Modeling the Dependence Structure between Bonds and Stocks: A Multivariate Copula Approach

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Abstract

We apply a copula-GARCH approach to modeling the joint distribution of excess returns of four major assets: one year and ten year Treasury bonds and S&P 500 and Nasdaq indices. Three approaches are attempted in building the multivariate copula for the dependence structure of multiple assets: (1) n-dimensional normal copula and n-dimensional Student's t copula, (2) hierarchical Archimedean copula and (3) mixed copula. To ensure better convergence in estimation, we adopt a two-stage estimation procedure where marginal distributions are estimated separately in the first stage and then the copula is estimated in the second. Our results show that normal copula and Student's t copula yield higher log-likelihoods while hierarchical Archimedean copula and mixed copula yield lower log-likelihoods. We plot overall and rolling correlation, exceedance correlation and tail dependence of the data and find significant time-varying features with correlations between stocks and bonds, and with asymmetric exceedance correlation among all assets. With a time-varying dependence structure, our Student's t copula yields the highest log-likelihood and generates the most consistent dependence patterns with the data.

1 Introduction

During the last decade we have witnessed a large number of applications of copula theory in financial modeling and this popularity of copula mainly results from its capability of decomposing the joint distribution of random variables into marginal distributions of individual variables and the copula form which links the margins. Accordingly, the task of finding a proper joint distribution becomes to find a copula form which features a proper dependence structure given that marginal distributions of individual variables are properly specified. Yet there are still two issues we are confronted with when using copula theory to model multiple financial time series. The first issue is how to choose a copula form that best describes the data. As we know, different copula forms feature different aspects of dependence structure between random variables. Some copulas may fit one particular aspect of the data very well but do not have a very good overall fit, while some may have a very good overall fit but do not fit a particular aspect of the data very well. Therefore, what criteria to use

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when we choose from copula candidates becomes a major question remaining to be fully addressed. Meanwhile, how to build a multivariate copula which is sufficiently flexible to simultaneously account for the dependence structure for each pair of random variables in the joint distribution is still quite challenging.

When deciding on which copula form to use, many authors look at the overall fit of the data while still considering the fitness in certain aspects of the data. For instance, Patton (2004) employes a group of most frequently used copulas to model the dependence structure between excess returns of "large cap" and "small cap" stock indices. When addressing dependence issues, he mainly focuses on the asymmetric dependence between the two stock indices, that is, the fact that returns of two assets are more correlated when in the market downturn than when in the upturn. To measure this asymmetric dependence, he uses exceedance correlation, which was formerly suggested by Longin and Solnik (2001) and Ang and Chen (2002), and shows that rotated Gumbel copula provides a good fit of empirical exceedance correlation of the two indices. Meanwhile, his estimation results show that rotated Gumbel copula yields the highest log-likelihood among all the copula candidates and then rotated Gumbel copula is chosen to model the bivariate distribution of the two indices. It is worthwhile to note that a good fit in terms of exceedance correlation is neither sufficient nor necessary for a copula to yield the highest log-likelihood. Generally, authors tend to choose the copula with the best overall fit of the data. Nevertheless, when certain aspects of the data have more importance, for example, certain investment strategies might require better fit in one aspect of rather than the overall fit of the data, the task then becomes to find a particular copula which best fits that particular aspect of the data. Therefore, the weightings assigned to different aspects of the data when selecting copula models should be determined by the ultimate application of the modeled joint distribution[1].

Meanwhile, finding a flexible n-dimensional copula is still a challenging task. Normal copula and Student's t copula are the two copulas with more flexible parameters for the dependence structure of each pair of the variables. However, some authors argue that normal copula and Student's t copula may not fit the multiple financial series so well. Patton (2004) shows that rotated Gumbel copula outperforms both normal and Student's t copulas in describing asymmetric dependence structure of the two stock indices. Moreover, Archimedean copulas are shown to better fit bivariate financial time series in some cases, but when it is built in high dimensions, it has only one parameter describing the dependence structure between each pair of random variables, which makes the dependence structure too restrictive for high dimensional data.

This paper models the joint distribution of excess returns of four major assets-one year and ten year Treasury bonds and S&P 500 and Nasdaq indices by a multivariate copula approach. As in many previous works, the modeling framework adopted here is a copula-GARCH model. In particular, we use a GARCH specification together with a skewed t distribution to model the marginal distribution of individual assets and then build up a proper n-dimensional copula to link the margins. We apply three approaches in building this n-dimensional copula. The first approach is n-dimensional normal copula and n-dimensional Student's t copula. We first pursue with these two copulas partially because they have their compact and flexible parameter structures and partially because they can serve as a benchmark to compare with the latter two approaches. Moreover, we make the correlation

[1]Information criteria, such AIC and BIC, are commonly used for model selection. However, to address the confidence level and the statistical power, we need to conduct the goodness-of-fit tests or out-of-sample tests. Moreover, we can construct certain metrics to directly measure how well the estimated distribution fits the empirical data.

matrix in the elliptical copulas time-varying in order to better describe the time-varying dependence structure. Secondly, we construct a hierarchical Archimedean copulas based on different candidates from Archimedean families. As shown in Nelsen (1998), Archimedean families have nice properties that make it easy to nest one copula into another copula to form a hierarchical structure. We use three Archimedean families in our paper-Clayton, Frank and Gumbel copulas and use them respectively to build up their own hierarchical copula. As a third appoach of generating flexibility in modeling distinct dependence structures among random variables, we pursue with a mixed copula strategy. A mixed copula is formed by summing up a group of weighted copulas where each copula features dependence between one pair of variables and the sum of the weights is equal to unity. This strategy is enlightened by Tsafack (2006) who uses each copula component to account for the dependence structures of two pairs of variables simultaneously.

Our data set ranges from October 11, 1984 to October 28, 2005 and is sampled in daily frequency. We plot exceedance correlation and tail dependence of all possible pairs of variables. We find that empirical (unconditional) dependence asymmetries between two stock indices are not as significant as documented in previous studies and those between two bonds are also mild. Meanwhile, the (unconditional) dependence between bonds and stocks appears to be quite weak. We also plot correlation, exceedance correlation and tail dependence on rolling windows. We observe that rolling correlations for the pair of two stock indices and the pair of two bonds tend to fluctuate in the positive range, while the rolling correlation between any stock and any bond lies in the positive range for the first half period and then tends to fluctuate into the negative range for the rest of time. Moreover, the rolling exceedance correlations for each pair of variables are changing significantly across the time, while the rolling tail dependence tends to be relatively stable. All these plotted time-varying dependence measures indicate the need for a time-varying parameter structure in the copula function.

To avoid convergence difficulty of log-likelihood maximization problem, we adopt a two-stage estimation procedure where in the first stage we estimate marginal distributions of individual series separately by maximizing their own log-likelihood functions and in the second stage we plug the estimated margins in the copula and estimate copula parameters by maximizing the log-likelihood of the copula. In stage one, for two bonds, TARCH effects are not significant so we fit a GARCH(1,1) to the data, while TARCH effects are significant for stock data to which we fit a TARCH(1,1) accordingly. We use a skewed t distribution to model individual innovations. We also provide results under normal distribution assumptions for comparison purposes and our results show that skewed t distribution significantly improves the log-likelihood compared with normal distribution. For stage two, our results show that normal copula and Student's t copula yield higher log-likelihood than hierarchical copula and mixed copula, where hierarchical copula yields the lowest log-likelihood. Within the elliptical copulas, their time-varying versions even yield higher log-likelihood and the time-varying Student's t copula has the highest log-likelihood.

The modeling of dependence structure within and between the classes of stocks and bonds is very important for various financial modeling purposes, such as option pricing and portfolio optimization. In our four-asset model, besides the stable correlation within the each class of assets, what is mostly focused on by practitioners is the time-varying correlation between stocks and bonds, which fluctuates from positive range to negative range[2]. Moreover, our empirical measures indicate different patterns of asymmetric dependence between

[2] Generally, the positive correlation between stocks and bonds can be explained by "discount effects", while the negative correlation is due to "flight to safety". See Andersson, Krylova and Vaehaemaa (2008).

each pair of assets in terms of exceedance correlation and tail dependence. The time-varying Student's t copula stands out to be a good choice for modeling the conditional joint density with various non-normal features. In our case, the estimated time-varying correlation matrix in the copula function, even though not the same as the actual correlations of underlying variables in theory[3], can still replicate the plotted rolling correlations very well. In fact, we can also plot the actual correlation from our model by simulation procedures, which would be expected to have similar time-varying patterns with the former ones. Meanwhile, it indicates that our copula-GARCH framework does not specify explicit parameters which directly govern the correlation structure of underlying variables.

Even though its copula form features symmetric exceedance correlation, Student's t copula can still accomodate certain degree of asymmetric exceedance correlation as the exceedance correlation is determined by both copula form and the nested marginal distribution and skewed marginal distributions with symmetric copulas can still lead to asymmetric exceedance correlation. However, the limitations lie in the tail dependence. Within a multivariate Student's t copula, the tail dependence of each pair of variables is symmetric and is determined by the correlation coefficient and the degree-of-freedom parameter ν. As the degree-of-freedom parameter goes to infinity, the tail dependence goes to zero and the Student's t copula becomes normal copula. Therefore, Student's t copula tends to be restrictive in modeling asymmetric tail dependence among all the pairs of variables.

Our work is closely related to Goeij and Marquering (2004) and Tsafack (2006). The former authors model the conditional covariance between stock and bond markets returns by a multivariate GARCH approach. Using daily data of excess returns of one year and ten year Treasury bonds and S&P 500 and Nasdaq indices, they show strong evidence of heteroskedasticity and asymmetries in the covariance between stock and bond market returns. By a portfolio optimization excercise, they show that optimal portfolio shares can be substantially affected by asymmetries in covariance and asymmetric volatility timing can lead to sizable economic gains. The latter author models the dependence structure and extreme comovements of international equity and bond markets by a regime-switching copula-GARCH model. In one regime, he uses a n-dimensional normal copula to link the marginal distributions and in the other he uses a mixed copula of which each copula component features dependence structures of two particular pairs of variables. His empirical results show that dependence between international assets of the same type is high in both regimes while the dependence between equity and bond markets is low even within one country.

The paper is organized as follows. Section 2 gives a short literature review on recent applications of copulas to modeling financial time series. Section 3 introduces the model where we introduce copula theory, the copula-GARCH framework and the estimation procedure. In particular, we elaborate on the three strategies of building the multivariate copula and the corresponding estimation issues. Section 4 documents the data set we use and the dependence structures between the four time series, where we focus on time-varying and asymmetric dependence measures. Section 5 reports estimation results for the three modeling strategies and makes comparison between them in terms of overall fit of data and asymmetric dependence measures. Section 6 concludes.

[3] The correlation matrix in the Student's t copula form can be the same as the correlation of underlying variables only when the marginal distributions follow a t distribution with the same degress-of-freedom parameter with the copula.

2 Literature Review

Copula-GARCH models were previously proposed by Jondeau and Rockinger (2002) and Patton (2004, 2006). To measure time-varying conditional dependence between time series, the former authors use copula funtions with time-varying parameters as functions of predetermined variables, and model marginal distributions with an autoregressive version of Hansen's (1994) GARCH–type model with time-varying skewness and kurtosis. They show for many market indices, dependency increases after large movements and for some cases it increases after extreme downturns. Patton (2006) applies the copula-GARCH model to modeling the conditional dependence between exchange rates. He finds that Mark-dollar and Yen-dollar exchange rates are more correlated during depreciation against dollar than during appreciation periods. By a similar approach, Patton (2004) models the skewness and asymmetric dependence between "large cap" and "small cap" indices and examines the economic and statistical significance of the asymmetries for asset allocations in an out-of-sample setting. As in above literature, copulas are mostly used in modeling asymmetric dependence and tail dependence between times series. Among copula candidates, Gumbel's copula features higher dependence (correlation) at upper side with positive upper tail dependence and Rotated Gumbel's copula features higher dependence (correlation) at lower side with positive lower tail dependence. Hu (2006) studies the dependence structure between a number of pairs of major market indices by a mixed copula approach. Her copula is constructed by a weighted sum of three copulas-normal, Gumbel's and rotated Gumbel's copulas. Jondeau and Rockinger (2006) models the bivariate dependence between major stock indices by a Student's-t copula where the parameters are assumed to be modeled by a two-state Markov process.

The task of flexibly modeling dependence structure becomes more challenging for n-dimensional distributions. Tsafack (2006) builds up a complex multivariate copula to model four international assets (two international equities and two bonds). In his model, he assumes that the copula form has a regime-switching setup where in one regime he uses a n-dimensional normal copula and in the other he uses a mixed copula of which each copula component features the dependence structure of two pairs of variables. Savu and Trede (2006) develop a hierarchical Archimedean copula which renders more flexible parameters to characterize dependency between each pair of variables. In their model, each pair of closely related random variables is modeled by a copula of a particular Archimedean class and then these pairs are nested by copulas as well. The nice property of Archimedean family easily leads to the validity of the joint distribution constructed by this hierarchical structure. Zimmer and Trivedi (2006) apply trivariate hierachical Archimedean copulas to model sample selection and treatment effects with applications to the family health care demand.

Statistical goodness-of-fit tests can provide some guidance for selecting copula models. Chen *et al.* (2004) propose two simple goodness-of-fit tests for multivariate copula models, both of which are based on multivariate probability integral transform and kernel density estimation. One test is consistent but requires the estimation of the multivariate density function, hence is suitable for a small number of random variables, while the other may not be consistent but requres only kernel estimation of a univariate density function, hence is suitable for a large number of assets. Berg and Bakken (2006) propose a consistent goodness-of-fit test for copulas based on the probability integral transform and they incorporate in their test a weighting functionality which can increase influence of some specific areas of copulas.

Due to their parameter structure, the estimation of copula-GARCH models also suffers from "the curse of dimensionality"[4]. The exact maximum likelihood (ML) method works in theory. In practice, however, as the number of time series being modeled increases, the numerical optimization problem in the ML method will become formidable. Joe and Xu (1996) propose a two-step procedure, where in the first step only the parameters in the marginal distributions are estimated by the ML method and then the parameters of the copula structure are estimated by the ML method in the second one. This two-step method is called inference for the margins (IFM) method. Joe (1997) shows that under regular conditions the IFM estimator is consistent and has the property of asymptotic normality and Patton (2006) also shows that this two-step method yields asymptotically efficient and normal parameter estimates. Instead of estimating the parametric marginal distribution in the IFM method, we can estimate the marginals by using empirical distributions, which can avoid the problem of misspecification of the marginal distributions. This method is called Canonical Maximum Likelihood (CML) method by Cherubini *et al.* (2004). Hu (2006) uses this method and she names it as a semi-parametric method. Based on Genest *et al* (1995), she shows that CML estimator is consistent and has property of asymptotical normality. Moreover, copula model can also be estimated under a non-parametric framework. Deheuvels (1981) introduces the notion of empirical copula and shows that the empirical copula converges uniformly to the underlying true copula. Finally, Xu (2004) shows how the copula models can be estimated with a Bayesian approach. The author shows how a Bayesian approach can be used to account for estimation uncertainty in portfolio optimization based on a copula-GARCH model and she proposes to use a Bayesian MCMC algorithm to jointly estimate the copula models.

It is worth noting that we can model non-normal joint distributions by alternative approaches. Ang and Bekaert (2002) use a regime-switching model to account for asymmetric exceedance correlations between asset returns. Goeij and Marquering (2004) uses multivariate GARCH with time-varying conditional covariance matrix to model the asymmetric dependence between stock and bond markets returns. Simiarly as the copula-GARCH models, Lee and Long (2006) prospose a copula-based multivariate GARCH model with uncorrelated dependent errors, which are generated by a linear combination of dependent random variables.

3 The Model

The copula-GARCH framework generally consists of two components: the first component is to model the marginal distribution of individual time series by GARCH specifications with a certain distribution (a skewed t distribution is used in our paper), and the second is to find a proper copula form to link the margins to best capture the dependence structure between individual time series. In this section, we briefly introduce copula theory, illustrate the copula-GARCH framework, where copula theory is adapted to the conditional case and then explain the estimation procedure. In particular, we focus on the three approaches of constructing multivariate copulas.

[4]For a detailed survey on the estimations of copula-GARCH model, see Chapter 5 of Cherubini et al. (2004).

3.1 Copula

A copula is a multivariate distribution function with uniform marginal distributions as its arguments, and its functional form links all the margins to form a joint distribution of multiple random variables[5]. Copula theory is mainly based on the work of Sklar(1959) and we state the Sklar's theorem for continuous marginal distributions as follows.

Theorem 1 *Let $F_1(x_1), ..., F_n(x_n)$ be given marginal distribution functions and continuous in $x_1, ...x_n$ respectively. Let H be the joint distribution of $(x_1, ..., x_n)$. Then there exists a unique copula C such that*

$$H(x_1, ..., x_n) = C(F_1(x_1), ..., F_n(x_n)), \forall (x_1, ..., x_n) \in \overline{\mathbb{R}}^n. \tag{1}$$

Conversely, if we let $F_1(x_1), ..., F_n(x_n)$ be continuous marginal distribution functions and C be a copula, then the function H defined by Equation (1) is a joint distribution function with marginal distributions $F_1(x_1), ..., F_n(x_n)$.

The above theory allows us to decompose a multivariate distribution function into marginal distributions of each random variable and the copula form linking the margins. Conversely, it also implies that to construct a multivariate distribution, we can first find a proper marginal distribution for each random variable, and then obtain a proper copula form to link the margins. Depending on which dependence measure used, the copula function mainly, not exclusively, governs the dependence structure between individual variables. Hence, after specifying marginal distributions of each variable, the task of building a multivariate distribution solely becomes to choose a proper copula form which best describes the dependence structure between variables.

Differentiating equation (1) with respect to $(x_1, ..., x_n)$ leads to the joint density function of random variables in terms of copula density. It is given as

$$h(x_1, ..., x_n) = c(F_1(x_1), ..., F_n(x_n)) \prod_{i=1}^{n} f_i(x_i), \forall (x_1, ..., x_n) \in \overline{\mathbb{R}}^n \tag{2}$$

where $c(F_1(x_1), ..., F_n(x_n))$ is the copula density form and $f_i(x_i)$ is the density function for variable i. Equation (2) implies that the log-likelihood of the joint density can be decomposed into components which only involve each marginal density and a component which involves copula parameters. It provides a convenient structure for a two-stage estimation, which will be illlustrated in details in following sections.

To better fit the data, we usually assume the moments of distributions of random variables are time-varying and depend on past variables. Therefore, the distribution of random variables at time t becomes a conditional one and then the above copula theory needs to be extended to a conditional case. It is given as follows[6].

Theorem 2 *Let Ω_{t-1} be the information set up to time t and let $F_1(x_{1,t}|\Omega_{t-1}), ..., F_n(x_{n,t}|\Omega_{t-1})$ be continuous marginal distribution functions conditional on Ω_{t-1}. Let H be the joint distribution of $(x_1, ..., x_n)$ conditional on Ω_{t-1}. Then there exists a unique copula C such that*

[5] See Nelsen (1998) and Joe(1997) for a formal treatment of copula theory, and Bouye et al. (2000), Cherubini et al. (2004) and Embrechts et al. (2002) for applications of copula theory in finance.

[6] See Patton (2004).

$$H(x_1, ..., x_n|\Omega_{t-1}) = C(F_1(x_1|\Omega_{t-1}), ..., F_n(x_n|\Omega_{t-1})|\Omega_{t-1}), \forall(x_1, ..., x_n) \in \overline{\mathbb{R}}^n \quad (3)$$

Conversely, if we let $F_1(x_{1,t}|\Omega_{t-1}), ..., F_n(x_{n,t}|\Omega_{t-1})$ *be continuous conditional marginal distribution functions and* C *be a copula, then the function* H *defined by Equation (3) is a conditional joint distribution function with conditional marginal distributions* $F_1(x_{1,t}|\Omega_{t-1}), ..., F_n(x_{n,t}|\Omega_{t-1})$.

It is worth noting that for the above theorem to hold, the information set Ω_{t-1} has to be the same for the copulas and all the marginal distributions. If different information sets are used, the conditional copula form on the right side of (3) may not be a valid distribution. Generally, the same information set used may not be relevant for each marginal distributions and the copula. For example, the marginal distributions or the copula may be only conditional on a subset of the universally used information set. At the very beginning of estimation of the conditional distributions, however, we should use the same information set based on which we can test for insigficant explanatory variables so as to stick to a relevant subset for each marginal distribution or the copula.

3.2 Modeling marginal distributions

Before building a copula model, we need to find a proper specification for marginal distributions of individual series, as misspecified marginal distributions automatically lead to a misspecifiied joint distribution. Let $x_{1,t}$, $x_{2,t}$, $x_{3,t}$ and $x_{4,t}$ denote excess returns of 1-year Treasury bond, 10-year Treasury bond, S&P 500 index and Nasdaq index which are considered in this paper. We first model the conditional mean of these four series as shown in equations (4), (8), (12) and (16), where $\eta_{1,t}$, $\eta_{2,t}$ $\eta_{3,t}$ and $\eta_{4,t}$ are innovation terms, and $\sigma_{1,t}$, $\sigma_{2,t}$, $\sigma_{3,t}$ and $\sigma_{4,t}$ are standard deviations of the four series respectively. For the two Treasury bonds, we assume that their excess returns depend on the lagged values of excess returns and yield spreads, as shown in equations (4) and (8) where $SPR1_t$ and $SPR10_t$ are yield spreads for 1-year Treasury bond and 10-year Treasury bond. For the two stock indices, we only assume that their excess returns depend on their own lagged values as shown in equations (12) and (16), and one could add more exogenous variables, such as risk-free rates and default spread, to possibly improve the explanatory power. We estimate the conditional mean for the four series with up to 14 lags of each explanatory variable and retain statistically significant variables for the specifications of conditional means.

Equations (5), (9), (13) and (17) show the specifications of the conditional variance for the four series, where $1(\eta_{i,t-1} < 0)$ is an indicator function which equals one when $\eta_{i,t-1} < 0$ and zero otherwise. To account for the asymmetric effects of innovations on the conditional variance, that is, the fact that the excess returns of the assets tend to be more volatile when the asset market is in the downturn than when in the upturn, we assume a TARCH(1,1) sepcification for each asset. From our estimation results (shown in the latter part of the paper), we find that TARCH effects are not statistically significant for the two Treasury bonds but significant for the two stock indices. Consequently, we specify a GARCH(1,1) setup for the bonds and a TARCH(1,1) setup for the stock indices.

To model the conditional higher moments of the series, we follow Hansen (1994) and Jondeau and Rockinger (2003) who assume a skewed t distribution for the innovation terms of GARCH specifications and find that the skewed t distribution fits financial time series better than normal distribution. Accordingly, we assume $\eta_{i,t} \sim ST(\eta_{i,t}|\nu_{i,t}, \lambda_{i,t})$ with zero mean and unitary variance where $\nu_{i,t}$ is the degrees-of-freedom parameter and $\lambda_{i,t}$ is the skewness parameter. As shown in equations (6), (10), (14) and (18), we assume the degrees-of-freedom parameter $\nu_{i,t}$ is time-varying and depends on the lagged values of explanatory variables in a nonlinear form. The function $K(\cdot)$ takes a form of $K(x) = 4.0001 + x^2$ making $\nu_t > 4$ to ensure the third and fourth moments to exist (see Jondeau and Rockinger (2003)). Similarly, equations (7), (11), (15) and (19) show the specifications for the time-varying skewness parameter $\lambda_{i,t}$. $\lambda_{i,t}$ depends on the lagged values of explanatory variables in a non-linear form and the function $\Lambda(\cdot)$ takes the form of $\Lambda(x) = (1 - e^{(-x)})/(1 + e^{(-x)})$ to ensure $\lambda_{i,t} \in (-1, 1)$. We assume that explanatory variables in $K(\cdot)$ and $\Lambda(\cdot)$ are lagged residuals and yield spreads for the bonds and only lagged residuals for the stocks.

1-year bond

$$x_{1,t} = \alpha_1 x_{1,t-1} + \alpha_2 SPR1_{t-1} + \alpha_3 SPR1_{t-2} + \sigma_{1,t}\eta_{1,t}, \tag{4}$$

$$\sigma^2_{1,t} = \alpha_4 + \alpha_5 \sigma^2_{1,t-1} + \alpha_6 \sigma^2_{1,t-1}\eta^2_{1,t-1}, \tag{5}$$

$$\nu_{1,t} = K(\alpha_7 + \alpha_8 \sigma_{1,t-1}\eta_{1,t-1} + \alpha_9 SPR1_{t-1}), \tag{6}$$

$$\lambda_{1,t} = \Lambda(\alpha_{10} + \alpha_{11}\sigma_{1,t-1}\eta_{1,t-1} + \alpha_{12} SPR1_{t-1}). \tag{7}$$

10-year bond

$$x_{2,t} = \alpha_{13} x_{2,t-1} + \alpha_{14} x_{2,t-7} + \alpha_{15} SPR10_{t-1} + \alpha_{16} SPR10_{t-2} + \sigma_{2,t}\eta_{2,t}, \tag{8}$$

$$\sigma^2_{2,t} = \alpha_{17} + \alpha_{18}\sigma^2_{2,t-1} + \alpha_{19}\sigma^2_{2,t-1}\eta^2_{2,t-1}, \tag{9}$$

$$\nu_{2,t} = K(\alpha_{20} + \alpha_{21}\sigma_{2,t-1}\eta_{2,t-1} + \alpha_{22} SPR10_{t-1}), \tag{10}$$

$$\lambda_{2,t} = \Lambda(\alpha_{23} + \alpha_{24}\sigma_{2,t-1}\eta_{2,t-1} + \alpha_{25} SPR10_{t-1}). \tag{11}$$

S&P 500 index

$$x_{3,t} = \alpha_{26} x_{3,t-2} + \alpha_{27} x_{3,t-3} + \alpha_{28} x_{3,t-12} + \sigma_{3,t}\eta_{3,t}, \tag{12}$$

$$\sigma^2_{3,t} = \alpha_{29} + \alpha_{30}\sigma^2_{3,t-1} + \alpha_{31}\sigma^2_{3,t-1}\eta^2_{3,t-1} + \alpha_{32}\sigma^2_{3,t-1}\eta^2_{3,t-1}1(\eta_{3,t-1} < 0), \tag{13}$$

$$\nu_{3,t} = K(\alpha_{33} + \alpha_{34}\sigma_{3,t-1}\eta_{3,t-1}), \tag{14}$$

$$\lambda_{3,t} = \Lambda(\alpha_{35} + \alpha_{36}\sigma_{3,t-1}\eta_{3,t-1}). \tag{15}$$

Nasdaq index

$$x_{4,t} = \alpha_{37} x_{4,t-1} + \alpha_{38} x_{4,t-2} + \alpha_{39} x_{4,t-4} + \alpha_{40} x_{4,t-12} + \alpha_{41} x_{4,t-13} + \sigma_{4,t}\eta_{4,t}, \tag{16}$$

$$\sigma^2_{4,t} = \alpha_{42} + \alpha_{43}\sigma^2_{4,t-1} + \alpha_{44}\sigma^2_{4,t-1}\eta^2_{4,t-1} + \alpha_{45}\sigma^2_{4,t-1}\eta^2_{4,t-1}1(\eta_{4,t-1} < 0), \tag{17}$$

$$\nu_{4,t} = K(\alpha_{46} + \alpha_{47}\sigma_{4,t-1}\eta_{4,t-1}), \tag{18}$$

$$\lambda_{4,t} = \Lambda(\alpha_{48} + \alpha_{49}\sigma_{4,t-1}\eta_{4,t-1}). \tag{19}$$

3.3 Modeling dependence structure

When the number of random variables is greater than two, it is challenging to find an ideal copula form which best characterizes the dependence structure of each pair of variables simultaneously. Nevertheless, we pursue with the following modeling strategies: (1) we use the two elliptical copulas-normal copula and Student's t copula, and we focus on assigning a time-varying correlation matrix for the copulas; (2) we use a hierarchical copula framework to link the margins of random variables together and (3) we construct a copula by mixing a group of copulas where each copula characterizes the dependence structure of each pair of variables.

3.3.1 Two elliptical copulas

Normal copula and Student's t copula are the two from elliptical families, which are frequently used in modeling the joint distribution of random variables. We try these two copulas to provide a comparison with the other two approaches. Let Φ^{-1} denote the inverse of the standard univariate normal distribution Φ and $\Phi_{\Sigma,n}$ be n-dimensional normal distribution with correlation matrix Σ. Hence, the n-dimensional normal copula is

$$C(\mathbf{u};\Sigma) = \Phi_{\Sigma,n}(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_n)), \tag{20}$$

and its density form is

$$c(\mathbf{u};\Sigma) = \frac{1}{\sqrt{\det(\Sigma)}} \exp\left\{-\frac{\mathbf{x}'(\Sigma^{-1} - \mathbf{I}_n)\mathbf{x}}{2}\right\}. \tag{21}$$

where $x = (\Phi^{-1}(u_1), ..., \Phi^{-1}(u_n))'$. It can be shown via Sklar's theorem that the normal copula generates the standard joint normal distribution if and only if the margins are standard normal distributions.

On the other hand, let T_ν^{-1} be the inverse of the standardized univariate Student's t distribution T_ν with degrees of freedom $\nu > 2$ and $T_{R,\nu}$ be the n-dimensional standardized Student's t distribution with the correlation matrix R and degrees of freedom parameter ν. Then the n-dimensional Student's t copula is

$$C(\mathbf{u};R,\nu) = T_{R,\nu}(T_\nu^{-1}(u_1), ..., T_\nu^{-1}(u_n)), \tag{22}$$

and its density function is

$$c(\mathbf{u};R,\nu) = \frac{\Gamma(\frac{\nu+n}{2})\left[\Gamma(\frac{\nu}{2})\right]^{n-1}}{\sqrt{\det(R)}\left[\Gamma(\frac{\nu+1}{2})\right]^{n}} \left(1 + \frac{\boldsymbol{\varsigma}'R^{-1}\boldsymbol{\varsigma}}{\nu}\right)^{-\frac{\nu+n}{2}} \prod_{i=1}^{n}\left(1 + \frac{\varsigma_i^2}{\nu}\right)^{\frac{\nu+1}{2}} \tag{23}$$

where $\varsigma = (\varsigma_1, ..., \varsigma_n)'$ with $\varsigma_i = T_\nu^{-1}(u_i)$.

As shown in the later section, the dependence measures among the four major assets tend to be time-varying over the long time horizon, which implies the time-varying copula parameters. Borrowing from the dynamic conditional correlation (DCC) structure of the multivariate GARCH models, we can specify the time-varying parameter structure in the copula as follows[7].

For a normal copula, the time-varying correlation matrix, the only group of parameters, is governed by

$$Q_t = S(1-\alpha-\beta) + \alpha(\mathbf{x}_{t-1}\mathbf{x}'_{t-1}) + \beta Q_{t-1}, \tag{24}$$

where S is the unconditional correlation matrix of the x_t, and α and β are positive and satisfy the condition $\alpha+\beta<1$. We assign $Q_0 = S$ and the dynamics of Q_t is given by (24). Let $q_{i,j,t}$ be the i,j element of the matrix Q_t and the i,j element of the conditional correlation matrix Σ_t can be calculated as

$$\rho_{i,j,t} = \frac{q_{i,j,t}}{\sqrt{q_{i,i,t}q_{j,j,t}}}. \tag{25}$$

Moreover, the specification of (24) garantees that the conditional correlation matrix Σ_t is positive definite.

For a Student's t copula, we need to model not only the correlation matrix but the degree-of-freedom parameter as well. With the constant degree-of-freedom parameter, the time-varying correlation matrix R_t is also given by (24) while in this case R_t is a function of the degree-of-freedom parameter.

3.3.2 Hierarchical copula

Archimedean copulas are also a good choice for modeling bivariate distributions. For high dimensional cases, however, they tend to be too restrictive as they imply exactly the same dependence structure across all the pairs of variables. To make Archimedean copulas less restrictive in describing dependence structures of n-dimensional distributions, we can resort to a hierarchical structure. We can divide the four assets considered in our paper into two pairs: the first pair consists of the two Treasury bonds and then the second includes the two stock indices. Ideally, we can find a proper bivariate copula to model each pair of assets respectively and then nest these two copulas into another bivariate copula to form a joint distribution of the four assets. Let $C_1(u_1,u_2)$ and $C_2(u_3,u_4)$ be the two copulas governing the two pairs of assets and then the final joint distribution can be given as

$$F(x_1,x_2,x_3,x_4) = C_3(C_1(u_1,u_2), C_2(u_3,u_4)) \tag{26}$$

where $u_i = F_i(x_i)$ and its density form is

$$f(x_1,x_2,x_3,x_4) = c_3(C_1(u_1,u_2), C_2(u_3,u_4))c_1(u_1,u_2)c_2(u_3,u_4)\prod_{i=1}^{4} f_i(x_i) \tag{27}$$

However, equation (26) does not always hold for any copulas and for certain choices of C_1 and C_2, C_3 will not satisfy the definition of copulas[8].

Nevertheless, Archimedean copulas have certain properties that facilitate constructing hierarchical copulas as shown in equation (26). We briefly give its definition as follows[9].

[7] Please see Engle and Sheppard (2001) and Engle (2002) for details on the multivariate DCC-GARCH models.

[8] See 3.4 of Nelson(1998)

[9] See Nelson(1998) for more details.

Definition 3 *A n-dimensional Archimedean copula is defined by*

$$C(u_1, ..., u_n) = \phi^{-1}(\phi(u_1) + ... + \phi(u_n)) \tag{28}$$

where ϕ is a generator function with its inverse as ϕ^{-1}. $\phi : [0,1] \to [0, \infty]$ is a continuous, strictly decreasing convex function with $\phi(1) = 0$ and $\phi(0) = \infty$ and $\phi^{-1} : [0, \infty] \to [0, 1]$ is completely monotonic on $[0, \infty]$.

The Archimedean copulas we consider in this paper are Frank, Clayton and Gumbel copulas and they are three mostly used Archimedean copula families with single dependence parameter. Following Savu and Trede (2006), we choose one of the three families and consistently use it for C_1, C_2 and C_3 in equation (26). Let ρ_1, ρ_2 and ρ_3 be the dependence parameters of C_1, C_2 and C_3 respectively and we impose the restriction $\rho_3 < \rho_1$ and $\rho_3 < \rho_2$ to ensure that C_3 generates a distribution function. It is worth noting that we could also choose from more Archimedean copula families and use different family for the hierarchical structure at the same time. However, the dependence parameter restrictions that ensure the validity of the distribution function have yet to be derived. We leave this point for future research.

Therefore, we use $C_1(u_1, u_2)$ and $C_2(u_3, u_4)$ to model the dependence structures of the two bonds and the two stock indices respectively and then use C_3 to model the dependence between the bonds and the stocks. Obviously, our nesting structure in equation (26) leads to the same dependence structure for any pair of bond and stock. Additionally, the empirical dependence between the two bonds and that between the two stock indices are both generally stronger than that between bond markets and stock markets. This is consistent with our dependence parameter restrictions.

3.3.3 Mixed copula

As the third approach, we build up a mixture of copulas where each copula bears a certain weight and features the dependence structure between one particular pair of variables. In particular, with four random variables, we need six copulas to characterize all the dependence relations. In each copula, we assume that only one pair of variables has a dependence structure and the other variables are all independent with each other and with the dependent pair. For instance, we first model the dependence between x_1 and x_2 by a copula $C_1(u_1, u_2)$ and then construct a copula with four variables by multiplying $C_1(u_1, u_2)$ with u_3 and u_4. $C_1(u_1, u_2)u_3u_4$ is a copula by Theorem 3.4.3 on page 86 of Nelson (1998) and in this copula x_3 and x_4 are independent with each other and independent with x_1 and x_2. Similarly, we can construct the other five copulas as $C_2(u_3, u_4)u_1u_2$, $C_3(u_1, u_4)u_2u_3$, $C_4(u_1, u_3)u_2u_4$, $C_5(u_2, u_4)u_1u_3$ and $C_6(u_2, u_3)u_1u_4$. Hence, the mixture of copulas C_M can be given as

$$\begin{aligned} C_M(u_1, u_2, u_3, u_4; \boldsymbol{\pi}, \boldsymbol{\rho}) &= \pi_1 C_1(u_1, u_2; \rho_1)u_3u_4 + \pi_2 C_2(u_3, u_4; \rho_2)u_1u_2 \\ &\quad + \pi_3 C_3(u_1, u_4; \rho_3)u_2u_3 + \pi_4 C_4(u_1, u_3; \rho_4)u_2u_4 \\ &\quad + \pi_5 C_5(u_2, u_4; \rho_5)u_1u_3 + \pi_6 C_6(u_2, u_3; \rho_6)u_1u_4 \end{aligned} \tag{29}$$

where $\pi = [\pi_1 \ \pi_2 \ \pi_3 \ \pi_4 \ \pi_5, \pi_6]'$ with $\pi_i \geq 0$ for $i = 1, ..., 6$ and $\sum\limits_{i=1}^{6} \pi_i = 1$ accounts for the weights for each copula and $\rho = [\rho_1 \ \rho_2 \ \rho_3 \ \rho_4 \ \rho_5 \ \rho_6]'$ is the vector of parameters in each copula. Accordingly, the density form c_M is given by

$$
\begin{aligned}
c_M(u_1,u_2,u_3,u_4;\boldsymbol{\pi},\boldsymbol{\rho}) &= \pi_1 c_1(u_1,u_2;\rho_1)+\pi_2 c_2(u_3,u_4;\rho_2) \\
&\quad +\pi_3 c_3(u_1,u_4;\rho_3)+\pi_4 c_4(u_1,u_3;\rho_4) \\
&\quad +\pi_5 c_5(u_2,u_4;\rho_5)+\pi_6 c_6(u_2,u_3;\rho_6) \qquad (30)
\end{aligned}
$$

where c_i is the density form of C_i for $i = 1, ..., 6$.

This approach is originally enlightened by Tsafack (2006) where as one component of his model, he uses a mixture of copulas to model the joint distribution of four international assets. In his model, each component of the mixed copula simultaneously characterizes the dependence structure of two pairs of variables. In our model, however, each copula component features the dependence structure of only one pair of variables so that each weighting parameter will fully account for the weight of the dependence of that pair of variables.

3.4 Estimation

Let $X_t = [x_{1,t}, x_{2,t}, x_{3,t}, x_{4,t}, SPR1_t, SPR10_t]'$ be a vector containing all the observed variables at time t and the terms $\sigma^2_{i,t}$ and $\sigma^2_{i,t}\eta^2_{i,t}$ are not included as they can be expressed in terms of current and lagged values of X_t. Then let $\underline{X}_t = [X'_t, X'_{t-1}, ..., X'_{-m}]'$ be a vector containing all the past observed variables up to time t where m is the maximum number of lags of variables in equations (4) to (19). Let $\Theta = [\theta, \gamma_1, \gamma_2, \gamma_3, \gamma_4]$ be the set of parameters in the joint distribution where θ is the set of parameters in the copula and γ_i is the set of parameters in the marginal distributions. Then the conditional *cdf* of four asset excess returns at time t is

$$F(x_{1,t}, x_{2,t}, x_{3,t}, x_{4,t}|\underline{X}_{t-1}, \Theta) = C(u_{1,t}, u_{2,t}, u_{3,t}, u_{4,t}|\underline{X}_{t-1}, \theta) \qquad (31)$$

where $C(\cdot|\underline{X}_{t-1}, \theta)$ is the conditional copula and $u_i = F_i(x_i|\underline{X}_{t-1}, \gamma_i)$ is the conditional *cdf* of the margins. Differentiating both sides with respect to $x_{1,t}, x_{2,t}, x_{3,t}, x_{4,t}$ leads to the density function as

$$f(x_{1,t}, x_{2,t}, x_{3,t}, x_{4,t}|\underline{X}_{t-1}, \Theta) = c(u_{1,t}, u_{2,t}, u_{3,t}, u_{4,t}|\underline{X}_{t-1}, \theta)\prod_{i=1}^{4} f_i(x_{i,t}|\underline{X}_{t-1}, \gamma_i) \qquad (32)$$

where $c(\cdot|\underline{X}_{t-1}, \theta)$ is the density of the conditional copula and $f_i(\cdot|\underline{X}_{t-1}, \gamma_i)$ is the conditional density of the margins. Accordingly, the log-likelihood of the sample is given by

$$L(\Theta) = \sum_{t=1}^{T} \log f(x_{1,t}, x_{2,t}, x_{3,t}, x_{4,t}|\underline{X}_{t-1}, \Theta). \qquad (33)$$

With equation (32), the log-likelihood can be written as

$$L(\theta, \gamma_{1,t}, \gamma_{2,t}, \gamma_{3,t}, \gamma_{4,t}) = \sum_{t=1}^{T} \log c(u_{1,t}, u_{2,t}, u_{3,t}, u_{4,t}|\underline{X}_{t-1}, \theta) + \sum_{t=1}^{T}\sum_{i=1}^{4} f_i(x_{i,t}|\underline{X}_{t-1}, \gamma_{i,t}) \qquad (34)$$

where we add time script to γ_i to make it time-varying. Although the simultaneous estimation of all the parameters leads to efficiency, the large number of parameters usually cause the maximization procedure hard to converge. Hence, we pursue with a two-stage estimation procedure as in previous studies. Let $\gamma_{i,t} = [m_{i,t}, \sigma^2_{i,t}, \nu_{i,t}, \lambda_{i,t}]'$ for $i = 1, 2, 3$ and 4 where $m_{i,t}, \sigma^2_{i,t}, \nu_{i,t}$ and $\lambda_{i,t}$ denote the conditional mean, the conditional variance, degrees-of-freedom and the skewness of the marginal distribution. And the two-stage procedure can be illustrated as follows.

Stage one:

$$\widehat{m}_{i,t} = \arg\min \sum_{t=1}^{T} (x_{i,t} - m_{i,t})^2, \text{ for } i = 1, 2, 3 \text{ and } 4 \tag{35}$$

$$\begin{aligned}
[\widehat{\sigma}^2_{i,t}, \widehat{\nu}_{i,t}, \widehat{\lambda}_{i,t}]' &= \arg\max \sum_{t=1}^{T} \log f_i(x_{i,t}, \widehat{m}_{i,t}, \sigma^2_{i,t}, \nu_{i,t}, \lambda_{i,t} | \underline{X}_{t-1}) \\
&= \arg\max \sum_{t=1}^{T} \left[\log f_{ST}(\frac{x_{i,t} - \widehat{m}_{i,t}}{\sigma_{i,t}}, \nu_{i,t}, \lambda_{i,t} | \underline{X}_{t-1}) + \log \frac{1}{\sigma_{i,t}} \right] \\
&\text{for } i = 1, 2, 3 \text{ and } 4.
\end{aligned} \tag{36}$$

Stage two:

$$\widehat{\theta} = \arg\max \sum_{t=1}^{T} \log c(\widehat{u}_{1,t}, \widehat{u}_{2,t}, \widehat{u}_{3,t}, \widehat{u}_{4,t} | \underline{X}_{t-1}; \theta) \tag{37}$$

In stage one, we first estimate the conditional mean $m_{i,t}$ by OLS estimator as in equation (35) where we only assume the residuals (generally, $\sigma_{i,t}\eta_{i,t}$) are uncorrelated with the explanatory variables to ensure the estimation consistency. Plugging the estimated conditional mean into the marginal density, we estimate $[\widehat{\sigma}^2_{i,t}, \widehat{\nu}_{i,t}, \widehat{\lambda}_{i,t}]'$ by maximum likelihood estimator as shown in equation (36). We see that the marginal density of excess returns can be written in forms of the density of the innovation term in excess return equation (generally, $\eta_{i,t}$), which is assumed to have a skewed t distribution. Accordingly, $f_{ST}(\cdot | \underline{X}_{t-1})$ is the density function of the skewed t distribution conditional on $\underline{X}_{t-1}$. $\widehat{\gamma}_{i,t}$ is a consistent estimate of the time-varying parameters in the marginal distribution of excess returns.

With $\widehat{\gamma}_{i,t}$ calculated, we can calculate $\widehat{u}_{i,t}$ for $i = 1, 2, 3$ and 4 as follows

$$\begin{aligned}
\widehat{u}_{i,t} &= F_i(x_{i,t}, \widehat{\gamma}_{i,t} | \underline{X}_{t-1}) \\
&= F_{ST}(\frac{x_{i,t} - \widehat{m}_{i,t}}{\widehat{\sigma}_{i,t}}, \widehat{\nu}_{i,t}, \widehat{\lambda}_{i,t} | \underline{X}_{t-1}), \text{ for } i = 1, 2, 3 \text{ and } 4.
\end{aligned} \tag{38}$$

In equation (38), $F_{ST}(\cdot | \underline{X}_{t-1})$ is the *cdf* of the skewed t distribution conditional on $\underline{X}_{t-1}$. Hence, in stage two, θ can be consistently estimated by maximizing the log-likelihood of the copula density as in equation (37). One thing worth noting is that $\gamma_{i,t}$ is assumed to be functions of lagged explanatory variables as shown in equations (8)-(19), we actually estimate parameters in those functions and then calculate $\widehat{\gamma}_{i,t}$. Similarly, some copula parameters in θ have restricted range and are assumed to be certain functions of some arbitrary parameters. Again, we first estimate those arbitrary parameters and then calculate estimates of those copula parameters. We will show estimation details in section 4.

3.4.1 Estimating correlation matrices in normal copula and Student's t copula

In stage two, if we estimate normal copula or Student's t copula still by numerical maximization, the correlation matrix often becomes near singular during the maximum searching procedure which makes its inverse imprecise so as to ruin the convergence. However, it turns out that for normal copula the correlation matrix can be estimated analytically instead of by numerical maximization. The log-likelihood for normal copula in stage two is given by

$$L(\Sigma) = -\frac{T}{2}\log(\det(\Sigma)) - \frac{1}{2}\sum_{t=1}^{T}\mathbf{x}'(\Sigma^{-1} - \mathbf{I}_4)\mathbf{x} \tag{39}$$

where $x = (\Phi^{-1}(u_1), ..., \Phi^{-1}(u_n))'$ and the correlation matrix Σ is the only set of unknown parameters in normal copula. We derive the first order conditions with respect to Σ^{-1} as follows.

$$\frac{\partial L(\Sigma)}{\partial \Sigma^{-1}} = \frac{T}{2}\Sigma - \frac{1}{2}\sum_{t=1}^{T}\mathbf{x}'\mathbf{x}. \tag{40}$$

and setting (40) equal to zero yields the estimator of Σ.

$$\widehat{\Sigma} = \frac{1}{T}\sum_{t=1}^{T}x'x \tag{41}$$

Then for Student's t copula, the log-likelihood of the copula density in stage two is given by

$$\begin{aligned} L(R,\nu) \quad = \quad & -\frac{T}{2}\log(\det(R)) + T\log(\Gamma(\frac{\nu+4}{2})) + 3T\log(\Gamma(\frac{\nu}{2})) - 4T\log(\frac{v+1}{2}) \\ & -\frac{\nu+4}{2}\sum_{t=1}^{T}\log(1+\frac{\boldsymbol{\varsigma}_t' R^{\mathbf{-1}}\boldsymbol{\varsigma}_t}{\nu}) + \frac{\nu+1}{2}\sum_{t=1}^{T}\sum_{i=1}^{4}(1+\frac{\varsigma_{i,t}^2}{\nu}) \end{aligned} \tag{42}$$

where $\varsigma_t = (\varsigma_{1,t}, ..., \varsigma_{n,t})'$ with $\varsigma_{i,t} = T_\nu^{-1}(u_{i,t})$ and the unknown parameters consist of the correlation matrix R and the degrees-of-freedom parameter ν. It is impossible to derive analytical estimates for R and ν. Following Chen *et al.* (2004), however, we can use the sample correlation matrix of $(\varsigma_{1,t}, ..., \varsigma_{n,t})'$ as $\widehat{R}$, which is a function of the degrees-of-freedom parameter ν. Then we substitute R for $\widehat{R}$ in equation (42) and maximize the log-likelihood $L(\widehat{R}, \nu)$ with respect to ν.

We pursue with similar approaches to estimate the time-varying dependence structures. For the normal copula, the log-likelihood becomes

$$L(\Sigma_t) = -\frac{1}{2}\sum_{t=1}^{T}\log(\det(\Sigma_t)) - \frac{1}{2}\sum_{t=1}^{T}\mathbf{x}'(\Sigma_t^{-1} - \mathbf{I}_4)\mathbf{x}. \tag{43}$$

The dynamics Σ_t is driven by (24), where we set $S = Q_0 = \widehat{\Sigma}$ which is estimated by (41). Then we can estimate α and β by maximizing the log-likelihood (43) and with $\widehat{\alpha}$, $\widehat{\beta}$ and $\widehat{\Sigma}$, we can calculate and plot Σ_t. Similarly, the log-likelihood of the Student's t copula with time-varying dependence structure is given by

$$
\begin{aligned}
L(R_t, \nu) \quad = \quad & -\frac{1}{2}\sum_{t=1}^{T}\log(\det(R_t)) + T\log(\Gamma(\frac{\nu+4}{2})) + 3T\log(\Gamma(\frac{\nu}{2})) - 4T\log(\frac{\upsilon+1}{2})) \qquad (44) \\
& -\frac{\nu+4}{2}\sum_{t=1}^{T}\log(1+\frac{\boldsymbol{\varsigma}_t' {R_t}^{-\mathbf{1}}\boldsymbol{\varsigma}_t}{\nu}) + \frac{\nu+1}{2}\sum_{t=1}^{T}\sum_{i=1}^{4}(1+\frac{\varsigma_{i,t}^2}{\nu}).
\end{aligned}
$$

where R_t is driven by (24). We estimate R from (42) and set $S = Q_0 = \widehat{R}$. Since R_t and ς_t are functions of the degree-of-freedom parameter ν, by maximizing the log-likelihood (44), we eventually estimate α, β and ν.

3.4.2 Estimating hierarchical copula

With hierarchical copula, equation (37) in stage two becomes

$$
\begin{aligned}
[\widehat{\rho}_1, \widehat{\rho}_2, \widehat{\rho}_3] \quad = \quad & \arg\max \sum_{t=1}^{T} \log c_3(C_1(\widehat{u}_{1,t}, \widehat{u}_{2,t}|\underline{X}_{t-1}, \rho_1), C_2(\widehat{u}_{3,t}, \widehat{u}_{4,t}|\underline{X}_{t-1}, \rho_2)|\underline{X}_{t-1}, \rho_3) \\
& + \sum_{t=1}^{T} c_1(\widehat{u}_{1,t}, \widehat{u}_{2,t}|\underline{X}_{t-1}, \rho_1) + \sum_{t=1}^{T} c_2(\widehat{u}_{3,t}, \widehat{u}_{4,t}|\underline{X}_{t-1}, \rho_2) \qquad (45)
\end{aligned}
$$

where ρ_1, ρ_2 and ρ_3 are the dependence parameters of C_1, C_2 and C_3 respectively. We actually estimate these three parameters as follows

$$
\begin{aligned}
\widehat{\rho}_1 \quad & = \quad \arg\max \sum_{t=1}^{T} c_1(\widehat{u}_{1,t}, \widehat{u}_{2,t}|\underline{X}_{t-1}, \rho_1), \qquad (46) \\
\widehat{\rho}_2 \quad & = \quad \arg\max \sum_{t=1}^{T} c_2(\widehat{u}_{3,t}, \widehat{u}_{4,t}|\underline{X}_{t-1}, \rho_2), \qquad (47) \\
\widehat{\rho}_3 \quad & = \quad \arg\max \sum_{t=1}^{T} \log c_3(C_1(\widehat{u}_{1,t}, \widehat{u}_{2,t}|\underline{X}_{t-1}, \widehat{\rho}_1), C_2(\widehat{u}_{3,t}, \widehat{u}_{4,t}|\underline{X}_{t-1}, \widehat{\rho}_2)|\underline{X}_{t-1}, \rho_3). \qquad (48)
\end{aligned}
$$

With $\widehat{u}_{1,t}, \widehat{u}_{2,t}, \widehat{u}_{3,t}$ and $\widehat{u}_{4,t}$ calculated from stage one, we can estimate ρ_1 and ρ_2 as shown in equations (46) and (47) and then estimate ρ_3 in equation (48).

3.4.3 Estimating mixed copula

In stage two, before we estimate the parameters π and ρ by maximizing the log-likelihood of copula densities, we need first to decide on specific copula forms for $c_i(\cdot)$ in equation (30). For each pair of variables, we try a list of commonly used copula forms to estimate the bivariate distribution of the pair and choose the one which yields the highest log-likelihood as the copula form to be used in equation (30). The copula candidates we use in this paper are normal, Student's t, Plackett, Frank, Clayton, Rotated Clayton, Gumbel and rotated Gumbel copulas. One thing worth noting is that we only use the copula forms determined by those bivariate copula estimations, but the parameters in each chosen bivariate copula will be jointly estimated again in stage two.

Therefore, equation (37) in stage two can be written as

$$[\widehat{\boldsymbol{\pi}}_t, \widehat{\boldsymbol{\rho}}_t] = \arg\max \sum_{t=1}^{T} \log c(\widehat{u}_{1,t}, \widehat{u}_{2,t}, \widehat{u}_{3,t}, \widehat{u}_{4,t}; \boldsymbol{\pi}_t, \boldsymbol{\rho}_t | \underline{X}_{t-1}) \tag{49}$$

where we attach π and ρ with subscript t to indicate the possibility of making these parameters or some of them time-varying. Again, there is no analytical solution for this maximization problem and we will resort to numerical methods to find the estimate values.

4 Data

4.1 Data documentation and descriptive statistics

Our data set includes daily excess returns on two Treasury bonds and two stock indices-1-year Treasury bond, 10-year Treasury bond, S&P 500 index and Nasdaq index, denoted by x_1 x_2 x_3 and x_4 respectively. The excess returns are calculated by differencing the returns of assets by the risk free rate approximated by the three-month Treasury bond rate. Please see the Appendix for the details of daily return calculations and adjustments for weekends and holidays. The time span of the data set is October 11, 1984 to October 28, 2005 and it consists of 5257 observations. The stock market data used for calculating stock excess returns are closing stock indexes which are obtained from http://finance.yahoo.com, while the bond data are daily constant maturity interest rate series which are obtained from the Federal Reserve Bank in St. Louis. Table 1 provides a summary of the descriptive statistics of the daily excess returns on those four assets. We find that all the assets have slightly positive excess returns and the stock excess returns tend to have higher volatilities than the bonds. Moreover, we find that for most series the skewness is significantly different from zeros and the kurtosis is much larger than three for all the series. It implies that empirical distribution may not be well described by the normal distribution. In the figure (1), we plot excess returns on these four assets and we clearly observe that the volatility clustering, that is, the large returns tend to be followed by large returns of either sign. Those observations suggest that it is reasonable to use GARCH specifications and non-normal distributions to model the individual series.

	1-year bond	10-year bond	S&P 500	NASDAQ
Mean	0.0027	0.0177	0.0241	0.0312
Maximum	0.7911	4.8041	9.0828	14.1571
Minimum	-0.34	-2.7163	-20.526	-11.4054
Std. Dev.	0.0568	0.4666	1.0681	1.4157
Skewness	0.9301	-0.017	-1.4267	-0.0151
Kurtosis	15.8637	6.9465	33.6029	11.5635
Jarque-Bera	37004.17	3411.849	206924.8	16063.63
Observations	5257	5257	5257	5257

Table 1: Descriptive statistics for excess returns on stock indexes and bonds. The time span is October 11, 1984 to October 28, 2005 and all returns are daily returns in percentages.

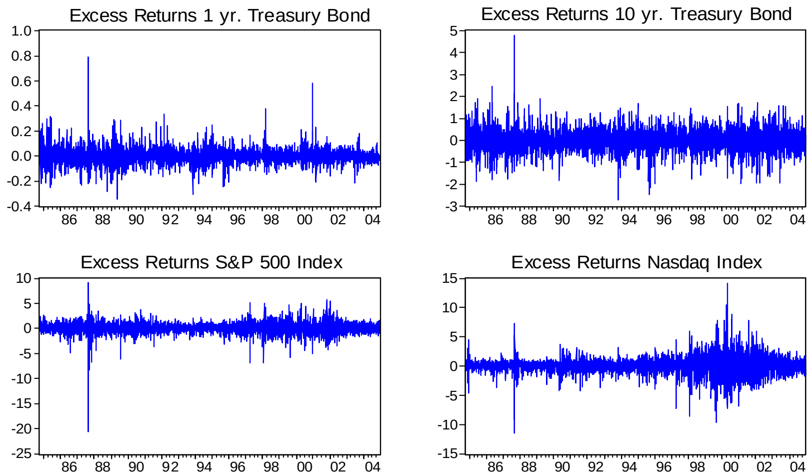

Figure 1: This figure shows the plotted excess returns of the four assets. The time span is October 11, 1984 to October 28, 2005 and all returns are daily returns in percentages.

4.2 Dependence measures

Most dependence measures, such as linear correlation, rank correlation and Kendall's τ, measure dependence between a pair of random variables over their entire supports. Hence, dependence asymmetry will be averaged out by those measures if there exits any. To measure the possible dependence asymmetry, we adopt exceedance correlation and tail dependence coefficient (TDC) as in previous studies. Longin and Solnik (2001) and Ang and Chen (2002) suggest using exceedance correlation to measure asymmetric dependence, that is, linear correlation between the variables on a subset of the support and they find that there is clear asymmetric dependence between the stock returns. In particular, this dependence asymmetry implies that two stock returns tend to be more dependent when the market is in the downturn than when in the upturn.

Exceedance correlation can be defined as follows.

$$\widetilde{\rho}(q) \equiv \begin{cases} \operatorname{corr}\left[X, Y | X \leq Q_x(q) \cap Y \leq Q_y(q)\right], & \text{for } q \leq 0.5 \\ \operatorname{corr}\left[X, Y | X > Q_x(q) \cap Y > Q_y(q)\right], & \text{for } q > 0.5, \end{cases} \tag{50}$$

where $Q_x(q)$ and $Q_y(q)$ are the qth percentiles of X and Y respectively. Among the frequently used copulas, Normal, Student's t, Plackett and Frank copulas belong to symmetric ones and feature a synmmetric exceedance, which might not fit well the data with significant asymmetric exceedance correlation. Meanwhile, Clayton and Gumbel copulas are asymmetric and they feature higher correlations at upper quantiles and lower quantiles respectively, while their rotated counterparts feature the opposite asymmetric correlation[10].

To define TDC, we have to first define tail dependence functions (TDF). The lower TDF $\tau^L(q)$ and the upper TDF $\tau^U(q)$ are given as

[10]It is worth noting that the exceedance correlation depends not only on the copula but on the marginal distribution as well. We can certainly find the case where a symmetric copula with skewed marginal distributions leads to asymmetric exceedance correlation.

$$\tau^L(q) \equiv \Pr[X \leq Q_x(q)|Y \leq Q_y(q)], \text{ for } q \in (0, 0.5] \tag{51}$$

$$\tau^U(q) \equiv \Pr[X > Q_x(1-q)|Y > Q_y(1-q)], \text{ for } q \in (0, 0.5] \tag{52}$$

Hence, TDC is simply the limits of the TDFs when q goes to zero, and it is given by

$$\tau^L = \lim_{q \to 0} \tau^L(q), \tag{53}$$

$$\tau^U = \lim_{q \to 0} \tau^U(q) \tag{54}$$

Within our copula-GARCH framework, exceedance correlation not only depends on the copula form but also on marginal distributions of individual variables. Hence, we plot exceedance correlations of both excess returns and transformed residuals (margins u in copula). Figure (2) shows plotted exceedance correlations for all possible pairs out of the four assets in terms of excess returns. First, for two stock indices, the correlation asymmetry is mild, which is not consistent with previous studies. For instance, Patton (2004) plots exceedance correlation between "large cap" and "small cap" indices from January 1954 to Decemeber 1999 at weekly frequency and there are clear asymmetries of exceedance correlation for both raw excess returns and transformed residuals. One thing worth noting is that if we plot exceedance correlation for the first subperiod (Oct. 12, 1984-Oct. 15, 1996) of our data set, we find clear asymmetric correlation as in Patton (2004). However, for the second subperiod (Oct. 16, 1996-Oct. 28, 2005), correlation asymmetry is not obvious, which makes exceedance correlation for the whole data less asymmetric. Secondly, for two Treasury bonds, there is some degree of asymmetry where in the upturn from 50 percentile to 90 percent the correlation between the two bonds are constantly around 0.6, while from 50 to 10 percentile the correlation gradually decreases from 0.6 to 0.4. For the dependence between stock indices and Treasury bonds, the correlations are slightly positive and for some pairs, the correlation goes to slightly negative at tails.

Moreover, figure (3) shows plotted exceedance correlations for all possible pairs out of the four assets in terms of transformed residuals (margins u in copula). Generally, plotted exceedance correlations for transformed residuals are similar as those for the raw excess returns. In particular, correlation asymmetry between two stock indices is mild, and correlations between stocks and bonds are weak.

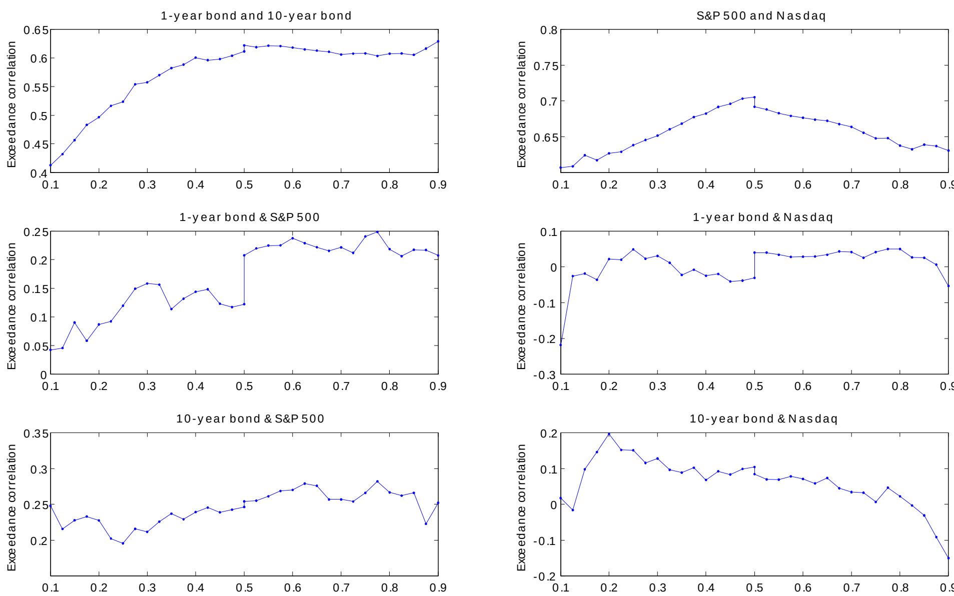

Figure 2: This figure shows the estimated exceedance correlations between each possible pair of assets.

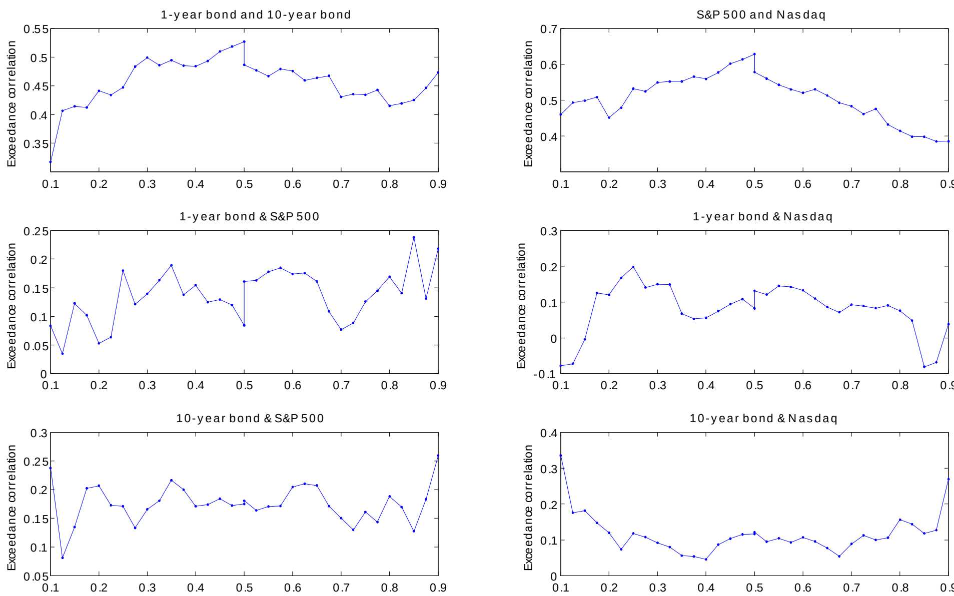

Figure 3: This figure shows the estimated exceedance correlations between each possible pair of assets in terms of transformed residuals (u).

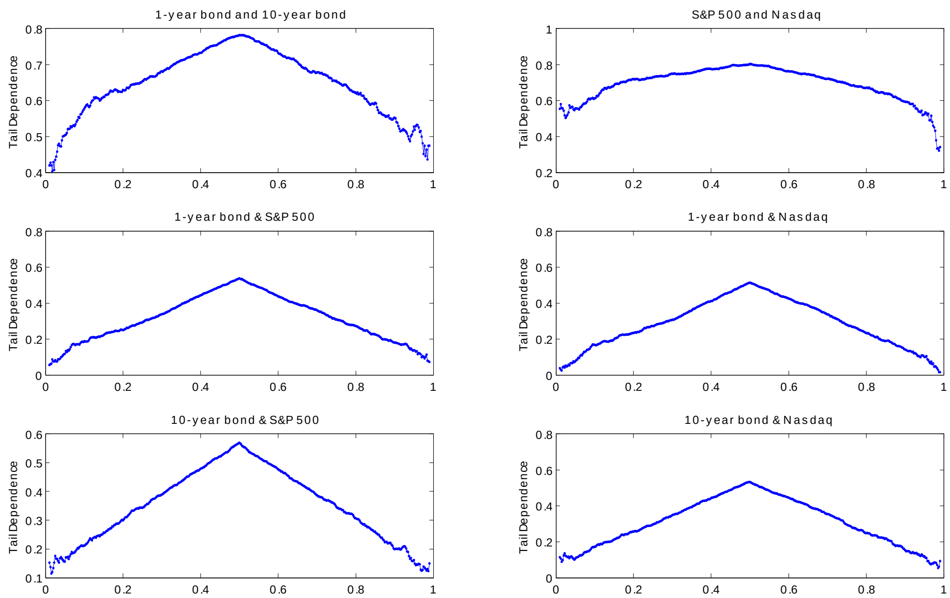

Figure 4: This figure shows the estimated tail dependence coefficients between each possible pair of assets in terms of transformed residuals (u).

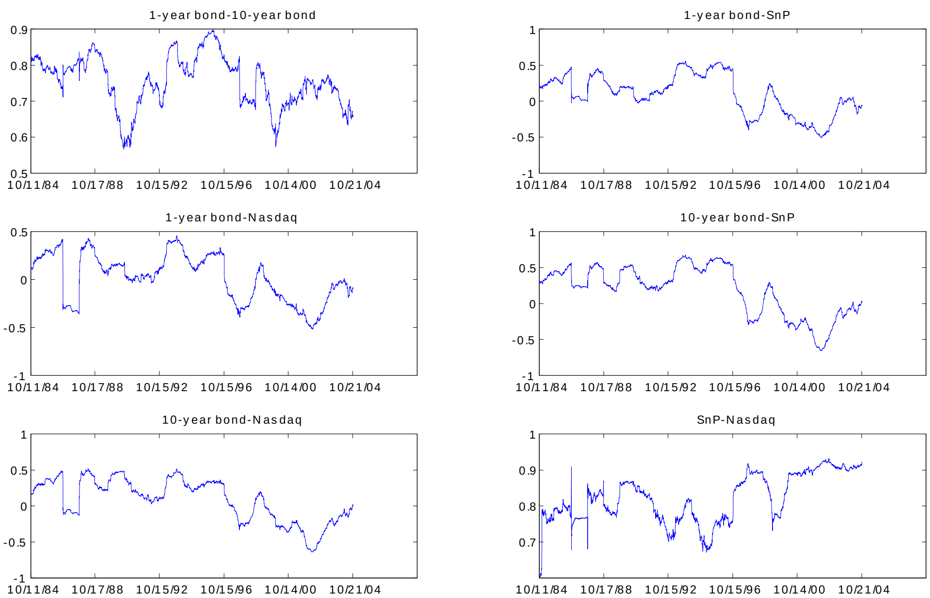

Figure 5: Plots of time-varying correlations calculated based 251 day rolling windows.

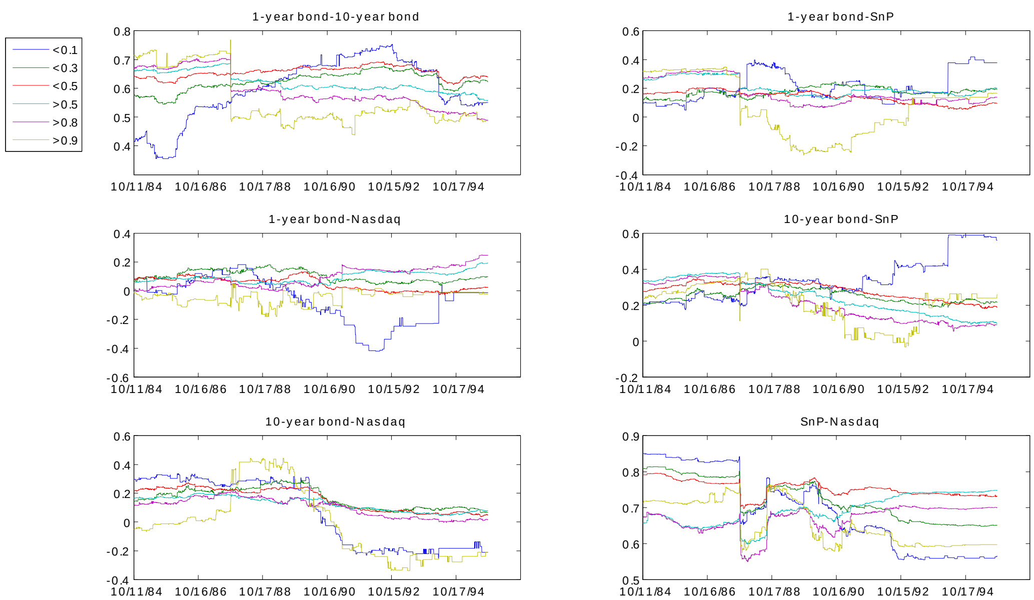

Figure 6: Plots of time-varying exceedence correlations based on 10-year rolling windows.

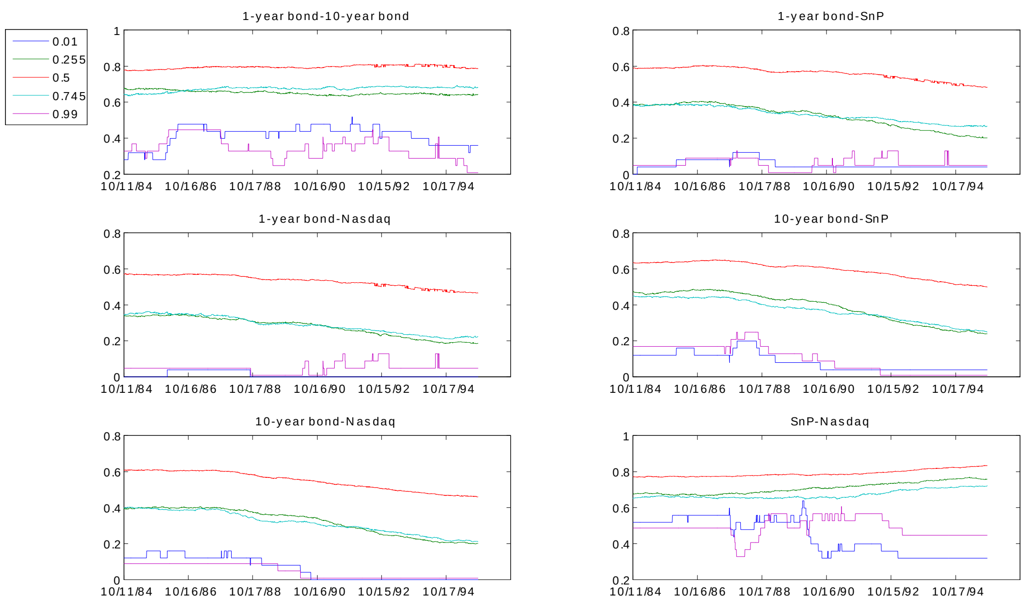

Figure 7: Plots of time-varying quantile dependence based on 10-year rolling windows.

Since tail dependence only depends on the copula, we only plot tail dependence of transformed residuals for all possible pairs out of the four assets as shown in figure (4). For two stock indices, tail dependence is symmetric over most of the interval $[0, 1]$ but asymmetric at two tails where TDC is about 0.6 at the left end and 0.3 at the right. It is consistent with the previous findings about the asymmetric excedence of the two stock indices. For the two bonds, tail dependence is generally symmetric over the whole interval while at the left ends TDC tends to be about 0.4 and at the right end TDC tends to be a bit higher as 0.45. Finally, for the pairs between stocks and bonds, tail dependence is symmetric and at the two ends TDC converges to zero.

The dependence between the stock indices is significant and the mild dependence asymmetry can be identified where higher correlation happens at the market downturn and lower correlation exists at the upturn. Similarly, the two bonds carries a significant dependence with slightly higher correlation at the upturn. Both stock indices and bonds have slightly asymmetric positive tail dependence, which are consistent with their own asymmetric exceedance correlations. In contrast, the dependence between any stock and any bond tends to be relatively weak, with the exceedance correlations ranging from -0.2 to 0.2. Moreover, the tail dependence between any stock and bonds tends to be zero.

As the time span of the data set lasts for about twenty years and the dynamics of financial series are driven by many changeable factors, it is quite reasonable to assume that the structure of the model is determined by different regimes or the parameters of the model are time-varying. With this respect, we plot the correlation, exceedance correlation and quantile dependence with rolling windows for the four series. Figure (5) shows the linear correlation between each pair of excess returns with 1-year (251 trading days) rolling window. We find that the correlation between the two stock indices and the two bonds is relatively stable and fluctuates between 0.6 and 0.9, while the correlation between each pair of the stock and bond fluctuates in the range of 0 to 0.5 in the first half period of the data set and starts to go into the range of -0.5 to 0 for the rest[11]. Figures (6) and (7) respectively plot the exceedance correlation and tail dependence (quantile dependence at different values) between each pair of excess returns based on 10-year rolling windows[12]. The asymmetry of exceedance correlation changes significantly across time and it exhibits all the patterns including higher correlation in the upper half support, symmetry and higher correlation in the lower half support. The tail dependence tends to be relatively stable across time. Specifically, the tail dependence between the two stocks and the two bonds are significantly positive, while the tail dependence between each pair of stock and bond tends to be very close to zero. All the above dependence plots based on rolling windows indicate that the (unconditional) dependence structure of the financial variables tends to change significantly across time. Hence, it is quite plausible to assume a time-varying dependence structure for the conditional joint density of the financial variables.

[11] Andersson, Krylova and Vaehaemaa (2008) show that higher inflation expectation leads to positive correlation between bonds and stocks, while the negative correlation coincides with lower inflation expectation and higher uncertainty ("flight to safety").

[12] We need relatively longer length of data to plot the exceedence correlation and tail dependence.

5 Empirical results

5.1 Results for marginal distributions

In this section, we briefly report estimation results for marginal distributions of the four individual series. The conditional mean of the individual series is estimated by OLS as shown in Table (2). For the two Treasury bonds, we find that the conditional mean of 1-year bond depends on its own value at 1 lag and its spread at 1 and 2 lags while the conditional mean of the 10-year bond depends on its own value at 1 and 7 lags and its spread at 1 and 2 lags. For the two stock indices, S&P 500 index depends on its own values at 2, 3 and 12 lags while Nasdaq index depends on its own values at 1, 2, 4, 12 and 13 lags. For the four series, the constant term is statistically insignificant. Moreover, we conduct Box-Pierce test and Breusch-Godfrey test for possible serial correlation and cannot reject the null hypothesis that the residuals are uncorrelated for all the four series. Unsatisfactorily, we find that the R-squared is very low. Nevertheless, we believe that our estimation of the conditional mean is sufficiently robust for further estimation of the higher moments of the series.

In Table (3), we show estimation results for the marginal distribution of residuals. Generally, we applied TARCH(1,1) setup to the residuals to account for time-varying conditional variance and asymmetric influence of innovations on the conditional variance. To better describe the data, we assume the innovations have a skewed t distribution, while for comparison purposes we also provide estimation results for normal distribution. For the two Treasury bonds, TARCH effects are not statistically significant and then we use a GARCH(1,1) setup, while for the two stock indices TARCH effects are statistically significant. In the skewed t distribution, we assume that the degrees-of-freedom parameter and skewness paramters are time-varying. For all the four series, we find that skewed t distribution universally yields higher log-likelihoods than normal distribution.

5.2 Results for copulas

5.2.1 Normal copula and Student's t copula

The left part of Table (4) reports estimation results for normal copula. The correlation matrix Σ is estimated by sample correlation and based on standard errors we find that all the coefficients in $\widehat{\Sigma}$ are statistically significant. Since marginal distributions of individual series are not normal, these estimated correlation coefficients do not exactly measure the correlation of each pair of variables. Nevertheless, they still measure the dependence structure between each pair of variables and the high values of the coefficients between the two bonds and between the two stocks suggest the high dependence within each market, while the low values between the bond and the stock indicate the low dependence between the two markets.

The results for the Student's t distribution are shown in the right part of Table (4) where correlation matrix R and the degrees-of-freedom ν are estimated. Again, based on standard errors, we find that all the estimates are statistically significant. The correlation matrix R has quite similar values as correlation matrix Σ of normal copula, while Student's t copula yields higher log-likelihoods and lower AIC and BIC than normal copula.

Featuring symmetric dependence structure and zero tail dependence, the normal copula should fit the dependence between any pair of a bond and a stock quite well as implied by the plotted exceedance correlations and TDCs between bonds and stocks in figures (2) to

1 year Treasury bond			10 year Treasury bond			S&P 500 index			Nasdaq index		
Variable	Coefficient	Robust SE	Variable	Coefficient	Robust SE	Variable	Coefficient	Robust SE	Variable	Coefficient	Robust SE
ONEYRBD(-1)	0.1226	0.0332	TENYRBD(-1)	0.0994	0.0278	S&P(-2)	-0.0433	0.0264	NAS(-1)	0.0527	0.0209
YIELDSPRD1(-1)	0.0899	0.0353	TENYRBD(-7)	0.0423	0.0135	S&P(-3)	-0.0376	0.0204	NAS(-2)	-0.0335	0.0218
YIELDSPRD1(-2)	-0.0857	0.0345	YIELDSPRD10(-1)	0.4512	0.2464	S&P(-12)	0.0423	0.0155	NAS(-4)	0.0319	0.0218
			YIELDSPRD10(-2)	-0.4431	0.2459				NAS(-12)	0.0647	0.0192
R-squared	0.0117					R-squared	0.0046		NAS(-13)	0.0485	0.0221
			R-squared	0.0080							
Q-statistics (12 lags)	p-value					Q-statistics (12 lags)	p-value		R-squared	0.0114	
13.0042	0.3687		Q-statistics (12 lags)	p-value		7.7064	0.8076				
			14.2740	0.2835					Q-statistics (12 lags)	p-value	
									6.0052	0.9158	
LM Test: (12 lags)			LM Test: (12 lags)			LM Test: (12 lags)			LM Test: (12 lags)		
F-statistic	p-value		F-statistic	p-value		F-statistic	p-value			F-statistic	p-value
1.4979	0.1028		1.2603	0.2351		1.5078	0.1132		1.097	0.3574	

Table 2: Estimation results for marginal distributions: conditional mean

1 year bond			10 year bond		
	Coefficient	Robust SE		Coefficient	Robust SE
Normal distribution			Normal distribution		
Variance			Variance		
Constant	3.82E-05	1.63E-05	Constant	0.0045	0.0014
ARCH(-1)	0.0651	0.0183	ARCH(-1)	0.0346	0.0075
GARCH(-1)	0.9260	0.0192	GARCH(-1)	0.9381	0.0118
Log likelihod	8099.9		Log likelihod	-3260.5	
Skewed t distribution			Skewed t distribution		
Variance			Variance		
Constant	3.37E-05	1.08E-05	Constant	0.0038	0.0010
ARCH(-1)	0.0478	0.0094	ARCH(-1)	0.0407	0.0060
GARCH(-1)	0.9430	0.0111	GARCH(-1)	0.9420	0.0086
Nu			Nu		
Constant	-0.4438	0.2640	Constant	1.0356	0.2764
resid(-1)	-2.2971	2.9921	resid(-1)	-0.4921	0.4764
yieldsprd(-1)	1.2193	0.4185	yieldsprd(-1)	0.1735	0.1338
Lambda			Lambda		
Constant	-0.0167	0.0231	Constant	-0.0494	0.2967
resid(-1)	2.5605	0.5624	resid(-1)	0.2938	0.1069
yieldsprd(-1)	-0.0842	0.0541	yieldsprd(-1)	-0.0360	0.1395
Log likelihood	8465.1		Log likelihood	-3094.5	

S&P			Nasdaq		
	Coefficient	Robust SE		Coefficient	Robust SE
Normal distribution			Normal distribution		
Variance			Variance		
Constant	0.0181	0.0080	Constant	0.0170	0.0052
ARCH(-1)	0.0178	0.0068	ARCH(-1)	0.0666	0.0135
ARCH(-1)(<0)	0.1078	0.0362	ARCH(-1)(<0)	0.0954	0.0267
GARCH(-1)	0.9130	0.0247	GARCH(-1)	0.8785	0.0212
Log likelihod	-6940.6		Log likelihod	-7618	
Skewed t distribution			Skewed t distribution		
Variance			Variance		
Constant	0.0117	0.0037	Constant	0.0101	0.0025
ARCH(-1)	0.0220	0.0059	ARCH(-1)	0.0635	0.0100
ARCH(-1)(<0)	0.0844	0.0202	ARCH(-1)(<0)	0.0705	0.0150
GARCH(-1)	0.9276	0.0124	GARCH(-1)	0.9012	0.0121
Nu			Nu		
Constant	1.5249	0.2880	Constant	-1.7775	0.2113
resid(-1)	-0.4433	0.7385	resid(-1)	0.3906	0.1195
Lambda			Lambda		
Constant	-0.1220	0.0315	Constant	-0.344	0.0372
resid(-1)	0.1473	0.0359	resid(-1)	0.0056	0.0351
Log likelihood	-6769.8		Log likelihood	-7469.4	

Table 3: Estimation results for marginal distributions: conditional variance, skewness and kurtosis.

(4). Yet the normal copula may lose a bit of good fit in modeling the dependence within the two bonds or the two stocks due to their mild asymmetric correlations and positive tail dependence. Meanwhile, the Student's t copula fits well the symmetric dependence between any pair of a bond and a stock, but not their zero tail dependence, while it fits well the positive tail dependence between the two bonds or the two stocks, but not their asymmetric correlations. As the dependence asymmetry and the non-zero tail dependence are weak, both normal copula and Student's t copula yield high log-likelihoods.

Now let us turn to the estimation results for the time-varying correlation matrix of the two elliptical copulas. Table (5) shows the estimates of the parameters which govern the dynamics of the correlation matrix, which are all statistically significant based on standard errors. The estimates of α and β are quite similar for both copulas, and the estimate of degree-of-freedom parameter in the Student's t copulas is clearly higher than its counterpart with constant correlation matrix. With time-varying correlation matrix, the two elliptical copulas yield higher log-likelihood and lower AIC and BIC than their counterpart with constant correlation matrix, and within the two copulas Student's t copulas has relatively higher log-likelihood and lower AIC and BIC. Figure (8) shows the plots of the estimated time-varying correlation matrix for the two elliptical copulas. These time-varying correlation matrices are the correlation matrices of the copula forms but are not exactly the actual correlations of the underlying asset returns. Nevertheless, these time-varying correlations in the copulas forms still have very similar patterns with the actual correlations on rolling windows plotted in figure (5).

5.2.2 Hierarchical copula

In table (6), we report estimation results for hierarchical copulas which are constructed by Frank, Clayton and Gumbel copulas respectively. For all the three hierarchical copulas, we find that the achieved log-likelihoods at the first level (C_1 and C_2) are much higher than that at the second level (C_3). This is due to the fact that at the first level C_1 and C_2 measure the dependence within two bonds and within two stocks respectively and it is much stronger than dependence between bonds and stocks, which is measured by C_3 at the second level. The dependence parameters in all the copulas are assumed to be time-varying and the estimated dependence parameters at the first level (C_1 and C_2) are universally greater than those at the second level (C_3), which is consistent with our dependence parameter restrictions.

Fank copula features symmetric dependence structure, hence it may better fit the dependence between any pair of a bond and a stock, and not fit so well the asymmetric dependence within the two bonds or the two stocks. Featuring higher correlation in the downturn and positive tail dependence at the left tail, Clayton may better fit the dependence between the two stocks, while Gumbel copula features higher correlation in the upturn and positive tail dependence at the right tail and it may better fit the dependence between the two bonds. We find that Gumbel copula yields the highest log-likelihood and lowest AIC and BIC among the three hierarchical copulas.

It is worth noting that we could improve the model fit by modeling each pair of two assets using one particular copula that features their particular dependence structure. For instance, we could use a Clayton copula to model the two stocks for their higher correlation in the downturn and positive lower tail dependence, use a Gumbel copula to model the two bonds for their slightly higher correlation in the upturn and positive lower tail dependence, and use a Frank copula to nest the two copulas to account for the symmetric dependence

Normal copula					Student's t copula				
	$\widehat{\Sigma}$					$\widehat{R}$			
	1 year bond	10 year bond	S P 500 index	Nasdaq index		1 year bond	10 year bond	S P 500 index	Nasdaq index
1 year bond					1 year bond				
10 year bond	0.7610				10 year bond	0.7611			
	(0.0062)					(0.0056)			
S P 500 index	0.1056	0.1817			S P 500 index	0.0987	0.1775		
	(0.0152)	(0.0156)				(0.0145)	(0.0151)		
Nasdaq index	0.0476	0.0974	0.8190		Nasdaq index	0.0434	0.0940	0.8234	
	(0.0145)	(0.0152)	(0.0046)			(0.0141)	(0.0148)	(0.0041)	
log-likelihood	5340.2					ν			
						9.0067			
AIC	-2.0348					(0.6062)			
BIC	-2.0273								
					log-likelihood	5548.1			
					AIC	-2.113			
					BIC	-2.105			

Table 4: Estimation results for marginal distributions: conditional variance, skewness and kurtosis.

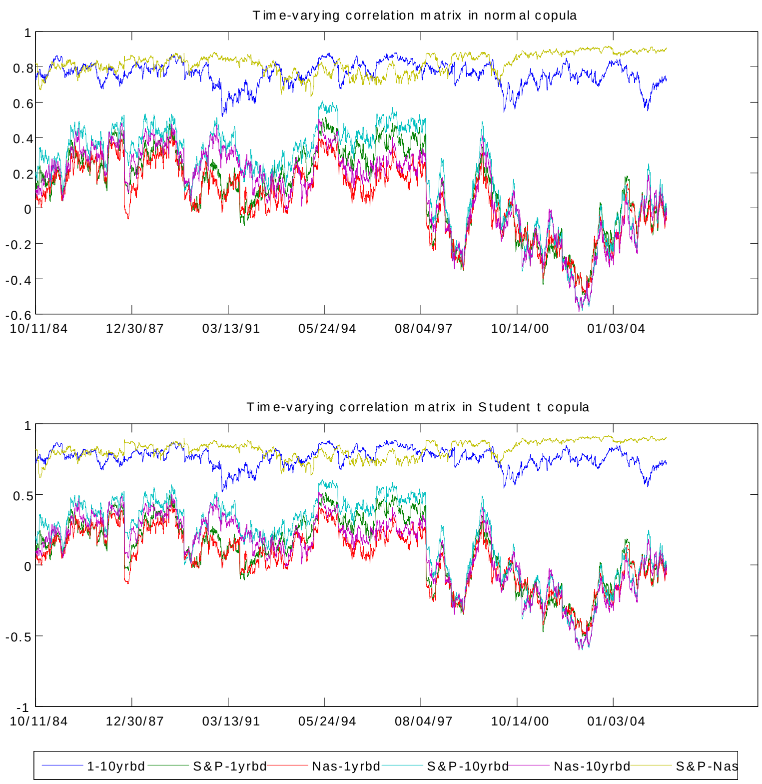

Figure 8: Plots of estimated time-varying correlation matrix in the two elliptical copulas.

Normal copula		
	Estimate	Robust SE
α	0.0166	0.0023
β	0.0046	0.0014
Log likelihood	5827	
AIC	-2.2219	
BIC	-2.2169	
Student's t copula		
	Estimate	Robust SE
α	0.0159	0.0022
β	0.0049	0.0014
ν	11.1843	0.8273
Log likelihood	5993	
AIC	-2.285	
BIC	-2.28	

Table 5: Estimation of time-varying dependence structure in the two elliptical copulas.

between the two markets. However, we need to prove the validity of the constructed copula before we pursue with this procedure. More work could be done in this direction.

5.2.3 Mixed copula

In stage two of the estimation for mixed copulas, we need first to choose copula forms for the copulas $C1$ to $C6$. Our copula candidates include normal, Student's t, Plackett, Frank, Clayton, Rotated Clayton, Gumbel and Rotated Gumbel copulas of which the first four feature a symmetric dependence structure and the rest features an asymmetric one. Table (7) shows the log-likelihoods yield by using all the copula candidates for $C1$ to $C6$. We find that all the copula candidates universally yield much higher log-likelihoods for $C1$ to $C2$ than those for $C3$ to $C6$, which is consistent with the fact that the dependence within stocks and that within bonds are both generally much stronger than that between bonds and stocks. Secondly, among all the copula candidates, we find that Student's t copula universally yields the highest log-likelihood for $C1$ to $C6$. Due to this observation, we select Student's t copula for the estimation of $C1$ to $C6$.

Since they yield quite low log-likelihoods, the weights of $C3$ to $C6$ tend to be zero when we maximize mixed copula with respect to the weights. Accordingly, we only choose $C1$ to $C2$ to construct our mixed copula and table (8) shows the estimation results. We find that copula $C1$ accounts for a lower weight as 0.37 and $C2$ accounts for a higher weight as 0.63, which suggest that the dependence between the two stock indices accounts for more weights for this mixed copula.

One point worth noting is that we can extend the mixed copula by including all the copulas from $C1$ to $C6$ and making their weights time-varying. By making the weights depend on the lagged explainatory variable, we may better fit the data where the weights of $C3$ to $C6$ tend to be more significant in some periods and less significant in others. Similarly, a regime-switching specification can also be applied in this case. More work could be done in this direction.

5.2.4 Comparison of three approaches

Among the three approaches, normal copula and Student's t copula yield the highest log-likelihood where the latter yields even higher log-likelihood than the former. It is consistent with our plotted asymmetric dependence between the series, as the plotted dependence asymmetries are not very significant and normal copula and Student's t copula capture this aspect of the data sufficiently well. With the two elliptical copulas, our estimation results show that time-varying correlation matrix can better describe the dynamics of the dependence structure and further increase the log-likelihood. Hence, the time-varying Student's t copula yields the highest log-likelihood for our data set.

Hierarchical copula				
Frank copula				
Copula	Log likelihood	AIC	BIC	Number of parameters
c1	2133.4	-0.8120	-0.8058	5
c2	2772.4	-1.0566	-1.0529	3
c3	206.45	-0.0772	-0.0722	4
Clayton copula				
Copula	Log likelihood	AIC	BIC	Number of parameters
c1	1835.4	-0.6983	-0.6921	5
c2	2498.1	-0.9519	-0.9482	3
c3	174.68	-0.0641	-0.0578	5
Gumbel copula				
Copula	Log likelihood	AIC	BIC	Number of parameters
c1	2188.6	-0.8331	-0.8268	5
c2	2760.5	-1.0521	-1.0483	3
c3	235.64	-0.0884	-0.0833	4

Table 6: Estimation of hierarchical copulae: Frank copula, Clayton copula and Gumbel copula.

Log likelihood						
	C_1	C_2	C_3	C_4	C_5	C_6
Symmetric copulas						
Normal	2276.5	2969.4	40.683	90.953	25.03	95.222
Student's t	2376.6	3047.8	70.754	137.87	81.459	193.97
Plackett	2226.2	1317.3	46.985	84.88	26.726	88.903
Frank	2133.4	2772.4	45.397	89.803	24.906	95.798
Asymmetric copulas						
Clayton	1835.4	2498.1	23.758	64.491	32.172	94.064
Rotated Clayton	1781.4	2205	26.516	73.902	24.703	91.874
Gumbel	2188.6	2760.5	35.149	93.216	35.452	120.25
Rotated Gumbel	2229.7	2953.5	29.902	82.818	42.842	124.29

Table 7: This table shows the log-likelihoods yield by using all the copula candidates.

Mixed copula			
	Coefficient	SE	RobustSE
π_1	0.37322	0.016067	0.015215
π_2	0.62678	0.01751	0.015218
log-likelihood	3570.3		
AIC	-1.3614		
BIC	-1.3589		

Table 8: This table shows the estimated weights of the mixed copula, together with the log-likelihood, AIC and BIC.

Meanwhile, it is worthwhile to note that Student's t copula also has certain limitations on its dependence structure. Even though its copula form features symmetric exceedance correlation, Student's t copula can still accomodate certain degree of asymmetric exceedance correlation as the exceedance correlation is determined by both copula form and the nested marginal distribution and skewed marginal distributions with symmetric copulas can still leads to asymmetric exceedance correlation. However, the limitations lie in the tail dependence. Within a multivariate Student's t copula, the tail dependence of each pair of variables is symmetric and is determined by the correlation coefficient and the degree-of-freedom parameter ν As well known, as the degree-of-freedom parameter goes to infinity, the tail dependence goes to zero and the Student's t copula becomes normal copula. Therefore, Student's t copula tends to be restrictive in modeling asymmetric tail dependence among each pairs of variables.

Moreover, with time-varying normal and Student's t copulas we can plot the time-varying correlation matrix which replicates very well the actual correlation between the underlying variables. However, our copula-GARCH framework does not carry explicit parameters which directly govern the actual correlation matrix of the underlying variables and we need resort to simulations to derive the actual correlation structure.

Even though hierarchical copula yields significantly low log-likelihood in our case, it is still a valid and intuitive way of building up a joint distribution. With other data sets or more choices of Archimedean copulas, it may lead to a better overall fit of the data. One thing worth noting is that in our case given the copulas we use the weak dependence between bonds and stocks at the second level of the hierarchical copula appears to be the main reason for low log-likelihoods.

The mixed copula approach is another good way of incorporating dependence structures of all possible pairs of variables. Under a mixed copula, we can choose any copula form for each pair of variables, while in hierarchical copula we have more restrictive choices of copula forms as the chosen copulas have to satisfy certain conditions for the resulted copula to be a valid distribution function. Nevertheless, the weighted structure of mixed copula is still a compromised way of incorporating all dependence relations of variables.

6 Conclusion

This paper attempts three approaches in building up a multivariate copula in a copula-GARCH framework for modeling the joint distribution of multiple asset excess returns. Our assets include one year and ten year Treasury bonds, and S&P 500 and Nasdaq indices. As for asymmetric dependence, we plot (uncondidtional) exceedance correlation and tail dependence between these series. We find that empirical dependence asymmetries between two stock indices are not as significant as documented in previous studies and those between two bonds are also mild. Meanwhile, the (unconditional) dependence between bonds and stocks appears to be quite weak. We also plot the correlations, exceedance correlations and tail dependence on rolling windows. We observe that rolling correlations for the pair of two stock indices and the pair of two bonds tend to fluctuate in the positive range, while the rolling correlation between any stock and any bond lie in the positive range in the first half time period and then tend to fluctuate into the negative range for the rest of time span. Moreover, the rolling exceedance correlations for each pair of variables are changing significantly across the time, while the rolling tail dependence tends to be relatively stable. All these plotted time-varying dependence measures indicate the need of assigning time-varying parameter structure in the copula function.

We construct our multivariate joint distribution by using normal copula, Student's t copula, hierarchical Archimedean copulas and mixed copulas and estimation is conducted by a two-step procedure. Our results show that normal copula and Student's t copula yield higher log-likelihood than hierarchical copula and mixed copula, where hierarchical copula yields the lowest log-likelihood. Within the elliptical copulas, their time-varying versions even yield higher log-likelihood and the time-varying Student's t copula tends to have the highest log-likelihood.

The time-varying Student's t copula stands out to be a good choice for modeling the conditional joint density with various non-normal features. In our case, the estimated time-varying correlation matrix in the copula function, even though not the same as the actual correlations of underlying variables, can still replicate the plotted rolling correlations very well. In fact, we can also plot the actual correlation from our model by simulation procedures, which would be expected to have similar time-varying patterns with the former ones. Meanwhile, it indicates that our copula-GARCH framework does not specify explicit parameters which directly govern the correlation structure of underlying variables.

Even though its copula form features symmetric exceedance correlation, Student's t copula can still accomodate certain degree of asymmetric exceedance correlation as the exceedance correlation is determined by both copula form and the nested marginal distribution and skewed marginal distributions with symmetric copulas can still leads to asymmetric exceedance correlation. However, the limitations lie in the tail dependence. Within a multivariate Student's t copula, the tail dependence of each pair of variables is symmetric and is determined by the correlation coefficient and the degree-of-freedom parameter ν. Therefore, Student's t copula tends to be restrictive in modeling different tail dependence among each pairs of variables.

Hierarchical copula and mixed copula yield relatively lower log-likelihoods, but they are still an intuitive and valid way of building a n-dimensional copula. Given the Archimedean copulas we use, the weak dependence between bonds and stocks at the second level of the hierarchical copula appears to be the main reason for low log-likelihoods. With other data sets or more choices of Archimedean copulas, it may lead to a better overall fit of the data. We could improve the model's fit by modeling each pair of two assets using one particular copula that features their particular dependence structure after building the validity of the resulted hierarchical copula. For the mixed copula, we could include all the copulas accounting for dependence between each pair of variables and make their weights time-varying and we may better fit the data where the weights can be more significant in some periods and less significant in others. Moreover, a regime-switching specification can be used based on the mixed copula. Nevertheless, the weighted structure of mixed copula is still a compromised way of incorporating all dependence relations of variables.

A Calculations of the bond and stock returns

To calcuate the bond excess returns, we use the daily constant maturity interest rate series which are obtained from the federal bank in St. Louis. We follow Jones, Lamont and Lumsdaine (1998) to calculate the bond returns and we follow the notations in the appendix B of Goeij and Marquering (2004). The coupon of the U.S. Treasury bonds is paid semiannually and the coupon on the hypothetical bonds is half the stated coupon yield. So the price of the hypothetical bond at the beginning of the holding period is equal to its face value. Using the next day's yield augmented with the accrued interest rate, we calculate an end-of-period price on this bond as

$$P_{n-\#hd,t+1} = \sum_{i=1}^{2n-1} \frac{\frac{1}{2}y_{n,t}}{(1+\frac{1}{2}y_{n,t+1})^i} + \frac{1+\frac{1}{2}y_{n,t}}{(1+\frac{1}{2}y_{n,t+1})^{2n}} + \frac{\text{\#holding days}}{365} y_{n,t},$$

where $P_{n-\#hd,t+1}$ is the price of the bond at the end of the holding period, n is the number of years the bond is referring to, t is the time, and $y_{n,t}$ is the yield of an n-year bond at time t . Hence the return of the bond for the holding period can be calculated as

$$r_{t+1} = P_{n-\#hd,t+1} - 1.$$

Then we use the three-month interest rate as the risk-free rate that accrues over the holding period, to calcuate the excess return of the bond:

$$r^e_{t+1} = r_{t+1} - \frac{\text{\#holding days}}{365} y_{3mo,t} \, .$$

For the stock excess returns, we obtain the closing indexes from http://finance.yahoo.com. The stock return of the holding period can be calculted as

$$r_{index,t+1} = \frac{P_{index,t+1} - P_{index,t}}{P_{index,t}}$$

where $P_{index,t}$ is the price of the index at time t . Hence using the three-month interest rate as the risk-free rate, the excess return of the stock of the holding period can be calculated as

$$r^e_{index,t+1} = r_{index,t+1} - \frac{\text{\#hodling days}}{365} y_{3mo,t}.$$

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