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The fractal structure of exchange rates: measurement and forecasting

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Abstract

Recent research has demonstrated that exchange rates have fractal properties. The mathematical definition of fractality is non-integer dimensionality. Statistically, this corresponds to intermittency, or inhomogeneity in a time series. Fractality in financial markets originates in systems with large numbers of equations. It is therefore stochastic fractality, which is separate and distinct from deterministic chaos. Other nonlinear models that have been used to forecast exchange rates, such as the ARCH group, do not exhibit fractality. At the same time, there are crucial differences between the fractal processes observed in financial markets and the multifractals identified in physics. In physics, fractals show strong scaling symmetries. In financial markets, this is not the case. Scaling symmetries are weak, and in the long run, exchange rates evolve toward a non-fractal state. Fractality in exchange rates is consistent with structural econometric models: differentials in rates of return across national boundaries generate both nonlinear variability and intermittency. Empirical tests demonstrate that out of 21 daily exchange rates, all scale as fractals. Short-term interest rates and interest rate differentials also show evidence of fractality. A forecasting algorithm based on state transitions is proposed. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

It is widely recognized that financial market series exhibit non-linearity (Granger and Terasvirta, 1993; Campbell et al., 1997). Recent research has demonstrated that in foreign exchange markets, this non-linear variability shows fractal properties (Mantegna and Stanley, 1995, 1996, 1997, 1998; Schmitt et al., 1999). Statistically, this corresponds to intermittency, or inhomogeneity in a time series. Until the last few years, the term fractal was associated mainly with deterministic chaos, which originates from a small number of generating equations. Starting in the 1980s, however, a new class of stochastic fractals was identified, mainly from studies of turbulence in physics. Termed 'multifractals', these are processes originating in high-dimensional systems (Schertzer and Lovejoy, 1991; Schertzer et al., 1991; see also Bunde and Havlin, 1995). While the mechanisms that generate multifractals in physics do not necessarily correspond to economic processes, the concept of stochastic fractality is directly applicable in econometrics.

This paper has two objectives. First, it is demonstrated that exchange rates scale as fractals. The determinants of exchange rates/interest rates and differentials in real rates of return, are also found to have fractal properties. Second, a forecasting algorithm based on state transitions is proposed. The idea underlying this is that the basic property of fractals is intermittency. The state transition method can be used to model the occurrence of irregular periods of extreme volatility. A further finding here is that fractality is consistent with economic theory on exchange rates. Fractality can result from well-known models of exchange rate determination, in which currencies vary as a function of differentials in real rates of return across national boundaries. In this sense, there is no inherent conflict between the empirical finding of fractality, and the structural equations used in econometric models.

This paper is organized as follows. Section 2 describes the main fractal statistics, and proposes a generating model. Sections 3 and 4 describe fractal scaling and parameter estimation. Section 5 summarizes the data. Section 6 conducts empirical tests and Section 7 proposes a forecasting model.

At the outset, it is useful to explicitly set out the difference between fractals and other non-linear models. The mathematical definition of fractality is non-integer dimensionality. Models such as ARIMA and the ARCH group are non-fractal, i.e. their fractal dimension is an integer. While they can reproduce some of the features of financial markets, they do not generate intermittency. At the same time, there is a crucial difference between the stochastic fractality found in foreign exchange markets and the multifractals identified in physics. Multifractals exhibit strong scaling symmetries: there are proportionality relationships over a wide range of time scales. In financial markets, scaling symmetries are much weaker, and hold only over shorter intervals. For this reason, processes of this type are here referred to as stochastic fractals, rather than multifractals.

2. The fractal statistics

The basic fractal parameters, denoted H, α, C, measure nonstationarity, the probability distribution, and the degree of intermittency. The statistic H, which varies between 0 and 1, is familiar from numerous papers on fractional processes (Granger and Joyeux, 1980; Hosking, 1981, 1984; Gray et al., 1989). As discussed in Feder (1988), the variance of fractionally integrated processes is proportional to t^{2H}, while the autocorrelation is a function of $2^{(2H-1)}-1$. The case of $H=0.5$ corresponds to integration of order zero, or $I(0)$; the process is short memory in the sense that the dependence between time points decays rapidly. In the case of non-linear series, processes for which $H=0.5$ are said to be short memory in mean and distribution (Granger and Terasvirta, 1993). Values of $0.5<H<1$ characterize persistent, or long-memory series, which correspond to $(0<I<0.5)$. Values of $H<0.5$ characterize antipersistent or turbulent processes, which correspond to the case of $(-0.5<I<0)$. In essence, H quantifies the non-conservation of the mean.

Although H describes only stationary processes, these by definition include the differentials of integrated series. The relationship between H and degree of integration is therefore straightforward. For the first difference of a non-stationary integrated series, H is equal to $I+0.5$; the degree of integration for the level series can therefore be estimated as $H-0.5+1$ (Beran, 1992).

The coefficient α determines the class of the probability distribution. Let Y_t be a time series. Since most economic series are non-stationary integrated processes, Y_t will be assumed to have these properties, and the following discussion will refer to the log-difference. The probability that a given observation will be greater than a given value, κ, can be expressed as: $\Pr(|\ln Y_t-\ln Y_{t-1}|>\kappa)\approx\kappa^{-\alpha}$ (where $\kappa\gg 1$). When $\alpha=2$, the distribution is the lognormal. In the case of $1<\alpha<2$, the distribution is wider. As α decreases toward 1, the series exhibits greater dispersal around the mean; the probability of extreme deviations is higher. The special case of $\alpha=1$ yields the Cauchy distribution. Another special case, $\alpha=0$, yields divergence of all statistical moments. The intervening case of $0<\alpha<1$ yields another class of processes that are less unstable than the case of $1<\alpha<2$. In essence, when $0<\alpha<1$, integration tends to smooth the process. Conversely, when $1<\alpha<2$, the probability of extreme events is higher, and integration does not smooth out fluctuations (see also Mandelbrot, 1964; Fama, 1965)[1].

For the most part, ARCH-type and state-transition models show $\alpha<2$. Fitting ARCH parameterizations to exchange rates invariably shows $1<\alpha<2$ for the fitted model, which is consistent with the exchange rate data. Instead, the key difference between stochastic fractals and other non-linear models lies with the fractal dimension.

The coefficient C is the *fractal codimension*. Let D be the embedding dimension of the space that encompasses the series; for a line $D=1$, for a plane $D=2$ and for

[1] In some cases, as argued in Schmitt et al. (1999), the probability distribution cannot be described by a single value, but only a function. For instance, the probability of extreme deviations may differ from the probability of fluctuations in the 'intermediate' range. In this case, α describes only the class of the probability distribution.

a three-dimensional object $D=3$. Let L be a characteristic length, in this instance a time scale. Let N be the size of the set associated with this scale. The relationship between the size of the set and the characteristic scale is given by the fractal dimension, denoted d: $N(L) \approx L^{d}$. The codimension is defined as $C=D-d$. The series is homogenous when $C=0$, i.e. it fills the embedding space, and inhomogenous or fractal when $C \neq 0$. Expressed another way, for fractal processes, d is a fractional rather than an integer value, and $C \neq 0$.

Let τ be a time scale running from 1 to T, let λ be a ratio of scales, and let γ be a function, termed the order of singularities. Then it is possible to express the distribution of the fluctuations as a function of the codimension.

$$\Pr(|\ln Y_t - \ln Y_{t-1}| \geq \lambda^{\gamma}) \approx \lambda^{-C(\gamma)} \tag{1a}$$

and solving for $C(\gamma)$ shows that this is negatively related to the probability that $|\ln Y_t - \ln Y_{t-1}| \geq \gamma$. This relationship is associated with a relationship for the statistical moments, in this instance the mean value, denoted μ.

$$\mu(|\ln Y_t - \ln Y_{t-1}|^{q}) \approx \lambda^{\zeta(q)} \tag{1b}$$

where q is a series of scaling coefficients, and ζ is a function. The case of $q=1$ simplifies to the mean log-difference. The function $\zeta(q)$ is related to three fractal parameters, H, α and $C(1)$. The universal form for $\zeta(q)$ is given in Eq. (6) below. Because the mean value of the process is related to $C(1)$ via $\zeta(q)$, the parameter $C(1)$ is the codimension of the mean. Specifically, $C(1)$ is a function of the first derivative of $\zeta(q)$ at the origin; this relationship is given in Eqs. (7a)–(7d). $C(1)$ expresses the intermittency or inhomogeneity of the mean value of the process. In other words, the probability of occurrences of the statistical moments at any given time scale is a negative function of the codimension. As $C(1)$ increases toward the embedding dimension, the process becomes less homogenous and more intermittent. For a further discussion, see Lavallee et al. (1991).

It is this property — a non-integer dimension or equivalently a non-zero codimension — that sets fractals apart from other non-linear models. To demonstrate this, a series of simulation experiments were run. One hundred series were generated for each of the following four types of time-series models:

1. the ARIMA and fractional ARIMA (ARFIMA) models;
2. the generalized ARCH (GARCH) model of Engle (1982) and Bollerslev (1986);
3. state transition models, such as those of Hamilton (1989) and Terasvirta (1994), and the self-exciting threshold autoregression (SETAR) model of Tong (1990); and
4. the state transition ARCH (ST-ARCH) model of Hamilton (1994), Gray (1996), and Ang and Bekaert (1998).

In each instance, the series spanned 7500 datapoints, comparable to the length of the exchange rate series. Different values were used for the ARCH variance, and the state transitions were set at irregular intervals. The fractal statistics were then estimated, using the procedures described below. In the ARIMA and GARCH models, the estimate of $C(1)$ was always zero, although the values of α usually lay well below 2 and often approached unity. In essence, the ARCH models generate wide-tailed probability distributions, but do not yield fractality. In most of the state transition and ST-ARCH models, the null hypothesis that $C(1)=0$ could not be rejected. There were however cases in which state transition models produced fractality. When multiple state transitions were used, and transitions were imposed at frequent but irregular intervals, it was possible to obtain values of $C(1)$ in the range of up to 0.02. This is however a somewhat lower degree of fractality than is found in exchange rates. State transitions may contribute to an explanation of fractality, but they are not the main cause.

If widely-used time series models do not account for fractality, it is reasonable to ask if there is an equation derived from economic theory that can. Both economic theory and empirical studies argue that exchange rates are determined primarily by differentials in real rates of return and relative prices across national boundaries, in other words, terms like $[(\Sigma_j \ln R_{jt} - \ln \pi^{\mathrm{e}}_t) - (\Sigma_j \ln R_{j\mathrm{F}t} - \ln \pi^{\mathrm{e}}_{\mathrm{F}t})]$ and $(\ln P_{Tt} - \ln P_{T\mathrm{F}t})$, where R is the rate of return, π is the inflation rate, the superscript e denotes expectations, the subscript F denotes foreign, P_T is the price of traded goods, and the summation is across j classes of financial assets (see Levich, 1985, for a review of the early literature).

It is not widely recognized that these differentials can exhibit fractality. However, Mandelbrot (1997) notes that taking ratios of stochastic processes can high degrees of volatility. Random decreases in the denominator can yield explosive increases in the ratio; similarly, random increases in the denominator yield violent contractions. Even if both the numerator and denominator are non-fractal, fluctuations in either term can lead to large, discontinuous changes in the value of the ratio. The result is a process with a wide probability distribution and a high probability of intermittent extreme events. This mechanism is consistent with the finding in prior studies that fractality results from multiplicative interactions between time series (since taking ratios is multiplication by reciprocals).

To test this, a second set of simulation experiments were run. Again, 100 realizations each of ARIMA, GARCH and ST-ARCH models were used. Then ratios of these processes were taken. In three-fourths of the cases tested, the ratio of the ARIMA series showed $C(1)>0$. The ratios of the GARCH models showed fractality in 70% of cases, and the ratios of the ST-ARCH models also showed fractality in nearly three-fourths of the tests. Typically, in these experiments, $1<\alpha<2$, and $0.02<C(1)<0.15$. These values are actually quite close to the parameters estimated for the exchange rate and interest rate differential data. More complex models were then developed. Weighted averages of several ratios were taken, and the weights were allowed to vary in a non-linear manner. These processes were then integrated and tested. The more complex models — weighted aggregations of several ratios — also scaled as fractals, with $0.04<C(1)<0.20$ in three-fifths of the cases.

It is straightforward to generalize this to econometric models of exchange rates. Let EX_t be the exchange rate. A plausible generating equation is of the form:

$$
\begin{aligned}
&(\ln \mathrm{EX}_t - \ln \mathrm{EX}_{t-1}) \\
&= a_0 + a_1(\Sigma_j \ln \mathrm{R}_{jt} - \Sigma_j \ln \mathrm{R}_{j\mathrm{F}t}) + a_2(\ln \pi^{\mathrm{e}}_t - \ln \pi^{\mathrm{e}}_{\mathrm{F}t}) + a_3(\ln \mathrm{P}_{Tt} - \ln \mathrm{P}_{T\mathrm{F}t}) \\
&\quad + a_4 \Xi_t + \varepsilon_t
\end{aligned} \tag{2}
$$

where Ξ is a vector of other factors and ε is a residual. Clearly, the dynamics associated with this type of equation are complex and non-linear. Fractality arises because the process of taking log-differences generates chaotic behavior. The utility of this formulation is that it reconciles econometric models with statistical evidence on fractality. It argues that fractality results directly from structural exchange rate models. A ratio of stochastic processes — a reasonable model for the differential in rates of return — generates a fractal, even if neither the numerator nor the denominator are fractals.

There is of course another possible source of fractality. The term Ξ includes exogenous disturbances such as price shocks, and global crises. These factors are by definition intermittent processes. In essence, fractality is produced both by the process of taking ratios and by the superimposition of aperiodic shocks.

3. Fractal scaling

The three fractal parameters can be calculated through a scaling procedure (Schertzer and Lovejoy, 1991). Let q be a series of exponents, which can be either positive or negative, and is not necessarily limited to integer values. Let T be the largest time scale of interest. Then one can compute scalings for the absolute value of the first difference of ln Y_t over increasing time horizons: $\mu(|\ln Y_t - \ln Y_{t-1}|^q)$, … $\mu(|\ln Y_t - \ln Y_{t-T}|^q)$. The basis for this procedure is the notion of scaling symmetries, which are intrinsic in certain types of multifractals. Within the time frame delimited by T, the mean value of $|\ln Y_t - \ln Y_{t-1}|^q$ is proportional at all orders of scaling (values of q) to the mean absolute value of the rate of change at the outer scale, T, multiplied by the ratio of the smallest (t) to the largest time scales, raised to an exponent, ζ, which is a function of q.

$$\mu(|\ln Y_t - \ln Y_{t-1}|^q) \approx \mu(|\ln Y_t - \ln Y_{t-T}|^q)[(t/T)^{\zeta(q)}] \tag{3}$$

In effect, if a process shows an average rate of change on one time scale ($t < T$), it will show comparable rates of variation at longer time horizons when scaled by the factor $[(t/T)^{\zeta(q)}]$. Even if scaling symmetries are much weaker in economic time series, this procedure can be used to obtain the fractal parameters. To estimate $\zeta(q)$, let $V_\tau =$ a time series running from 1 to T, and compute the scaling:

$$\ln V_\tau = \ln[\mu(|\ln Y_t - \ln Y_{t-1}|^q)], \ldots \ln[\mu(|\ln Y_t - \ln Y_{t-T}|^q)] \tag{4}$$

Then regress ln V_τ on the log of the time scale, ln τ_t.

$$\ln V_\tau = a_0 + a_1 \ln \tau_t \tag{5}$$

The regression coefficient a_1 provides an estimate of $\zeta(q)$. The same procedure can be used to estimate the scaling function $\zeta(q)$ for all values of q. Again following Schertzer and Lovejoy (1991), the theoretical distribution of $\zeta(q)$ is given by:

$$\zeta(q) = qH - \{[C(1)/(\alpha - 1)](q^{\alpha} - q)\} \quad \text{when } \alpha \neq 1;$$

$$qH - [C(1)q \ln q] \quad \text{when } \alpha = 1 \tag{6}$$

Eq. (6) implies that the slope of $\zeta(q)$ will depend on the generator of the process. In the standard normal, $H = 0.5$, $\alpha = 2.0$, $C(1) = 0$ and $\zeta(q) = qH$. The scaling curve is a linear trend. In the case of non-linear non-fractal processes such as ARCH, $0 < \alpha < 2$ but $C(1) = 0$, so that the scaling curve is still linear, although the slope varies as a function of H and α. In the fractal case, however, $C(1) \neq 0$. As $C(1)$ increases above zero and α decreases toward 1, and $\zeta(q)$ becomes progressively more curved at higher orders of scaling. The degree of curvature is a function of the probability distribution and the inhomogeneity, and in this sense, a measure of the turbulence of the process.

4. Estimation of the parameters

Estimation of H is based on Eq. (6). In the case of $q = 1$, the quantity $-\{[C(1)/(\alpha - 1)](q^{\alpha} - q)\}$ equals to zero. By construction, when $q = 1$, $\zeta(q) = H$. One advantage of this method of estimating H is that it is robust to non-linearity. Several methods have previously been used to test for fractional degrees of integration, but most of these implicitly assume a linear generating process (Geweke and Porter-Hudak, 1983; Fox and Taqqu, 1986; Sowell, 1992). Their performance on non-linear series may be biased (Cheung, 1993). The scaling method is also less vulnerable to bias than the rescaled range method of Hurst (1951), which may also perform poorly in the case of non-linearity (Lo, 1991).

Two methods of estimating α and $C(1)$ have been proposed. The first, which is computationally simpler, uses the first derivative of $\zeta(q)$ at the origin, denoted $\zeta'(0)$ (Schertzer et al., 1997). This term is equal to $\zeta(1)$ plus a second term, which is a power-law function of α.

$$\zeta'(0) = \zeta(1) + \delta^{\alpha}[C(1)/(\alpha - 1)] \tag{7a}$$

where δ is a constant. Now multiply both sides by q.

$$\mathrm{q}\zeta'(0) = \zeta(q) + \delta q^{\alpha}[C(1)/(\alpha - 1)] \tag{7b}$$

Subtracting $\zeta(q)$ from both sides yields:

$$[\mathrm{q}\zeta'(0) - \zeta(q)] = \delta q^{\alpha}[C(1)/(\alpha - 1)] \tag{7c}$$

which in turn yields the log-linear regression:

$$\ln[q\zeta'(0) - \zeta(q)] = a_0 + a_1 \ln q\delta \tag{7d}$$

The coefficient a_1 estimates α, while the antilog of the intercept is equal to $[C(1)/(\alpha - 1)]$, so that $C(1)$ can be calculated as $[\exp(a_0)](\alpha - 1)$. Note that the range of uncertainty in $C(1)$ is proportional to the standard error of both the intercept and the second regression coefficient a_1. The first derivative method is effective only for $1 < \alpha < 2$, which is the case here. In the case of $0 < \alpha < 1$, the coefficients can be estimated through the double trace method of Lavallee et al. (1992) and Schmitt et al. (1992, 1993).

5. The data

Ideally, the data should meet two criteria: it should be both highly resolved and span a long period. Prior studies have determined that identifying the properties of macroeconomic series usually requires a period encompassing several full business cycles (Perron, 1991). At the same time, tests for fractality require a large number of observations. Accordingly, the exchange rate data is at a daily frequency, and dates from the advent of flexible exchange rates.

Time series for the following 18 currencies were obtained from the Federal Reserve: Australia, Austria, Belgium, Canada, Denmark, Finland, France, Germany, Ireland, Italy, Japan, The Netherlands, New Zealand, Norway, Spain, Switzerland, Sweden and the United Kingdom. In addition, three composite rates were used, the Federal Reserve G-10 Index, the Federal Reserve Major Currencies Index (MCI) and the dollar-EMS rate, i.e. the bilateral rate against the European Currency Unit (Ecu). In each case, the exchange rate is the value at 12:00 h on each trading day. Dollar-gold convertibility was suspended on August 15, 1971, and trading between the dollar and the major European currencies was halted for several days thereafter. Accordingly, the time series begin on August 23, 1971, unless otherwise indicated, and run until December 31, 1998. The dollar-EMS rate begins December 28, 1978. The G-10 index and the Japanese yen begin on August 31, 1971. The Finnish markaa begins September 9, 1971. The MCI and the Spanish peseta begin January 2, 1973.

For the most part, the relationship between the European currencies and the dollar can be characterized as a managed float. Since the European currencies have frequently been linked, through a pegging arrangement in August 1971 to March 1973, through the 'snake' in 1976–1978 and through the EMS from December 1978 until the formation of the Euro, their individual range of variation against the dollar has been constrained by their relationship with each other. Australia and New Zealand were fixed against the dollar until late 1974, adopted crawling pegs against a trade-weighted index of foreign currencies until the early 1980s, and went over to pure floating in the mid-1980s.

Fractality in exchange rates can arise even if interest rates are non-fractal, but interest rate differentials and even interest rates themselves may also be fractal processes. Testing differentials in real rates of return requires using lower frequency data. While daily interest rates are available in some instances, rates have typically varied more on a week-to-week than on a day-to-day basis. At a weekly frequency, the flexible exchange rate period may be too brief to evaluate the properties of interest rates. Accordingly, the tests are run for interest rate data going back to the 1950s. Table 1 describes the time series used. Due to problems of data availability and comparability, the tests are limited to the United States and Germany. In the United States, the tests are run for the Federal funds rate and the 90-day Treasury bill rate. In Germany, the tests are run for the Lombard rate, beginning in 1954, and the 90-day money market rate, beginning in 1959.

The real interest rate is defined as $(R_t - \pi_t^{\mathrm{e}})$ where R_t is the nominal rate of return and π_t^{e} is expected inflation. The inflation rate is the consumer price index. Since this is available only at a monthly frequency, it is interpolated to a weekly frequency using a Kalman filter. The expected inflation rate is the actual inflation rate smoothed by regression on leads and lags. The real interest rate differential is specified as $[(R_{\mathrm{US}} - \pi_{\mathrm{US}}^{\mathrm{e}}) - (R_{\mathrm{GE}} - \pi_{\mathrm{GE}}^{\mathrm{e}})]$. If the ratio rather than the difference is used, the results are similar. Four differentials are tested, the nominal and real differentials between the Federal funds and Lombard rates, and the 90-day Treasury bill rate and 90-day German money market rate.

6. The estimated parameters

Before estimating $\zeta(q)$, the maximum scale (T) must be specified. This cannot be assumed to be asymptotic, and as the sequel will demonstrate, evidence of fractality dissipates over long horizons. It is therefore preferable to estimate the parameters over a range of scales. The results across 60–1200 values were extremely similar. The findings for $T = 200$ are reported. For the weekly interest rate data, the outer scale was set at 60, or slightly more than a year.

Table 2 reports the estimated parameters. The value of H on the log-difference lies in a range of 0.53–0.63, with the majority between 0.55 and 0.59. This of course means that the level series is $I(1.03)$ to $I(1.13)$, an order of integration slightly higher than unity. By implication, the rate of change shows persistence. For the nominal interest rate series, H lies in a range of 0.53–0.57. The real rates and the differentials on the other hand show much lower values of H, implying fractional

Table 1
The interest rate data

United States
90-day Treasury bill rate, weekly, January 8, 1960 to December 31, 1998 (2034 observations)
Federal funds rate, weekly, July 7, 1954 to December 31, 1998 (2322 observations)
Source: Federal Reserve Board
Real rates: nominal rates less smoothed CPI, interpolated to a weekly frequency
Germany
Lombard rate, weekly, July 7, 1954 to December 31, 1998 (2322 observations)
90-Day money market rate, weekly, December 1, 1959 to December 31, 1998 (2039 observations)
Source: Deutsche Bundesbank
Real rates: nominal rates less smoothed CPI, interpolated to a weekly frequency

Table 2
The fractal parameters

Series	H	α	$C(1)$
1. Daily exchange rates			
Composite rates			
Federal Reserve MCI	0.57	1.21	0.026
Federal Reserve G-10	0.59	1.21	0.027
Dollar-ECU	0.57	1.24	0.026
Individual country rates			
Australia	0.56	1.76	0.044
Austria	0.55	1.21	0.052
Belgium	0.56	1.13	0.036
Canada	0.53	1.15	0.055
Denmark	0.56	1.11	0.036
Finland	0.58	1.32	0.076
France	0.58	1.15	0.048
Germany	0.59	1.21	0.031
Ireland	0.55	1.15	0.052
Italy	0.60	1.39	0.070
Japan	0.63	1.34	0.072
Netherlands	0.56	1.19	0.021
New Zealand	0.56	1.78	0.088
Norway	0.55	1.13	0.045
Spain	0.56	1.57	0.115
Switzerland	0.55	1.08	0.033
Sweden	0.56	1.25	0.068
UK	0.59	1.10	0.039
2. Weekly interest rate data			
United States			
Federal funds rate	0.54	1.88	0.145
90-day Treasury bill rate	0.58	1.49	0.107
Real Federal funds rate	0.45	1.16	0.092
Real 90-day Treasury bill rate	0.46	1.14	0.093
Germany			
Lombard rate	0.54	1.99	0.046
Money market rate	0.55	1.35	0.011
Real Lombard rate	0.33	1.55	0.091
Real money market rate	0.34	1.09	0.045
Differentials			
Nominal differential, Federal funds — Lombard	0.44	1.29	0.091
Nominal differential, 90-day Treasury bill rate — German money market rate	0.45	1.29	0.061
Real differential, Federal funds — Lombard	0.34	1.09	0.048
Real differential, 90-day Treasury bill rate — German money market rate	0.35	1.21	0.071

integration at orders less than unity. The nominal differential shows $H = 0.44$–0.45; the real differential shows $H = 0.34$–0.35. Real interest rate differentials are more turbulent than exchange rates themselves; evidently, other factors induce greater persistence in currencies.

Table 2 also reports α and $C(1)$. The values of α for the G10 index, the MCI index and the dollar-EMS rate are fairly close, 1.21–1.24. Since these are composite rates, the rates for individual countries can be expected to be dispersed around these values, and this is in fact the case. The distribution of coefficients is, however, quite skewed. The estimates of α lie for the most part slightly below the values for the composite rates, but there are individual cases of much higher values. Australia and New Zealand both show α higher than the sample mean, most likely because of the different exchange rate regimes they adopted.

For the three composite rates, $C(1)$ lies in a narrow range of 0.026–0.029. This may be caused by the fact that taking a weighted average of currencies reduces inhomogeneity. For most of the individual countries, the degree of intermittency is higher, up to a maximum of $C(1) = 0.117$. The possibility that $C(1) = 0$ was evaluated using the standard error, which implies a range of uncertainty of ± 0.015. The codimension is clearly non-zero in every instance.

Tests were also run for longer time horizons, using ensemble averaging. Beyond $T = 3000$ there was evidence of parameter shifts. The value of α decreased toward unity, and in several cases, actually fell below 1. The values of $C(1)$ fell to zero in all but three cases. In essence, as the time horizon increases, the probability distribution shifts, and evidence of fractality diminishes. Fractality in exchange rates lasts for roughly a full business cycle, but breaks down at longer horizons.

Interest rates in general show narrower probability distributions. The two administered rates — Federal funds and Lombard — show values of α close to 2, with $C(1) = 0.145$ and $C(1) = 0.0460$, respectively. The short-term market rates however show α in a range of 1.35–1.45. All the real interest rates show wide probability distributions — α in a range of 1.14 to 1.16 (United States) to 1.08 to 1.55 (Germany). The interest rates differentials show only slightly larger values (α in a range of 1.21 to 1.29). This may be due to causality from the American rate to the German rate. Interest rate reaction equations indicate that the Bundesbank has to some degree reacted to Federal Reserve decisions (Fair, 1984: 179–181). The interest rates all show strong evidence of fractality, with values of $C(1)$ as high as 0.145.

7. The forecasting model

It has been suggested that for these multifractal processes, scaling symmetries can be exploited to develop forecasting algorithms (Schertzer et al., 1997: 456–457). The scaling symmetry approach can be implemented as follows. The proportionality relationship given in Eq. (3) for mean absolute rates of change can be generalized for $q = 1$ to a relationship between simple log differences. For the predicted rate of change, $(\ln Y_{t+1} - \ln Y_t) = \vartheta_1(\ln Y_t - \ln Y_{t-1})$, while for observed rates of change $(\ln Y_t - \ln Y_{t-1}) = \vartheta_2(\ln Y_t - \ln Y_{t-2})$, $(\ln Y_t - \ln Y_{t-1}) = \vartheta_3(\ln Y_t - \ln Y_{t-3})$, $(\ln Y_t - \ln Y_{t-1}) = \vartheta_n(\ln Y_t - \ln Y_{t-n})$. The value of ϑ_1 is unobserved, but can be extrapolated using lagged values. The proportionality coefficients are obtained using the ratios: $\vartheta_2 = [(\ln Y_t - \ln Y_{t-1})/(\ln Y_t - \ln Y_{t-2})]$, $\vartheta_n = [(\ln Y_t - \ln Y_{t-1})/(\ln Y_t - \ln Y_{t-n})]$. A simple forecasting model is then given by an autoregression, with one or more lags of the scale ratios on the RHS.

However, if scaling symmetries are weak, this approach may be of little value. Instead, a more promising idea is to model the intermittency of the series using state transitions. Let S_{Et} be a state variable corresponding to extreme fluctuations. This is similar to the approach in Shiyun et al. (1999), which uses a Markov-switching method to capture transitions between states of low, intermediate and extreme volatility. Since there is no accepted definition of an extreme event, this was set somewhat arbitrarily at 2.5 times the mean absolute rate of change. The conditional probability of a transition to an extreme state can be estimated using a logit regression. This yields the following forecasting model.

$$\ln Y_t = a_0 + a_1 \ln Y_{t-1} + a_2 S_{Et} \ln Y_{t-1} \tag{8}$$

$$S_{Et} = a_0 + a_1 \ln Y_{t-1} + a_2 S_{Et-1} + a_3(\ln Y_{t-1} - \ln Y_{t-2}) \tag{9}$$

The logit regression is usually found to forecast better when it includes both levels and differences on the RHS. It is also possible to combine the two approaches, yielding:

$$\ln Y_t = a_0 + a_1 \ln Y_{t-1} + a_2 \vartheta_{2t-1} + a_3 S_{Et} \ln Y_{t-1} + a_4 S_{Et} \vartheta_{2t-1} \tag{10}$$

The following forecasting experiments were run. The basic model was a first-order autoregression. Ideally, a structural model would be preferable, such as regressing on real interest rate differentials, but data limitations preclude this here. Various scaling symmetry models were then tested, using several combinations of scale ratios. In addition, a series of ARCH-type models were estimated. The best performing model was usually a GARCH, in which a generalized error distribution was modified to allow for wider tails (Nelson, 1991). Finally, the simple state transition model and the state transition model with scale ratios were tested.

The models were estimated over the first 500 data points, and then forecast iteratively over the remaining values. The forecast in each instance is for one period ahead. After each forecast, the model is reestimated using a Kalman filter, which allows the regression coefficients to change in a non-linear way as the model is updated[2].

[2] In the general form of the Kalman filter, the model is given by $Y_t = X_t\beta_t + u_t$, where Y_t is a time series, X_t is the matrix of RHS variables, β is the vector of states at time t, and the variance of u_t is υ_t. The evolution of β over time is governed by the process $\beta_t = \Psi_t\beta_{t-1} + v_t$. In effect, Ψ is the matrix of transition probabilities. In this instance, Ψ_t was set equal to the identity matrix, so that the evolution of β_t follows a random walk. The error terms v_t and u_t are independent of each other. Denote the covariance matrix of β_t by Σ_t, let the prime denote the transpose, and let Ω_t be a matrix. The updating algorithm is given by:

$$\Omega_t = \Psi_t\Sigma_{t-1}\Psi'_t + v_t;$$

$$\Sigma_t = \Omega_t - \Omega_t(X'_tX_tS_tX'_t + \upsilon_t)^{-1}X_t\Omega_t;$$

$$\beta_t = \Sigma_t\beta_{t-1} + \Sigma_tX'_t\upsilon_{t-1}(Y_t - X_t\Psi_t\beta_{t-1}).$$

Table 3 presents the results of the forecasting experiments. While ten fractal equations were estimated, to save space only the optimal one is reported. The model based on scaling symmetries is not supported. In every instance, the RMSE is raised. Otherwise, the results are generally supportive of a state transition approach. In seven cases a model based on state transitions, either with or without scale ratios, achieves the lowest RMSE. In an additional seven cases, the state transition model ties the AR1 and GARCH models. What is striking about these experiments is how close the results are: the AR1, GARCH and ST all achieve about the same degree of predictive accuracy. On the other hand, in five cases the AR1 and GARCH models achieve the best results, and in two cases, the AR1 achieves the smallest forecast error. In sum, the state transition approach is unambiguously preferable in one-third of cases, and at least as effective as other methods in an additional one-third.

Given that exchange rates are fractals rather than random walks or ARCH processes, it is reasonable to ask why the AR1 and GARCH models are able to forecast as well as they do. Part of the reason lies with the ability of the Kalman filter to at least partially capture changes in state. The answer may also lie in the fact that they parameterize some of the inhomogeneity in the data. When the ARCH functions used in the forecasting models are scaled, they are found to show a non-zero codimension. Similarly, the time-varying AR1 parameter estimated by the Kalman filter also tests positive for fractality. Despite the fact that these models are non-fractal, when fit to the data they do capture some of the intermittency.

8. Conclusions

The tests demonstrate that exchange rates have fractal properties, and that fractality is pervasive. It is a characteristic of foreign exchange markets across a broad range of countries, and over time spans as long as a full business cycle. Further, all the interest rates and interest rate differentials showed evidence of fractality. The fractal properties of financial markets may explain the difficulty that forecasters have often encountered in trying to predict exchange rates with models that do not incorporate any notion of inhomogeneity. The absence of strong scaling symmetries in exchange rates, however, means that models based on scale ratios have very little predictive power. Instead, the optimal approach is probably to include state transition terms in order to model intermittency.

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Table 3
Model forecasting performance[a]

Model	Scale ratios included	Length of distributed lag	RMSE
1. Federal Reserve MCI			
AR1	–	–	2.754×10^{-3}
GARCH	–	–	2.754×10^{-3}
Scale ratios	ϑ_2 to $\vartheta_{,6}$	1	2.771×10^{-3}
ST	–	–	2.752×10^{-3b}
ST, scale ratios	ϑ_2 to ϑ_3	1	2.752×10^{-3b}
2. Federal Reserve G10 Index			
AR1	–	–	3.637×10^{-3b}
GARCH	–	–	3.637×10^{-3b}
Scale ratios	ϑ_2 to ϑ_4	1	3.641×10^{-3}
ST	–	–	3.637×10^{-3b}
ST, scale ratios	ϑ_2 to ϑ_3	1	3.643×10^{-3}
3. Dollar–Ecu			
AR1	–	–	5.136×10^{-3}
GARCH	–	–	5.136×10^{-3}
Scale ratios	ϑ_2 to ϑ_3	1	5.148×10^{-3}
ST	–	–	5.127×10^{-3}
ST, scale ratios	ϑ_2 to ϑ_3	1	5.118×10^{-3b}
4. Australia			
AR1	–	–	3.298×10^{-3b}
GARCH	–	–	3.299×10^{-3}
Scale ratios	ϑ_2 to ϑ_3	1	3.311×10^{-3}
ST	–	–	3.298×10^{-3b}
ST, scale ratios	ϑ_2 to ϑ_3	1	3.299×10^{-3}
5. Austria			
AR1	–	–	4.870×10^{-3b}
GARCH	–	–	4.870×10^{-3b}
Scale ratios	ϑ_2 to ϑ_3	1	4.895×10^{-3}
ST	–	–	4.877×10^{-3}
ST, scale ratios	ϑ_2 to ϑ_3	1	4.978×10^{-3}
6. Belgium			
AR1	–	–	4.627×10^{-3b}
GARCH	–	–	4.627×10^{-3b}
Scale ratios	ϑ_2 to ϑ_5	1	4.731×10^{-3}
ST	–	–	4.630×10^{-3}
ST, scale ratios	ϑ_2 to ϑ_3	1	4.632×10^{-3}
7. Canada			
AR1	–	–	1.850×10^{-3}
GARCH	–	–	1.851×10^{-3}
Scale ratios	ϑ_2 to ϑ_3	1	1.852×10^{-3}
ST	–	–	1.849×10^{-3b}
ST, scale ratios	ϑ_2 to ϑ_3	1	1.853×10^{-3}
8. Denmark			
AR1	–	–	4.624×10^{-3b}
GARCH	–	–	4.626×10^{-3}
Scale ratios	ϑ_2 to ϑ_4	1	4.628×10^{-3}
ST	–	–	4.627×10^{-3}
ST, scale ratios	ϑ_2 to ϑ_3	1	4.627×10^{-3}

Table 3 (Continued)

Model	Scale ratios included	Length of distributed lag	RMSE
9. Finland			
AR1	–	–	4.090×10^{-3b}
GARCH	–	–	4.090×10^{-3b}
Scale ratios	ϑ_2 to ϑ_3	1	4.097×10^{-3}
ST	–	–	4.090×10^{-3b}
ST, scale ratios	ϑ_2 to ϑ_3	1	4.091×10^{-3}
10. France			
AR1	–	–	4.445×10^{-3b}
GARCH	–	–	4.445×10^{-3b}
Scale Ratios	ϑ_2	4	4.536×10^{-3}
ST	–	–	4.445×10^{-3b}
ST, Scale Ratios	ϑ_2	1	4.448×10^{-3}
11. Germany			
AR1	–	–	4.697×10^{-3}
GARCH	–	–	4.698×10^{-3}
Scale ratios	ϑ_2 to ϑ_5	1	4.716×10^{-3}
ST	–	–	4.692×10^{-3}
ST, scale ratios	ϑ_2 to ϑ_4	1	4.690×10^{-3b}
12. Ireland			
AR1	–	–	4.415×10^{-3b}
GARCH	–	–	4.415×10^{-3b}
Scale ratios	ϑ_2	1	4.439×10^{-3}
ST	–	–	4.436×10^{-3}
ST, scale ratios	ϑ_2	1	4.438×10^{-3}
13. Italy			
AR1	–	–	4.183×10^{-3}
GARCH	–	–	4.182×10^{-3}
Scale ratios	ϑ_2 to $\vartheta_{,3}$	1	4.195×10^{-3}
ST	–	–	4.179×10^{-3b}
ST, scale ratios	ϑ_2 to ϑ_3	1	4.218×10^{-3}
14. Japan			
AR1	–	–	4.440×10^{-3}
GARCH	–	–	4.436×10^{-3}
Scale ratios	ϑ_2	5	4.547×10^{-3}
ST	–	–	4.375×10^{-3b}
ST, scale ratios	ϑ_2	1	4.378×10^{-3}
15. Netherlands			
AR1	–	–	4.594×10^{-3b}
GARCH	–	–	4.594×10^{-3b}
Scale ratios	ϑ_2 to ϑ_5	1	4.599×10^{-3}
ST	–	–	4.596×10^{-3}
ST, scale ratios	ϑ_2 to $\vartheta_{,3}$	1	4.598×10^{-3}

Table 3 (Continued)

Model	Scale ratios included	Length of distributed lag	RMSE
16. New Zealand			
AR1	–	–	3.521×10^{-3b}
GARCH	–	–	3.523×10^{-3}
Scale ratios	ϑ_2 to ϑ_4	1	3.546×10^{-3}
ST	–	–	3.531×10^{-3}
ST, scale ratios	ϑ_2	1	3.544×10^{-3}
17. Norway			
AR1	–	–	4.097×10^{-3b}
GARCH	–	–	4.097×10^{-3b}
Scale ratios	ϑ_2 to ϑ_3	1	4.102×10^{-3}
ST	–	–	4.097×10^{-3b}
ST, scale ratios	ϑ_2 to ϑ_3	1	4.100×10^{-3}
18. Spain			
AR1	–	–	4.219×10^{-3b}
GARCH	–	–	4.219×10^{-3b}
Scale ratios	ϑ_2 toϑ_5	1	4.226×10^{-3}
ST	–	–	4.219×10^{-3b}
ST, scale ratios	ϑ_2 to $\vartheta_{,3}$	1	4.221×10^{-3}
19. Switzerland			
AR1	–	–	5.433×10^{-3b}
GARCH	–	–	5.433×10^{-3b}
Scale ratios	ϑ_2 to ϑ_3	1	5.435×10^{-3}
ST	–	–	5.434×10^{-3}
ST, scale ratios	ϑ_2	1	5.435×10^{-3}
20. Sweden			
AR1	–	–	3.994×10^{-3b}
GARCH	–	–	3.994×10^{-3b}
Scale ratios	ϑ_2	4	4.001×10^{-3}
ST	–	–	3.994×10^{-3b}
ST, scale ratios	ϑ_2	1	3.996×10^{-3}
21. United Kingdom			
AR1	–	–	4.297×10^{-3}
GARCH	–	–	4.297×10^{-3}
Scale ratios	ϑ_2 to ϑ_3	1	4.302×10^{-3}
ST	–	–	4.261×10^{-3}
ST, scale ratios	ϑ_2 to ϑ_3	1	4.255×10^{-3b}

[a] ST refers to the simple state transition model. ST, scale ratios refers to the state transition model with scale ratios. Scale ratio ϑ_2 denotes $[(Y_t - Y_{t-1})/[(Y_t - Y_{t-2})]$.

[b] Denotes the minimum RMSE.

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