.15}$$

where $\mathcal{D}^{\mathbb{Q}}[F_t]$ is the infinitesimal generator applied to F, i.e., the conditional expectation of dF_t under $\mathbb{Q}$.

[5]See also Zhang and Zhu (2006), Lu and Zhu (2010), and Branger et al. (2016) for VIX futures pricing models.

This implies that

$$
\begin{aligned}
b_t^* &= r_f C_t^* + \frac{\partial C^*}{\partial \eta}\lambda_t^\eta - \frac{\partial C^*}{\partial \eta}\gamma(\eta_t) - \frac{\partial C^*}{\partial F}\mathcal{D}^{\mathbb{Q}}[F_t] \\
&= r_f C_t^* + \frac{\partial C^*}{\partial \eta}\lambda_t^\eta - \frac{\partial C^*}{\partial \eta}\gamma(\eta_t),
\end{aligned}
\tag{2.16}
$$

where the second line follows from the fact that F_t is a martingale under $\mathbb{Q}$, so that its drift is equal to zero.

Combining (2.16) with (2.14), we obtain the equation for the delta-hedged VIX option gain:

$$
\begin{aligned}
\Pi_{t,t+\tau}^* &= C_{t+\tau}^* - C_t^* - \int_t^{t+\tau} \frac{\partial C^*}{\partial F} dF_s - \int_t^{t+\tau} r_f C_s^* ds \\
&= \int_t^{t+\tau} \frac{\partial C^*}{\partial \eta}\lambda_s^\eta ds + \int_t^{t+\tau} \frac{\partial C^*}{\partial \eta}\phi\sqrt{\eta_s} dW_s^3.
\end{aligned}
\tag{2.17}
$$

The delta-hedged VIX option gain in Equation (2.17) is the counterpart to the delta-hedged equity option gain in Equation (2.7). The difference comes from the fact that the short stock position serving as the hedge in the case of equity options is funded at r_f, while for VIX futures the hedging position is zero cost.

Taking expectations, we can derive the corresponding expected gain on delta-hedged VIX options:

$$
\frac{\mathbb{E}_t\left[\Pi_{t,t+\tau}^*\right]}{F_t} = \frac{1}{F_t}\int_t^{t+\tau} \mathbb{E}_t\left[\frac{\partial C_s^*}{\partial \eta_s}\lambda_s^\eta\right] ds. \tag{2.18}
$$

Similar to index options, the expected gain on delta-hedged VIX options depends on the market price of volatility-related risks and the option exposure to these risks. However, in contrast to index options, at least for short maturities τ, VIX options are mainly exposed to volatility-of-volatility risks. Their exposure to these risks, captured by $\frac{\partial C_s^*}{\partial \eta_s}$, is expected to be positive, so that the average level of the expected gains on VIX options and its dependence on the volatility-of-volatility factor is governed by the market price of the volatility risks λ^{η}. Unlike the expected gains for index options in (2.10), the expected gains for VIX options do not generally admit a linear factor structure because the option price is no longer homogeneous in the underlying asset value. We provide a numerical example below to support our arguments, and then consider the relative importance of the risks in the data.

3 Illustrative Numerical Example

3.1 Calibration

We calibrate the model to match key asset-pricing moments in the data. The $\mathbb{Q}$-dynamics of the model are given in Equations (2.2) and (2.9). As it is common in the literature we choose $\theta(V_t) = \kappa_V(\bar{V} - V_t)$ and $\gamma(\eta_t) = \kappa_\eta(\bar{\eta} - \eta_t)$. For simplicity, we assume that the three Brownian motions W^1, W^2 and W^3 are uncorrelated.

We choose parameter values for r_f, κ_V, $\bar{V}$, λ_V, κ_η, $\bar{\eta}$, λ_η, and ϕ. The parameters can be found in Table 1. We choose r_f and $\bar{V}$ to match the unconditional level of risk-free rate and the variance of the S&P500 index in our sample period, respectively. We choose λ_V, $\bar{\eta}$, and κ_V to target the level, the variance, and the persistence of VIX^2, respectively. These moments can be computed in closed form, because VIX^2 is linear in the state variables, as shown in Appendix B.1. Finally, we calibrate λ_η, ϕ, and κ_η to match the level, the variance, and the persistence of $VVIX^2$. We run a simulation study to compute model-implied moments of $VVIX^2$, since they are not available in closed form.

3.2 Expected gains on delta-hedged VIX options

To calculate model-implied delta-hedged option gains, we calculate prices of VIX futures and VIX options on a grid, as described in Appendix B.1. We use Equation (2.18) and numerically solve the integral

$$\int_t^{t+\tau} \mathbb{E}_t \left[\frac{\partial C_s^*}{\partial \eta_s} \lambda_s^\eta \right] ds. \tag{3.1}$$

Details about the numerical procedure are given in Appendix B.2.

Figure 1 shows expected delta-hedged option gains divided by the futures price as a function of η for different levels of the variance V_t. For the upper graph we choose $\lambda_\eta = -5$, i.e. we assume that the market price of volatility-of-volatility risk is negative. For the lower graph we assume $\lambda_\eta = 5$. For the strategies shown here, we always choose at-the-money options. When the investor sets up a strategy in state (V_0, η_0), the strike price of the option he buys is equal to the VIX futures price in state (V_0, η_0), i.e., to $F_t(V_0, \eta_0, \tau_F)$.

Given a negative λ_η, we find that the expected gains are all negative.[6] This is because the derivatives of the VIX options with respect to η are positive for all grid points and for all times to maturity. We also find that expected option gains are larger in absolute terms for higher values of η. Since we truncate the process for V at zero, this effect is more pronounced for low values of V: A high volatility of volatility not only increases the value of the VIX option because the option price is convex in V, but also because it increases the upside potential of the VIX without increasing the downside potential, due to the lower bound at zero. Expected option gains are not exactly linear in η_t since the derivative of VIX option prices with respect to η is not constant but decreasing in η (see Figure 2).

With a positive λ_η expected delta-hedged option gains are positive. In this case, all VIX option prices are still increasing in the volatility of volatility. We can conclude that the

[6] Quantitatively, the expected delta-hedged VIX option gains normalized by the VIX futures price, computed at the average values of the volatility states, are about -0.1% for two-day returns, or -1.5% time-aggregated to a monthly horizon. These values are close to the empirical estimates presented in Section 5.

sign of the expected option gain is pinned down by the sign of the market price of volatility of volatility risk, and that the expected delta-hedged VIX option gains are increasing, in absolute terms, in the volatility of volatility.

3.3 Delta computations

A key input into the computations of delta-hedged option returns is the delta of the option. A typical approach in the literature is to approximate the unknown "true" deltas by the Black-Scholes implied ones; see Bakshi and Kapadia (2003), Duarte and Jones (2007), and our subsequent analysis. In this section, we use our numerical model to compute and compare the "true" and Black-Scholes deltas, and evaluate the adequacy of the approximation for the VIX options.

The Black-Scholes delta for a call option on a futures contract is given by $\Delta = e^{-r_f \tau_C} N(d_1)$ where N denotes the stadnard normal cumulative distribution function, and d_1 is given as

$$d_1 = \frac{log(F_t(V,\eta)) - log(K) + \frac{1}{2} IV^2 \tau_C}{IV \sqrt{\tau_C}}. \tag{3.2}$$

Here, IV denotes the implied volatility of the underlying. We use the observed option and futures prices and solve the Black Scholes formula for IV.

The Black-Scholes model assumes that the option's underlying follows a geometric Brownian motion. In our model V_t does not follow such a process. As a consequence, neither the VIX nor VIX futures prices follow a proportional process. In particular, the variance of the VIX futures price is largely driven by η, while its level is largely driven by V.

To evaluate the quality of the approximation, we compute the true delta from the model, $\frac{\partial C^*}{\partial F}$. The two prices C^* and F are both functions of V and η. An innovation in F can be due to an innovation in V or in η, and these two innovations may lead to different innovations in

C^*. The goal is to set up a portfolio that is neutral to innovations in the variance V, such that the portfolio is in the end only exposed to innovations in the volatility of volatility. It is therefore appropriate to consider the derivative in the V-*direction* only. We use the chain rule to come up with

$$C^*_{Fv} = \frac{\partial C^* / \partial V}{\partial F / \partial V} \tag{3.3}$$

as a proxy for $\frac{\partial C^*}{\partial F}$. C^*_{Fv} quantifies how much the VIX option price changes after a change in F that is due to a change in V.

Figure 3 shows both variants of delta as functions of the state variables V and η. The plotted deltas are for options with a strike price of $F(\bar{V}, \bar{\eta})$. We find that the two variants of delta have a similar shape, but C^*_{FV} is closer to 0 when the option is deep in the money and closer to 1 when the option is far out of the money. The Black Scholes delta of the at-the-money option is equal to 0.63 while C^*_{FV} of the at-the-money option is 0.60. In the empirical part of our paper we use mostly at-the-money options and test if the results are robust to variations in deltas. The impact of using either delta on model-implied option gains is analyzed further in Appendix B.3.

4 Variance Measures

4.1 Construction of Variance Measures

The VIX index is a model-free, forward-looking measure of implied volatility in the U.S. stock market, published by the Chicago Board Options Exchange (CBOE). The square of the VIX index is defined as in Equation (2.12) where $\tau = \frac{30}{365}$. Carr and Madan (1998), Britten-Jones and Neuberger (2000), and Jiang and Tian (2005) show that VIX^2 can be computed from the prices of call and put options with the same maturity at different strike prices:

$$VIX_t^2 = \frac{2e^{r_f \tau}}{\tau} \left[\int_0^{S_t^*} \frac{1}{K^2} P_t(K) dK + \int_{S_t^*}^{\infty} \frac{1}{K^2} C_t(K) dK \right], \tag{4.1}$$

where K is the strike price, C_t and P_t are the put and call prices, S_t^* is the fair forward price of the S&P500 index, and r_f is the risk-free rate. The VIX index published by the CBOE is discretized, truncated, and interpolated across the two nearest maturities to achieve a constant 30-day maturity.[7] Jiang and Tian (2005) show through simulation analysis that the approximations used in the VIX index calculation are quite accurate.

Since February 2006, options on the VIX have been trading on the CBOE, which give investors a way to trade the volatility of volatility. As of Q3 2012, the open interest in front-month VIX options was about 2.5 million contracts, which is similar to the open interest in front-month S&P500 index option contracts.

We calculate our measure of the implied volatility of volatility using the same method as the VIX, applied to VIX options instead of S&P500 options. The index, which has since been published by the CBOE as the "VVIX index" in 2012 and back-filled, is calculated as:

$$VVIX_t^2 = \frac{2e^{r_f \tau}}{\tau} \left[\int_0^{F_t} \frac{1}{K^2} P_t^*(K) dK + \int_{F_t}^{\infty} \frac{1}{K^2} C_t^*(K) dK \right], \tag{4.2}$$

where F_t is the VIX futures price, and C_t^*, P_t^* are the prices of call and put options on the VIX, respectively.[8] The squared VVIX is calculated from a portfolio of out-of-the-money call and put options on VIX futures contracts. It captures the implied volatility of VIX futures returns over the next 30-days, and is a model-free, forward-looking measure of the implied

[7] More details on the exact implementation of the VIX can be found in the white paper available on the CBOE website: http://www.cboe.com/micro/vix

[8] The official index is back-filled until 2007. We apply the same methodology and construct the index for an additional year back to 2006. The correlation between our measure of the VVIX and the published index is over 99% in the post-2007 sample. Our empirical results remain essentially unchanged if we restrict our sample to only the post-2007 period.

volatility of volatility.

In addition to the implied volatilities, we can also compute the realized volatilities for the stock market and the VIX. The construction here follows Barndorff-Nielsen and Shephard (2004) using high-frequency, intraday data.[9] Realized variance is defined as the sum of squared high-frequency log returns over the trading day:

$$RV_t = \sum_{j=1}^{N} r_{t,j}^2. \tag{4.3}$$

Barndorff-Nielsen and Shephard (2004) show that RV_t converges to the quadratic variation as $N \to \infty$. We follow the standard approach of considering 5 minute return intervals. A finer sampling frequency results in better asymptotic properties of the realized variance estimator, but also introduces more market microstructure noise such as the bid-ask bounce discussed in Heston, Korajczyk, and Sadka (2010). Liu, Patton, and Sheppard (2015) show that the 5 minute realized variance is very accurate, difficult to beat in practice, and is typically the ideal sampling choice in most applications combining accuracy and parsimony.

We estimate two realized variance measures, one for the S&P500 and one for the VIX. For the S&P500, we use the S&P500 futures contract and the resulting realized variance will be denoted RV^{SPX}. For the VIX, we use the spot VIX index and denote the resulting realized variance of the VIX by RV^{VIX}, which is our measure of the physical volatility of volatility. For robustness, we also entertain an alternative measure of the realized variance RV^{VX*} which is computed using the 5-minute VIX futures returns. The sample for this measure is shorter, and starts in July 2012.

[9]The data are obtained from http://www.tickdata.com.

4.2 Variance Dynamics

All of our variables are at the monthly frequency. The implied variance measures are given by the index values at the end of the month, and the realized variance measures are calculated over the past month and annualized. The sample for the benchmark measures runs from February 2006 to December 2016.

Table 2 presents summary statistics for the implied and realized variance measures. While the average level of the VIX is about 22%, the average level of the VVIX is much higher at about 88%, which captures the fact that VIX futures returns are much more volatile than market returns: volatility, itself, is very volatile. $VVIX^2$ is also much more volatile and less persistent than VIX^2, with an AR(1) coefficient of 0.33 compared to 0.81 for VIX^2. The VVIX exhibits relatively low correlation with the VIX, with a correlation coefficient of about 0.27. The mean of realized variance for S&P500 futures returns is 0.024, which corresponds to an annualized volatility of 15.5%. S&P500 realized variance is persistent and quite strongly correlated to the VIX index (correlation coefficient of 0.88) and much more weakly correlated to the VVIX index (correlation coefficient about 0.26). The realized variance of VIX is strongly related to the VVIX index (correlation of 0.61), and to a much lesser extent, the VIX index (correlation of 0.18).

In the last row of Table 2 we also consider the realized variance of VIX futures returns, since the VIX futures contract (not the spot VIX) is the underying asset for VIX options and there is no simple cost-of-carry relationship between VIX futures and spot VIX. Compared to spot VIX realized variance, VIX futures realized variance is lower on average, less volatile, more persistent, and more correlated with VIX. Recall, however, that our data for VIX futures is from 2012m7 to 2016m12 because of the availability of tick data.

Figure 4 shows the time-series of the VIX and VVIX from February 2006 to December 2016. There are some common prominent moves in both series, such as the a peak during the financial crisis. Notably, however, the VVIX also peaks during other times of economic uncertainty, such as the summer of 2007 (quant meltdown, beginning of the subprime crisis), May 2010 (Eurozone debt crisis, flash crash), August 2011 (U.S. debt ceiling crisis), and August 2015 (sell-off driven by the Chinese stock market crash). The movements in VIX during these events are far smaller than the spikes in the VVIX. The suggests that the VVIX captures important uncertainty-related risks in the aggregate market, distinct from the VIX itself.

In Figure 5 we present time-series plots for both S&P500 and VIX realized variances. As shown in Panel A, the realized and implied variances of the stock market follow a similar pattern, and S&P500 realized variance is nearly always below the implied variance. There is a large spike in both series around the financial crisis in October 2008, at which point realized variance exceeded implied variance. The difference between the mean of VIX^2 and RV^{SPX} is typically interpreted as a variance premium, which is the difference between end-of-month model-free, forward-looking implied variance calculated from S&P500 index options and the realized variance of S&P500 futures returns over the past month. Unconditionally, the average level of the VIX (22%) is greater than the average level of the S&P500 realized volatility (15.6%), so that the variance premium is positive, consistent with the evidence in Bollerslev et al. (2009) and Drechsler and Yaron (2011). This also implies a negative market price of volatility risk.

Panel B of the Figure shows the time series of the realized and implied volatility of the VIX index. Generally, the implied volatility tends to increase at times of pronounced spikes in the realized volatility. The implied volatility is also high during other times of economic distress and uncertainty, such as May 2010 (Eurozone debt crisis and flash crash), and August 2011 (U.S. debt ceiling crisis). During normal times, the VVIX is above the VIX realized variance, although during times of extreme distress we see the realized variance of VIX can exceed the VVIX. The average level of the VVIX (87.8%) is greater than the average level of the VIX realized volatility (81.8%), so that the volatility-of-volatility premium is also positive.[10] As we see in Table 2, VIX futures realized volatility is lower than VIX index realized volatility, so the actual magnitude of the volatility-of-volatility risk premium is even larger when compared to VIX futures. Similar to our discussion of the variance premium, this evidence suggests that investors dislike volatility-of-volatility risks, and the market price of these risks is negative.

In addition to unconditional moments, we can also analyze the conditional dependence of volatility on volatility of volatility. Specifically, we consider the predictability of future realized variances by the VIX and VVIX, in spirit of Canina and Figlewski (1993), Christensen and Prabhala (1998), and Jiang and Tian (2005) who use option implied volatilities to predict future realized volatilities. Following that approach, we consider both univariate and multivariate encompassing regressions to assess the predictability of future realized variances by the VIX and VVIX.

In our main specification, the dependent variable is the realized variance (RV) over the next month, for both the S&P500 and the VIX. Univariate regressions test whether each implied volatility measure (the VIX or the VVIX) can forecast future realized variances; multivariate encompassing regressions compare the relative forecasting importance of the VIX and VVIX and whether one implied volatility measure subsumes the information content of the other. The univariate regressions are restricted versions of the corresponding multivariate encompassing regression. For the S&P500 the regression is given as

$$RV_{t+1}^{SPX} = \beta_0 + \beta_1 VIX_t^2 + \beta_2 VVIX_t^2 + \beta_3 RV_t^{SPX} + \epsilon_{t+1}. \tag{4.4}$$

[10] Song (2013) shows that the average level of his VVIX measure, computed using numerical integration rather than the model-free VIX construction, is lower than the average realized volatility of VIX. One of the key differences between his and our computations is the frequency of returns used in the realized variance computations. Consistent with the literature, we rely on 5-minute returns to compute the realized variances, while Song (2013) uses daily returns.

Similarly for the VIX, we have:

$$RV_{t+1}^{VIX} = \beta_0 + \beta_1 VIX_t^2 + \beta_2 VVIX_t^2 + \beta_3 RV_t^{VIX} + \epsilon_{t+1}. \tag{4.5}$$

Our benchmark results are presented for all variables calculated in annualized variance units.

The first regression in Panel A of Table 3 shows that the VIX can forecast future realized variance of S&P500 returns. This is consistent with the findings of Jiang and Tian (2005). The VVIX can also forecast future S&P500 realized variance somewhat, although the statistical significance is weaker than that of the VIX and the magnitude of the regression coefficient is several times smaller. In the encompassing regression, we see that the VIX dominates the VVIX in forecasting future S&P500 realized variance. A one standard deviation increase in VIX^2 is associated with a 0.6 standard deviation increase in the realized variance of S&P500 returns next month. The coefficient on the VIX does not change much when we include the VVIX. Including lags of the realized variances themselves do not materially change the results. These results are consistent with our model specification.

Panel B of Table 3 shows our predictability results for VIX realized variance, which is our proxy for physical volatility of volatility. The VIX is not significantly related to future VIX realized variation; in fact, the point estimate is negative. The t-statistic is nearly zero, and the adjusted R^2 is below zero. In contrast, the VVIX is a significant predictor of future VIX realized variation. The regression coefficient for the VVIX is about 0.6 in a univariate regression, and is 0.7 in the multivariate regression. A one standard deviation increase in the current value of $VVIX^2$ is associated with more than 0.2 standard deviation increase in next month realized variance of VIX. In Panel C, we use VIX futures realized return volatility as the proxy for physical volatility-of-volatility and also find that VVIX is a significant predictor of future realized variance which drives out the VIX in a multivariate regression.

The empirical evidence suggests that fluctuations in the volatility of volatility are not directly related to the level of the volatility itself. This is consistent with our two-volatility model specification in Section 2. In many reduced form and structural models, the volatility of volatility is directly linked to the level of the volatility. For example, Heston (1993) models volatility as following a Cox, Ingersoll, and Ross (1985) square-root process. In that case, the level of volatility itself should forecast future realized volatility of volatility. The evidence in the data does not support this assumption, and supports richer dynamics of the volatility process with separate movements in the volatility of volatility.

5 Evidence from Options

In this section we analyze the implications of equity and VIX option price dynamics for the pricing of volatility and volatility-of-volatility risks in the data. Our model suggests that the market prices of volatility and volatility-of-volatility risks determine the key properties of the cross-section and time-series of delta-hedged equity and VIX option gains. Specifically, if the market prices of volatility and volatility-of-volatility risks are negative, the average delta-hedged equity and VIX option gains are also negative. In the cross-section, the average returns are more negative for option strategies which have higher exposure to the volatility and volatility-of-volatility risks. Finally, in the time series higher volatility and volatility of volatility predicts more negative gains in the future. We evaluate these model predictions in the data, and find a strong support that both volatility and volatility-of-volatility risks are priced in the option markets, and have negative market prices of risks.

5.1 Delta-Hedged Option Gains

We consider discrete-time counterparts to the continuously-rebalanced delta-hedged gains in Equations (2.7) and (2.17):

$$\begin{aligned}
\Pi_{t,t+\tau} &= \underbrace{C_{t+\tau} - C_t}_{\text{option gain/loss}} \underbrace{- \sum_{n=0}^{N-1} \Delta_{t_n} \left(S_{t_{n+1}} - S_{t_n} \right)}_{\text{delta hedging gain/loss}} + \underbrace{\sum_{n=0}^{N-1} r_f \left(\Delta_{t_n} S_{t_n} - C_t \right) \frac{\tau}{N}}_{\text{risk-free rate}}, \\
\Pi^*_{t,t+\tau} &= \underbrace{C^*_{t+\tau} - C^*_t}_{\text{option gain/loss}} \underbrace{- \sum_{n=0}^{N-1} \Delta_{t_n} \left(F_{t_{n+1}} - F_{t_n} \right)}_{\text{delta hedging gain/loss}} - \underbrace{\sum_{n=0}^{N-1} r_f C^*_t \frac{\tau}{N}}_{\text{risk-free rate}}.
\end{aligned} \tag{5.1}$$

Δ_{t_n} indicates the option delta at time t_n, e.g., for a call option, $\Delta_{t_n} = \frac{\partial C_{t_n}}{\partial S_{t_n}}$, and N is the number of trading days in the month. This discrete delta-hedging scheme is also used in Bakshi and Kapadia (2003) and Bertsimas et al. (2000).

At the close of each option expiration, we look at the prices of all options with non-zero open interest and non-zero trading volume. We take a long position in the option, and hedge each day using the Δ according to the Black-Scholes model, with the net investment earning the risk-free interest rate appropriately.[11] To minimize the effect of recording errors, we discard options that have implied volatilities below the 1st percentile or above the 99th percentile. All options have exactly one calendar month to maturity; S&P500 options expire on the third Friday of every month, while VIX options expire on the Wednesday that is 30 days away from the third Friday of the following month.

Table 4 shows average index and VIX delta-hedged option gains in our sample. We separate options by call or put, and group each option into four bins by moneyness to obtain eight bins for both S&P500 and VIX options. The column $\frac{\Pi}{S}$ gives the delta-hedged option

[11] This requires an estimate of the implied volatility of the option, which may require an option price. We use implied volatilities directly backed out from market prices of options whenever possible; if an option does not have a quoted price on any intermediate date, we fit a cubic polynomial to the implied volatility curve given by options with quoted prices, and back out the option's implied volatility. This is similar to typical option position risk management done by professional traders.

gain scaled by the initial index level, and the column $\frac{\Pi}{C}$ gives the delta-hedged option gain scaled by the initial option price, which can be interpreted more readily as a "return" in the traditional sense.

Panel A of Table 4 shows that on average out-of-the-money delta-hedged S&P500 call options have significantly negative returns. Likewise, delta-hedged put options on the S&P500 also have significantly negative returns at all levels of moneyness. To guard against potential outliers in option returns, we show that the results are similar using medians rather than average values; in percentages, delta-hedged index put options are negative about 80% of the time. This evidence is largely consistent with Bakshi and Kapadia (2003). In the model, negative average returns on delta-hedged index puts imply that volatility and/or volatility-of-volatility risks have a negative market price of risk. S&P500 option gains display mild positive serial correlation, which we will account for in our time-series predictive regressions in the later sections.

Panel B of Table 4 shows average returns for delta-hedged VIX options. The average delta-hedged VIX option returns are negative and statistically significant in all bins except for out-of-the-money puts and in-the-money calls, which are marginally significant. Call options lose more money as they become more out of the money, regardless of whether we are scaling by the index or by the option price. Estimates of the loss for call options ranges from -0.25% of the index value for in-the-money VIX calls to -1.20% of the index value for out-of-the-money VIX calls. When viewed as a percentage of the option price, delta-hedged VIX calls return about -1% per month at the money, and -20% out-of-the-money. The call option returns are negative 60% to 80% of the time. The results for VIX put options are similar. In the model, negative average returns for delta-hedged VIX options imply that investors dislike volatility-of-volatility risks, and are systematically paying a premium hedge against increases in the volatility of volatility. This suggests that the price of the volatility-of-volatility risk is negative. VIX option gains exhibit weak negative serial correlation, which we also account for in our time-series predictive regressions in the later sections. For both the S&P500 and VIX, the delta-hedged option gains are quite volatile.

In the next section, we provide further direct evidence by controlling for the exposures of the delta-hedged option positions to the underlying risks.

5.2 Cross-Sectional Evidence

As shown in the previous section, the average delta-hedged option gains are negative for S&P500 and VIX options. Our model further implies (see Equations (2.10) and (2.18)) that options with higher sensitivity to volatility and volatility-of-volatility risks should have more negative gains. To compute the estimates of option exposures to the underlying risks, we follow the approach of Bakshi and Kapadia (2003) which relies on using the Black and Scholes (1973) model to proxy for the true option betas.

Specifically, to compute the proxy for the option beta to volatility risk, we consider the vega of the option:

$$\frac{\partial C}{\partial \sigma} = S\sqrt{\frac{\tau}{2\pi}} e^{-\frac{d_1^2}{2}} \propto e^{-\frac{d_1^2}{2}}, \tag{5.2}$$

where $d_1 = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{S}{K} + \left(r_f - q + \frac{\sigma^2}{2}\right)\tau\right]$, q is the dividend yield, and σ is the implied volatility of the option. This approach allows us to compute proxies for the exposures of equity options to volatility risks, and of VIX options to the volatility-of-volatility risks.

To illustrate the relation between the moneyness of the option and the vega-measured exposure of options to volatility risks, we show the option vega as a function of option moneyness in Figure 6. Vega represents an increase in the value of the option as implied volatility increases by one percentage point. Higher volatility translates into higher future profits from delta-hedging due to the convexity effect; hence both call and put options have strictly positive vegas. Further, since the curvature of option value is the highest for at-the-money options, at-the-money options have the highest vega in the cross-section, and thus the largest exposure to volatility risks. An alternative way to proxy for the option sensitivity to volatility risks is to use the "gamma" of the option, which represents the second derivative of the option price to the underlying stock price: $\Gamma = \frac{\partial^2 C}{\partial S^2}$. As shown in Figure 6, the shape of the vega and gamma functions are almost identical, hence, the implied cross-sectional dispersion in volatility betas by moneyness are very similar as well.

To capture the sensitivity of option prices to the volatility of volatility, we compute the Black and Scholes (1973) second partial derivative of the option price with respect to the volatility, which is known in "volga" for "volatility gamma". Volga is calculated as:

$$\frac{\partial^2 C}{\partial \sigma^2} = S\sqrt{\frac{\tau}{2\pi}} e^{-\frac{d_1^2}{2}} \left(\frac{d_1 d_2}{\sigma}\right) = \frac{\partial C}{\partial \sigma} \left(\frac{d_1 d_2}{\sigma}\right), \tag{5.3}$$

where $d_2 = d_1 - \sigma\sqrt{\tau}$. Figure 7 shows the plot of volga as a function of the moneyness of the option. Volga is positive, and exhibits twin peaks with a valley around at-the-money. At-the-money options are essentially pure bets on volatility, and are approximately linear in volatility (see Stein (1989)). Therefore, the volga is the lowest for at-the-money options. Deep-out-of-the-money options and deep-in-the-money options do not have much sensitivity to volatility of volatility either, since for the former it is a pure directional bet, and for the latter the option value is almost entirely comprised of intrinsic value. Options that are somewhat away from at-the-money are most exposed to volatility-of-volatility risks.

Table 5 shows our cross-sectional evidence from the regressions of average option returns on our proxies of options' volatility and volatility-of-volatility betas. Panel A shows univariate and multivariate regressions of delta-hedged S&P500 option gains scaled by the index on the sensitivities of the options to volatility and volatility-of-volatility risks. Our encompassing regression for delta-hedged S&P500 options is:

$$GAINS^i_{t,t+\tau} = \frac{\Pi^i_{t,t+\tau}}{S_t} = \tilde{\lambda}_1 VEGA^i_t + \tilde{\lambda}_2 VOLGA^i_t + \gamma_t + \epsilon^i_{t,t+\tau}. \tag{5.4}$$

Since each date includes multiple options, as in Bakshi and Kapadia (2003) we allow for a date-specific component in $\Pi^i_{t,t+1}$ due to the option expirations, i.e., we include time-fixed effects represented by γ_t. Conceptually, our approach is related to Fama and MacBeth (1973) regressions. Instead of estimating risk betas in the first stage, due to the non-linear structure of option returns, we measure the exposures from economically motivated proxies for the risk sensitivities.[12]

The results in Panel A show that both volatility and the volatility of volatility are priced in the cross-section of delta-hedged S&P500 option returns. While univariate estimates for vega are insignificantly different from zero, univariate estimates for volga are significantly negative at -0.005 with a t-statistic of -4.28. In multivariate regressions, once we control for the effect of volga, the loading on vega becomes signicantly negative as we expect from the volatility premium. This demonstrates that especially in a period where we observe large spikes in volatility, it is important to control for exposure to volatility-of-volatility to properly see the impact of exposure to volatility on delta-hedged S&P500 option returns.

Panel B of Table 5 presents cross-sectional results for delta-hedged VIX options. As Equation (2.17) shows, delta-hedged VIX options are mainly exposed to volatility-of-volatility risks, and the vega for VIX options captures the sensitivity of VIX options to innovations in the volatility of volatility. The coefficient on vega of -1.23 is negative and statistically significant. Thus, in the cross-section of both S&P500 options and VIX options, we find strong evidence of a negative price of volatility-of-volatility risk.

[12]Song and Xiu (2016) demonstrate an alternative method of estimating risk sensitivities nonparametrically using local linear regression methods.

5.3 Time-Series Evidence

In the model, time-variation in the expected delta-hedged option gains is driven by V and η, and the loadings are determined by the market prices of volatility and volatility-of-volatility risk. We group options into the same bins as we used for average returns in Table 4, and average the scaled gains within each bin, so that we have a time-series of option returns for each moneyness bin. To examine the contribution of both risks for the time-variation in expected index option payoffs, we consider the following regression:

$$GAINS^i_{t,t+\tau} = \frac{\Pi^i_{t,t+\tau}}{S_t} = \beta_0 + \beta_1 VIX^2_t + \beta_2 VVIX^2_t + \gamma GAINS^i_{t-\tau} + u^i + \epsilon^i_{t+\tau}, \tag{5.5}$$

where we include fixed effects u^i to account for the heterogeneity in the sensitivity of options in different moneyness bins to the underlying risks. We regress the delta-hedged option gain scaled by the index from expiration to expiration on the value of the VIX and VVIX indices at the end of the earlier expiration; in other words, we run one-month ahead predictive regressions of delta-hedged option returns on the VIX and VVIX. We include lagged gains to adjust for serial correlation in the residuals, following Bakshi and Kapadia (2003).

Panel A of Table 6 shows the regression results for the index options. The univariate regression of delta-hedged S&P500 option gains on VIX^2 yields a negative coefficient which is statistically significant, consistent with Bakshi and Kapadia (2003). $VVIX^2$ also predicts future S&P500 option gains with a negative coefficient, and in a multivariate regression on both VIX and VVIX we find both loadings are negative and statistically significant.

It is important to point out that the true loadings on the volatility factors are time-varying, and generally depend on the underlying volatility and volatility-of-volatility states (see, e.g., Equation (2.11)). As shown in Bakshi and Kapadia (2003), these loadings are insensitive to volatility for at-the-money options. However, away from the money, the loadings may depend on the variance risk factors, so that the relation between delta-hedged gains and the variance-related risk factors is non-linear. To guard against finding a spurious coefficient on the VVIX, we repeat the exercise using only at-the-money options, and find very similar results.

Panel B of Table 6 shows the corresponding evidence for VIX options. The VVIX negatively and significantly predicts future VIX option gains. This is consistent with a negative market price of volatility-of-volatility risk. The results remain very similar for only at-the-money options.

6 Robustness

6.1 Alternative Variance Specifications

Our results for the predictability of realized by implied variance are robust to alternative specifications of volatility. Specifically, we consider regressing in volatility units or log-volatility units, rather than variance. The robustness regressions follow the form:

$$\sqrt{RV^{x}_{t,t+1}} = \beta_0 + \beta_1 VIX_t + \beta_2 VVIX_t + \epsilon_{t+1} \tag{6.1}$$

$$\ln \sqrt{RV^{x}_{t,t+1}} = \beta_0 + \beta_1 \ln VIX_t + \beta_2 \ln VVIX_t + \epsilon_{t+1} \tag{6.2}$$

where x refers to SPX or VIX.

In Table 7, we see that the point estimates and significance are very close to our baseline specification in variance units.

6.2 Sensitivity to Jump Risk Measures

The evidence in our paper highlights the roles of the volatility and volatility-of-volatility factors, which are driven by smooth Brownian motion shocks. In principle, the losses on delta-hedged option portfolios can also be attributed to large, discontinuous movements (jumps) in the stock market and in the market volatility. In this section we verify that our empirical evidence for the importance of the volatility-related factors is robust to the inclusion of jump measures considered in the literature.

Specifically, we consider three measures of jump risks, which we construct for the S&P500 and the VIX. Our first jump measure corresponds to the slope of the implied volatility curve:

$$\begin{aligned} SLOPE^{SPX} &= \sigma^{SPX}_{OTM} - \sigma^{SPX}_{ATM}, \\ SLOPE^{VIX} &= \sigma^{VIX}_{OTM} - \sigma^{VIX}_{ATM}. \end{aligned} \tag{6.3}$$

The OTM contract for the S&P500 options is defined as a put option with a moneyness closest to 0.9, and for VIX options as a call option with a moneyness closest to 1.1. In both cases, the ATM option has moneyness of 1. These slopes are positive for both index and VIX options. Positive slope of the index volatility smile is consistent with the notion of negative jumps in market returns (see e.g. Bates (2000), Pan (2002), Eraker, Johannes, and Polson (2003)), while the fact that the implied volatility curve for VIX options slopes upwards is consistent with positive volatility jumps (Drechsler and Yaron (2011) and Eraker and Shaliastovich (2008), among others). In this sense, these slope measures help capture the variation in the market and volatility jumps in the economy.

Our second jump measure incorporates the whole cross-section of option prices, beyond just the slope of the smile. It is based on the model-free risk-neutral skewness of Bakshi et al. (2003):

$$SKEW(t, t+\tau) = \frac{e^{r_f \tau} W_{t,t+\tau} - 3\mu_{t,t+\tau} e^{r_f \tau} V_{t,t+\tau} + 2\mu_{t,t+\tau}^3}{\left[e^{r_f \tau} V_{t,t+\tau} - \mu_{t,t+\tau}^2\right]^{3/2}}, \tag{6.4}$$

where $V_{t,t+\tau}, W_{t,t+\tau}, X_{t,t+\tau}$ are given by the prices of the volatility, cubic, and quartic contracts, and $\mu_{t,t+\tau}$ is function of them. Importantly, these measures are computed model-free using only observed option prices. The details for the computations are provided in the Appendix C.

Finally, our third measure of jump risks is based on high-frequency index and VIX data, rather than option prices. It corresponds to the realized jump intensity, and relies on the bipower variation methods in Barndorff-Nielsen and Shephard (2004), Huang and Tauchen (2005), and Wright and Zhou (2009). Specifically, while the realized variance defined in (4.3) captures both the continuous and jump variation, the bipower variation, defined as

$$BV_t = \frac{\pi}{2}\left(\frac{M}{M-1}\right)\sum_{j=2}^{M} |r_{t,j-1}||r_{t,j}| \tag{6.5}$$

measures the amount of continuous variation returns. Hence, we can use the following test statistic to determine if there is a jump on any given day:

$$J_t = \frac{\frac{RV_t - BV_t}{RV_t}}{\sqrt{\frac{\theta}{M}\max 1, \frac{QV_t}{BV_t^2}}}, \tag{6.6}$$

where $\theta = \left(\frac{\pi}{2}\right)^2 + \pi - 5$, and QV_t is the quad-power quarticity defined in Huang and Tauchen (2005) and Barndorff-Nielsen and Shephard (2004). The test statistic is standard normally distributed. We flag the day as having a jump if the probability exceeds 99.9% both for index returns and for the VIX. These cut-offs imply an average frequency of jumps of once every two months for the index, and about three jumps a month for the VIX. This is broadly consistent with the findings of Tauchen and Todorov (2011), who find that VIX jumps tend to happen much more frequently than S&P500 jumps. Over a month, we sum up all the days where we have a jump, and we define our jump intensity measure on a monthly level as:

$$RJ_t = \frac{1}{T}\sum_{i=0}^{T-1} J_{t+i},$$

where T is the number of trading days in the given month.

We use the jump statistics to document the robustness of the link between the volatility and volatility-of-volatility factors and options gains. We consider a regression of the form

$$GAINS^i_{t,t+\tau} = \beta_0 + \beta_1 VIX^2_t + \beta_2 VVIX^2_t + \beta_3 JUMP_t + \gamma GAINS^i_{t-\tau} + u^i + \epsilon^i_{t+\tau} \quad (6.7)$$

where $JUMP_t$ is one of the above jump risk proxies. We use index jump measures for index gains, and VIX jump measures for VIX gains.

Table 8 displays our results. Both for S&P500 and VIX options, controlling for $SLOPE$ does not change the ability of VIX and $VVIX$ to predict future delta-hedged option gains. Both factors are still significant, and the point estimates $\hat{\beta}_1$ and $\hat{\beta}_2$ remain largely unchanged. $SLOPE$ itself is not significant at conventional levels for S&P500 options but significantly positive for VIX options. The risk-neutral skewness also does not affect the predictive ability of VIX and $VVIX$, although, again, it is significant for VIX options. The two estimates of β_3 have the correct signs, since skewness is negative for S&P500 options and positive for VIX options; this is broadly similar to the findings of Bakshi and Kapadia (2003). Finally, we control for realized jump intensity RJ and we see a similar result where the statistical significance of VIX and $VVIX$ as well as their point estimates are largely unchanged. RJ seems to marginally predict future S&P500 option gains; for VIX options, however, RJ does not seem to predict future VIX option gains.

Hence, our evidence suggests that the VIX and VVIX have a significant impact on option returns even in the presence of stock market and volatility jumps. We leave a formal treatment of jumps for future research.

6.3 Sub-samples

To further investigate robustness of our results, we split our data into two sub-periods of roughly equal length, a pre-2012 and a post-2012 period. Tables 9 and 10 show the results in the two sub-periods. As we can see from the tables, the sub-period results are very similar to the full sample results in Table 4, and show large negative average S&P500 and VIX option gains, consistent with investors pricing volatility and volatility-of-volatility risks with negative market prices of risk.

6.4 Delta computations

We have used delta hedging to insulate our option returns from changes in the direction of the underlying, so that we can better focus on the volatility and volatility-of-volatility risks. We use Black-Scholes deltas to determine the appropriate hedge. While we have used simulation to examine the potential impact of using Black-Scholes deltas on our results, we also provide additional empirical robustness in a manner similar to Coval and Shumway (2001) by perturbing the deltas used in the option hedging. In Tables 11 and 12, we set the deltas to 0.95 and 1.05 times the value from the Black-Scholes model, respectively. We see that our results are not sensitive to the choice of using Black-Scholes deltas for option hedging and are quite robust.

7 Conclusion

Using S&P500 and VIX options data, we show that a time-varying volatility of volatility is a separate risk factor which affects option returns, above and beyond market volatility risks. We measure volatility risks using the VIX index, and volatility-of-volatility risk using the VVIX index. The two indices, constructed from the index and VIX option data, capture the ex-ante risk-neutral uncertainty of investors about future market returns and VIX innovations, respectively. The VIX and VVIX have separate dynamics, and are only weakly related in the data: the correlation between the two series is under 30%. On average, risk-neutral volatilities identified by the VIX and VVIX exceed the realized physical volatilities of the corresponding variables in the data. Hence, the variance premium and variance-of-variance premium for VIX are positive, which suggests that investors dislike variance and variance-of-variance risks.

We show the pricing implications of volatility and volatility-of-volatility risks using options market data. Average delta-hedged option gains are negative, which suggests that investors pay a premium to hedge against innovations in not only volatility but also the volatility of volatility. In the cross-section of both delta-hedged S&P500 options and VIX options, options with higher sensitivities to volatility-of-volatility risk earn more negative returns. In the time-series, higher values of the VVIX predict more negative delta-hedged option returns, for both S&P500 and VIX options.

Our findings are consistent with a no-arbitrage model which features time-varying market volatility and volatility-of-volatility factors. The volatility factors are priced by the investors, and in particular, volatility and volatility of volatility have negative market prices of risks.

A Delta-Hedged Equity Options

The state vector is $x_t = \begin{bmatrix} S_t & V_t & \eta_t \end{bmatrix}'$. Under the linear risk premium structure, $\lambda_t^V = \lambda^V V_t$ and $\lambda_t^\eta = \lambda^\eta \eta_t$. Note that since C_t is homogeneous of degree 1 in the underlying S_t and the strike price K, $\frac{\partial C}{\partial V}$ and $\frac{\partial C}{\partial \eta}$ are also homogeneous of degree 1 in S_t and K. Define a pair of functions:

$$\begin{aligned} g_1(x_t) &= \lambda_t^V \frac{\partial C_t}{\partial V_t} \\ g_2(x_t) &= \lambda_t^\eta \frac{\partial C_t}{\partial \eta_t} \end{aligned} \tag{A.1}$$

We can re-write Equation (2.8) as:

$$\mathbb{E}_t \left[\Pi_{t,t+\tau} \right] = \mathbb{E}_t \left[\int_t^{t+\tau} g_1(x_u) du \right] + \mathbb{E}_t \left[\int_t^{t+\tau} g_2(x_u) du \right]. \tag{A.2}$$

Define operators $\mathcal{L}$ and Γ such that:

$$\begin{aligned} \mathcal{L}[.] \; dt &= \frac{\partial [.]}{\partial S} \mu_t S_t dt + \frac{\partial [.]}{\partial V} \theta(V_t) dt + \frac{\partial [.]}{\partial \eta} \gamma(\eta_t) dt + \frac{\partial [.]}{\partial t} dt \\ &+ \frac{1}{2} \frac{\partial^2 [.]}{\partial S^2} [dS_t, dS_t] + \frac{1}{2} \frac{\partial^2 [.]}{\partial V^2} [dV_t, dV_t] + \frac{1}{2} \frac{\partial^2 [.]}{\partial \eta^2} [d\eta_t, d\eta_t] \\ &+ \frac{\partial^2 [.]}{\partial S \partial \eta} [dS_t, d\eta_t] + \frac{\partial^2 [.]}{\partial S \partial V} [dS_t, dV_t] + \frac{\partial^2 [.]}{\partial V \partial \eta} [dV_t, d\eta_t] \\ \Gamma[.] &= \left[\frac{\partial [.]}{\partial S} S_t \sqrt{V_t}, \; \frac{\partial [.]}{\partial V} \sqrt{\eta_t}, \; \frac{\partial [.]}{\partial \eta} \phi \sqrt{\eta_t} \right]. \end{aligned} \tag{A.3}$$

Then, for $u > t$, Itô's Lemma implies that:

$$g_1(x_u) = g_1(x_t) + \int_t^u \mathcal{L} g(x_{u'}) du' + \int_t^u \Gamma g(x_{u'}) dW_{u'}.$$

The integral in the first expectation on the right-hand side of Equation (A.2) becomes:

$$\begin{aligned} \int_t^{t+\tau} g_1(x_u) du &= \int_t^{t+\tau} \left[g_1(x_t) + \int_t^u \mathcal{L} g(x_{u'}) du' + \int_t^u \Gamma g(x_{u'}) dW_{u'} \right] du \\ &= g_1(x_t) \tau + \frac{1}{2} \mathcal{L} g_1(x_t) \tau^2 + \frac{1}{6} \mathcal{L}^2 g_1(x_t) \tau^3 + ... + \text{Itô Integrals} \\ &= \sum_{n=0}^{\infty} \frac{\tau^{1+n}}{(1+n)!} \mathcal{L}^n g_1(x_t) + \text{Itô Integrals}, \end{aligned}$$

and likewise for the second integral in (A.2). We can use this to re-write (A.2) as:

$$\begin{aligned}\mathbb{E}_t\left[\Pi_{t,t+\tau}\right] &= \mathbb{E}_t\left[\int_t^{t+\tau} g_1(x_u)du\right] + \mathbb{E}_t\left[\int_t^{t+\tau} g_2(x_u)du\right] \\ &= \sum_{n=0}^{\infty}\frac{\tau^{1+n}}{(1+n)!}\mathcal{L}^n\left[g_1(x_t)\right] + \sum_{n=0}^{\infty}\frac{\tau^{1+n}}{(1+n)!}\mathcal{L}^n\left[g_2(x_t)\right].\end{aligned} \tag{A.4}$$

Note that $g_1(x_t) = \alpha_1(V_t, \tau; K)S_t$, and $g_2(x_t) = \alpha_2(\eta_t, \tau; K)S_t$. By Lemma 1 of Bakshi and Kapadia (2003), $\mathcal{L}^n[g_1(x_t)]$ and $\mathcal{L}^n[g_2(x_t)]$ will also be proportional to S_t, which implies that:

$$\mathcal{L}^n\left[g_1(x_t)\right] = \lambda^V V_t \Phi_{t,n}^V S_t \;\; \forall n$$

$$\mathcal{L}^n\left[g_2(x_t)\right] = \lambda^\eta \eta_t \Phi_{t,n}^\eta S_t \;\; \forall n.$$

Therefore, we have:

$$\begin{aligned}\mathbb{E}_t\left[\Pi_{t,t+\tau}\right] &= \sum_{n=0}^{\infty}\frac{\tau^{1+n}}{(1+n)!}\mathcal{L}^n\left[g_1(x_t)\right] + \sum_{n=0}^{\infty}\frac{\tau^{1+n}}{(1+n)!}\mathcal{L}^n\left[g_2(x_t)\right] \\ &= S_t\left[\lambda^V\beta_t^V V_t + \lambda^\eta\beta_t^\eta\eta_t\right],\end{aligned}$$

which implies that:

$$\frac{\mathbb{E}_t\left[\Pi_{t,t+\tau}\right]}{S_t} = \lambda^V\beta_t^V V_t + \lambda^\eta\beta_t^\eta\eta_t, \tag{A.5}$$

where the sensitivities to the risk factors are given by:

$$\begin{aligned}\beta_t^V &= \sum_{n=0}^{\infty}\frac{\tau^{1+n}}{(1+n)!}\Phi_{t,n}^V > 0 \\ \beta_t^\eta &= \sum_{n=0}^{\infty}\frac{\tau^{1+n}}{(1+n)!}\Phi_{t,n}^\eta > 0.\end{aligned} \tag{A.6}$$

The betas are positive since $\frac{\partial C_t}{\partial V_t} > 0$ and $\frac{\partial C_t}{\partial \eta_t} > 0$.

B Delta-Hedged VIX Options

Unlike the expected gains for index options in (2.10), the expected gains for VIX options do not generally admit a linear factor structure because the option price is no longer homogeneous in the underlying asset value. We provide a numerical example below to investigate the patterns in delta-hedged VIX option gains and how they are related to the volatility of volatility.

B.1 Simulation study for model calibration

The squared VIX can be calculated in closed form within the model and is given by

$$VIX_t^2 = A + B\,V_t \tag{B.1}$$

where the coefficients A and B are

$$A = \frac{\kappa_V \bar{V}}{\kappa_V + \lambda^V}\left[1 - \frac{1 - e^{-(\kappa_V + \lambda^V)\tau}}{(\kappa_V + \lambda^V)\tau}\right], \qquad B = \frac{1 - e^{-(\kappa_V + \lambda^V)\tau}}{(\kappa_V + \lambda^V)\tau} \tag{B.2}$$

To calculate the time series moments of $VVIX^2$, we proceed as follows: We consider a grid of states (V_t, η_t), where the grid spans values between 0 and 3 times the steady state of both variables. The grid points serve as starting values for N paths of the state variables that we draw according to a discretized version of the dynamics in Equation (2.2). In particular, since η_t follows a square-root process, we draw η_{t+1} given η_t from a non-central χ^2-distribution

$$\eta_{t+\Delta t}^n = \frac{(1 - e^{-(\kappa_\eta + \lambda_\eta)\Delta t})\phi^2}{4(\kappa_\eta + \lambda_\eta)}\,\chi_d^2\left(\frac{4\kappa_\eta e^{-\kappa_\eta \Delta t}}{(1 - e^{-(\kappa_\eta + \lambda_\eta)\Delta t})\phi^2}\,\eta_t^n\right)(\omega_{\eta,t+\Delta t}^n) \tag{B.3}$$

for each $n = 1, \ldots, N$, with d degrees of freedom where

$$d = \frac{4\kappa_V}{\phi^2}\,\frac{\kappa_\eta + \lambda_\eta}{\kappa_V + \lambda_V}\,\bar{\eta}. \tag{B.4}$$

The increments in V are approximately Gaussian with

$$V_{t+\Delta t}^n = \mathcal{N}\Big(V_t^n + [\kappa_V \bar{V} - (\kappa_V + \lambda^V)V_t^n]\Delta t, \Delta t\ \eta_t^n\Big)(\omega_{V,t+\Delta t}^n). \tag{B.5}$$

To ensure non-negative realizations of the variance, we truncate the normal distribution at zero. To make the simulation results more reliable in small samples N, we use the same pseudo random numbers $(\omega_{\eta,t+\Delta t}^n, \omega_{V,t+\Delta t}^n)$ for each grid point.

We calculate the prices of VIX futures that mature in $t + \tau_F$. We assume a time to maturity τ_F of one month and choose $\Delta t = \frac{\tau_F}{30}$. For a given parameterization and a particular vector of initial state variables (V_t, η_t), we simulate the distribution of $V_{t+\tau_F}$ and $\eta_{t+\tau_F}$. We then calculate the time t futures price by

$$F_t(V_t, \eta_t, \tau_F) = \frac{1}{N}\sum_{n=1}^{N}\sqrt{A + B\ V_{t+\tau_F}^n} \tag{B.6}$$

The next step is calculate a grid of prices of VIX options that also mature in one month. These options are written on VIX futures that have a time to maturity of one month at the maturity of the option. To do so, we simulate state variables under $\mathbb{Q}$ as described above and calculate the VIX futures prices at maturity of the option using the grid of futures prices calculated earlier. Since the terminal values of the drawn state variables are between the points of the futures prices grid, we interpolate (or extrapolate if a state variable is outside the grid) the futures prices. The resulting futures price of the n-th path given initial values of (V_t, η_t) is denoted by $F_{t+\tau_C}^n(V_t, \eta_t, \tau_F)$. The call and put prices are

$$\begin{aligned} C_t^*(V_t, \eta_t, \tau_C, K) &= \frac{1}{N}e^{-r_f \tau_C}\sum_{n=1}^{N}(F_{t+\tau_C}^n(V_{t+\tau_C}, \eta_{t+\tau_C}, \tau_F) - K)^+, \\ P_t^*(V_t, \eta_t, \tau_C, K) &= \frac{1}{N}e^{-r_f \tau_C}\sum_{n=1}^{N}(K - F_{t+\tau_C}^n(V_{t+\tau_C}, \eta_{t+\tau_C}, \tau_F))^+ \end{aligned} \tag{B.7}$$

where K denotes the strike price. We use a grid of strike prices that ranges across values between $0.4\bar{K}$ and $1.6\bar{K}$, where $\bar{K}$ is the steady state futures price $F_t(\bar{V}, \bar{\eta}, \tau_C + \tau_F)$.

To calculate $VVIX^2$, we use VIX option prices calculated above and proceed similarly to the procedure that is applied to data. We numerically solve the integral

$$VVIX_t^2(V_t,\eta_t) = \frac{2e^{r_f\,\tau_C}}{\tau_C}\Big[\int_0^{F_t(V_t,\eta_t,\tau_C+\tau_F)} \frac{1}{K^2} P_t^*(V_t,\eta_t,\tau_C,K)dK \\ ... + \int_{F_t(V_t,\eta_t,\tau_C+\tau_F)}^{\infty} \frac{1}{K^2} C_t^*(V_t,\eta_t,\tau_C,K)dK\Big] \tag{B.8}$$

using our grid of strike prices as discretization. Futures prices $F_t(\cdot,\cdot,\tau_C+\tau_F)$ with longer maturity are calculated as described above.

Finally, we draw one long path of the state variables under $\mathbb{P}$. For that purpose, we proceed as described above but use the $\mathbb{P}$-dynamics of the state variables:

$$\eta_{t+\Delta t} = \frac{(1-e^{-\kappa_\eta \Delta t})\phi^2}{4\kappa_\eta}\,\chi_d^2\left(\frac{4\kappa_\eta e^{-\kappa_\eta\Delta t}}{(1-e^{-\kappa_\eta\Delta t})\phi^2}\,\eta_t\right)(\omega^0_{\eta,t+\Delta t}), \tag{B.9}$$

with d degrees of freedom where

$$d = \frac{4\kappa_\eta}{\phi^2}\,\bar{\eta} \tag{B.10}$$

and

$$V_{t+\Delta t} = \mathcal{N}\Big(V_t + \kappa_V(\bar{V}-V_t)\Delta t, \Delta t\;\eta_t\Big)(\omega^0_{V,t+\Delta t}). \tag{B.11}$$

Using our grid of VIX and $VVIX$, we can now study the time series properties of the model-implied quantities and compare them with the empirical moments.

B.2 Numerical procedure to solve Equation (3.1)

To solve the integral in Equation (3.1) numerically, we approximate the continuous path of $\frac{\partial C_s^*}{\partial \eta_s}$ by a discretization with step size Δt:

$$\int_t^{t+\tau} \mathbb{E}_t\left[\frac{\partial C_s^*}{\partial \eta_s}\lambda_s^\eta\right] ds \approx \frac{1}{N}\sum_{n=1}^{N}\sum_{i=0}^{I-1} \frac{\partial C^*_{t+i\Delta t}(V^n_{t+i\Delta t},\eta^n_{t+i\Delta t},\tau_C - i\Delta t, K)}{\partial \eta_{t+i\Delta t}}\lambda^\eta_{t+i\Delta t}\Delta t, \tag{B.12}$$

where $I\Delta t = \tau$. We implement this equation in the following way: First, we compute grids of VIX option prices according to the procedure outlined in Appendix B.1 also for shorter times to maturity, in particular, $\tau_C - i\Delta t$ for $i = 0, \ldots, I$. Second, we approximate the derivative of C^* with respect to η using finite differences, i.e., at the grid point (V_{m_V}, η_{m_η}), we use

$$\frac{\partial C^*(V_{m_V}, \eta_{m_\eta})}{\partial \eta_{m_\eta}} \approx \frac{C^*(V_{m_V}, \eta_{m_\eta+1}) - C^*(V_{m_V}, \eta_{m_\eta-1})}{\eta_{m_\eta+1} - \eta_{m_\eta-1}}. \tag{B.13}$$

Third, we again simulate innovations in the state variables under $\mathbb{P}$. After each simulation step, we end up in states which are between two grid points or even outside the grid. Because we do not have derivatives at these points, we proceed as follows. We start with the calculation of expected option gains over the last time step from $t + \tau_C - \Delta t$ to $t + \tau_C$. This gives us a grid of expected option gains over this short period. We then simulate increments in the state variables in the period from $t + \tau_C - 2\Delta t$ to $t + \tau_C - \Delta t$, starting again with the usual (V, η)-grid and add up expected gains over this period with expected gains over the final period given the state that is drawn using the scheme that approximates the $\mathbb{P}$ dynamics (see Section 3.1). We do so by interpolating the grid produced before. We use the same procedure and iterate back until time t.

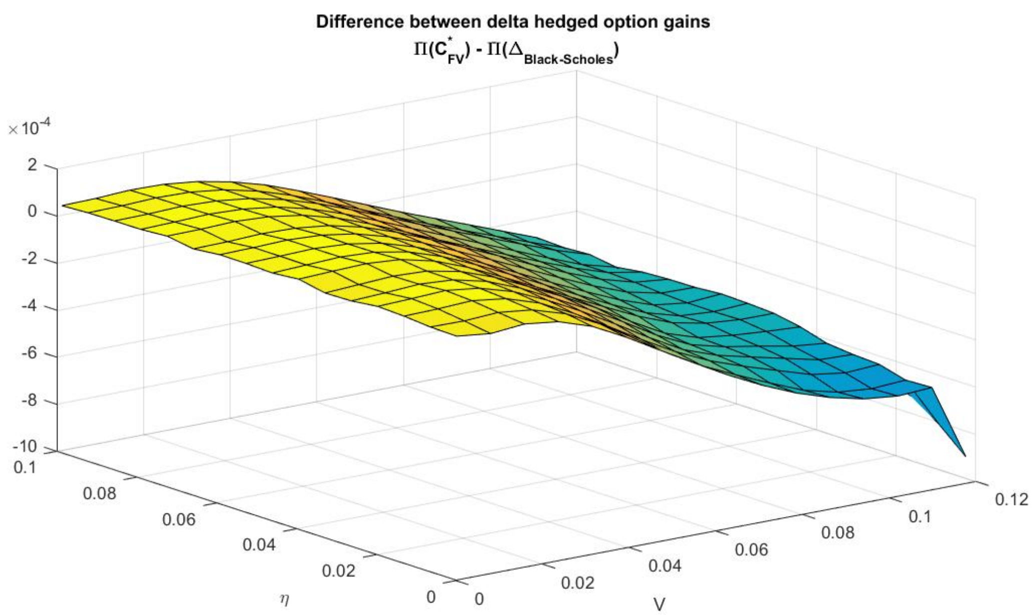
B.3 Spread in trading strategy using different deltas

Instead of using Equation (2.18) to calculate expected option gains, one could alternatively simulate realized gains from the trading strategy and take the average across realizations. For this purpose, we would have to calculate deltas of the VIX options and set up a delta hedged portfolio. This procedure, however, has the disadvantage that we need much larger samples to obtain stable results, because of the fluctuations in the Brownian motion W_3 that are averaged out when taking the expectation (see Equation (2.17)). While this task is very costly in terms of computation time, one can look at realized delta hedged option gains path by path and study the difference between gains when different deltas are used for setting up the hedge portfolio.

We now compare the difference between gains from our trading strategy when using the Black-Scholes delta or the "true" delta C^*_{FV} (see Section 3) for setting up the hedge portfolio. Even with noisy estimates of the expected gains, the difference between both gains is informative because we use the same paths of the state variables in the simulation of the two strategies. Thus, the sampling error widely cancels out when considering the difference.

Figure B.1 shows the difference between delta hedged option gains when using either delta. For moderate values of V the difference is close to zero. When V is high, the strategy that uses the Black-Scholes delta yields less negative returns than the strategy that uses C^*_{FV}. In the light of this result, we can conclude that using Black-Scholes deltas is a rather conservative strategy. In our empirical exercise, we would expect even more pronounced gains when using C^*_{FV}, which is, however, not observable in the data.

Figure B.1: Difference Between Delta-Hedged Option Gains



The figure plots the difference between the delta-hedged option gains using the model-implied versus the Black-Scholes delta. All options are at-the-money.

C Risk-Neutral Skewness

The prices of the volatility, cubic, and quartic contracts $V_{t,t+\tau}, W_{t,t+\tau}, X_{t,t+\tau}$ are given

$$
\begin{aligned}
V_{t,t+\tau} &= \int_{S_t}^{\infty} \frac{2(1-\log\frac{K}{S_t})}{K^2} C_t(t+\tau;K)dK + \int_0^{S_t} \frac{2(1+\log\frac{S_t}{K})}{K^2} P_t(t+\tau;K)dK,\\
W_{t,t+\tau} &= \int_{S_t}^{\infty} \frac{6\log\frac{K}{S_t} - 3(\log\frac{K}{S_t})^2}{K^2} C_t(t+\tau;K)dK\\
&\quad - \int_0^{S_t} \frac{6\log\frac{S_t}{K} + 3(\log\frac{S_t}{K})^2}{K^2} P_t(t+\tau;K)dK,\\
X_{t,t+\tau} &= \int_{S_t}^{\infty} \frac{12(\log\frac{K}{S_t})^2 - 4(\log\frac{K}{S_t})^3}{K^2} C_t(t+\tau;K)dK\\
&\quad + \int_0^{S_t} \frac{12(\log\frac{S_t}{K})^2 + 4(\log\frac{S_t}{K})^3}{K^2} P_t(t+\tau;K)dK,
\end{aligned}
$$

and $\mu_{t,t+\tau} = e^{r_f\tau} - 1 - \frac{e^{r_f\tau}}{2}V_{t,t+\tau} - \frac{e^{r_f\tau}}{6}W_{t,t+\tau} - \frac{e^{r_f\tau}}{24}X_{t,t+\tau}$.

To construct these measures, we use out-of-the-money options to mitigate liquidity concerns. Following Shimko (1993), each day we interpolate the Black-Scholes implied volatility curve at the observable strikes using a cubic spline, and then calculate option prices to compute the above moments. We construct these measures for both S&P500 options and VIX options. Our implied volatility slope and risk-neutral skewness measures are calculated using options with the same maturity as our test assets.

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Tables and Figures

Table 1: **Model parameters**

Variance	$\bar{V}$	κ_V		λ_V
	$.1979^2$	2.4982		-4.8
Vol-of-vol	$\bar{\eta}$	κ_η	ϕ	λ_η
	.0329	13.097	.0694	-5
Risk-free rate	r_f			
	.0102			
Times to maturity	Δt	τ	τ_F	τ_C
	1/360	2/360	30/360	10/360

The table shows the calibrated parameters for the model. Variance and vol-of-vol refers to the parameters of the variance and volatility-of-volatility dynamics, respectively. Times-to-maturity indicate the time intervals at annual frequency.

Table 2: **Summary Statistics**

variable	mean	std.	AR(1)	corr. VIX^2	corr. $VVIX^2$
VIX^2	0.048	0.053	0.812	1.000	0.269
$VVIX^2$	0.771	0.214	0.333	0.269	1.000
RV^{SPX}	0.024	0.050	0.631	0.880	0.258
RV^{VIX}	0.670	0.727	0.182	0.182	0.609
RV^{VX*}	0.306	0.181	0.284	0.726	0.640

The table shows summary statistics for the implied and realized variances of S&P500 and the VIX index. The implied variances are computed from the option data: VIX^2 corresponds to $\left(\frac{VIX}{100}\right)^2$ and $VVIX^2$ stands for $\left(\frac{VVIX}{100}\right)^2$. Realized variances are annualized and computed using 5-minute data on S&P500 returns (RV^{SPX}); log VIX index innovations (RV^{VIX}); and VIX futures returns (RV^{VX*}). The data are monthly from 2006m2 to 2016m12; VIX futures high-frequency data are from 2012m7 to 2016m12.

Table 3: **Predictability of Realized Variance Measures**

	VIX^2		$VVIX^2$		R^2_{adj}
	slope	t-stat.	slope	t-stat.	
S&P500 Index					
$RV^{SPX}_{t,t+1}$	0.604	[4.82]			39.42
			0.039	[1.48]	2.01
	0.605	[4.53]	-0.001	[-0.10]	38.95
VIX Index					
$RV^{VIX}_{t,t+1}$	-0.607	[-0.57]			-0.35
			0.612	[3.56]	2.49
	-1.384	[-1.34]	0.704	[4.05]	2.67
VIX Futures					
$RV^{VX*}_{t,t+1}$	3.215	[3.07]			2.55
			0.243	[2.24]	7.19
	-0.935	[-0.46]	0.282	[2.34]	5.49

The table shows the evidence from the projections of future realized variances of S&P500 and the VIX index on the current implied variances. Numbers in brackets indicate Newey-West t-statistics with 6 lags. The data are monthly from 2006m2 to 2016m12; VIX futures high-frequency data are from 2012m7 to 2016m12.

Table 4: **Delta-Hedged Option Gains**

	moneyness	$\frac{\Pi}{S}(\%)$ mean	t-stat.	median	$\% < 0$	std.	AR(1)	$\frac{\Pi}{C}(\%)$ mean	t-stat.	median
S&P500										
Call	0.950 to 0.975	0.07	[3.23]	0.00	50%	0.55	0.36	1.61	[3.71]	0.01
	0.975 to 1.000	0.04	[2.63]	-0.02	52%	0.50	0.19	2.38	[3.72]	-0.79
	1.000 to 1.025	-0.04	[-2.64]	-0.05	57%	0.44	0.09	-2.45	[-1.74]	-5.90
	1.025 to 1.050	-0.08	[-6.51]	-0.05	62%	0.38	0.14	-32.98	[-9.00]	-22.50
Put	0.950 to 0.975	-0.14	[-7.86]	-0.18	81%	0.54	0.35	-19.27	[-9.01]	-28.74
	0.975 to 1.000	-0.16	[-9.89]	-0.19	75%	0.50	0.20	-10.24	[-7.93]	-16.89
	1.000 to 1.025	-0.22	[-13.17]	-0.22	75%	0.48	0.04	-8.77	[-13.99]	-10.85
	1.025 to 1.050	-0.25	[-12.11]	-0.23	78%	0.47	0.05	-5.84	[-12.72]	-5.97
VIX										
Call	0.800 to 0.900	-0.25	[-1.66]	-0.59	62%	2.26	-0.03	-1.38	[-1.54]	-3.12
	0.900 to 1.000	-0.77	[-3.83]	-1.26	70%	2.95	-0.09	-6.76	[-3.80]	-12.40
	1.000 to 1.100	-0.84	[-3.54]	-1.37	71%	3.24	-0.17	-10.03	[-3.34]	-16.29
	1.100 to 1.200	-1.20	[-5.04]	-1.52	77%	3.16	-0.13	-19.01	[-4.81]	-25.49
Put	0.800 to 0.900	-0.50	[-1.67]	-0.90	64%	2.69	-0.06	-12.76	[-1.55]	-25.18
	0.900 to 1.000	-0.80	[-3.73]	-1.29	71%	3.05	-0.14	-10.91	[-3.08]	-21.59
	1.000 to 1.100	-0.84	[-3.42]	-1.45	70%	3.31	-0.18	-6.13	[-3.17]	-11.02
	1.100 to 1.200	-1.17	[-4.86]	-1.31	77%	3.16	-0.14	-5.53	[-4.77]	-6.45

The table shows average delta-hedged option gains on the S&P500 and VIX options across their moneyness. Options have one month to maturity, are grouped into an equal-weighted portfolio inside the moneyness bin, and are held till expiration. The delta-hedge is computed using the Black-Scholes formula, with daily rebalancing and the margin difference earning the risk-free rate. The delta-hedged option gains Π are scaled either by the index or by the option price. The t-statistics are testing the null that the delta-hedged option gain is equal to zero. The $\% < 0$ column shows the fraction of observations with negative gains. The data are monthly from 2006m2 to 2016m12.

Table 5: **Delta-Hedged Option Gains by Volatility Risk Sensitivities**

	Vega $= \frac{\partial C}{\partial \sigma}$		Volga $= \frac{\partial^2 C}{\partial \sigma^2}$	
	slope	t-stat.	slope	t-stat.
SPX Options				
$\frac{\Pi_{t,t+1}}{S_t}$	0.009	[0.61]		
			-0.005	[-4.28]
	-0.104	[-4.40]	-0.013	[-6.21]
VIX Options				
$\frac{\Pi_{t,t+1}}{S_t}$	-1.23	[-3.35]		

The table shows the evidence from the cross-sectional regressions of the delta-hedged S&P500 and VIX option gains on the vega and volga of the options. Regressions include time fixed effects, and option characteristics are computed using the Black-Scholes formula. Number is brackets indicate Newey-West t-statistics. The data are monthly from 2006m2 to 2016m12.

Table 6: **Predictability of Delta-Hedged Option Gains**

	VIX^2		$VVIX^2$		$GAINS_{t-1}$	
	slope	t-stat.	slope	t-stat.	slope	t-stat.
SPX Options						
$\frac{\Pi_{t,t+\tau}}{S_t}$	-0.69	[-3.42]			0.24	[4.08]
			-0.26	[-2.87]	0.23	[4.16]
	-0.50	[-3.41]	-0.23	[-2.75]	0.26	[4.20]
SPX Options: ATM only						
$\frac{\Pi_{t,t+\tau}}{S_t}$	-0.53	[-2.81]			0.16	[3.43]
			-0.25	[-2.41]	0.18	[3.22]
	-0.35	[-2.28]	-0.24	[-2.29]	0.19	[3.38]
VIX Options						
$\frac{\Pi_{t,t+\tau}}{S_t}$			-1.01	[-3.38]	-0.10	[-7.07]
VIX Options: ATM only						
$\frac{\Pi_{t,t+\tau}}{S_t}$			-0.77	[-4.43]	-0.11	[-6.89]

The table shows the evidence of predictability of future delta-hedged S&P500 and VIX option gains by the implied volatility and volatility-of-volatility measures. The regressions across all monenyness bins include moneyness fixed effects; the regressions for the ATM options are for the moneyness ranges between 0.975 to 1.025 for the index and 0.9 to 1.1 for the VIX options. Lag gains are included to correct for serial correlation of the residuals. Number is brackets indicate Newey-West t-statistics. The data are monthly from 2006m2 to 2016m12.

Table 7: **Predictability of Realized Variance Measures: Alternate Specifications**

	VIX		$VVIX$		$\ln VIX^2$		$\ln VVIX^2$	
	slope	t-stat.	slope	t-stat.	slope	t-stat.	slope	t-stat.
S&P500								
$\sqrt{RV_{t,t+1}^{SPX}}$	0.723	[6.37]						
			0.128	[1.63]				
	0.732	[5.98]	-0.023	[-0.52]				
$\ln RV_{t,t+1}^{SPX}$					0.964	[8.70]		
							0.528	[1.38]
					0.986	[8.26]	-0.215	[-1.07]
VIX								
$\sqrt{RV_{t,t+1}^{VIX}}$	0.200	[-0.57]						
			0.687	[5.10]				
	-0.495	[-1.42]	0.789	[5.15]				
$\ln RV_{t,t+1}^{VIX}$					-0.069	[-0.68]		
							0.752	[5.18]
					-0.161	[-1.57]	0.873	[4.80]

The table shows the evidence from the alternative projections of future realized variances of S&P500 and the VIX index on the current implied volatility and volatility of volatility measures. The realized and implied variances are modified to be expressed in standard deviation or log units. Numbers in brackets indicate Newey-West t-statistics with 6 lags. The data are monthly from 2006m2 to 2016m12.

Table 8: **Predictability of Delta-Hedged Option Gains: Robustness to Jump Measures**

	VIX^2		$VVIX^2$		$SLOPE$		$SKEW$		RJ		$GAINS_{t-1}$	
	slope	t-stat.	slope	t-stat.	slope	t-stat.	slope	t-stat.	slope	t-stat.	slope	t-stat.
SPX Options												
$\frac{\Pi_{t,t+\tau}}{S_t}$	-0.64	[-2.49]	-0.21	[-3.25]	-1.44	[-1.02]					0.27	[4.00]
	-0.49	[-2.58]	-0.23	[-2.92]			-0.01	[-0.15]			0.26	[4.06]
	-0.49	[-3.44]	-0.23	[-2.75]					0.01	[1.81]	0.26	[4.20]
VIX Options												
$\frac{\Pi_{t,t+\tau}}{S_t}$			-0.98	[-3.19]	2.55	[2.60]					-0.10	[-7.14]
			-0.76	[-2.17]			0.34	[3.64]			-0.10	[-7.58]
			-1.39	[-3.59]					0.03	[1.12]	-0.08	[-3.55]

The table shows the evidence of predictability of future delta-hedged S&P500 and VIX option gains by the implied volatility and volatility-of-volatility measures, controlling for the jump risk measures. The cross-sectional regressions include moneyness fixed effects, and lag gains are included to correct for serial correlation of the residuals. The jump measures are as follows. $SLOPE$ is the slope of the implied volatility smile, calculated as Black-Scholes implied volatility of an out-of-the-money ($\frac{K}{S} = 0.9$) minus the implied volatility of an at-the-money put option ($\frac{K}{S} = 1$) for S&P500 options, and between a $\frac{K}{S} = 1.1$ call option and an at-the-money call option for the VIX options. RJ is the realized jump variation calculated using high-frequency S&P500 futures tick data for the S&P500 and VIX tick data for the VIX. $SKEW$ is the model-free measure of risk-neutral skewness. Number is brackets indicate Newey-West t-statistics. The data are monthly from 2006m2 to 2016m12.

Table 9: **Delta-Hedged Option Gains: Pre-2012 Sub-Sample**

	moneyness	$\frac{\Pi}{S}(\%)$ mean	t-stat.	median	$\% < 0$	std.	AR(1)	$\frac{\Pi}{C}(\%)$ mean	t-stat.	median
S&P500										
Call	0.950 to 0.975	0.12	[2.67]	-0.04	54%	0.76	0.40	2.48	[2.94]	-0.85
	0.975 to 1.000	0.04	[1.12]	-0.02	53%	0.64	0.26	1.61	[1.52]	-0.94
	1.000 to 1.025	-0.08	[-2.92]	-0.11	61%	0.58	0.11	-5.83	[-3.00]	-11.16
	1.025 to 1.050	-0.13	[-4.92]	-0.07	63%	0.51	0.12	-28.22	[-5.74]	-17.53
Put	0.950 to 0.975	-0.12	[-3.31]	-0.25	81%	0.77	0.38	-12.24	[-3.23]	-28.10
	0.975 to 1.000	-0.19	[-5.98]	-0.31	75%	0.66	0.24	-9.36	[-4.68]	-18.05
	1.000 to 1.025	-0.27	[-8.64]	-0.34	72%	0.62	0.05	-9.64	[-9.24]	-12.92
	1.025 to 1.050	-0.30	[-8.58]	-0.30	76%	0.57	0.06	-6.51	[-8.80]	-7.33
VIX										
Call	0.800 to 0.900	-0.62	[-2.96]	-0.65	68%	2.16	-0.10	-3.74	[-3.10]	-3.66
	0.900 to 1.000	-1.49	[-5.63]	-2.01	75%	2.62	-0.21	-13.00	[-5.48]	-15.61
	1.000 to 1.100	-1.02	[-3.16]	-1.46	73%	2.90	-0.10	-11.97	[-2.72]	-20.79
	1.100 to 1.200	-1.38	[-4.25]	-1.60	80%	2.89	-0.08	-22.24	[-3.72]	-29.28
Put	0.800 to 0.900	-0.65	[-1.75]	-1.37	64%	2.64	-0.10	-14.14	[-1.28]	-34.73
	0.900 to 1.000	-1.40	[-4.94]	-1.96	74%	2.77	-0.21	-19.34	[-3.98]	-32.81
	1.000 to 1.100	-0.97	[-2.82]	-1.56	74%	3.05	-0.13	-7.01	[-2.48]	-12.75
	1.100 to 1.200	-1.29	[-3.88]	-1.38	79%	2.87	-0.07	-6.12	[-3.68]	-7.49

The table shows average delta-hedged option gains on the S&P500 and VIX options across their moneyness, in the pre-2012 sub-sample. Options have one month to maturity, are grouped into an equal-weighted portfolio inside the moneyness bin, and are held till expiration. The delta-hedge is computed using the Black-Scholes formula, with daily rebalancing and the margin difference earning the risk-free rate. The delta-hedged option gains Π are scaled either by the index or by the option price. The t-statistics are testing the null that the delta-hedged option gain is equal to zero. The $\% < 0$ column shows the fraction of observations with negative gains. The data are monthly from 2006m2 to 2011m12.

Table 10: **Delta-Hedged Option Gains: Post-2012 Sub-Sample**

	moneyness	$\frac{\Pi}{S}(\%)$ mean	t-stat.	median	% < 0	std.	AR(1)	$\frac{\Pi}{C}(\%)$ mean	t-stat.	median
Panel A: S&P500										
Call Options	0.950 to 0.975	0.03	[2.05]	0.01	46%	0.27	-0.07	0.88	[2.40]	0.31
	0.975 to 1.000	0.05	[3.21]	-0.01	51%	0.35	-0.14	3.00	[3.83]	-0.40
	1.000 to 1.025	0.00	[0.21]	-0.03	54%	0.29	-0.08	0.22	[0.11]	-3.70
	1.025 to 1.050	-0.05	[-5.08]	-0.04	62%	0.20	0.17	-36.85	[-6.95]	-32.35
Put Options	0.950 to 0.975	-0.15	[-13.37]	-0.14	81%	0.26	0.02	-24.72	[-10.37]	-29.33
	0.975 to 1.000	-0.14	[-9.75]	-0.15	75%	0.32	-0.03	-10.96	[-6.49]	-15.59
	1.000 to 1.025	-0.17	[-11.76]	-0.18	77%	0.31	-0.11	-8.06	[-10.69]	-9.39
	1.025 to 1.050	-0.20	[-9.91]	-0.21	82%	0.31	-0.01	-5.11	[-9.87]	-5.39
Panel B: VIX										
Call Options	0.800 to 0.900	0.07	[0.36]	-0.25	57%	2.30	-0.05	0.64	[0.50]	-1.57
	0.900 to 1.000	-0.17	[-0.61]	-0.87	66%	3.08	-0.08	-1.63	[-0.65]	-6.79
	1.000 to 1.100	-0.71	[-2.07]	-1.21	69%	3.49	-0.23	-8.52	[-2.07]	-13.85
	1.100 to 1.200	-1.05	[-3.06]	-1.32	75%	3.36	-0.19	-16.37	[-3.10]	-19.43
Put Options	0.800 to 0.900	-0.25	[-0.50]	-0.86	65%	2.79	0.22	-10.54	[-0.85]	-20.76
	0.900 to 1.000	-0.26	[-0.85]	-0.90	68%	3.19	-0.13	-3.43	[-0.68]	-12.19
	1.000 to 1.100	-0.74	[-2.15]	-1.18	66%	3.50	-0.22	-5.46	[-2.06]	-8.49
	1.100 to 1.200	-1.08	[-3.15]	-1.26	76%	3.38	-0.19	-5.08	[-3.15]	-5.49

The table shows average delta-hedged option gains on the S&P500 and VIX options across their moneyness, in the post-2012 sub-sample. Options have one month to maturity, are grouped into an equal-weighted portfolio inside the moneyness bin, and are held till expiration. The delta-hedge is computed using the Black-Scholes formula, with daily rebalancing and the margin difference earning the risk-free rate. The delta-hedged option gains Π are scaled either by the index or by the option price. The t-statistics are testing the null that the delta-hedged option gain is equal to zero. The % $<$ 0 column shows the fraction of observations with negative gains. The data are monthly from 2012m1 to 2016m12.

Table 11: **Delta-Hedged Option Gains: 0.95 $\times$ Δ_{BS} Delta**

		$\frac{\Pi}{S}(\%)$						$\frac{\Pi}{C}(\%)$		
	moneyness	mean	t-stat.	median	% < 0	std.	AR(1)	mean	t-stat.	median
S&P500										
Call	0.950 to 0.975	0.07	[3.63]	0.04	45%	0.51	0.36	1.70	[4.21]	0.81
	0.975 to 1.000	0.05	[3.29]	-0.01	52%	0.46	0.23	2.62	[4.38]	-0.30
	1.000 to 1.025	-0.03	[-2.50]	-0.05	57%	0.42	0.07	-2.08	[-1.54]	-4.97
	1.025 to 1.050	-0.08	[-6.86]	-0.05	62%	0.36	0.11	-31.52	[-8.87]	-22.31
Put	0.950 to 0.975	-0.14	[-7.24]	-0.20	83%	0.61	0.35	-20.09	[-8.85]	-31.90
	0.975 to 1.000	-0.17	[-9.01]	-0.22	77%	0.59	0.20	-11.02	[-7.68]	-20.48
	1.000 to 1.025	-0.23	[-11.36]	-0.26	75%	0.60	0.05	-9.34	[-12.36]	-13.36
	1.025 to 1.050	-0.28	[-10.16]	-0.31	74%	0.62	0.06	-6.47	[-10.75]	-7.71
VIX										
Call	0.800 to 0.900	-0.38	[-2.17]	-0.83	61%	2.65	-0.05	-2.13	[-2.03]	-4.46
	0.900 to 1.000	-0.90	[-4.06]	-1.70	69%	3.26	-0.12	-7.80	[-3.94]	-14.21
	1.000 to 1.100	-0.94	[-3.65]	-1.65	73%	3.50	-0.15	-11.05	[-3.36]	-20.47
	1.100 to 1.200	-1.32	[-5.38]	-1.73	79%	3.25	-0.13	-20.90	[-5.11]	-29.08
Put	0.800 to 0.900	-0.44	[-1.51]	-0.73	63%	2.62	0.07	-11.33	[-1.44]	-23.15
	0.900 to 1.000	-0.73	[-3.52]	-1.03	70%	2.95	-0.14	-10.13	[-2.99]	-16.30
	1.000 to 1.100	-0.74	[-3.14]	-0.79	66%	3.20	-0.18	-5.48	[-2.97]	-6.53
	1.100 to 1.200	-1.05	[-4.31]	-0.86	69%	3.18	-0.12	-4.97	[-4.30]	-4.87

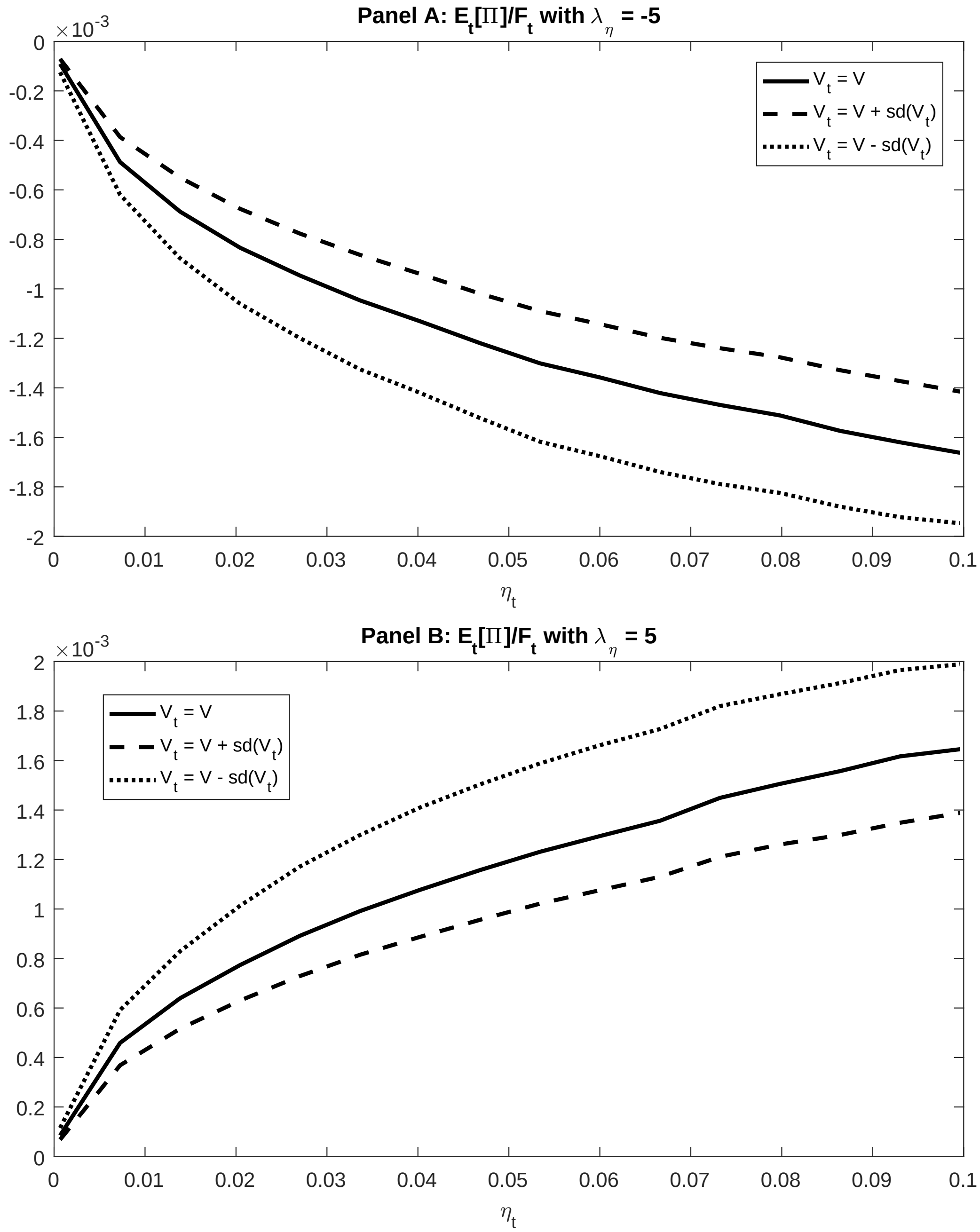
The table shows average delta-hedged option gains on the S&P500 and VIX options across their moneyness, when the Black-Scholes delta is reduced to 0.95 $\times$ Δ_{BS}. Options have one month to maturity, are grouped into an equal-weighted portfolio inside the moneyness bin, and are held till expiration. The delta-hedge is computed using the Black-Scholes formula, with daily rebalancing and the margin difference earning the risk-free rate. The delta-hedged option gains Π are scaled either by the index or by the option price. The t-statistics are testing the null that the delta-hedged option gain is equal to zero. The % $<$ 0 column shows the fraction of observations with negative gains. The data are monthly from 2006m2 to 2016m12.

Table 12: **Delta-Hedged Option Gains: 1.05 $\times$ Δ_{BS} Delta**

	moneyness	$\frac{\Pi}{S}$(%) mean	t-stat.	median	% < 0	std.	AR(1)	$\frac{\Pi}{C}$(%) mean	t-stat.	median
S&P500										
Call	0.950 to 0.975	0.07	[2.68]	-0.04	56%	0.64	0.29	1.51	[3.03]	-0.86
	0.975 to 1.000	0.04	[1.96]	-0.07	56%	0.57	0.13	2.15	[2.98]	-2.61
	1.000 to 1.025	-0.04	[-2.66]	-0.08	60%	0.48	0.10	-2.82	[-1.87]	-9.11
	1.025 to 1.050	-0.08	[-6.06]	-0.05	63%	0.40	0.14	-34.43	[-8.96]	-25.11
Put	0.950 to 0.975	-0.13	[-8.46]	-0.16	79%	0.48	0.34	-18.45	[-9.05]	-25.34
	0.975 to 1.000	-0.15	[-10.66]	-0.17	72%	0.43	0.19	-9.47	[-7.97]	-13.93
	1.000 to 1.025	-0.20	[-14.58]	-0.19	76%	0.40	0.01	-8.20	[-14.76]	-8.48
	1.025 to 1.050	-0.22	[-13.33]	-0.20	80%	0.38	0.01	-5.21	[-13.55]	-5.15
VIX										
Call	0.800 to 0.900	-0.11	[-0.77]	-0.45	57%	2.26	0.06	-0.63	[-0.71]	-2.54
	0.900 to 1.000	-0.63	[-3.29]	-0.99	67%	2.84	-0.03	-5.73	[-3.37]	-9.49
	1.000 to 1.100	-0.75	[-3.26]	-1.04	68%	3.12	-0.16	-9.01	[-3.17]	-12.89
	1.100 to 1.200	-1.08	[-4.56]	-1.39	77%	3.14	-0.12	-17.11	[-4.38]	-21.77
Put	0.800 to 0.900	-0.56	[-1.79]	-1.19	67%	2.80	0.06	-14.20	[-1.63]	-32.17
	0.900 to 1.000	-0.87	[-3.85]	-1.60	70%	3.20	-0.13	-11.69	[-3.12]	-24.20
	1.000 to 1.100	-0.93	[-3.59]	-1.94	73%	3.51	-0.17	-6.79	[-3.28]	-13.46
	1.100 to 1.200	-1.30	[-5.18]	-1.72	79%	3.28	-0.14	-6.10	[-5.02]	-8.40

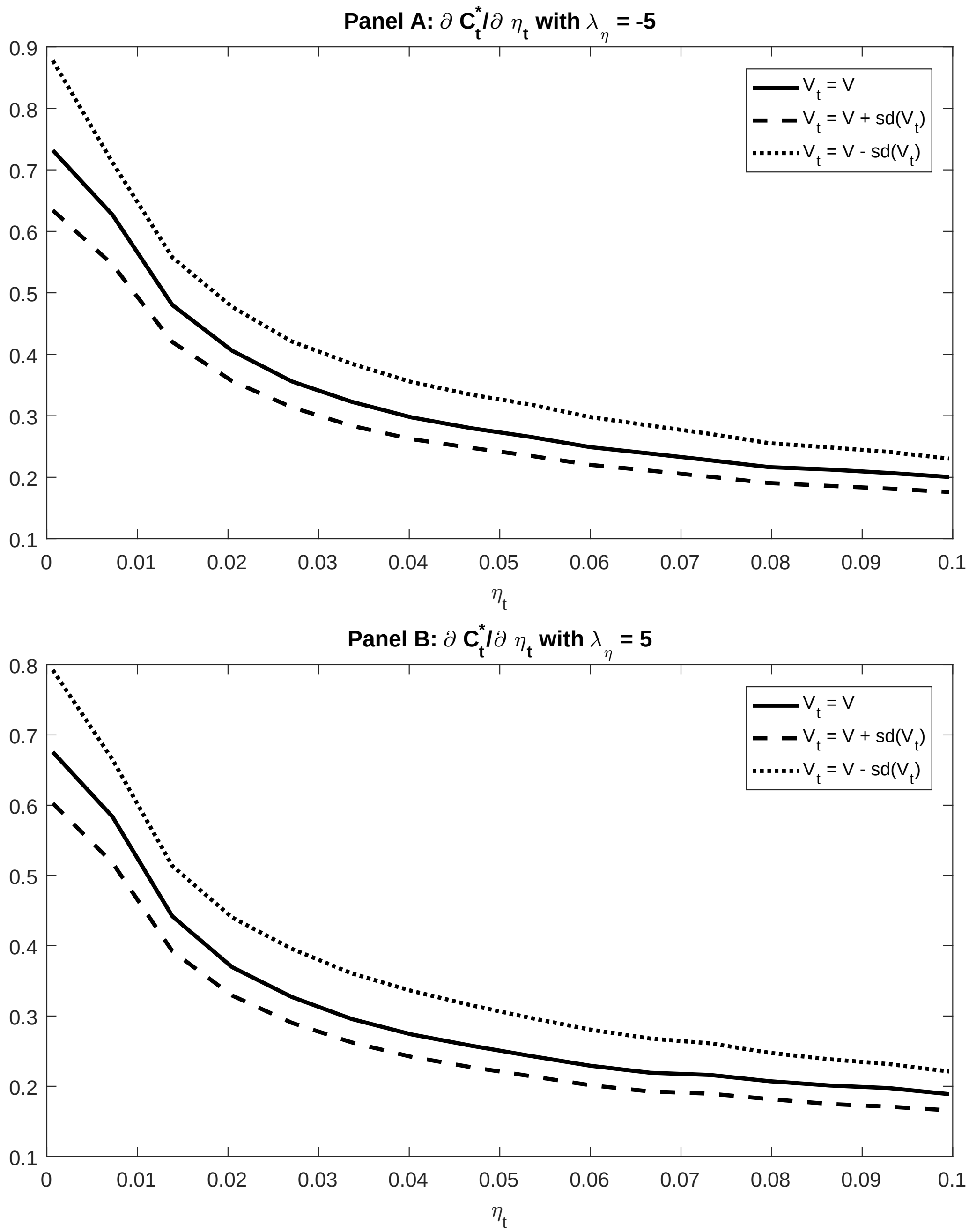
The table shows average delta-hedged option gains on the S&P500 and VIX options across their moneyness, when the Black-Scholes delta is increased to 1.05 $\times$ Δ_{BS}. Options have one month to maturity, are grouped into an equal-weighted portfolio inside the moneyness bin, and are held till expiration. The delta-hedge is computed using the Black-Scholes formula, with daily rebalancing and the margin difference earning the risk-free rate. The delta-hedged option gains Π are scaled either by the index or by the option price. The t-statistics are testing the null that the delta-hedged option gain is equal to zero. The % $<$ 0 column shows the fraction of observations with negative gains. The data are monthly from 2006m2 to 2016m12.

Figure 1: Model-implied Expected Delta-hedged VIX Option Gains



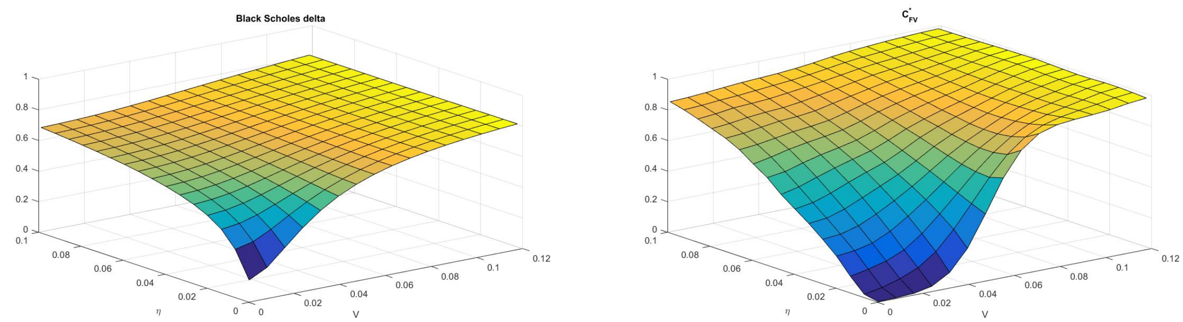
The figures show model-implied expected delta-hedged option gains as a function of the volatility-of-volatility η for different levels of the volatility state V_t. The upper plot shows gains for a negative market price of volatility-of-volatility risks, $\lambda_\eta = -5$. The lower plot shows gains for a positive market price of volatility-of-volatility risks $\lambda_\eta = 5$. All options are at the money.

Figure 2: Model-implied Sensitivities of VIX Option Prices



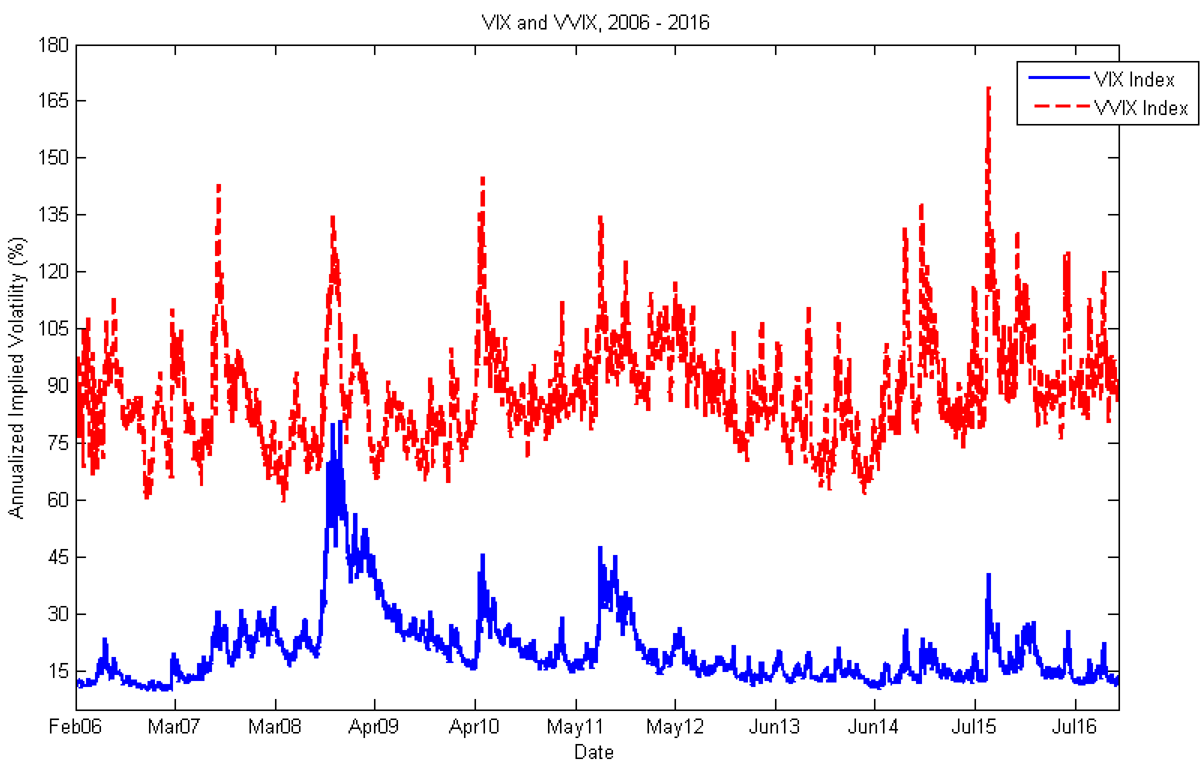
The figure shows model-implied sensitivities (derivatives) of at-the-money VIX option prices to the volatility-of-volatility η as a function of the state variable η for different levels of V_t. The upper plot shows gains for a negative market price of volatility-of-volatility risks, $\lambda_\eta = -5$. The lower plot shows gains for a positive market price of volatility-of-volatility risks $\lambda_\eta = 5$.

Figure 3: Black-Scholes and Model-implied Deltas



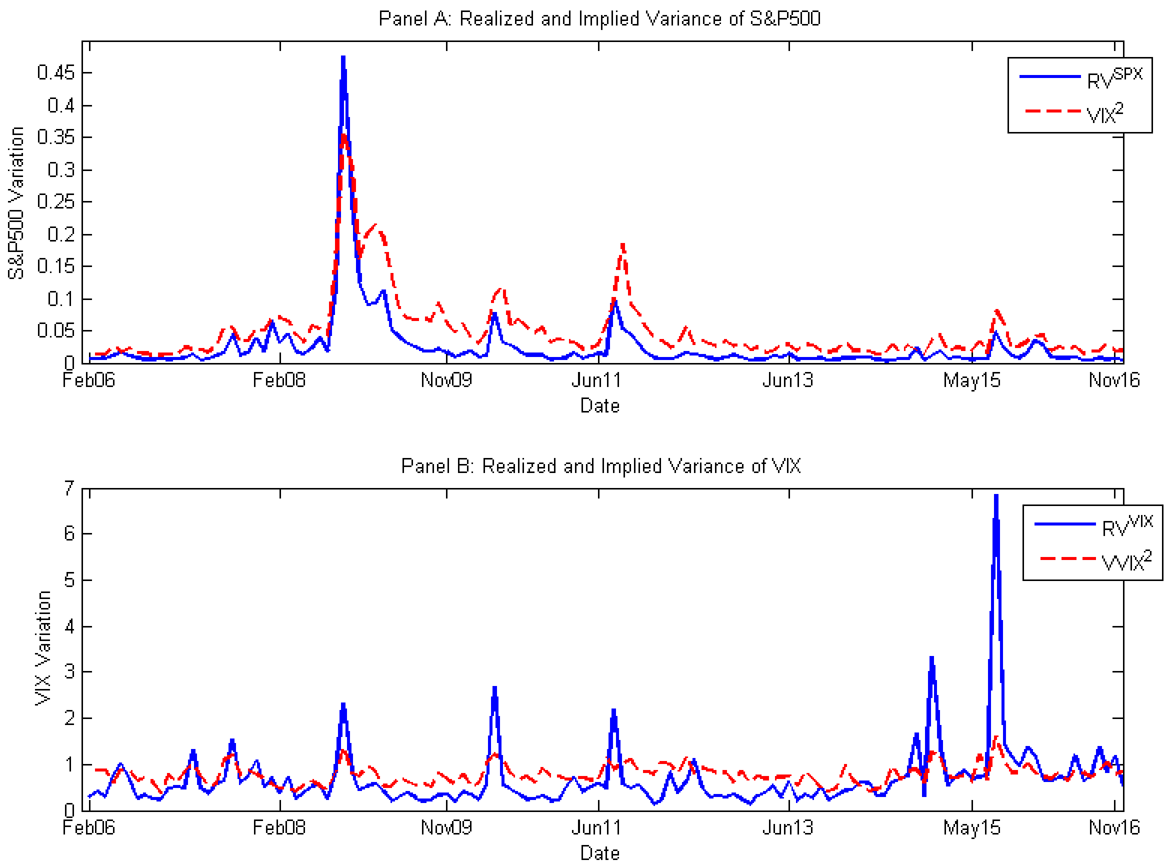
The figure plots the Black-Scholes delta of a VIX option (left panel) and the model-implied delta (right panel) as functions of the volatility and volatility of volatility state variables V and η.

Figure 4: Volatility and Volatility-of-Volatility Measures



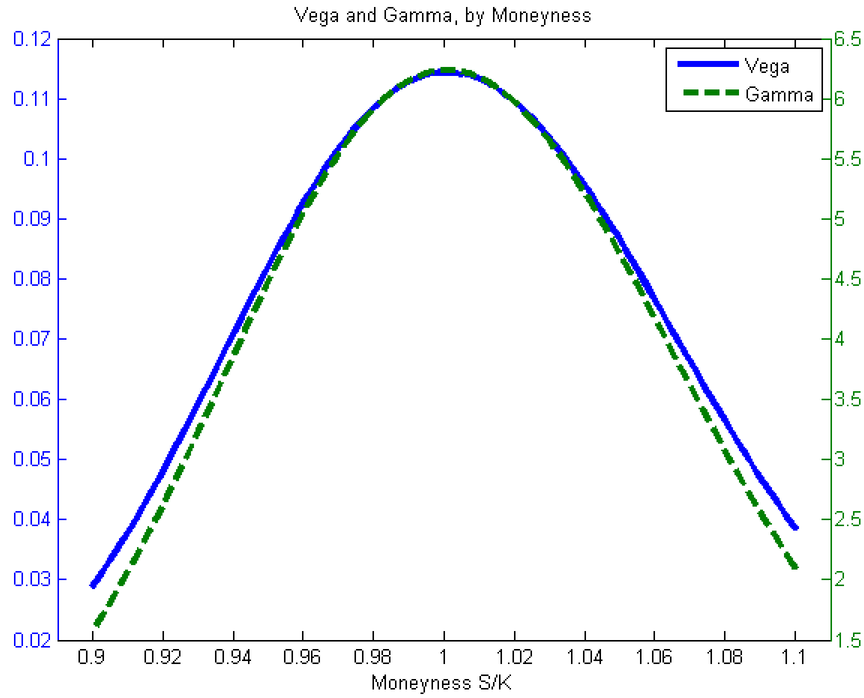
The figure shows the time-series of the volatility and volatility-of-volatility measures. The solid blue line is the VIX and the dashed red line is the VVIX. Monthly data from 2006m2 to 2016m12

Figure 5: Realized and Implied Variance Measures



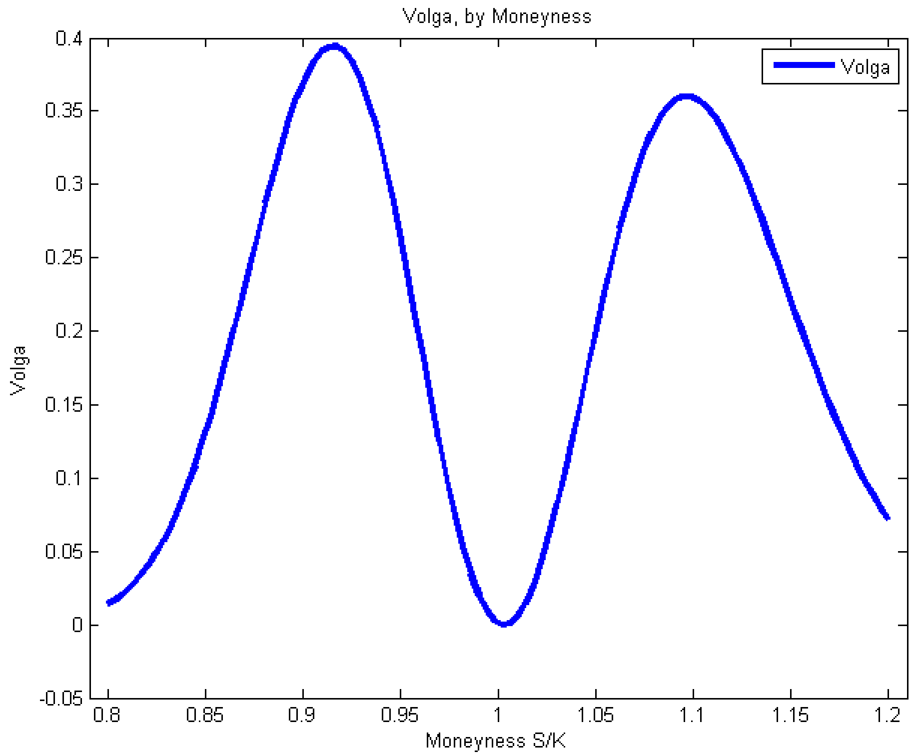
The figure shows the time-series of the realized and implied variances for the S&P500 (top panel) and the VIX (bottom panel). The blue solid lines are the realized variances, and red dashed lines are the implied variances. All measures are in annualized variance units. Monthly data from 2006m2 to 2016m12

Figure 6: Vega and Gamma by Moneyness



The figure shows the Black-Scholes vega and gamma of the option by option moneyness for average levels of volatility.

Figure 7: Volga by Moneyness



The figure shows the Black-Scholes volga of the option by option moneyness for average levels of volatility.