INFERENCE AND ARBITRAGE:

THE IMPACT OF STATISTICAL ARBITRAGE ON STOCK PRICES

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Abstract

What are the trade-offs that statistical arbitrageurs face? What is the impact of statistical arbitrage on equilibrium asset prices? This paper models the impact of arbitrageurs on stock prices when arbitrageurs have to learn about the long-run behavior of the stock price process. The arbitrageur conditions the investment strategy on the observation of price and volume. The learning process of the statistical arbitrageurs leads to an optimal trading strategy that can be upward sloping in prices. The presence of privately informed investors makes the equilibrium price dependent on the history of trading volume. The response of prices to news is nonlinear, and little news can have large effects when it is very informative for traders.

1 Introduction

Statistical arbitrageurs employ a variety of investment strategies to take advantage of mispriced assets. The common feature of these strategies is that temporary deviations of prices from their long term level are exploited. The principal difficulty for the arbitrageur is to distinguish permanent movements in prices from temporary fluctuations in prices due to supply and demand shocks. In this paper, the inference problem of statistical arbitrageurs that learn to distinguish between temporary and permanent deviations of prices is modeled explicitly.

The arbitrageurs trade against two classes of investors: noise traders and fundamental traders. The noise traders are causing deviations of prices from the fundamental value that the statistical arbitrageurs exploit. The fundamental traders have an informational advantage over the statistical arbitrageurs, but they have a short time horizon and are risk averse so that their information is not fully reflected in the equilibrium price. Statistical arbitrageurs are assumed to be unconstrained, risk neutral and have a long-term investment horizon. They condition their trading strategies on the history of prices and trading volume. In order to assess the value of an asset, the arbitrageur needs to estimate a model, i.e. needs to learn from past observations of data on prices and volume. This model guides the arbitrageur in distinguishing price changes of an asset due to permanent from transitory disturbances. Arbitrageurs are assumed to have non-normal priors about the price process. This assumption is capturing the fact that many arbitrageurs utilize highly nonlinear arbitrage strategies.

Arbitrageurs face a trade-off between an **inference effect** and an **arbitrage effect**.

When prices increase, the arbitrageurs have an incentive to sell the asset, as it becomes more expensive given beliefs about the true value of the asset. However, a higher price also makes it more likely that the true expected payoff is high, which leads to an updating of beliefs. This is the inference effect, which makes the hedge fund's trades upward sloping in price of the risky for a certain range of parameters and prices.

Intuitively, the trading strategy of the arbitrageurs can be upward sloping in prices for the following reason. As prices increase, statistical arbitrageurs infer that the long-run average price must be higher than previously thought and increase their asset holdings. In other regions of the price, a drop represents a buy opportunity for statistical arbitrageurs. Innovations to the price have different impacts on the arbitrageur's trading strategy depending on the level of prices. In the range of prices where the arbitrage effect dominates the inference effect, the statistical arbitrageurs learns a lot about the relative likelihood of the high or the low state, which makes the price move drastically. Small disturbances due to noise or fundamental information makes prices move very strongly in these ranges. When prices are very low or very high, not much is learned from new information, and the price reacts very little to either noise or news.

The absence of a simple relationship between the arrival of information and movements in prices has lead many to question the relevance of informational sources for movements in asset prices. Instead, it is often argued that noise traders or irrational speculators are causing movements without news and are mitigating the impact of new information. In the framework presented here, the stochastic structure leads to a nonlinear relationship between new information and prices that has such pricing behavior as a consequence. Little fundamental news moves prices dramatically at times, whereas big pieces of news have little impact on prices at other times.

Recently, a number of authors have studied the impact of arbitrageurs, hedge funds, or convergence traders on equilibrium prices. Xiong (2001) studies convergence traders that are wealth constrained. When prices drop sharply, the arbitrageurs' wealth drops, which can amplify drops in prices. In the current paper, arbitrageurs are unconstraint and risk-neutral, and the amplification effect is due to learning. Gromb and Vayanos (2002) investigate the welfare implications of margin requirements in segmented markets.[1] They show that financial constraints can lead to too much or too little risk taking behavior. In the (unconstrained) first-best of the model considered here, the absence of private information would always lead to less volatile prices, and the equilibrium is constrained efficient. Abreu and Brunnermeier (2002) study the coordination problem of arbitrageurs in reaction to a bubble. Even though rational arbitrageurs know that there is a bubble in asset prices, they do not know when the other arbitrageurs will start to trade against the market. Once coordination happens, the bubble crashes.[2] [3]

My model is driven by asymmetric information between the risk-neutral arbitrageurs

[1]Liu and Longstaff (2000) also study the effect of margin constraints on arbitrage.

[2]Goetzmann, Ingersoll, Ross (1998) study the incentives provided by high water mark provisions in the hedge fund industry.

[3]For a description of the technology bubble see Ofek and Richardson (2002). For a systematic account of extreme movements in stock prices see Gabaix et al. (2002 a,b).

and informed, risk-averse investors. The focus of the paper is on the relationship between fundamental news and prices, and volume and prices. Romer (1993) also studies price movements in an asymmetric information setting. He shows that information aggregation can lead to a situation where information is suddenly revealed, even though no new fundamental information was revealed. Gennotte and Leland (1990) show in an asymmetric information model that the demand function for assets can become backward sloping, so that crashes can occur. In the model presented here, the demand function is never backward sloping, but it is nonlinear. The current paper is not a model of crashes, but one of amplification. The results on the relationship between volume and prices are an extension of results obtained by Brown and Jennings (1989), Grundy and McNichols (1989). Both of these papers study technical analysis in an asymmetric information framework, extending the intuitions outlined in Treynor and Ferguson (1985).

The nonlinearity that arises from the bimodal distribution is similar to the one studied by Veronesi (1999), where the (unobserved) drift of the dividend jumps between a high and a low state. This leads to an amplification mechanisms similar to the one studied here. The main difference is that hedge funds in the model studied here face asymmetric information, and that gives rise to interesting implications about volume.[4]

Limits to arbitrage due to irrationality are studied by Shleifer and Vishny (1997). They point out that arbitrageurs might have the largest arbitrage opportunities at times when they are the most constrained, as investors withdraw their capital when arbitrageurs loose

[4]Nonlinearities are very important for empirical work. Boudoukh et al. (2001) find that a nonlinear regression can explain a substantial amount of the variation of FCOJ prices uncovered by Roll (1984).

money from a position.

Recent empirical literature on hedge funds include Brunnermeier and Nagel (2002), Baker and Savasoglu (2002), Brown and Goetzmann (2001), Brown, Goetzmann and Park (1998), Wurgler and Zhuravskaya (2002), Mitchell and Pulvino (2002), Fung and Hsieh (2000, 2001).

The rest of the paper is organized as follows. In section 2, the pricing in a benchmark economy in which noise trader risk is priced in equilibrium is derived. In section 3, the concept of statistical arbitrage is defined, and it is demonstrated that there exist statistical arbitrage opportunities for infinitely lived arbitrageurs in the benchmark economy. In section 4, the partial equilibrium is analyzed. In particular, the optimal investment strategy when the arbitrageur has no price impact and takes the noisy equilibrium price as given is analyzed. In section 5, the general equilibrium with risk-neutral, infinitively lived arbitrageurs is analyzed. The main finding is here that the equilibrium pricing function is nonlinear: at normal times, arbitrageur are stabilizing prices as their investment strategy is downward sloping in prices. However, when the inference effect dominates the arbitrage effect, the arbitrageur has an upward sloping equilibrium asset holding. This makes the equilibrium price more sensitive to noise trader shocks. Section 6 concludes.

2 The Noise Trader Economy

In order to analyze the impact of statistical arbitrageurs on the equilibrium price, we start by studying a benchmark economy without arbitrageurs. The key ingredient of this economy is that noise trader risk is priced. The reason for noise to be priced is that the maximizing investors have a short time horizon. In particular, I study an OLG economy where the maximizing agents live instantaneously. The model is a continuous time analog to DeLong, Shleifer, Summers, Waldmann (1990a).

There are two assets in the economy, a risk free bond that pays continuously compounded interest rate r, and that is in infinitively elastic supply. There is also a dividend paying stock with price P_t. Each stock yields dividends according to the following process:

$$dD_t = \mu dt + \sigma^D dZ_t^D$$

The drift of the dividend μ is assumed to be constant. The term Z_t^D denotes a Brownian motion. σ^D is the instantaneous standard deviation of the dividend.

For now, there are two types of agents in the economy. Noise traders demand the dividend in an amount that evolves according to the following process:

$$du_t = \theta\left(\bar{u} - u_t\right) dt + \sigma^u dZ_t^u \tag{1}$$

where Z_t^u denotes a Brownian motion that is independent of Z_t^D. The noise trader demand is reverting back to the long-run mean $\bar{u}$, at rate θ. When $\theta = 0$, the noise trader process is a random walk, and when $\theta \to \infty$, the noise trader process is i.i.d., i.e. the noise reverts back to $\bar{u}$ instantaneously.

The equilibrium price of the stock is assumed to be

$$dP_t = \eta_t dt + \sigma^P dZ_t^P$$

where $Z_t^P = \left[Z_t^u, Z_t^D\right]$ and $\sigma^P = \left[\sigma^{uP}, \sigma^{DP}\right]'$.

The economy is interpreted as the continuous time limit of an OLG economy with the following set-up. In each period, a new generation of investors is born, earns a non-random labor income, invests the labor income according to CARA utility with coefficient α, and consumes when old. The demand of these investors is then:

$$y_t = \frac{\eta_t + D_t - rP_t}{r\alpha\sigma^{P2}} \tag{2}$$

The demand function is the standard demand of a CARA investor when the price process has constant drift, as demonstrated in Merton (1990). When the drift η_t is time varying and the investor is long-lived, the demand (2) is not optimal, as a long-lived investor hedges time-variation in the drift rate. However, I want to derive an equilibrium where noise trader risk is priced in equilibrium, so that there is a role for long-lived arbitrageurs.

The equilibrium price in the economy is such that the market for the asset clears at all times. In particular, the equilibrium price is such that investors demand y_t plus the noise traders demand u_t equal total supply:

$$y_t + u_t = S \tag{3}$$

From this market clearing condition, it follows that the pricing function is linear in D_t and u_t. The derivation of the pricing function is done in the appendix. The equilibrium price when $\sigma^D > 0$ is:

$$P_t = \frac{D_t}{r} + \frac{\mu}{r^2} - \alpha \left(\sigma^P\right)^2 (S - \bar{u}) + \frac{r\alpha \left(\sigma^P\right)^2}{r+\theta} (u_t - \bar{u}) \tag{4}$$

where

$$\begin{aligned}
\sigma^{PD} &= \sigma^D / r \\
\sigma^{Pu} &= \frac{r+\theta}{r\alpha\sigma^u} \left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + (2\sigma^D/r)^2} \right) \\
\left(\sigma^P\right)^2 &= \left(\sigma^{PD}\right)^2 + \left(\sigma^{Pu}\right)^2
\end{aligned}$$

There are two remarks that are important. First, in the limit when $\sigma^D = 0$, there are two equilibria. One is the limiting case of the equilibrium from the proposition, and the other one is the "fundamental" equilibrium with $P_t = r^{-1}D_t + r^{-2}\mu$. These two equilibria correspond to the two equilibria from DeLong, Shleifer, Summers, Waldmann (1990a). When the dividend is risky, only the equilibrium with noise trader risk is possible. The second remark is that the equilibrium pricing function with long-lived investors is would be $P_t = r^{-1}D_t + r^{-2}\mu - \alpha \left(\sigma^D\right)^2 S$, i.e. only dividend risk, and no noise risk would be priced.

Taking the derivative of the pricing function gives:

$$dP_t = \left(\frac{\mu}{r} + \frac{r\alpha \left(\sigma^P\right)^2}{r+\theta} \theta (\bar{u} - u_t) \right) dt + \sigma^P dZ^P$$

The noise trader risk leads to time-varying drift of the equilibrium price process. When noise trader demand is high, it is pushing up prices contemporaneously. However, it is expected to lead to lower prices in the future, as noise trader demand is mean reverting.

3 Statistical Arbitrage

In contrast to the investors from the previous section, statistical arbitrageurs are assumed to have an infinite time horizon. The arbitrageurs are also risk-neutral, so that their objective function is:

$$\lim_{T\to\infty} \max_{\{A_t\}} E_t \left[e^{-rT} W_T\right] \tag{5}$$

Statistical arbitragers are assumed to have no endowment of wealth. The dynamic budget constraint is:

$$dW_t = r\left(W_t - A_t P_t\right) dt + A_t dP_t + A_t D_t dt$$

where A_t denotes the number of stocks held by the arbitrageur.

Definition 1 *A statistical arbitrage at time t is an investment strategy $\underset{T\to\infty}{lim} \{A_s^*\}_{s=t}^T$ that satisfies the following conditions:*

1) No wealth initial wealth $W_t = 0$.

2) The arbitrage portfolio $\{A_s^*\}$ generates a path of wealth W_s such that the expected payoff of terminal wealth asymptotes to a positive limit: $\underset{T\to\infty}{lim} E_t \left[e^{-rT} W_T\right] > 0$.

3) The variance of the arbitrage wealth converges at rate t : $\underset{t\to\infty}{lim} \frac{Var_t\left[e^{-r(T-t)} W_T\right]}{T} \to 0$

This definition of a statistical arbitrage is an extension of the standard definition of an arbitrage, extended to the infinite horizon. Note the similarity to the APT of Ross (1976). Whereas in the APT, there is no asymptotic arbitrage in the cross-section of stocks as the number of stocks asymptotes to infinity, in the definition here it is assumed that there is no arbitrage in the time-dimension of each stock as time asymptotes to infinity.

In the benchmark economy, there exist statistical arbitrage opportunities. To understand the nature of a long-lived arbitrageur in this economy, start by considering buying one stock and holding it forever. Solving the budget constraint for $A_s = 1$ for all s, and with zero initial wealth $W_t = 0$, we find:

$$e^{-r(T-t)}W_T = \int_t^T d\left(e^{-r(s-t)}P_s\right) + \int_t^T e^{-r(s-t)}D_s ds$$

Solving the budget constraint for the price process gives:

$$\begin{aligned} E_t\left[e^{-r(T-t)}W_T\right] &= r\alpha\left(\sigma^P\right)^2\left(\frac{1-e^{-r(T-t)}}{r}(S-\bar{u}) - \frac{1-e^{-r(T-t)}}{r+\theta}(u_t-\bar{u})\right) \\ &\to r\alpha\left(\sigma^P\right)^2\left(\frac{S-\bar{u}}{r} + \frac{\bar{u}-u_t}{r+\theta}\right) \quad \text{as } T\to\infty \end{aligned}$$

Therefore, as long as noise traders have driven the price sufficiently low such that the condition $(u_t - \bar{u}) < r^{-1}(r+\theta)(S-\bar{u})$ is satisfied is condition 2. for a statistical arbitrage satisfied.

The variance of discounted wealth for holding 1 stock, financed by borrowing is:

$$Var_t\left[e^{-r(T-t)}W_T\right] = \left(\sigma^P\right)^2\frac{1-e^{-2r(T-t)}}{2r} \to \frac{\left(\sigma^P\right)^2}{2r} \quad \text{as } \ T\to\infty$$

Therefore, condition 3) for a statistical arbitrage is satisfied with a buy-and-hold strategy, as $Var_t\left[e^{-r(T-t)}W_T\right]/t \to 0$.

The equilibrium price process derived in equation (4) admits statistical arbitrage opportunities once arbitrageurs with an objective function as in equation (5)are introduced. The mean reversion in expected returns allows the arbitrager to have positive expected payoffs with no initial wealth and a declining variance. In general equilibrium, under the assumptions made so far, the only possible price in the presence of arbitrageurs is $P_t = r^{-1}D_t + r^{-2}\mu$.

However, we will introduce uncertainty about the long run level of the price. This will introduce an inference problem for the arbitrageur.

4 Inference and Arbitrage in Partial Equilibrium

The previous section demonstrated that there are statistical arbitrage opportunities in the benchmark noise trader economy. This section is concerned with the optimal investment strategy of arbitrageurs, when the arbitrageur faces parameter uncertainty. The idea is that the arbitrageur is unsure whether buying or selling a stock is the optimal investment strategy as the long-run mean of the noise-trader risk is unknown. For certain levels of noise trader demand, the arbitrageur is unsure whether going long or short in the stock is optimal as it is unknown whether the deviation of price from the "fundamental" level is caused by temporary or permanent disturbance, i.e. whether it is caused by u_t or by $\bar{u}$.

The expected wealth from holding a stock at the current level of the noise trader shock is:

$$E_t\left[e^{-rT}W_T\right] \to r\alpha\left(\sigma^P\right)^2\left(\frac{S-\bar{u}}{r}+\frac{\bar{u}-u_t}{r+\theta}\right)$$

The relevant parameter for asymptotic profitability of buying a stock is therefore dependent on the level of mean reversion of the noise trader demand $\bar{u}$. Uncertainty about the true level of the long-run average of noise trader demand therefore introduces uncertainty about the profitability of going long or short in the stock.

It is now assumed that the long-run average of the noise can take 2 values, 0 or $\bar{U}$. Under this assumption, there is a range of noise in which the arbitrageur is unsure about the arbitrage:

$$\frac{S\left(r+\theta\right)}{r}-\frac{\theta}{r}\bar{U}<u_t<\frac{S\left(r+\theta\right)}{r}$$

When u_t is between these two critical values, the arbitrageur cannot not be sure whether the observed price level is an arbitrage or not. When $u_t<\frac{S(r+\theta)}{r}-\frac{\theta}{r}\bar{U}$, the arbitrageur is sure that noise trader demand is unusually low, and the asset must be undervalued. When $u_t>\frac{S(r+\theta)}{r}$, the arbitrageur knows for sure that the stock is overpriced due to the noise-trader risk. In between these two boundaries, the arbitrageur must make a best guess as to what the appropriate investment strategy is.

Denoting the expectation of u_t conditional on the history of prices by m_t, the optimal filter for the noise is:

$$dm_t=\theta\left(\bar{u}-m_t\right)dt+\sigma_t^m dZ_t^m$$

The definition of $\sigma_t^m dZ_t^m$ is given in the appendix. The likelihood function of $\bar{u}$, conditional on the history $m^t=\{m_\tau:0\leq\tau\leq t\}$ can be computed from Küchler and Sørensen (1997):

$$\Pr\left(\bar{u}|m^t\right)=\exp\left(\frac{\theta\left(m_0-\bar{u}\right)^2}{2\sigma_0^{m2}}-\frac{\theta\left(m_t-\bar{u}\right)^2}{2\sigma_t^{m2}}-\frac{1}{2}\theta^2\int\frac{\left(u_s-\bar{u}\right)^2}{\sigma_s^{u2}}ds+\frac{1}{2}\theta t\right)$$

The Radon-Nikodym derivative of changing the measure of m_t from $\bar{u}=\bar{U}$ to $\bar{u}=\bar{U}$ is then:

$$\phi_t=\frac{\Pr\left(\bar{u}=\bar{U}|m^t\right)}{\Pr\left(\bar{u}=0|m^t\right)}=\exp\left[\theta\bar{U}\left(\frac{m_t}{\sigma_t^{m2}}-\frac{m_0}{\sigma_0^{m2}}+\theta\int_0^t\frac{m_s-\bar{U}/2}{\sigma_s^{u2}}ds+\frac{\sigma_t^{m2}-\sigma_0^{m2}}{2\sigma_t^{m2}\sigma_0^{m2}}\bar{U}\right)\right] \quad (6)$$

There are two important elements in this expression.

The probability of the high state, conditional on the estimate of noise trader shocks is then according to Bayes rule:

$$\pi_t=\frac{\pi_0\phi_t}{\pi_0\phi_t+1-\pi_0}$$

The evolution of π can be found by taking the time derivative ofπ:

$$d\pi_t = \pi_t (1 - \pi_t) \frac{\theta \bar{U}}{\sigma_t^{m2}} \left(\theta \left(\bar{u} - \bar{U}\pi_t \right) dt + \sigma_t^m dZ_t^m \right)$$

Using this expression, the critical condition for condition 2) can be expressed under the assumption that the arbitrageurs do not know the true level of average noise trader demand $\bar{u}$:

$$E_t \left[e^{-r(T-t)} W_T \right] \to \frac{S - \bar{U}\pi_t}{r} + \frac{\bar{U}\pi_t - m_t}{r + \theta} \quad \text{as } T \to \infty$$

The important feature of figure 1 is that the sensitivity of expected discounted wealth with respect to the estimated level of noise trader demand is different according to the level of estimated noise trader demand m_t. For very high and very low levels of estimated noise trader demand, the expected discounted wealth level is not very sensitive to changes in the level of noise. However, in an intermediate range, the arbitrageur learns strongly about the profitability of an arbitrage opportunity, and the expected discounted wealth is very sensitive to changes in the estimated level of noise.

[**figure 1**]

The feature of the learning process that is not visible from **figure 1** is that the location of the kink in the expected wealth depends on the history of the estimated noise shocks m. This feature can be seen from the likelihood function 6. A history of positive shocks to m makes it more likely that the high level of average noise trader demand is the true one, i.e.

that the critical value of the price at which going long in the stock provides an arbitrage opportunity is actually lower.

The filtering problem as it is derived has one interesting property. Assuming that the statistical arbitrageur only conditions on the history of the price, it can be verified that learning about the long-run mean of noise $\bar{u}$ is the same learning problem as learning about he drift of the dividend μ, up to a linear transformation. In particular, application of theorem 12.7. of Liptser and Shiryaev (2000), assuming that μ instead of $\bar{u}$ is unknown leads to the following property:

$$E\left[\bar{u}|\mathcal{F}_t^P\right] = \frac{r+\theta}{\theta\alpha r^2\sigma^{P2}} E\left[\mu|\mathcal{F}_t^P\right]$$

where $\mathcal{F}_t^P$ denotes the filtration generated by the history of prices up to time t. As long as the statistical arbitrageur only learns from the price, it is thus indistinguishable whether the learning is about the drift of the dividend, or the long-run average of the noise trader demand. In this partial equilibrium analysis, both of these constants change the long-run level of the price, up to the multiplication by $\frac{r+\theta}{\theta\alpha r^2\sigma^{P2}}$.

If instead of learning from price, the arbitrageurs learn from volume, we get the following learning mechanism. As a proxy for volume, define the variable v_t as:

$$v_t = \frac{y_t^2 + u_t^2}{4} = \frac{1}{2}u_t^2$$

Using Itô's lemma gives the following process for v:

$$dv_t = \theta\left(\bar{u} - \sqrt{v_t} + \frac{1}{2\theta}\sigma^{u2}\right)dt + \sigma^u dZ_t^u$$

Learning about the profitability of an arbitrage opportunity from volume is similar to learning from price. Applying again theorem 12.7 from Liptser and Shiryaev (2000), the expectation of $\bar{u} = \bar{U}$, conditional on the history of v_t is

$$d\pi_t^v = \pi_t^v \left(1 - \pi_t^v\right) \frac{\theta \bar{U}}{\sigma^{u2}} \left[\left(\bar{u} - \pi_t^v \bar{U}\right) dt + \sigma^u dZ_t^u\right]$$

Learning from volume is inheriting many of the properties of learning from price. In particular, the likelihood function displays a similar nonlinearity as the one in the case of learning from the price, and the critical condition for expected wealth has a similar shape as the one displayed in figure 1.

5 The Impact of Arbitrageurs in General Equilibrium

Having studied the investment strategy of the arbitrageurs in partial equilibrium when the noise trader risk is priced, I will now study the general equilibrium. The key trade-off between inference and arbitrage that causes the expected discounted wealth from holding a stock to have a nonlinear form will translate into a nonlinear price. It is shown that the economics behind this nonlinearity are the stock holdings of the arbitrageur. The equilibrium asset holdings of arbitrageurs are determined by 2 opposing forces, the inference and the arbitrage effect. When the inference effect dominates the arbitrage effect, the equilibrium asset holdings of the arbitrageur are upward sloping in prices.

With free entry of arbitrageurs, the equilibrium price must be such that the expected payoff from a zero wealth investment yields an expected discounted payoff of 0 asymptotically:

$$\lim_{T\to\infty} E_t\left[e^{-rT}W_T\right] = 0$$

Following Santos and Woodford (1997), I also impose the transversality condition:

$$\lim_{T\to\infty} E_t\left[e^{-rT}P_T\right] = 0$$

The equilibrium price is then the discounted future payoff of dividends:

$$P_t = E_t\left[\int_t^{\infty} e^{-rs}D_s ds\right]$$

The equilibrium price is therefore such that noise trader risk is not priced. I assume that the arbitrageurs face uncertainty about the drift of the dividend payoff. The drift can take two values, $\mu = \left\{0, \bar{D}\right\}$. The equilibrium price is then:

$$P_t = \frac{D_t}{r} + \frac{E_t\left[\mu\right]}{r^2}$$

In general equilibrium, the statistical arbitrageurs set the price, so that there is no information in the price that is of value to the arbitrageurs. Learning about the drift of the dividend is like learning about the right level of the price. With a 0 drift, the correct level of the price is D_t/r. With a drift of $\bar{D}$, the correct level of the price is $D_t/r + \bar{D}/r^2$. This is similar to the set-up in the previous section. Just as previously, the arbitrageurs need to learn the correct level of price. Whereas before, they needed to infer the right level of average noise trader demand, they now have to infer the correct level of the dividend drift.

I assume that the short-horizon investors know the drift μ of the dividend. In order to make the model tractable, I assume that the noise trader process is still given by 1, but that the u now refers to the belief of the noise traders about the true level of the drift of the dividend. In particular, the drift of the price process is $r^{-1}\mu + r^{-2}E_t(d\pi_t)$. Whereas informed investors know μ, the noise traders have stochastic belief about μ.

I will start by analyzing the economy when the arbitrageurs can observe the demand schedule submit by the noise traders plus the informed traders. This is a very similar set-up to Kyle (1985). The difference to Kyle is the stochastic structure of the shocks: whereas Kyle assumes that μ is normally distributed, it is assumed here that μ has a bimodal distribution.

I will show that the statistic that the arbitrageurs can infer from observing the demand of the rest of the market is:

$$x_t = \mu + u_t$$

From the specification of the noise trader process 1, the process for x is therefore:

$$\begin{aligned} dx_t &= \theta(\bar{x} - x_t)\,dt + \sigma^u dZ^u \\ \bar{x} &= \bar{u} + \mu \end{aligned}$$

Furthermore, corresponding to the different drifts μ, the variables $\bar{X} = \bar{u} + \bar{D}$, and $X = \bar{u}$ are defined. From Küchler and Sørensen (1997), the likelihood function of x_t conditional on $\bar{x}$ and a path of $x^t = \{x_\tau : 0 \leqslant \tau \leqslant t\}$ is:

$$f\left(x_t|x^t,\bar{x}\right) = \exp\frac{\theta}{2\sigma_u^2}\left(-\left(x_t-\bar{x}\right)^2 + \left(x_0-\bar{x}\right)^2 - \theta\int_0^t \left(x_s-\bar{x}\right)^2 ds + t/\sigma_u^2\right)$$

from this likelihood function, the Radon-Nikodym derivatives of changing the measures from

$\bar{x} = \bar{X}$ to $\bar{x} = X$ can be computed. I normalize $X = 0$:

$$\phi_t = \exp \frac{\theta \bar{X}}{\sigma^{u2}} \left[x_t - x_0 + \theta \int_0^t (x_s - \bar{x}/2)\, ds \right]$$

The relative likelihood of $\bar{x} = \bar{X}$ versus $\bar{x} = 0$ therefore depends on two main statistics. First, the current level of x_t minus the initial level x_0. This is a level effect: x_t is likely to be higher when the mean reversion level of x is higher. The second term is depending on the history of realizations of x_t. In particular, the average time spend above the long-run mean $\bar{x}$.

The Bayesian estimate of $\bar{x}$ is then:

$$\Pr\left[\bar{x} = \bar{X} | x^t\right] = \pi_t = \frac{\phi_t}{1 - 1/\pi_0 + \phi_t}$$

where $x^t = \{x_\tau : 0 \leqslant \tau \leqslant t\}$. I normalize the $\bar{u} + \mu$ such that $X = 0$. Then, the law of motion of π is:

$$d\pi_t = \pi_t (1 - \pi_t) \theta \bar{X} (\sigma^u)^{-2} \left[\left(\bar{x} - \bar{X}\pi \right) dt + \sigma^u dZ^u \right]$$

In terms of the observation of the x_t, this can be rewritten as follows:

$$d\pi_t = \pi_t (1 - \pi_t) \theta \bar{D} (\sigma^u)^{-2} \left[dx_t + \theta \left(\bar{X}\pi - x_t \right) dt \right]$$

Under the arbitrageurs information set, the process of π is a martingale. However, under the investors information set, as investors know $\bar{X}$, the process of π has a positive drift if $\bar{x} = \bar{X}$ and has a negative drift if $\bar{x} = 0$.

Using the notation

$$dP_t = \eta_t^P dt + \sigma_t^P dZ$$

and taking the derivative of the pricing function we find:

$$\begin{aligned}\eta_t &= r^{-1}\mu + r^{-2}\pi_t\left(1-\pi_t\right)\theta\bar{X}\left(\sigma^u\right)^{-2}\left(\bar{x}-\bar{X}\pi\right)\\ \left(\sigma_t^P\right)^2 &= r^{-2}\left[\sigma_D^2 + r^{-2}\pi_t^2\left(1-\pi_t\right)^2\theta^2\bar{X}^2\left(\sigma^u\right)^{-2}\right]\end{aligned}$$

The demand from the investors is then:

$$\frac{\eta_t + D_t - rP_t}{r\alpha\left(\sigma_t^P\right)^2} = \frac{r^{-1}\left(\mu-\bar{D}\right)\pi + r^{-2}\pi_t\left(1-\pi_t\right)\frac{\theta\bar{D}}{\sigma^{u2}}\left(\mu-\bar{D}\pi\right)}{r\alpha\left(\sigma_t^P\right)^2} = \frac{\mu-\bar{D}\pi}{r\alpha\tilde{\sigma}_t^2}$$

where

$$\tilde{\sigma}_t^2 = \frac{\left[r^{-1}\sigma_D + r^{-2}\pi_t\left(1-\pi_t\right)\theta\bar{X}\left(\sigma_u\right)^{-1}\right]^2}{\left[r^{-1} + r^{-2}\pi_t\left(1-\pi_t\right)\theta\bar{X}\left(\sigma^u\right)^{-2}\right]}$$

The demand of the noise traders together with the informed investors is then:

$$\frac{x_t - \bar{X}\pi_t}{r\alpha\tilde{\sigma}_t^2}$$

As the arbitrageurs set the price, their holding is the supply S minus the demand of the noise traders plus informed investors. The asset holdings are represented in figure 3 as a function of price. The key feature is that the equilibrium asset holdings of the arbitrageur are downward sloping for high and low levels of the price, and upward sloping for intermediate levels. The upwardsloping portion of the investment strategy corresponds to the relatively steep section of the pricing function (see figure 3). The intuition is that there is an inference and an arbitrage effect. When prices are very high or very low, the arbitrageur is a contrarian investor. However, for an intermediate range, the arbitrageur learns from the information that is contained in prices: as prices increase, the arbitrageur increases equilibrium asset holdings, as the arbitrageur updates his beliefs about the long-run level of prices.

[figure 2]

[figure 3]

6 Conclusion

This paper analyzes a model with statistical arbitrageurs that are risk-neutral, unconstrained, and have a long-horizon. The arbitrageur's problem is that it is not known whether the unconditional payoff of the trading strategy is zero or positive. The key assumption is that there are only two states: in the partial equilibrium part of the paper, it is assumed that the average noise trader demand is either 0 or positive. In the general equilibrium section, it is assumed that the drift of the dividend could be 0 or positive. This twin-peaked distribution leads to interesting implications for the pricing function, and illustrates the arbitrageurs fundamental trade-off. Throughout the paper it is assumed that the arbitragers only learn from the observation of prices and volume, not from the observation of dividends.

The arbitrageurs face a trade-off between inference and arbitrage: low demand for the stock means that prices are driven lower, relative to fundamental valuation. However, the arbitrageur can learn from demand shocks as well. In a certain range of prices, this learning is dominating the arbitrage effect, and the arbitrageur has upward sloping stock holdings. Depending on the relative strength of the inference and the arbitrage effect, arbitrageurs thus have either downward sloping or upward sloping asset holdings as a function of price.

The history of demand matters for prices. Prices can drop (increase) sharply when enough selling (buying) pressure has accumulated in the market. A situation is possible where the arbitrageur absorbs all the selling from the market, and prices do not change. Suddenly, the arbitrageur realizes that the cumulative sales from the rest of the market must be very informative, the arbitrageur starts to sell heavily, and prices drop sharply.

A Appendix

A.1 Equilibrium Price in the Noise trader Economy (Equation 4):

Proof. Let's start with the guess that the drift of the price is a linear function of the noise trader demand::

$$\eta_t = C_0 + C_1 u_t$$

From the market clearing condition 3, we find:

$$P_t = r^{-1}\left(C_0 - r\alpha\sigma^{P2}S\right) + r^{-1}D_t + r^{-1}\left(C_1 + r\alpha\sigma^{P2}\right)u_t \tag{7}$$

Differentiating and replacing for the dividend and the noise trader process gives:

$$dP_t = r^{-1}\left(C_1 + r\alpha\sigma^{P2}\right)\theta\left(\bar{u} - u_t\right)dt + r^{-1}\mu dt + r^{-1}\sigma^D dZ^D + r^{-1}\left(C_1 + r\alpha\sigma^{P2}\right)\sigma^u dZ^u$$

Matching coefficients for the drift term:

$$C_0 = r^{-1}\mu + \tfrac{r\alpha\theta}{r+\theta}\left(\sigma^P\right)^2\bar{u} \quad C_1 = -\tfrac{r\alpha\theta}{r+\theta}\left(\sigma^P\right)^2$$

Replacing back into 7 gives the pricing function:

$$P_t = \frac{D_t}{r} + \frac{\mu}{r^2} + \alpha\sigma^{P2}\left[\frac{\theta}{r+\theta}\left(\bar{u} - u_t\right) + \left(u_t - S\right)\right] \tag{8}$$

Differentiating the pricing function 8, and matching the volatility gives the following equation:

$$0 = \left(\sigma^P\right)^4\left(\frac{r\alpha}{r+\theta}\right)^2\left(\sigma^u\right)^2 - \left(\sigma^P\right)^2 + \left(\frac{1}{r}\right)^2\left(\sigma^D\right)^2$$

Using the independence between innovations to D and innovations to u that implies that $\left(\sigma^{Pu}\right)^2 + \left(\sigma^{PD}\right)^2 = \left(\sigma^{P}\right)^2$ we get:

$$\begin{aligned} \sigma^{PD} &= \sigma^{D}/r \\ \sigma^{Pu} &= \frac{r+\theta}{r\alpha\sigma^{u}}\left(\frac{1}{2}+\frac{1}{2}\sqrt{1+\left(2\sigma^{D}/r\right)^2}\right) \end{aligned}$$

Which concludes the proof of the proposition. ■

A.2 Optimal Filtering of Noise Trader Demand:

Proof. The filtering problem is the following. The arbitrageur observes the price that evolves according to the process:

$$dP_t = \left(\frac{\mu}{r} + \frac{r\alpha\sigma^{P2}}{r+\theta}\theta\left(\bar{u}-u_t\right)\right)dt + \sigma^{uP}dZ^{u} + \sigma^{PD}dZ^{D}$$

The arbitrageur wants to learn about the current level of noise trader demand from observing the price process. The unobserved noise trader demand evolves according to:

$$du_t = \theta\left(\bar{u}-u_t\right)dt + \sigma^{u}dZ^{u}$$

All the conditions for Theorem 12.7 from Liptser and Shiryaev (2000) are satisfied, so that the filtered process can be directly computed. Denote the filtration generated by P buy $\mathcal{F}_t^P = \{P_\tau : \tau < t\}$, the conditional expectation $m_t = E\left[u_t|\mathcal{F}_t^P\right]$ and the forecast error

$\gamma_t = E\left[\left(m_t - E\left[u_t|\mathcal{F}_t^P\right]\right)^2|\mathcal{F}_t^P\right]$, then we find:

$$\begin{aligned} dm_t &= \theta\left(\bar{u} - m_t\right)dt + \sigma_t^m dZ^m \\ \sigma_t^m dZ^m &= \Gamma_t dP_t - \Gamma_t\left(\frac{\mu}{r} + \frac{r\alpha\left(\sigma^P\right)^2}{r+\theta}\theta\left(\bar{u} - m_t\right)\right)dt \\ \Gamma_t &= \left(\sigma^{uP}\right)^{-2}\left(\sigma^u\sigma^{uP} + \gamma_t r\alpha\theta\left(r+\theta\right)^{-1}\left(\sigma^P\right)^2\right) \\ d\gamma_t &= -2\theta\gamma_t + \left(\sigma^u\right)^2 - \left(\sigma^{uP}\right)^{-2}\left(\sigma^u\sigma^{uP} + \gamma A_1\right)^2 \end{aligned}$$

■

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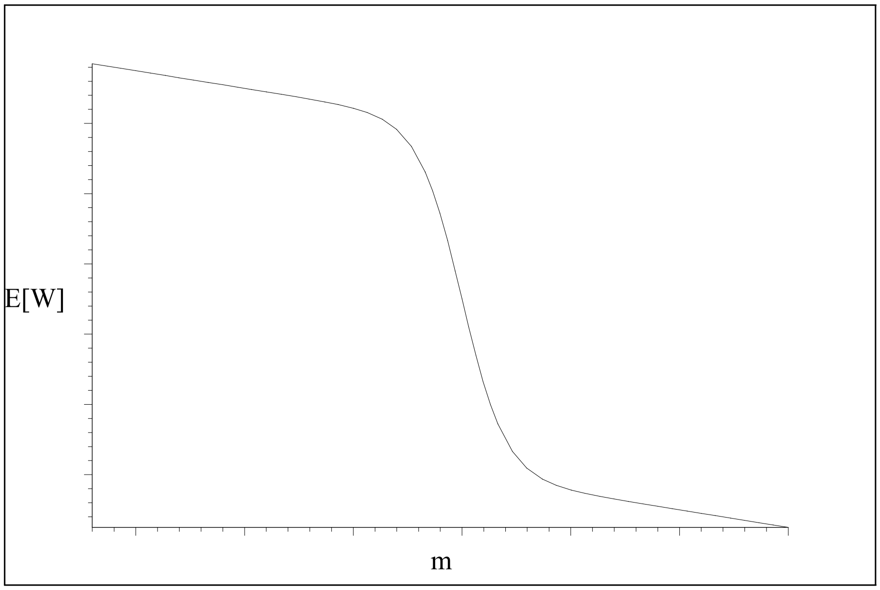

Figure 1: Expected Wealth from Arbitrage

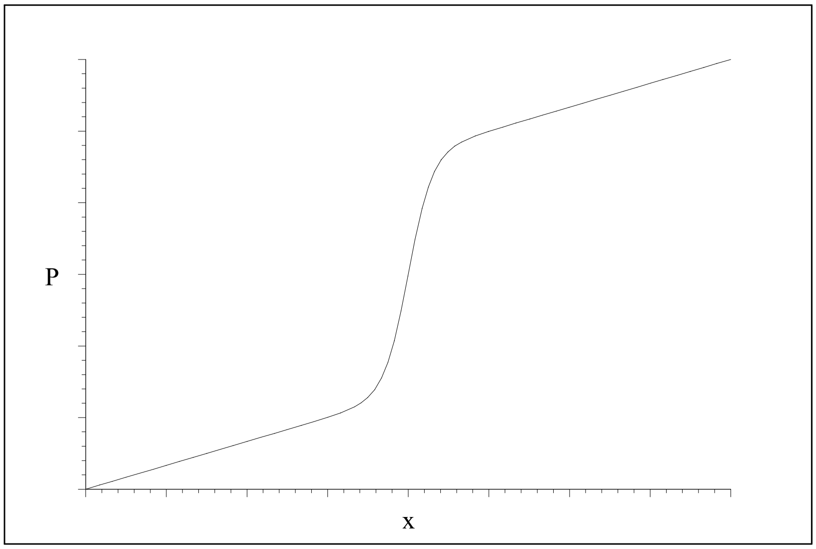

Figure 2: Equilibrium Price P(x)

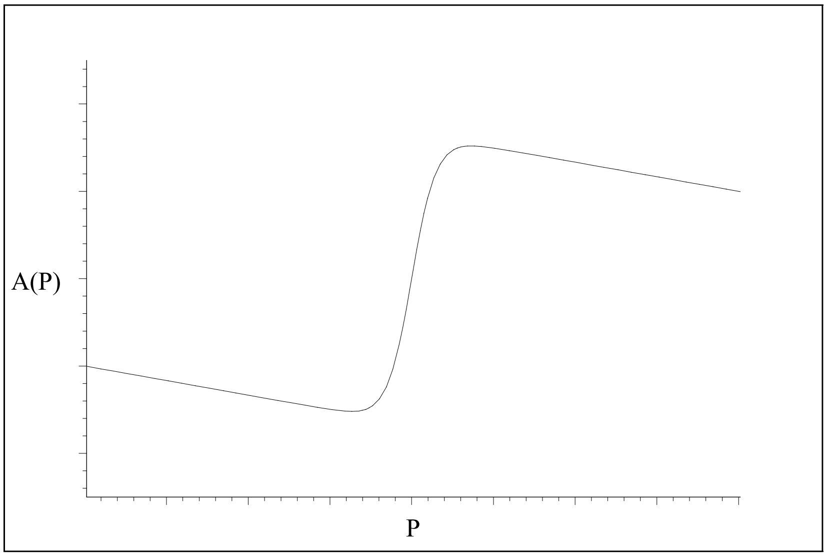

Figure 3: The Arbitrageur's Stock Holdings