Analytic value function for pairs trading strategy with a Lévy-driven Ornstein-Uhlenbeck process

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Abstract

This paper studies the performance of pairs trading strategy under a specific spread model. Based on the empirical evidence of mean reversion and jumps in the spread between pairs of stocks, we assume that the spread follows a Lévy-driven Ornstein-Uhlenbeck process with two-sided jumps. To evaluate the performance of a pairs trading strategy, we propose the expected return per unit time as the value function of the strategy. Significantly different from the current related works, we incorporate an excess jump component into the calculation of return and time cost. Further, we obtain the analytic expression of strategy value function, where we solve out the probabilities of crossing thresholds via the Laplace transform of first passage time of the Lévy-driven Ornstein-Uhlenbeck process in one-sided and two-sided exit problems. Through numerical illustrations, we calculate the value function and optimal thresholds for a spread model with symmetric jumps, reveal the non-negligible contribution of incorporating the excess jumps into the value function, and analyze the impact of model parameters on the strategy performance.

Keywords: Pairs trading; Lévy-driven Ornstein-Uhlenbeck process; Optimal thresholds; Two-sided exit problem.

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1 Introduction

Pairs trading is a popular statistical arbitrage strategy, which emerged in 1980s from a quantitative group at Morgan Stanley and has been widely used in the financial industry since then. A typical pairs trading strategy aims to construct a market neutral portfolio, made up of a long-short position in two highly related securities with a predetermined ratio. Hence, it can effectively eliminate the exposure to the major market movement. A pairs trading investor believes that the spread between two highly related assets should dynamically move around its long-term equilibrium. When the spread excesses a certain threshold, the entry timing of the position is triggered, and the investor longs the undervalued asset while short the overvalued one. When the spread reverts to a given distance to its equilibrium, the investor receives the signal of the exit timing, and closes the long-short position to make a profit. For the original and seminal studies of pairs trading strategy, readers can refer to Gatev et al. (1999), Vidyamurthy (2004), Gatev et al. (2006) and Ehrman (2006).

There are three popular methods of constructing a pairs trading strategy: distance method, co-integration method and stochastic spread method, see Krauss (2017) for a comprehensive survey. In this paper, we focus on the third method, which should answer the following three important questions on how to:

1. Model the spread;

2. Design the trading strategy;

3. Specify and analytically formulate the value function of a strategy for performance evaluation.

For the first question, the Brownian motion-driven Ornstein-Uhlenbeck (OU hereafter) process is a general choice, see, for example, Elliott et al. (2005), Bertram (2010), Zeng and Lee (2014), Göncü and Akyildirim (2016), and Bai and Wu (2017). However, the assumption of continuous sample paths for spread process is not consistent with a series of empirical observations. To overcome these deficiencies, various modifications in terms of jump processes come up, such as generalized OU process driven by a Lévy process of finite activity in Larsson et al. (2013), Lévy-driven OU process with generalized hyperbolic distributed marginal in Göncü and Akyildirim (2016), doubly mean-reverting process using conditional modeling in Liu et al. (2017), Lévy-driven OU process with Gaussian distributed marginal in Stübinger and Endres (2018). Recently, Endres and Stübinger (2019) study an OU process driven by double exponential jump-diffusion model (OUDEJ hereafter), and in pursuit of analytical tractability, they use two independent Poisson processes to generate the upward and downward jumps.

For designing a pairs trading strategy, the core elements are trading thresholds, which are related to the entry and exit timing. Elliott et al. (2005) propose that a strategy would be performed at a constant threshold and unwound at a fixed time later . In Gregory et al. (2010), the thresholds are set as one standard deviation from the long-term mean. Göncü and Akyildirim (2016) generalize the exit thresholds of conventional pairs trading strategy such that the investor should close the position at a distance above or below the long-term mean. Bai and Wu (2017) consider a strategy with two thresholds symmetric about the long-term mean.

In order to evaluate the performance of a pairs trading strategy, we should specify a value function, combining profitability and time cost in terms of thresholds and spread model. Bertram (2010) regard the higher profit and less trading time as two independent value functions with respect to (w.r.t. hereafter) their means and variances. Leung and Li (2015) incorporate a stop-loss constraint into the expected liquidation value. Both of Zeng and Lee (2014) and Endres and Stübinger (2019) specify the value function as the expected return per unit time, where the return is a constant as the difference of two trading thresholds.

To obtain the analytic expression of value function, the most common method is the Laplace transform of first passage times. On the Laplace transform of first passage times of generalized OU processes, some relevant literature studies the one-sided jump cases, such as OU process with non-positive jumps in Hadjiev (1985), Novikov (2004), and Patie (2005), OU process driven by a compound Poisson process with uniform or exponential distributed marginal in Novikov et al. (2005), OU process driven by a driftless subordinator in Loeffen and Patie (2010). Besides, some results on OU processes with two-sided jumps have also been obtained. See, for example, OU processes driven by compound Poisson processes with double exponential jumps in Jacobsen and Jensen (2007), Lévy-driven OU processes in Borovkov and Novikov (2008) where the positive jumps are exponentially distributed and the two-sided jumps can happen simultaneously, OU processes driven by α-stable and variance gamma distributed jumps in Obuchowski and Wyłomańska (2013), OUDEJ process in Zhou et al. (2017).

In this paper, we extend the existing literature in three aspects. First, we use OUDEJ process (c.f. Zhou et al. (2017)) to characterize the spread process in the pairs trading, involving both properties of mean reversion and two-sided jumps. In contrast with Endres and Stübinger (2019) which regard the return as the difference of two constant thresholds, we incorporate the jump size of spread into the calculation of strategy profitability. Based on this, we come up with an overall value function as the expected return per unit time. The incorporation of jumps brings essential difficulty to the calculation of value function, since there exists a nonzero probability of overshoot or undershoot when the spread crosses the thresholds. Second, we obtain the explicit expression of the Laplace transform of two-sided first passage time of OUDEJ process. By Jacobsen and Jensen (2007), we can not obtain the probabilities of crossing the thresholds in each side directly from the Laplace transform since the integral therein diverges as the variable of Laplace transform tends to 0. We overcome this technical difficulty in Theorem 5 by introducing new auxiliary functions via the Laplace transform, obtaining crossing probabilities as well as expectations of first passage times.

Third, we obtain the optimal thresholds numerically through the derived explicit expressions rather than Monte Carlo methods. Besides, we get insights of the relationship between strategy performance and model parameters, and highlight the effects of incorporating jumps into the value function.

The structure of this paper is as follows: Section 2 introduces the pairs trading strategy and specifies a value function based on the tradeoff between profitability and time cost. Section 3 derives the analytic value function explicitly. Section 4 obtains the optimal thresholds numerically under an OUDEJ model with some symmetric characteristics, and analyzes the impact of jumps and model parameters on the strategy performance. Section 5 contains conclusions. The related mathematical preliminaries of jump test and OUDEJ process, as well as the explicit expression of Laplace transform of two-sided first passage time are presented in Appendix.

2 Pairs trading

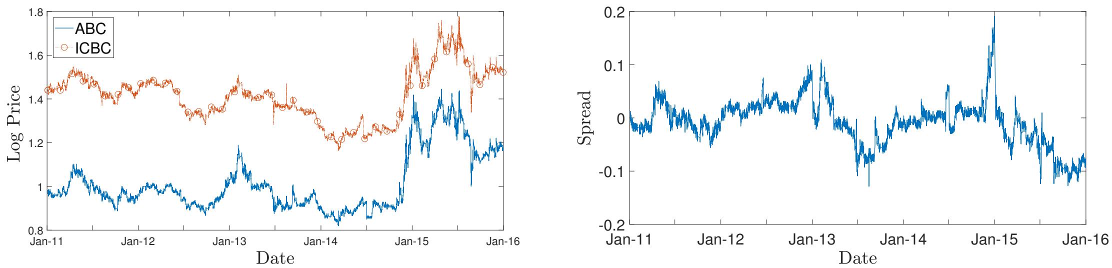
In this section, we set up key elements of the pairs trading strategy, which answer the three questions at the beginning of Section 1. We first characterize the spread as an OUDEJ process (c.f. Zhou et al. (2017)), then specify the form of trading signals, and finally propose a value function to evaluate the strategy performance.

2.1 Model evaluation

In previous empirical studies, jumps of asset returns have been revealed. Gilder (2009) examines transaction prices of 72 US equities and an exchange traded fund (ETF) from January 2002 to December 2006, and finds that total number of jumps identified for each security is relatively large and the sizes of the detected jumps are distributed symmetrically, using the intraday jump test in Andersen and Bollerslev (1997). Stübinger and Endres (2018) analyze the oil sector of the S&P 500 constituents from 1998 to 2015, and show the existence of jumps through a jump analysis.

Further, OU process is found insufficient for modeling the spread dynamics in pairs trading. Göncü and Akyildirim (2016) consider daily futures prices for 23 different commodities from July 1997 to June 2015, and conclude that the normality is clearly rejected for all AR(1) corrected residuals. Endres and Stübinger (2019) test minute-by-minute prices of the S&P 500 index constituents from January 1998 to December 2015. They show that Lévy-driven OU models are more suitable for spreads, following the method in Abdelrazeq (2015).

In view of the foregoing, empirical features of the spread of two stocks have a significant influence on the pairs trading. To obtain the facts of jump part of the spread movement, we pick up ABC (Agricultural Bank of China) and ICBC (Industrial and Commercial Bank of China) in Chinese A share stock market as the chosen pair, and collect data of 1-minute prices from January 2011 to December 2015. Figure 1a shows that two stocks have similar return trends, and Figure 1b further represents the mean reversion and the two-sided jumps properties of their spread. Here, we adopt the popular BN-S methodology in Barndorff-Nielsen and Shephard (2004) and Gilder (2009) to statistically test the existence of jumps in spread of ABC-ICBC, and technical details are given in Appendix A. Given 5% significance level, we reject the null hypothesis of no jumps in 139 days out of 1214 days in total.



(a) Log 1-minute prices of ABC and ICBC

(b) Spread of ABC and ICBC

Figure 1: Trends of ABC and ICBC from January 2011 to December 2015: Left subplot for the log prices and right subplot for the spread, whose investment ratios are determined by the ratio of initial log prices in each year.

In conclusion, we adopt a double exponential jump-diffusion-driven OU process as our spread model. As a Lévy process, double exponential jump-diffusion model receives considerable attention in the literature, see Kou and Wang (2003), Ramezani and Zeng (2007), Kou et al. (2017) and Endres and Stübinger (2019) among others. As stated in Kou (2002), double exponential jump-diffusion model can reproduce the leptokurtosis feature of the return distribution and is internally self-consistent. In addition, its high peak and heavy tails can model both overreaction and underreaction to outside news in market. In empirical test, Ramezani and Zeng (1998) analyze daily returns for 6 New York Stock Exchange listed stocks from January 1991 to December 1992, and suggest that double exponential jump-diffusion model fits stock data better than the normal jump-diffusion model.

2.2 Model setup

In a pairs trading strategy, we denote by $S_{1,t}$ and $S_{2,t}$ the prices of two considered stocks at time t. The spread process X_t is defined as the difference in log-returns $X_t = \ln\left(\frac{S_{1,t}}{S_{1,0}}\right) - \beta \ln\left(\frac{S_{2,t}}{S_{2,0}}\right),\ t \geqslant 0$, where β is the investment ratio of the long-short position, usually exogenous in process X_t.

We assume that X_t is an OUDEJ process

$$dX_t = \kappa\left(\alpha - X_t\right) dt + dL_t,\ t \geqslant 0, \tag{1}$$

with mean reversion speed $\kappa > 0$, mean reversion level $\alpha \in \mathbb{R}$, and Lévy process $L = (L_t)_{t \geq 0}$. Besides, the initial value X_0 may be random. According to Section 2.1, we assume Lévy process as a double exponential jump-diffusion model

$$L_t = \mu t + \sigma W_t + \sum_{k=1}^{N_t} Y_k, \tag{2}$$

where $\mu \in \mathbb{R}$, $\sigma > 0$, $W = (W_t)_{t \geqslant 0}$ is a standard Brownian motion, $(N_t)_{t \geqslant 0}$ denotes a Poisson process with constant jump intensity λ, and $\{Y_k,\ k = 1, 2, \ldots\}$ is a sequence of independent and identically distributed double exponential random variables with probability density function

$$f_Y(y) = p\gamma_1 e^{-\gamma_1 y} \mathbb{1}\{y \geqslant 0\} + (1-p)\gamma_2 e^{\gamma_2 y} \mathbb{1}\{y < 0\}.$$

Here, $\mathbb{1}\{A\}$ is the indicator function for any event A, $p \in [0,1]$ is the upward jump probability, and $\gamma_1, \gamma_2 > 0$ are the rate parameters.

2.3 Trading signals

Given the spread process in (1), the investor should decide the proper timings to enter and exit the trading. First, we define the entry timing ν of a pairs trading strategy as

$$\nu(a,d) := \inf\{t \geqslant 0 : X_t \notin (d,a)\}, \tag{3}$$

where a and d are predetermined trading thresholds satisfying $d < a$. The definition (3) implies two circumstances for the investor to open the position. Specifically, when $X_t \geqslant a$ (resp. $X_t \leqslant d$), i.e., stock S_1 is relatively overvalued (resp. undervalued) than stock S_2, the investor would short (resp. long) a portfolio $\mathcal{P}$ which is made up of one share of S_1 in long position and β shares of S_2 in short position. With the belief that the deviated spread will revert to its long-term mean in the future, after opening the position, the investor would wait and close the position at a predetermined exit timing when the deviated spread returns to make a profit. Then, depending on whether $X_\nu \geqslant a$ or $X_\nu \leqslant d$, the exit timing η is defined as

$$\eta(a,b,c,d) := \begin{cases} \inf\{t > \nu : X_t \leqslant b\}, & \text{if } X_\nu \geqslant a, \\ \inf\{t > \nu : X_t \geqslant c\}, & \text{if } X_\nu \leqslant d. \end{cases} \tag{4}$$

Here, the exit thresholds b, c in (4) should be constant levels between $[d,a]$, rendering a positive return in this trade.

For the sake of simplicity, we introduce equivalent expressions of ν and η in (3)-(4) via the first passage times of X_t. For the above entry thresholds a and d, we define the one-sided and two-sided first passage times of X_t as

$$\begin{aligned} &\tau_a^+(X) := \inf\{t \geqslant 0 : X_t \geqslant a\}, \quad \tau_d^-(X) := \inf\{t \geqslant 0 : X_t \leqslant d\}, \\ &\tau_{a,d}(X) := \tau_a^+(X) \wedge \tau_d^-(X) = \inf\{t > 0 : X_t \geqslant a \text{ or } X_t \leqslant d\}. \end{aligned} \tag{5}$$

For convenience, when X_0 is a constant x, we simply denote $\tau_a^+(X), \tau_d^-(X)$ and $\tau_{a,d}(X)$ by τ_a^+, τ_d^- and $\tau_{a,d}$, respectively. Based on the notations in (5), we can express the entry and exit timing in (3)-(4) as $\nu(a,d) = \tau_{a,d}$, and $\eta(a,b,c,d) = \tau_b^-(\hat{X}^a)\mathbb{1}\{X_{\tau_{a,d}} \geqslant a\} + \tau_c^+(\check{X}^d)\mathbb{1}\{X_{\tau_{a,d}} \leqslant d\}$, where

$$\hat{X}^a := X \circ \theta_{\tau_a^+} \text{ for } a > x \text{ and } \check{X}^d := X \circ \theta_{\tau_d^-} \text{ for } d < x \tag{6}$$

denote the paths of X_t initial at stopping times τ_a^+ and τ_d^-, respectively. Here, the shift transformation θ_s for $s > 0$ in (6) is defined by $(\theta_s \omega)(t) := \omega(s+t),\ t > 0$, for any path $\omega(\cdot)$.

2.4 Strategy value function

In a pairs trading strategy, it is desirous to maximize return and minimize time cost simultaneously. However, higher return and less time cost seldom meet together, which requires a tradeoff between them. Enlightened by Leung and Li (2015) and Endres and Stübinger (2019), given the initial value[1] $X_0 = x$ in (1), we consider a value function expressed as the expected return per unit time

$$v_x(a,b,c,d) := \frac{\mathbb{E}_x\left[r(a,b,c,d)\right]}{\mathbb{E}_x\left[L(a,b,c,d)\right]}. \tag{7}$$

Here, the return $r(a,b,c,d)$ and the time cost $L(a,b,c,d)$ are specified within a trading cycle involving entry and exit timing, both of which are only determined by the thresholds a, b, c, d given the process X_t with its initial value x. We pursuit a pairs trading strategy by controlling the thresholds a, b, c, d, which is an optimization problem in terms of the value function (7) as

$$\begin{aligned} &\max_{\{a,b,c,d\}} \quad v_x(a,b,c,d) \\ &\text{subject to} \quad b, c \in [d,a]. \end{aligned} \tag{8}$$

Later, we will derive the analytic form of the value function in (7), which is the key step to solve the problem (8).

2.4.1 Profitability

A complete pairs trading consists of a one-round entry and exit. The return of given strategy is the absolute difference of spread values at the entry timing $\nu(a,d)$ in (3) and the exit timing $\eta(a,b,c,d)$ in (4), i.e.,

$$r(a,b,c,d) := |X_\nu - X_\eta|. \tag{9}$$

In previous works modeling the spread with jumps such as Göncü and Akyildirim (2016) and Endres and Stübinger (2019), the return (9) is the difference of two constant trading thresholds. Here, in our model setup, the two-sided jumps may result in an overshoot $X_{\tau_a^+} - a$ (resp.an undershoot $d - X_{\tau_d^-}$) when X_t upcrosses the threshold a (resp. downcrosses threshold d), which incurs an excess return. Thus, the return $r = r(a,b,c,d)$ in (9) should take the effect of jumps into consideration. The specific form of return r depends on the initial value $X_0 = x$.

[1] It should be noted that the initial value x can take any values in theory and practice. Since pairs trading is an consecutive process, the value of X_ν or X_η is exactly the initial value of next trading. Due to the double exponential jumps in the OUDEJ process, the overshoot/undershoot allows X_ν and X_η to exceed the predetermined thresholds by any distance, so the initial value of next trade can take any value theoretically. In practice, the investor should adjust investment ratio β dynamically, so the initial value of spread can change with strategy adjustment.

When $x \geqslant a$, the investor immediately shorts one share of portfolio $\mathcal{P}$ and closes the position once X_t downcrosses b, yielding the return

$$r = (x-b)\,\mathbb{1}\left\{X_{\tau_b^-} = b\right\} + \left(x - X_{\tau_b^-}\right)\mathbb{1}\left\{X_{\tau_b^-} < b\right\} = x - b + \left(b - X_{\tau_b^-}\right)\mathbb{1}\left\{X_{\tau_b^-} < b\right\},$$

which is the summation of a fixed part $x-b$ and a possible excess return $b - X_{\tau_b^-}$ from downcrossing by jump at the exit timing.

Analogically, when $x \leqslant d$, the return r can be similarly represented as

$$r = (c-x)\,\mathbb{1}\left\{X_{\tau_c^+} = c\right\} + \left(X_{\tau_c^+} - x\right)\mathbb{1}\left\{X_{\tau_c^+} > c\right\} = c - x + \left(X_{\tau_c^+} - c\right)\mathbb{1}\left\{X_{\tau_c^+} > c\right\}.$$

When $x \in (d,a)$, the investor would not open the position until the spread X_t exits the interval (d,a). After that, the spread process in the pairs trading strategy is adjusted to be $X \circ \theta_{\tau_{a,d}}$ starting at time 0. When the spread first upcrosses threshold a at the entry timing, that is, $\tau_a^+ < \tau_d^-$, the investor opens the position by selling the portfolio $\mathcal{P}$, and unwounds the trading once the spread downcrosses the threshold b, from which the return is

$$r_a = a - b + \left(X_{\tau_a^+} - a\right)\mathbb{1}\left\{X_{\tau_a^+} > a\right\} + \left[b - \hat{X}^a_{\tau_b^-(\hat{X}^a)}\right]\mathbb{1}\left\{\hat{X}^a_{\tau_b^-(\hat{X}^a)} < b\right\}. \tag{10}$$

Similarly, when $\tau_d^- < \tau_a^+$, the return can be represented as

$$r_d = c - d + \left(d - X_{\tau_d^-}\right)\mathbb{1}\left\{X_{\tau_d^-} < d\right\} + \left[\check{X}^d_{\tau_c^+(\check{X}^d)} - c\right]\mathbb{1}\left\{\check{X}^d_{\tau_c^+(\check{X}^d)} > c\right\}. \tag{11}$$

Here, $\hat{X}^a$ in (10) and $\check{X}^d$ in (11) are defined in (6).

We summarize the above results into the following conclusion on the representations of return.

Proposition 1. *Given the thresholds a,b,c,d and initial value $X_0 = x$, the return $r(a,b,c,d)$ in (7) of a one-round pairs trading takes the following form*

$$r(a,b,c,d) = \begin{cases} x - b + \left(b - X_{\tau_b^-}\right)\mathbb{1}\left\{X_{\tau_b^-} < b\right\}, & \text{if } x \geqslant a, \qquad (12)\\ r_a \mathbb{1}\left\{\tau_a^+ < \tau_d^-\right\} + r_d \mathbb{1}\left\{\tau_d^- < \tau_a^+\right\}, & \text{if } d < x < a, \qquad (13)\\ c - x + \left(X_{\tau_c^+} - c\right)\mathbb{1}\left\{X_{\tau_c^+} > c\right\}, & \text{if } x \leqslant d, \qquad (14)\end{cases}$$

where r_a and r_d are defined in (10) and (11) respectively.

2.4.2 Time cost

In this section, we define the trading cycle similarly as in Zeng and Lee (2014) to measure time cost, which consists of two consecutive periods explained as follows. In the waiting period, the investor would wait the signal of entry timing to open the position. We should note that this first period would disappear if the initial value $x \notin (d,a)$, which triggers the entry timing immediately. In the holding period, the investor holds the portfolio and waits the signal of exit timing to close the position. Hence, the length of a trading cycle $L(a,b,c,d)$ is the sum of a pair of waiting time and holding time. Similar to the derivation of the representation of return in Section 2.4.1, we express $L(a,b,c,d)$ under different initial values as follows.

Proposition 2. *Given the thresholds a, b, c, d and initial value $X_0 = x$, the length of a trading cycle $L(a, b, c, d)$ in (7) takes the following form*

$$L(a,b,c,d) = \begin{cases} \tau_b^-, & \text{if } x \geqslant a, & (15) \\ \tau_{a,d} + \tau_b^-\left(\hat{X}^a\right)\mathbb{1}\left\{\tau_a^+ < \tau_d^-\right\} + \tau_c^+\left(\check{X}^d\right)\mathbb{1}\left\{\tau_d^- < \tau_a^+\right\}, & \text{if } d < x < a, & (16) \\ \tau_c^+, & \text{if } x \leqslant d, & (17) \end{cases}$$

where $\hat{X}^a$ and $\check{X}^d$ are defined in (6).

Finally, we substitute the representations of return $r(a, b, c, d)$ in Proposition 1 and the length of a trading cycle $L(a, b, c, d)$ in Proposition 2 into the value function (7), and obtain the concrete expression of value function as

$$v_x(a,b,c,d) = \begin{cases} \dfrac{x-b+\mathbb{E}_x\left[b-X_{\tau_b^-}; X_{\tau_b^-}<b\right]}{\mathbb{E}_x\left[\tau_b^-\right]}, & \text{if } x \geq a, \\ \dfrac{\mathbb{E}_x\left[r_a;\tau_a^+<\tau_d^-\right]+\mathbb{E}_x\left[r_d;\tau_d^-<\tau_a^+\right]}{\mathbb{E}_x\left[\tau_{a,d}\right]+\mathbb{E}_x\left[\tau_b^-\left(\hat{X}^a\right);\tau_a^+<\tau_d^-\right]+\mathbb{E}_x\left[\tau_c^+\left(\check{X}^d\right);\tau_d^-<\tau_a^+\right]}, & \text{if } d < x < a, \\ \dfrac{c-x+\mathbb{E}_x\left[X_{\tau_c^+}-c; X_{\tau_c^+}>c\right]}{\mathbb{E}_x\left[\tau_c^+\right]}, & \text{if } x \leqslant d, \end{cases} \quad (18)$$

where r_a and r_d are defined in (10) and (11) respectively, and we write $\mathbb{E}[X;A] = \mathbb{E}[X\mathbb{1}\{A\}]$ for any random variable X and event A.

Remark 1. Every expectation value in (18) contains jumps. On the one hand, the overshoot and undershoot induce excess return, and randomize the initial value of spread process in the holding period. On the other hand, the distributions first passage times are significantly different from the pure continuous case.

Remark 2. In the value function (18), we ignore the influence of transaction cost and slippage to simplify our analysis. It should be noted that the two-sided jumps are totally different from these two concepts. Specifically, jumps are inherent in the spread model, which can impact on the timing of crossing thresholds and bring a random size to the return meanwhile, while transaction cost and slippage only occur after the determination of the trading signal.

3 Analytic value function

In this section, we aim to derive the analytic form of the value function in (18). For ease of exposition, we first introduce some notations in Section 3.1, then derive the explicit expressions of $\mathbb{E}_x[r(a, b, c, d)]$ and $\mathbb{E}_x[L(a, b, c, d)]$ in the two subsequent parts.

3.1 Auxiliary functions

In this part, we introduce some basic functions. For $q > 0$, $x \in \mathbb{R}$ and $i = 1, 2, 3, 4$, we define

$$
\begin{aligned}
\psi(x,q) &:= |x|^{\frac{q}{\kappa}-1}\, e^{-\frac{\sigma^2}{4\kappa}x^2+\frac{\mu}{\kappa}x}\, |x+\gamma_1|^{\frac{p\lambda}{\kappa}}\, |x-\gamma_2|^{\frac{(1-p)\lambda}{\kappa}}, \\
F_i(x,q) &:= \int_{\Gamma_i} \psi(z,q) e^{-(x-\alpha)z}\, dz, \\
G_i(x,q) &:= \int_{\Gamma_i} \frac{\gamma_1}{z+\gamma_1} \psi(z,q) e^{-(x-\alpha)z}\, dz, \\
H_i(x,q) &:= -\int_{\Gamma_i} \frac{\gamma_2}{z-\gamma_2} \psi(z,q) e^{-(x-\alpha)z}\, dz,
\end{aligned} \tag{19}
$$

where $\Gamma_1 = (-\infty, -\gamma_1)$, $\Gamma_2 = (-\gamma_1, 0)$, $\Gamma_3 = (0, \gamma_2)$ and $\Gamma_4 = (\gamma_2, \infty)$. We define the following summations

$$
F_{1,4}(x,q) := F_1(x,q) + F_4(x,q)\,,\ G_{1,4}(x,q) := G_1(x,q) + G_4(x,q)\,. \tag{20}
$$

Suppose $q = 0$ in (19) and (20), we define

$$
\begin{aligned}
&F_i(x) := F_i(x,0), \quad G_i(x) := G_i(x,0)\,,\ H_i(x) := H_i(x,0)\,,\ \text{for } i = 1,2,3,4, \\
&F_{1,4}(x) := F_{1,4}(x,0)\,,\ G_{1,4}(x) := G_{1,4}(x,0)\,.
\end{aligned} \tag{21}
$$

In later derivations, we need to handle the partial derivatives of $F_i(x,q)$, $G_i(x,q)$ and $H_i(x,q)$ w.r.t. q, but these partial derivatives diverge as $q \downarrow 0$ for $i = 2,3$. Hence, we seek to derive alternative representations of functions in (19). From the fact that $\lim_{z\to 0} |z|\, \psi(z,q) e^{-(x-\alpha)z} = 0$ and $\lim_{z\to \gamma_k} |z|\, \psi(z,q) e^{-(x-\alpha)z} = 0, k = 1,2$, the integration by parts yields that for $i = 2,3$,

$$
F_i(x,q) = (-1)^{\mathbf{1}_{\{i=2\}}} \frac{\kappa}{q} \int_{\Gamma_i} e^{-\frac{\sigma^2}{4\kappa}z^2+\frac{\mu}{\kappa}z} (z+\gamma_1)^{\frac{p\lambda}{\kappa}} (\gamma_2 - z)^{\frac{(1-p)\lambda}{\kappa}} e^{-(x-\alpha)z} d\,|z|^{\frac{q}{\kappa}}.
$$

Then, we define

$$
F_2^*(x,q) = \frac{q}{\kappa} F_2(x,q) \text{ and } F_3^*(x,q) = -\frac{q}{\kappa} F_3(x,q),\ q > 0, \kappa > 0. \tag{22}
$$

Similarly, we define

$$
G_i^*(x,q) := (-1)^{\mathbf{1}_{\{i=3\}}} \frac{q}{\kappa} G_i(x,q) \text{ and } H_i^*(x,q) := (-1)^{\mathbf{1}_{\{i=3\}}} \frac{q}{\kappa} H_i(x,q),\ q > 0, \kappa > 0. \tag{23}
$$

Further, we adopt the same notation as that in (20)-(21), denote by[2]

$$
\begin{aligned}
&F_i^*(x) = F_i^*(x,0) \text{ for } i = 2,3,\ G_3^*(x) = G_3^*(x,0),\ H_2^*(x) = H_2^*(x,0), \\
&G_2^*(x) := \lim_{q\downarrow 0} G_2^*(x,q),\ \ H_3^*(x) := \lim_{q\downarrow 0} H_3^*(x,q),
\end{aligned} \tag{24}
$$

and define

$$
F_{2,3}^*(x) := F_2^*(x) + F_3^*(x)\,,\ G_{2,3}^*(x) := G_2^*(x) + G_3^*(x)\,. \tag{25}
$$

Throughout this paper, for any function f belongs to $\{F_i, G_i, H_i, i = 1,4\}$, $\{F_i^*, G_i^*, H_i^*, i = 2,3\}$ or $\left\{F_{1,4}(x), G_{1,4}(x), F_{2,3}^*(x), G_{2,3}^*(x)\right\}$, the operation $\dot{f} := \frac{\partial}{\partial q} f\Big|_{q=0}$ denotes the partial derivative of f w.r.t. q evaluated at $q = 0$.

[2]The existence of $G_2^*(x)$ and $H_3^*(x)$ is verified in Appendix C.1.

3.2 Analytic form of expected return

Suppose that the spread process X_t evolves as the OUDEJ process in (1) satisfying $X_0 = x$, based on the formulation in (12)-(14), we derive the analytic form of expected return in (9) as follows.

Theorem 1. (i) *For $X_0 = x \notin (d, a)$, the expected returns in (12) and (14) are*

$$\mathbb{E}_x\left[r\left(a,b,c,d\right)\right] = \begin{cases} x - b + \frac{1}{\gamma_2}\frac{F_4(b)-F_4(x)}{F_4(b)-H_4(b)}, & \text{if } x \geqslant a, \\ c - x + \frac{1}{\gamma_1}\frac{F_1(c)-F_1(x)}{F_1(c)-G_1(c)}, & \text{if } x \leqslant d, \end{cases}$$

where $F_i(\cdot)$, $G_i(\cdot)$, $H_i(\cdot)$ for $i = 1, 4$ are defined in (21).

(ii) *For $X_0 = x \in (d, a)$, the expected return in (13) is*

$$\begin{aligned}
\mathbb{E}_x\left[r\left(a,b,c,d\right)\right] &= \mathbb{P}_x\left(X_{\tau_{a,d}} \geqslant a\right)(a-b) + \mathbb{P}_x\left(X_{\tau_{a,d}} \leqslant d\right)(c-d) \\
&+ \frac{1}{\gamma_2}\left[\mathbb{P}_x\left(X_{\tau_{a,d}} = a\right)\frac{F_4(b)-F_4(a)}{F_4(b)-H_4(b)} + \mathbb{P}_x\left(X_{\tau_{a,d}} > a\right)\frac{F_4(b)-G_4(a)}{F_4(b)-H_4(b)} + \mathbb{P}_x\left(X_{\tau_{a,d}} < d\right)\right] \\
&+ \frac{1}{\gamma_1}\left[\mathbb{P}_x\left(X_{\tau_{a,d}} = d\right)\frac{F_1(c)-F_1(d)}{F_1(c)-G_1(c)} + \mathbb{P}_x\left(X_{\tau_{a,d}} < d\right)\frac{F_1(c)-H_1(d)}{F_1(c)-G_1(c)} + \mathbb{P}_x\left(X_{\tau_{a,d}} > a\right)\right],
\end{aligned}$$

where the probabilities $\mathbb{P}_x\left(X_{\tau_{a,d}} = a\right)$, $\mathbb{P}_x\left(X_{\tau_{a,d}} > a\right)$, $\mathbb{P}_x\left(X_{\tau_{a,d}} = d\right)$, and $\mathbb{P}_x\left(X_{\tau_{a,d}} < d\right)$ are explicitly given by (B.35) in Appendix B.3.

Remark 3. The significant difference between Theorem 1 and the related works lie in the expected excess return, which is the average jump size multiplied by the undershoot/overshoot probability at the exit timing. Here, the calculation of corresponding probabilities is one of the theoretical difficulties of our model.

Proof. We decompose the excess return into the product of jump size and overshoot/undershoot probabilities. The case for one-sided exit problems is obtained in Jacobsen and Jensen (2007). To extend their conclusion to the two-sided problems, we are confronted with the problem that a type of integral diverges as the variable in Laplace transform tends to 0. To overcome this difficulty, we exploit the new auxiliary functions defined in Section 3.1.

(i) When $X_0 = x \geqslant a$, from the representation of return in (12), the expected return is given by

$$\mathbb{E}_x\left[r\left(a,b,c,d\right)\right] = x - b + \mathbb{E}_x\left[b - X_{\tau_b^-}; X_{\tau_b^-} < b\right] = x - b + \mathbb{E}_x\left[b - X_{\tau_b^-}\middle| X_{\tau_b^-} < b\right]\mathbb{P}_x\left(X_{\tau_b^-} < b\right).$$

The conditional memoryless property (B.3) in Lemma 1 further yields that

$$\mathbb{E}_x\left[r\left(a,b,c,d\right)\right] = x - b + \frac{1}{\gamma_2}\mathbb{P}_x\left(X_{\tau_b^-} < b\right) = x - b + \frac{1}{\gamma_2}\frac{F_4(b)-F_4(x)}{F_4(b)-H_4(b)},$$

where in the last equation we use the fact $\mathbb{P}_x\left(X_{\tau_b^-} < b\right) = \frac{F_4(b)-F_4(x)}{F_4(b)-H_4(b)}$ in (B.8) of Theorem 4.

Similarly, when $X_0 = x \leqslant d$, it follows from (14), (B.2) and (B.7) that

$$\mathbb{E}_x\left[r\left(a,b,c,d\right)\right] = c - x + \frac{1}{\gamma_1}\frac{F_1(c)-F_1(x)}{F_1(c)-G_1(c)}.$$

(ii) When $X_0 = x \in (d,a)$, the expected return in (13) is calculated as

$$
\begin{aligned}
\mathbb{E}_x\left[r\left(a,b,c,d\right)\right] =& \left(a-b\right)\mathbb{P}_x\left(X_{\tau_{a,d}} \geqslant a\right) + \left(c-d\right)\mathbb{P}_x\left(X_{\tau_{a,d}} \leqslant d\right)\\
&+\mathbb{E}_x\left[X_{\tau_a^+} - a; X_{\tau_{a,d}} > a\right] + \mathbb{E}_x\left[b - \hat{X}^a_{\tau_b^-\left(\hat{X}^a\right)}; \tau_a^+ < \tau_d^-, \hat{X}^a_{\tau_b^-\left(\hat{X}^a\right)} < b\right]\\
&+\mathbb{E}_x\left[d - X_{\tau_d^-}; X_{\tau_{a,d}} < d\right] + \mathbb{E}_x\left[\check{X}^d_{\tau_c^+\left(\check{X}^d\right)} - c; \tau_a^+ < \tau_d^-, \check{X}^d_{\tau_c^+\left(\check{X}^d\right)} > c\right].
\end{aligned}
\tag{26}
$$

In the following, we explicitly calculate each term in (26). $\mathbb{P}_x\left(X_{\tau_{a,d}} \geqslant a\right)$ and $\mathbb{P}_x\left(X_{\tau_{a,d}} \leqslant d\right)$ in (26) can be calculated by (B.35).

The third term in (26) satisfies

$$
\begin{aligned}
&\mathbb{E}_x\left[X_{\tau_a^+} - a; X_{\tau_{a,d}} > a\right] = \mathbb{E}_x\left[X_{\tau_a^+} - a\middle| X_{\tau_{a,d}} > a\right]\mathbb{P}_x\left(X_{\tau_{a,d}} > a\right)\\
=&\mathbb{E}_x\left\{\mathbb{E}_x\left[X_{\tau_a^+} - a\middle| X_{\tau_a^+} > a\right]\middle| X_{\tau_{a,d}} > a\right\}\mathbb{P}_x\left(X_{\tau_{a,d}} > a\right) = \frac{1}{\gamma_1}\mathbb{P}_x\left(X_{\tau_{a,d}} > a\right),
\end{aligned}
\tag{27}
$$

where the last equation follows from (B.2). In analogy to (27), the fifth term in (26) follows that

$$
\mathbb{E}_x\left[d - X_{\tau_d^-}; X_{\tau_{a,d}} < d\right] = \frac{1}{\gamma_2}\mathbb{P}_x\left(X_{\tau_{a,d}} < d\right).
\tag{28}
$$

By the fact that $\mathbb{1}\left\{\tau_a^+ < \tau_d^-\right\} = \mathbb{1}\left\{\min\limits_{t \leqslant \tau_a^+} X_t > d\right\} \in \mathcal{F}_{\tau_a^+}$, tower rule and strong Markov property, the fourth term in (26) could be expressed as

$$
\begin{aligned}
&\mathbb{E}_x\left[b - \hat{X}^a_{\tau_b^-\left(\hat{X}^a\right)}; \tau_a^+ < \tau_d^-, \hat{X}^a_{\tau_b^-\left(\hat{X}^a\right)} < b\right]\\
=&\mathbb{E}_x\left\{\mathbb{E}_x\left[b - \hat{X}^a_{\tau_b^-\left(\hat{X}^a\right)}; \hat{X}^a_{\tau_b^-\left(\hat{X}^a\right)} < b\middle| \mathcal{F}_{\tau_a^+}\right]; \tau_a^+ < \tau_d^-\right\}\\
=&\frac{1}{\gamma_2}\mathbb{E}_x\left[\mathbb{P}_{X_{\tau_a^+}}\left(X_{\tau_b^-} < b\right); X_{\tau_{a,d}} = a\right] + \frac{1}{\gamma_2}\mathbb{E}_x\left[\mathbb{P}_{X_{\tau_a^+}}\left(X_{\tau_b^-} < b\right); X_{\tau_{a,d}} > a\right],
\end{aligned}
\tag{29}
$$

where the last equation follows from the conditional memoryless property (B.3). By conditional memoryless property (B.2) and the smoothing law of conditional expectation, we have

$$
\begin{aligned}
\mathbb{E}_x\left[\mathbb{P}_{X_{\tau_a^+}}\left(X_{\tau_b^-} < b\right); X_{\tau_{a,d}} = a\right] =& \mathbb{E}_x\left[\mathbb{P}_a\left(X_{\tau_b^-} < b\right); X_{\tau_{a,d}} = a\right]\\
=&\mathbb{P}_x\left(X_{\tau_{a,d}} = a\right)\mathbb{P}_a\left(X_{\tau_b^-} < b\right)
\end{aligned}
\tag{30}
$$

and

$$
\begin{aligned}
\mathbb{E}_x\left[\mathbb{P}_{X_{\tau_a^+}}\left(X_{\tau_b^-} < b\right); X_{\tau_{a,d}} > a\right] =& \mathbb{P}_x\left(X_{\tau_{a,d}} > a\right)\mathbb{E}_x\left[\mathbb{P}_{X_{\tau_a^+}}\left(X_{\tau_b^-} < b\right)\middle| X_{\tau_{a,d}} > a\right]\\
=&\mathbb{P}_x\left(X_{\tau_{a,d}} > a\right)\int_0^\infty \mathbb{P}_{z+a}\left(X_{\tau_b^-} < b\right)\gamma_1 e^{-\gamma_1 z}\,dz.
\end{aligned}
\tag{31}
$$

Substituting (30) and (31) into (29), we obtain that

$$
\mathbb{E}_x\left[b - \hat{X}^a_{\tau_b^-\left(\hat{X}^a\right)}; \tau_a^+ < \tau_d^-, \hat{X}^a_{\tau_b^-\left(\hat{X}^a\right)} < b\right]
$$

$$
\begin{aligned}
=&\frac{1}{\gamma_2}\mathbb{P}_x\left(X_{\tau_{a,d}}=a\right)\mathbb{P}_a\left(X_{\tau_b^-}<b\right)+\frac{1}{\gamma_2}\mathbb{P}_x\left(X_{\tau_{a,d}}>a\right)\int_0^\infty \mathbb{P}_{z+a}\left(X_{\tau_b^-}<b\right)\gamma_1 e^{-\gamma_1 z}\,dz\\
=&\frac{1}{\gamma_2}\mathbb{P}_x\left(X_{\tau_{a,d}}=a\right)\frac{F_4(b)-F_4(a)}{F_4(b)-H_4(b)}+\frac{1}{\gamma_2}\mathbb{P}_x\left(X_{\tau_{a,d}}>a\right)\frac{F_4(b)-G_4(a)}{F_4(b)-H_4(b)},
\end{aligned}\tag{32}
$$

where the last line follows from the results of (B.8) as well as

$$
\int_0^\infty \mathbb{P}_{z+a}\left(X_{\tau_b^-}<b\right)\gamma_1 e^{-\gamma_1 z}\,dz=\int_0^\infty\frac{F_4(b)-F_4(z+a)}{F_4(b)-H_4(b)}\gamma_1 e^{-\gamma_1 z}\,dz=\frac{F_4(b)-G_4(a)}{F_4(b)-H_4(b)}.
$$

Similarly, for the last term in (26), we have

$$
\begin{aligned}
&\mathbb{E}_x\left[\check{X}^d_{\tau_c^+(\check{X}^d)}-c;\tau_d^-<\tau_a^+,\check{X}^d_{\tau_c^+(\check{X}^d)}>c\right]\\
=&\frac{1}{\gamma_1}\mathbb{E}_x\left[\mathbb{P}_{X_{\tau_d^-}}\left(X_{\tau_c^+}>c\right);\tau_d^-<\tau_a^+\right]\\
=&\frac{1}{\gamma_1}\mathbb{P}_x\left(X_{\tau_{a,d}}=d\right)\frac{F_1(c)-F_1(d)}{F_1(c)-G_1(c)}+\frac{1}{\gamma_1}\mathbb{P}_x\left(X_{\tau_{a,d}}<d\right)\frac{F_1(c)-H_1(d)}{F_1(c)-G_1(c)}.
\end{aligned}\tag{33}
$$

The result follows by substituting (27)-(28), (32) and (33) into (26), which completes the proof. □

3.3 Analytic form of expected length of a trading cycle

In Section 2.4.2, we introduce the length of a trading cycle which measures the average time cost in a one-round pairs trading. We recall that the trading circle therein is decomposed into a waiting period and a holding period. In this part, we derive the expectation of the length of a trading cycle through handling these two parts respectively.

Theorem 2. **(i)** *For $X_0=x\notin(d,a)$, the expected length of a trading cycle is*

$$
\mathbb{E}_x\left[L\left(a,b,c,d\right)\right]=\begin{cases}\mathbb{E}_x\left[\tau_b^-\right], & \text{if } x\geqslant a,\\ \mathbb{E}_x\left[\tau_c^+\right], & \text{if } x\leqslant d,\end{cases}
$$

where $\mathbb{E}_x\left[\tau_c^+\right]$ and $\mathbb{E}_x\left[\tau_b^-\right]$ are given by (B.13) (B.14) derived in Theorem 5 respectively.

(ii) *For $X_0=x\in(d,a)$, the expected length of a trading cycle is*

$$
\begin{aligned}
\mathbb{E}_x\left[L\left(a,b,c,d\right)\right]=&\mathbb{E}_x\left[\tau_{a,d}\right]+\mathbb{P}_x\left(X_{\tau_{a,d}}=a\right)\mathbb{E}_a\left[\tau_b^-\right]+\mathbb{P}_x\left(X_{\tau_{a,d}}>a\right)\int_0^\infty\mathbb{E}_{z+a}\left[\tau_b^-\right]\gamma_1 e^{-\gamma_1 z}\,dz\\
&+\mathbb{P}_x\left(X_{\tau_{a,d}}=d\right)\mathbb{E}_d\left[\tau_c^+\right]+\mathbb{P}_x\left(X_{\tau_{a,d}}<d\right)\int_0^\infty\mathbb{E}_{d-z}\left[\tau_c^+\right]\gamma_2 e^{-\gamma_2 z}\,dz.
\end{aligned}\tag{34}
$$

Here, $\mathbb{E}_x\left[\tau_{a,d}\right]$ is given by (B.37), the distribution of $X_{\tau_{a,d}}$ is calculated in (B.35), the expectations of one-sided first passage times $\mathbb{E}_a\left[\tau_b^-\right]$ and $\mathbb{E}_d\left[\tau_c^+\right]$ are given by (B.13) and (B.14), and the integrations in (34) can be explicitly calculated as

$$
\begin{aligned}
&\int_0^\infty\mathbb{E}_{z+a}\left[\tau_b^-\right]\gamma_1 e^{-\gamma_1 z}\,dz=\check{\zeta}_a^{(1)}\left(b\right)+\check{\zeta}_a^{(2)}\left(b\right),\\
&\int_0^\infty\mathbb{E}_{d-z}\left[\tau_c^+\right]\gamma_2 e^{-\gamma_2 z}\,dz=\hat{\zeta}_d^{(1)}\left(c\right)+\hat{\zeta}_d^{(2)}\left(c\right),
\end{aligned}\tag{35}
$$

where $\check{\zeta}_a^{(i)}(b)$ *and* $\hat{\zeta}_d^{(i)}(c)$ *for* $i=1,2$ *are respectively solved by*

$$\begin{pmatrix} F_3^*(b) & H_3^*(b) \\ F_4(b) & H_4(b) \end{pmatrix} \begin{pmatrix} \check{\zeta}_a^{(1)}(b) \\ \check{\zeta}_a^{(2)}(b) \end{pmatrix} = \begin{pmatrix} [\dot{F}_3^*(b) - \dot{H}_3^*(b)]\frac{G_4(a)-H_4(b)}{F_4(b)-H_4(b)} + \dot{H}_3^*(b) - \dot{G}_3^*(a) \\ [\dot{F}_4(b) - \dot{H}_4(b)]\frac{G_4(a)-H_4(b)}{F_4(b)-H_4(b)} + \dot{H}_4(b) - \dot{G}_4(a) \end{pmatrix}, \tag{36}$$

and

$$\begin{pmatrix} F_1(c) & G_1(c) \\ F_2^*(c) & G_2^*(c) \end{pmatrix} \begin{pmatrix} \hat{\zeta}_d^{(1)}(c) \\ \hat{\zeta}_d^{(2)}(c) \end{pmatrix} = \begin{pmatrix} [\dot{F}_1(c) - \dot{G}_1(c)]\frac{H_1(d)-G_1(c)}{F_1(c)-G_1(c)} + \dot{G}_1(c) - \dot{H}_1(d) \\ [\dot{F}_2^*(c) - \dot{G}_2^*(c)]\frac{H_1(d)-G_1(c)}{F_1(c)-G_1(c)} + \dot{G}_2^*(c) - \dot{H}_2^*(d) \end{pmatrix}. \tag{37}$$

Here, $F_i(\cdot)$*,* $G_i(\cdot)$ *and* $H_i(\cdot)$ *for* $i=1,4$ *are given by (21),* $F_j^*(\cdot)$*,* $G_j^*(\cdot)$ *and* $H_j^*(\cdot)$ *for* $j=2,3$ *are given by (24).*

Proof. In this proof, we decompose the expected length of a trading cycle into the expectation of first passage times and crossing probabilities. The derivation of former expectation exploits the techniques in Appendix B, and the latter quantity has been derived in Theorem 1.

(i) When $X_0 = x \notin (d,a)$, from (15) and (17) in Proposition 1, $\mathbb{E}_x[L(a,b,c,d)]$ immediately reduces to $\mathbb{E}_x[\tau_b^-]$ or $\mathbb{E}_x[\tau_c^+]$, which are explicitly derived in Appendix B.2.

(ii) When $X_0 = x \in (d,a)$, it follows from (16) that

$$\mathbb{E}_x[L(a,b,c,d)] = \mathbb{E}_x[\tau_{a,d}] + \mathbb{E}_x\left[\tau_b^-\left(\hat{X}^a\right); \tau_a^+ < \tau_d^-\right] + \mathbb{E}_x\left[\tau_c^+\left(\check{X}^d\right); \tau_d^- < \tau_a^+\right]. \tag{38}$$

Here, $\mathbb{E}_x[\tau_{a,d}]$ is calculated by (B.37). Thus we first calculate the second term in RHS of (38). By the strong Markov property, we notice that

$$\mathbb{E}_x\left[\tau_b^-\left(\hat{X}^a\right); \tau_a^+ < \tau_d^-\right] = \mathbb{E}_x\left\{\mathbb{E}_x\left[\tau_b^-\left(\hat{X}^a\right)\Big|\mathcal{F}_{\tau_a^+}\right]; \tau_a^+ < \tau_d^-\right\} = \mathbb{E}_x\left\{\mathbb{E}_{X_{\tau_a^+}}\left[\tau_b^-\right]; \tau_a^+ < \tau_d^-\right\}$$
$$= \mathbb{P}_x\left(X_{\tau_{a,d}} = a\right)\mathbb{E}_a\left[\tau_b^-\right] + \mathbb{P}_x\left(X_{\tau_{a,d}} > a\right)\int_0^\infty \mathbb{E}_{z+a}\left[\tau_b^-\right]\gamma_1 e^{-\gamma_1 z}\,dz, \tag{39}$$

where the last equation considers two cases of crossing the threshold continuously and by jump, and follows from the conditional memoryless property (B.2). To precede, we take $x = z+a$ and $d = b$ in (B.14), and integrate both sides w.r.t. z from 0 to ∞. Then (B.18) yields that

$$\begin{pmatrix} F_3^*(b) & H_3^*(b) \\ F_4(b) & H_4(b) \end{pmatrix} \begin{pmatrix} \int_0^\infty \check{\pi}_b^{(1)}(z+a)\gamma_1 e^{-\gamma_1 z}\,dz \\ \int_0^\infty \check{\pi}_b^{(2)}(z+a)\gamma_1 e^{-\gamma_1 z}\,dz \end{pmatrix}$$
$$= \begin{pmatrix} [\dot{F}_3^*(b) - \dot{H}_3^*(b)]\int_0^\infty \mathbb{P}_{z+a}\left(X_{\tau_b^-} = b\right)\gamma_1 e^{-\gamma_1 z}\,dz + \dot{H}_3^*(b) - \int_0^\infty \dot{F}_3^*(z+a)\gamma_1 e^{-\gamma_1 z}\,dz \\ [\dot{F}_4(b) - \dot{H}_4(b)]\int_0^\infty \mathbb{P}_{z+a}\left(X_{\tau_b^-} = b\right)\gamma_1 e^{-\gamma_1 z}\,dz + \dot{H}_4(b) - \int_0^\infty \dot{F}_4(z+a)\gamma_1 e^{-\gamma_1 z}\,dz \end{pmatrix}. \tag{40}$$

Denote $\check{\zeta}_a^{(1)}(b) := \int_0^\infty \mathbb{E}_{z+a}\left[\tau_b^-; X_{\tau_b^-} = b\right]\gamma_1 e^{-\gamma_1 z}\,dz, \check{\zeta}_a^{(2)}(b) := \int_0^\infty \mathbb{E}_{z+a}\left[\tau_b^-; X_{\tau_b^-} < b\right]\gamma_1 e^{-\gamma_1 z}\,dz$, and it suffices to prove that both the RHS of (36) and (40) are equal. We first notice that given the definitions of $F_4(x)$, $G_4(x)$ in (21), and $F_3^*(x)$, $G_3^*(x)$ in (24), if we take their partial derivatives w.r.t. q, perform the corresponding integration, and let $q \downarrow 0$, then we obtain that

$$\int_0^\infty \dot{F}_3^*(z+a)\gamma_1 e^{-\gamma_1 z}\,dz = \dot{G}_3^*(a) \text{ and } \int_0^\infty \dot{F}_4(z+a)\gamma_1 e^{-\gamma_1 z}\,dz = \dot{G}_4(a). \tag{41}$$

Then from the probability of a continuous downcrossing in (B.8), we also have

$$\int_0^\infty \mathbb{P}_{z+a}\left(X_{\tau_b^-} = b\right)\gamma_1 e^{-\gamma_1 z}\,dz = \frac{G_4(a) - H_4(b)}{F_4(b) - H_4(b)}. \tag{42}$$

Thus, substituting (41) and (42) into (40), we find that $\left(\check{\zeta}_a^{(1)}(b), \check{\zeta}_a^{(2)}(b)\right)^\top$ indeed satisfies the linear system (36).

For the third term in (38), by the strong Markov property and conditional memoryless property (B.3), we obtain that

$$\begin{aligned}&\mathbb{E}_x\left[\tau_c^+\left(\check{X}^d\right); \tau_d^- < \tau_a^+\right] = \mathbb{E}_x\left\{\mathbb{E}_{X_{\tau_d^-}}\left[\tau_c^+\right]; \tau_d^- < \tau_a^+\right\} \\ =&\mathbb{P}_x\left(X_{\tau_{a,d}} = d\right)\mathbb{E}_d\left[\tau_c^+\right] + \mathbb{P}_x\left(X_{\tau_{a,d}} < d\right)\int_0^\infty \mathbb{E}_{d-z}\left[\tau_c^+\right]\gamma_2 e^{-\gamma_2 z}\,dz.\end{aligned} \tag{43}$$

Analogously, using (B.13) and (B.17), we can verify that $\int_0^\infty \mathbb{E}_{d-z}\left(\tau_c^+\right)\gamma_2 e^{-\gamma_2 z}\,dz$ in (43) can be evaluated via (35) and (37), where $\hat{\zeta}_d^{(1)}(c) := \int_0^\infty E_{d-z}\left[\tau_c^+; X_{\tau_c^-} = c\right]\gamma_2 e^{-\gamma_2 z}\,dz$ and $\hat{\zeta}_d^{(2)}(c) := \int_0^\infty E_{d-z}\left[\tau_c^+; X_{\tau_c^-} > c\right]\gamma_2 e^{-\gamma_2 z}\,dz$. The result in (34) follows by plugging (39) and (43) into (38). $\square$

3.4 Optimal problems under symmetric thresholds cases

In this section, we study a special subclass of OUDEJ model in symmetric case, and obtain a simplified form of the value function (18).

Intuitively, in practice the selected two stocks in a pairs trading strategy are often in the same sector or in a common industrial chain. Thus their return movements tend to be similar and it is reasonable to assume that their spread has a symmetric trend movement and jump structure. Corresponding to our model setup, for the special symmetric case, we assume that $p = 0.5$, $\mu = \alpha = 0$, $\gamma_1 = \gamma_2 =: \gamma$ in (1) and (2), which generalizes the model of Zeng and Lee (2014) by adding two-sided jumps. Under this symmetric parameter assumption, the spread model reduces to

$$\begin{gathered}dX_t = -\kappa X_t dt + dL_t, \text{ where } L_t = \sigma W_t + \sum_{k=1}^{N_t} Y_k \text{ for } t \geqslant 0, \\ \text{and p.d.f. of } Y_k \text{ is } f_Y(y) = \frac{1}{2}\gamma e^{-\gamma|y|}.\end{gathered} \tag{44}$$

The functions in (19) and (21) can be simplified as, for $q > 0$, $x \in \mathbb{R}$ and $i = 1, 2, 3, 4$,

$$\begin{aligned}\psi(x,q) &= |x|^{\frac{q}{\kappa}-1} e^{-\frac{\sigma^2}{4\kappa}x^2}\left|x^2-\gamma^2\right|^{\frac{\lambda}{2\kappa}}, \\ F_i(x) &= \int_{\Gamma_i}\frac{1}{|z|}e^{-\frac{\sigma^2}{4\kappa}z^2 - xz}\left|z^2-\gamma^2\right|^{\frac{\lambda}{2\kappa}}\,dz, \\ G_i(x) &= \int_{\Gamma_i}\frac{\gamma}{z+\gamma}\frac{1}{|z|}e^{-\frac{\sigma^2}{4\kappa}z^2 - xz}\left|z^2-\gamma^2\right|^{\frac{\lambda}{2\kappa}}\,dz, \\ H_i(x) &= -\int_{\Gamma_i}\frac{\gamma}{z-\gamma}\frac{1}{|z|}e^{-\frac{\sigma^2}{4\kappa}z^2 - xz}\left|z^2-\gamma^2\right|^{\frac{\lambda}{2\kappa}}\,dz,\end{aligned} \tag{45}$$

where $\Gamma_1 = (-\infty, -\gamma)$, $\Gamma_2 = (-\gamma, 0)$, $\Gamma_3 = (0, \gamma)$ and $\Gamma_4 = (\gamma, \infty)$. Then we have that in (21), for $x \in \mathbb{R}$,

$$\begin{aligned}F_{1,4}(x) &= F_1(x) + F_4(x) = \int_\gamma^\infty \frac{1}{z}e^{-\frac{\sigma^2}{4\kappa}z^2}\left(e^{xz}+e^{-xz}\right)\left(z^2-\gamma^2\right)^{\frac{\lambda}{2\kappa}}\,dz, \\ G_{1,4}(x) &= G_1(x) + G_4(x) = \int_\gamma^\infty \frac{1}{z}\gamma e^{-\frac{\sigma^2}{4\kappa}z^2}\left(\frac{e^{xz}}{-z+\gamma} + \frac{e^{-xz}}{z+\gamma}\right)\left(z^2-\gamma^2\right)^{\frac{\lambda}{2\kappa}}\,dz.\end{aligned}$$

As for the device of trading strategy, we adopt the symmetric entry thresholds in the spirit of Zeng and Lee (2014). The following proposition shows under spread model (44), when the entry thresholds are symmetric about initial value $X_0 = 0$, the optimal exit thresholds are also symmetric.

Proposition 3. *Conditional on the spread process evolves as (44) with initial value $X_0 = 0$ and the entry thresholds satisfy $d = -a$, the optimal exit thresholds b^* and c^* for maximizing $v(a, b, c, -a)$ should be $b^* = -c^*$.*

Proof. First, we notice that in the symmetric context, the function $\psi(x, q)$ in (45) satisfies $\psi(x, q) = \psi(-x, q)$ for $x \in \mathbb{R}$, and some straightforward calculations yield that for $i = 1, 2, 3, 4$,

$$\begin{aligned} F_i(-x, q) &= F_{5-i}(x, q), G_i(-x, q) = H_{5-i}(x, q), \\ \dot{F}_i(-x, q) &= \dot{F}_{5-i}(x, q), \dot{G}_i(-x, q) = \dot{H}_{5-i}(x, q). \end{aligned} \tag{46}$$

Similarly, the symmetry relation (46) also holds for functions F^*, G^*, H^* defined by (22) - (24). Then, we define $\mathbb{E}\left[r(a, b, c, -a)\right]$[3]$= R_1(a, b) + R_2(c, -a)$, where

$$\begin{aligned} R_1(a, b) &:= \mathbb{P}\left(X_{\tau_{a,-a}} \geqslant a\right)(a - b) + \frac{1}{\gamma}\mathbb{P}\left(X_{\tau_{a,-a}} = a\right)\frac{F_4(b) - F_4(a)}{F_4(b) - H_4(b)} \\ &+ \frac{1}{\gamma}\mathbb{P}\left(X_{\tau_{a,-a}} > a\right)\frac{F_4(b) - G_4(a)}{F_4(b) - H_4(b)} + \frac{1}{\gamma}\mathbb{P}\left(X_{\tau_{a,-a}} > a\right), \\ R_2(c, -a) &:= \mathbb{P}\left(X_{\tau_{a,-a}} \leqslant -a\right)(c + a) + \frac{1}{\gamma}\mathbb{P}_x\left(X_{\tau_{a,-a}} = -a\right)\frac{F_1(c) - F_1(-a)}{F_1(c) - G_1(c)} \\ &+ \frac{1}{\gamma}\mathbb{P}\left(X_{\tau_{a,-a}} < -a\right)\frac{F_1(c) - H_1(-a)}{F_1(c) - G_1(c)} + \frac{1}{\gamma}\mathbb{P}\left(X_{\tau_{a,-a}} < -a\right). \end{aligned}$$

Under the symmetric parameter setting, (B.35) and (46) yield that $\mathbb{P}\left(X_{\tau_{a,-a}} = a\right) = \mathbb{P}\left(X_{\tau_{a,-a}} = -a\right)$ and $\mathbb{P}\left(X_{\tau_{a,-a}} > a\right) = \mathbb{P}\left(X_{\tau_{a,-a}} < -a\right)$, then we have $R_2(c, -a) = R_1(a, -c)$. Hence, $\mathbb{E}\left[r(a, b, c, -a)\right] = R_1(a, b) + R_1(a, -c)$.

Similarly, we define $\mathbb{E}\left[L(a, b, c, -a)\right] = L_1(a, b) + L_1(a, -c)$, where for any $y < a$,

$$L_1(a, y) := \mathbb{E}\left[\tau_{a,-a}; \tau_a^+ < \tau_{-a}^-\right] + \mathbb{P}\left(X_{\tau_{a,-a}} = a\right)\mathbb{E}_a\left[\tau_y^-\right] + \mathbb{P}\left(X_{\tau_{a,-a}} > a\right)\int_0^\infty \mathbb{E}_{z+a}\left[\tau_y^-\right]\gamma e^{-\gamma z}\,dz.$$

Given the above decompositions, the value function can be represented as

$$v(a, b, c, -a) = \frac{R_1(a, b) + R_1(a, -c)}{L_1(a, b) + L_1(a, -c)}.$$

As $\left.\frac{R_1(a,-c)}{L_1(a,-c)}\right|_{c=-b} = \frac{R_1(a,b)}{L_1(a,b)}$ for any $b \in (-a, a)$, if we denote $b^* := \arg\max_b \frac{R_1(a,b)}{L_1(a,b)}$, then

$$-b^* = \arg\max_c \frac{R_1(a, -c)}{L_1(a, -c)}. \tag{47}$$

Thus, it suffices to show that $(b^*, -b^*) = \underset{(b,c)}{\arg\max}\, v(a, b, c, -a)$. It follows from the definition of b^* and (47) that $\frac{R_1(a,b^*)}{L_1(a,b^*)} \geqslant \frac{R_1(a,b_1)}{L_1(a,b_1)}$ and $\frac{R_1(a,b^*)}{L_1(a,b^*)} \geqslant \frac{R_1(a,-c_1)}{L_1(a,-c_1)}$ for any (b_1, c_1), so

[3] In the following, when $x = 0$, we drop the subscript of $\mathbb{E}_x[\cdot]$, $\mathbb{P}_x(\cdot)$ and $v_x(\cdot)$.

$$v\left(a,b^{*},-b^{*},-a\right)=\frac{R_{1}\left(a,b^{*}\right)}{L_{1}\left(a,b^{*}\right)}\geqslant\frac{R_{1}\left(a,b_{1}\right)+R_{1}\left(a,-c_{1}\right)}{L_{1}\left(a,b_{1}\right)+L_{1}\left(a,-c_{1}\right)}=v\left(a,b_{1},c_{1},-a\right),$$

which completes the proof. □

In Proposition 3, we illuminate the rationality of the symmetric thresholds setup in theory. In the following, we assume that $d=-a$ and $c=-b$ in Section 2.3. Furthermore, the value function can be greatly simplified to be characterized by thresholds a and b. The expressions of expected return and length of a trading cycle can be separated by continuous paths and jumps respectively, highlighting the effects of jumps. The following theorem summarizes the results.

Theorem 3. *Suppose the spread process evolves as (44) with initial value $X_0=x\in(-a,a)$ and the entry thresholds satisfy $d=-a$, then we have the expected return*

$$\mathbb{E}_{x}\left[r\left(a,b,-b,-a\right)\right]=a-b+\frac{1}{\gamma}p_{a}\frac{F_{4}(b)-F_{4}(a)}{F_{4}(b)-H_{4}(b)}+\frac{1}{\gamma}\left(1-p_{a}\right)\left[1+\frac{F_{4}(b)-G_{4}(a)}{F_{4}(b)-H_{4}(b)}\right],\tag{48}$$

and the expected length of a trading cycle is

$$\mathbb{E}_{x}\left[L\left(a,b,-b,-a\right)\right]=\mathbb{E}_{x}\left[\tau_{a,-a}\right]+p_{a}\mathbb{E}_{a}\left[\tau_{b}^{-}\right]+\left(1-p_{a}\right)\int_{0}^{\infty}\mathbb{E}_{z+a}\left[\tau_{b}^{-}\right]\gamma e^{-\gamma z}\,dz.\tag{49}$$

Here, p_a in (48) and (49) is the probability of crossing the thresholds a or $-a$ continuously, calculated as $p_a=\frac{\int_{\gamma}^{\infty}\frac{1}{z}\left(z^{2}-\gamma^{2}\right)^{\frac{\lambda}{2\kappa}}e^{-\frac{\sigma^{2}}{4\kappa}z^{2}}\left[\left(e^{xz}+e^{-xz}\right)+\gamma\left(\frac{e^{az}}{z-\gamma}-\frac{e^{-az}}{z+\gamma}\right)\right]dz}{\int_{\gamma}^{\infty}\left(z^{2}-\gamma^{2}\right)^{\frac{\lambda}{2\kappa}}e^{-\frac{\sigma^{2}}{4\kappa}z^{2}}\left(\frac{e^{az}}{z-\gamma}+\frac{e^{-az}}{z+\gamma}\right)dz}$, $\mathbb{E}_a\left[\tau_b^-\right]$ in (49) is given by (B.14), and $\mathbb{E}_x\left[\tau_{a,-a}\right]$ in (49) is calculated as

$$\mathbb{E}_{x}\left[\tau_{a,-a}\right]=\frac{\left[G_{2,3}^{*}\left(a\right)-F_{2,3}^{*}\left(a\right)\right]J_{1}\left(a\right)-\left[G_{1,4}\left(a\right)-F_{1,4}\left(a\right)\right]J_{2}\left(a\right)}{F_{1,4}\left(a\right)G_{2,3}^{*}\left(a\right)-F_{2,3}^{*}\left(a\right)G_{1,4}\left(a\right)},\tag{50}$$

where the functions $F_{1,4},G_{1,4}$ are defined in (20), $F^*_{2,3},G^*_{2,3}$ are defined in (25). Besides, $J_1(a)=p_a\dot{F}_{1,4}(a)+(1-p_a)\dot{G}_{1,4}(a)-\dot{F}_{1,4}(x)$, and $J_2(a)=p_a\dot{F}^*_{2,3}(a)+(1-p_a)\dot{G}^*_{2,3}(a)-\dot{F}^*_{2,3}(x)$.

Remark 4. We consider a further special case of Theorem 3 where all of the initial value x, threshold a and b equal to zero, that is, the investor short(resp. long) one share of portfolio once the spread is larger(resp. less) than 0. For an OU spread process, both of the return and length of a trading cycle equal to 0. Here, for our OUDEJ process, the expected return is $\frac{1}{\gamma}(1-p_a)\left[1+\frac{F_4(a)-G_4(a)}{F_4(a)-H_4(a)}\right]>0$, and the expected length of a trading cycle is $(1-p_a)\int_0^\infty\mathbb{E}_{z+a}\left[\tau_a^-\right]\gamma e^{-\gamma z}\,dz>0$. Hence, the value function reduces to

$$\frac{1}{\gamma}\frac{1+\frac{F_{4}(a)-G_{4}(a)}{F_{4}(a)-H_{4}(a)}}{\int_{0}^{\infty}\mathbb{E}_{z+a}\left[\tau_{a}^{-}\right]\gamma e^{-\gamma z}\,dz},$$

which totally results from jumps .

Proof. We calculate $\mathbb{E}_x\left[r\left(a,b,-b,-a\right)\right]$ at first. Following from (46), (48) is the reduced case of Theorem 1, where $p_a=\mathbb{P}_x\left(X_{\tau_{a,-a}}=a\right)+\mathbb{P}_x\left(X_{\tau_{a,-a}}=-a\right)$. To calculate p_a, we sum the equations for $i=1$ and $i=4$ in (B.35), and (46) yields that (c.f. (20))

$$F_{1,4}\left(x\right)=p_{a}F_{1,4}(a)+\left(1-p_{a}\right)G_{1,4}(a),$$

from which we solve p_a as

$$p_a = \frac{F_{1,4}(x) - G_{1,4}(a)}{F_{1,4}(a) - G_{1,4}(a)}.$$

For the expected length of a trading cycle, we assume the symmetric thresholds assumption in (B.17) and (B.18), and it follows that for any $z \geqslant 0$,

$$\hat{\pi}_{-b}^{(1)}(-a-z) = \check{\pi}_{b}^{(2)}(a+z) \text{ and } \hat{\pi}_{-b}^{(2)}(-a-z) = \check{\pi}_{b}^{(1)}(a+z).$$

Substituting the above relations into (B.13) and (B.14), we have $\mathbb{E}_{a+z}(\tau_b^-) = \mathbb{E}_{-a-z}(\tau_{-b}^+)$ for any $z \geqslant 0$, which implies that

$$\mathbb{E}_a\left[\tau_b^-\right] = \mathbb{E}_{-a}\left[\tau_{-b}^+\right] \text{ and } \int_0^\infty \mathbb{E}_{a+z}\left[\tau_b^-\right]\gamma e^{-\gamma z}\,dz = \int_0^\infty \mathbb{E}_{-a-z}\left[\tau_{-b}^+\right]\gamma e^{-\gamma z}\,dz. \tag{51}$$

The result (49) follows from substituting (51) into (34).

To obtain the expectation of the two-sided first passage time $\mathbb{E}_x\left[\tau_{a,-a}\right]$ in (49), we sum the first and fourth equations of linear system (B.38) to obtain that

$$F_{1,4}(a)\left(\dot{\pi}_0^{(1)} + \dot{\pi}_0^{(3)}\right) + G_{1,4}(a)\left(\dot{\pi}_0^{(2)} + \dot{\pi}_0^{(4)}\right) = p_a\dot{F}_{1,4}(a) + (1-p_a)\dot{G}_{1,4}(a) - \dot{F}_{1,4}(x), \tag{52}$$

and sum the second and third equations in (B.38) to obtain that

$$F_{2,3}^*(a)\left(\dot{\pi}_0^{(1)} + \dot{\pi}_0^{(3)}\right) + G_{2,3}^*(a)\left(\dot{\pi}_0^{(2)} + \dot{\pi}_0^{(4)}\right) = p_a\dot{F}_{2,3}^*(a) + (1-p_a)\dot{G}_{2,3}^*(a) - \dot{F}_{2,3}^*(x), \tag{53}$$

where π_q is defined in (B.21). We can solve $\dot{\pi}_0^{(1)} + \dot{\pi}_0^{(3)}$ and $\dot{\pi}_0^{(2)} + \dot{\pi}_0^{(4)}$ from (52)-(53) and yield (50) via $\mathbb{E}_x\left[\tau_{a,-a}\right] = -\left(\dot{\pi}_0^{(1)} + \dot{\pi}_0^{(3)}\right) - \left(\dot{\pi}_0^{(2)} + \dot{\pi}_0^{(4)}\right)$. □

4 Numerical analysis

In this section, we address the following three numerical issues in the pairs trading employing the analytic value function (18) obtained in Section 3:

1. The impact of trading thresholds on the value function for a given pair;
2. The effects of incorporating jumps into the model setup and value function;
3. The sensitivity analysis of model parameters on strategy performance.

Throughout this section, we follow the symmetric assumption as in Section 3.4 for model parameters. Specifically, we assume that the spread process X_t is given by (1) and (2) with $X_0 = 0$ and the benchmark parameter setting[4] as

$$\kappa = 0.2,\ \alpha = 0,\ \mu = 0,\ \sigma = 0.1,\ \lambda = 5,\ p = 0.5,\ \gamma_1 = \gamma_2 = 20. \tag{54}$$

[4]By calculating the first-order and second-order moments of X_t, we can show that the benchmark parameter setting agrees with the common sense in market. Specifically, from ABC-ICBC spread minute-by-minute data in 2011 analyzed in Section 2.1, we calculate the daily mean and standard derivation as 0 and 0.3. In the other hand, from the results (2.3) in Zhou et al. (2017), we can calculate the expectation and variance of X_t.

Further, we adopt the symmetric assumption on the thresholds, i.e., letting $a = -d$ and $b = -c$ in the study of the above three questions. In this context, the pairs trading strategy is completely characterized by the threshold a and the difference of thresholds $d_{ab} := a - b$, since the value function defined in (7) can be rewritten as $v_0 (a, a - d_{ab}, d_{ab} - a, -a)$, denoted by $v(a, d_{ab})$ for simplicity. Accordingly, we denote the return and length of a trading cycle by $r(a, d_{ab})$ and $L(a, d_{ab})$.

4.1 Strategy performance

In this section, we study the strategy performance in terms of the expected return $\mathbb{E}[r(a, d_{ab})]$, the expected length of a trading cycle $\mathbb{E}[L(a, d_{ab})]$ as well as their ratio $v(a, d_{ab})$ under distinct values of trading thresholds. Specifically, we choose that $a \in [0.001, 0.5]$ and $d_{ab} \in [0.001, \min\{0.5, 2a\}]$ (so as to make $-b, b \in [-a, a]$) with step size selected as 0.001 respectively. Under the parameter setting (54), $v(a, d_{ab})$ reaches its maximum 0.045 when $a = 0.054$ and $d_{ab} = 0.046$. Hence, in this context the investor should adopt the thresholds $a = 0.054$, $b = 0.008$, $c = -0.008$, $d = -0.054$. Figure 2 shows the variation of value function $v(a, d_{ab})$ w.r.t. a and d_{ab}. When we fix d_{ab} (resp. a) and adjust a (resp. d_{ab}), we find $v(a, d_{ab})$ first increases until reaches the maximum, then turns to decrease monotonously, and the optimal a (resp. d_{ab}) maximizing the value function increases along with the rise of d_{ab} (resp. a).

We take sight into the variation of the expected return $\mathbb{E}[r(a, d_{ab})]$ and the expected length of a trading cycle $\mathbb{E}[L(a, d_{ab})]$ respectively. As shown in the upper subplots in Figure 3, $\mathbb{E}[r(a, d_{ab})]$ increases as each of a and d_{ab} rises, and tends to the limit $d_{ab} + 1/\gamma$, which is the summation of the threshold difference and expectation of the jump size. The lower subplots in Figure 3 illustrates that the slope of the curve of $\mathbb{E}[L(a, d_{ab})]$ w.r.t. a or d_{ab} increases gradually from a relatively small value. Hence, with either a or d_{ab} increasing, the rise of $\mathbb{E}[r(a, d_{ab})]$ first dominates the rise of $\mathbb{E}[L(a, d_{ab})]$ to raise the value function, then the relation reverses to bring down the value function.

$$
\begin{aligned}
\mathbb{E}_x(X_t) &= \lim_{q \downarrow 0} \frac{\partial}{\partial q} \ln \left[\mathbb{E}_x \left(e^{q X_t} \right) \right] = e^{-\kappa t} x + \left(1 - e^{-\kappa t}\right) \left[\alpha + \frac{1}{\kappa} \left(\mu + \frac{\lambda p}{\gamma_1} - \frac{\lambda (1-p)}{\gamma_2} \right) \right], \\
Var_x(X_t) &= \lim_{q \downarrow 0} \frac{\partial^2}{\partial q^2} \ln \left[\mathbb{E}_x \left(e^{q X_t} \right) \right] - \mathbb{E}_x^2 (X_t) = \left(1 - e^{-2\kappa t}\right) \left[\frac{\sigma^2}{2\kappa} + \frac{\lambda}{\kappa} \left(\frac{p}{\gamma_1^2} + \frac{1-p}{\gamma_2^2} \right) \right].
\end{aligned}
$$

When t is large enough, the asymptotic expectation and standard derivation are

$$
\lim_{t \to \infty} \mathbb{E}_x(X_t) = \alpha + \frac{1}{\kappa} \left(\mu + \frac{\lambda p}{\gamma_1} - \frac{\lambda (1-p)}{\gamma_2} \right), \text{ and } \lim_{t \to \infty} SD_x(X_t) = \left[\frac{\sigma^2}{2\kappa} + \frac{\lambda}{\kappa} \left(\frac{p}{\gamma_1^2} + \frac{1-p}{\gamma_2^2} \right) \right]^{1/2}.
$$

Using empirical results, we set the benchmark parameters subject to that the asymptotic mean and standard derivation equal to 0 and 0.3, respectively.

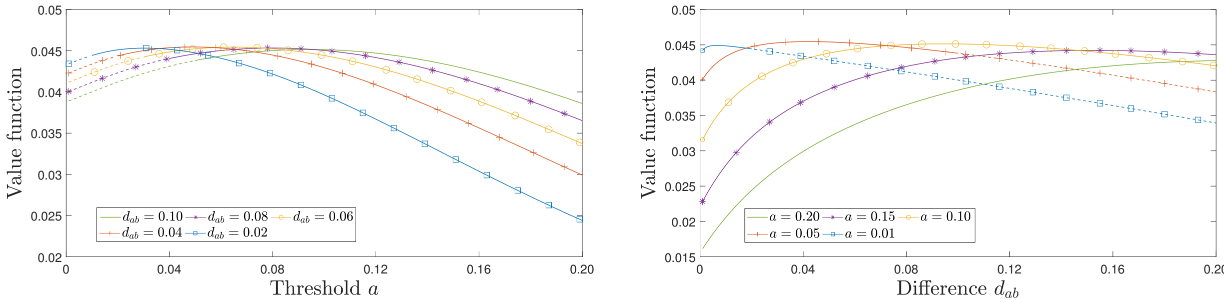


Figure 2: Variation of the value function with threshold a and difference d_{ab}: Left subplot for the sectional view with fixed a, and right subplot for the sectional view with fixed d_{ab}, respectively. We impose the assumption that $b, c \in [d, a]$, and in order to facilitate the observation, we use the dashed line to represent the results for thresholds which do not satisfy this restriction (the same below).

4.2 Effects of incorporating jumps

In this section, we focus on the effects of incorporating jumps on both of the value function (18) and the expectation of profitability defined in Proposition 1. We first calculate the value function with optimal trading thresholds under continuous OU process, and compare it with the counterpart under OUDEJ model. Then, in order to highlight the contribution of jumps in the profitability, we calculate the excess return and excess return ratio caused by jump under different parameter settings.

4.2.1 Effects of jumps on the value function

In this part, we analyze the effect of jumps through the value functions equipped with optimal thresholds under two spread processes (OUDEJ and OU processes). Recalling that in (8), the optimal threshold under OUDEJ model is obtained by maximizing the value function defined in (7). Here, for comparison, we consider another optimal threshold by assuming the spread process X_t follows an OU process (a special case of OUDEJ process without jumps). We then substitute these two optimal thresholds into the value function (18) which is governed by the OUDEJ process. Obviously, the optimal threshold from OU process is inferior to the former one from OUDEJ process. The comparison between the two results reveals the impact of jumps on the value function.

To conduct the above numerical analysis, it is essential to calculate the value function (18) in the context of OU process. We notice that $\mathbb{E}_x [r(a, b, c, d)]$ under OU process equals to the difference between thresholds, so it suffices to calculate $\mathbb{E}_x [L(a, b, c, d)]$, which is summarized by Theorem 7 in Appendix B.4. Then under different parameter settings, we adjust the diffusion parameter in the OU model to make its first- and second- order moments equal to those of corresponding OUDEJ process, and use this modified parameters to obtain the optimal thresholds under OU process.

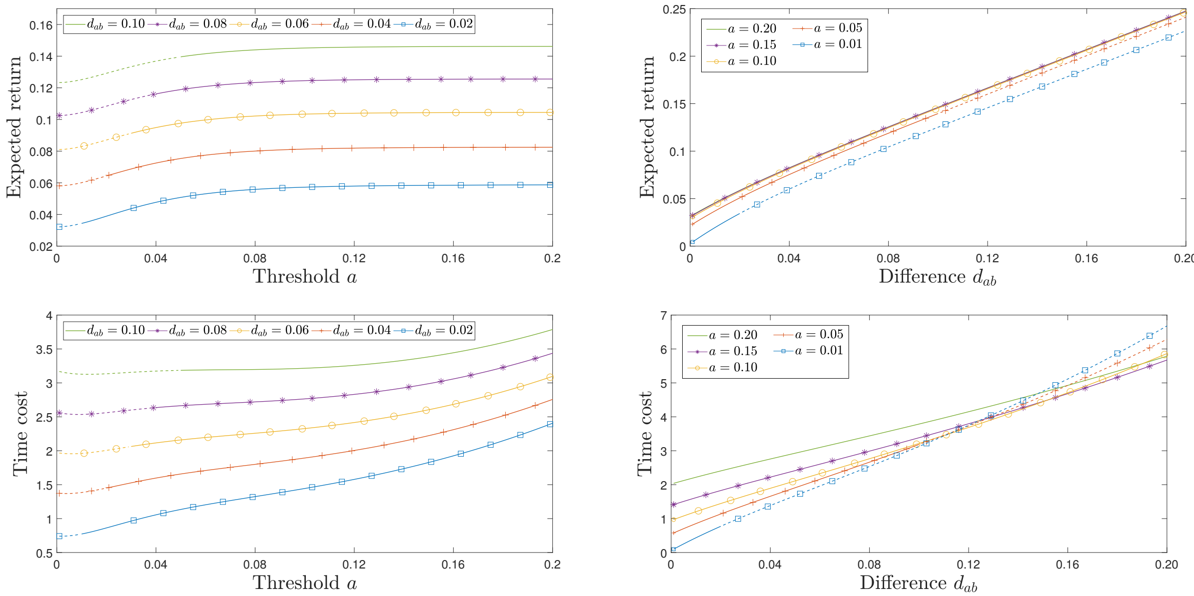


Figure 3: Variation of the expected return (upper subplots) and the expected length of a trading cycle (lower subplots) w.r.t. threshold a and difference d_{ab}.

In Figure 4, We plot the variation of value function (18) calculated with the two threshold settings specified above. The discrepancy between value functions under two models becomes larger when the diffusion parameter σ in OUDEJ model is relatively small, or either one of the mean reversion speed κ, the jump size $1/\gamma$ and jump intensity λ is relatively large. For example, in the upper right subplot in Figure 4, when we take $\gamma = 10$, the value function in terms of optimal threshold set under OUDEJ process is 5.45% higher than the counterpart under OU process.

4.2.2 Contribution of jumps on the profitability

In this section, we analyze the contribution of jumps on improving profitability for our trading strategy, which justifies the necessity of extending the definition of expected return as the difference between thresholds in earlier works such as Endres and Stübinger (2019).

As stated in Theorem 1, the jumps can lead to an excess return, $r_e(a, d_{ab}) := r(a, d_{ab}) - d_{ab}$, beyond the width of trading interval in the pairs trading strategy. We use the expected excess return $\mathbb{E}_x[r_e]$ to measure the contribution of jumps, and also consider the expected excess return ratio as $\mathbb{E}_x[r_r(a, d_{ab})] := \mathbb{E}_x[r_e(a, d_{ab})/r(a, d_{ab})] = \mathbb{E}_x[(r(a, d_{ab}) - d_{ab})/r(a, d_{ab})]$.

Figure 5 illustrates the contribution of jumps on the profitability under different values of threshold a and difference d_{ab} with the benchmark parameter (54). The upper subplots in Figure 5 show that the increase in either threshold a or difference d_{ab} raises $\mathbb{E}[r_e(a, d_{ab})]$, since the probability of crossing by jump rises in both cases. We further observe that with a and d_{ab} increasing, $\mathbb{E}[r_e(a, d_{ab})]$

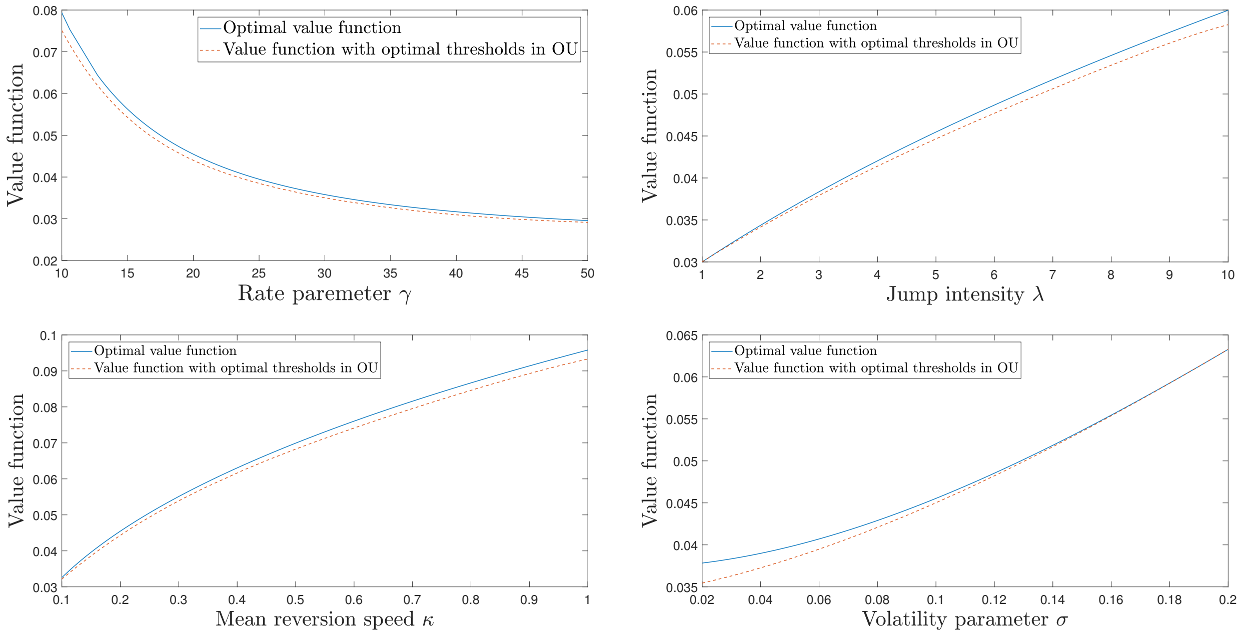


Figure 4: Value functions under optimal thresholds in OU and OUDEJ model.

tends to the expectation of jump size $1/\gamma$, which is 0.05 in our context. The reason is that when the threshold interval is wide enough to be far away from the mean reversion level α, most likely the mean reversion property would enable the spread revert before crossing the thresholds, so the crossing of threshold is almost caused by jump.

The expected excess return ratio is exhibited in the lower subplots of Figure 5. On the whole, jumps contribute to at least 10% of the strategy return, and the proportion is over 50% when the difference of thresholds d_{ab} narrows. Especially, in the optimal strategy, 44.54% of expected return comes from the jumps. Hence, in practice it is meaningful for the pairs trading investor to exploit the effects of jumps.

Figure 6 shows the contribution of jumps on the profitability w.r.t. different parameter settings, similarly conducted as in Section 4.2.1. For the jump parameters γ and λ, both the expected excess return $\mathbb{E}\left[r_e\left(a, d_{ab}\right)\right]$ and the expected excess return ratio $\mathbb{E}\left[r_r\left(a, d_{ab}\right)\right]$ increase when $1/\gamma$ or λ rises, and $\mathbb{E}\left[r_r\left(a, d_{ab}\right)\right]$ nearly maintains above 10%, which is the ratio of $1/\gamma$ and the maximum value of d_{ab}. For parameters σ and κ, $\mathbb{E}\left[r_e\left(a, d_{ab}\right)\right]$ decreases when they become larger, as the probability of crossing by jump decreases[5]. In contrast, $\mathbb{E}\left[r_r\left(a, d_{ab}\right)\right]$ significantly declines when σ increases, and rises slightly when κ increases. It should be noted that $\mathbb{E}\left[r_e\left(a, d_{ab}\right)\right]$ and $\mathbb{E}\left[r_r\left(a, d_{ab}\right)\right]$ present opposite changing directions for κ, which is different from the other three parameters. Indeed, the increase in $\mathbb{E}\left[r_r\left(a, d_{ab}\right)\right]$ as κ rises is because that the difference of optimal thresholds narrows more significantly than the expected excess return. In conclusion, the above analyses reveal the

[5]This point will be explained in detail in next section.

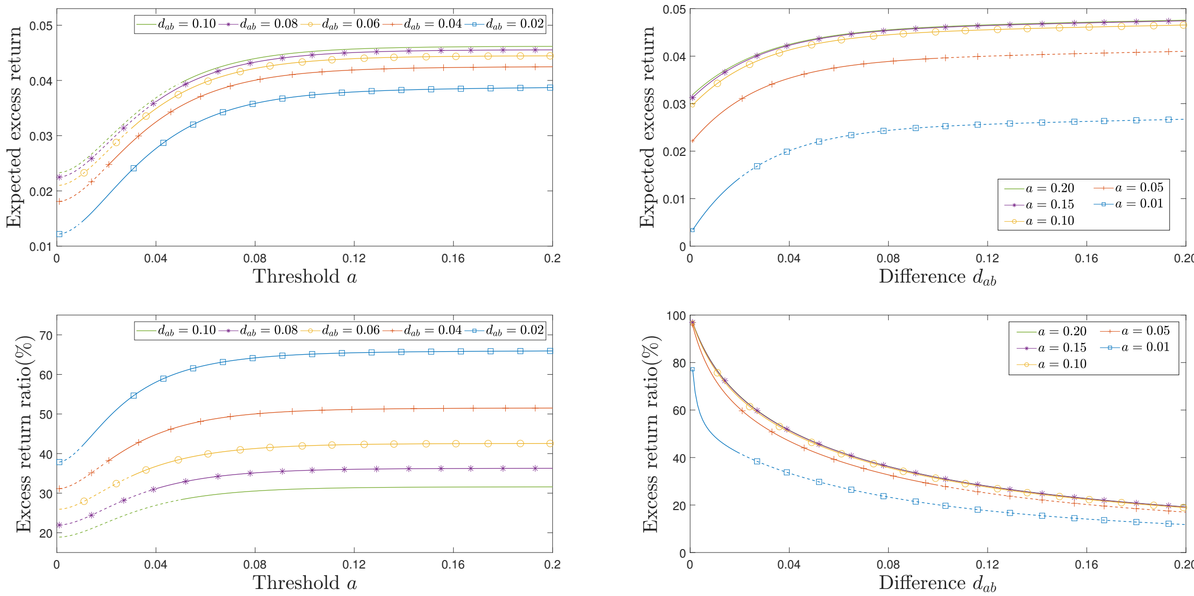


Figure 5: Contribution of jumps on profitability: Upper subplots for the expected excess return and lower subplots for the ratio of expected excess return in the total expected return.

indispensable role of jumps on the profitability under various circumstances.

4.3 Sensitivity analysis

The performance of pairs trading strategy is much affected by the model parameters of the spread dynamics, which characterize the underlying pair. In our model setup (1) and (2) under symmetric assumptions, κ measures the mean reversion speed of the spread, σ measures the random fluctuations caused by the continuous movement in the spread process, $1/\gamma$ is the expectations of positive and negative jump size respectively, and λ reflects the frequency of two-sided jumps. In this section, we make a detailed analysis on how each parameter mentioned above influences the optimal trading thresholds a and b, and the corresponding strategy performance such as the expected return $\mathbb{E}\left[r\left(a, d_{ab}\right)\right]$, the expected length of a trading cycle $\mathbb{E}\left[L\left(a, d_{ab}\right)\right]$, and their ratio $v\left(a, d_{ab}\right)$. To perform the sensitivity analysis for a given parameter, we fix other parameters as the benchmark parameters setting in (54).

4.3.1 Rate parameter γ

The numerical results in Figure 7 show how the jump size rate γ affects the performance of pairs trading strategy. As γ increases from 10 to 100, all of the optimal thresholds a and b, the difference d_{ab}, the expected return $\mathbb{E}\left[r\left(a, d_{ab}\right)\right]$, the expected length of trading cycle $\mathbb{E}\left[L\left(a, d_{ab}\right)\right]$ and $v(a, d_{ab})$ decrease monotonically. Intuitively, $1/\gamma$ is the expectation of two-sided jump size. Smaller γ can

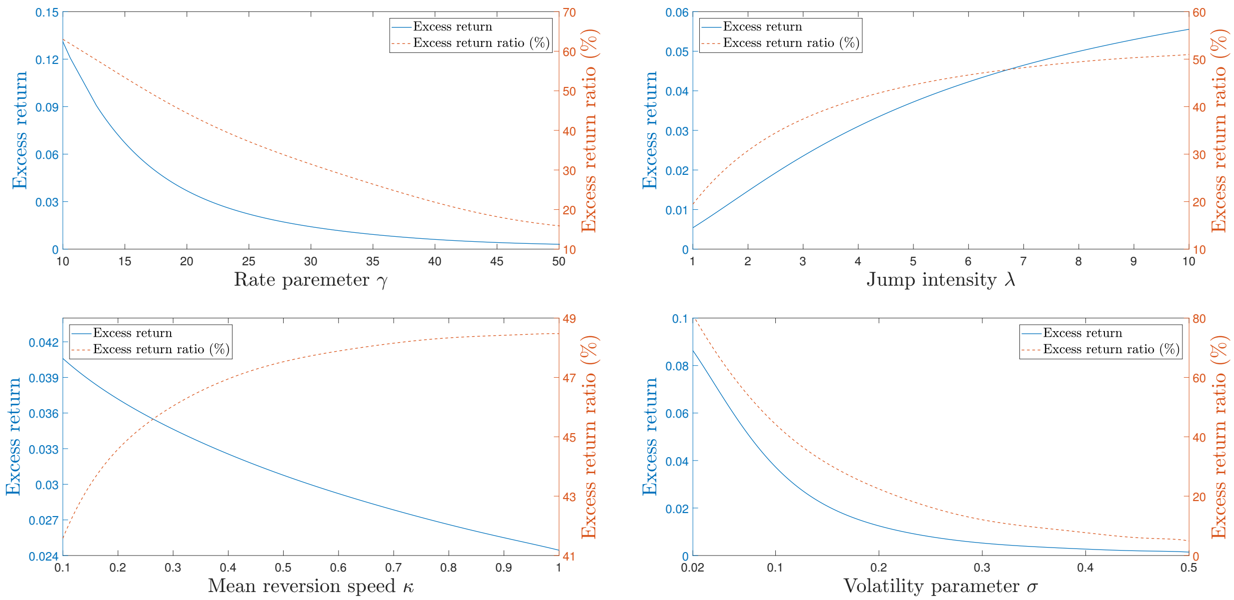


Figure 6: Contribution of jumps on profitability with respect to different parameter settings.

generate a larger return through the overshoot or undershoot in occurrence of crossing the thresholds by jump. Meanwhile, the broadening thresholds occur from the investor's objective to rise the probability of crossing the thresholds by jump, so as to increase the expected return. Evidently, larger discrepancy between thresholds produces longer trading time. From the left plot in Figure 7, we see that when γ decreases, the rising expected return dominates longer trading time, and thus increases the value function. In conclusion, the investor should select stock pairs with larger jump size, so as to make use of more excess return caused by jump of the spread to maximize the value function.

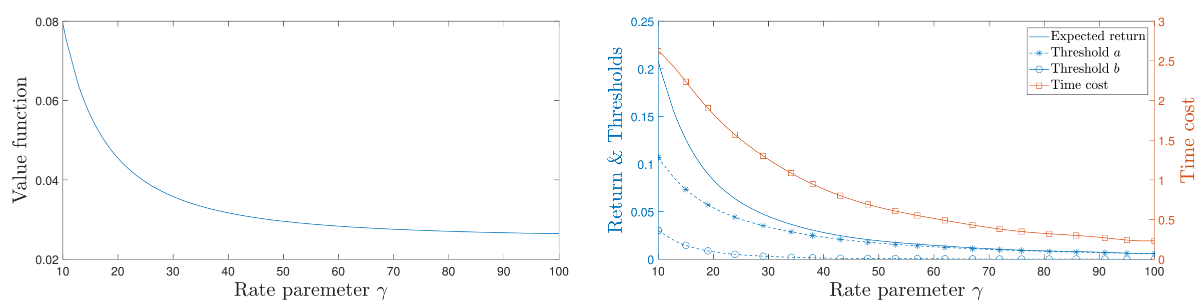


Figure 7: Sensitivity of rate parameter γ: Left subplot for the value function, and right subplot for the optimal threshold a, b, the expected return and the expected length of a trading cycle (time cost).

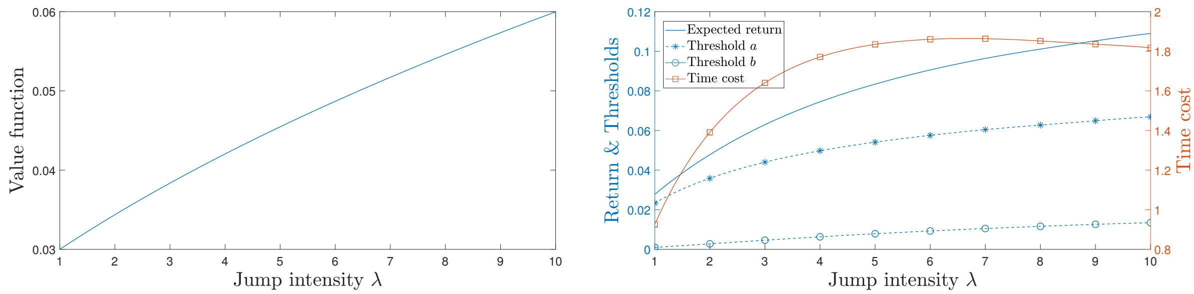


Figure 8: Sensitivity of jump intensity λ: Left subplot for the value function, and right subplot for the optimal threshold a, b, the expected return and the time cost.

4.3.2 Jump intensity λ

Figure 8 shows that the value function $v(a, d_{ab})$ is a monotonically increasing function of jump intensity λ. Besides, as λ increases from 1.0 to 10.0, the optimal thresholds a, b, the difference d_{ab}, $\mathbb{E}\left[r\left(a, d_{ab}\right)\right]$ and $\mathbb{E}\left[L\left(a, d_{ab}\right)\right]$ increase monotonically. Larger λ induces more jumps, expanding the extent of random fluctuations caused by the two-sided jumps. Hence when λ increases, the optimal thresholds a and b are raised to increase the probabilities to jump across the thresholds, so as to increase the excess return caused by jump. Correspondingly, wider threshold interval rises the trading time to some extent. Since the increment of excess return dominates the increasing length of a trading circle, the ratio $v(a, d_{ab})$ rises. Thus, stock pairs with more jump frequency should be preferred in the pairs trading strategy, which can result in more frequent trades with excess return by jump.

4.3.3 Mean reversion speed κ

Figure 9 illustrates the variation of strategy performance with mean reversion speed κ. As κ increases from 0.1 to 1.0, the optimal thresholds a and b, the difference d_{ab}, the expected return $\mathbb{E}\left[r\left(a, d_{ab}\right)\right]$ and the expected length of a trading cycle $\mathbb{E}\left[L\left(a, d_{ab}\right)\right]$ decrease monotonically, while $v(a, d_{ab})$ increases monotonically. As κ measures the speed of mean reversion, larger κ would make the spread revert faster when it deviates from its long-term mean α at a large distance, resulting in a smaller deviated range of the spread. Hence, the trading thresholds should be closer to the long-term mean in pursuit of less time cost. Although $\mathbb{E}\left[r\left(a, d_{ab}\right)\right]$ also decreases as the thresholds are closer to α, smaller $\mathbb{E}\left[L\left(a, d_{ab}\right)\right]$ dominates the decreasing return, which finally raises the ratio $v(a, d_{ab})$. In other words, the investor had better select stock pairs with stronger mean reversion speed, in order to strive for higher trading frequency at the cost of an insignificant reduction in profitability.

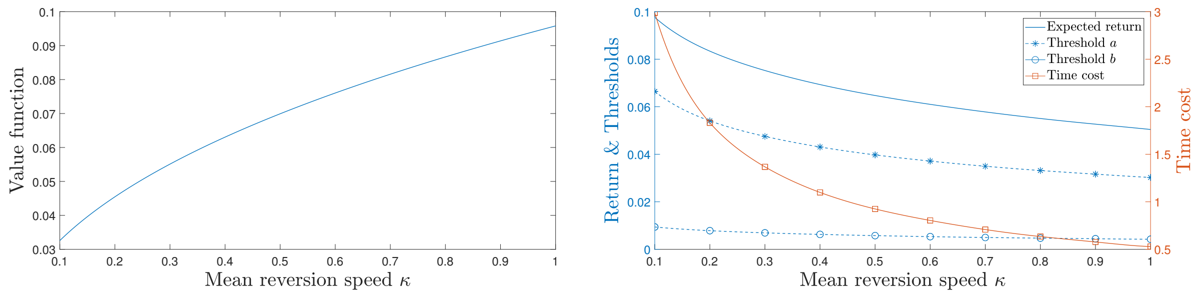


Figure 9: Sensitivity of mean reversion speed κ: Left subplot for the value function, and right subplot for the optimal threshold a, b, the expected return and the time cost.

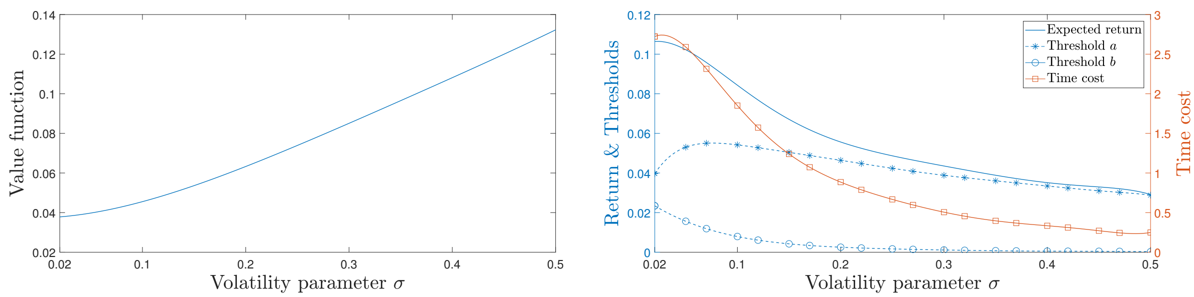


Figure 10: Sensitivity of diffusion parameter σ from 0.02 to 0.5: Left subplot for the value function, and right subplot for the optimal threshold a, b, the expected return and the time cost.

4.3.4 Diffusion parameter σ

The variation of strategy performance with diffusion parameter σ is shown in Figure 10. As σ increases from 0.02 to 0.5, the expected return $\mathbb{E}\left[r\left(a, d_{ab}\right)\right]$ and the expected length of a trading cycle $\mathbb{E}\left[L\left(a, d_{ab}\right)\right]$ decrease monotonically, while $v\left(a, d_{ab}\right)$ increases monotonically. Since σ measures the extent of random fluctuations caused by Brownian motion, larger σ enhances the oscillation of the spread process. Thus the optimal threshold interval is narrowed in order to further reduce $\mathbb{E}\left[L\left(a, d_{ab}\right)\right]$. Although this will also reduce $\mathbb{E}\left[r\left(a, d_{ab}\right)\right]$, the saving of time dominates and thus finally increases the ratio $v(a, d_{ab})$.

In conclusion, for the parameters γ and λ related to jumps, the rise of either $1/\gamma$ or λ contributes to more expected return at the cost of a relative small increment of expected length of a trading cycle, which produces higher $v(a, d_{ab})$. In contrast, for the parameters κ and σ, the rise of either parameter can induce an optimal strategy with higher $v(a, d_{ab})$, by shortening the trading time at the cost of relatively small reduction of expected return. In reality, such parameters can be fitted from market data and used to select optimal stock pairs. Thus, the investor could exploit these parameter information to construct more effective pairs of stocks for profit.

5 Conclusions

In this paper, we derive an analytic value function for the evaluation of pairs trading strategy, modeling the spread as a Lévy-driven OU process with two-sided jumps. The effect of jump size is embedded into the value function and represented as a new part with theoretical obstacle. Under a symmetric jump case, we obtain the optimality of symmetric thresholds of pairs strategy, and illustrate that the jump component can significantly affect the strategy performance.

Through numerical analysis, we get optimal thresholds by the analytic value function instead of Monte Carlo method extensively employed in literature. Further, we obtain precisely the contribution of the jump part in the value function. By the sensitivity analysis, we find that higher jump frequency and amplitude result in higher value function, where more profit dominates increasing length of a trading cycle. However, the sensitivities of volatility and reversion parameter still exist and are consistent with that of pure continuous models.

For further research, a large-scale empirical analysis may be considered, where the effect of jump could be highlighted. Second, the regime switching model may be explored to reflect different market conditions. Third, the theoretical results on the OUDEJ model could be applied in other fields, such as option pricing.

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Appendix A BN-S methodology

We use the BN-S methodology to test the evidence of the jump term in the stock spread model. The BN-S methodology is first proposed in Barndorff-Nielsen and Shephard (2004), and is by far the most developed and widely used method in the jumps testing. Suppose that on any trading day t, there are M returns $r_{t,j} = p\left(t-1+\frac{j}{M}\right) - p\left(t-1+\frac{j-1}{M}\right)$, for $j = 1, 2, \cdots, M$. The BN-S methodology measures discrepancy between realized variance RV_t and bipower variation BV_t, where

$$RV_t = \sum_{j=1}^{M} r_{t,j}^2, BV_t = \frac{\pi}{2}\frac{M}{M-1}\sum_{j=2}^{M} |r_{t,j-1}| \, |r_{t,j}| \, . \tag{A.1}$$

From (A.1) we define the following relative jump measure $RJ_t = (RV_t - BV_t)/RV_t$. After studentizing RJ_t, the test statistic is defined as

$$z_t = \frac{RJ_t}{\sqrt{\frac{1}{M}\left(\frac{\pi^2}{4}+\pi-5\right)\max\left(1, \frac{TP_t}{BV_t^2}\right)}},$$

where the tripower quarticity $TP_t = \frac{M^2}{M-2}\left[\Gamma\left(\frac{7}{6}\right)\Gamma\left(\frac{1}{2}\right)\right]^{-3}\sum_{j=3}^{M}|r_{t,j-2}r_{t,j-1}r_{t,j}|^{4/3}$. Then, under the null hypothesis of no jumps on day t, z_t converges in distribution to a standard normal random variable. Hence, we reject the null hypothesis at α confidence level if $|z_t| > \Phi^{-1}_{1-\frac{\alpha}{2}}$, where Φ is the cumulative normal distribution function.

Appendix B Preliminaries on OUDEJ

To calculate the analytic value function in (18), we need to obtain the probabilities of OUDEJ process to cross the thresholds continuously and by jump, and the expectations of one-sided and two-sided first passage times. All these results are related to the Laplace transform of first passage time of OUDEJ process. In this Appendix section, we collect all the background mathematical preliminaries to derive this Laplace transform and obtain the corresponding results.

Appendix B.1 Conditional memoryless property

To facilitate the derivation of Laplace transforms of first passage times of OUDEJ process in (1), we first study its conditional memoryless property in this section. Kou and Wang (2003) derive the conditional memoryless property and conditional independence of the double exponential jump-diffusion process (2), which is the sum of independently drifted Brownian motion and compound Poisson process. However, in our context, the generalized OU Process cannot be decomposed in the same way as Kou and Wang (2003). Hence, we are enlightened by Pitman (1981) and Kijima and Siu (2014) to make path decomposition, and derive the conditional memorylessness of our model.

Lemma 1. *The OUDEJ process in (1) exhibits the conditional memoryless property, namely, for any $y > 0$,*

$$\mathbb{P}_x\left(X_{\tau_a^+}-a>y\middle|\, X_{\tau_a^+}>a\right)=e^{-\gamma_1 y}, \tag{B.2}$$

$$\mathbb{P}_x\left(d-X_{\tau_d^-}>y\middle|\, X_{\tau_d^-}<d\right)=e^{-\gamma_2 y}. \tag{B.3}$$

Proof. Denote the arrival times of the Poisson process N by $\{T_n, n=1,2,\ldots\}$ and $\mathcal{F}_t=\sigma(X_s : s\leqslant t)$, $\mathbb{P}_n := \mathbb{P}_x\left(T_n=\tau_a^+\leqslant t, X_{\tau_a^+}-a>y\right)$ for $n=1,2,\ldots$. Since $\{T_n=\tau_a^+\}\subset\left\{\max_{0\leqslant s<T_n} X_s<a\right\}\in\mathcal{F}_{T_n-}$, it follows that

$$\begin{aligned}
\mathbb{P}_n=&\mathbb{P}_x\left(\max_{0\leqslant s<T_n} X_s<a, X_{T_n}-a>y, T_n\leqslant t\right)\\
=&\mathbb{E}_x\left\{\mathbb{P}_x\left(X_{T_n}-a>y\middle|\, \mathcal{F}_{T_n-}, T_n\right)\mathbb{1}\left\{\max_{0\leqslant s<T_n} X_s<a, T_n\leqslant t\right\}\right\}\\
=&\mathbb{E}_x\left\{\mathbb{P}_x\left(X_{T_n-}+Y_{N_{T_n}}-a>y\middle|\, \mathcal{F}_{T_n-}, T_n\right)\mathbb{1}\left\{\max_{0\leqslant s<T_n} X_s<a, T_n\leqslant t\right\}\right\}.
\end{aligned}$$

Under the OUDEJ process, $Y_{N_{T_n}}=X_{T_n}-X_{T_n-}$ follows a double-exponential distribution, so

$$\begin{aligned}
\mathbb{P}_n=&e^{-\gamma_1 y}\mathbb{E}_x\left\{pe^{-\gamma_1(a-X_{T_n-})}\mathbb{1}\left\{\max_{0\leqslant s<T_n} X_s<a, T_n\leqslant t\right\}\right\}\\
=&e^{-\gamma_1 y}\mathbb{E}_x\left\{\mathbb{P}_x\left(X_{T_n}-a>0\middle|\, \mathcal{F}_{T_n-}, T_n\right)\mathbb{1}\left\{\max_{0\leqslant s<T_n} X_s<a, T_n\leqslant t\right\}\right\}\\
=&e^{-\gamma_1 y}\mathbb{P}_x\left(X_{T_n}-a>0, T_n=\tau_a^+\leqslant t\right).
\end{aligned}$$

Note that the event $\{X_{\tau_a^+}>a\}$ only occurs at the arrival times of N, and we have

$$\mathbb{P}_x\left(\tau_a^+\leqslant t, X_{\tau_a^+}-a>y\right)=\sum_{n=1}^{\infty}\mathbb{P}_n=e^{-\gamma_1 y}\mathbb{P}_x\left(\tau_a^+\leqslant t, X_{\tau_a^+}-a>0\right).$$

Then by similar deductions to that of Proposition 2.1 in Kou and Wang (2003), we can finally obtain the conditional memoryless property for the OUDEJ process. □

Appendix B.2 Results on one-sided exit problem

In this section, we consider the one-sided exit problem of OUDEJ process in (1). Based on the Laplace transforms of first passage times, we first calculate the probabilities of the overshoot and undershoot, and then we obtain the expectations of one-sided first passage times.

Appendix B.2.1 Probabilities of the overshoot and undershoot

In Zhou et al. (2017), they derive the joint Laplace transforms of τ_a^+ and the overshoot $X_{\tau_a^+}-a$ as well as τ_d^- and undershoot $d-X_{\tau_d^-}$ for the OUDEJ process. As we are only concerned about the distributions of first passage times, we obtain the Laplace transforms of first passage times by simply taking $\xi=0$ and $\rho=0$ in Lemma 2.1 and Lemma 2.2 of Zhou et al. (2017) as follows.

Lemma 2. *For $x < a$, $q > 0$,, $0 < p \leqslant 1$, the Laplace transform of τ_a^+ defined in (5) is solved by*

$$\mathbb{E}_x\left[e^{-q\tau_a^+}\right] = \mathbb{E}_x\left[e^{-q\tau_a^+}; X_{\tau_a^+} = a\right] + \mathbb{E}_x\left[e^{-q\tau_a^+}; X_{\tau_a^+} > a\right], \tag{B.4}$$

with the two expectations in RHS of (B.4) can be solved by

$$\begin{pmatrix} F_1(a,q) & G_1(a,q) \\ F_2(a,q) & G_2(a,q) \end{pmatrix} \begin{pmatrix} \mathbb{E}_x\left[e^{-q\tau_a^+}; X_{\tau_a^+} = a\right] \\ \mathbb{E}_x\left[e^{-q\tau_a^+}; X_{\tau_a^+} > a\right] \end{pmatrix} = \begin{pmatrix} F_1(x,q) \\ F_2(x,q) \end{pmatrix}, \tag{B.5}$$

where $F_i(\cdot,q)$ and $G_i(\cdot,q)$ for $i = 1,2$ are defined in (19).

Lemma 3. *For $x > d$, $q > 0$,, $0 \leqslant p < 1$, the Laplace transform of τ_d^- defined in (5) is solved by*

$$\mathbb{E}_x\left[e^{-q\tau_d^-}\right] = \mathbb{E}_x\left[e^{-q\tau_d^-}; X_{\tau_d^-} = d\right] + \mathbb{E}_x\left[e^{-q\tau_d^-}; X_{\tau_d^-} = d\right], \tag{B.6}$$

with the two expectations in RHS of (B.6) can be solved by

$$\begin{pmatrix} F_3(d,q) & H_3(d,q) \\ F_4(d,q) & H_4(d,q) \end{pmatrix} \begin{pmatrix} \mathbb{E}_x\left[e^{-q\tau_d^-}; X_{\tau_d^-} = d\right] \\ \mathbb{E}_x\left[e^{-q\tau_d^-}; X_{\tau_d^-} < d\right] \end{pmatrix} = \begin{pmatrix} F_3(x,q) \\ F_4(x,q) \end{pmatrix},$$

where $F_i(\cdot,q)$ and $H_i(\cdot,q)$ for $i = 3,4$ are defined in (19).

Further, the monotone convergence theorem yields the probabilities for X_t to cross the threshold continuously and by jump by letting $q \downarrow 0$ in Lemma 2 and Lemma 3.

Theorem 4. *For $x < a$ and $0 < p \leqslant 1$, the probabilities of a continuous upcrossing and an overshoot are*

$$\mathbb{P}_x\left(X_{\tau_a^+} = a\right) = \frac{F_1(x) - G_1(a)}{F_1(a) - G_1(a)} \text{ and } \mathbb{P}_x\left(X_{\tau_a^+} > a\right) = \frac{F_1(a) - F_1(x)}{F_1(a) - G_1(a)}. \tag{B.7}$$

For $x > d$ and $0 \leqslant p < 1$, the probabilities of a continuous downcrossing and an undershoot are

$$\mathbb{P}_x\left(X_{\tau_d^-} = d\right) = \frac{F_4(x) - H_4(d)}{F_4(d) - H_4(d)} \text{ and } \mathbb{P}_x\left(X_{\tau_d^-} < d\right) = \frac{F_4(d) - F_4(x)}{F_4(d) - H_4(d)}. \tag{B.8}$$

Here, $G_1(\cdot)$, $H_4(\cdot)$ and $F_i(\cdot)$ for $i = 1,4$ are defined in (21).

Proof. It follows directly from the monotone convergence theorem that

$$\mathbb{P}_x\left(X_{\tau_a^+} = a\right) = \lim_{q\downarrow 0}\mathbb{E}_x\left[e^{-q\tau_a^+}; X_{\tau_a^+} = a\right] \text{ and } \mathbb{P}_x\left(X_{\tau_a^+} > a\right) = \lim_{q\downarrow 0}\mathbb{E}_x\left[e^{-q\tau_a^+}; X_{\tau_a^+} > a\right].$$

Since $F_1(a,q)$ and $G_1(a,q)$ are uniformly convergent as $q \downarrow 0$, we obtain that in (B.5)

$$F_1(a)\mathbb{P}_x\left(X_{\tau_a^+} = a\right) + G_1(a)\mathbb{P}_x\left(X_{\tau_a^+} > a\right) = F_1(x). \tag{B.9}$$

However, the function $F_2(a,q)$ and $G_2(a,q)$ diverge as $q \downarrow 0$. Fortunately, we can exploit the fact

$$\mathbb{P}_x\left(X_{\tau_a^+} > a\right) + \mathbb{P}_x\left(X_{\tau_a^+} = a\right) = 1. \tag{B.10}$$

Thus, the two upcrossing probabilities in (B.7) follow from (B.9) and (B.10). Analogically, the downcrossing probabilities in (B.8) can be similarly solved by

$$\begin{cases} H_4(d)\mathbb{P}_x\left(X_{\tau_d^-} < d\right) + F_4(d)\mathbb{P}_x\left(X_{\tau_d^-} = d\right) = F_4(x), \\ \mathbb{P}_x\left(X_{\tau_d^-} < d\right) + \mathbb{P}_x\left(X_{\tau_d^-} = d\right) = 1, \end{cases}$$

which concludes the proof. $\square$

Remark 5. Here we consider two special cases with only one-sided jumps in the driven process L_t. Recall the notations in Section 3.1, for sake of convenience, we define two new integral intervals

$$\Gamma_5 := \Gamma_1 \cup \Gamma_2 \text{ and } \Gamma_6 := \Gamma_3 \cup \Gamma_4, \tag{B.11}$$

and define for $i = 5, 6$,

$$F_i(x, q) := \int_{\Gamma_i} \psi(z, q) e^{-(x-\alpha)z} \, dz.$$

When $p = 0$, the OUDEJ model reduces to a generalized OU process with only negative jumps, and the results above can not hold since the method of integration by parts (2.14) in Zhou et al. (2017) can not be applied directly. Here, we replace the integral interval Γ_1 by Γ_5 in the proof of Lemma 2.1 in Zhou et al. (2017), and repeat the derivation therein to get the Laplace transform of τ_a^+ from

$$F_5(x, q) = F_5(a, q) \, \mathbb{E}_x \left[e^{-q\tau_a^+}; X_{\tau_a^+} = a \right] = F_5(a, q) \, \mathbb{E}_x \left[e^{-q\tau_a^+} \right]. \tag{B.12}$$

Similarly, when $p = 1$, the OUDEJ model has only positive jumps, and we obtain the Laplace transform of τ_d^- from

$$F_6(x, q) = F_6(d, q) \, \mathbb{E}_x \left[e^{-q\tau_d^-}; X_{\tau_d^-} = d \right] = F_6(d, q) \, \mathbb{E}_x \left[e^{-q\tau_d^-} \right].$$

Appendix B.2.2 Expectations of first passage times

In this part we focus on the expectations of one-sided first passage times, which measure how long the spread will reach the given thresholds on average. Obviously, the expectations of one-sided first passage times τ_a^+ and τ_d^- defined in (5) can be represented by

$$\mathbb{E}_x \left[\tau_a^+ \right] = \hat{\pi}_a^{(1)}(x) + \hat{\pi}_a^{(2)}(x), \text{ for } x < a, \tag{B.13}$$

$$\mathbb{E}_x \left[\tau_d^- \right] = \check{\pi}_d^{(1)}(x) + \check{\pi}_d^{(2)}(x), \text{ for } x > d, \tag{B.14}$$

where

$$\hat{\pi}_a^{(1)}(x) := \mathbb{E}_x \left[\tau_a^+; X_{\tau_a^+} = a \right] \text{ and } \hat{\pi}_a^{(2)}(x) := \mathbb{E}_x \left[\tau_a^+; X_{\tau_a^+} > a \right], \text{ for } x > a, \tag{B.15}$$

and

$$\check{\pi}_d^{(1)}(x) := \mathbb{E}_x \left[\tau_d^-; X_{\tau_d^-} = d \right] \text{ and } \check{\pi}_d^{(2)}(x) := \mathbb{E}_x \left[\tau_d^-; X_{\tau_d^-} > d \right] \text{ for } x > d. \tag{B.16}$$

We can solve $\left(\hat{\pi}_a^{(1)}(x), \hat{\pi}_a^{(2)}(x) \right)^\intercal$ and $\left(\check{\pi}_d^{(1)}(x), \check{\pi}_d^{(2)}(x) \right)^\intercal$, so as to derive $\mathbb{E}_x[\tau_a^+]$ and $\mathbb{E}_x\left[\tau_d^-\right]$ in (B.13)-(B.14) using the following conclusion.

Theorem 5. *The vectors* $\left(\hat{\pi}_a^{(1)}(x), \hat{\pi}_a^{(2)}(x) \right)^\intercal$ *and* $\left(\check{\pi}_d^{(1)}(x), \check{\pi}_d^{(2)}(x) \right)^\intercal$ *defined in (B.15)-(B.16) are solved by*

$$\begin{pmatrix} F_1(a) & G_1(a) \\ F_2^*(a) & G_2^*(a) \end{pmatrix} \begin{pmatrix} \hat{\pi}_a^{(1)}(x) \\ \hat{\pi}_a^{(2)}(x) \end{pmatrix} = \begin{pmatrix} [\dot{F}_1(a) - \dot{G}_1(a)] \mathbb{P}_x \left(X_{\tau_a^+} = a \right) + \dot{G}_1(a) - \dot{F}_1(x) \\ [\dot{F}_2^*(a) - \dot{G}_2^*(a)] \mathbb{P}_x \left(X_{\tau_a^+} = a \right) + \dot{G}_2^*(a) - \dot{F}_2^*(x) \end{pmatrix} \tag{B.17}$$

and

$$\left(\begin{array}{cc} F_3^*(d) & H_3^*(d) \\ F_4(d) & H_4(d) \end{array}\right)\left(\begin{array}{c} \check{\pi}_d^{(1)}(x) \\ \check{\pi}_d^{(2)}(x) \end{array}\right)=\left(\begin{array}{c} \left[\dot{F}_3^*(d)-\dot{H}_3^*(d)\right]\mathbb{P}_x\left(X_{\tau_d^-}=d\right)+\dot{H}_3^*(d)-\dot{F}_3^*(x) \\ \left[\dot{F}_4(d)-\dot{H}_4(d)\right]\mathbb{P}_x\left(X_{\tau_d^-}=d\right)+\dot{H}_4(d)-\dot{F}_4(x) \end{array}\right) \quad \text{(B.18)}$$

respectively. Here $\mathbb{P}_x\left(X_{\tau_a^+}=a\right)$ *and* $\mathbb{P}_x\left(X_{\tau_d^-}=d\right)$ *are given by (B.7) and (B.8).*

Proof. First we prove the equation (B.17). We substitute (22) and (23) into (B.5) to obtain that

$$F_1(x,q)=F_1(a,q)\mathbb{E}_x\left[e^{-q\tau_a^+};X_{\tau_a^+}=a\right]+G_1(a,q)\mathbb{E}_x\left[e^{-q\tau_a^+};X_{\tau_a^+}>a\right], \quad \text{(B.19)}$$

$$F_2^*(x,q)=F_2^*(a,q)\mathbb{E}_x\left[e^{-q\tau_a^+};X_{\tau_a^+}=a\right]+G_2^*(a,q)\mathbb{E}_x\left[e^{-q\tau_a^+};X_{\tau_a^+}>a\right]. \quad \text{(B.20)}$$

Then we take partial derivatives on both sides of (B.19) and (B.20) w.r.t. q, and let $q\downarrow 0$ to obtain

$$F_1(a)\hat{\pi}_a^{(1)}(x)+G_1(a)\hat{\pi}_a^{(2)}(x)=\left[\dot{F}_1(a)-\dot{G}_1(a)\right]\mathbb{P}_x\left(X_{\tau_a^+}=a\right)+\dot{G}_1(a)-\dot{F}_1(x),$$

$$F_2^*(a)\hat{\pi}_a^{(1)}(x)+G_2^*(a)\hat{\pi}_a^{(2)}(x)=\left[\dot{F}_2^*(a)-\dot{G}_2^*(a)\right]\mathbb{P}_x\left(X_{\tau_a^+}=a\right)+\dot{G}_2^*(a)-\dot{F}_2^*(x),$$

which verifies that $\hat{\pi}_a^{(1)}(x)$ and $\hat{\pi}_a^{(2)}(x)$ defined by (B.15) indeed satisfy (B.17).

Similarly, we can show that $\check{\pi}_d^{(1)}(x)$ and $\check{\pi}_d^{(2)}(x)$ defined in (B.16) are solved by (B.18), which is omitted here for brevity. This completes the proof. □

Appendix B.3 Results on two-sided exit problem

As stated in the Section 2.3, when the initial value of the spread is between the upper threshold and the lower threshold, we should study the two-sided exit problem of X_t, which generalizes the one-sided exit problem derived in Appendix Appendix B.2.

For ease of exposition, we define

$$\pi_q(x)=\left(\pi_q^{(1)}(x),\pi_q^{(2)}(x),\pi_q^{(3)}(x),\pi_q^{(4)}(x)\right)^{\intercal}, \quad \text{(B.21)}$$

where

$$\begin{aligned} &\pi_q^{(1)}(x)=\mathbb{E}_x\left[e^{-q\tau_{a,d}};X_{\tau_{a,d}}=a\right],\ \pi_q^{(2)}(x)=\mathbb{E}_x\left[e^{-q\tau_{a,d}};X_{\tau_{a,d}}>a\right],\\ &\pi_q^{(3)}(x)=\mathbb{E}_x\left[e^{-q\tau_{a,d}};X_{\tau_{a,d}}=d\right],\ \pi_q^{(4)}(x)=\mathbb{E}_x\left[e^{-q\tau_{a,d}};X_{\tau_{a,d}}<d\right]. \end{aligned} \quad \text{(B.22)}$$

The following result derives $\pi_q(x)$ in (B.21), which fully characterizes the Laplace transform of $\tau_{a,d}$ through $\mathbb{E}_x\left[e^{-q\tau_{a,d}}\right]=\sum_{i=1}^4\pi_q^{(i)}(x)$.

Theorem 6. *For $q>0$ and $d<x<a$, the vector $\pi_q(x)$ in (B.21) is solved by*

$$M_q\pi_q(x)=F_M(x,q), \quad \text{(B.23)}$$

where

$$M_q=\left(\begin{array}{cccc} F_1(a,q) & G_1(a,q) & F_1(d,q) & H_1(d,q) \\ F_2(a,q) & G_2(a,q) & F_2(d,q) & H_2(d,q) \\ F_3(a,q) & G_3(a,q) & F_3(d,q) & H_3(d,q) \\ F_4(a,q) & G_4(a,q) & F_4(d,q) & H_4(d,q) \end{array}\right),$$

and $F_M(x,q) = (F_1(x,q), F_2(x,q), F_3(x,q), F_4(x,q))^\intercal$, *with* $F_i(\cdot,q), G_i(\cdot,q), H_i(\cdot,q)$ *for* $i = 1,2,3,4$ *defined in (19).*

Proof. Consider the function $h_i^q(x)$ defined by

$$h_i^q(x) = \begin{cases} G_i(a,q), & x > a, \\ F_i(x,q), & d \leqslant x \leqslant a, \\ H_i(d,q), & x < d, \end{cases} \tag{B.24}$$

for $q > 0$ and $i = 1,2,3,4$. It's easy to see that $h_1^q(x,q)$ and $\partial h_1^q(x)/\partial x$ are bounded and differentiable for $x \in [d,a]$. Denote by $\mathcal{A}$ the generator of X, then for $d \leqslant x \leqslant a$, we have

$$\mathcal{A}h_1^q(x) = [\mu + \kappa(\alpha - x)] \frac{\partial}{\partial x} F_1(x,q) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} F_1(x,q) + \lambda \int_{-\infty}^{\infty} h_1^q(x+z) f_Y(z)\, dz - \lambda F_1(x,q), \tag{B.25}$$

where

$$\frac{\partial}{\partial x} F_1(x,q) = -\int_{\Gamma_1} z\psi(z,q) e^{-(x-\alpha)z}\, dz,\ \frac{\partial^2}{\partial x^2} F_1(x,q) = \int_{\Gamma_1} z^2 \psi(z,q) e^{-(x-\alpha)z}\, dz, \tag{B.26}$$

$$\begin{aligned} \int_{-\infty}^{\infty} h_1^q(x+z) f_Y(z)\, dz = {} & H_1(d,q) \int_{-\infty}^{d-x} (1-p)\gamma_2 e^{\gamma_2 z}\, dz + G_1(a,q) \int_{a-x}^{\infty} p\gamma_1 e^{-\gamma_1 z}\, dz \\ & + \int_{d-x}^{0} F_1(x+z,q)(1-p)\gamma_2 e^{\gamma_2 z} dz + \int_0^{a-x} F_1(x+z,q) p\gamma_1 e^{-\gamma_1 z}\, dz. \end{aligned} \tag{B.27}$$

Besides, given the fact $\lim_{z\downarrow -\infty} z\psi(z,q)e^{-(x-\alpha)z} = 0$ and $\lim_{z\uparrow -\gamma_1} z\psi(z,q)e^{-(x-\alpha)z} = 0$, integration by parts gives that

$$\kappa(x-\alpha)\int_{\Gamma_1} z\psi(z,q)e^{-(x-\alpha)z}\, dz = \kappa \int_{\Gamma_1} \left(\frac{\partial \psi(z,q)}{\partial z} z + \psi(z,q) \right) e^{-(x-\alpha)z}\, dz. \tag{B.28}$$

Substitute (B.26)-(B.28) into (B.25), and straightforward algebraic calculation yields that

$$\begin{aligned} & \mathcal{A}h_1^q(x) - q h_1^q(x) \\ = & \int_{\Gamma_1} e^{-(x-\alpha)z} \left[\kappa \frac{\partial \psi(z,q)}{\partial z} z + \psi(z,q) \left(\kappa + \frac{\sigma^2}{2} z^2 - \mu z - \lambda p \frac{z}{\gamma_1 + z} + \lambda(1-p)\frac{z}{\gamma_2 - z} - q \right) \right] dz \\ = & 0, \end{aligned} \tag{B.29}$$

where the last line follows from the definition of $\psi(z,q)$ in (19). Itô's formula and dominated convergence theorem yield that

$$h_1^q(x) = \lim_{t\to\infty} \mathbb{E}_x \left[e^{-q(\tau_{a,d}\wedge t)} h_1^q\left(X_{\tau_{a,d}\wedge t}\right) \right] = \mathbb{E}_x \left[e^{-q\tau_{a,d}} h_1^q\left(X_{\tau_{a,d}}\right) \right] \text{ for } d \leqslant x \leqslant a.$$

Conditional on the position of $X_{\tau_{a,d}}$, the value of $h_1^q\left(X_{\tau_{a,d}}\right)$ is specified in (B.24), so the above equation reduces to

$$\begin{aligned} F_1(x,q) = {} & F_1(a,q)\,\mathbb{E}_x\left[e^{-q\tau_{a,d}}; X_{\tau_{a,d}} = a\right] + G_1(a,q)\,\mathbb{E}_x\left[e^{-q\tau_{a,d}}; X_{\tau_{a,d}} > a\right] \\ & + F_1(d,q)\,\mathbb{E}_x\left[e^{-q\tau_{a,d}}; X_{\tau_{a,d}} = d\right] + H_1(d,q)\,\mathbb{E}_x\left[e^{-q\tau_{a,d}}; X_{\tau_{a,d}} < d\right]. \end{aligned} \tag{B.30}$$

Similarly, for $i = 2,3,4$, we can show that

$$\begin{aligned} F_i(x,q) = {} & F_i(a,q)\,\mathbb{E}_x\left[e^{-q\tau_{a,d}}; X_{\tau_{a,d}} = a\right] + G_i(a,q)\,\mathbb{E}_x\left[e^{-q\tau_{a,d}}; X_{\tau_{a,d}} > a\right] \\ & + F_i(d,q)\,\mathbb{E}_x\left[e^{-q\tau_{a,d}}; X_{\tau_{a,d}} = d\right] + H_i(d,q)\,\mathbb{E}_x\left[e^{-q\tau_{a,d}}; X_{\tau_{a,d}} < d\right]. \end{aligned} \tag{B.31}$$

Thus formula (B.23) follows from (B.30) and (B.31). □

Remark 6. Given Theorem 6 and conditional independence property stated in Appendix B.1, we can derive the joint Laplace transform of $\tau_{a,d}$ defined in (5) and the overshoot and undershoot as

$$\begin{aligned}&\mathbb{E}_x\left[e^{-q\tau_{a,d}-\rho\left(X_{\tau_{a,d}}-a\right)};X_{\tau_{a,d}}>a\right]\\=&\mathbb{E}_x\left[e^{-q\tau_{a,d}};X_{\tau_{a,d}}>a\right]\mathbb{E}_x\left[e^{-\rho\left(X_{\tau_{a,d}}-a\right)};X_{\tau_{a,d}}>a\right]=\frac{\gamma_1}{\rho+\gamma_1}\pi_q^{(2)}\left(x\right),\end{aligned}\tag{B.32}$$

and

$$\begin{aligned}&\mathbb{E}_x\left[e^{-q\tau_{a,d}-\xi\left(d-X_{\tau_{a,d}}\right)};X_{\tau_{a,d}}<d\right]\\=&\mathbb{E}_x\left[e^{-q\tau_{a,d}};X_{\tau_{a,d}}<d\right]\mathbb{E}_x\left[e^{-\xi\left(d-X_{\tau_{a,d}}\right)};X_{\tau_{a,d}}<d\right]=\frac{\gamma_2}{\xi+\gamma_2}\pi_q^{(4)}\left(x\right),\end{aligned}\tag{B.33}$$

where in the last equations of (B.32) and (B.33), we employ the property (B.2) and (B.3) respectively.

To calculate the probabilities for X_t to cross the thresholds continuously and by jump, we represent $F_i\left(\cdot,q\right), G_i\left(\cdot,q\right)$ and $G_i\left(\cdot,q\right)$ for $i=2,3$ using their "$*$ versions" in (B.23), and make some simplification to obtain that

$$M_q^*\pi_q\left(x\right)=F_M^*\left(x,q\right),\tag{B.34}$$

where

$$M_q^*=\begin{pmatrix}F_1(a,q) & G_1(a,q) & F_1(d,q) & H_1(d,q)\\ F_2^*(a,q) & G_2^*(a,q) & F_2^*(d,q) & H_2^*(d,q)\\ F_3^*(a,q) & G_3^*(a,q) & F_3^*(d,q) & H_3^*(d,q)\\ F_4(a,q) & G_4(a,q) & F_4(d,q) & H_4(d,q)\end{pmatrix},$$

with functions $F_i^*\left(\cdot,q\right), G_i^*(\cdot,q)$ and $G_i^*\left(\cdot,q\right)$ for $i=2,3$ defined in (22) and (23), and $F_M^*\left(x,q\right)=\left(F_1(x,q),F_2^*(x,q),F_3^*(x,q),F_4(x,q)\right)^\intercal$. Then the probabilities for $X_{\tau_{a,d}}$ to cross the thresholds continuously and by jump are solved by the following linear system

$$M_0^*\begin{pmatrix}\mathbb{P}_x\left(X_{\tau_{a,d}}=a\right)\\ \mathbb{P}_x(X_{\tau_{a,d}}>a)\\ \mathbb{P}_x\left(X_{\tau_{a,d}}=d\right)\\ \mathbb{P}_x\left(X_{\tau_{a,d}}<d\right)\end{pmatrix}=\begin{pmatrix}F_1(x)\\ F_2^*(x)\\ F_3^*(x)\\ F_4(x)\end{pmatrix},\tag{B.35}$$

where $M_0^*:=\lim_{q\downarrow 0}M_q^*$.

Now we focus on the calculation of $\mathbb{E}_x\left[\tau_{a,d}\right]$. Denote by

$$\dot{\pi}_0\left(x\right):=\left(\dot{\pi}_0^{(1)}\left(x\right),\dot{\pi}_0^{(2)}\left(x\right),\dot{\pi}_0^{(3)}\left(x\right),\dot{\pi}_0^{(4)}\left(x\right)\right)^\intercal,\tag{B.36}$$

where $\dot{\pi}_0^{(i)}\left(x\right):=\frac{\partial}{\partial q}\pi_q^{(i)}\left(x\right)\Big|_{q=0}$ such that

$$\begin{aligned}&\dot{\pi}_0^{(1)}\left(x\right)=-\mathbb{E}_x\left[\tau_{a,d};X_{\tau_{a,d}}=a\right],\ \dot{\pi}_0^{(2)}\left(x\right)=-\mathbb{E}_x\left[\tau_{a,d};X_{\tau_{a,d}}>a\right],\\&\dot{\pi}_0^{(3)}\left(x\right)=-\mathbb{E}_x\left[\tau_{a,d};X_{\tau_{a,d}}=d\right],\ \dot{\pi}_0^{(4)}\left(x\right)=-\mathbb{E}_x\left[\tau_{a,d};X_{\tau_{a,d}}<d\right].\end{aligned}$$

Then it follows that

$$\mathbb{E}_x\left[\tau_{a,d}\right] = -\sum\nolimits_{i=1}^{4} \dot{\pi}_0^{(i)}\left(x\right). \tag{B.37}$$

To calculate $\dot{\pi}_0\left(x\right)$ in (B.36) so as to evaluate $\mathbb{E}_x\left[\tau_{a,d}\right]$ in (B.37), we take the partial derivative on both sides of (B.34) w.r.t. q and let $q \downarrow 0$, then we can solve $\dot{\pi}_0\left(x\right)$ from

$$M_0^*\dot{\pi}_0\left(x\right) = -\dot{M}_0^*\pi_0\left(x\right) + \dot{F}_M, \tag{B.38}$$

where $\dot{M}_0^* := \lim_{q\downarrow 0}\frac{\partial}{\partial q}M_q^*$, and $\dot{F}_M := \frac{\partial}{\partial q}F_M^*\Big|_{q=0} = \left(\dot{F}_1(x,0), \dot{F}_2^*(x,0), \dot{F}_3^*(x,0), \dot{F}_4(x,0)\right)^{\intercal}$.

Appendix B.4 Exit problems under one-sided jumps and no jumps

The methodology in Appendix B.3 can also be applied to OU processes with one-sided jumps and Brownian Motion-driven OU process. For sake of convenience, we define

$$G_i(x,q) := \int_{\Gamma_i}\frac{\gamma_1}{z+\gamma_1}\psi(z,q)e^{-(x-\alpha)z}\,dz,\ \ H_i(x,q) := -\int_{\Gamma_i}\frac{\gamma_2}{z-\gamma_2}\psi(z,q)e^{-(x-\alpha)z}\,dz, \tag{B.39}$$

for $i = 5,6$ with Γ_5 and Γ_6 defined in (B.11).

Appendix B.4.1 Case of one-sided jumps

For the generalized OU processes with only positive exponential jumps with mean $1/\gamma_1$, we employ the same technique as in the proof in Theorem 6. The main difference lies in that here we only use three integral intervals Γ_1, Γ_2, and Γ_6 to construct functions satisfying (B.29). It is easy to check that for $i = 1,2,6$,

$$\begin{aligned} F_i\left(x,q\right) &= F_i(a,q)\mathbb{E}_x\left[e^{-q\tau_{a,d}};X_{\tau_{a,d}} = a\right] + F_i(d,q)\mathbb{E}_x\left[e^{-q\tau_{a,d}};X_{\tau_{a,d}} = d\right] \\ &\quad + H_i(d,q)\mathbb{E}_x\left[e^{-q\tau_{a,d}};X_{\tau_{a,d}} < d\right]. \end{aligned}$$

Thus, in this context the Laplace transform of $\tau_{a,d}$ satisfies

$$\begin{pmatrix} F_1(a,q) & G_1(a,q) & F_1(d,q) \\ F_2(a,q) & G_2(a,q) & F_2(d,q) \\ F_6(a,q) & G_6(a,q) & F_6(d,q) \end{pmatrix}\begin{pmatrix} \mathbb{E}_x\left(e^{-q\tau_{a,d}};X_{\tau_{a,d}} = a\right) \\ \mathbb{E}_x\left(e^{-q\tau_{a,d}};X_{\tau_{a,d}} > a\right) \\ \mathbb{E}_x\left(e^{-q\tau_{a,d}};X_{\tau_{a,d}} = d\right) \end{pmatrix} = \begin{pmatrix} F_1(x,q) \\ F_2(x,q) \\ F_6(x,q) \end{pmatrix},$$

where $F_i(\cdot,q), G_i(\cdot,q), i = 3,4,5$ are defined in (19) and (B.39).

Similarly, for the generalized OU processes with only negative exponential jumps with mean $1/\gamma_2$, the Laplace transform of $\tau_{a,d}$ satisfies

$$\begin{pmatrix} F_3(a,q) & F_3(d,q) & H_3(d,q) \\ F_4(a,q) & F_4(d,q) & H_4(d,q) \\ F_5(a,q) & F_5(d,q) & H_5(d,q) \end{pmatrix}\begin{pmatrix} \mathbb{E}_x\left(e^{-q\tau_{a,d}};X_{\tau_{a,d}} = a\right) \\ \mathbb{E}_x\left(e^{-q\tau_{a,d}};X_{\tau_{a,d}} = d\right) \\ \mathbb{E}_x\left(e^{-q\tau_{a,d}};X_{\tau_{a,d}} < d\right) \end{pmatrix} = \begin{pmatrix} F_3(x,q) \\ F_4(x,q) \\ F_5(x,q) \end{pmatrix},$$

where $F_i(\cdot,q)$ and $H_i(\cdot,q)$ for $i = 3,4,5$ are defined through (19) and (B.39).

Based on the above derivations, the joint Laplace transform $\mathbb{E}_x\left[e^{-q\tau_{a,d}-\rho(X_{\tau_{a,d}}-a)};X_{\tau_{a,d}} > a\right]$ and $\mathbb{E}_x\left[e^{-q\tau_{a,d}-\xi(d-X_{\tau_{a,d}})};X_{\tau_{a,d}} < d\right]$ can be derived in a similar manner as in (B.32) and (B.33).

Appendix B.4.2 Case of no jumps

For the OU processes driven by Brownian motion, the Laplace transform of $\tau_{a,d}$ satisfies

$$\begin{pmatrix} F_5(a,q) & F_5(d,q) \\ F_6(a,q) & F_6(d,q) \end{pmatrix} \begin{pmatrix} \mathbb{E}_x\left(e^{-q\tau_{a,d}}; X_{\tau_{a,d}}=a\right) \\ \mathbb{E}_x\left(e^{-q\tau_{a,d}}; X_{\tau_{a,d}}=d\right) \end{pmatrix} = \begin{pmatrix} F_5(x,q) \\ F_6(x,q) \end{pmatrix}, \tag{B.40}$$

where

$$F_i^*(x,q) := \frac{q}{\kappa} F_i(x,q) \text{ and } F_i^*(x) := F_i^*(x,0). \tag{B.41}$$

Under the symmetry assumption $d=-a$, some algebraic derivations can show that (B.40) yields the same Laplace transform of $\tau_{a,d}$ as in Darling et al. (1953), i.e.,

$$\mathbb{E}_x\left(e^{-q\tau_{a,-a}}\right) = \frac{F_6(-x,q)+F_6(x,q)}{F_6(-a,q)+F_6(a,q)} = e^{\frac{\kappa}{2\sigma^2}(x^2-a^2)} \frac{D_{-\frac{q}{\kappa}}\left(-\frac{\sqrt{2\kappa}}{\sigma}x\right)+D_{-\frac{q}{\kappa}}\left(\frac{\sqrt{2\kappa}}{\sigma}x\right)}{D_{-\frac{q}{\kappa}}\left(-\frac{\sqrt{2\kappa}}{\sigma}a\right)+D_{-\frac{q}{\kappa}}\left(\frac{\sqrt{2\kappa}}{\sigma}a\right)},$$

where $D_\zeta = \frac{1}{\Gamma(-\zeta)} e^{-\frac{x^2}{4}} \int_0^\infty z^{-\zeta-1} e^{-\frac{z^2}{2}-xz}\, dz$ is Weber function (c.f. Whittaker and Watson (1996)).

Since the OU process is a special case of OUDEJ without jumps, we can adopt a similar methodology to calculate the expected length of a trading cycle under OU process.

Theorem 7. *For $X_0 = x \in (d,a)$, the expected length of a trading cycle under OU process is*

$$\mathbb{E}_x\left[L(a,b,c,d)\right] = \mathbb{E}_x\left[\tau_{a,d}\right] + \frac{F_6^*(x)-F_6^*(d)}{F_6^*(a)-F_6^*(d)} \cdot \frac{\dot{F}_6^*(b)-\dot{F}_6^*(a)}{F_6^*(b)} + \frac{F_6^*(a)-F_6^*(x)}{F_6^*(a)-F_6^*(d)} \cdot \frac{\dot{F}_5^*(c)-\dot{F}_5^*(d)}{F_5^*(c)},$$

where $\mathbb{E}_x\left[\tau_{a,d}\right] = -\dot{\pi}_0^{(1)}(x) - \dot{\pi}_0^{(3)}(x)$ with $\dot{\pi}_0^{(1)}(x)$ and $\dot{\pi}_0^{(3)}(x)$ satisfying

$$\begin{pmatrix} F_5^*(a) & F_5^*(d) \\ F_6^*(a) & F_6^*(d) \end{pmatrix} \begin{pmatrix} \dot{\pi}_0^{(1)}(x) \\ \dot{\pi}_0^{(3)}(x) \end{pmatrix} = \begin{pmatrix} \left[\dot{F}_5^*(d)-\dot{F}_5^*(a)\right]\frac{F_6^*(x)-F_6^*(d)}{F_6^*(a)-F_6^*(d)} - \dot{F}_5^*(d)+\dot{F}_5^*(x) \\ \left[\dot{F}_6^*(d)-\dot{F}_6^*(a)\right]\frac{F_6^*(a)-F_6^*(x)}{F_6^*(a)-F_6^*(d)} - \dot{F}_6^*(d)+\dot{F}_6^*(x) \end{pmatrix}.$$

Here, $F_5^*(\cdot)$ and $F_6^*(\cdot)$ are defined in (B.41) and $\dot{F}_i^*(x) := \lim_{q\downarrow 0} \frac{\partial}{\partial q} F_i^*(x)$ for $i=5,6$.

Proof. When $X_0 = x \in (d,a)$, it follows from (16) that

$$\mathbb{E}_x\left[L(a,b,c,d)\right] = \mathbb{E}_x\left[\tau_{a,d}\right] + \mathbb{E}_x\left[\tau_b^-\left(\hat{X}^a\right); \tau_a^+ < \tau_d^-\right] + \mathbb{E}_x\left[\tau_c^+\left(\check{X}^d\right); \tau_d^- < \tau_a^+\right]. \tag{B.42}$$

By the strong Markov property, the second term in (B.42) is calculated as

$$\mathbb{E}_x\left[\tau_b^-\left(\hat{X}^a\right); \tau_a^+ < \tau_d^-\right] = \mathbb{E}_x\left\{\mathbb{E}_x\left[\tau_b^-\left(\hat{X}^a\right)\middle|\mathcal{F}_{\tau_a^+}\right]; \tau_a^+ < \tau_d^-\right\} = \mathbb{P}_x\left(X_{\tau_{a,d}}=a\right)\mathbb{E}_a\left[\tau_b^-\right]. \tag{B.43}$$

Similarly, the third term in (B.42) can be given by

$$\mathbb{E}_x\left[\tau_c^+\left(\check{X}^d\right); \tau_d^- < \tau_a^+\right] = \mathbb{P}_x\left(X_{\tau_{a,d}}=d\right)\mathbb{E}_d\left[\tau_c^+\right]. \tag{B.44}$$

From (B.42), (B.43), and (B.44), it suffices to calculate $\mathbb{P}_x\left(X_{\tau_{a,d}}=a\right)$, $\mathbb{P}_x\left(X_{\tau_{a,d}}=d\right)$, $\mathbb{E}_a\left[\tau_b^-\right]$ and $\mathbb{E}_d\left[\tau_c^+\right]$. To obtain the former two probabilities, we note from (B.40) that

$$F_6(a,q)\mathbb{E}_x\left[e^{-q\tau_{a,d}}; X_{\tau_{a,d}}=a\right] + F_6(d,q)\mathbb{E}_x\left[e^{-q\tau_{a,d}}; X_{\tau_{a,d}}=d\right] = F_6(x,q), \tag{B.45}$$

and let $q\downarrow 0$ in (B.45) according to the monotonous convergence theorem. To overcome the problem that $F_6(x,q)$ diverges as $q\downarrow 0$, we replace $F_6(\cdot,q)$ using $F_6^*(\cdot,q)$ defined by (B.41) in (B.45) and let $q\downarrow 0$. Combining with the fact that $\mathbb{P}_x\left(X_{\tau_{a,d}}=a\right)+\mathbb{P}_x\left(X_{\tau_{a,d}}=d\right)=1$, we obtain that

$$F_6^*(x) = F_6^*(a)\mathbb{P}_x\left(X_{\tau_{a,d}} = a\right) + F_6^*(d)\mathbb{P}_x\left(X_{\tau_{a,d}} = d\right) = \left[F_6^*(a) - F_6^*(d)\right]\mathbb{P}_x\left(X_{\tau_{a,d}} = a\right) + F_6^*(d),$$

from which the distribution of $X_{\tau_{a,d}}$ follows that

$$\mathbb{P}_x\left(X_{\tau_{a,d}} = a\right) = \frac{F_6^*(x) - F_6^*(d)}{F_6^*(a) - F_6^*(d)} \text{ and } \mathbb{P}_x\left(X_{\tau_{a,d}} = d\right) = \frac{F_6^*(a) - F_6^*(x)}{F_6^*(a) - F_6^*(d)}. \tag{B.46}$$

To derive $\mathbb{E}_d\left[\tau_c^+\right]$ in (B.44), the monotonous convergence theorem yields that $\mathbb{E}_d\left[\tau_c^+\right] = -\lim_{q\downarrow 0}\dot{\mathbb{E}}_d\left[e^{-q\tau_c^+}\right]$. To derive $\dot{\mathbb{E}}_d\left[e^{-q\tau_c^+}\right]$, we take $x = d$ and $a = c$ in (B.12), and represent $F_5(x,q)$ using $F_5^*(x,q)$ defined by (B.41) to obtain that

$$F_5^*(d,q) = F_5^*(c,q)\,\mathbb{E}_d\left[e^{-q\tau_c^+}\right].$$

We then take partial derivatives on both sides of the above equation w.r.t. q and let $q \downarrow 0$, it follows from the definition (B.41) that $\dot{F}_5^*(d) = \dot{F}_5^*(c) + F_5^*(c)\lim_{q\downarrow 0}\mathbb{E}_x\left[e^{-q\tau_c^+}\right]$, from which we have

$$\mathbb{E}_d\left[\tau_c^+\right] = \frac{\dot{F}_5^*(c) - \dot{F}_5^*(d)}{F_5^*(c)}. \tag{B.47}$$

Analogically, we also have in (B.43)

$$\mathbb{E}_a\left[\tau_b^-\right] = \frac{\dot{F}_6^*(b) - \dot{F}_6^*(a)}{F_6^*(b)}. \tag{B.48}$$

To derive $\mathbb{E}_x\left[\tau_{a,d}\right]$, i.e., the first term in (B.42), we utilize the decomposition

$$\mathbb{E}_x\left[\tau_{a,d}\right] = -\dot{\pi}_0^{(1)}(x) - \dot{\pi}_0^{(3)}(x), \tag{B.49}$$

where $\pi_0^{(1)}(x)$ and $\pi_0^{(3)}(x)$ are defined in (B.22). It follows directly from (B.40) that for $i = 5, 6$,

$$F_i^*(x,q) = F_i^*(a,q)\mathbb{E}_x\left[e^{-q\tau_{a,d}}; X_{\tau_{a,d}} = a\right] + F_i^*(d,q)\mathbb{E}_x\left[e^{-q\tau_{a,d}}; X_{\tau_{a,d}} = d\right], \tag{B.50}$$

and we take partial derivatives on both sides of (B.50) w.r.t. q, and let $q \downarrow 0$ to obtain

$$F_i^*(a)\dot{\pi}_0^{(1)}(x) + F_i^*(d)\dot{\pi}_0^{(3)}(x) = \left[\dot{F}_i^*(d) - \dot{F}_i^*(a)\right]\mathbb{P}_x\left(X_{\tau_{a,d}} = a\right) - \dot{F}_i^*(d) + \dot{F}_i^*(x).$$

for $i = 5, 6$. From the above linear system we can solve $\dot{\pi}_0^{(1)}(x)$ and $\dot{\pi}_0^{(3)}(x)$ and the expectation (B.49) follows immediately.

Substituting (B.46)-(B.49) into (B.42), we verify this theorem. □

Appendix C Auxiliary results

Appendix C.1 Existence of $G_2^*(x)$ and $H_3^*(x)$ in (24)

Proof. Let $H_3(x,q) = \int_{\Gamma_3}|z|^{\frac{q}{\kappa}-1}g(z)\,dz$, where $g(z) = \gamma_2 e^{-\frac{\sigma^2}{4\kappa}z^2 + \frac{\mu}{\kappa}z}\,|z+\gamma_1|^{\frac{p\lambda}{\kappa}}\,|z-\gamma_2|^{\frac{(1-p)\lambda}{\kappa}-1}e^{-(x-\alpha)z}$ satisfies $\lim_{z\to 0} g(z) = \gamma_1^{\frac{p\lambda}{\kappa}}\gamma_2^{\frac{(1-p)\lambda}{\kappa}} =: A$. Since we are concerned about the property of $H_3(x,q)$ when q

tends to 0, there exists a sufficient small q such that $\sqrt{q} < \gamma_2$. Then we divide the integral interval Γ_3 by $\sqrt{q}$ and represent $qH_3(x,q)$ as

$$qH_3(x,q) = q\int_0^{\sqrt{q}} |z|^{\frac{q}{\kappa}-1}\, A\, dz + q\int_0^{\sqrt{q}} |z|^{\frac{q}{\kappa}-1} \left(g\left(z\right) - A\right)\, dz + \int_{\sqrt{q}}^{\gamma_2} |z|^{\frac{q}{\kappa}}\, \frac{q}{z} g\left(z\right)\, dz. \tag{C.51}$$

The first term of the RHS in (C.51) is calculated as

$$q\int_0^{\sqrt{q}} |z|^{\frac{q}{\kappa}-1}\, A\, dz = A\kappa q^{\frac{q}{2\kappa}}.$$

By L′Hôspital's rule, $\frac{g(z)-A}{z} = O\left(1\right)$, the second term of the RHS in (C.51) reads

$$q\int_0^{\sqrt{q}} |z|^{\frac{q}{\kappa}-1} \left(g\left(z\right) - A\right)\, dz = q\int_0^{\sqrt{q}} |z|^{\frac{q}{\kappa}}\, \frac{g\left(z\right) - A}{z}\, dz = O\left(1\right).$$

The third term of the RHS in (C.51) satisfies

$$q\int_{\sqrt{q}}^{\gamma_2} |z|^{\frac{q}{\kappa}}\, \frac{q}{z} g\left(z\right)\, dz \leqslant \int_{\sqrt{q}}^{\gamma_2} |z|^{\frac{q}{\kappa}}\, q^{\frac{3}{2}} g\left(z\right)\, dz \to 0,\ \ q \downarrow 0.$$

Substituting the above three terms into (C.51) and from $q^{\frac{q}{2\kappa}} \to 1$ as $q \downarrow 0$, we obtain that

$$qH_3(x,q) = A\kappa q^{\frac{q}{2\kappa}} + O\left(1\right) = O\left(1\right).$$

Thus, $\lim\limits_{q\downarrow 0} qH_3(x,q)$ is finite. Analogically, we can also prove that $\lim\limits_{q\downarrow 0} qG_2(x,q)$ is finite. $\square$