The Quadrant Probabilities of Paired Financial Time Series

Jimin Lin*

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Abstract

This paper introduces the quadrant probability as a new instrument to evaluate the correlation and co-movement between a pair of financial time series. We pair several stocks and indices and examine the pattern of the corresponding quadrant probabilities. To better understand the behavior of the quadrant probabilities, we start from using the centric bivariate normal probability to simulate two correlated time series, and then we generalize it into the non-centric case. In the non-centric case, we construct a method to estimate the quadrant probability series with rolling linear regressions and numerical integration. We find that our approach not only reproduces the pattern of the observed quadrant probability series but also holds sensitivity with respect to the correlation between the two time series. In other words, our approach successfully captures the correlation and the co-movement pattern of the pair of time series.

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*Department of Applied Mathematics, University of Washington, Seattle, WA 98195. E-mail: jmlin@uw.edu.

1 Introduction

Two correlated financial time series tend to present a pattern of co-movement. Such behavior facilitates various profitable trading strategies, such as mean reverting strategies based on pairs trading of stocks (see Leung and Li (2015, 2016); Huang and Wang (2016)). Other than stocks, it is also common to see pairs trading strategies performed on exchange-traded funds. ETFs have become popular and convenient tools for investors who would like to gain exposures on baskets of underlying securities. Many ETFs track a specific index, such as stock index, bond index or commodity index. Some ETFs are well correlated, and their spreads are mean reverting, which motivates the construction of cointegrated portfolios (see Leung and Santoli (2016); Simonson (2017)). There are studies focusing on the index-tracking ability of ETFs. Time series models can be fitted to account for the deviations of ETFs from their underlying assets (see Abdou (2017)). Stochastic models and dynamic strategies can be developed to perform very well in index tracking (see Ward (2017)). Corresponding to the index tracking optimization, some benchmarks are constructed to quantitatively evaluate the tracking errors (see Guo and Leung (2015); Leung and Ward (2015)). Also, it is worth noting that the co-movement of stock is common to be asymmetric (see Jiang et al. (2017)). Inspired by these thoughts, we construct an approach to gain insight into a pair of two correlated financial series.

In this paper, we introduce the quadrant probabilities as a new instrument to evaluate the correlation and co-movement of a pair of time series. We define the quadrant probabilities as the four kinds of probabilities of whether the two time series move in the same direction or different directions. Intuitively, two positively correlated time series are more likely to move upwards or downwards together than to move in different directions. On the contrary, the probability for two negatively correlated time series to move in the opposite directions should be higher than the probability for them to increase or decrease simultaneously. Motivated by this idea, we pair several index series and stock series and calculate the quadrant probabilities of the four cases: 1. both series one and series two increase. 2. series one increases and series two decreases. 3. series one decreases and series two increases. 4. both series one and series two decrease. By selecting time windows of different lengths and doing the rolling calculation, we obtain four new time series consisting of the quadrant probabilities. Our analysis shows the connection between the correlation of the pair of time series and the patterns of the corresponding quadrant probability series. To better understand the behaviors of the quadrant probability series, we first simulate and study the centric quadrant bivariate normal probabilities. Next, we consider the non-centric quadrant bivariate normal probabilities and try to forecast the one day forward future values of them. We use rolling linear regressions on the pair of time series, and take the regression parameters as the input to estimate the future quadrant probabilities. Because there is no explicit solution for the double integral of the probability density function of bivariate normal distributions, we implement the numerical integration to approximate it. We present the result that our estimated quadrant probabilities series in comparison with the observed ones. Finally, we compute the statistics of the estimated quadrant probabilities.

The rest of the paper is organized as follows. In Section 2 we present the methodology for calculating the quadrant probability and the quadrant probability series with data of stocks and indices. In Section 3 we conduct the simulation of quadrant probability series with centric bivariate normal distributions. In Section 4 we investigate the non-centric case and apply rolling linear regressions and numerical integration to simulate and predict the quadrant probability series on the real data. We also show the statistics of the simulated quadrant probability series.

2 Pairs of Stocks and Indices

2.1 Correlation

In this section, we select GSPC, SPY, SSO, QQQ, VIX, VXX, USO, GLD, MSFT and AAPL for study. We choose the daily adjusted close price from yahoo finance from the listing date of the indices and stocks to the date 2017-12-29, and we use the arithmetic returns. Firstly we check the correlations among these stocks and indices.

As can be seen in the Figure 1, the returns of GSPC, SPY, and SSO are highly correlated with each other. Their correlation coefficients are over 98%. It is evident since both the SPY and SSO are ETFs of the S&P 500 index. Also, GSPC and SPY have a high positive correlation of 82% with the QQQ. SSO has a higher correlation of 92% with QQQ. As for the VIX and VXX, the GSPC, SPY and SSO are about 6% to 9% more negatively correlated with VXX than with VIX, while the QQQ is 18% more negatively correlated with VXX than with VIX. The USO is 41% correlated with GSPC, 33% with QQQ, -30% with VIX and -33% with VXX. It has 23% to 26% positive correlations with GLD, MSFT, and APPL. The GLD does not have strong correlations with these indices and stocks. Both MSFT and APPL are positively correlated with the S&P 500 index or ETFs and the QQQ and negatively correlated with the volatility index or future. However, MSFT has stronger correlations with these indices than APPL does.

2.2 The Quadrant Probabilities

We pair the 10 indices and stocks with each other and get 45 pairs of time series. Noticing that some time series have different time horizons, we select the time range from the first day when both of the series are available to the date 2017-12-29. Then we calculate the percent of times when the two series of a pair move in the same or different directions. We define the quadrant probabilities with notation $(++, +-, -+, --)$ as follows:

$$
\begin{aligned}
++ &= \mathbb{E}\left[\mathbb{1}_{\{r_1>0, r_2>0\}}\right] \\
+- &= \mathbb{E}\left[\mathbb{1}_{\{r_1>0, r_2<0\}}\right] \\
-+ &= \mathbb{E}\left[\mathbb{1}_{\{r_1<0, r_2>0\}}\right] \\
-- &= \mathbb{E}\left[\mathbb{1}_{\{r_1<0, r_2<0\}}\right]
\end{aligned}
$$

Here r_1 is the first time series and r_2 is the second time series. $\mathbb{1}_{\{.\}}$ is the indicator. For instance, $\mathbb{1}_{\{r_1>0, r_2>0\}}$ is:

$$
\mathbb{1}_{\{r_1>0, r_2>0\}} = \begin{cases} 1, & \text{if } r_1 > 0 \text{ and } r_2 > 0 \\ 0, & \text{otherwise} \end{cases}
$$

Figure 2 shows the proportions of the quadrant probabilities. Each horizontal bar with four colored fractions represents a pair of time series and its proportions of $(++, +-, -+, --)$. By observing the border between the $+-$ (the blue fraction) and the $-+$ (the green fraction), we could know which index or stock goes positive or negative more frequently. Visually, if the border is on the left of the 0.5 point, then the left-hand side index or stock of the pair goes negative more often rather than going positive, and vise versa. As an alternative way to present the correlation between two time series, it is consistent with the correlation matrix in Figure 1. It also indicates an asymmetric pattern of the correlation.

For pairs of the GSPC, SPY and SSO, the $++$ and $--$ take up almost all the proportions, which means that they move in the same direction most of the time. Also, the $+-$ and $-+$ of the GSPC-SSO pair and the SPY-SSO pair are narrower than that of the GSPC-SPY pair, which aligns with the value of correlation coefficients of theirs. What's more, we can also judge by the border that GSPC, SPY, and SSO are more often to increase rather than decrease.

The VIX and VXX also have high proportions of $++$ and $--$. It is a more common case where they go down together since the $--$ fraction is the largest. Also, by observing the border, we can know that both of them decrease more often than increase. Pairs containing VIX or VXX have larger $+-$ and $-+$ fractions, which means they move in the different directions in the most of time. What's more, if we compare the size of the $-+$ and $+-$, we can find it often happens where the VIX and VXX decrease while the other indices and stocks increase at the same time.

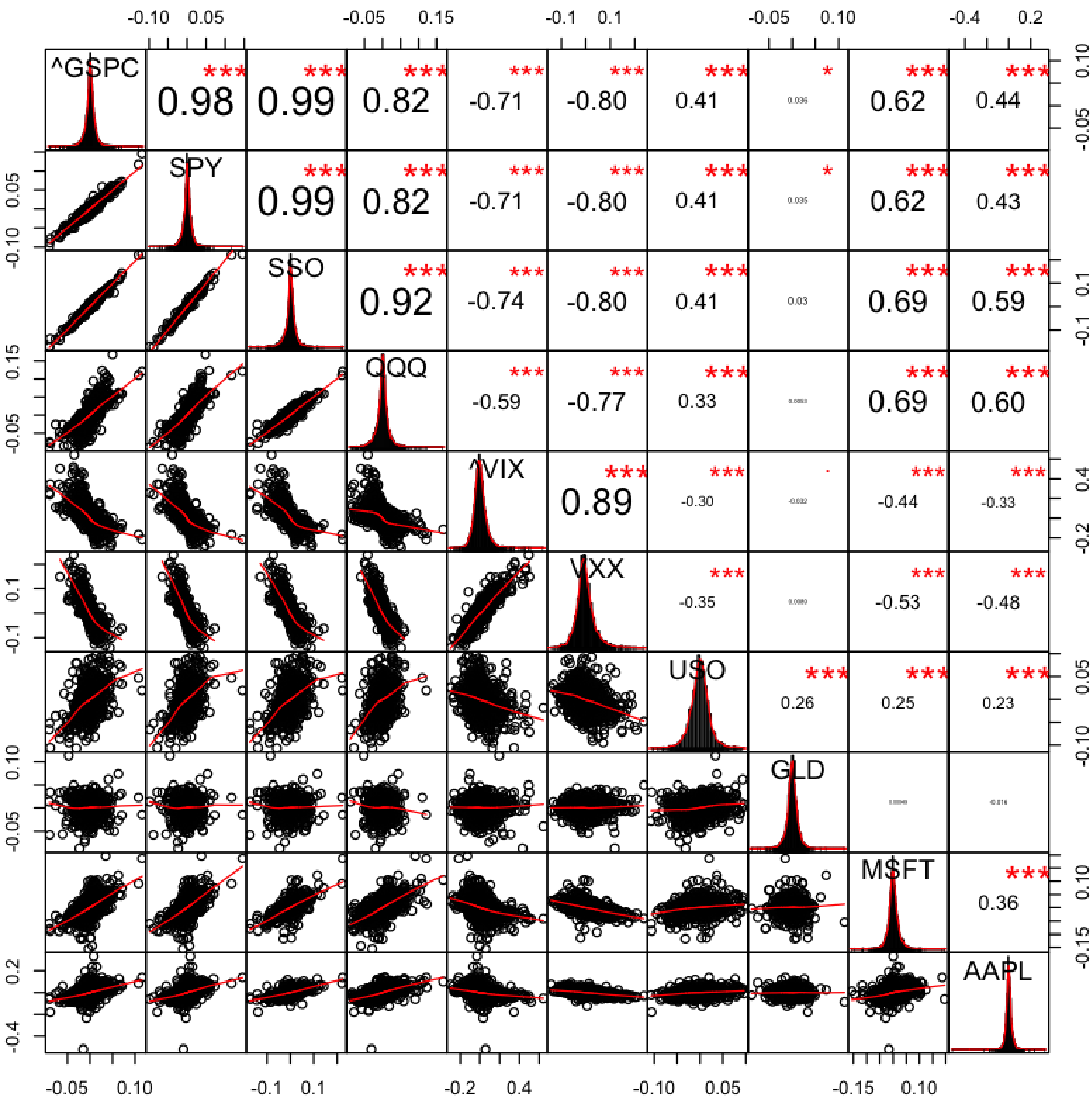


Figure 1: Correlation plot. Subplots in the diagonal are the histograms of each stock or index. The lower left triangle are the scatter plots of each pair of two stocks or indices, and the upper right triangle are the corresponding correlation coefficient of each pair.

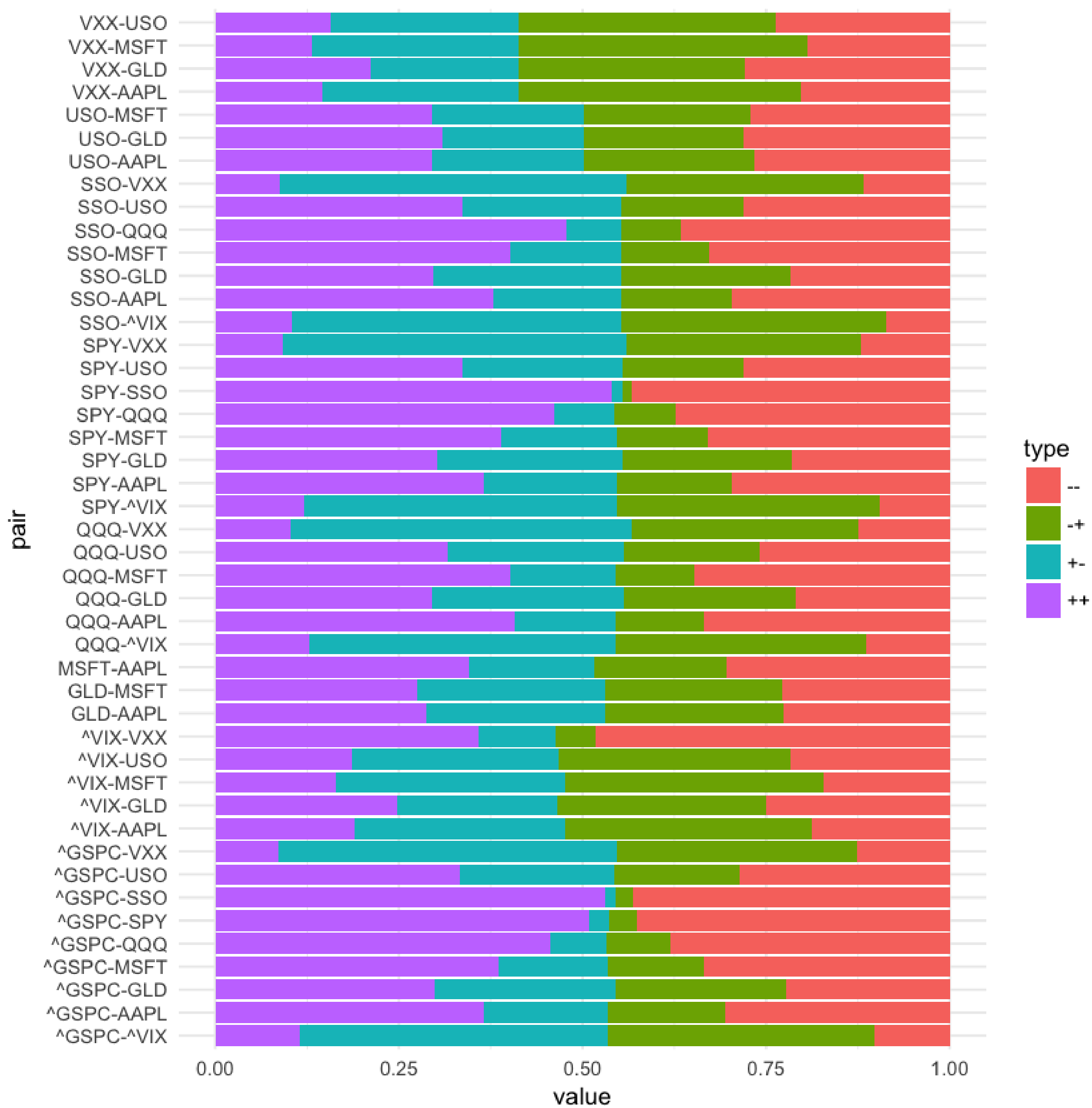


Figure 2: Quadrant probabilities $(++, +-, -+, --)$. The time horizon of each pair of time series runs from when both of them are available to 2017-12-29. The x-axis shows the percentage from 0 to 1. Along the y-axis, each horizontal bar with four colored fractions represents a pair and its proportions of $(++, +-, -+, --)$. The proportions of $(++, +-, -+, --)$ are represented by the purple fraction, the blue fraction, the green fraction and the red fraction. $++$ denotes the percent of the time when both time series increase. $+-$ denotes the percent of the time when the first series increases and the second decreases. $-+$ denotes the percent of the time when the first series decreases and the second increases. $--$ denotes the percent of the time when both time series decrease.

Take a look at the pairs of GSPC, SPY, and SSO with stocks, the ++ is the most substantial fraction, and the border excesses the 0.5 point, which means that GSPC, SPY, and SSO and stock increase together but the former are more often to increase than the latter.

2.3 Time Series of the Quadrant Probabilities

In the previous section, we use the entire time horizon to calculate the quadrant probabilities for each pair. To look in details of the behaviors of the four probabilities, instead using the entire period, we apply rolling time window to compute the quadrant probabilities. Therefore, we obtain four time series that consist of the quadrant probabilities, which are denoted as $\{++_t\}$, $\{+-_t\}$, $\{-+_t\}$, $\{--_t\}$ and defined as following:

$$\begin{aligned} ++_t &= \mathbb{E}\left[\mathbb{1}^{(t-w,t]}_{\{r_1>0,r_2>0\}}\right] \\ +-_t &= \mathbb{E}\left[\mathbb{1}^{(t-w,t]}_{\{r_1>0,r_2<0\}}\right] \\ -+_t &= \mathbb{E}\left[\mathbb{1}^{(t-w,t]}_{\{r_1<0,r_2>0\}}\right] \\ --_t &= \mathbb{E}\left[\mathbb{1}^{(t-w,t]}_{\{r_1<0,r_2<0\}}\right] \end{aligned}$$

Here $t \in [t_0, T]$ denotes the time interval. For each pair of time series, t_0 is the first date when both of the time series are available, and T is the last date 2017-12-29. $(t-w,t]$ is the time window, and w is the width of it. We let the width of the window be 21, 63, 126 and 252 days, which correspond to the days of one month, one quarter, one half of a year and one year.

The Figure 3 shows the four series generated with 126 days non-overlapping window. We can see that, in the GSPC-SPY, GSPC-SSO, SPY-SSO plots, the $\{+-_t\}$ and $\{-+_t\}$ series have pretty low values and show a trend to decrease. It indicates that the correlations among GSPC, SPY and SSO are becoming stronger. Also, the up and down movements of ++ and – series are likely to present the momentum-reverse pattern.

It is true that VIX and VXX decrease together for the most of time. We can also find that, in the pair of other series vs. VIX or VXX, the $\{+-_t\}$ series has the highest values in general, while in the pair of VIX or VXX vs. other series, the $\{-+_t\}$ series is usually the largest. So it is a typical situation where VIX or VXX drops down while other index or stock rises at the same time.

Recall the Figure 1 and compare it with the Figure 3, it is intuitive and generally correct that the larger the value of the correlation coefficient between a pair of two series, the wider the gap between the $\{++_t\}$, $\{--_t\}$ and the $\{+-_t\}$, $\{-+_t\}$ series.

3 Simulation with Centric Bivariate Normal Probability

To better understand the properties of the $\{++_t\}$, $\{+-_t\}$, $\{-+_t\}$, $\{--_t\}$ series, we implement simulation approach. We generate two correlated series R_1 and R_2 using standard normal random variables $X_1, X_2 \sim N(0,1)$ by:

$$\begin{aligned} R_1 &= X_1 \\ R_2 &= \rho X_1 + \sqrt{1-\rho^2} X_2 \end{aligned}$$

Both of R_1 and R_2 have the length of 100000. For comparison, we control ρ ranged from $[-0.9, 0.9]$ with interval 0.1.

3.1 Quadrant Probabilities of Centric Bivariate Normal Random Variables

The Figure 4 shows the proportion of the quadrant probabilities $(++, +-, -+, --)$ of the entire sample. The y-axis denotes the pairs (r_1, r_2). For example, the first row "rho=0.9" represents the

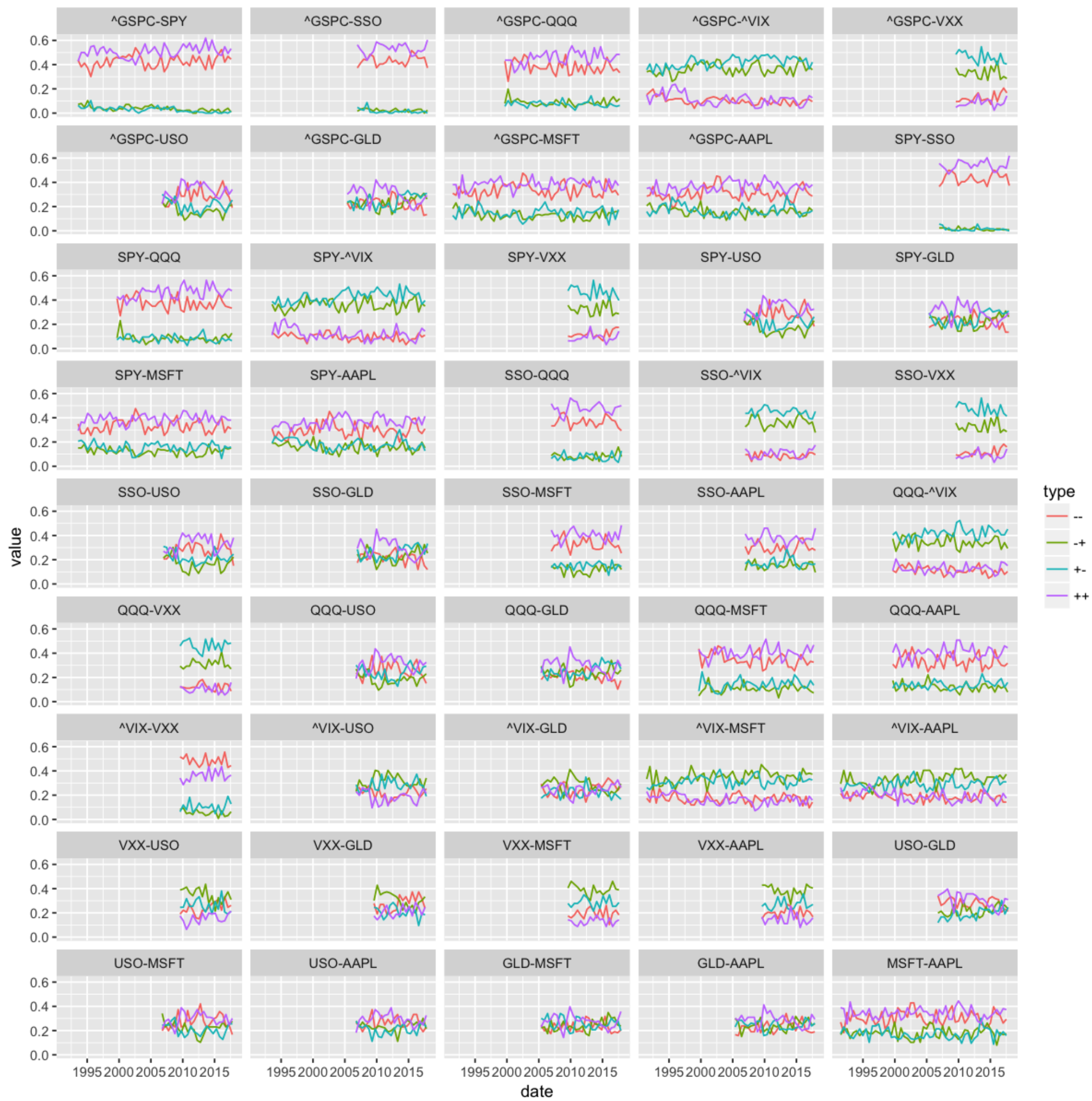


Figure 3: The quadrant probability series $\{++_t\}$, $\{+-_t\}$, $\{-+_t\}$, $\{--_t\}$. To have a cleaner and smoother look of the $\{++_t\}$, $\{+-_t\}$, $\{-+_t\}$, $\{--_t\}$ series, in this figure we use non-overlapping time window of 126 days. Each subplot shows the four quadrant probability series of each pair of stock or index. For all the subplots, the x-axis goes from 1991-01-02 to 2017-12-29. Notice that some series are shorter than other series because the their data only came into existence later than 1991-01-02. The y-axis shows the value of each probability series, which is a percentage from 0 to 1.

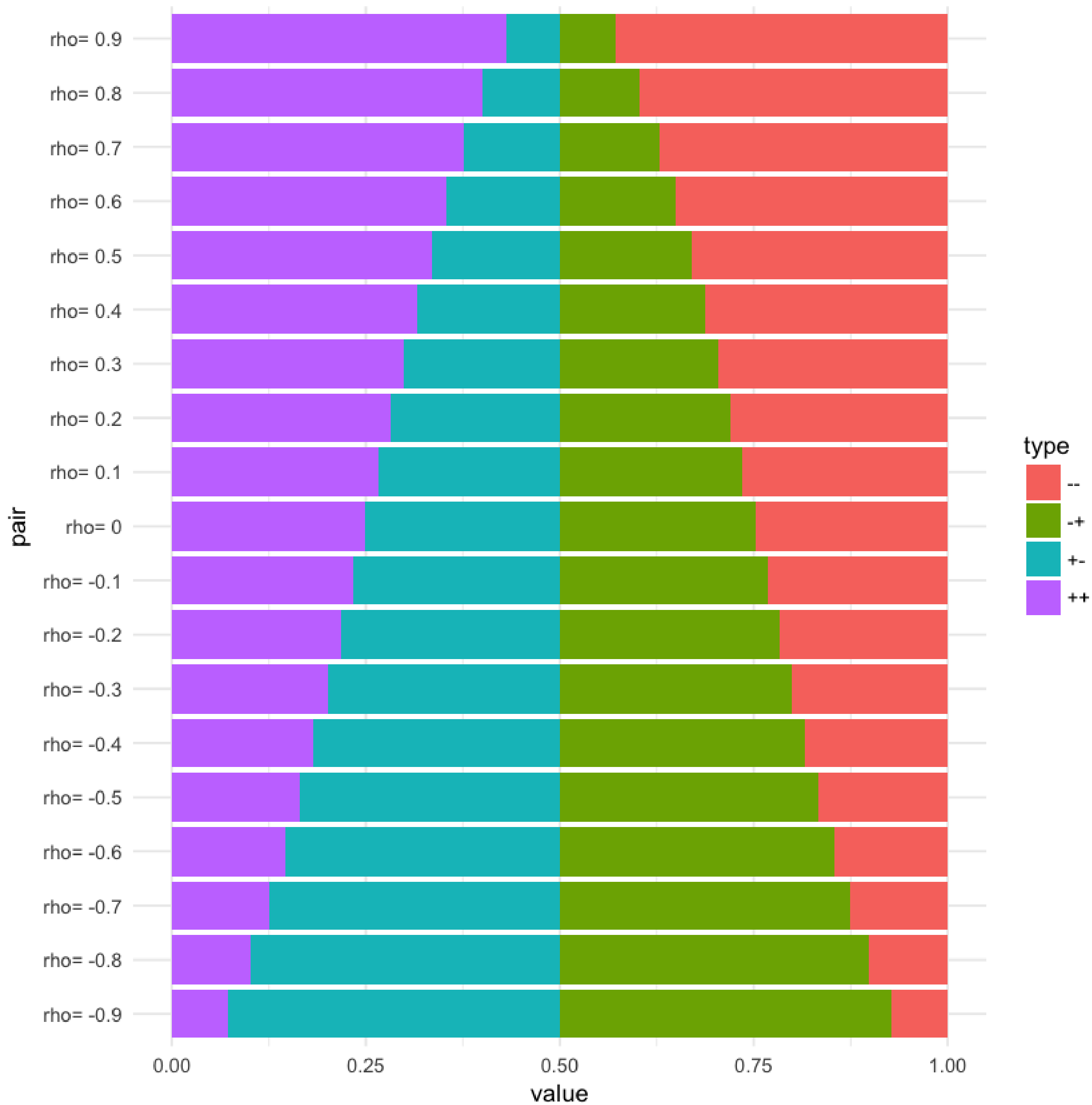


Figure 4: Proportion of the simulated quadrant probabilities $(++, +-, -+, --)$. x-axis shows the percentage from 0 to 1. Along the y-axis the horizontal bars with 4 colored fractions represents pairs with different correlation coefficients ρ from -0.9 to 0.9 and their proportions of $(++, +-, -+, --)$. The proportions of $(++, +-, -+, --)$ are represented correspondingly by the purple fraction, the blue fraction, the green fraction and the red fraction.

proportion of $(++, +-, -+, --)$ of the pair $(r_1, r_2|\rho = 0.9)$. The x-axis is the value of stacked proportion. We can see the Figure 4 is symmetric. The larger the positive value of ρ, the higher proportions of the $++$, and $--$ series, hence the lower proportions of the $+-$, $-+$ series, vice versa.

Chandramouli and Ranganathan (1999) gives the explicit formula of the quadrant probabilities of centric bivariate normal random variables with ρ. The standardized bivariate normal distribution takes $\sigma_1 = \sigma_2 = 1$ and $\mu_1 = \mu_2 = 0$:

$$p(r_1, r_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} exp\Big(-\frac{r_1^2 - 2\rho r_1 r_2 + r_2^2}{2(1-\rho^2)}\Big)$$

The analytic quadrant probabilities are:

$$\begin{aligned} P(r_1 < 0, r_2 < 0) &= P(r_1 > 0, r_2 > 0) \\ &= \int_0^{\infty}\int_0^{\infty} p(r_1, r_1) dr_1 dr_2 \\ &= \frac{1}{4} + \frac{sin^{-1}(\rho)}{2\pi} \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} P(r_1 > 0, r_2 < 0) &= P(r_1 < 0, r_2 > 0) \\ &= \int_0^{\infty}\int_{-\infty}^{0} p(r_1, r_1) dr_1 dr_2 \\ &= \frac{cos^{-1}(\rho)}{2\pi} \end{aligned} \tag{3.2}$$

Figure 5 presents the relation of quadrant probabilities calculated according to Equation (3.1) and Equation (3.2) with respect to the correlation coefficient. We can see that with the increase of ρ from -1 to 1, the probability that R_1 and R_2 move in the same direction increases and the probability that R_1 and R_2 move in different directions decreases.

3.2 The Simulated Quadrant Probability Series

To see the change of the $\{++_t\}$, $\{+-_t\}$, $\{-+_t\}$, $\{--_t\}$ series with the correlation coefficient ρ, we choose non-overlapping window of one year's length (252 days). The length of each series is 396.

In the Figure 6 the x-axis is the time index, and the y-axis is the value of series $\{++_t\}$, $\{+-_t\}$, $\{-+_t\}$, $\{--_t\}$. We can see clearly that from $\rho = -0.9$ to $\rho = 0.9$, the $\{+-_t\}$ and $\{-+_t\}$ series decrease while the $\{++_t\}$ and $\{--_t\}$ series increase. Then the two types of series crossover when $rho = 0$, and $\{++_t\}$ $\{--_t\}$ series keep rising and become greater than $\{+-_t\}$ $\{-+_t\}$ series.

It shows that when r_1 and r_2 have a positive correlation, the probability that they move in the same direction ($++$ and $--$) is higher than that they move in different directions ($+-$, $-+$), vice versa. The more significant the correlation, the larger the differences between the two kinds of quadrant probabilities.

In the simulation, the values of series are stable. Also, $\{++_t\}$ and $\{--_t\}$ have the same properties, so do the $\{+-_t\}$ and $\{-+_t\}$ series. However, with the real stocks data, $\{++_t\}$ and $\{--_t\}$, $\{+-_t\}$ and $\{-+_t\}$ are much volatile, and they are not symmetric either. That leads us to apply the non-centric bivariate normal probability in Section 4.

3.3 Statistics of the Quadrant Probability Series and Their Rolling Correlations

In the real scenario, we have to take account of the length and the smoothness of the data, so it is of necessity to apply overlapping window. Therefore, we recalculate the $\{++_t\}$, $\{+-_t\}$, $\{-+_t\}$,

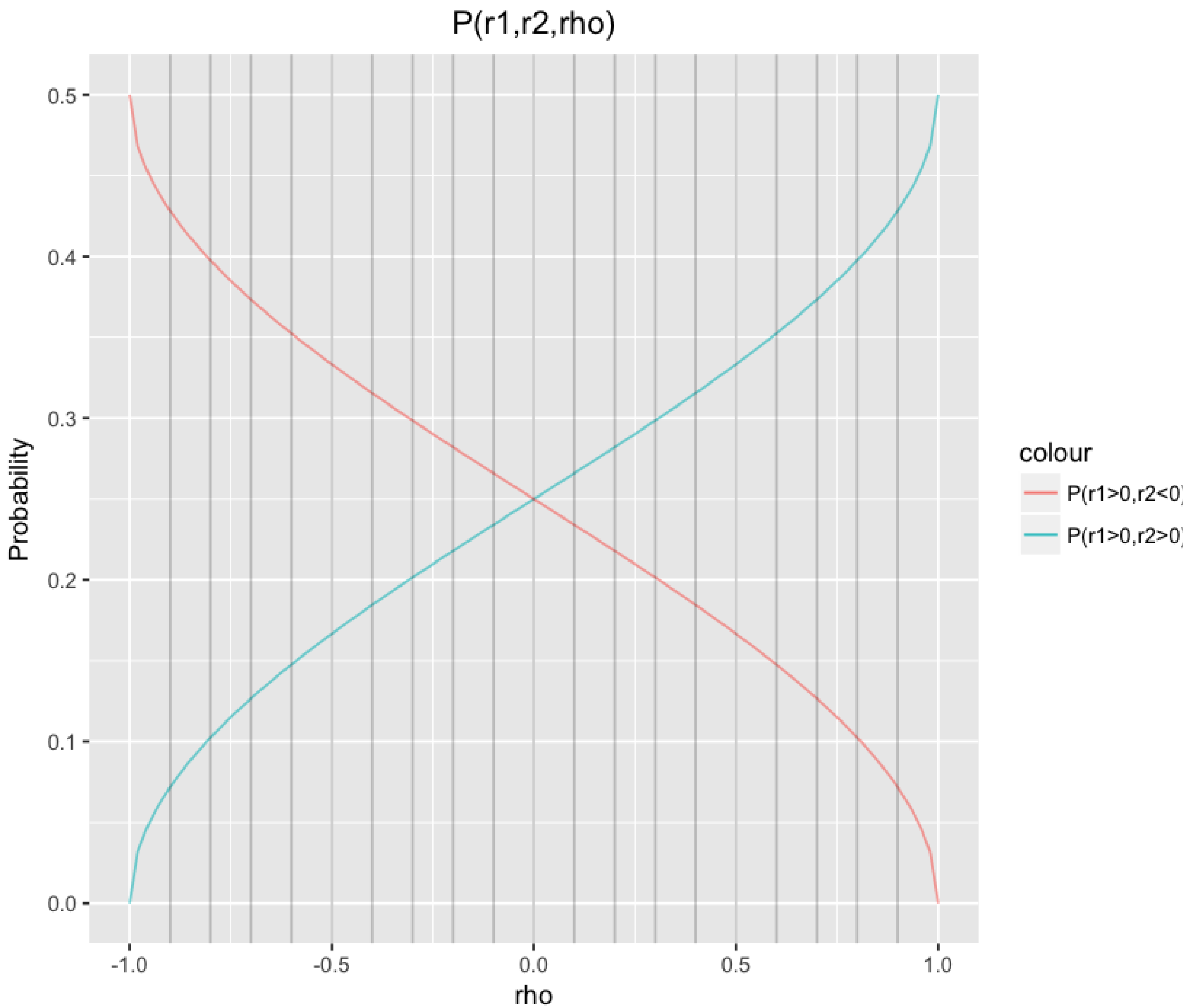


Figure 5: Quadrant probabilities. The x-axis is the correlation coefficient from -1 to 1. The y-axis shows the corresponding quadrant probability $P(r_1 > 0, r_2 > 0)$ and $P(r_1 > 0, r_2 < 0)$. In addition, $P(r_1 > 0, r_2 > 0) = P(r_1 < 0, r_2 < 0)$, and $P(r_1 > 0, r_2 < 0) = P(r_1 < 0, r_2 > 0)$.

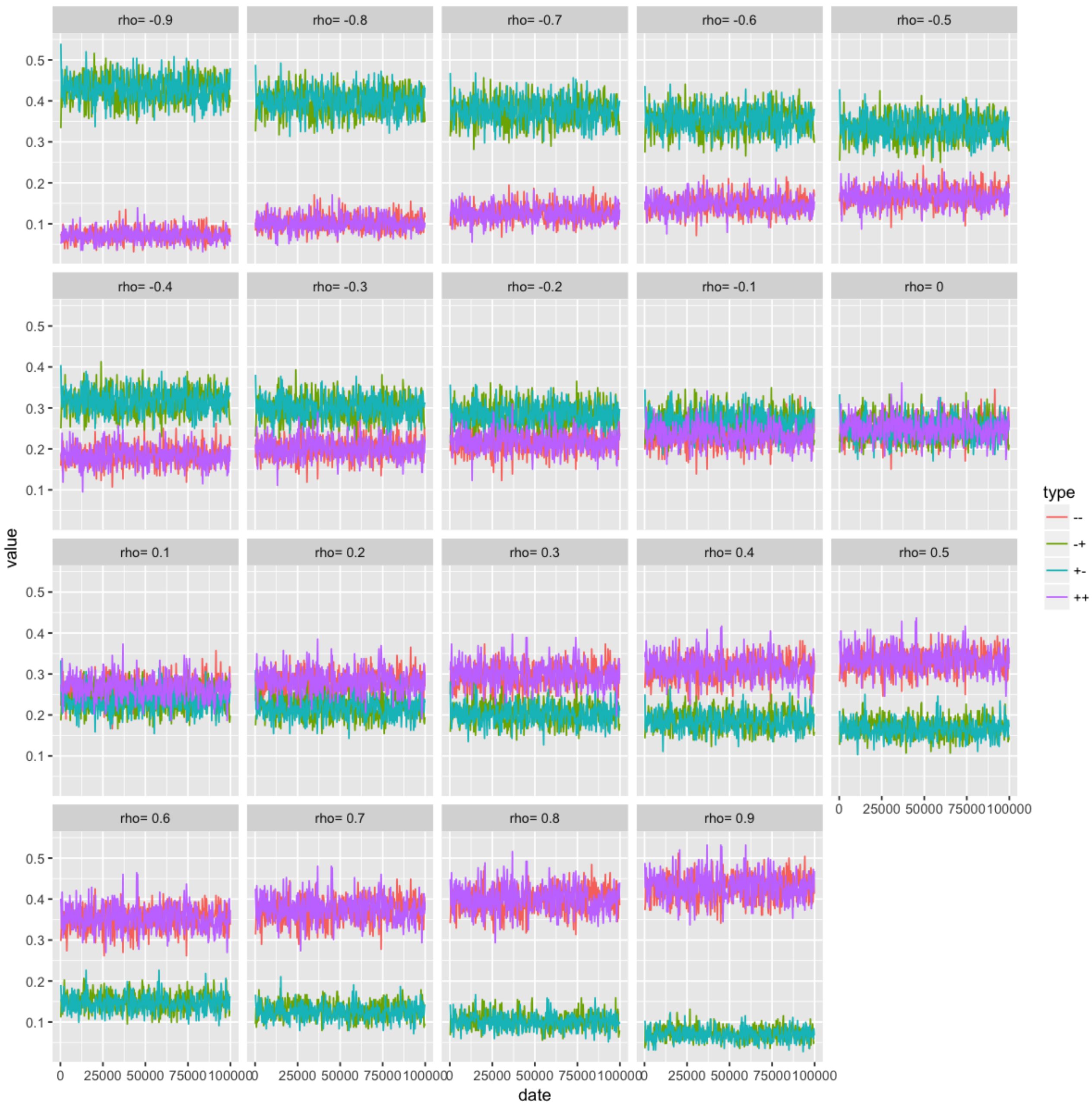


Figure 6: The simulated $\{++_t\}$, $\{+-_t\}$, $\{-+_t\}$, $\{--_t\}$ series. The x-axis is the time index. The y-axis shows the percentage from 0 to 1. From $\rho = -0.9$ to $\rho = 0.9$, the $\{+-_t\}$ and $\{-+_t\}$ series decrease while the $\{++_t\}$ and $\{--_t\}$ series increase. Then the two types of series crossover when $\rho = 0$, and the $\{++_t\}$ and $\{--_t\}$ series keep rising and become greater then the $\{+-_t\}$ and $\{-+_t\}$ series.

$\{--_t\}$ series and their rolling correlations with 252 days overlapping window, and plot the average values of these series.

In the Figure 7, the subplots in the left-hand side give the mean values of time series, and those in the right-hand side present the standard deviations. The first row provides the means and standard deviations of the $\{++_t\}$, $\{+-_t\}$, $\{-+_t\}$, $\{--_t\}$ series, and the second row is for the rolling correlations of theirs. The light-colored lines are the simulated value, and the solid-colored lines are the smoothed ones.

For the $\{++_t\}$, $\{+-_t\}$, $\{-+_t\}$, $\{--_t\}$:

1. The means of $\{++_t\}$, $\{+-_t\}$, $\{-+_t\}$, $\{--_t\}$ are almost the same to the theoretical values of the quadrant probabilities. For the different random data sets, the average values of $\{++_t\}$, $\{+-_t\}$, $\{-+_t\}$, $\{--_t\}$ are consistent.

2. The standard deviations are symmetric. When ρ rises from -0.9 to 0.9, the standard deviation of $\{++_t\}$ and $\{--_t\}$ increases from about 0.0175 to about 0.0325, while the standard deviation of $\{+-_t\}$ and $\{-+_t\}$ drops from near 0.0325 to 0.0175. The reason why $\{++_t\}$ is higher than $\{--_t\}$ and why $\{-+_t\}$ is higher than $\{+-_t\}$ might due to the sampling problem. Intuitively, the property of $\{++_t\}$ and $\{--_t\}$, $\{+-_t\}$ and $\{-+_t\}$ should be identical.

For the rolling correlations, because now we have a larger sample, so the means and standard deviations are much smoother than those of the non-overlapping window. We can see that:

1. The average value of rolling correlations is negative. As ρ rises from -0.9 to 0.9, $+-/-+$ and $++/--$ monotonically change between about -0.7 and -0.1, while the other correlation series are convex and drop from below -0.2 to -0.3, and then return to -0.2 again.

2. As for the standard deviation, both the $+-/-+$ and $++/--$ show the same particular pattern. When R_1 and R_2 have a positive correlation, the $++/--$ not only has the most negative value but also has its standard deviation decreasing sharply. When $\rho = 0.9$, $++/--$ has a mean of about -0.7 and standard deviation of -0.3. Symmetrically, the $++/--$ has a large negative value of -0.7 and small standard deviation of 0.3 when $rho = -0.9$.

4 Simulation with Non-centric Bivariate Normal Probability

In the previous section, we have discussed the quadrant probabilities under the centric bivariate normal distribution. In the real scenario, two time series are most likely to have different means and variances. In other words, two time series are usually non-centric bivariate normally distributed. Therefore, in this section, we consider a more general method to calculate the non-centric bivariate normal probability.

4.1 The Bivariate Normal Probability in Non-centric Case

If two random variables $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma_2^2)$ have a correlation coefficient of ρ, and we take the regression formula

$$X_2 = a + bX_1 + \epsilon$$

where ϵ is independent and $\epsilon \sim N(0, \delta^2)$. Then

$$\mu_2 = b\mu_1 \qquad \sigma_2^2 = b^2\sigma_1^2 + \delta^2$$

so the distribution of X_2 is

$$X_2 \sim N(a + b\mu_1, b^2\sigma_1^2 + \delta^2)$$

The joint distribution of X_1 and X_2 is

$$X_1, X_2 \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

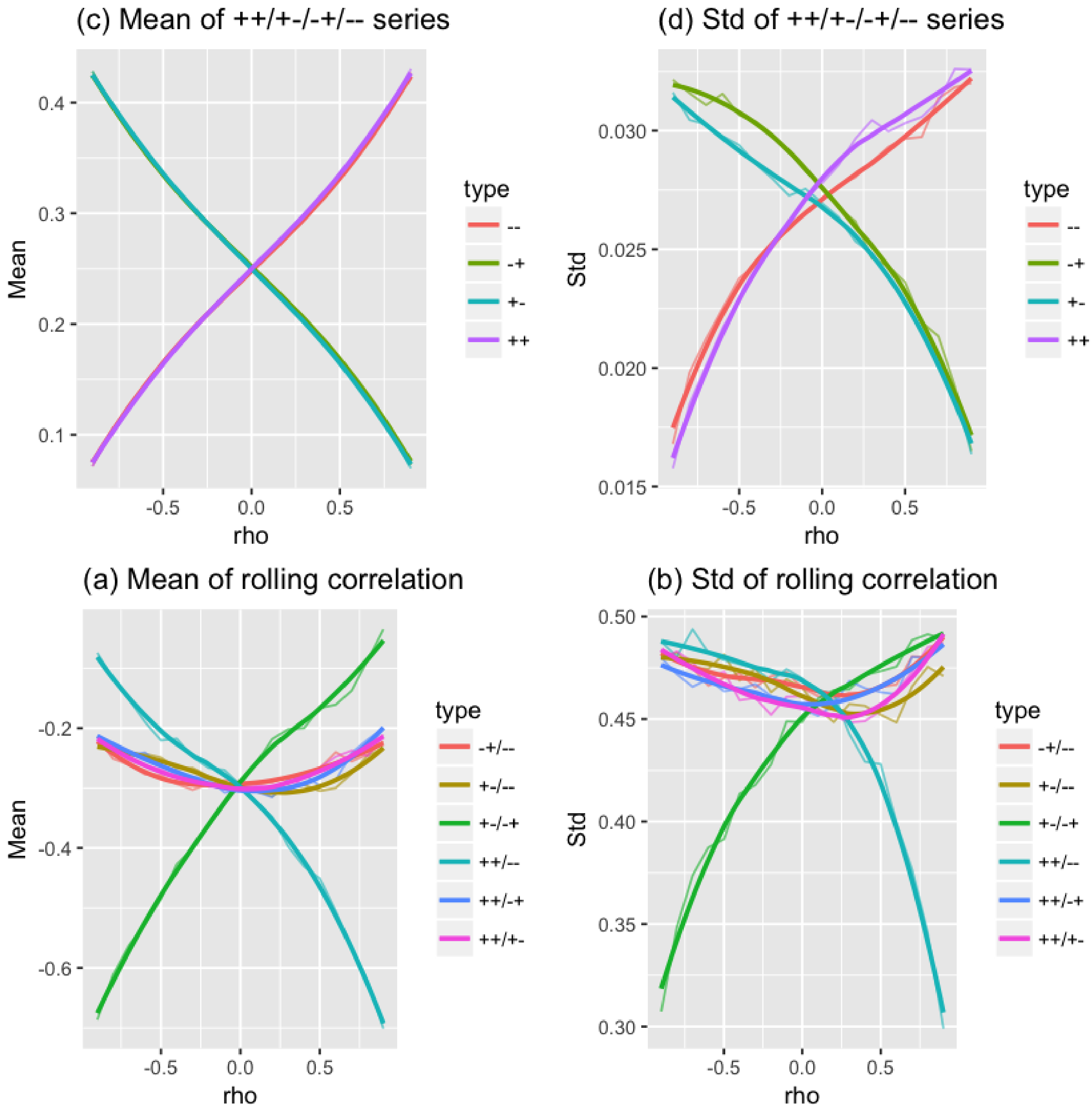


Figure 7: Means and standard deviations. The x-axis is the correlation coefficient assigned from -0.9 to 0.9, and the y-axis is the corresponding value of mean and standard deviation of the $\{++_t\}$, $\{+-_t\}$, $\{-+_t\}$, $\{--_t\}$ series' and their rolling correlation series'. The left-hand side of subplot shows the means of time series and the right-hand side gives the standard deviations. The first row shows the means and standard deviations of the $\{++_t\}$, $\{+-_t\}$, $\{-+_t\}$, $\{--_t\}$ series, and the second row is for the rolling correlations of theirs. The light-colored lines are the simulated value, and the solid-colored lines are the smoothed ones

where

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ a + b\mu_1 \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sqrt{b^2\sigma_1^2+\delta^2} \\ \rho\sigma_1\sqrt{b^2\sigma_1^2+\delta^2} & b^2\sigma_1^2+\delta^2 \end{pmatrix}, \quad \rho = \frac{b\sigma_1}{\sqrt{b^2\sigma_1^2+\delta^2}},$$

Therefore, the probability density function of X_1, X_2 is given as

$$\phi(x_1, x_2, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}\sigma_1\sqrt{b^2\sigma_1^2+\delta^2}} \exp\left(-\frac{\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-a-b\mu_1)^2}{b^2\sigma_1^2+\delta^2} - \frac{2\rho(x-\mu_1)(x_2-a-b\mu_1)}{\sigma_1\sqrt{b^2\sigma_1^2+\delta^2}}}{2(1-\rho^2)} \right)$$

Recall the centric case in Section 3, the quadrant probabilities can be given analytically by:

$$P(x_1 < \mu_1, x_2 < a + b\mu_1) = P(x_1 > \mu_1, x_2 > a + b\mu_1) = \frac{1}{4} + \frac{sin^{-1}\left(\frac{b\sigma_1}{\sqrt{b^2\sigma_1^2+\delta^2}}\right)}{2\pi}$$

$$P(x_1 > \mu_1, x_2 < a + b\mu_1) = P(x_1 < \mu, x_2 > a + b\mu_1) = \frac{cos^{-1}\left(\frac{b\sigma_1}{\sqrt{b^2\sigma_1^2+\delta^2}}\right)}{2\pi}$$

4.2 Approximating the Quadrant Probability by Numerical Integration

In our study, we care where the two time series move in the same or different directions. The probability for a pair of series to increase together is

$$P(x_1 > 0, x_2 > 0) = \int_0^{+\infty} \int_0^{+\infty} \phi\left(x_1, x_2, \rho\right) dx_1 dx_2$$

This integral cannot be solved analytically. We refer to the study of Genz (2004) on numerical computation of the non-centric quadrant probability of bivariate normal distribution. It shows that the bivariate standard normal probability for $P(x > h, y > k)$

$$P(x_1 > h, x_2 > k) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_h^{+\infty} \int_k^{+\infty} \exp\left(-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1-\rho^2)} \right) dx_1 dx_2 \tag{4.1}$$

can be approximated by

$$L(h, k, \rho) = \Phi(-h)\Phi(-k) + \frac{1}{2\pi} \int_0^{\rho} \frac{1}{\sqrt{1-v^2}} \exp\left(-\frac{h^2 + k^2 - 2vhk}{2(1-v^2)} \right) dv \tag{4.2}$$

To adapt the formula to our case, we standardize the two random variables. Denote:

$$y_1 = \frac{x_1 - \mu_1}{\sigma_1}, \quad y_2 = \frac{x_2 - a - b\mu_1}{\sqrt{b^2\sigma_1^2+\delta^2}}$$

and

$$h = -\frac{\mu_1}{\sigma_1}, \quad k = -\frac{a + b\mu_1}{\sqrt{b^2\sigma_1^2+\delta^2}}, \quad \rho = \frac{b\sigma_1}{\sqrt{b^2\sigma_1^2+\delta^2}}$$

Then the Equation (4.1) can be approximated with Equation (4.2) as

$$\begin{aligned} P\left(x_1 > 0, x_2 > 0\right) &= P\left(y_1 > h, y_2 > k\right) \\ &\approx L(h, k, \rho) \end{aligned}$$

Therefore, we can compute the numerical integration of the four quadrant probabilities, which are give by:

$$\begin{aligned} P\left(x_1 > 0, x_2 > 0\right) &\approx L(h, k, \rho) \\ P\left(x_1 > 0, x_2 < 0\right) &\approx L(h, -k, -\rho) \\ P\left(x_1 < 0, x_2 > 0\right) &\approx L(-h, k, -\rho) \\ P\left(x_1 < 0, x_2 < 0\right) &\approx L(-h, -k, \rho) \end{aligned} \tag{4.3}$$

4.3 The Simulated Quadrant Probability Series

We firstly select MSFT and GSPC to verify our method, using the regression model:

$$X_2 = a + bX_1 + \epsilon \tag{4.4}$$

First, we fit the model (4.4) with the entire time horizon from when both of time series are available to the date 2017-12-29. According to the discussion in the previous section, we assume that $\epsilon \sim N(0, 1)$, and X_1, X_2 are jointly normal distributed.

After estimating the regression parameters $\hat{a}$ and $\hat{b}$, we can obtain the five parameters μ_1, σ_1, μ_2, σ_2, ρ, δ and input them into the Formulas (4.3) and Equation (4.2) to calculate the quadrant probabilities. The Table 1 shows the comparison of the observed $(++, +-, -+, --)$ and estimated ones.

Next we apply the rolling time window to generate the $\{++_t\}$, $\{+-_t\}$, $\{-+_t\}$, $\{--_t\}$ series as we did in Section 2. Let the length of window be 252, and overlapping. For the observed $\{++_t\}$, $\{+-_t\}$, $\{-+_t\}$, $\{--_t\}$ series, they are calculated as we did in Section 2 to count the proportion of times when $(X_1 > 0, X_2 > 0)$, $(X_1 > 0, X_2 < 0)$, $(X_1 < 0, X_2 > 0)$, $(X_1 < 0, X_2 < 0)$ within the window.

We compute the estimated $\{++_t\}$, $\{+-_t\}$, $\{-+_t\}$, $\{--_t\}$ series with the following procedure: first, fit the model $X_2 = a + bX_1 + \epsilon$ with the 252 observations in the window and obtain the parameters $\mu_1, \sigma_1, \mu_2, \sigma_2, \rho, \delta$. Second, input the parameters into the Formulas (4.3) and (4.2) to calculate the estimated $++_t, +-_t, -+_t, --_t$. Repeat the procedure along the moving window and we can obtain the estimated $\{++_t\}$, $\{+-_t\}$, $\{-+_t\}$ and $\{--_t\}$ series.

The Figure 8 is the comparison plot of the observed $\{++_t\}$, $\{+-_t\}$, $\{-+_t\}$, $\{--_t\}$ series and the estimated $\{++_t\}$, $\{+-_t\}$, $\{-+_t\}$, $\{--_t\}$ series of GSPC-MSFT. We can find that the patterns of the series in the two plots are almost the same. However, the estimated series are less volatile than the observed series and its gap of $\{++_t\}$ and $\{--_t\}$ series, and $\{+-_t\}$ $\{-+_t\}$ series are narrower as well.

Figure 13 gives extra examples of estimated and observed $\{++_t\}$, $\{+-_t\}$, $\{-+_t\}$, $\{--_t\}$ series. As we can see, the more correlated the two series are, the closer are the estimated series to the observed.

To compare the estimated series with the observed, we take the differences of the observed series and the estimated series:

$$\begin{aligned} ++_t^{diff} &= ++_t^{observed} - ++_t^{estimated} \\ +-_t^{diff} &= +-_t^{observed} - +-_t^{estimated} \\ -+_t^{diff} &= -+_t^{observed} - -+_t^{estimated} \\ --_t^{diff} &= --_t^{observed} - --_t^{estimated} \end{aligned}$$

	$++^{obs}$	$+-^{obs}$	$-+^{obs}$	$--^{obs}$	$++^{est}$	$+-^{est}$	$-+^{est}$	$--^{est}$
ˆGSPC-ˆVIX	0.115	0.4196	0.3621	0.1034	0.1374	0.3759	0.3743	0.1124
ˆGSPC-GLD	0.2993	0.2463	0.2317	0.2227	0.2675	0.2432	0.2453	0.2441
ˆGSPC-MSFT	0.3847	0.1498	0.1313	0.3342	0.3726	0.1408	0.1455	0.3412
ˆGSPC-AAPL	0.3662	0.1683	0.1607	0.3048	0.338	0.1753	0.1815	0.3052
ˆGSPC-ORCL	0.3799	0.1547	0.1442	0.3212	0.3509	0.1625	0.1661	0.3205
ˆGSPC-GOOG	0.3869	0.1602	0.1316	0.3212	0.3639	0.1475	0.1587	0.3299
ˆGSPC-BBBY	0.3659	0.1712	0.1463	0.3166	0.3403	0.1723	0.1721	0.3154
ˆGSPC-MMM	0.3902	0.1444	0.142	0.3234	0.3695	0.1439	0.1463	0.3404
ˆGSPC-GE	0.403	0.1316	0.1186	0.3468	0.3918	0.1215	0.1236	0.3631
ˆVIX-GLD	0.2469	0.219	0.2842	0.2499	0.2584	0.2554	0.2543	0.2319
ˆVIX-MSFT	0.1641	0.313	0.352	0.171	0.1932	0.3185	0.3249	0.1634
ˆVIX-AAPL	0.1904	0.2867	0.3365	0.1864	0.214	0.2976	0.3055	0.1829
ˆVIX-ORCL	0.1717	0.3054	0.3524	0.1705	0.205	0.3066	0.312	0.1764
ˆVIX-GOOG	0.1602	0.304	0.3584	0.1774	0.1983	0.3149	0.3243	0.1625
ˆVIX-BBBY	0.1719	0.3063	0.3403	0.1815	0.2065	0.3058	0.3059	0.1819
ˆVIX-MMM	0.1779	0.2992	0.3543	0.1686	0.1925	0.3191	0.3232	0.1651
ˆVIX-GE	0.1616	0.3155	0.36	0.1629	0.1838	0.3278	0.3315	0.1568
GLD-MSFT	0.2745	0.2566	0.2408	0.2281	0.2636	0.2492	0.2503	0.237
GLD-AAPL	0.2884	0.2427	0.243	0.226	0.2683	0.2444	0.2597	0.2276
GLD-ORCL	0.2796	0.2514	0.2424	0.2266	0.2634	0.2493	0.2498	0.2375
GLD-GOOG	0.2781	0.253	0.239	0.2299	0.2654	0.2473	0.2546	0.2327
GLD-BBBY	0.2648	0.2663	0.2375	0.2314	0.2502	0.2625	0.2499	0.2374
GLD-MMM	0.2866	0.2445	0.2493	0.2196	0.2657	0.247	0.2495	0.2377
GLD-GE	0.266	0.2651	0.2436	0.2254	0.2534	0.2593	0.2483	0.239
MSFT-AAPL	0.3456	0.1704	0.1813	0.3027	0.327	0.1911	0.1925	0.2894
MSFT-ORCL	0.3586	0.1575	0.1655	0.3184	0.343	0.1751	0.174	0.3079
MSFT-GOOG	0.3447	0.1721	0.1738	0.3094	0.3442	0.1703	0.1784	0.3071
MSFT-BBBY	0.3173	0.1969	0.1948	0.2909	0.3111	0.2051	0.2012	0.2825
MSFT-MMM	0.3365	0.1795	0.1957	0.2883	0.318	0.2001	0.1977	0.2842
MSFT-GE	0.3377	0.1783	0.1839	0.3001	0.3315	0.1866	0.1839	0.2981
AAPL-ORCL	0.3406	0.1863	0.1835	0.2896	0.3224	0.1972	0.1946	0.2859
AAPL-GOOG	0.3513	0.1825	0.1673	0.299	0.3514	0.1794	0.1711	0.298
AAPL-BBBY	0.3141	0.2111	0.1981	0.2767	0.2999	0.219	0.2124	0.2686
AAPL-MMM	0.3239	0.203	0.2083	0.2648	0.3041	0.2155	0.2117	0.2688
AAPL-GE	0.3261	0.2008	0.1955	0.2776	0.3142	0.2053	0.2012	0.2793
ORCL-GOOG	0.3397	0.1834	0.1789	0.2981	0.337	0.1775	0.1855	0.3
ORCL-BBBY	0.3263	0.1959	0.1858	0.2919	0.3114	0.2049	0.2009	0.2827
ORCL-MMM	0.3381	0.186	0.1941	0.2818	0.3097	0.2072	0.206	0.277
ORCL-GE	0.3378	0.1863	0.1838	0.2921	0.3221	0.1949	0.1932	0.2898
GOOG-BBBY	0.321	0.1976	0.1822	0.2993	0.3122	0.2103	0.1891	0.2883
GOOG-MMM	0.3459	0.1727	0.189	0.2924	0.3366	0.1859	0.1787	0.2987
GOOG-GE	0.3331	0.1854	0.1783	0.3031	0.3245	0.1981	0.178	0.2995
BBBY-MMM	0.3318	0.1804	0.2	0.2878	0.3149	0.1975	0.2007	0.2869
BBBY-GE	0.3298	0.1824	0.1911	0.2967	0.3167	0.1956	0.1963	0.2914
MMM-GE	0.3625	0.1697	0.1591	0.3087	0.3444	0.1713	0.1709	0.3133

Table 1: Observed $(++, +-, -+, --)$ vs estimated $(++, +-, -+, --)$.

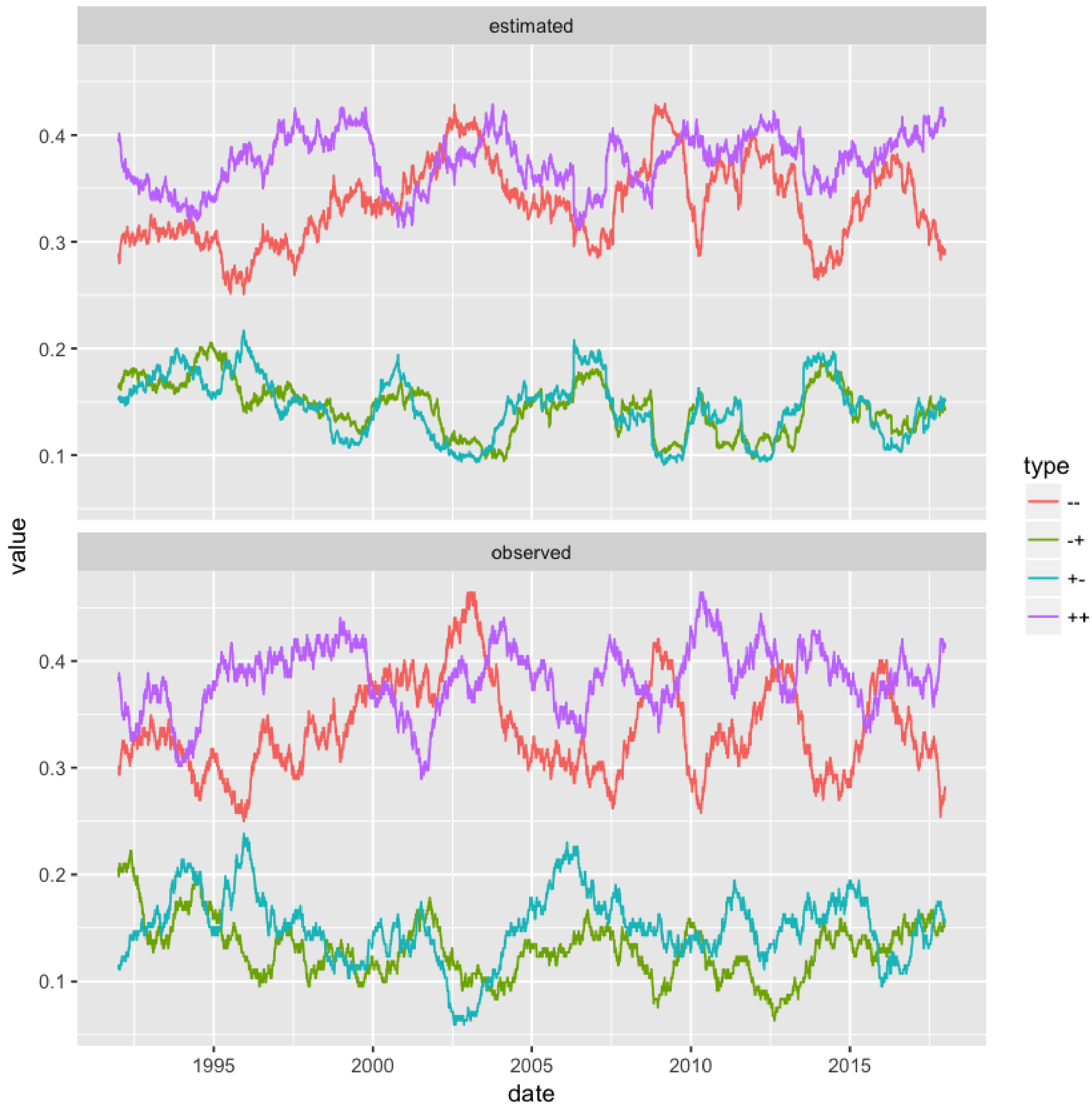


Figure 8: Estimated $\{++_t\}$, $\{+-_t\}$, $\{-+_t\}$, $\{--_t\}$ series vs observed series of GSPC-MSFT. The time horizon is from 1991-01-03 to 2017-12-29. The x-axis is the time index and the y-axis shows the value of each probability series, which is percentage from 0 to 1. Notice that the patterns of the series in the two plots are almost the same. However, the estimated series are less volatile than the observed series and its gap of $\{++_t\}$ and $\{--_t\}$ series, and $\{+-_t\}$ $\{-+_t\}$ series are narrower as well

The Figure 9 shows the difference between estimated and observed series. We can see that the four different series have constant mean and variance and the differences between the observed value and the estimated value are within ± 0.04 most of the time.

The Figure 10 is the histogram of the differences. Notice that the $\{++_t^{diff}\}$ and $\{+-_t^{diff}\}$ series are slightly right skewed while the $\{-+_t^{diff}\}$ and $\{--_t^{diff}\}$ series are slightly left skewed.

4.4 Statistics of the Estimated Quadrant Probabilities

After we apply the model to estimate the quadrant probabilities $(++, +-, -+, --)$, we are also interested to know the statistics of them. Thus we do simulations to generate the $(++, +-, -+, --)$ and calculate their means and standard deviations.

Starting with $X_1 \sim N(\mu_1, \sigma_1)$, we can either simulate $X_2\ N(\mu_2, \sigma_2)$ with correlation ρ, or choose the regression coefficients as inputs to simulate $X_2 = a + bX_1 + \epsilon$. The two methods are equivalent. To have a better control of parameters for the simulation, we choose the first method so that we can adjust the sets of parameters $\mu_1, \sigma_1, \mu_2, \sigma_2, \rho$.

Further, we can reduce the number of parameters without losing generality with standardization. We let $h = \mu_1/\sigma_1$ and $k = \mu_2/\sigma_2$. Then $P(X1 > 0, X2 > 0)$ becomes $P(Z1 > h, Z2 > k)$, so the number of parameters are reduced to only three - h, k and ρ.

We simulated 1000 samples of a length of 6882, which equals to the trading days from 1991-01-03 to 2017-12-29. Figure 11 is the histogram of the simulated $(++, +-, -+, --)$ with $\rho = 0.6$. The mean values of estimated $(++, +-, -+, --)$ are 0.395, 0.146, 0.146 and 0.313. They are symmetrically distributed, and the standard deviations are about 0.055.

Figure 12 is the 3D plot to visualize the standard deviations of the simulated quadrant probability series. Noticing that $-\mu/\sigma$ of our selected stocks and indices is around -0.05 to -0.02, correspondingly we choose h and k in $[-0.1, 0.1]$ and fix $\rho = 0.2, 0.4, 0.6, 0.8$. The 3D plot shows that the $\sigma(++)$ goes from 0.006 to 0.0055 with h, k increasing from -0.1 to 0.1.

5 Conclusion

This paper provides a new methodology for evaluating the correlation and co-movement of a pair of time series. With the entire time horizon, we compute the quadrant probabilities $(++, +-, -+, --)$ to present the correlation between two time series. Further, when we apply rolling time windows to compute $(++, +-, -+, --)$, we construct the quadrant probability series $\{++_t\}$, $\{+-_t\}$, $\{-+_t\}$ and $\{--_t\}$. The pattern of $\{++_t\}$, $\{+-_t\}$, $\{-+_t\}$ and $\{--_t\}$ reveals the co-movement of the pair of time series. If the correlation is significant, the $\{++_t\}$ and $\{--_t\}$ will greatly diverge from the $\{+-_t\}$ and $\{-+_t\}$. Meanwhile, we notice there are cases where $\{++_t\}$ could be greater than$\{--_t\}$, which indicates the situation where the two time series tend to increase simultaneously rather than dropping together. Inspired by the pattern of the quadrant probability series, we construct a method to forecast such series with the real data set. We implement rolling linear regression on a pair of time series, such as GSPC-MSFT showed in Section 4. With the regression parameters, we estimate the future quadrant probabilities with numerical integration. The accuracy of the estimated $\{++_t\}$, $\{+-_t\}$, $\{-+_t\}$, $\{--_t\}$ would vary with respect to the correlation ρ of the two time series. Generally, the estimated quadrant probability series would be less volatile than the observed series. Finally, the statistics of the estimated series shows the standard deviations of the quadrant probabilities are negatively affected by the μ_1, σ_1, μ_2, σ_2 and rho. For future research, the quadrant probability could serve as a useful measurement for various topics. For example, we could implement it to measure how well an ETF tracks an index. Similarly, we could study how two similar ETFs behave in the perspective of quadrant probabilities. It could also be used to evaluate how an individual company from a certain sector co-moves with respect to a price index.

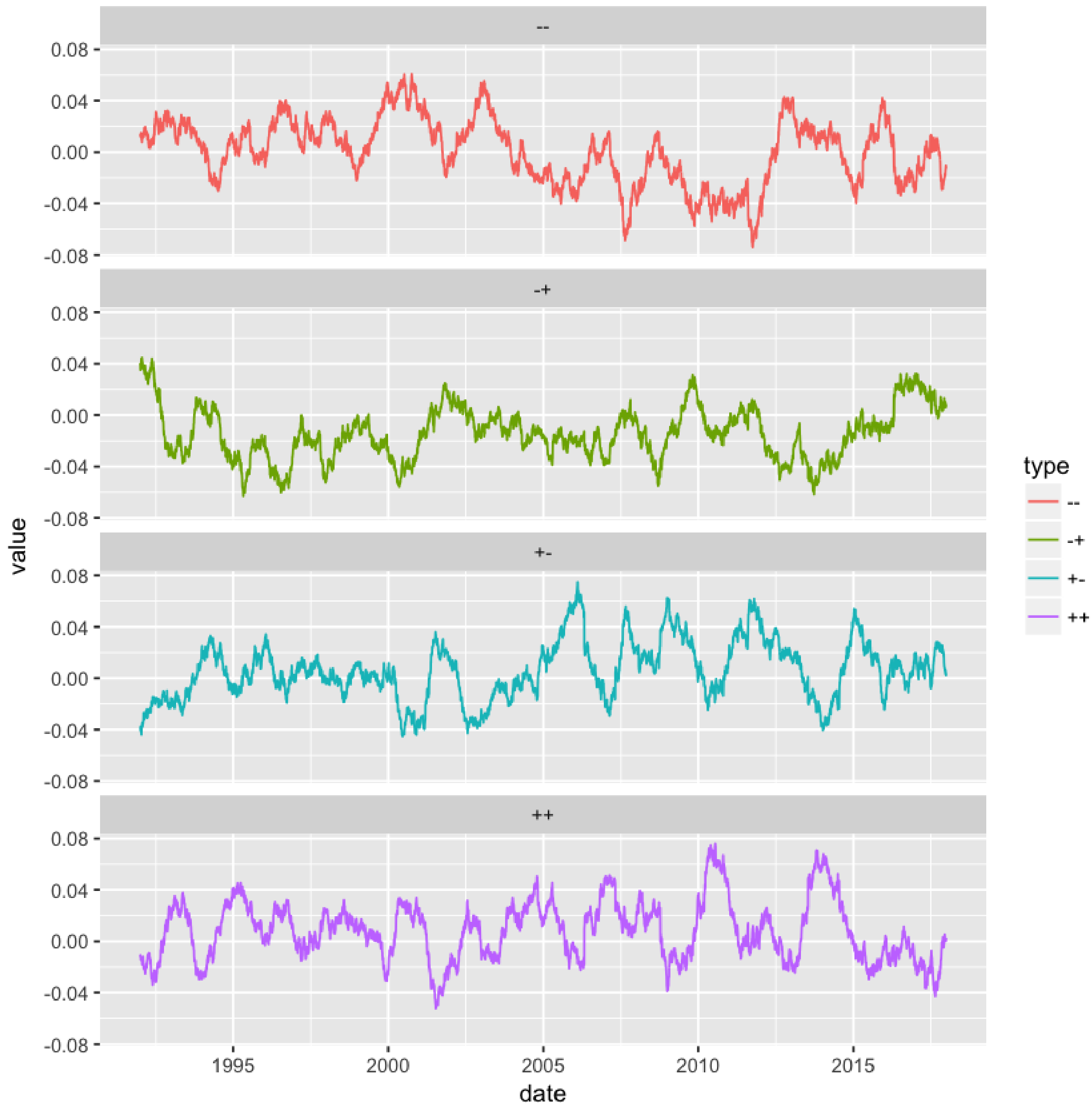


Figure 9: Difference series of observed and estimated $\{++_t\}$, $\{+-_t\}$, $\{-+_t\}$, $\{--_t\}$ series of GSPC-MSFT. The time horizon is from 1991-01-03 to 2017-12-29. The x-axis is the time index and the y-axis shows the value of each difference series from 0 to 1. We can see that the four difference series have constant mean and variance and the differences between the observed value and the estimated value are within ± 0.04 most of the time

Figure 10: Histogram of difference series of observed and estimated $\{++_t\}$, $\{+-_t\}$, $\{-+_t\}$, $\{--_t\}$ series of GSPC-MSFT. We can see that the $\{++_t^{diff}\}$ and $\{+-_t^{diff}\}$ series are slightly right skewed while the $\{-+_t^{diff}\}$ and $\{--_t^{diff}\}$ series are slightly left skewed.

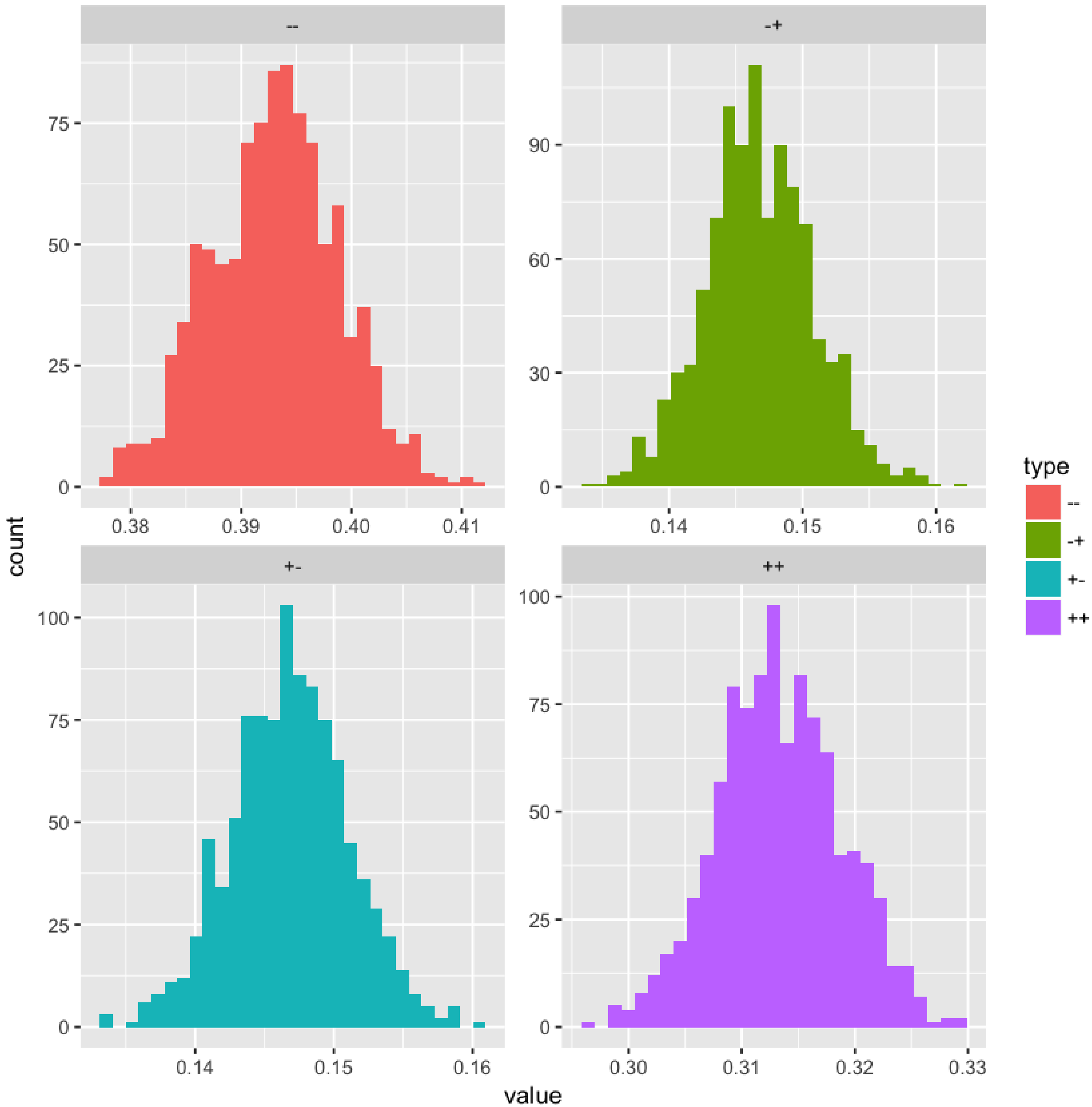
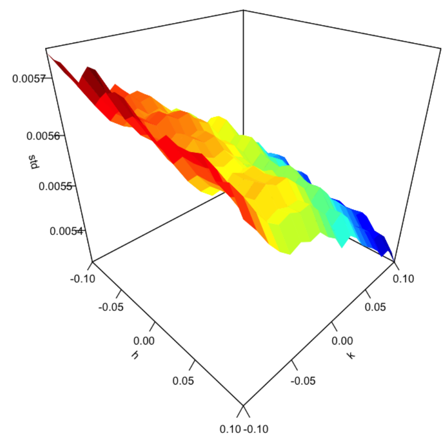
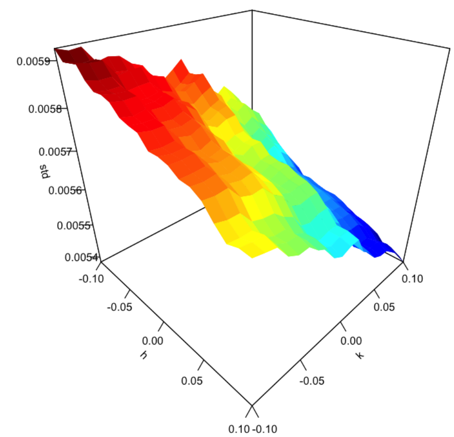


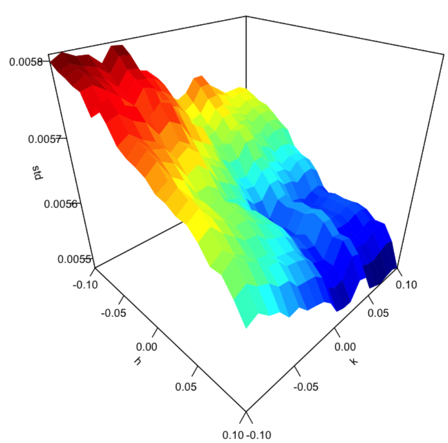
Figure 11: Histogram of simulated quadrant probabilities $(++, +-, -+, --)$ at $\rho = 0.6$. It is calculated from simulated 1000 samples of length of 6882, which equals to the trading days from 1991-01-03 to 2017-12-29. The mean values of estimated $(++, +-, -+, --)$ are 0.395, 0.146, 0.146 and 0.313, and the standard deviations are about 0.055. In this case both the $++$ and $--$ are greater than $+-$ and $-+$, but it is worth noting that $++$ is greater than $--$, which shows a asymmetric pattern. In other words, the two paired time series tend to increase together rather than decreasing together



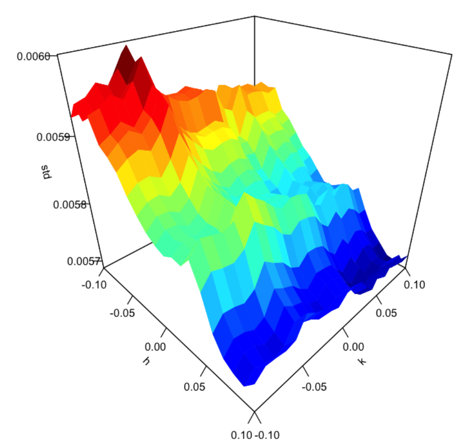
(a) σ_{++}, $\rho = 0.2$



(b) σ_{++}, $\rho = 0.4$



(c) σ_{++}, $\rho = 0.6$



(d) σ_{++}, $\rho = 0.8$

Figure 12: Standard deviation of ++ at $\rho \in \{0.2, 0.4, 0.6, 0.8\}$. We simulate 1000 samples of length of 6882 - which equals to the trading days from 1991-01-03 to 2017-12-29 - to presents the changes of standard deviations of ++ with respect to the parameters h, k and ρ. The horizontal axis are the values of h and k from -0.1 to 0.1. The vertical axis is the value of σ_{++}. We can see that, for each ρ, with the increase of h and k, the standard deviations of ++ decrease correspondingly.

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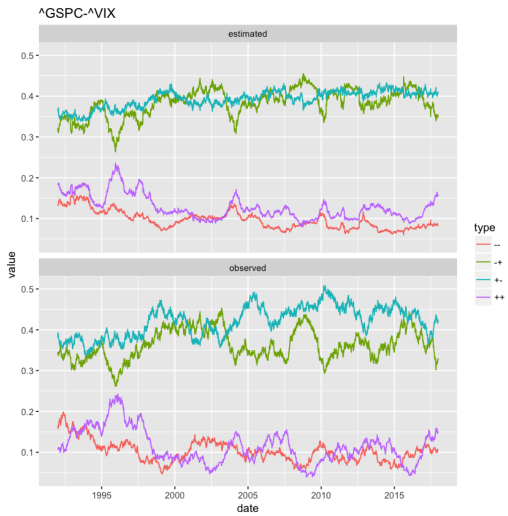
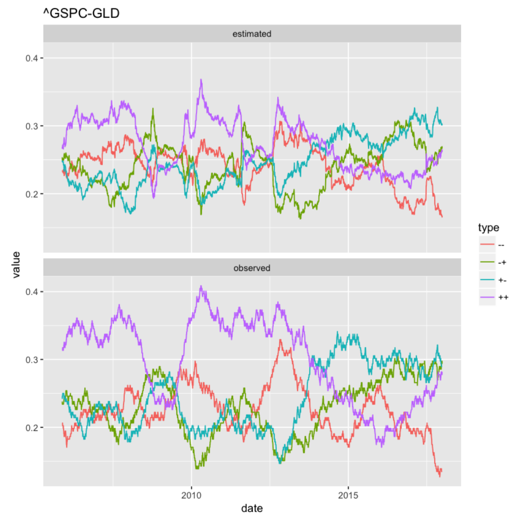
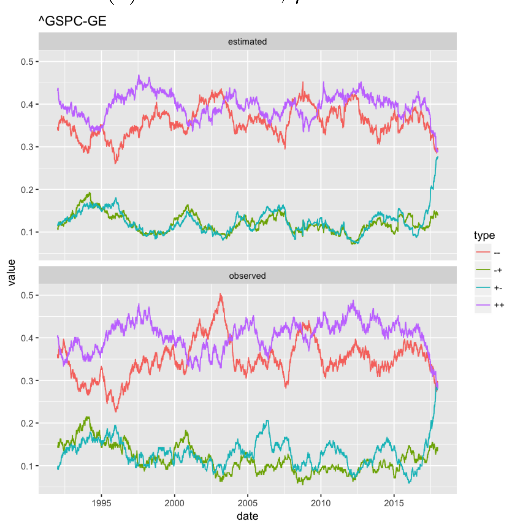


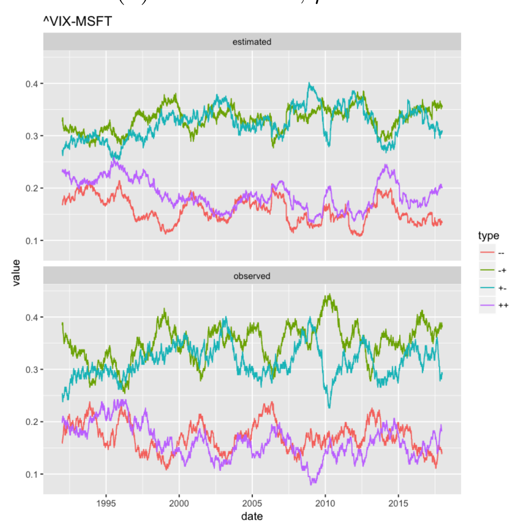
(a) GSPC-VIX, $\rho = -0.71$



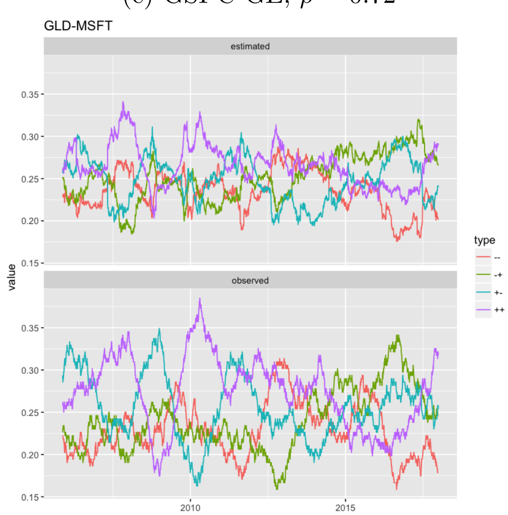
(b) GSPC-GLD, $\rho = 0.04$



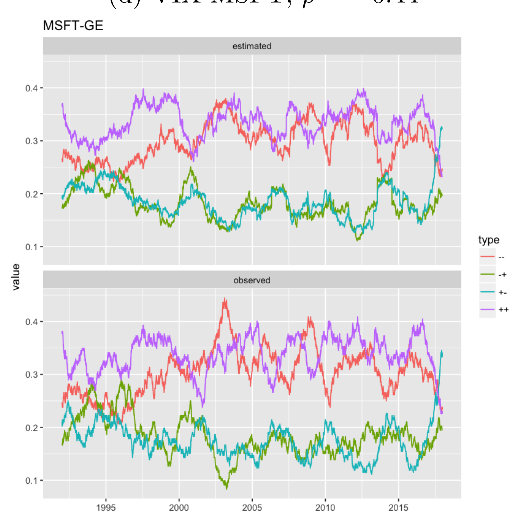
(c) GSPC-GE, $\rho = 0.72$



(d) VIX-MSFT, $\rho = -0.44$



(e) GLD-MSFT, $\rho = 0.00$



(f) MSFT-GE, $\rho = 0.39$

Figure 13: Extra examples of estimated $\{++_t\}$, $\{+-_t\}$, $\{-+_t\}$, $\{--_t\}$ series vs observed quadrant probability series.