Convenience Yield-Based Pricing of Commodity Futures*

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First draft: July 23, 2008
This draft: July 17, 2010

ABSTRACT

This paper proposes a convenience yield-based pricing for commodity futures, which embeds the incompleteness of commodity futures markets in convenience yield. By using the pricing method, we conduct empirical analyses of crude oil, heating oil, and natural gas futures traded on the NYMEX in order to assess the incompleteness of energy futures markets. We show that the fluctuation from incompleteness is partly owed to the fluctuation from convenience yield. In addition, it is shown that the additional Sharpe ratio, which represents the degree of market incompleteness and is also used for derivative pricing written on energy prices, is obtained from the NYMEX data. Then, we apply the implied market price of risk to the pricing of Asian call option on crude oil futures. As an empirical example, we try to compute the call option price using the parameters estimated from crude oil futures prices.

Key words: Convenience yield, stochastic discount factor, incomplete markets, commodity futures
JEL Classification: C51, G12, Q40

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*Views expressed in this paper are those of the author and do not necessarily reflect those of J-POWER. All remaining errors are mine. The author thanks Matt Davison, Toshiki Honda, Sebastian Jaimungal, Matteo Manera, Ryozo Miura, Nobuhiro Nakamura, Kazuhiko Ōhashi, Marcel Prokopczuk, Luca Taschini, and all seminar participants in the 31st JAFEE Conference and the Bachelier Finance Society 2010 Sixth World Congress for their helpful comments.

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1. Introduction

Convenience yield is often used to describe the value to hold commodities as is explained in e.g., Geman (2005). For example, the storage of natural gas is conducted as a preemptive strategic action for power companies to keep stable power generation from gas-fired power plants in response to the electricity demand. It implies that the storage of natural gas produces any positive value for the generator, which is represented as a convenience yield. More importantly, convenience yield is useful to represent the linkage between spot and futures prices on futures curves because it is also used to explain upward and downward sloping futures curves referred to as contango and backwardation. On the other hand, asset pricing theory offers a concept of stochastic discount factor to determine financial instrument prices like futures prices written on spot prices. Hence, the relationship between spot and futures prices is expressed by twofold ways: convenience yield and stochastic discount factor. Putting two concepts together, convenience yield may play an alternative role of a stochastic discount factor. This paper highlights the relationship between convenience yield and stochastic discount factor and offers a commodity pricing representation based on convenience yield.

Risk neutral valuation often used in financial markets is applied to commodity derivative pricing as in e.g., Bjerksund (1991), Schwartz (1997), Schwartz and Smith (2000), Yan (2002), Casassus and Collin-Dufresne (2005), Korn (2005), Eydeland and Wolyniec (2003), among others. Recently, Trolle and Schwartz (2009) developed a very sophisticated commodity price model using the HJM type forward cost of carry with stochastic volatility while unfortunately they assume risk neutral measure not only for spot prices but also for cost of carry and unspanned volatility. However, it is generally difficult to price a commodity product such as commodity futures using the risk neutral valuation because commodity markets are incomplete due to their illiquidity. Hence, we have a further elaborate task to select a stochastic discount factor (SDF) in evaluating their value. A familiar tool to select SDF is a utility-based approach where SDF, i.e., market price of risk, is assigned to the unspanned risk and the derivative prices are uniquely determined. For example, Davis (2001) and Cao and Wei (2000) characterize the prices of weather derivatives, which are generally categorized in commodity derivatives, using utility functions and optimal consumptions. However, this method deeply depends on the selection of the utility function and the optimal consumption. To avoid these imperfections and incorporate the market incompleteness into commodity futures pricing, the good-deal bounds (GDB) is developed by Cochrane and Saa-Requejo (2000). The point of the GDB pricing stems from the introduction of a restriction on the variance of the SDF: a certain upper bound characterized by the maximum Sharpe ratio is always required to be more than or equal to Sharpe ratios of all assets in the market. This binds the possible variance range of the SDF to an interval based on the maximum Sharpe ratio. As an example to price incomplete market assets, Kanamura and Ohashi (2009) applied GDB to weather derivatives. GDB may be useful to price incomplete market assets in the sense that it does not rely on the utility function and optimal consumption like utility-based models. However, the method is somewhat dissatisfactory because the maximum Sharpe ratio that binds SDF is unknown and thus must be given exogenously. A pricing scheme for commodities is desirable if the Sharpe ratio, i.e., the restriction on SDF, is given by using commodity market data. Here, we consider that two-way concept on intertemporal relationship between spot and futures prices, i.e., convenience yield can implicitly determine SDF, will be useful to determine the Sharpe ratio. Consequently, convenience yield can offer the Sharpe ratio, i.e., market price of risk other than spot market price of risk, that we employ for commodity derivative pricing. This pricing scheme is referred to as "a convenience yield-based pricing." This paper proposes the convenience yield-based pricing for commodity futures, which embeds the incompleteness of commodity futures markets in convenience yield.

By using the pricing method, we conduct empirical analyses of crude oil, heating oil, and natural gas futures traded on the NYMEX in order to assess the incompleteness of energy futures markets. We show that the Sharpe ratio, which represents the degree of the market incompleteness and is also used for derivative pricing written on energy prices, is obtained from the NYMEX data. Finally, we apply the market price of risk embedded in energy futures markets to the pricing of Asian call option on crude oil futures. As an empirical example, we calculate the call option price using the parameters estimated from crude oil futures prices.

This paper is organized as follows. Section 2 proposes the convenience yield-based pricing for commodity futures. Section 3 conducts empirical studies to examine the incompleteness of energy futures markets. Section 4 applies the empirical results to the pricing of Asian call option on crude oil futures. Section 5 concludes.

2. The Convenience Yield-Based Pricing for Commodity Futures

Gibson and Schwartz (1990) introduce a two-factor model for commodity spot prices, which represents a well-known price model in commodity markets. We start with this basic spot price model to obtain commodity futures prices. Spot prices and convenience yields follow

$$\frac{dS_t}{S_t} = (\mu - \delta_t)dt + \sigma_1 dw_t, \tag{1}$$

$$d\delta_t = \kappa(\alpha - \delta_t)dt + \sigma_2 du_t, \tag{2}$$

where $E_t[dw_t du_t] = \rho dt$. In Gibson and Schwartz (1990), convenience yield is treated as an important factor characterizing the relationship between spot and futures prices.

Then we address the modeling of futures prices. While Schwartz (1997) introduced risk neutral measure to price commodity futures, the ambiguity may remain in the existence of such probability measure. Hence, we chose more comprehensive representation of the futures prices by using stochastic discount factor (SDF). The futures prices F_t^T at time t with maturity T are in general represented as follows:

$$F_t^T = E_t\left[\frac{\frac{\Lambda_T}{\Lambda_t} S_T}{E_t\left[\frac{\Lambda_T}{\Lambda_t}\right]}\right], \tag{3}$$

where SDF is denoted by Λ_t at time t and interest rate is assumed to be constant (see e.g., Campbell, Lo, and MacKinlay (1997)).

In order to obtain the futures prices, we try to characterize the stochastic discount factor in equation (3). As is well known, commodity markets may demonstrate incompleteness because of the illiquidity. Following Cochrane and Saa-Requejo (2000) which can generally express the incompleteness of the market, we assume that Λ_t is expressed by

$$\frac{d\Lambda_t}{\Lambda_t} = -r dt - \phi dw_t - \nu dz_t, \tag{4}$$

where ν represents the incompleteness of the market in the sense of unspanned part by spot markets. The point is that the market risk nests on twofold risk of spot market and its orthogonal part. Note that $\nu = \sqrt{A^2 - \phi^2}$ where A represents the Sharpe ratio exogenously given in Cochrane and Saa-Requejo (2000).

Next, let us consider how the incompleteness, i.e., the orthogonal part to spot price risk, is described. Untraded convenience yield explains upward and downward sloping futures curves referred to as contango and backwardation, which includes illiquid delivery month commodity futures. Since convenience yield is useful to represent the linkage between spot and futures prices as is illustrated by two different points on futures curves, convenience yield may be a key to represent incompleteness of commodity markets. On the other hand, asset pricing theory offers a concept of stochastic discount factor to characterize futures prices written on spot prices. Putting two ideas together, convenience yield can play an alternative role of a stochastic discount factor. Thus, we assume that the fluctuation due to convenience yield (du_t) is spanned by both of complete and incomplete parts (dw_t and dz_t, resp.):

$$du_t = \rho dw_t + \sqrt{1-\rho^2} dz_t. \tag{5}$$

By using Ito's lemma to equation (1), we obtain

$$S_T = S_t e^{(\mu-\alpha-\frac{1}{2}\sigma_1^2)(T-t)+\frac{1}{\kappa}(1-e^{-\kappa(T-t)})(\delta_t-\alpha)+\int_t^T(\sigma_1+\frac{\sigma_2\rho}{\kappa}(1-e^{-\kappa(T-s)}))dw_s+\int_t^T(\frac{\sigma_2\sqrt{1-\rho^2}}{\kappa}(1-e^{-\kappa(T-s)}))dz_s}. \tag{6}$$

Again by using Ito's lemma to equation (4), we obtain

$$\frac{\Lambda_T}{\Lambda_t} = e^{-(r+\frac{1}{2}\phi^2+\frac{1}{2}\nu^2)(T-t)-\int_t^T\phi dw_s-\int_t^T\nu dz_s}. \tag{7}$$

Injecting equations (6) and (7) into equation (3), we have the futures price as follows:

$$F_t^T = S_t e^{\Upsilon(t,T)-\Omega(t,T)\delta_t}, \tag{8}$$

$$\Upsilon(t,T) = (r-\alpha+\frac{\sigma_2^2}{2\kappa^2}-\frac{\sigma_1\sigma_2\rho}{\kappa}+\phi\frac{\sigma_2\rho}{\kappa}+\frac{\nu\sigma_2\sqrt{1-\rho^2}}{\kappa})(T-t)+\frac{\sigma_2^2}{4\kappa^3}(1-e^{-2\kappa(T-t)})$$
$$+(\alpha\kappa+\rho\sigma_1\sigma_2-\frac{\sigma_2^2}{\kappa}-\phi\sigma_2\rho-\nu\sigma_2\sqrt{1-\rho^2})\frac{1-e^{-\kappa(T-t)}}{\kappa^2}, \tag{9}$$

$$\Omega(t,T) = \frac{1-e^{-\kappa(T-t)}}{\kappa}. \tag{10}$$

The representation is referred to as convenience yield-based pricing for commodity futures (CY-based pricing) because convenience yield is used as if a stochastic discount factor to connect between spot and futures prices. The point of this representation stands on the inclusion of incompleteness parameter ν into spot-futures price relationship. It leads to the advantage that the incomplete market price of risk ν can directly be estimated from the data without using the exogenous Sharpe ratio. Thus, we obtained a futures pricing representation not using risk neutral measure in complete market setting, but using incompleteness parameter ν embedded in convenience yield.

If commodity futures are traded with high liquidity and all maturity date products are transacted in the market, i.e., T is not restricted as certain point of time, futures price risk would be completely spanned in the market and ν might represent market price of futures risk unspanned by spot price risk. However, the liquidity of futures markets may be quite low in a certain delivery time, in particular longer maturity, say 10 years. In addition, T is limited, say, 1-, 2-, 3- months, and so forth. Hence ν includes incomplete part neither spanned by existing spot nor futures prices and incomplete part spanned by low liquidity futures products. In this sense, we injected all the other market price of risk other than spot markets, in particular futures price risk both for high or low liquidity traded products and untraded delivery months or years products, into convenience yields. Thus, the CY-based pricing has an advantage to model all delivery futures products irrelevant to the listed or unlisted delivery months as well as irrelevant to the liquidity. In addition, futures markets provide the futures prices that have no trading volume. The information of ν implied from futures markets will be beneficial to know how the overall maximum Sharpe ratio expands when non traded assets like new derivative instruments written on the same underlying commodities are introduced into the market. This is because it is expected that risk premium for illiquid futures products including no traded futures is the same degree as the risk premium for newly traded derivative products within the same underlying commodity. That is, additional derivative market price of risk will be enveloped by ν. Thus, ν implied from futures markets may be applicable to the other associated derivative pricing.

Then we examine the relationship between our pricing representation and existing models. Comparing our pricing representation to Bjerksund (1991), the market price of convenience yield risk λ in Bjerksund (1991) holds for

$$\lambda = \phi\rho + \nu\sqrt{1-\rho^2}. \tag{11}$$

Note that for Schwartz (1997), the relation is changed into $\frac{\lambda}{\sigma_2} = \phi\rho + \nu\sqrt{1-\rho^2}$. These relationships suggest that the market price of convenience yield risk is found to be split into two parts: complete market price of risk and incomplete market price of risk, which are weighed using the correlation between spot price returns and convenience yields. Hence, we can find for sure that the market price of convenience yield risk denoted by λ includes untraded and unspanned risk of ν by using the relationship between SDF and convenience yield. In addition, the breakdown of market price of risk can facilitate the treatment of the incompleteness not only for empirical analyses of commodity markets, but also for derivative pricing written on commodity futures. Our focus is to bring a new idea to the pricing representation by introducing incomplete market price of risk embedded in convenience yields. In this sense, we offer the other interpretation of Bjerksund (1991) and Schwartz (1997) models not relying on risk neutral measure. In addition, our results may be consistent with Attaoui and Six (2009) where orthogonal decomposition for market price of convenience yield risk is conducted. However, our approach is different in the sense that convenience yield is directly connected to stochastic discount factor on futures price formulae.

3. Empirical Studies for Energy Prices

3.1. Data

In this study, we use the daily closing prices of WTI crude oil (WTI), heating oil (HO), and natural gas (NG) futures traded on the NYMEX. Each futures product includes six delivery months – from one month to six months. The covered time period is from April 3, 2000 to March 31, 2008. The data are obtained from Bloomberg. Summary statistics for WTI, HO, and NG futures prices are provided in Tables 1, 2, and 3, respectively. These tables indicate that WTI, HO, and NG have common skewness characteristics. The skewness of WTI, HO, and NG futures prices is positive, meaning that the distributions are skewed to the right.

[INSERT TABLE 1 ABOUT HERE]

[INSERT TABLE 2 ABOUT HERE]

[INSERT TABLE 3 ABOUT HERE]

Then we examine mean reversion of futures price spreads. Here we define price spreads by the differences of the logs of futures prices: $SP_t^{ij} = \log F_t^i - \log F_t^j$ $(i < j)$. By doing so, the influence of spot prices is offset when the futures prices are expressed by the multiplication of spot price and the exponential of convenience yield function as in equation (8). In addition, SP_t^{ij} may correspond to the level of convenience yield while time dependent coefficient $\Omega(t,T)$ may exist. Hence, SP_t^{ij} examination will detect the characteristics of convenience yield as the first order approximation. We estimate futures price spreads model using

$$SP_t^{ij} = \rho_0 + \rho_1 SP_{t-1}^{ij} + \eta_t, \tag{12}$$

where i and j represent different two delivery months selected from 1-month to 6-months $(i < j)$, respectively. Table 4 reports the estimation results using 1- and 2-month crude oil, heating oil, natural gas futures price spreads, respectively. According to estimated ρ_1's and the corresponding standard errors in the table, price spreads demonstrate mean reversion. It implies that there exists another mean-reverting process in futures price spreads other than the spot price process. Taking into account the price spread definition and constant $\Omega(t,T)$ because of generic futures products, i.e., approximately constant $T-t$, convenience yields for energy futures prices may correspond to the mean reverting process other than spot price process, which supports the model structure using convenience yield.

[INSERT TABLE 4 ABOUT HERE]

3.2. Parameter estimation

We estimate the model parameters employing the Kalman filter (KF). To simplify the calculation, we log transform the spot price S_t into new variable x_t:

$$dx_t = (\mu - \frac{1}{2}\sigma_1^2 - \delta_t)dt + \sigma_1 dz_t. \tag{13}$$

KF consists of time and measurement update equations. On one hand, since x and δ in equations (13) and (2), respectively are time updated, these equations represent the linear time update equations in the KF system. We discretize the continuous-time model for x in equation (13) into

$$x_t = x_{t-1} - \Delta t \delta_t + (\mu - \frac{1}{2}\sigma_1^2)\Delta t + \sigma_1 \varepsilon_t \equiv f_1(x_{t-1}, \delta_{t-1}, \varepsilon_t). \tag{14}$$

Similarly, the continuous-time model for δ in equation (2) into the following:

$$\delta_t = (1 - \kappa \Delta t)\delta_{t-1} + \kappa \alpha \Delta t + \sigma_2 \eta_t \equiv f_2(x_{t-1}, \delta_{t-1}, \eta_t). \tag{15}$$

On the other hand, the measurement update equation in the KF system is obtained from the futures-spot price relationship. We define the log of F_t^T by the new variable y_t ($y_t = \ln F_t^T$), and discretize equation (8) into the following:

$$y_t = x_t - \Omega(t,T)\delta_t + \Upsilon(t,T) + \xi_t \equiv h_1(x_t, \delta_t, \xi_t). \tag{16}$$

Following Welch and Bishop (2004), time and measurement update equations are expressed by

$$\begin{pmatrix} x_t \\ \delta_t \end{pmatrix} = \begin{pmatrix} \tilde{x}_t \\ \tilde{\delta}_t \end{pmatrix} + A_t \begin{pmatrix} x_{t-1} - \hat{x}_{t-1} \\ \delta_{t-1} - \hat{\delta}_{t-1} \end{pmatrix} + W_t \begin{pmatrix} \varepsilon_t \\ \eta_t \end{pmatrix}, \tag{17}$$

$$y_t = h_1(\tilde{x}_t, \tilde{\delta}_t, 0) + B_t \begin{pmatrix} x_t - \tilde{x}_t \\ \delta_t - \tilde{\delta}_t \end{pmatrix} + V_t \xi_t, \tag{18}$$

where $\tilde{x}_t = f_1(\hat{x}_{t-1}, \hat{\delta}_{t-1}, 0)$, $\tilde{\delta}_t = f_2(\hat{x}_{t-1}, \hat{\delta}_{t-1}, 0)$, $A_t = \begin{pmatrix} 1 & -\Delta t \\ 0 & 1-\kappa\Delta t \end{pmatrix}$, $W_t = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$,
$B_t = \begin{pmatrix} 1 & -\frac{1}{\kappa}(1-e^{-\kappa(T-t)}) \end{pmatrix}$, $V_t = 1$, $V[\varepsilon_t, \eta_t] = Q_t = \begin{pmatrix} \Delta t & \gamma\Delta t \\ \gamma\Delta t & \Delta t \end{pmatrix}$, and
$V[\xi_t] = R_t = \mathrm{diag}[m_1, m_2, m_3, m_4, m_5, m_6]$ (Diagonal matrix) where $m_i \geq 0$ for $i = 1$ to 6.

Tables 5 and 6 show the complete set of the KF equations which include time and measurement update equations so as to calculate the a priori estimate error covariance matrix (Φ_t^-) and the a posteriori estimate error covariance matrix (Φ_t), respectively.

[INSERT TABLE 5 ABOUT HERE]

[INSERT TABLE 6 ABOUT HERE]

Note that we define the a priori estimate error and the covariance by $e_t^- \equiv \begin{pmatrix} x_t - \hat{x}_t^- \\ \delta_t - \hat{\delta}_t^- \end{pmatrix}$ and $\Phi_t^- \equiv E[e_t^- e_t^{-T}]$, and that we also define the a posteriori estimate error and the covariance by $e_t \equiv \begin{pmatrix} x_t - \hat{x}_t \\ \delta_t - \hat{\delta}_t \end{pmatrix}$ and $\Phi_t \equiv E[e_t e_t^T]$ where K_t is the Kalman gain.

Using the recursive updates of time and measurement update equations as in Tables 5 and 6, measurement errors ($\tilde{e}_{y_t}$) and the covariance matrices (Σ_t) are given by

$$\tilde{e}_{y_t} = y_t - h_1(\hat{x}_t^-, \hat{\delta}_t^-, 0), \tag{19}$$

$$\Sigma_t = B_t \Phi_t^- B_t^T + V_t R_t V_t^T. \tag{20}$$

Using the measurement errors and the covariance matrices, the parameters (Θ) in equations (1) and (2) are estimated by the maximum likelihood method

$$\hat{\Theta} = \arg\min_{\Theta} \sum_{t=1}^{N} \ln|\Sigma_t| + \sum_{t=1}^{N} \tilde{e}_{y_t} \Sigma_t^{-1} \tilde{e}_{y_t}^T, \tag{21}$$

where $\Theta = (\mu, \sigma_1, \kappa, \alpha, \sigma_2, \rho, \nu, m_1, m_2, m_3, m_4, m_5, m_6)$. Note that risk free rate r is set to 6%.

We estimated parameters Θ for WTI crude oil, heating oil, and natural gas futures as reported in Tables 7, 8, and 9, respectively. The parameters except m_3, m_4, and m_5 for WTI are statistically significant according to Table 7. Since ρ is estimated as 0.857 which is different from 1, it is shown that the fluctuation from incompleteness is partly owed to the fluctuation from convenience yield in equation (5). In addition, the positive ρ is consistent with the view for convenience yield as an embedded timing option to the commodity as discussed in Geman (2005). Then, according to Cochrane (2001), negative ν generates price upper boundary for the relevant derivative products. It implies that the market incompleteness due to futures trading including illiquid delivery months may request additional positive risk premium for crude oil derivatives comparing to spot market price of risk. While futures market evolution made the Sharpe ratio increased by negative ν in the orthogonal direction to dw_t, the absolute value regarding ν is relevant because the incompleteness due to the illiquidity can produce positive and negative risk premium.[1] In this case, the incompleteness of crude oil market is calculated as $|\nu| = 1.404$ using the market data. Hence, we could directly obtain the incomplete market price of risk without using the exogenous Sharpe ratio. Then, we examine the results for heating oil. The parameters except α, m_2, m_3, and m_4 are statistically significant according to Table 8. Since ρ is statistically significant as 0.745, it is also shown that the fluctuation of incompleteness partly stems from the fluctuation of convenience yield in equation (5). Additionally taking into account that it is smaller than the WTI correlation, the orthogonal fluctuation for HO may contribute more to the convenience yield fluctuation than WTI in equation (5). Since ν is statistically significant, the incompleteness of heating oil market could be obtained using the market data. Finally examining the results for natural gas in Table 9, the parameters except μ and α are statistically significant. In addition to the existence of incompleteness from convenience yield, we also obtained the incomplete natural gas market price of risk using the market data.

[INSERT TABLE 7 ABOUT HERE]

[INSERT TABLE 8 ABOUT HERE]

[INSERT TABLE 9 ABOUT HERE]

Let us discuss the incompleteness of energy futures markets by comparing to the complete market price of risk. The comparisons between ϕ and ν are reported in Table 10. Note that completeness parameter ϕ is calculated as $\phi = \frac{\mu - r}{\sigma_1}$ where we assume $r = 0.06$. "A" represents the Sharpe ratio defined as the square of the sum of two squared market prices of risks. For crude oil, the incomplete market price of risk (1.404) is a little greater than the complete market price of risk (0.924). It suggests that the crude oil market should be almost equally spanned by both

[1] In addition, while ν is obtained as negative value, taking into account $\frac{d\Lambda_t}{\Lambda_t} = -rdt - \phi dw_t - (-\nu)dz_t$, due to the symmetry of dz_t, i.e., $dz_t = -dz_t$ by definition, the absolute value of ν represents the degree of the incompleteness of the market, i.e., incomplete market price of risk.

complete and incomplete markets. The Sharpe ratio is calculated as 1.681. It implies that the pricing of derivative instruments written on crude oil prices just requires to introduce the Sharpe ratio of about twice (1.681) as large as the complete market price of risk (0.924) based on the GDB in Cochrane (2001). For heating oil, the Sharpe ratio takes a value of 1.365 with $|\phi| = 0.883$ while the Sharpe ratio for natural gas is obtained as 0.807 with $|\phi| = 0.302$.[2] It is shown that the portion of complete parts from crude oil and heating oil in the whole Sharpe ratio is bigger than natural gas. This may be consistent with the fact that oil-related markets such as crude oil and heating oil futures are more liquid than natural gas.

Since ν is larger than ϕ for all three products, it is found that the unspanned risk by spot prices asks for larger reward, i.e., higher Sharpe ratio, than the spanned risk by spot prices. It implies that unspanned part including commodity futures price risk may play an important role in the stochastic discount factor. It may correspond to the roots of commodity futures market development to reduce higher volatility in the spot prices. In particular, for natural gas which will be expected the highest volatility of the three, ν is around twice as large as ϕ. It can be explained by the reason such that higher volatility risk of natural gas prices may be more spanned by the unspanned part including the futures price risk. Additionally, according to these results it was shown that the Sharpe ratio used for derivative pricing written on energy prices may be calculated from almost 0.8 to 1.7 as long as we use the NYMEX data.

[INSERT TABLE 10 ABOUT HERE]

4. Application of CY-Based Pricing to Energy Derivatives

4.1. Partial differential equation

The previous section obtained incomplete market price of risk (ν) implied from futures markets. ν will be useful to price the newly introduced derivative products written on the same underlying asset because the illiquid futures products are taken as the same to the newly introduced derivative in the sense that both have no trading volume. Thus, ν may represent how the introduction of new products expands the whole market Sharpe ratio and may be applicable to the derivative pricing. We conduct the pricing of Asian call option on energy futures prices using good-deal bounds of Cochrane and Saa-Requejo (2000). The point is that ν is endogenously given by the

[2] If we do not allow insignificance of μ, the corresponding Sharpe ratio is calculated as 0.751.

parameters which have already estimated in Section 3, while ν is arbitrarily given in good-deal bounds framework. In general, GDB pricing is expressed by

$$\underline{C}_t = E_t \int_{s=t}^{T} \frac{\underline{\Lambda}_s}{\underline{\Lambda}_t} x_s ds + E_t \left(\frac{\underline{\Lambda}_T}{\underline{\Lambda}_t} x_T \right), \tag{22}$$

$$\frac{d\underline{\Lambda}_t}{\underline{\Lambda}_t} = -rdt - \phi dw_t \mp \nu dz_t, \tag{23}$$

where $x_s ds$ and x_T represent continuous dividends and a terminal payoff and $\mp$ represents lower and upper price boundaries, respectively.

We for simplicity assume that the average i-month futures price from times 0 to T is given by

$$I = \frac{1}{T} \int_0^T F^i(S, \delta, t) dt. \tag{24}$$

Note that i is constant because we use generic futures prices and thus we neglect the superscript i below. We denote a price boundary by $\underline{C}(S, \delta, I, t)$. We have

$$\frac{d\underline{C}}{\underline{C}} = \mu_{\underline{C}} dt + \sigma_{\underline{C}w} dw + \sigma_{\underline{C}z} dz. \tag{25}$$

GDB pricing is transformed into

$$\mu_{\underline{C}} - r + \sigma_{\underline{C}w} \mp \sigma_{\underline{C}z} \nu = 0, \tag{26}$$

where $\mp$ represents lower and upper price boundaries, respectively. Note that $x_s = 0$.

By applying Ito's lemma to $\underline{C}(S, \delta, I, t)$, we have

$$\begin{aligned}
\mu_{\underline{C}} = \frac{1}{\underline{C}} \Bigg\{ & \frac{\partial \underline{C}}{\partial t} + (\mu - \delta) S \frac{\partial \underline{C}}{\partial S} + \kappa(\alpha - \delta) \frac{\partial \underline{C}}{\partial \delta} + \frac{1}{2} \sigma_1^2 S^2 \frac{\partial^2 \underline{C}}{\partial S^2} \\
& + \frac{1}{2} \sigma_2^2 \frac{\partial^2 \underline{C}}{\partial \delta^2} + \rho \sigma_1 \sigma_2 S \frac{\partial^2 \underline{C}}{\partial \delta \partial S} + \frac{1}{T} F \frac{\partial \underline{C}}{\partial I} \Bigg\}, \\
\sigma_{\underline{C}w} = \frac{1}{\underline{C}} & \left\{ \mu S \frac{\partial \underline{C}}{\partial S} + \rho \sigma_2 \frac{\partial \underline{C}}{\partial \delta} \right\}, \\
\sigma_{\underline{C}z} = \frac{1}{\underline{C}} & \left\{ \sigma_2 \sqrt{1 - \rho^2} \frac{\partial \underline{C}}{\partial \delta} \right\}.
\end{aligned}$$

The GDB upper and lower price boundaries of an Asian energy derivative are given as the solution of the following PDE by injecting $\mu_{\underline{C}}$, $\sigma_{\underline{C}w}$, and $\sigma_{\underline{C}z}$ into equation (26):

$$\begin{aligned}
&-r\underline{C}+\frac{\partial \underline{C}}{\partial t}+\frac{1}{2}\sigma_1^2 S^2\frac{\partial^2 \underline{C}}{\partial S^2}+\frac{1}{2}\sigma_2^2\frac{\partial^2 \underline{C}}{\partial \delta^2}+\rho\sigma_1\sigma_2 S\frac{\partial^2 \underline{C}}{\partial\delta\partial S}+\frac{dI}{dt}\frac{\partial \underline{C}}{\partial I}\\
&=(\delta-r)S\frac{\partial \underline{C}}{\partial S}+\left(\phi\rho\sigma_2-\kappa(\alpha-\delta)+k\nu\sigma_2\sqrt{1-\rho^2}\mathrm{sgn}\left(\frac{\partial \underline{C}}{\partial\delta}\right)\right)\frac{\partial \underline{C}}{\partial\delta},
\end{aligned} \tag{27}$$

with the terminal payoff:

$$\underline{C}(S,\delta,I,T)=f(I_T), \tag{28}$$

where $k=-1$ and $+1$ generate the upper and lower price boundaries, respectively.

To obtain the GDB prices of the Asian call option, we set the payoff at maturity to be $f(I_T)=\max(I_T-K,0)$ and, following Ingersoll (1987), $\frac{dI}{dt}$ to be

$$dI=\frac{1}{T}F(S,\delta,t)dt. \tag{29}$$

4.2. Asian call option price

We computed Asian call option prices written on 1-month crude oil futures prices assuming that the strike price is 70 USD, initial convenience yield is zero, and interest rate is set to 6 %. The results are reported in Figure 1 and Table 11. Figure 1 suggests that both upper and lower risk premiums are small enough comparing with the level of the option prices because the three price curves are almost seen on the same line. It may be easily used for practitioners in the sense that the option price is obtained using very small price premiums. Then in order to examine the detailed risk premiums, the upper and lower risk premiums are calculated in Table 11. According to Table 11, upper and lower risk premiums take the same value one another irrelevant to initial futures prices. For example, upper and lower premiums are calculated as 0.05 at 80 USD initial futures price. It may imply that both of the seller and buyer of the option request the same risk premium to the crude oil market incompleteness. In addition, we observe that both premiums slightly increase in initial futures prices. When futures prices are high, it is expected that the volatility is also high due to the inverse leverage effect often observed in energy markets. Taking into account that the upper price boundary tends to be selected due to negative ν and that call option premium increases in the volatility, the risk premium may increase in futures prices. Like these, we were able to obtain the risk premium based on the incompleteness of energy futures market implied from the CY-Based pricing we proposed.

[INSERT FIGURE 1 ABOUT HERE]

[INSERT TABLE 11 ABOUT HERE]

Finally, we examine the sensitivity of the option prices to δ. The upper price boundary is reported in Figure 2 because it tends to be selected in this market due to negative ν. The mesh is taken by the same convenience yield and the same spot prices. As we can see, when convenience yield is small under a constant spot price, i.e., large futures price, the upper option boundary takes large option value. For energy prices, it is well-known that there exists the inverse leverage effect which suggests that the volatility is high during high energy price periods as in Geman (2005). It requires the large risk premium for this call option written on futures prices due to the high volatility as in Figure 2. Thus, it is shown that the initial level of convenience yield affects the premium of the option price.

[INSERT FIGURE 2 ABOUT HERE]

5. Conclusion

This paper has proposed a convenience yield-based pricing for commodity futures, which embeds the incompleteness of commodity futures markets in convenience yield. The characteristics of the pricing representation stem from splitting market price of convenience yield risk into complete and incomplete market parts orthogonal each other, which can easily treat the incompleteness of commodity markets. In addition, by using the pricing method we have conducted empirical analyses of crude oil, heating oil, and natural gas futures traded on the NYMEX in order to assess the incompleteness of energy futures markets. We have shown that the fluctuation from incompleteness is partly owed to the fluctuation from convenience yield. In addition, it was shown that the additional Sharpe ratio, which represents the degree of the market incompleteness and is also used for derivative pricing written on energy prices, is obtained from the NYMEX data. Finally, we applied the market price of risk embedded in energy futures markets to the pricing of Asian call option on crude oil futures.

This paper only dealt with energy futures due to the availability of data. The concept in this paper can be extended to other commodity futures like agricultural futures. These empirical studies may be the next direction for our future researches.

References

Attaoui, S., and P. Six, 2009, Commodity derivatives pricing with an endogenous convenience yield market price of risk, Working paper, SSRN.

Bjerksund, P., 1991, Contingent Claims Evaluation when the Convenience Yield is Stochastic: Analytical Results, Working paper, Norwegian School of Economics and Business Administration.

Campbell, J., A. Lo, and A. C. MacKinlay, 1997, *The Econometrics of Financial Markets*. (Princeton University Press Princeton, New Jersey).

Cao, M., and J. Wei, 2000, Equilibrium Valuation of Weather Derivatives, Working paper, University of Toronto.

Casassus, J., and P. Collin-Dufresne, 2005, Stochastic Convenience Yield Implied from Commodity Futures and Interest Rates, *Journal of Finance* 60, 2283–2331.

Cochrane, J. H., 2001, *Asset Pricing*. (Princeton University Press New Jersey).

Cochrane, J. H., and J. Saa-Requejo, 2000, Beyond Arbitrage: Good-Deal Asset Price Bounds in Incomplete Markets, *Journal of Political Economy* 108, 79–119.

Davis, M., 2001, Pricing weather derivatives by marginal value, *Quantitative Finance* 1, 1–4.

Eydeland, A., and K. Wolyniec, 2003, *Energy and Power Risk Management: New Developments in Modeling, Pricing, and Hedging*. (John Wiley & Sons, Inc. Hoboken).

Geman, H., 2005, *Commodities and Commodity Derivatives*. (John Wiley & Sons Ltd West Sussex).

Gibson, R., and E. Schwartz, 1990, Stochastic Convenience Yield and the Pricing of Oil Contingent Claims, *Journal of Finance* 45, 959–976.

Ingersoll, J. E. Jr., 1987, *Theory of Financial Decision Making*. (Rawman & Littlefield Publishers Maryland).

Kanamura, T., and K. Ohashi, 2009, Pricing summer days options by good-deal bounds, *Energy Economics* 31, 289–297.

Korn, O., 2005, Drift Matters: An Analysis of Commodity Derivatives, *The Journal of Futures Markets* 25, 211–241.

Schwartz, E., and J. Smith, 2000, Short-Term Variations and Long-Term Dynamics in Commodity Prices, *Management Science* 46, 893–911.

Schwartz, E. S., 1997, The Stochastic Behaviour of Commodity Prices: Implication for Valuation and Hedging, *Journal of Finance* 52, 923–973.

Trolle, A. B., and E. S. Schwartz, 2009, Unspanned Stochastic Volatility and the Pricing of Commodity Derivatives, *Review of Financial Studies*, *forthcoming*.

Welch, G., and G. Bishop, 2004, An Introduction to the Kalman Filter, Working paper, University of North Carolina at Chapel Hill.

Yan, X., 2002, Valuation of Commodity Derivatives in a New Multi-Factor Model, *Review of Derivatives Research* 5, 251–271.

Figures & Tables

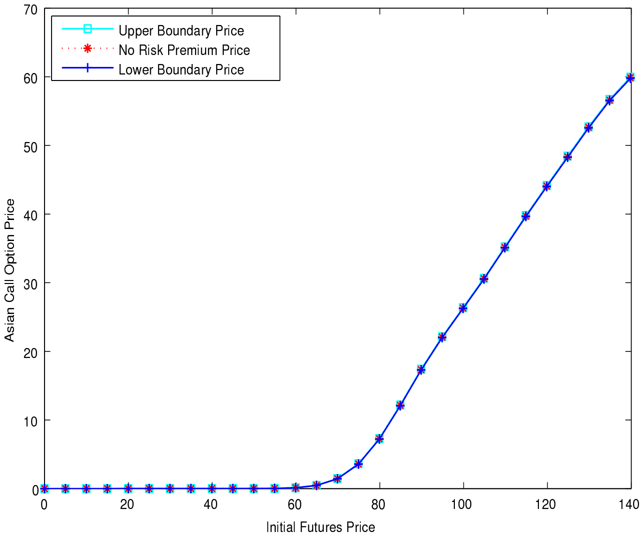

Figure 1. Asian Call Option Prices (K=70, δ=0)

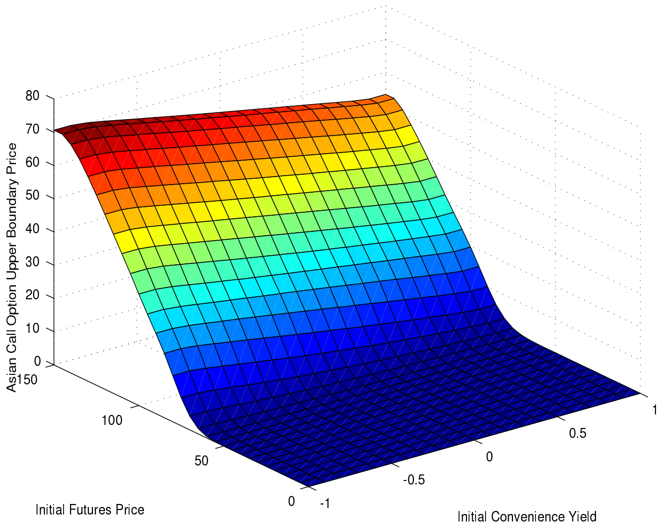

Figure 2. Sensitivity Analysis of Call Option Prices to δ

	WTI_1	WTI_2	WTI_3	WTI_4	WTI_5	WTI_6
Mean	45.96	46.03	45.99	45.87	45.71	45.55
Median	37.21	36.47	35.91	35.44	34.99	34.53
Maximum	110.33	109.17	107.94	106.90	106.06	105.44
Minimum	17.45	17.84	18.06	18.27	18.44	18.60
Std. Dev.	20.66	20.92	21.14	21.32	21.49	21.65
Skewness	0.81	0.76	0.72	0.69	0.67	0.66
Kurtosis	2.79	2.59	2.44	2.33	2.24	2.17

Table 1. Basic Statistics of WTI Crude Oil Futures Prices

	HO_1	HO_2	HO_3	HO_4	HO_5	HO_6
Mean	127.86	128.27	128.35	128.15	127.82	127.44
Median	101.90	99.83	98.53	96.25	94.02	91.97
Maximum	314.83	306.45	301.55	301.05	301.10	301.50
Minimum	49.99	51.31	51.71	51.96	51.52	50.87
Std. Dev.	59.54	60.28	60.98	61.52	61.93	62.30
Skewness	0.73	0.68	0.65	0.64	0.63	0.62
Kurtosis	2.58	2.37	2.21	2.11	2.04	2.00

Table 2. Basic Statistics of Heating Oil Futures Prices

	NG_1	NG_2	NG_3	NG_4	NG_5	NG_6
Mean	6.01	6.16	6.27	6.31	6.35	6.36
Median	5.94	6.11	6.19	6.09	6.11	6.18
Maximum	15.38	15.43	15.29	14.91	14.67	14.22
Minimum	1.83	1.98	2.08	2.18	2.26	2.33
Std. Dev.	2.27	2.32	2.36	2.34	2.33	2.30
Skewness	0.90	0.90	0.90	0.73	0.60	0.39
Kurtosis	4.75	4.72	4.57	3.83	3.28	2.43

Table 3. Basic Statistics of Natural Gas Futures Prices

Products	WTI		HO		NG	
Parameters	ρ_0	ρ_1	ρ_0	ρ_1	ρ_0	ρ_1
Estimates	4.470×10^{-5}	0.962	-2.870×10^{-5}	0.957	-1.045×10^{-3}	0.959
Standard errors	1.010×10^{-4}	0.009	1.320×10^{-4}	0.015	3.540×10^{-4}	0.006
Log likelihood	7614		7199		5679	
AIC	-15225		-14395		-11354	
SIC	-15214		-14384		-11343	

Table 4. Mean Reversion of Energy Futures Price Spreads

$$\hat{x}_t^- = f_1(\hat{x}_{t-1}, \hat{\delta}_{t-1}, 0) \quad \text{(A 1)}$$

$$\hat{\delta}_t^- = f_2(\hat{x}_{t-1}, \hat{\delta}_{t-1}, 0) \quad \text{(A 2)}$$

$$\Phi_t^- = A_t \Phi_{t-1} A_t^T + W_t Q_t W_t^T \quad \text{(A 3)}$$

Table 5. KF Time Update Equations

$$K_t = \Phi_t^- B_t^T (B_t \Phi_t^- B_t^T + V_t R_t V_t^T)^{-1} \quad \text{(A 4)}$$

$$\hat{x}_t = \hat{x}_t^- + K_t(y_t - h_1(\hat{x}_t^-, \hat{\delta}_t^-, 0)) \quad \text{(A 5)}$$

$$\hat{\delta}_t = \hat{\delta}_t^- + K_t(y_t - h_1(\hat{x}_t^-, \hat{\delta}_t^-, 0)) \quad \text{(A 6)}$$

$$\Phi_t = (I - K_t B_t)\Phi_t^- \quad \text{(A 7)}$$

Table 6. KF Measurement Update Equations

Parameters	μ	σ_1	κ	α	σ_2	ρ	ν
Estimates	0.563	0.544	1.629	0.093	0.636	0.857	-1.404
(Std. Err.)	0.000	0.001	0.001	0.002	0.000	0.002	0.000
Parameters	m_1	m_2	m_3	m_4	m_5	m_6	
Estimates	2.197×10^{-4}	1.628×10^{-5}	1.000×10^{-6}	1.000×10^{-6}	1.000×10^{-5}	7.753×10^{-6}	
(Std. Err.)	1.854×10^{-5}	3.004×10^{-6}	1.893×10^{-6}	1.312×10^{-6}	2.747×10^{-5}	3.655×10^{-6}	
Loglike	5.461×10^{4}						
AIC	-1.092×10^{5}						
SIC	-1.092×10^{5}						

Table 7. Parameter Estimation of Crude Oil Futures

Parameters	μ	σ_1	κ	α	σ_2	ρ	ν
Estimates	0.568	0.575	1.358	0.069	0.883	0.745	-1.041
(Std. Err.)	0.196	0.017	0.062	0.249	0.034	0.081	0.234
Parameters	m_1	m_2	m_3	m_4	m_5	m_6	
Estimates	3.619×10^{-4}	1.000×10^{-5}	2.936×10^{-5}	1.000×10^{-5}	1.181×10^{-4}	6.920×10^{-4}	
(Std. Err.)	6.725×10^{-5}	1.889×10^{-5}	1.810×10^{-5}	2.531×10^{-5}	3.485×10^{-5}	1.719×10^{-4}	
Loglike	4.325×10^{4}						
AIC	-8.648×10^{4}						
SIC	-8.651×10^{4}						

Table 8. Parameter Estimation of Heating Oil Futures

Parameters	μ	σ_1	κ	α	σ_2	ρ	ν
Estimates	0.361	0.995	0.617	-0.416	2.061	0.829	-0.749
(Std. Err.)	0.299	0.022	0.089	0.660	0.081	0.012	0.252
Parameters	m_1	m_2	m_3	m_4	m_5	m_6	
Estimates	3.314×10^{-3}	5.368×10^{-5}	1.198×10^{-3}	1.033×10^{-3}	3.278×10^{-5}	2.884×10^{-3}	
(Std. Err.)	1.199×10^{-4}	2.124×10^{-5}	3.595×10^{-5}	3.133×10^{-5}	1.393×10^{-5}	9.812×10^{-5}	
Loglike	3.083×10^{4}						
AIC	-6.163×10^{4}						
SIC	-6.166×10^{4}						

Table 9. Parameter Estimation of Natural Gas Futures

	$\lvert\phi\rvert$	$\lvert\nu\rvert$	$A=\sqrt{\nu^2+\phi^2}$
Crude oil	0.924	1.404	1.681
Heating oil	0.883	1.041	1.365
Natural gas	0.302	0.749	0.807

Table 10. Incompleteness of Energy Futures Markets

Initial Futures Prices	70	80	90	100	110	120	130
Upper Boundary Price	1.47	7.31	17.40	26.36	35.23	44.15	52.71
No Risk Premium Price (NRPP)	1.45	7.25	17.34	26.30	35.15	44.07	52.62
Lower Boundary Price	1.43	7.20	17.27	26.24	35.08	44.00	52.54
Upper Premium (UP)	0.02	0.05	0.07	0.06	0.07	0.08	0.08
Lower Premium (LP)	0.02	0.05	0.07	0.06	0.08	0.08	0.08
UP/NRPP (%)	1.08	0.72	0.39	0.24	0.21	0.17	0.16
LP/NRPP (%)	1.07	0.72	0.39	0.23	0.21	0.17	0.16

Table 11. Asian Call Option Risk Premium