A Two-Factor Model for Commodity Prices and Futures Valuation*

Diana Ribeiro† Stewart Hodges‡

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Abstract

This paper develops a new reduced form two-factor model for commodity spot prices and futures valuation. This models extends Schwartz's (1997)[20] two-factor model by adding two new features. First we replace the Ornstein-Uhlenbeck process for the convenience yield by a Cox-Ingersoll-Ross (CIR) process. This ensures that our model is arbitrage free while Schwartz's model does not rule out arbitrage possibilities. Second, we introduce a time-varying volatility for the spot price process. In particular, we consider the spot price volatility is proportional to the square root of the convenience yield level. This implicitly implies that the spot price volatility depends on inventory levels of the commodity as predicted by the theory of storage. We empirically test both models using weekly crude oil futures data from 5^{th} of March 1999 to the 15^{th} of October 2003. In both cases, we estimate the models's parameters using the Kalman filter.

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†Doctoral Researcher at the Warwick Business School, University of Warwick, Coventry, CV4 7AL, United Kingdom, phone +44 (0)24 76524465, fax: +44 (0)24 7652 7167 and email: D.Ribeiro@wbs.ac.uk

‡Director of the Financial Options Research Center (FORC), Warwick Business School, University of Warwick, Coventry CV4 7AL, United Kingdom, phone: +44 (0)24 76523606, fax: +44 (0)24 7652 4176 and email forcsh@wbs.warwick.ac.uk

1 Introduction

The recent development of energy markets has highlighted the necessity of developing new models to evaluate energy futures and other financial contracts. The reduced form class of models dominates the current literature and practice. Basically, this approach considers the spot price and the convenience yield as separate stochastic processes with constant correlation. These models are particularly attractive from practitioner's perspective since they provide closed form solutions to evaluate futures and some other derivatives contracts. This in turn allows for a relatively easy calibration and computational implementation of these models.

This class of models was first introduced by Brennan and Schwartz (1985) [3] where the spot commodity price follows a geometric Brownian motion and the convenience yield is treated as a dividend yield. This specification is inappropriate since it does not take into account the mean-reversion property of the spot commodity prices and neglects the inventory-dependence property of the convenience yield. Gibson and Schwartz (1990)[11] introduce a two-factor, constant volatility model where the spot price and the convenience yield follow a joint stochastic process with constant correlation. Specifically, the spot price follows a geometric Brownian motion and the convenience yield follows a mean reverting stochastic process of the Ornstein-Uhlenbeck type. These two state variables are only linked through a coefficient of correlation. Schwartz (1997)[20] introduces variation of this model where the convenience yield is brought into the spot price process as a dividend yield. Schwartz (1997)[20] - model 3, Miltersen and Schwartz (1998)[18] and Hilliard and Reis (1998)[13] add a third stochastic factor to the model to account for stochastic interest rates. Nevertheless, the inclusion of stochastic interest rates in the commodity price models does not have a significant impact in the pricing of commodity options and futures in practice. Accordingly, interest rate can be assumed deterministic.

Although these multi-factor models generate a rich set of dynamics for the commodity term structure and represent prevailing tools for derivatives pricing, they also present a number of problems. First, these models do not guarantee that the convenience yield is always well defined and thus may become negative, possibly allowing for arbitrage opportunities. More specifically, arbitrage-free arguments require that the discounted futures prices net of carrying costs cannot be greater than the contemporaneous spot prices (c.f., e.g., Hull (2000)[15]). By not ruling out negative values for the convenience yield, this arbitrage argument may be violated. Secondly, these models present other mis-specification problems due to the fact that both the spot price and the convenience yield have constant volatility and correlation. Accordingly, they do not allow the variance of the spot and futures, and the correlation between them, to depend on the level of the price or convenience yield, as suggested by the theory of storage. These mis-specifications of the models generate severe option mispricings, as pointed out by Pirrong (1998)[19], Clewlow and Strickland (2000)[6] and Routledge, Seppi and Spatt (2000)[5].

In this paper, we introduce a new reduced form model for commodity spot prices and futures valuation which builds on and extends the reduced form models of Gibson and Schwartz (1990)[11] and Schwartz (1997)[20]. We develop a two-factor model where spot prices and instantaneous convenience yield follow a joint stochastic process with constant correlation. Our model introduces two significant additions to Schwartz's (1997)[20] two factor model: it rules out arbitrage possibilities and it considers time varying spot and convenience yield volatilities. Namely, the spot price follows a geometric Brownian motion where the convenience yield is treated as an exogenous dividend yield and the volatility is proportional to the square root of the instantaneous convenience yield level. The instantaneous convenience yield follows a Cox-Ingersoll-Ross (CIR) which precludes negative values and makes the volatility proportional to the square root of the instantaneous convenience yield level. This ensures that our model does not allow cash-and-carry arbitrage possibilities.

We obtain a closed-form solution for the prices of futures prices of the exponential affine form[1]. We solve the partial differential equation (PDE) for futures prices by supposing that the solution has a general exponential affine form. By replacing this general affine form into the initial PDE we obtain a system of two ordinary differential equations (ODEs) with initial conditions consistent with the futures price initial condition. We find that each of these ODEs has a unique closed form solution. These in turn provide the solution for the initial PDE satisfied by the futures prices[2]. This affine relationship is tractable and offers empirical advantages. In particular, the linear relationship between the logarithm of the futures price and the underlying state variables allows the use of the Kalman filter in the estimation of the parameters of the model. Spot prices data are not easily obtained in most of the commodity markets and therefore futures prices with closest maturity are used as a proxy for the commodity price level. Additionally, the instantaneous convenience yield is not observable but it can be derived

[1]Since we assume that the interest rate is constant, the futures and forward prices are the same.

[2]A similar approach has been applied in the interest rate models literature such as Hull and White (1990) [14] Brown and Schaefer (1994)[4] and Duffie and Kan (1994)[8] and (1996)[9].

from the relationship between the spot and futures prices with closest maturity. On the other hand, futures prices are widely observed and traded in diverse markets. The non-observability of the state variables remains one of the main difficulties in modelling commodity spot prices and contracts. Due to the non-observability of the state variable, the linearity of the logarithm of the futures prices in the state variables and the Markovian property of these, the Kalman filter seems to be the most appropriate technique to estimate the model's parameters, following a similar methodology to Schwartz (1997)[20]. The basic principle of Kalman filter is the use of temporal series of observable variables to reconstitute the value of the non-observable variables. Accordingly, by observing futures prices, we can estimate the parameter values for the spot price and convenience yield.

We apply the Kalman filter method to estimate the parameters of our model using light crude oil futures data for the period from 17^{th} of March 1999 to 15^{th} of October 2003. Additionally, we also apply the Kalman filter to Schwartz's (1997)[20] two factor model and compare the results. The empirical results show that our model outperform Schwartz's (1997) model but not significantly, in term of pricing errors. The performance of both models is approximately the same in terms of pricing errors. In both cases the analysis reveals a very strong mean reversion and a high value of average convenience yield. This indicates that the oil market is predominantly in pronounced backwardation during this period of time. Additionally, both the spot price and the convenience yield's reveal high volatility. We can interpret this market behavior as a clear consequence of the recent turmoil world events since September 2001 culminating with the Golf war. These incidents have clearly increased the markets volatility in general and the uncertainty of the oil supply in particular.

The remaining of this paper is organized as follows. Section 2 develops the two-factor model and derives the corresponding partial differential equation for futures valuation. Section 3 describes the empirical work, including the state-space formulation of the model, the data used and the empirical results. Section 4 concludes.

2 Valuation Model

In this section we present a new two-factor model for commodity prices and derive the corresponding formulas for pricing futures contracts. This model has two stochastic factors. The first factor is the spot price, which follows as a geometric Brownian motion with a time-varying volatility, which is proportional to the square root of the instantaneous convenience yield level. The second factor is the convenience yield, which follows a Cox-Ingersoll-Ross (CIR) stochastic process as described in Cox, Ingersoll and Ross (1985) [16]. This process precludes negative convenience yields and implies that the absolute variance of the convenience yield increases when the convenience yield itself increases. We assume that both stochastic processes have constant correlation. The correlation between the two stochastic factors and the direct proportionality of the spot price's and the convenience yield's volatilities to the square root of the instantaneous convenience yield level implicitly reflect the effect of the commodity inventories levels on the spot price and convenience yield behavior. According to the theory of storage, when the inventory level of the commodity decreases, the spot price increases since the commodity is scarce and therefore convenience yield should also increase. This is because the futures prices are less elastic than spot price to supply and demand conditions variation in the market. As a result, when the commodity is scarce, the futures price will not increase as much as the spot price and therefore we observe backwardation in the commodity's forward curve. Similarly, the market is in contango when storage levels are high. In addition, the volatility of the spot prices is higher for high levels of the convenience yield, since high volatility signals low inventory levels and increases supply uncertainty. This uncertainty in the market also affects similarly the volatility of the convenience yield itself.

The model we present is very tractable since it allows for a closed form solution for futures prices. In particular we obtain linear relation between the logarithm of futures prices and the underlying factors. These properties will be crucial in the empirical work that follows.

We assume that he spot price and the instantaneous convenience yield follow the joint stochastic process:

$$dS = (\mu - \delta)Sdt + \sigma_1\sqrt{\delta}Sdz_1 \quad (1)$$

$$d\delta = (\alpha(m - \delta))dt + \sigma_2\sqrt{\delta}dz_2 \quad (2)$$

where:

- μ is the total expected return on the spot commodity price;
- σ_1 represents the constant of proportionality between the total spot price volatility and the square root of the instantaneous convenience yield;

- σ_2 represents the constant of proportionality between the total instantaneous convenience yield volatility and the square root of the instantaneous convenience yield;
- α is the instantaneous convenience yield's speed of mean reversion;
- m is the convenience yield long-run mean, that is, the level to which δ reverts as t goes to infinity;
- dz_1 and dz_2 are increments to standard Brownian motion and are correlated with $dz_1 dz_2 = \rho dt$, ρ being constant;

$$dz_1 dz_2 = \rho dt \tag{3}$$

The probability density of the convenience yield at time t conditional on its value at current time t is a non-central chi-square (see Cox, Ingersoll and Ross (1985) [16]). The conditional moments of δ at time t under the objective measure are given by:

$$E[\delta_t|\delta_{t-dt}] = m(1 - e^{-\alpha dt}) + \delta_{t-1} e^{-\alpha dt} \tag{4}$$

$$Var[\delta_t|\delta_{t-dt}] = m\left(\frac{\sigma_2^2}{2\alpha}\right)(1 - e^{-\alpha dt})^2 + \delta_{t-dt}\left(\frac{\sigma_2^2}{\alpha}\right)(e^{-\alpha dt} - e^{-2\alpha dt}) \tag{5}$$

By defining $X = \ln S$ and applying Ito's Lemma, the process for the log price is given by:

$$dX = \left(\mu - \left(1 + \frac{1}{2}\sigma_1^2\right)\delta\right)dt + \sigma_1\sqrt{\delta}dz_1 \tag{6}$$

Under the risk neutral measure, the stochastic processes that drive the the state variables becomes[3]:

$$\begin{aligned} dS &= (r + c - \delta)Sdt + \sigma_1\sqrt{\delta}Sdz_1^* &(7)\\ d\delta &= (\alpha(m - \delta) - \lambda)dt + \sigma_2\sqrt{\delta}dz_2^* &(8) \end{aligned}$$

where:

[3]See Gibson and Schwartz (1990) [11]

- r is the risk-free (constant) interest rate;
- c is the (constant) marginal cost of storage;
- λ is the (constant) market price of risk for the convenience yield;
- σ_1, σ_2, α and m are as before;
- dz_1^* and dz_2^* are increments to standard Brownian motion under the risk-neutral measure and are correlated with $dz_1^* dz_2^* = \rho dt$, ρ as before;

The process for the log price then becomes:

$$dX = \left((r+c) - \left(1 + \frac{1}{2}\sigma_1^2\right)\delta\right)dt + \sigma_1\sqrt{\delta}dz_1^* \tag{9}$$

The interest rate, r and the marginal storage cost, c are assumed constant. The commodity is modelled as an asset that pays a stochastic dividend yield δ. Since the convenience yield risk cannot hedged, the risk-adjusted convenience yield process will have a market price of risk, λ associated with it, which we assume to be constant[4]. Equation (7) is an extension of a standard process for the commodity process allowing for a stochastic convenience yield and a time varying volatility, which is proportional to the square root of the time-varying stochastic convenience yield.

By assuming that the instantaneous convenience yield follows a CIR process we ensure that our model is arbitrage-free because it precludes negative values. This assumption ensures that the discounted futures prices net of carrying costs cannot be greater than the discounted contemporaneous spot prices (c.f., e.g. Hull (2000) [15]). Considering that $\tau = T - t$ represents time to maturity the arbitrage-free condition can be written as:

$$F(\tau) \leq S_t \exp\{(r+c)(\tau)\} \tag{10}$$

where

- $F(\tau)$ is the forward price at time t, for delivery of a commodity at time $T > t$;
- S_t is the spot price of the commodity at time t;

[4]See Gibson and Schwartz (1990)[11] and Schwartz (1997)[20]

- c is the (constant) proportion of the spot price which defines the marginal cost of storage;
- r is risk-free (constant) interest rate;

It can be proved that if the instantaneous convenience yield is always non-negative the arbitrage-free condition in equation (10) is satisfied[5].

The futures prices must satisfy the partial differential equation (PDE):

$$\begin{aligned}\frac{1}{2}\sigma_1^2\delta S^2 + \rho\sigma_1\sigma_2\delta S F_{s\delta} + ((r+c)-\delta)SF_S + \\ (\alpha(m-\delta)-\lambda)\,F_\delta - F_\tau = 0\end{aligned} \tag{11}$$

subject to the boundary condition $F(S,\delta,0) = S$. This PDE suggests an exponential affine form solution:

$$F(S,\delta,\tau) = Se^{A(\tau)-B(\tau)\delta} \tag{12}$$

Equivalently, the logarithm of the futures prices is given by:

$$\ln F(S,\delta,\tau) = \ln S + A(\tau) - B(\tau)\delta \tag{13}$$

This satisfies (12) and the boundary condition when

$$\frac{1}{2}\sigma_2^2 B^2 + (\alpha - \rho\sigma_1\sigma_2)B - 1 + B_\tau = 0 \tag{14}$$

and

$$(r+c) + (\lambda - \alpha m)B - A_\tau = 0 \tag{15}$$

with

$$A(0) = 0; \qquad B(0) = 0. \tag{16}$$

It follows that if (14) and (15) are solved subject to the boundary conditions in (16), equation (12) provides the price of a futures contract maturing at

[5]Note that Schwartz (1997) [20] does ensure non-negative convenience yield given that the stochastic convenience yield in his two-factor model follows an Ornstein-Uhlenbeck. This may generate arbitrage possibilities in his model.

time T, where $B(\tau)$ and $A(\tau)$ are, respectively[6]:

$$B(\tau) = \frac{2(1-e^{-k_1\tau})}{k_1 + k_2 + (k_1 - k_2)e^{-k_1\tau}} \tag{17}$$

and

$$A(\tau) = (r+c)\tau + (\lambda - \alpha m)\int_t^T B(q)dq \tag{18}$$

where:

$$\begin{aligned}
\int_t^T B(q)dq &= \frac{2}{k_1(k_1+k_2)} \ln\left[\frac{(k_1+k_2)e^{k_1\tau} + k_1 - k_2}{2k_1}\right] + && (19)\\
& \quad \frac{2}{k_1(k_1-k_2)} \ln\left[\frac{k_1 + k_2 + (k_1-k_2)e^{-k_1\tau}}{2k_1}\right] && (20)
\end{aligned}$$

where

$$\begin{aligned}
k_1 &= \sqrt{k_2^2 + 2\sigma_2^2} && (21)\\
k_2 &= (\alpha - \rho\sigma_1\sigma_2) && (22)
\end{aligned}$$

3 Empirical Estimation of the Joint Stochastic Process

In this section we estimate and empirically test both our model and the Schwartz's (1997) two factor model. Data for most of the spot commodity prices are extremely difficult to obtain price for most of the commodities and the instantaneous convenience yield is calculated using two distinct futures values with close maturity. On the other hand, we are able to observe daily several futures prices at different maturities. This non-observability and the linear relationship between futures prices and the state variables in the model clearly suggest that the Kalman filter is the most appropriate technique to estimate the model's parameters.

The principle of Kalman filter is to use a time series of observable variables and to infer the value of the non-observable variables. This technique is

[6]See derivation of the solutions in appendix A

suitable whenever there is a linear dependency of the observable variables upon the state variables and when the later are Markovian processes. Kalman filter is a technique which has become increasingly popular in Finance and has been applied to both Gaussian and CIR type interest rate models as well and in commodity futures valuation in Schwartz (1997) [20]. Affine models are particularly suited for estimating using Kalman filter because of their linear structure. In the context of interest rate models Gaussian examples can be found in Babbs and Nowman (1999)[1] and Lund (1997)[17], who estimate a two-factor generalized Vasicek model. In the CIR case, there are examples due to Ball and Torous (1996)[2], Duan and Simonato (1995)[7] and again Lund (1997). Schwartz (1997) [20] two-factor model belongs to the Gaussian class while our model fits in the CIR class.

The state form is applied to a multivariate time series of observable variables, which in our case are a futures prices time series at several different maturities. These observed variables are related to the state vector which consist of the state variables, which in our model are the spot price and the instantaneous convenience yield via the *measurement equation*. The measurement equation is then given by equation (13) by adding serially and cross sectionally uncorrelated disturbances with mean zero and variance to take into account for the irregularities of the observations. In the Kalman filter, the non-observable state variables are generated by first-order Markov processes which correspond to the discrete-time form of equations 8 and 9. The later are arranged in a vector, which forms the *transition equation*. See Harvey (1989)[12] for a detailed description of this method.

We calibrate Schwartz's (1997)[20] two factor model using exactly the same methodology as described in his article. To calibrate our model we follow the same steps but we need to take into account an important difference between the two empirical models. The state-space form of Schwartz's model is Gaussian while the state-space form of our model is non-Gaussian, given that we do not have constant volatility.

For a Gaussian state-space model, the Kalman filter provides an optimal solution to prediction, updating and evaluating the likelihood function. The Kalman filter recursion is a set of equations which allows an estimator to be updated once a new observation becomes available. The Kalman filter first forms an optimal predictor of the unobserved state variable vector given its previously estimated value. This prediction is obtained using the distribution of unobserved state variables, conditional on the previous estimated values. These estimates for the unobserved state variables are then updated using the information provided by the observed variables. Prediction errors, obtained as a by-product of the Kalman filter, can then be used to evaluate the likelihood function.

When the state-space model is non-Gaussian, the Kalman filter can still be applied and the resulting filter is quasi optimal. This filter is then used to obtain a quasi-likelihood function and the estimates obtained is linearly optimal. This approximation is needed because of the non-Gaussian nature of the problem, which can be compared to linearizing a non-linear function in the typical Kalman filtering applications. Duan and Simonato (1995)[7] and Geyer and Pichler (1998)[10] apply Kalman Filter to estimate and test exponential-affine term structure models for both the Gaussian and non-Gaussian cases. For a detailed discussion see Duan and Simonato (1995)[7] and Harvey (1989)[12].

It is also important to mention that the CIR process also differs from standard Kalman filter application because of the non-negative constraint on the convenience yield. Following Duan and Simonato (1995)[7] and Geyer and Pichler (1998)[10] we modify the standard Kalman filter by simply replacing any negative element of the convenience yield estimate with zero.

3.1 State Space Formulation

From the valuation formula given by equations (12), (17) and (18), the measurement equation can be written as:

$$Y_t = d_t + Z_t[X_t, \delta_t]' + \varepsilon_t, \qquad t = 1, ..., N \tag{23}$$

where:

- $Y_t = [\ln F(\tau_i)]$, for $i = 1, ..., n$ is a $n \times 1$ vector of observations where $F(\tau_i)$ is the observed futures price at time t for maturity τ_i. At each time t we observe n futures prices which correspond to n different maturities;

- $d_t = [A(\tau_i)]$ for $i = 1, ..., n$ is a $n \times 1$ where $A(\cdot)$ is given by equation (18).

- $Z_t = [1, -B(\tau_i)]$, for $i = 1, ..., n$ is a $n \times 2$ matrix where $B(\cdot)$ is calculated according to equation (17);

- ε_t is a $n \times 1$ is $n \times 1$vector of serially uncorrelated disturbances with $E[\varepsilon_t] = 0$, $Var[\varepsilon_t] = H_t$. This vector is introduced to account for possible errors in the data. The covariance matrix H_t is taken to be diagonal for computational simplicity;

The transition equation is given by:

$$[X_t, \delta_t]' = c_t + Q_t[X_t, \delta_t]' + \eta_t, \quad t = 1, ..., NT \tag{24}$$

where:

- $$c_t = \begin{bmatrix} \mu\Delta t \\ m(1 - e^{-\alpha\Delta t}) \end{bmatrix} \tag{25}$$

- $$Q_t = \begin{bmatrix} 1 & -(1 + \frac{1}{2}\sigma_1^2)\Delta t \\ 0 & 1 + e^{-\alpha\Delta t} \end{bmatrix} \tag{26}$$

- η_t is a 2×1 vector of serially uncorrelated disturbances with $E[\eta_t] = 0$ and $Var[\eta_t] = V_t$;

The covariance matrix of η_t is given by:

$$V_t = \begin{bmatrix} \sigma_1^2 \Delta t \delta_{t-dt} & \rho\sigma_1\sqrt{\Delta t}\sqrt{\delta_{t-dt}}\sqrt{Var[\delta_t|\delta_{t-1}]} \\ \rho\sigma_1\sqrt{\Delta t}\sqrt{\delta_{t-dt}}\sqrt{Var[\delta_t|\delta_{t-1}]} & Var[\delta_t|\delta_{t-1}] \end{bmatrix} \tag{27}$$

where:

$$Var[\delta_t|\delta_{t-1}] = m\left(\frac{\sigma_2^2}{2\alpha}\right)(1 - e^{-\alpha\Delta t})^2 + \delta_{t-1}\left(\frac{\sigma_2^2}{\alpha}\right)(e^{-\alpha\Delta t} - e^{-2\alpha\Delta t}) \tag{28}$$

The observation and state equation matrices Z_t, d_t, H_t, Q_t, c_t and V_t depend on the unknown parameters of the model. One of the main purposes of the Kalman filter implementation is to find estimates for these parameters. This can be done by maximizing the quasi likelihood function with respect to the unknown parameters through an optimization procedure.

Consider θ the vector of unknown parameters and $Y_t = \{y_t, y_{t-\Delta t}, \ldots, y_t, y_{t_0}\}$ the information vector at time t, which are not independent. We assume that the distribution of Y_t conditional on $Y_{t-\Delta t}$ under the objective measure is normal with mean $\hat{Y}_{t|t-\Delta t} = E[Y_t|Y_{t-\Delta t}]$ and covariance matrix F_t. The vector of prediction errors is given by $v_t = Y_t - \hat{Y}_{t|t-\Delta t}$ The logarithm of the *likelihood functionis* given by:

$$logL(Y;\theta) = -\frac{1}{2}\frac{t_{final} - t_0}{\Delta t}\log 2\pi - \frac{1}{2}\sum_t \log|F_t| - \frac{1}{2}\sum_t v_t F_t^{-1} v_t \tag{29}$$

Since both F_t and v_t depend upon θ, θ_t is chosen to maximize the quasi-likelihood function. This estimation procedure is recursive and it is calculated at each time t as part of the Kalman filter.

To calibrate Schwartz's (1997) two-factor model we use the state-space formulation as described in his paper.

3.2 Data

The data set used in this study consists on weekly observations of NYMEX light crude oil futures which covers the period from 17^{th} of March 1999 to 15^{th} of October 2003 (233 observations)[7]. At each observation we consider 7 contracts (n = 7) corresponding to 7 different maturities. Naturally, the time to maturity changes as we evolve in time and to force that the time to maturity remains within a narrow range t we role over the contracts during the period of observations. Table 1 describes the data used. We call F0 the contract closest maturity, F1 the second contract closest to maturity and so on.

Futures Contract	Mean Maturity (Standard Deviation)	Mean Price (Standard Deviation)
F0	0.044 (0.024)	26.875(4.714))
F1	0.127 (0.024)	26.538 (4.428)
F2	0.348 (0.024)	25.533 (3.861)
F3	0.598 (0.024)	24.566 (3.487)
F4	0.931 (0.024)	23.596 (3.135)
F5	1.181 (0.024)	23.002 (2.924)
F6	1.931 (0.024)	21.963 (2.513)

Table 1: Light Crude Oil Futures weekly data from 17^{th} of March 1999 to 15^{th} of October 2003

We first illustrate the individual evolution of the commodity spot prices and the convenience yield. In order to do so we define the proxies used for the spot price and the instantaneous convenience yield. Since spot crude oil prices are not available and the convenience yield is not observable it is necessary to define two proxies for these state variables in order to be able to calculate the futures prices formula given by equation (13). We adopt the

[7]The data was retrieved from the Internet on the 31^{st} of October 2003 from *Futures Guide* ^{TM}, http://www.futuresguide.com/index.php.. The original data set consists of daily observations. Weekly data was obtained by using every Wednesday (to avoid weekend effects) observation.

standard procedure which identifies the spot price with the settlement price of the crude oil futures contract with closest maturity. This corresponds to F0 in table 1. Following Gibson and Schwartz (1990) [11] we use the relationship between the futures and spot price where there is neither interest rate, storage costs nor convenience yield uncertainty, which is represented by equation (**??**). By using F0 and the contract with monthly adjacent maturity, that is, F_1, we approximate the annualized instantaneous convenience yield by:

$$\delta_{T-1,T} = (r + c) - 12 \ln \left[\frac{F(S,T)}{F(S,T-1)} \right] \tag{30}$$

where $\delta_{T-1,T}$ denotes the $T-1$ periods ahead annualized one month forward convenience yield, r denotes the (constant) riskless interest rate and c represents the marginal storage cost, which is proportional to the spot price. We consider the annual interest rate equal to 0.05 and calculate the storage cost as the minimum value necessary to ensure that the convenience yield is nonnegative for the whole sample period. We consider the annual marginal cost equal to 0.20.

In figures 1 and 2 we illustrate the individual evolution of the two state variables over the sample period[8]. These figures show that the convenience yield of light crude oil has a tendency to revert to its long-run mean and that it is an extremely hight volatile parameter, while the spot price is less volatile. This corroborates the choice of a mean-reverting process for the convenience yield. It is also clear that there is a strong positive correlation between both spot price and convenience yield.

[8]Although figure 2 shows four slightly negative values we do not consider these significant and therefore do not increase the storage cost. Obviously, it would me more realistic to consider a time variant cost. However, for simplicity, we keep it constant.

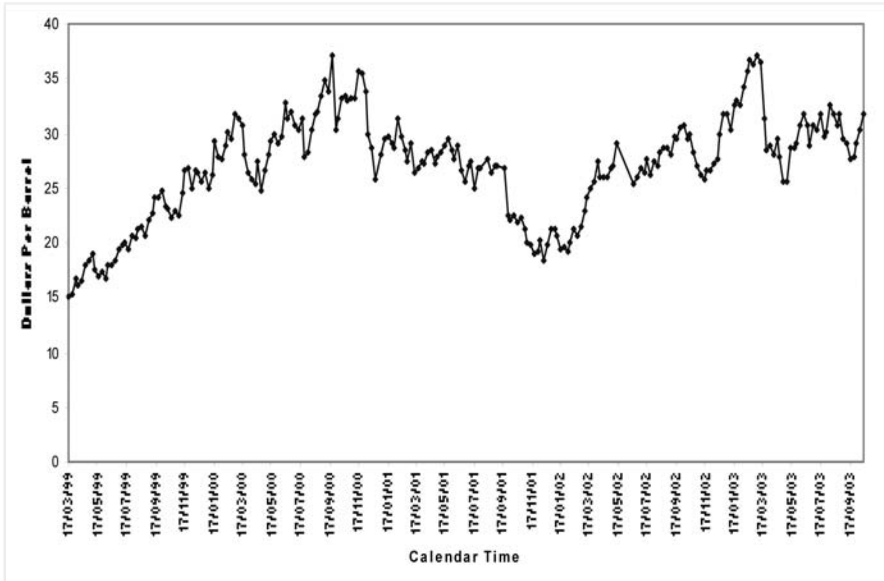

Figure 1: Evolution of Spot Crude Oil Price.

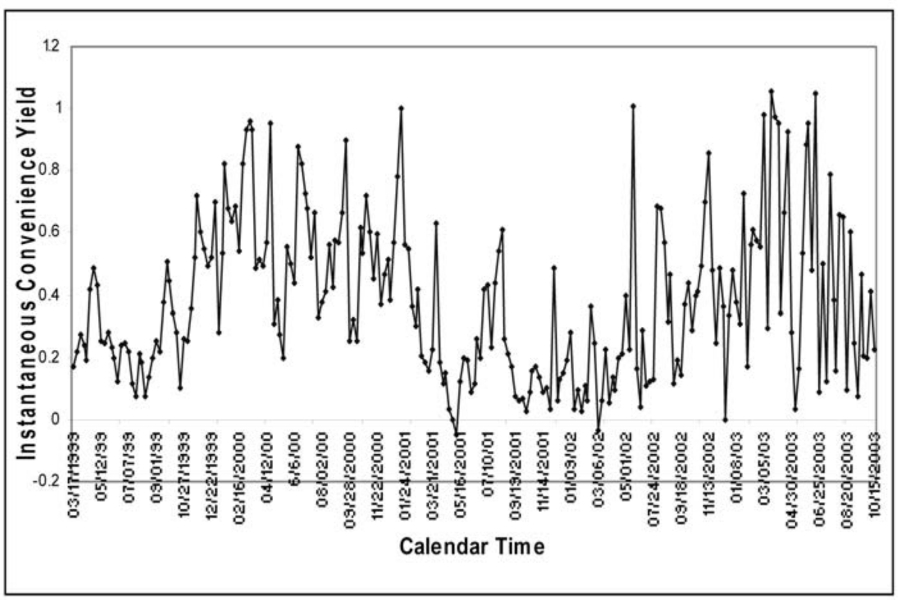

Figure 2: Evolution of oil convenience yield.

3.3 Empirical Results

Table 2 reports the estimation results for both our model and Schwartz's (1997)[20] two-factor model. The values obtained for the parameters are comparable for both cases. The most noticeable difference lies in the value of the long-run mean for the convenience yield, m. However, this difference is approximately 0.2 which is consistent with the storage cost value of 0.2 that we assume in our model.

The speed of mean reversion in the convenience yield equation, α, and the coefficient of correlation between the spot price and convenience yield, ρ are quite high and significant for both cases. These results are consistent with the empirical data illustrated in figures 1 and 2. The total expected return on the spot commodity, μ, and the market price of convenience yield, λ, are also positive and high. In particular, it is worth to mention the high value of average convenience yield. This indicates that during this period the market is predominantly in backwardation, a fact that becomes clear from observing figure 2. Additionally, both spot price and convenience yield volatilities are also high. This behavior in the crude oil market can be explained by the world events which took place after September 2001 and, in particular, the recent Golf war. These events lead to increasingly uncertainty in the world markets in general and in particular in the oil supply. This uncertainty naturally rises the value of having crude oil in storage, which implies a high convenience and a market in strong backwardation.

The measurement errors standard deviations which result from the Kalman filter are also displayed in table 2 and are denote by σ_{ε_1}, σ_{ε_2}, σ_{ε_3}, σ_{ε_4}, σ_{ε_5} and σ_{ε_6} and correspond to each of the futures contracts used, namely $F0$, $F1$, $F2$, $F3$, $F4$, $F5$ and $F6$ respectively. The magnitude of this errors is the same for both our model and Schwartz's (1997) two-factor model.

Parameters	**Our model**	**Schwartz's (1997)[20] two-factor model**
m	0.562 (0.017)	0.369 (0.018)
λ	1.627 (0.015)	1.900 (0.019)
α	6.302 (0.021)	6.334 (0.016)
σ_1	0.449 (0.010)	0.467 (0.012)
σ_2	0.739 (0.011)	0.748 (0.014)
ρ	0.922 (0.009)	0.828 (0.014)
μ	0.525 (0.012)	0.552 (0.008)
σ_{ε_0}	0.049 (0.008)	0.049 (0.009)
σ_{ε_1}	0.050 (0.009)	0.051 (0.010)
σ_{ε_2}	0.034 (0.004)	0.034 (0.005)
σ_{ε_3}	0.029 (0.003)	0.029 (0.004)
σ_{ε_4}	0.033 (0.005)	0.034 (0.005)
σ_{ε_5}	0.041 (0.007)	0.041 (0.008)
σ_{ε_6}	0.056 (0.009)	0.056 (0.009)

Table 2: Estimation results and standard errors for both our model and Schwartz's (1997) two-factor model using all the futures contracts F0, F1, F2, F3, F4, F5 and F6 from 05/03/99 to 15/10/03 from 17^{th} of March 1999 to 15^{th} of October 2003

Table 3 displays the mean pricing errors (MPE) and the root mean squared errors (RMSE) for all the observations. Both error measures are small and of the same order of magnitude for both our model and Schwartz's model. Note that the pricing errors for our model are smaller than for Schwartz's (1997) model but only by a small amount. It is important to note that the performance of both models decreases significantly as the time to maturity of the futures contract increases. This is illustrated in figure 3. This highlights the fact that both models become less effective as we increase the maturity of the futures contracts we are pricing.

Figures 4 and 5 illustrate the evolution of the forward curve of the market prices and both models.

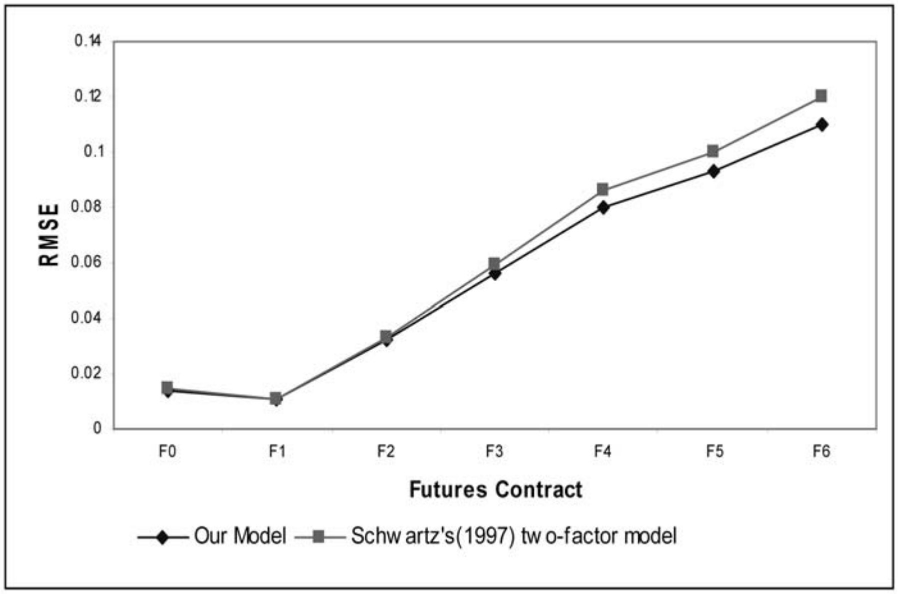

Figure 3: This figure illustrates the evolution of the RMSE of the futures contracts in the sample with increasing time to maturity for both our model and Schwartz's (1997)[20] two-factor model.

	Our Model		**Schwartz's (1997) Two-factor Model**	
Futures Contract	RMSE	MPE	RMSE	MPE
F0	0.014	-0.004	0.015	-0.004
F1	0.011	-0.004	0.011	-0.005
F2	0.032	0.014	0.033	0.016
F3	0.056	0.033	0.059	0.037
F4	0.080	0.049	0.086	0.057
F5	0.093	0.056	0.100	0.066
F6	0.110	0.050	0.120	0.068

Table 3: Summary statistics both our model's and Schwartz's (1997)[20] two-factor model's pricing errors in valuing futures contracts during the whole period from 05/03/99 to 15/10/03 from 17^{th} of March 1999 to 15^{th} of October 2003

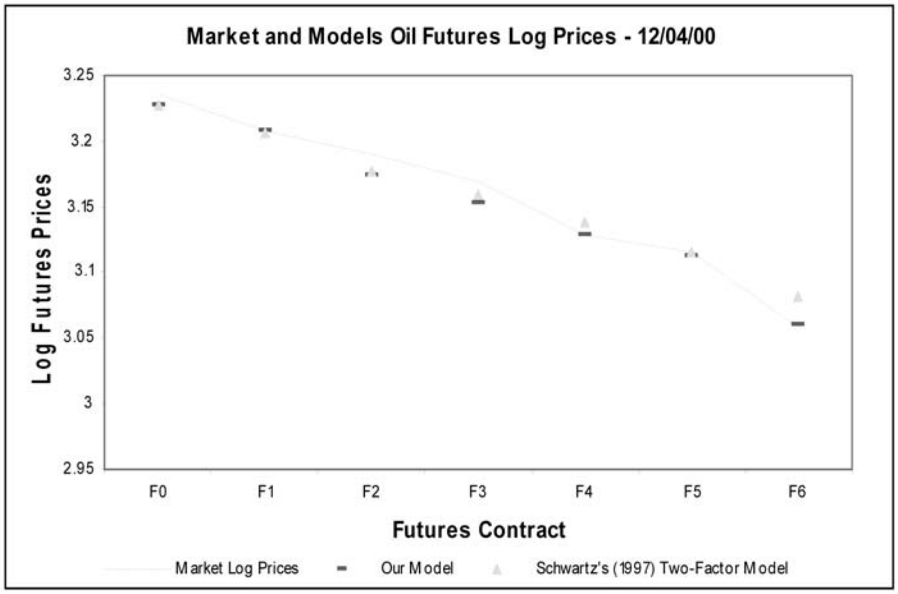

Figure 4: This figure illustrates the evolution of the forward curve for the market of futures prices and both our model and the Schwartz's (1997) two-factor model on the 12th of December 2000.

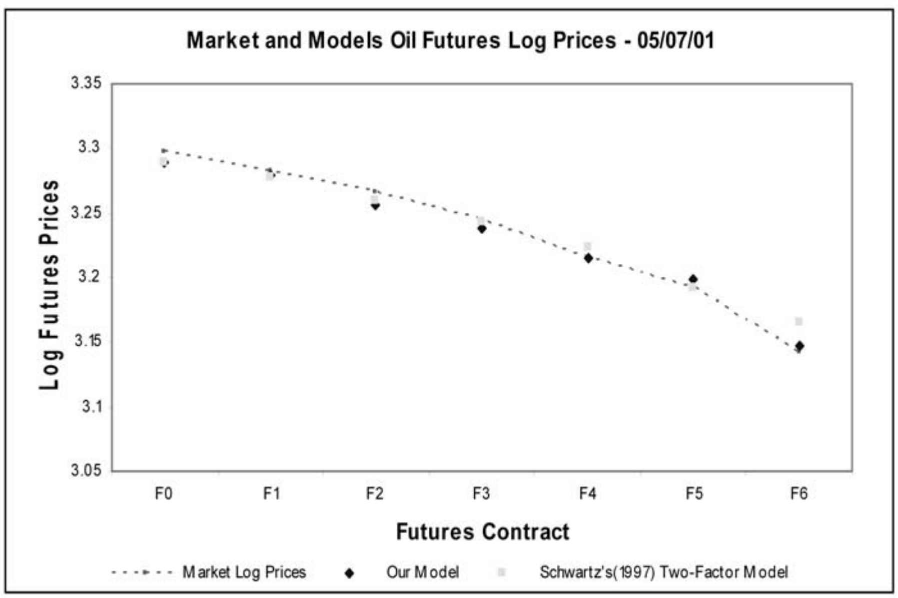

Figure 5: This figure illustrates the evolution of the forward curve for the market of futures prices and both our model and the Schwartz's (1997) two-factor model on the 5th of July 2001.

4 Conclusion

In this study we have presented a two-factor model for commodity prices and the corresponding futures valuation. This model extends Schwartz's (1997)[20] two factor model by adding two important features. First, the Ornstein-Uhlenbeck process for the convenience yield is replaced by a CIR process. This allows us to maintain the mean-reverting property of the convenience yield and additionally ensure that our model is arbitrage-free. Second, we consider both spot price and convenience yield volatilities proportional to the square root of the instantaneous convenience yield level. This implicitly reflects dependency between volatility of the state variables and commodity inventory levels, as predicted by the theory of storage.

Our model adds valuable characteristics to the existing reduced form models in the literature and outperforms Schwartz's model although not significantly. Both models achieve very good results when valuating short-term maturity data but fail to reproduce long-term futures prices. This suggests further extensions of these models for future work to improve the valuation of futures contracts with long maturities. This is particularly relevant to evaluate long-term investments on commodities.

Appendix A

In this appendix, we derive the solutions to the differential equations (14) and (15) with initial conditions (16)

Because the solution to equation (15) depends on the solution to equation (14) we star by solving the later.

Write $A_1 = \frac{1}{2}\sigma_2^2$ and $A_2 = \alpha - \rho\sigma_1\sigma_2$, then equation (14) becomes:

$$A_1 B^2 + A_2 B - 1 + B_\tau = 0, \quad \text{with} B(0) = 0 \tag{31}$$

Because this is a non-exact equation, we multiply both sides of the equation by the following integrating factor:

$$\mu(B) = \frac{A_1}{A_1 B^2 + A_2 B - 1} \tag{32}$$

Equation (14) then becomes:

$$A_1 + \frac{A_1}{A_1 B^2 + A_2 B - 1}\frac{dB}{dt} = 0, \quad \text{with} B(0) = 0 \tag{33}$$

The solution to this equation is

$$B(\tau) = \frac{2(1 - e^{-k_1\tau})}{k_1 + k_2 + (k_1 - k_2)e^{-k_1\tau}} \tag{34}$$

where

$$k_1 = \sqrt{k_2^2 + 2\sigma_2^2} \tag{35}$$
$$k_2 = (\alpha - \rho\sigma_1\sigma_2) \tag{36}$$

which is precisely equation (17).

Accordingly, the solution to equation (15) is:

$$A(\tau) = r\tau + (\lambda - \alpha m)\int_0^\tau B(q)dq \tag{37}$$

where:

$$\int_t^T B(q)dq = \frac{2}{k_1(k_1+k_2)} \ln\left[\frac{(k_1+k_2)e^{k_1\tau}+k_1-k_2}{2k_1}\right] + \tag{38}$$

$$\frac{2}{k_1(k_1-k_2)} \ln\left[\frac{k_1+k_2+(k_1-k_2)e^{-k_1\tau}}{2k_1}\right] \tag{39}$$

where k_1 and k_2 are as before.

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