VIX Option Pricing and CBOE VIX Term Structure: A New Methodology for Volatility Derivatives Valuation

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Abstract

The study proposes an arbitrage-free methodology of VIX term structure modeling that is tailored to handle the most actively traded VIX options. Under the model, the evolution of *future* VIX is completely determined by the volatility function of *forward* VIX *squared* normalized by VIX futures prices. A general volatility function with one- to three-factor models is used to examine their pricing for VIX options across strikes and maturities from February 24, 2006 to September 30, 2008. The primary contributions of this article are (i) taking as given **CBOE VIX Term Structure** to specify a general stochastic structure upon *forward* VIX, (ii) proposing an arbitrage-free closed-form solution to the VIX option value with the general volatility function that incorporates **mean-reversion** and **hump effects** to test two multifactor models, (iii) finding that the class of **multifactor** models outperforms the classes of one- and two-factor models, and (iv) finding that models with **hump volatility function** perform better than other models. Pricing with **three-factor models** gives the benefit of eliminating most moneyness and maturity biases, especially for ATM and OTM options. Correctly specified and calibrated, the single-factor **Hump** model may be superior to inappropriate multi-factor Exponential models in pricing.

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1. Introduction

The Chicago Board Options Exchange (CBOE) launched Volatility Index (VIX) futures on March 26, 2004 and VIX options on February 24, 2006. These were the first of an entire family of volatility products traded on exchanges. Since the calculation of their underlying VIX isolates expected volatility from other factors that could affect option prices such as dividends, interest rates, changes in the underlying price and time to expiration,[1] the VIX options and VIX futures offer a way for investors to buy and sell option volatility without having to deal with factors that have an impact on the value of an SPX option position. Currently, VIX options and VIX futures are among the most actively traded contracts at CBOE and the CBOE Futures Exchange (CFE), averaging close to 140,000 contracts combined per day. This increased trading activity has coincided with a growing sophistication among volatility traders and a better understanding of the inherent complexity of option implied volatility, behavior that depends on multiple factors such as the volatility "term structure" ─ implied volatility as a function of time to expiration.

Coinciding with the growth of futures and options on VIX, CBOE has calculated daily historical values for the **VIX Term Structure** (denoted *VIXTerm*)

[1] The VIX index calculations are based on and use real-time S&P 500 Index (SPX) option bid/ask quotes. The VIX calculation formula averages the weighted prices of at-the-money and out-of-the money puts and calls, which incorporates information from the volatility skew, to derive expected volatility.

dating back to 1992. The *VIXTerm* is a representation of SPX option implied volatility that involves applying the VIX formula to particular SPX option expirations to construct a term structure for fair variance. The generalized VIX formula has been modified to reflect business time to expiration.[2] As such, investors will be able to use *VIXTerm* to track the movement of the SPX option implied volatility in the listed contract months. The *VIXTerm* of various maturities allows us to infer a complete initial term structure of *forward* VIX that is contemporaneous with the prices of VIX futures and VIX options.

Current prices of both VIX futures and VIX options reflect the market's expectation of the VIX level at expiration, so-called *forward* VIX. The *forward* VIX, rather than the spot VIX itself, is consequently the underlying of VIX options. Lin (2007) also demonstrates the fair value to VIX futures is the *forward* VIX. Hence, the best tradable proxy for the underlying of VIX options is the VIX futures.

The literature on the price behavior of VIX spot and futures markets is growing fast (see, for example, Carr and Wu, 2006; Dupire, 2006; Zhang and Zhu, 2006; Zhu and Zhang, 2007; Lin, 2007; Dotsis, Psychoyios, and Skiadopoulos, 2007; Huskaj and

[2] The expiration of VIX futures and VIX options differs from other options expiration. While the options expire on the third Friday, the VIX options and VIX futures expire exactly 30 days before the next options expiration. The VIX options and VIX futures expire always on the Wednesday that is thirty days prior to the third Friday of the following calendar month. This is because the VIX index was designed to be a consistent 30-day benchmark of expected market volatility and the VIX index is calculated as a weighted average of options expiring on two different dates.

Nossman, 2009). Nevertheless, research on the valuation of VIX options is not concluded. Huskaj and Nossman (2009) assume that the VIX futures returns are captured by an inverse Gaussian distribution and develop models for the VIX futures and VIX options. Lin and Chang (2009) reconcile the growing literature on SPX price processes by investigating how much each generalization of the SPX price dynamics improves VIX option pricing and hedging. This study values VIX options based on the **CBOE VIX Term Structure** and current VIX futures prices.

This study presents the family of stochastic processes representing *future* VIX movements. In particular, our model precludes arbitrage opportunity between the SPX options market (converted to the square of *VIXTerm*) and the VIX futures market. In other words, this study assumes the effective integration between these two markets. Thus this family uniquely determines the *future* VIX process. The VIX options are then priced based on this dynamics of *future* VIX. Theoretical work in this paper has developed from one-factor to multi-factor models for pricing the VIX option.

A principal component analysis of changes in the *future* VIX that is constructed from daily *forward* VIX *squared* normalized by the VIX futures price over our sample period (February 24, 2006−September 30, 2008) reveals various factors that drive the evolution of the term structure. The first factor, interpreted as the “mean-reversion” factor, capturing decaying movements in the term structure, contributes 97.63% of the overall explained variance of *future* VIX changes. The second factor, interpreted as a "twist" factor in the yield curve, incorporating changes in the slope of term structure, contributes another 2.16% of the overall explained the variance of *future* VIX changes. Finally, the third factor is interpreted as "curvature", contributing further 0.12% of the overall explained variance of *future* VIX changes. This analysis suggests that most of the variation in returns on the *future* VIX can be explained by these three factors.

Like the Black (1976) model, claim prices are completely determined by a description of the volatility structure of *forward* VIX *squared* normalized by VIX futures prices. This study considers one- to three-factor models from two classes of volatility functions. The first class consists of VIX models with exponential volatility function parameterized as in the Vasicek (1977) model. This class of models intends to capture the downward and upward decaying term structure volatilities, denoted "*mean-reversion*" or "*Exponential*". The second class is indicated by the hump shape of volatility structure as in the Zeto (2002) model, denoted "*Hump*".

Our sample covers the period February 24, 2006 to September 30, 2008. Model parameters are estimated from VIX options across a full range of strikes and maturities and their contemporaneous VIX futures prices and *VIXTerm*. In-sample and out-of-sample pricing performance for each model are evaluated, and models with identical factors and models in the same volatility family are presented with option moneyness and maturity. The pricing results of all models are summarized as follows. First, three-factor models generally outperform one- and two-factor models. Second, the hump volatility function performs much better than pure mean-reversion volatility functions, and they are able to remove most of the moneyness and maturity biases. Third, the study demonstrates that out-of-sample pricing errors cannot be significantly reduced by adding an identical stochastic factor into a one- or two-factor model. The implications of our results are twofold. First, pricing VIX options is possible, but requires using multi-factor models. Second, a correctly specified and calibrated one-factor Hump model may replace the inappropriate Exponential multifactor models for pricing VIX options.

The remainder of the paper is organized as follows. Section 2 presents the contributions of this paper to existing literature. Section 3 characterizes the *future* VIX process and derives arbitrage-free prices of VIX options. Section 4 specifies the volatility functions governing the dynamics of *future* VIX. Section 5 presents the data. Section 6 estimates model parameters. Section 7 tests VIX options pricing formulas against the market data. Section 8 analyzes the association between forecasting errors and possible factors. Finally, Section 9 concludes.

2. Contributions to Existing Literature

A large part of the implied-volatility term structure literature has focused on equity options (see for example, Poterba and Summers, 1986; Stein, 1989; Diz and Finucane, 1993; Campa and Chang, 1995; Poteshman, 2001; Byoun, Kwok and Park, 2003; Mixon, 2007). More recently, volatility derivative data have also been used to analyze these models, and this paper contributes to that part of the literature. To indicate in more detail our contribution to the existing literature on volatility derivatives, this paper distinguishes three different aspects of the analyses: (i) the methodology used here, (ii) the models that are considered in this study, and (iii) the data.

This paper presents a general theory and a unifying framework for understanding arbitrage pricing theory in this context. In relation to the term structure of implied volatilities, arbitrage pricing theory has two purposes. The first, is to eliminate arbitrary arbitrage opportunities between the SPX option market, that gives *VIXTerm* of varying maturities, and the VIX futures market from a multifactor state economy. The second, is to price VIX options, taking as given the prices of the *VIXTerm* and VIX futures. The primary theoretical contribution of this paper is a new methodology for solving the first and second problems.

The methodology is new in volatility derivatives because (i) it imposes a multifactor structure directly on the evolution of the *future* VIX, (ii) it does not require an "inversion of the term structure" to eliminate the market prices of risk from VIX derivatives values, and (iii) it incorporates alternate volatility functions influencing the term structure of implied volatilities. The model can be used to consistently price and hedge all contingent claims (American or European) on the term structure, and it is derived from necessary and (more importantly) sufficient conditions for the absence of arbitrage.

The model in this paper takes as given the initial *forward* VIX curve and the current price of VIX futures. The study then specifies a general continuous-time stochastic process for their relative evolution across time. In particular, the study compares two popular choices for the *future* VIX volatilities: mean-reversion and hump volatility functions. These functions are also specified from one- to three-factor models. To ensure that the process is consistent with an arbitrage free economy, the study uses the insights of Brace et al. (1997) to characterize the conditions on the *future* VIX process such that there exists a unique, equivalent forward probability measure. Under these conditions, markets are complete and contingent claim valuation is then a straightforward application of the methods in Miltersen et al. (1997). The study illustrates this approach with VIX options.

The third aspect contributing to the existing literature is the use of VIX options as samples to evaluate models. End-of-day options on VIX across a full range of strikes and maturities are used so that we can compare models in terms of their cross-sectional ability to capture volatility surface dynamics. Our pricing framework for VIX options is the first time in literature to examine volatility derivatives in this way.

3. The Model

Given a family $\mathrm{F}_t^{\mathrm{VIX}}(T)$ of VIX futures prices and the collection of **CBOE VIX Term Structure**, $VIXTerm(t,T)$, this study focuses on the modeling of a finite family of *future* VIX that are associated with a pre-specified collection of settlement dates.

Consider a VIX *term structure* associated with a predetermined collection of settlement dates $0 \le T_0 < T_1 < \ldots < T_m$. Let us write $\tau_j = T_{j+1} - T_j$ for $j = 0,\ldots,m-1$, so that $T_{j+1} = T_0 + \sum_{i=0}^{j} \tau_i$ for every $j = 0,\ldots,m-1$. Since τ_j is not necessarily constant, the assumption of a fixed accrual period τ_0 is not imposed, and thus a model will be more suitable for **CBOE VIX Term Structure**. In addition, CBOE calculates **VIX Term Structure** data using a "business day" convention to measure time to expiration, as well as the "calendar day" convention used in the VIX Index itself. Therefore, the following proposition uses historic **CBOE VIX Term Structure** observations to compute a time series history of *forward* VIX *squared*, $\mathrm{VIX}(t,T_j;\tau_j)^2$, across all maturities T_j that is annualized using actual calendar days.

Proposition 1. *For any* $j=0,\ldots,m-1$ *and* $\forall t\in[0,T_j]$, *forward VIX squared,* $\mathrm{VIX}(t,T_j;\tau_j)^2$ *that starts at* T_j *and ends at* T_{j+1}, *is defined as the annualized integrated variance of the SPX returns over* $[T_j,T_{j+1}]$ *multiplied by 10,000. Specifically, forward VIX squared is obtained by setting*

$$\mathrm{VIX}(t,T_j;\tau_j)^2=\frac{1}{\tau_j^{cal}}[VIXTerm(t,T_{j+1})^2\cdot(T_{j+1}-t)-VIXTerm(t,T_j)^2\cdot(T_j-t)]$$

where $VIXTerm(t,\cdot)$ *is the* ***CBOE VIX Term Structure***. τ_j^{cal} *indicates the annualized calendar days between* T_j *and* T_{j+1}, *and* (T_l-t) *for* $l=j$ *or* $j+1$ *is measured in annualized trading days.*

The proof of Proposition 1 is given in Appendix A. It is noticeable that the volatilities that comprise the **VIX Term Structure** data are calculated by applying the VIX formula to a single strip of SPX options having the same expiration date. However, SPX options expire on the third Friday of the contract month, and options and futures on VIX settle on Wednesday, that is 30 calendar days before the third Friday of the calendar month immediately following the month in which the contract expires. In addition, in VIX futures and VIX options the accrual period (day-count fraction) is $\tau_0 \equiv 22/365$ on the trading-day basis (converted to $\tau_0^{cal} = 30/365$ on the calendar-day basis) for $j = 0,\ldots,m-1$, and thus it is also essential to have a model of *forward* τ_0-VIX *squared* that is capable of mimicking this important real-life feature .

The *forward* τ_0-VIX *squared* for all VIX option maturities T_j^o can be determined from the set of *forward* VIX *squared* by setting $\forall \tau_j = \tau_0 \equiv 22/365$, $T_j = T_j^o$ and $T_{j+1} = T_j^o + \tau_0$ in Proposition 1. To do this, we need to back out both $VIXTerm(t,T_j^o)$ and $VIXTerm(t,T_j^o + \tau_0)$ from observed $VIXTerm(t,T_j)$ and $VIXTerm(t,T_{j+1})$ using the formula in the CBOE VIX White paper:

$$
\begin{aligned}
& VIXTerm(t,T_j^o + \tau_0) = \sqrt{(T_j^o + \tau_0 - t)^{-1}} \\
& \times \sqrt{\tau_j \frac{\tau_{j+1} - (T_j^o + \tau_0 - t)}{\tau_{j+1} - \tau_j} \cdot VIXTerm(t,T_j)^2 + \tau_{j+1} \frac{(T_j^o + \tau_0 - t) - \tau_j}{\tau_{j+1} - \tau_j} \cdot VIXTerm(t,T_{j+1})^2} \\
& for \quad T_j \le (T_j^o + \tau_0) \le T_{j+1} \\
& VIXTerm(t,T_j^o) = \sqrt{(T_j^o - t)^{-1}} \\
& \times \sqrt{\tau_j \frac{\tau_{j+1} - (T_j^o - t)}{\tau_{j+1} - \tau_j} \cdot VIXTerm(t,T_j)^2 + \tau_{j+1} \frac{(T_j^o - t) - \tau_j}{\tau_{j+1} - \tau_j} \cdot VIXTerm(t,T_{j+1})^2} \\
& for \quad T_j \le T_j^o \le T_{j+1}
\end{aligned}
$$

Using Proposition 1, *forward* τ_0-VIX *squared* that starts at T_j^o and ends at $T_j^o + \tau_0$ is then given by

$$[\mathrm{VIX}(t,T_j^o;\tau_0)]^2 = \frac{1}{\tau_0^{cal}}[VIXTerm(t,T_j^o+\tau_0)^2 \cdot (T_j^o+\tau_0-t) - VIXTerm(t,T_j^o)^2 \cdot (T_j^o - t)] \tag{1}$$

The next step is to introduce the notion of a forward martingale measure Q^{F_j} associated with VIX futures. The corresponding Brownian motions are denoted by ω^{F_j}. By definition, the square of VIX *term structure*, $[VIXTerm(t,T_j)]^2$ for $\forall t \in [0,T_j]$ and $\forall j \in [0,m-1]$, is replicable by a portfolio of SPX options. From Proposition 1, *forward* VIX *squared*, $[\mathrm{VIX}(t,T_j;\tau_0)]^2$, can be thus regarded as a "tradable" asset.[3] It is then apparent that the ratio $z(t,T_j)$ of *forward* VIX *squared* over the VIX futures price is set to satisfy a martingale process associated with a forward measure Q^{F_j}, that uses VIX futures as the numéraire. Specifically, for $\forall t \in [0,T_j]$ and $\forall T_{j+1} - T_j = \tau_0$

$$z(t,T_j) = \frac{[\mathrm{VIX}(t,T_j;\tau_0)]^2}{\mathrm{F}_t^{\mathrm{VIX}}(T_j)} \sim Q^{F_j} - martingale \tag{2}$$

where $\mathrm{F}_t^{\mathrm{VIX}}(T_j)$ is the time-*t* price of VIX futures with expiry T_j. It is also useful to

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[3] Baxter and Rennie (1996) show how to deal with the contingent claim valuation on non-tradable underlying processes. Given that non-tradable X_t but a deterministic function of X_t, $Y_t = f(X_t)$, is tradable, the market price of risk for Y_t can be derived by changing the probability measure from actual measure to the risk-neutral measure. As a result, the claims on X_t can be priced via the risk-neutral process for X_t. Similarly, although VIX is non-tradable, the square of *VIXTerm* can be replicated by a static portfolio of traded SPX options and thus can be regarded as tradable. As a result, the square of *forward* VIX defined by the difference between the square of *VIXTerm*(t,T_1) and the square of *VIXTerm*(t,T_2) is "tradable" using Baxter and Rennie's (1996) argument.

observe that $z(T_j,T_j) = \mathrm{F}_{T_j}^{\mathrm{VIX}}(T_j) = \mathrm{VIX}(T_j,T_j;\tau_0)$ is the spot VIX at futures expiry T_j. For a certain adapted process $\sigma_k(\cdot,T_j)$, it is evident that the process z necessarily follows a martingale under the forward probability measure Q^{F_j}. That is,

$$\frac{dz(t,T_j)}{z(t,T_j)} = \sum_{k=1}^{d} \sigma_k(t,T_j) \cdot d\omega_k^{F_j}(t) \tag{3}$$

or, alternatively, in its integral form

$$z(T_j,T_j) = z(t,T_j)\exp\left[-\frac{1}{2}\sum_{k=1}^{d}\left(\int_t^{T_j} \sigma_k^2(u,T_j)\cdot du\right) + \sum_{k=1}^{d}\left(\int_t^{T_j} \sigma_k(u,T_j)\cdot d\omega_k^{F_j}(u)\right)\right] \tag{4}$$

where $\omega_1^{F_j},\ldots,\omega_d^{F_j}$ are independent one-dimensional Brownian motions under Q^{F_j}. Since $\mathrm{F}_{T_j}^{\mathrm{VIX}}(T_j) = \mathrm{VIX}(T_j,T_j;\tau_0)$, equation (5) also gives us

$$\mathrm{VIX}(T_j,T_j;\tau_0) = \frac{[\mathrm{VIX}(t,T_j;\tau_0)]^2}{\mathrm{F}_t^{\mathrm{VIX}}(T_j)} \cdot \exp\left[-\frac{1}{2}\sum_{k=1}^{d}\int_t^{T_j} \sigma_k^2(u,T_j)\cdot du + \sum_{k=1}^{d}\int_t^{T_j} \sigma_k(u,T_j)\cdot d\omega_k^{F_j}(u)\right] \tag{5}$$

Given the availability of *forward* VIX *squared* and current prices of VIX futures, equation (5) indicates that the evolution of *future* VIX, $\mathrm{VIX}(T_j,T_j;\tau_0)$, is completely determined by the volatility function.

Based on equation (5), this study derives closed-form expressions for the fair value of the VIX option. This study therefore concludes that based on data availability of *forward* VIX *squared*, $\mathrm{VIX}(t,T_j;\tau_0)^2$, and VIX futures prices, $\mathrm{F}_t^{\mathrm{VIX}}(T_j)$, it is enough to find the fair price of the VIX option for the settlement date T_j.

Further, this study assumes that the volatility of *future* VIX is a deterministic function; that is, we place ourselves within the Gaussian framework. The following Proposition provides a generic pricing formula for a European VIX call and put options in the Gaussian *future* VIX measure set-up.

Proposition 2. *Assume the Gaussian future VIX framework, so that the volatilities* $\sigma_k(t,T_j)$, *for* $k=1,\ldots,d$, *are deterministic and satisfy Novikov's condition, i.e.* $\mathrm{E}^{Q^{F_j}}\left[\exp\left(\frac{1}{2}\int_0^{T_j}\sigma_k^2(u,T_j)du\right)\right]<\infty$. *Then the arbitrage-free price at time* $t\le T_0$ *of a VIX call option with strike level K, settled at time* T_j, *equals*

$$\begin{aligned}C_t^{\mathrm{VIX}}(K,T_j)&=\mathrm{F}_t^{\mathrm{VIX}}(T_j)\,\mathrm{E}_t^{Q^{F_j}}\left[\frac{C_{T_j}^{\mathrm{VIX}}(K,T_j)}{\mathrm{F}_{T_j}^{\mathrm{VIX}}(T_j)}\right]\\&=\mathrm{F}_t^{\mathrm{VIX}}(T)\left[N(d_1)-K\frac{\mathrm{F}_t^{\mathrm{VIX}}(T_j)}{[\mathrm{VIX}(t,T_j;\tau_0)]^2}\cdot\exp\left(\sum_{k=1}^{d}\int_t^{T_j}\sigma_k^2(u,T_j)\cdot du\right)\cdot N(d_2)\right]\end{aligned}$$

where $\mathrm{F}_t^{\mathrm{VIX}}(T_j)$ *is the time-t price of VIX futures that mature at* T_j, $N(\cdot)$ is the cumulative density function of a standard normal-distributed *r.v.,*

$$d_1=\frac{\ln\left\{\dfrac{[\mathrm{VIX}(t,T_j;\tau_0)]^2}{\mathrm{F}_t^{\mathrm{VIX}}(T_j)}\right\}-\ln K-\dfrac{1}{2}\displaystyle\sum_{k=1}^{d}\int_t^{T_j}\sigma_k^2(u,T_j)du}{\sqrt{\displaystyle\sum_{k=1}^{d}\int_t^{T_j}\sigma_k^2(u,T_j)du}}\quad \textit{and}\quad d_2=d_1-\sqrt{\sum_{k=1}^{d}\int_t^{T_j}\sigma_k^2(u,T_j)du}\,.$$

Similarly, the pricing formula for a VIX put option with strike K and expiry T_j *is given by*

$$P_t^{\mathrm{VIX}}(K,T_j)=\mathrm{F}_t^{\mathrm{VIX}}(T_j)\left\{K\cdot\frac{\mathrm{F}_t^{\mathrm{VIX}}(T_j)}{[\mathrm{VIX}(t,T_j;\tau_0)]^2}\cdot\exp\left(\sum_{k=1}^{d}\int_t^{T_j}\sigma_k^2(u,T_j)du\right)\cdot N(-d_2)-N(-d_1)\right\}$$

The proof of Proposition 2 is given in Appendix B. The pricing formulas for VIX options can be completely specified by a volatility function, which is analogous to Black's model since for pricing options the only unknown is volatility and there is no requirement for estimating the market price of risk. In this framework, the deterministic volatility function $\sigma_k(t,T_j)$ can be freely specified and the model encompasses one- and multi-factor models. This model values all *forward* τ_0-VIX with varying maturities, which are calculated from the current *VIXTerm*, and can price all VIX contingent claims as arbitrage-free.

4. Specification of Volatility Function

Like the Black model, our VIX models require no assumptions about investor preferences. As a result, claim prices are completely determined by a description of the volatility structure that governs *future* VIX changes.

Forward prices of option volatility exhibit a "**term structure**", meaning that the prices of options expiring on different dates may imply different, albeit related, volatility estimates. The term structure of VIX is the curve of implied volatilities for periods extending from the current date to different future dates. Figure 1 represents the surface of **CBOE VIX Term Structure** Midpoints across the market close dates and option maturities over the period from January 2, 1992 to March 11, 2009.

[Figure 1 about here]

The initial *forward* VIX curve is then calculated from the set of **CBOE VIX Term Structure** $VIXTerm(t,T_j)$ for all T_j determined as the solution in Proposition 1. Figure 2 displays the historical *forward* VIX evolutions for all maturities. From this evolution, we can deduce the z curve implied by *forward* VIX *squared* normalized by VIX futures prices, as illustrated in Figure 3.

[Figure 2 about here]

[Figure 3 about here]

Given the evolution of the z curve, the study employs the principle components analysis (PCA) to estimate the volatility functions used in the VIX option pricing model.[4] For this analysis, the study uses changes in logarithmic z ratios. Figure 4 contains the histograms of daily changes in the six different maturity logarithmic z ratios over this sample period. A normal distribution is superimposed on each of these histograms to give a sense for the quality of the underlying normality assumption.

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[4] According to Jolliffe (2002) the central idea of PCA is to reduce the dimensionality of a data set which may consist of a large number of interrelated variables, while retaining as much as possible of the variation present in the data set. Thus principal component analysis is concerned with reducing the number of variables under consideration by using only the first few principal components, which explain the maximum amount of variance in the original variables by optimal least squares fitting of the covariance or correlation matrix of the original variables, and discarding those, which have small variances. Principal components are uncorrelated, so one can talk about each of the Principal components without referring to the others, each one makes an independent contribution to accounting for the variance of the original variables. In this study, the PCA method allows up to d factors given d z ratios. The hope is that only a small number of factors can explain most of the variation in z ratios.

[Figure 4 about here]

The covariance matrix of the various maturity logarithmic *z* ratios is the basis for a PCA using the deterministic volatility assumption. These volatility functions are then obtained from the principal components of the covariance matrix. A demonstration of this identification is presented in the Appendix C.

The first three factors account for 99.91 percent of the total variance, containing 97.63%, 2.16% and 0.12% percentage of the total variance explained by the first, second and third principal component, respectively. The first three volatility functions are graphed in Figure 5. The first factor is roughly a decaying movement. The second factor accounts for twisted movements in *z* ratios. Finally, the third factor emphasizes the curvature relation between the short and long *z* ratios.

[Figure 5 about here]

Based on PCA, this study examines two generalized volatility functions that allow *future* VIX to posit the characteristics of the "**term structure**", "**mean reversion**", and "**volatility of volatility**". The first function satisfies a pure-Gaussian economy and the diffusion volatility in this process allows mean reversion. This generalized volatility function is originally specified by Amin and Morton (1994) for interest rate derivatives:

$$\sigma_k^2(t,T_j) = \nu_k \, e^{-\eta_k (T_j - t)} \tag{6}$$

where $\sigma_k^2(t,T_j)$ is the volatility function governed by *k* independent factors; ν_k and η_k are volatility parameters and are assumed to be constant; $z(t,T_j) = [\mathrm{VIX}(t,T_j;\tau_0)]^2 / \mathrm{F}_t^{\mathrm{VIX}}(T_j)$ is the *forward* VIX *squared* normalized by the VIX futures price for a VIX option contracted at time *t* and maturing at time T_j. The specification of $\nu_k \exp[-\eta_k (T_j - t)]$ enables mean reversion where η_k is the coefficient of mean reversion.

Following Zeto (2002),[5] the second generalized function in *future* VIX is proposed in this study, in order to capture the hump shape of term structure volatility. That is,

$$\sigma_k^2\big(t,T_j\big) = \varphi_k + \big[\nu_k + \theta(T_j - t)\big]e^{-\eta_k(T_j - t)} \tag{7}$$

where φ_k, ν_k and η_k are strictly positive constants and $\sigma_k^2(t,T_j)$ is non-negative. φ_k governs the level of the term structure of volatilities which can be interpreted as a "long-run factor" since it uniformly shifts all maturity *future* VIX equally, ν_k controls the slope, and θ controls the curvature. The last parameter, η_k, is a damping or fitting parameter. This study extends Zeto (2002) from two-factor to three-factor volatility functions and Amin and Morton (1994) from one-factor to

[5] Zeto (2002) constructed three new volatility specifications of forward interest rates that allow for the humped term structure of volatilities.

three-factor volatility functions. Equations (7) and (8) can be written as one general volatility functions in the multifactor *future* VIX framework as follows,

$$\sigma^2(t,T_j) = \sum_{k=1}^{3}\{\varphi_k + [\nu_k + \theta(T_j - t)] \cdot e^{-\eta_k(T_j - t)}\} \qquad (8)$$

where $\{\varphi_k, \theta\} = \{0,0\}$ for decaying shapes in (6) or $\{\varphi_k, \theta\} \neq \{0,0\}$ for extra including a humped shape in (7).

Table 1 shows the models that will be tested in this study. In panels A, B and C, two one-factor, two-factor, and three-factor models in *future* VIX are presented and in each panel, models are categorized into two groups. In group 1, exponential models are proposed to capture mean reversion of volatility. In group 2, hump models are proposed to capture not only the hump feature, but also the mean-reversion effect. Table 2 presents the generic format for multifactor models tested in this study. This table also shows the models that were tested by Amin and Morton (1994), Zeto (2002), Gupta and Subrahmanyam (2005), Kuo and Paxson (2006), and Kuo and Lin (2007) for interest rate derivatives. However, the rest of the possible volatility models are ignored because they do not conform to the general form proposed in this study. The models unique in this study are part of two-factor and all three-factor models as well as the application of these models to the *future* VIX process.

[Table 1 about here]

[Table 2 about here]

From these volatility functions, we can diagnose whether *future* VIX implicit in VIX option prices posits the characteristics of the "term structure", "mean reversion", and "volatility of volatility".

5. Data

The sample period spans from February 24, 2006 through September 30, 2009. Daily midpoints of the last bid and last ask quotations for VIX options, daily mid-prices of CBOE VIX Term Structure, and daily settlement prices for VIX futures are obtained from the CBOE.

CBOE calculates VIX Term Structure data using a "business day" convention to measure time to expiration, as well as the "calendar day" convention used in the VIX index itself that is effective on VIX futures and VIX options. The VIX volatilities that comprise the VIX Term Structure data are calculated by applying the VIX formula to a single strip of SPX options having the same expiration date. But unlike the VIX index, VIX Term Structure data does not reflect constant-maturity volatility. Since SPX options mature on Fridays while options and futures on VIX settle on Wednesdays, that is 30 days before the third Friday of the calendar month immediately following the month in which the contract expires, this study utilizes the two VIX volatilities straddling a VIX option's expiration date to obtain the forward VIX, denoted "$\mathrm{VIX}(t,T_j;\tau_0)$". This is done for each contract and each day in the sample.

Several exclusion filters are applied to construct the VIX option price data. First, as options with less than six days to expiration may induce liquidity-related biases, they are excluded from the sample. Second, to mitigate the impact of price discreteness on option valuation, prices lower than \$3/8 are excluded. Finally, the VIX call option prices greater than the VIX futures prices with comparable maturities are excluded from the sample. Based on these criteria, 25,206 observations (approximately 23.51 percent of the original sample) are eliminated. A total of 82,017 records of joint futures and options prices on VIX and forward VIX are used for parameter estimation. Of these, 41,409 are calls and 40,608 are puts. Table 3 presents characteristics of the data sample across maturity and moneyness, where moneyness is defined as the VIX futures price divided by the option strike price, i.e. $\mathrm{F}_t^{\mathrm{VIX}}(T)/K$. Average VIX option prices range from \$0.5250 for deep out-of-the-money (DOTM) medium-term (MR) puts to \$16.0147 for deep in-the-money (DITM) MR puts. The average VIX futures prices range from \$17.2488 corresponding to the long-term (LR) at-the-money (ATM) puts to \$24.7907 for the short-term (SR) DOTM puts. The average forward VIX prices have the minimum value of $17.4977 within LR ATM puts and the maximum value of $24.0606 for SR DOTM puts. Except for SR DOTM puts, forward VIX is on average greater than VIX futures prices, which is consistent with the Jensen's inequality.

[Table 3 about here]

The average trading volume and open interest in the sixth and seventh rows indicate the trading activity and the total number of options contracts that are not closed or delivered on a particular day. Daily trading volume and open interest are useful for making sure that the options one trades are liquid, allowing one easily to enter and exit a trade, as well as ensure one gets the best possible price. When options have large open interest, it means they have a large number of buyers and sellers, and an active secondary market will increase the odds of getting option orders filled at good prices. So, all other things being equal, the bigger the open interest, the easier it will be to trade that option at a reasonable spread between the bid and ask. This is confirmed by the negative correlation between open interest and bid-ask spreads in the fifth row. In addition, a ratio of the trading volume of put options to call options can be used to gauge investor sentiment. For VIX options, a low volume of puts compared to calls indicates a bearish sentiment in the market.

The monyness-maturity characteristics of VIX options are very distinct from the distribution of equity options. Ideally, a purchase of equity volatility derivatives can reduce the downside risk and unattractive higher moment exposures of an investment in hedge funds or the S&P 500. In other words, if there is an interdependence between the S&P 500 and the VIX derivatives, then VIX derivatives may have a potential to improve the risk-reward relationship of investing in a stock market index. Our data show that in terms of trading volume and open interest investors trade calls (especially for ATM and OTM calls) more than puts, and SR options more than LR options, while in terms of numbers of observations investors trade DITM calls and puts most.

The "Black's implied volatility" (Black IV) in the fourth row shows that volatility smile and volatility term structure can be observed in the VIX options.[6] Like the Black model, our VIX model ensures that claim prices are determined through 'volatility' parameters, not through drifts or risk premia. However, in the Black model a single scalar carries all volatility information, whereas in our model the volatility function must describe the stochastic evolution of the entire term structure curve. Figure 6 demonstrates that Black's volatilities recovered from VIX options

[6] Given the assumption of one-factor constant volatility in Proposition 2, "Black's implied volatilities" can be calculated from market prices of VIX options.

vary from different categories of moneyness and maturity so that each model is fitted to volatility skew and volatility term structure.

[Figure 6 about here]

6. Implicit Parameter Estimation

To price VIX options across strikes and maturities, the parameters of the volatility functions must be estimated to construct the evolution of the *future* VIX. The vector of structural parameters Φ is backed out by minimizing the sum of the squared pricing errors between option model and market prices. The minimization is given by

$$\min_{\Phi} \sum_{t=1}^{N_T} \sum_{n=1}^{N_t} [C_n - C_n^*(\Phi)]^2 \tag{9}$$

where N_T is the number of trading days in the estimation sample, N_t is the number of options on day *t*, and C_n and C_n^* are the observed and model option prices, respectively. The model is estimated separately each month and thus Φ is assumed to be constant over a month. The assumption that the structural parameters are constant over a month is justified by an appeal to parameter stability (Bates, 1996; Eraker, 2004; Zhang and Zhu, 2006). Table 4 reports the monthly average of each estimated parameter series and the monthly-averaged in-sample mean squared errors for the one-factor, two-factor and three-factor exponential and hump volatility models, denoted “Exp-1f”, “Exp-2f”, “Exp-3f”, “Hump-1f”, “Hump-2f” and “Hump-3f”, respectively.

[Table 4 about here]

Several observations are in order. First, the estimated structural parameters for the volatility function generally differ across models. First, the estimates of the *long-run variance mean* (φ_1) for the Hump-1f, Hump-2f and Hump-3f models are 0.0224, 0.0530 and 0.0204, which are statistically significant different from zero, suggesting that the hump components explain a significant portion of the *future* VIX variance. Second, the estimates of the *variance mean reversion* (η_k) for the Exp-1f, Exp-2f and Exp-3f models are 6.32, {5.44, 4.60}, and {5.24, 4.65, 3.75}, which are relatively lower than those under the Hump-1f, Hump-2f and Hump-3f models. This suggests that the hump components explain a significant mean-reversion portion of the *future* VIX variance.

Third, the volatility curvature parameters (θ) under the one- and two-factor Hump models of 4.3043 and 4.0022 are greater than its three-factor Hump model counterpart of 1.1164, which is expected since the extra parameters (related to the volatility shape process) make the three-factor Hump model fit the data better than the one- and two-factor Hump models.

Fourth, the Exponential and Hump models attribute part of decaying shape to the slope (ν_k) of the term structure of volatilities. The slope coefficient is higher under the one- and two-factor Exponential models than Hump models, partly attributable to the existence of volatility humps. Note that a larger positive ν_k gives a reduced possibility for the asset z to jump up. A possible explanation for this observation is that the hump feature enhances the probability of a jump occurring once a large z price jump takes place. Hence, the Hump models can help to explain volatility mean reversion in financial markets.

Fifth, the hump feature enhances a negative correlation between asset prices z and their volatility in the Hump models. More specifically, negative correlation estimates of $corr(\Delta\sigma(t,T), \Delta z(t,T)/z(t,T))$ are –0.8248, –0.7623 and –0.6723 for the Hump-1f, Hump-2f and Hump-3f models, whereas –0.5112, –0.4211 and –0.3101 for the Exp-1f, Exp-2f and Exp-3f models, respectively. The negative conditional correlation between z returns and increments in volatility indicates the asymmetry of volatility across the z price level.

Together, these estimates show that, to the extent the pricing structure of option prices can be explained by each model, the Hump models' demands on the volatility process are the most stringent as they require both the highest variation (η_k) and the greatest covariance (in magnitude) with underlying returns z. The estimated implied variances $\sigma^2(t,T)$ are, however, generally greater among Exponential models. This is consistent with the observation that total variance attributed to the mean-reversion evolution is significantly reduced by incorporating a hump feature.

Finally, the fact that allowing multifactor volatilities to occur enhances the single-factor model's fit is illustrated by each model's average of the squared pricing errors between the market price and the model price for each option in an average month (*MSE*). The *MSE* is 6.40 for the Exp-1f model, 5.57 for the Exp-2f model, and 4.92 for the Exp-3f model. This pattern is repeated for the hump-shaped volatility models. Specifically, the *MSE* is 4.50 for the Hump-1f model, 4.36 for the Hump-2f model, and 4.26 for the Hump-3f model. The in-sample mean squared errors are consistently smaller for more complicated models within the same group. Therefore, the fitting error can be reduced by adding a factor.

Moreover, the Hump models produce much lower fitting errors (*MSE*) than the Exponential models, indicating that models with a hump feature provide the best fitting performance. The Akaike Information Criterion (*AIC*) and Schwarz Criterion (*SC*) values confirm the *MSE* results.[7] Thus, to price VIX options, the volatility

[7] The *AIC* and *SC* are calculated to assess the potential degree of overfitting; they reward goodness of fit, but penalize the use of more parameters. The preferred model is the one with the lowest *AIC* or *SC* value. Of course, overfitting is best detected by performing out-of-sample forecasts.

function should consider not only the mean-reversion effect but also the hump effect.

7. Out-of-Sample Pricing

This section provides a comparison of out-of-sample pricing. Out-of-sample pricing is carried out with the *previous month's* structural parameters and the *current day's* VIX futures and *forward* VIX prices to calculate the *current day's* VIX option model price. Following Dumas, Fleming, and Whaley (1998), we defined the pricing error (ε_i) outside the bid–ask spread as

$$\varepsilon_i = \begin{cases} Model\ Price - Bid\ Price, if\ Model\ Price < Bid\ Price \\ Model\ Price - Ask\ Price, if\ Model\ Price > Ask\ Price \\ 0\ ,\ otherwise \end{cases} \tag{10}$$

Three measures of goodness of fit are then employed to assess the out-of-sample pricing performance of the VIX option pricing models on the Exp-1f, Exp-2f, Exp-3f, Hump-1f, Hump-2f and Hump-3f specifications. These are the root mean squared pricing error (RMSE), the mean percentage pricing error (PE), and the mean absolute pricing error (MAE).

Table 5 reports RMSE, PE, and MAE values for several categories according to moneyness and time to expiration. Out of 36 moneyness-maturity combinations reported in Table 5, RMSE (MAE) is lower for the Hump models in 15 (20) cases than the Exponential models. The improvement is generally greater for call options under the Exp-2f model and put options under the Hump-3f model. From the panel of PE values, both Exponential and Hump models substantially underprice the call options and overprice the put options. The PE values show that the Exponential (Hump) pricing errors are on average smaller than the Hump (Exponential) pricing errors for call (put) options. For OTM options, the Exponential and Hump errors are an order of magnitude smaller. Pricing errors for both the Exponential and Hump models increase substantially as time to maturity increases. The MAE values in general confirm the RMSE results.

Out-of-sample performance for models is determined by the in-sample performance of models as well as the stability of volatility estimated by each model. Table 5 presents two distinct patterns from in-sample performance. First, with exceptions for call options under Hump models, multifactor models in general outperform one-factor models in the out-of-sample prediction error. Second, Hump (Exponential) models outperform Exponential (Hump) models for put (call) options. Nevertheless, out-of-sample prediction errors for all models are much smaller than the in-sample fitting errors, revealing that stability of volatility plays an important role in determining models out-of-sample performance.

[Table 5 about here]

Our objective in this study is to determine whether multifactor models are better able to price VIX options than one-factor models and whether a particular specification of volatility function is better capable to reduce the moneyness bias and the maturity bias, since each model is calibrated into options across strikes and maturities. Figure 7 presents out-of-sample prediction errors (mean absolute error) that are categorized based on the function of moneyness and maturity. The out-of-sample errors produced by the models with the same class of factors and by the models in the same volatility group are computed in Panels A and B, respectively.

The results in Figure 7 reveal that the class of three-factor models is more effective in reducing the moneyness bias (with one exception for *moneyness* > 1.30) and the maturity bias than the classes of one- and two-factor models. Three-factor models outperform two- and one-factor models, especially in call (put) options with DOTM (DITM) or with long expiry, indicating that when fitting to both volatility smile and volatility term structure, three-factor models are more accurate in pricing these options. In panel B, the group of Hump models produces lower error across option strikes (with one exception for *moneyness* > 1.30) and maturities than the group of Exponential models, indicating that these models are better in fitting to the skew of underlying asset distribution.

[Figure 7 about here]

Table 6 presents the out-of-sample pricing error correlations among one- and two-factor models. This table demonstrates that to improve pricing performance, it is inadequate to add a second stochastic factor to a one-factor model that has the same specification of volatility as the one-factor model. For example, the correlation between one- and two-factor exponential models is 0.96. These results are similar for Hump models, revealing that the choice of the second factor should be other than the original factor. To improve prediction performance, the two-factor model may be specified with one factor determining the level effect and the other factor describing the mean-reversion effect, rather than the two-factor exponential model, as specified in this study.

[Table 6 about here]

8. Regression Analysis

To identify the model performance in out-of-sample prediction, the market price is regressed against the model price, which is written as follows.

$$\text{Market Price} = \alpha + \beta \cdot \text{Model Price} + \varepsilon_i$$

where the market price is the bid (offer) price if the model price is less (greater) than the bid (offer) price, and the market price is set to the model price otherwise. A correct model should present α=0 and β=1. Table 7 reports the regression result for all models. The F-statistics of the joint test indicates that α=0 and β=1 are rejected for all models. A similar observation was found by Amin and Morton (1994), Zeto (2002), and Kuo and Paxson (2006). The β coefficients of the volatility functions in the three-factor Exponential and two-factor Hump models are higher than those in the other two models. This indicates that the three-factor Exponential and two-factor Hump models give a better estimation of the VIX option data. Particularly, these values for the Hump models are highest among all groups of models, showing that these models prices more correctly than other models.

[Table 7 about here]

To determine the cause of pricing error, the out-of-sample error (ε_i) is regressed against the option moneyness (*MON*) defined as the VIX futures price over the option's exercise price, the *z ratio* measured by the *forward* VIX *squared* normalized by the VIX futures price (*z*), time to maturity (*MAT*), the level of implied volatility ($\Sigma = \int_t^T \sigma^2(u, T)\, du$), and the change of implied volatility ($\Delta\Sigma$).

$$\varepsilon_i = \alpha + \beta_0 MON_i + \beta_1 z_i + \beta_2 MAT_i + \beta_3 \Sigma_i + \beta_4 \Delta\Sigma_i + u_i \tag{11}$$

Table 8 reports the results of the regression for all models, showing several patterns. First, the negative coefficients for most models show that the "smile" or "skew" effect is captured by *MON*. Second, the pricing error is affected by the relative level of *forward* VIX *squared* against the VIX futures price that is measured by z. If *future* VIX is mean-reverting, a low (greater) z is likely to be followed by a rise (decline) of z. This particularly affects the demand for out-of-money options and thus leads to greater pricing error. Third, the maturity or term structure effect is measured by the *MAT* variable. Since the group of Exponential models has not incorporated the curvature and long-run factor, larger coefficients of *MAT* in the three-factor Exponential model than the Hump group of models are observed.

Fourth, pricing errors are also affected by the level of implied volatility, measured by Σ. In a period of greater uncertainty, information asymmetry may be more severe than in periods of lower uncertainty and market makers may charge higher than normal prices for away-from-the-money options. Finally, the instability of implied volatility parameters measured by $\Delta\Sigma$ may have an impact on pricing errors for different models. Multifactor models with more structural parameters are likely to price better than single-factor models, but may suffer from the change of volatility.

[Table 8 about here]

9. Conclusion

This study proposes a *future* VIX approach that offers several advantages over the traditional term structure models of implied volatilities. First, *future* VIX models match the current **CBOE VIX term structure**. Second, like the Black model, these models require no assumptions about investor preferences. As a result, claim prices are completely determined by a description of the volatility structure of *future* VIX changes. In light of the advantages of the *future* VIX approach, this study examines two specific volatility models. The study also extends the models from one-factor to multi-factor economy. This is the first time to construct the VIX derivatives models and to test VIX options in this way.

Specifically, this study tests a general volatility function that incorporates **mean-reversion** and **hump** volatility functions in the *future* VIX framework using VIX options across strikes and maturities from February 24, 2006 to September 30, 2008. The primary empirical contribution of this study is to test not only one- and two-factor models but also the three-factor models.

Our results show that the class of three-factor models in general outperforms the classes of one- and two-factor models in in-sample fitting and out-of-sample prediction. A major benefit of using the three-factor models is to reduce most of the moneyness and maturity biases. The high correlations of out-of-sample errors between the one- and two-factor models with identical volatility functions imply that the second stochastic factor of a two-factor model should not be identical to the volatility function in the one-factor model. Although generally the class of three-factor models outperforms the classes of one- and two-factor models in out-of-sample pricing fit, the one-factor Hump model are as effective as the three-factor Exponential model, indicating that a correctly specified and calibrated one-factor Hump model can replace the three-factor Exponential model.

How many factors are needed to price VIX options? This has been a question for practitioners and academic researchers. Although various approaches have been applied to determine the number of factors to price their chosen data, our approach has market-driven interception. Unlike previous studies using historical data to determine the number of factors, this study uses liquid market option prices across a wide range of strikes and maturities to test the performance of single- and multi-factor models. The empirical results indicate that multi-factor models outperform one-factor models, particularly for away-from-the money options, but the pricing performance also depends upon the choice of models and the selection of second and third stochastic factors.

Appendix A The Proof of Proposition 1

Let $\sigma_S(t)$ be the instantaneous volatility of the SPX index return. By CBOE definition, $VIXTerm(t,T_j)$ and $VIXTerm(t,T_{j+1})$ are calculated as

$$\frac{VIXTerm(t,T_j)^2}{10{,}000} = \frac{1}{T_j - t}\mathrm{E}_t^Q\left(\int_t^{T_j} \sigma_S^2(u)du\right)$$
$$\frac{VIXTerm(t,T_{j+1})^2}{10{,}000} = \frac{1}{T_{j+1} - t}\mathrm{E}_t^Q\left(\int_t^{T_{j+1}} \sigma_S^2(u)du\right)$$

The *forward* VIX *squared*, $\mathrm{VIX}(t,T_j;\tau_j)^2$, is computed as

$$\begin{aligned}
&\frac{1}{\tau_j^{cal}}[VIXTerm(t,T_{j+1})^2 \cdot (T_{j+1}-t) - VIXTerm(t,T_j)^2 \cdot (T_j - t)] \\
&= \frac{10{,}000}{\tau_j^{cal}} \cdot \left[\mathrm{E}_t^Q\left(\int_t^{T_{j+1}} \sigma_S^2(u)du\right) - \mathrm{E}_t^Q\left(\int_t^{T_j} \sigma_S^2(u)du\right)\right] \\
&= \frac{10{,}000}{\tau_j^{cal}} \cdot \mathrm{E}_t^Q\left(\int_{T_j}^{T_{j+1}} \sigma_S^2(u)du\right) \\
&= \mathrm{VIX}(t,T_j;\tau_j)^2
\end{aligned}$$

, which gives Proposition 1.

Appendix B The Proof of Proposition 2

Under a forward measure Q^{F_j} that uses VIX futures as the numéraire, we have

$$\frac{C_t^{\mathrm{VIX}}(K,T_j)}{\mathrm{F}_t^{\mathrm{VIX}}(T_j)} = \mathrm{E}_t^{Q^{F_j}}\left[\frac{C_{T_j}^{\mathrm{VIX}}(K,T_j)}{\mathrm{F}_{T_j}^{\mathrm{VIX}}(T_j)}\right] = \mathrm{E}_t^{Q^{F_j}}(1_A) - K \cdot \mathrm{E}_t^{Q^{F_j}}[\mathrm{F}_{T_j}^{\mathrm{VIX}}(T_j)^{-1} \cdot 1_A]$$

where 1_A is an indicator either with value of 1 if $\mathrm{F}_{T_j}^{\mathrm{VIX}}(T_j) > K$ or with 0 otherwise.

From equation (6),

$$\mathrm{E}_t^{Q^{F_j}}(1_A) = \mathrm{Pr}_t^{Q^{F_j}}(A) = \mathrm{Pr}_t^{Q^{F_j}}(\ln \mathrm{F}_{T_j}^{\mathrm{VIX}}(T_j) > \ln K)$$

$$= \mathrm{Pr}_t^{Q^{F_j}}\left(\varepsilon < \frac{\ln\left\{\dfrac{[\mathrm{VIX}(t,T_j;\tau_0)]^2}{\mathrm{F}_t^{\mathrm{VIX}}(T_j)}\right\} - \ln K - \dfrac{1}{2}\sum_{k=1}^{d}\int_t^{T_j}\sigma_k^2(u,T_j)du}{\sqrt{\sum_{k=1}^{d}\int_t^{T_j}\sigma_k^2(u,T_j)du}} \right) = N(d_1)$$

where $d_1 = \dfrac{\ln\left\{\dfrac{[\mathrm{VIX}(t,T_j;\tau_0)]^2}{\mathrm{F}_t^{\mathrm{VIX}}(T_j)}\right\} - \ln K - \dfrac{1}{2}\sum_{k=1}^{d}\int_t^{T_j}\sigma_k^2(u,T_j)du}{\sqrt{\sum_{k=1}^{d}\int_t^{T_j}\sigma_k^2(u,T_j)du}}$. Further, the inverse of $\mathrm{F}_{T_j}^{\mathrm{VIX}}(T_j)$ is the same to the inverse of $\mathrm{VIX}(T_j,T_j;\tau_0)$. From equation (6), we have

$$\mathrm{E}_t^{Q^{F_j}}[\mathrm{F}_{T_j}^{\mathrm{VIX}}(T_j)^{-1}\cdot 1_A] = \frac{\mathrm{F}_t^{\mathrm{VIX}}(T_j)}{[\mathrm{VIX}(t,T_j;\tau_0)]^2}\mathrm{E}_t^{Q^{F_j}}\left[e^{+\frac{1}{2}\sum_{k=1}^{d}\int_t^{T_j}\sigma_k^2(u,T_j)\cdot du - \sum_{k=1}^{d}\int_t^{T_j}\sigma_k(u,T_j)\cdot d\omega_k^{F_j}(u)} \cdot 1_A \right]$$

$$= \frac{\mathrm{F}_t^{\mathrm{VIX}}(T_j)}{[\mathrm{VIX}(t,T_j;\tau_0)]^2} e^{\sum_{k=1}^{d}\int_t^{T_j}\sigma_k^2(u,T_j)\cdot du} \cdot \mathrm{E}_t^{Q^{F_j}}\left[e^{-\frac{1}{2}\sum_{k=1}^{d}\int_t^{T_j}\sigma_k^2(u,T_j)\cdot du + \sum_{k=1}^{d}\int_t^{T_j}[-\sigma_k(u,T_j)]\cdot d\omega_k^{F_j}(u)} \cdot 1_A \right]$$

Since the volatilities $\sigma_k(t,T_j)$ are assumed to satisfy Novikov's condition that is sufficient for application of Girsanov's theorem to define a new probability measure *R*:

$$\frac{dR}{dQ^F} = e^{-\frac{1}{2}\sum_{k=1}^{d}\int_t^{T_j}\sigma_k^2(u,T_j)\cdot du + \sum_{k=1}^{d}\int_t^{T_j}[-\sigma_k(u,T_j)]\cdot d\omega_k^{F_j}(u)}$$

where $d\omega_k^{F_j}(t) = d\omega_k^R(t) - \sigma_k(t,T_j)dt$ for $k = 1,\ldots,d$. That is,

$$\mathrm{E}_t^{Q^{F_j}}[\mathrm{F}_{T_j}^{\mathrm{VIX}}(T_j)^{-1}\cdot 1_A] = \frac{\mathrm{F}_t^{\mathrm{VIX}}(T_j)}{[\mathrm{VIX}(t,T_j;\tau_0)]^2} e^{\sum_{k=1}^{d}\int_t^{T_j}\sigma_k^2(u,T_j)\cdot du} \cdot \mathrm{E}_t^{Q^{F_j}}\left[\frac{dR}{dQ^{F_j}}\cdot 1_A\right]$$

$$= \frac{\mathrm{F}_t^{\mathrm{VIX}}(T_j)}{[\mathrm{VIX}(t,T_j;\tau_0)]^2} e^{\sum_{k=1}^{d}\int_t^{T_j}\sigma_k^2(u,T_j)\cdot du} \cdot \mathrm{E}_t^R(1_A)$$

Under the measure *R*, the *future* VIX process in equation (6) becomes

$$\mathrm{VIX}(T_j,T_j;\tau_0) = \frac{[\mathrm{VIX}(t,T_j;\tau_0)]^2}{\mathrm{F}_t^{\mathrm{VIX}}(T_j)} \cdot \exp\left[-\frac{3}{2}\sum_{k=1}^{d}\int_t^{T_j}\sigma_k^2(u,T_j)\cdot du + \sum_{k=1}^{d}\int_t^{T_j}\sigma_k(u,T_j)\cdot d\omega_k^R(u) \right]$$

Therefore, the value of $\mathrm{E}_t^R(1_A)$ is given by

$$\mathrm{E}_t^R(1_A) = \Pr_t^R(\ln \mathrm{F}_{T_j}^{\mathrm{VIX}}(T_j) > \ln K)$$

$$= \Pr_t^R\left(\varepsilon < \frac{-\ln K + \ln\left[\frac{[\mathrm{VIX}(t,T_j;\tau_0)]^2}{\mathrm{F}_t^{\mathrm{VIX}}(T_j)}\right] - \frac{3}{2}\sum_{k=1}^{d}\int_t^{T_j}\sigma_k^2(u,T_j)\cdot du}{\sqrt{\sum_{k=1}^{d}\int_t^{T_j}\sigma_k^2(u,T_j)\cdot du}} \right) = N(d_2)$$

where $d_2 = d_1 - \sqrt{\sum_{k=1}^{d}\int_t^{T_j}\sigma_k^2(u,T_j)\,du}$. Therefore, the time-$t$ fair value to the VIX call option is given by

$$C_t^{\mathrm{VIX}}(K,T_j) = \mathrm{F}_t^{\mathrm{VIX}}(T_j)\left[N(d_1) - K\cdot\frac{\mathrm{F}_t^{\mathrm{VIX}}(T_j)}{[\mathrm{VIX}(t,T_j;\tau_0)]^2}\cdot\exp\left(\sum_{k=1}^{d}\int_t^{T_j}\sigma_k^2(u,T_j)du\right)\cdot N(d_2)\right]$$

By put-call parity, we have the fair value to the VIX put option with strike K and expiry T_j.

$$P_t^{\mathrm{VIX}}(K,T_j) = \mathrm{F}_t^{\mathrm{VIX}}(T_j)\left\{ K\cdot\frac{\mathrm{F}_t^{\mathrm{VIX}}(T_j)}{[\mathrm{VIX}(t,T_j;\tau_0)]^2}\cdot\exp\left(\sum_{k=1}^{d}\int_t^{T_j}\sigma_k^2(u,T_j)du\right)\cdot N(-d_2) - N(-d_1)\right\}$$

Appendix C Volatility Function Estimation Using PCA

For d factors, one must observe at least d different $z(t,T)$, represented by the ($d \times 1$) vector at time t:

$$\begin{bmatrix} z(t,T_1) \\ \vdots \\ z(t,T_d) \end{bmatrix}$$

The basic ($d \times 1$) vector stochastic process $x(t)$ for logarithmic $z(t,T)$ that underlies the volatility estimation procedure is expressed as

$$\begin{bmatrix} lnz(t+\Delta, T_1) - lnz(t, T_1) \\ \vdots \\ lnz(t+\Delta, T_d) - lnz(t, T_d) \end{bmatrix} \approx \begin{bmatrix} \mu(t, T_1) \\ \vdots \\ \mu(t, T_d) \end{bmatrix} \Delta + \begin{bmatrix} \sum_{k=1}^{d} \sigma_k(t, T_1)\Delta\omega_1(t) \\ \vdots \\ \sum_{k=1}^{d} \sigma_k(t, T_d)\Delta\omega_d(t) \end{bmatrix}$$

with a d×1 mean vector, $[\mu(t, T_1), \cdots, \mu(t, T_d)]^T$, and a d×d covariance matrix Σ whose (i, j)-th element is

$$\sum_{k=1}^{d} \sigma_k(t, T_i)\sigma_k(t, T_j) \ .$$

Note that this evolution is under the actual probabilities $[\Delta\omega_1(t), \cdots, \Delta\omega_d(t)]^T$, a d×1 vector, that is approximated normally distributed with mean 0 and covariance matrix $I\Delta$ where I is the d×d identity matrix.

Using a time-series of N observations of logarithmic z ratios, we can obtain the d×d sample covariance matrix $\hat{\Sigma}$. Because this matrix is positive semidefinite, it can be decomposed as

$$\hat{\Sigma} = ALA^T$$

where the d×d matrix $A = (a_1, \cdots, a_d)$ gives the d eigenvectors a_k for $k = 1, \cdots, d$ of $\hat{\Sigma}$, and the d×d diagonal matrix $L = diag(\iota_1, \cdots, \iota_d)$ provides the d eigenvalues ι_k for $k = 1, \cdots, d$. This decomposition gives the estimates of the d volatility functions as

$$\begin{bmatrix} \sigma_k(t, T_1) \\ \vdots \\ \sigma_k(t, T_d) \end{bmatrix} = a_k \sqrt{\iota_k} \text{ for } k = 1, \dots, d \quad . \tag{C.1}$$

PCA can generate the principal components (eigenvectors) and eigenvalues of the covariance matrix $\hat{\Sigma}$ of the various maturity logarithmic z ratios. The volatility functions are then obtained from the expression (C.1).

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Table 1 Volatility Functions for VIX Models

Panel A: One-Factor Models		
1. $\sigma_1^2(t,T_j)=\nu_1 e^{-\eta_1(T_j-t)}$	1. $\int_t^{T_j}\sigma_1^2(u,T_j)\,du=\frac{\nu_1}{\eta_1}[1-e^{-\eta_1(T_j-t)}]$	Exp-1f
2. $\sigma_1^2(t,T_j)=\varphi_1+[\nu_1+\theta(T_j-t)]\,e^{-\eta_1(T_j-t)}$	2. $\int_t^{T_j}\sigma_1^2(u,T_j)\,du=\left(\frac{\nu_1}{\eta_1}+\frac{\theta}{\eta_1^2}\right)+\varphi_1(T_j-t)-\left[\left(\frac{\nu_1}{\eta_1}+\frac{\theta}{\eta_1^2}\right)+\frac{\theta}{\eta_1}(T_j-t)\right]e^{-\eta_1(T_j-t)}$	Hump-1f
Panel B: Two-Factor Models		
1. $\sigma_1^2(t,T_j)=\nu_1 e^{-\eta_1(T_j-t)}$ $\sigma_2^2(t,T_j)=\nu_2 e^{-\eta_2(T_j-t)}$	1. $\int_t^{T_j}\sigma_1^2(u,T_j)\,du=\frac{\nu_1}{\eta_1}[1-e^{-\eta_1(T_j-t)}]$ $\int_t^{T_j}\sigma_2^2(u,T_j)\,du=\frac{\nu_2}{\eta_2}[1-e^{-\eta_2(T_j-t)}]$	Exp-2f
2. $\sigma_1^2(t,T_j)=\varphi_1$ $\sigma_2^2(t,T_j)=[\nu_2+\theta(T_j-t)]e^{-\eta_2(T_j-t)}$	2. $\int_t^{T_j}\sigma_1^2(u,T_j)\,du=\varphi_1(T_j-t)$ $\int_t^{T_j}\sigma_2^2(u,T_j)\,du=\left(\frac{\nu_2}{\eta_2}+\frac{\theta}{\eta_2^2}\right)-\left[\left(\frac{\nu_2}{\eta_2}+\frac{\theta}{\eta_2^2}\right)+\frac{\theta}{\eta_2}(T_j-t)\right]e^{-\eta_2(T_j-t)}$	Hump-2f
Panel C: Three-Factor Models		
1. $\sigma_1^2(t,T_j)=\nu_1 e^{-\eta_1(T_j-t)}$ $\sigma_2^2(t,T_j)=\nu_2 e^{-\eta_2(T_j-t)}$ $\sigma_3^2(t,T_j)=\nu_3 e^{-\eta_3(T_j-t)}$	1. $\int_t^{T_j}\sigma_1^2(u,T_j)\,du=\frac{\nu_1}{\eta_1}[1-e^{-\eta_1(T_j-t)}]$ $\int_t^{T_j}\sigma_2^2(u,T_j)\,du=\frac{\nu_2}{\eta_2}[1-e^{-\eta_2(T_j-t)}]$ $\int_t^{T_j}\sigma_3^2(u,T_j)\,du=\frac{\nu_3}{\eta_3}[1-e^{-\eta_3(T_j-t)}]$	Exp-3f

2. $\sigma_1^2(t,T_j) = \varphi_1$ $\sigma_2^2(t,T_j) = \nu_2 e^{-\eta_2(T_j-t)}$ $\sigma_3^2(t,T_j) = \theta(T_j-t)\, e^{-\eta_3(T_j-t)}$	2. $\int_t^{T_j} \sigma_1^2(u,T_j)\, du = \varphi_1(T_j-t)$ $\int_t^{T_j} \sigma_2^2(u,T_j)\, du = \frac{\nu_2}{\eta_2}\left(1-e^{-\eta_2(T-t)}\right)$ $\int_t^{T_j} \sigma_3^2(u,T_j)\, du = \frac{\theta}{\eta_3^2} - \left[\frac{\theta}{\eta_3^2} + \frac{\theta}{\eta_3}(T_j-t)\right] e^{-\eta_3(T_j-t)}$	Hump-3f

Table 2 Generic Formats of Volatility Functions for Multifactor Models

$$\sigma^2(t,T_j) = \sum_{k=1}^{3} \{ \varphi_k + [\nu_k + \theta(T_j - t)]\, e^{-\eta_k (T_j - t)} \}$$

Model	φ_1	φ_2	φ_3	ν_1	ν_2	ν_3	η_1	η_2	η_3	θ
Exp-1f				√			√			
Hump-1f	√			√			√			√
Exp-2f				√	√		√	√		
Hump-2f	√				√			√		√
Exp-3f				√	√	√	√	√	√	
Hump-3f	√				√			√	√	√

Table 3 Sample Characteristics of VIX Options and VIX Futures

		Days to Expiration					
		Calls			*Puts*		
Moneyness		<60	60–180	>180	<60	60–180	>180
DOTM	Option price	0.5341	0.6028	0.7681	0.5569	0.5250	0.6186
	Futures price	19.2827	20.6080	17.8023	24.7907	23.0535	18.7887
	Forward VIX	19.4621	21.0588	18.0131	24.0606	23.5479	19.1502
	Black IV	0.2090	0.4524	0.6884	0.5514	0.3755	0.3593
	BA	0.1678	0.2115	0.2754	0.1413	0.2248	0.2719
	Open Interest	13,316	6,490	4,523	10,062	4,985	1,051
	Volume	1,888	263	174	1,001	210	39
	Observations	416	2,424	2,307	69	597	636
OTM	Option price	0.7243	1.0886	1.5453	0.6560	0.8141	1.0813
	Futures price	19.0954	20.0399	17.3586	22.3100	21.4832	17.8027
	Forward VIX	19.4997	20.5643	17.6463	22.5677	22.0537	18.0909
	Black IV	0.8522	1.0050	1.1627	0.4053	0.4507	0.4534
	BA	0.1191	0.2079	0.3002	0.1134	0.2041	0.2787
	Open Interest	30,255	8,085	2,793	15,335	5,071	2,721
	Volume	3,562	442	141	1,999	234	232
	Observations	1,782	3,552	1,990	506	2,054	1,247
AIM1	Option price	1.2597	1.9668	2.2981	1.1476	1.7793	2.0908
	Futures price	18.4449	19.8715	17.2509	18.9763	19.8397	17.4150
	Forward VIX	18.8035	20.4440	17.4988	19.3453	20.3606	17.6796
	Black IV	1.4046	1.1100	1.1307	0.6538	0.5375	0.5290
	BA	0.1305	0.2413	0.3518	0.1391	0.2460	0.3602
	Open Interest	35,740	10,277	2,751	14,632	5,918	3,035
	Volume	4,274	538	152	1,929	312	145
	Observations	1,944	3,033	1,805	1,424	2,371	1,535
ATM2	Option price	2.2632	3.0722	3.2432	2.7776	3.6639	3.7368
	Futures price	18.4335	19.8465	17.4182	18.2572	19.8638	17.2488
	Forward VIX	18.7846	20.3644	17.6848	18.6041	20.4362	17.4977
	Black IV	1.6406	1.3086	1.0115	0.9289	0.6386	0.6817
	BA	0.1824	0.3015	0.3876	0.2260	0.3372	0.4339
	Open Interest	21,029	7,667	2,812	8,015	1,843	378
	Volume	2,317	396	105	644	92	20
	Observations	1,650	2,373	1,530	2,034	3,028	1,810
ITM	Option price	3.7886	4.4885	4.2132	5.9688	6.9711	6.3101
	Futures price	19.1450	20.7651	17.8038	18.2120	20.0251	17.3540
	Forward VIX	19.4580	21.3484	18.0925	18.5539	20.5494	17.6385
	Black IV	1.7912	1.0934	0.7272	1.0721	0.7543	0.8200
	BA	0.2770	0.3825	0.4457	0.3278	0.4062	0.4823
	Open Interest	10,907	6,184	2,199	1,478	169	117
	Volume	682	140	178	48	13	3
	Observations	1,612	2,414	1,246	2,686	3,559	2,000
DITM	Option price	8.7552	8.6212	6.8750	14.7088	16.0147	13.4293
	Futures price	22.4329	21.9985	18.6789	18.1580	20.5640	17.7895
	Forward VIX	22.7561	22.5416	19.0581	18.4851	21.1034	18.0825
	Black IV	0.8216	0.5321	0.3766	1.1678	0.8618	1.0164
	BA	0.3915	0.4861	0.5514	0.3755	0.4697	0.5727
	Open Interest	3,852	2,106	1,332	160	64	24
	Volume	177	49	40	5	2	1
	Observations	3,681	5,204	2,424	5,230	6,764	3,041

Note. The reported figures are respectively the average VIX option price, the average VIX futures price, the average forward VIX, denoted "$\mathrm{VIX}(t,T_j;\tau_0)$" in the main text, the average Black implied volatility (Black IV), the average bid-ask spread of VIX options (BA), the average open interest of VIX options, the average trading volume of VIX options, and the total number of observations of VIX options for each moneyness–maturity category. The sample period extends from February 24, 2006 through September 30, 2008 for a total of 41,409 calls and 40,608 puts. *Moneyness* is defined as $\mathrm{F}_t^{\mathrm{VIX}}(T)/K$ where $\mathrm{F}_t^{\mathrm{VIX}}(T)$ is the time-t VIX futures price with expiry T and K is the option exercise price. DOTM (DITM), OTM (ITM), ATM1 (ATM2), ATM2 (ATM1), ITM (OTM) and DITM (DOTM) for calls (puts) are defined by *Moneyness* <0.70, 0.70–0.85, 0.85–1.00, 1.00–1.15, 1.15–1.30, and >1.30, respectively.

Table 4 Implied Parameters Estimation

Model Fit	Exp-1f	Hump-1f	Exp-2f	Hump-2f	Exp-3f	Hump-3f
MSE	6.3971	4.5030	5.5736	4.3622	4.9172	4.2636
AIC	4003	3225	3752	3148	3427	3100
SC	3820	3248	3775	3171	3461	3128
φ_1		0.0224*		0.0530**		0.0204*
		(1.95)		(4.45)		(1.87)
φ_2						
φ_3						
v_1	0.7538**	0.3881**	0.4086**		0.1764**	
	(8.67)	(20.43)	(10.50)		(7.88)	
v_2			0.2603**	0.3171**	0.1271**	0.5442**
			(7.90)	(10.48)	(7.72)	(17.63)
v_3					0.1888**	
					(7.33)	
η_1	6.3210**	10.2436**	5.4446**		5.2365**	
	(7.59)	(19.46)	(8.10)		(14.53)	
η_2			4.5967**	11.0358**	4.6548**	9.7083**
			(6.72)	(40.35)	(18.67)	(14.11)
η_3					3.7548**	8.0043**
					(14.67)	(18.63)
θ		4.3043**		4.0022**		1.1164**
		(57.26)		(90.46)		(11.25)
Implied Volatility (%)	37.83	27.64	36.24	25.49	29.37	27.77

Note. The values of *MSE* and estimated structural parameters reported here are their averages over 31 non-overlapping estimation months from February 24, 2006 to September 16, 2008, with a total of 40,569 calls and 39,754 puts. The reported *MSE* is the average of the squared pricing errors between the market price and the model price for each option in the sample. The Akaike information criterion (*AIC*) and Schwarz criterion (*SC*) not only reward goodness of fit, but also include a penalty that is an increasing function of the number of estimated parameters. This penalty discourages overfitting. The preferred model is the one with the lowest *AIC* or *SC* value. The *AIC* penalizes free parameters less strongly than does the *SC*. The formulas for the *AIC* and *SC* are

$$AIC = n\ln(MSE) + 2k \quad , \quad SC = n\ln(MSE) + k\ln(n)$$

where n is the number of observations, and k is the number of free parameters to be estimated. "Exp-1f" represents the one-factor Exponential model ($\sigma^2(t,T) = v_1 e^{-\eta_1(T-t)}$), "Exp-2f" is the two-factor Exponential model ($\sigma^2(t,T) = v_1 e^{-\eta_1(T-t)} + v_2 e^{-\eta_2(T-t)}$), and "Exp-3f" indicates the three-factor Exponential model ($\sigma^2(t,T) = v_1 e^{-\eta_1(T-t)} + v_2 e^{-\eta_2(T-t)} + v_3 e^{-\eta_3(T-t)}$). "Hump-1f" represents the one-factor Hump model ($\sigma^2(t,T) = \varphi_1 + [v_1 + \theta(T-t)]\, e^{-\eta_1(T-t)}$), "Hump-2f" is the two-factor Hump model ($\sigma^2(t,T) = \varphi_1 + [v_2 + \theta(T-t)]\, e^{-\eta_2(T-t)}$), and "Hump-3f" indicates the three-factor Hump model ($\sigma^2(t,T) = \varphi_1 + v_2 e^{-\eta_2(T-t)} + \theta(T-t)\, e^{-\eta_3(T-t)}$). The generic formats of volatility functions for multifactor models are expressed by

$$\sigma^2(t,T) = \sum_{k=1}^{3} \{ \varphi_k + [v_k + \theta(T-t)]\, e^{-\eta_k(T-t)} \}$$

where φ_k , v_k and θ control the level, the slope and the curvature of the term structure of volatilities, respectively. η_k is the coefficient of mean reversion. Implied volatility reports the total variance of z over the option's remaining life, i.e. $\int_t^T \sigma^2(u,T)du$. The figures within the parentheses are the t-statistics of parameter estimates. The symbols of ** and * indicate significance of t-statistics at the 1% and 5% levels, respectively.

Table 5 Out-of-Sample Root Mean Squared Errors (RMSE), Mean Absolute Pricing Errors (MAE) and Percentage Pricing Errors (PE)

			Days to Expiration					
			Calls			*Puts*		
Ofs error	*Model*	*Moneyness*	<60	60–180	>180	<60	60–180	>180
RMSE	Exp-1f	DOTM	0.4188	0.3849	0.7171	2.4352	1.2614	1.0973
		OTM	0.4832	0.5962	1.1343	1.3647	1.1193	1.8416
		ATM1	0.6453	0.8371	1.3863	1.1208	1.7241	2.5785
		ATM2	0.8123	0.9530	1.4831	1.6557	2.1138	3.215
		ITM	0.9643	1.0886	1.6269	2.1061	3.084	4.2632
		DITM	1.1602	1.1383	1.3774	2.9536	4.2124	6.2579
	Exp-2f	DOTM	0.4059	0.3697	0.7195	3.2968	1.1090	0.9110
		OTM	0.4554	0.5497	1.1133	1.6822	1.0554	1.8568
		ATM1	0.6034	0.7741	1.3651	1.2061	1.6232	2.3670
		ATM2	0.7720	0.8972	1.4781	1.7524	1.9695	3.1576
		ITM	0.9252	1.0330	1.6268	2.2012	2.8878	4.0450
		DITM	1.1462	1.1242	1.3727	3.0790	4.0024	5.8452
	Exp-3f	DOTM	0.4486	0.4198	0.7616	2.7917	0.8767	0.8117
		OTM	0.5253	0.6202	1.1762	1.4432	0.8741	2.2882
		ATM1	0.6716	0.8375	1.4090	1.0821	1.5014	2.6000
		ATM2	0.8249	0.9493	1.5112	1.6737	1.8575	3.5135
		ITM	0.9709	1.0718	1.6435	2.1243	2.7270	4.2991
		DITM	1.1549	1.1198	1.3741	2.9270	3.7771	5.4377
	Hump-1f	DOTM	0.4510	0.4243	0.7753	2.8594	0.8839	0.6143
		OTM	0.5224	0.6237	1.2103	1.4688	0.8895	2.3310
		ATM1	0.6681	0.8468	1.4500	1.0841	1.4805	2.6048
		ATM2	0.8201	0.9610	1.5578	1.6727	1.8231	3.5023
		ITM	0.9664	1.0725	1.6660	2.1154	2.6818	4.2617
		DITM	1.1524	1.1157	1.3628	2.9282	3.7154	5.1441
	Hump-2f	DOTM	0.4576	0.4382	0.7788	2.8594	0.7959	0.6742
		OTM	0.5465	0.6554	1.2233	1.4504	0.8176	2.3209
		ATM1	0.7049	0.8831	1.4675	1.0812	1.4498	2.6089
		ATM2	0.8529	0.9915	1.5708	1.6878	1.8085	3.5019
		ITM	0.9873	1.0962	1.6873	2.1188	2.6390	4.2606
		DITM	1.1491	1.1319	1.3667	2.9053	3.6468	5.0980
	Hump-3f	DOTM	0.4444	0.4356	0.7868	2.9069	0.8403	0.5365
		OTM	0.5154	0.6576	1.2427	1.4745	0.8493	2.7511
		ATM1	0.6830	0.9015	1.4961	1.0798	1.5012	2.9929
		ATM2	0.8563	1.0093	1.5908	1.6545	1.8491	3.9100
		ITM	0.9660	1.0857	1.6915	2.0760	2.6624	4.7152
		DITM	1.1329	1.1255	1.3677	2.8968	3.6419	5.0330
PE	Exp-1f	DOTM	-0.7083	-0.6013	-0.6003	1.2056	0.6647	0.7510
		OTM	-0.4598	-0.3903	-0.4649	0.2304	0.3772	0.8627
		ATM1	-0.1923	-0.2085	-0.3678	0.0862	0.1887	0.5725
		ATM2	-0.0089	-0.1042	-0.2583	-0.0139	0.0707	0.3995
		ITM	0.0117	-0.0309	-0.1825	-0.0050	0.0848	0.3663
		DITM	-0.0129	-0.0264	-0.0922	0.0162	0.0761	0.2896
	Exp-2f	DOTM	-0.7434	-0.5735	-0.5959	2.1415	0.7765	0.7380
		OTM	-0.4901	-0.3574	-0.4503	0.3735	0.4929	0.9042
		ATM1	-0.1964	-0.1844	-0.3584	0.0967	0.2764	0.5297
		ATM2	-0.0071	-0.0956	-0.2571	-0.0301	0.1024	0.3952
		ITM	0.0111	-0.0328	-0.1829	-0.0195	0.0921	0.3484

		DITM	-0.0148	-0.0329	-0.0950	0.0100	0.0824	0.2716
	Exp-3f	DOTM	-0.8588	-0.6846	-0.6758	1.3142	0.2554	0.3641
		OTM	-0.6100	-0.4449	-0.5086	-0.0325	0.1778	0.7904
		ATM1	-0.2788	-0.2295	-0.3856	-0.1281	0.0786	0.3503
		ATM2	-0.0294	-0.1097	-0.2672	-0.1345	-0.0153	0.3103
		ITM	0.0148	-0.0285	-0.1776	-0.0677	0.0158	0.2665
		DITM	-0.0031	-0.0193	-0.0786	-0.0080	0.0468	0.2072
	Hump-1f	DOTM	-0.8695	-0.6847	-0.6988	1.3997	0.2473	0.2068
		OTM	-0.6037	-0.4331	-0.5290	-0.0186	0.1692	0.6973
		ATM1	-0.2834	-0.2275	-0.4030	-0.1371	0.0772	0.2592
		ATM2	-0.0303	-0.1125	-0.2824	-0.1402	-0.0111	0.2643
		ITM	0.0152	-0.0303	-0.1836	-0.0677	0.0191	0.2307
		DITM	-0.0029	-0.0204	-0.0776	-0.0104	0.0443	0.1795
	Hump-2f	DOTM	-0.8882	-0.7196	-0.7057	1.3435	0.1227	0.2162
		OTM	-0.6493	-0.4799	-0.5405	-0.1505	0.0622	0.6416
		ATM1	-0.3211	-0.2525	-0.4120	-0.2212	-0.0018	0.2284
		ATM2	-0.0413	-0.1206	-0.2852	-0.1805	-0.0612	0.2386
		ITM	0.0171	-0.0283	-0.1837	-0.0852	-0.0130	0.2142
		DITM	0.0011	-0.0149	-0.0758	-0.0184	0.0299	0.1700
	Hump-3f	DOTM	-0.8449	-0.7111	-0.7238	1.4630	0.1831	0.0955
		OTM	-0.5762	-0.4637	-0.5524	-0.0425	0.0880	0.6828
		ATM1	-0.2672	-0.2498	-0.4214	-0.1131	0.0193	0.2140
		ATM2	-0.0287	-0.1235	-0.2906	-0.1175	-0.0413	0.2347
		ITM	0.0134	-0.0299	-0.1844	-0.0577	-0.0003	0.2145
		DITM	-0.0033	-0.0155	-0.0739	-0.0068	0.0331	0.1584
MAE	Exp-1f	DOTM	0.3429	0.3163	0.4857	1.1977	0.5698	0.6825
		OTM	0.3517	0.4561	0.8609	0.5940	0.5754	1.1772
		ATM1	0.4406	0.6180	1.0573	0.6564	0.8795	1.7157
		ATM2	0.5347	0.6804	1.1223	0.9884	1.2526	2.1858
		ITM	0.5995	0.7143	1.1738	1.2843	1.8133	2.9847
		DITM	0.6533	0.6811	0.9053	1.7831	2.7441	4.4667
	Exp-2f	DOTM	0.3502	0.3059	0.4848	1.5864	0.5200	0.5951
		OTM	0.3561	0.4268	0.8369	0.6291	0.5259	1.1041
		ATM1	0.4269	0.5771	1.0301	0.6163	0.8045	1.4842
		ATM2	0.5190	0.6449	1.1120	0.9340	1.1348	2.0301
		ITM	0.5825	0.6836	1.1732	1.2491	1.6943	2.7761
		DITM	0.6520	0.6803	0.9091	1.7264	2.6416	4.2646
	Exp-3f	DOTM	0.4018	0.3587	0.5320	1.3673	0.2868	0.4251
		OTM	0.4270	0.5021	0.9119	0.5293	0.3665	1.0089
		ATM1	0.4841	0.6389	1.0842	0.5934	0.6784	1.2348
		ATM2	0.5544	0.6879	1.1541	0.9763	1.0469	1.8814
		ITM	0.6146	0.7123	1.1892	1.2875	1.5691	2.4637
		DITM	0.6671	0.6560	0.8948	1.6949	2.4231	3.6338
	Hump-1f	DOTM	0.4054	0.3600	0.5476	1.3753	0.2692	0.3253
		OTM	0.4233	0.4957	0.9451	0.5340	0.3585	0.9260
		ATM1	0.4840	0.6407	1.1240	0.5904	0.6377	1.0952
		ATM2	0.5519	0.6951	1.1966	0.9736	0.9960	1.7707
		ITM	0.6139	0.7102	1.2060	1.2748	1.4985	2.3011
		DITM	0.6645	0.6553	0.8763	1.6895	2.3502	3.3345
	Hump-2f	DOTM	0.4132	0.3756	0.5502	1.3731	0.2480	0.3536
		OTM	0.4488	0.5327	0.9573	0.5448	0.3485	0.9052
		ATM1	0.5188	0.6760	1.1399	0.6151	0.6585	1.1040
		ATM2	0.5767	0.7199	1.2082	1.0137	1.0320	1.7849

	ITM	0.6291	0.7271	1.2198	1.3002	1.5162	2.3061
	DITM	0.6655	0.6616	0.8820	1.6928	2.3094	3.3252
Hump-3f	DOTM	0.3954	0.3727	0.5616	1.3681	0.2735	0.2857
	OTM	0.4103	0.5268	0.9784	0.5280	0.3481	0.9571
	ATM1	0.4843	0.6837	1.1678	0.5832	0.6671	1.1462
	ATM2	0.5619	0.7286	1.2281	0.9414	1.0221	1.8521
	ITM	0.6098	0.7198	1.2209	1.2350	1.4880	2.3783
	DITM	0.6495	0.6652	0.8737	1.6701	2.2804	3.2354

Note. The predictive performance of the six volatility estimates are examined using three alternative metrics based on the difference between the forecast and the actual VIX option price, ε_i. These are the root mean squared error (RMSE), the mean percentage error (PE), and the mean absolute error (MAE),

$$RMSE = \left[\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i^2\right]^2$$

$$PE = \frac{1}{n}\sum_{i=1}^{n}\frac{\varepsilon_i}{VIX\ option\ market\ price_i}$$

$$MAE = \frac{1}{n}\sum_{i=1}^{n}|\varepsilon_i| \ .$$

The sample period extends from March 22, 2006 through September 30, 2008 for a total of 41,229 calls and 40,397 puts. *Moneyness* is defined as $\mathrm{F}_t^{\mathrm{VIX}}(T)/K$ where $\mathrm{F}_t^{\mathrm{VIX}}(T)$ is the time-t VIX futures price with expiry T and K is the option exercise price. DOTM (DITM), OTM (ITM), ATM1 (ATM2), ATM2 (ATM1), ITM (OTM) and DITM (DOTM) for calls (puts) are defined by *Moneyness* <0.70, 0.70–0.85, 0.85–1.00, 1.00–1.15, 1.15–1.30, and >1.30, respectively.

Table 6 Pearson Correlation for Out-of-Sample Pricing Errors among Models

	Exp-1f	Hump-1f	Exp-2f	Hump-2f
Exp-1f	1			
Hump-1f	0.9096	1		
Exp-2f	0.9560	0.9592	1	
Hump-2f	0.8996	0.9960	0.9518	1

Note. The correlation of out-of-sample error is computed from the period of March 22, 2006 to Sept3mber 30, 2008.

Table 7 Regression Analysis

Model	α	β	R^2	p-value of F
One-Factor Model				
Exponential	1.0517**	0.8131**	0.8731	0.0000
Hump	1.0758**	0.8357**	0.8864	0.0000
Two-Factor Model				
Exponential	0.9971**	0.8185**	0.8840	0.0000
Hump	1.1018**	0.8397**	0.8864	0.0000
Three-Factor Model				
Exponential	1.0771**	0.8311**	0.8837	0.0000
Hump	1.1131**	0.8360**	0.8814	0.0000

Note. This table shows the result that market prices regress against model prices, given by

$$\text{Market Price} = \alpha + \beta \cdot \text{Model Price} + \varepsilon$$

The regression is carried out with the *previous month's* structural parameters and the *current day's* VIX futures and *forward* VIX prices to calculate the *current day's* VIX option model price. ** indicates that the *t* statistics (not reported here) are significance in the 95% confidence interval.

Table 8 Regression for Out-of-Sample Errors and Variables

Model	α	β_0	β_1	β_2	β_3	β_4	R^2(%)	p-value of F
One-Factor Model								
Exponential	0.0720	-0.7435**	-0.0419**	0.0042**	1.6948**	-0.8378**	6.45	0.0000
Hump	-2.5644**	-0.4144**	-0.0327**	0.0042**	3.3081**	-1.0053**	3.95	0.0000
Two-Factor Model								
Exponential	0.4851**	-0.8703**	-0.0330**	0.0038**	1.3882**	-0.6987**	4.87	0.0000
Hump	-2.3444**	-0.2751**	-0.0343**	0.0038**	3.1047**	-1.1649**	3.98	0.0000
Three-Factor Model								
Exponential	-1.8083**	-0.4325**	-0.0356**	0.0042**	2.7938**	-1.3022**	4.34	0.0000
Hump	-1.6943**	-0.2788**	-0.0353**	0.0036**	2.5763**	-0.8967**	3.93	0.0000

Note. The out-of-sample error (ε) is regressed against option moneyness (*MON*), *z ratio* (*z*), time to maturity (*MAT*), implied volatility (Σ) and the change of implied volatility ($\Delta\Sigma$).

$$\varepsilon_i = \alpha + \beta_0 \cdot MON_i + \beta_1 \cdot z_i + \beta_2 \cdot MAT_i + \beta_3 \cdot \Sigma_i + \beta_4 \cdot \Delta\Sigma_i + u_i$$

Moneyness is defined as $\mathrm{F}_t^{\mathrm{VIX}}(T)/K$ where $\mathrm{F}_t^{\mathrm{VIX}}(T)$ is the time-*t* VIX futures price with expiry *T* and *K* is the option exercise price. *z ratio* is measured by *forward* VIX *squared* normalized by the VIX futures price. This study uses the *previous month's* structural parameters and the *current day's* VIX option, VIX futures and *forward* VIX prices to calculate the *current day's* implied volatility (Σ). The change of implied volatility ($\Delta\Sigma$) is the difference in implied volatilities between date $t-1$ and date t. ** denotes significance of the coefficient at 95% confidence interval (*t* statistics are not reported here).

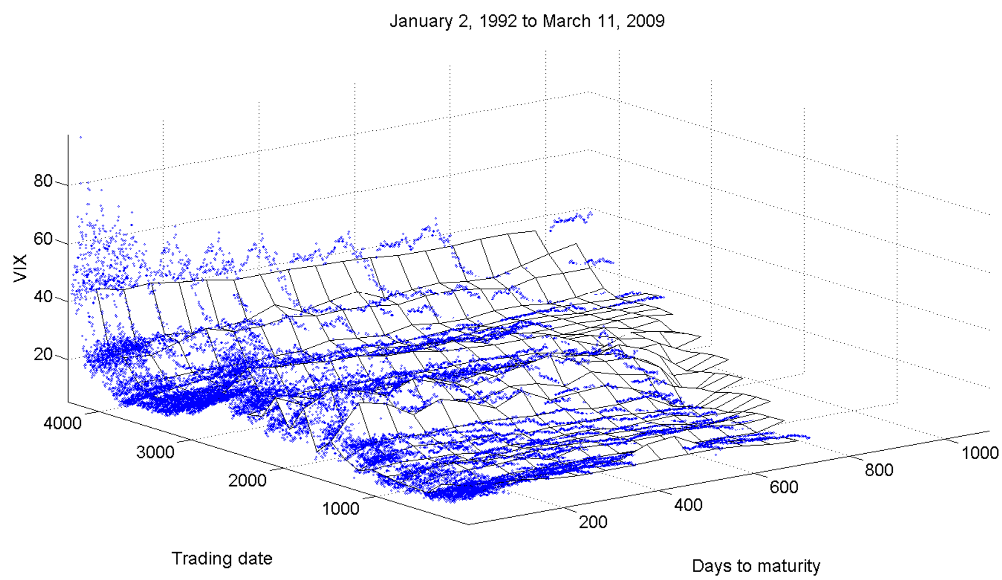


Figure 1 Spot VIX Curve Evolution. This chart represents the **CBOE VIX Term Structure** Midpoints as of the market close dates and days to option maturities over the period from January 2, 1992 to March 11, 2009. The round markers indicate actual observations.

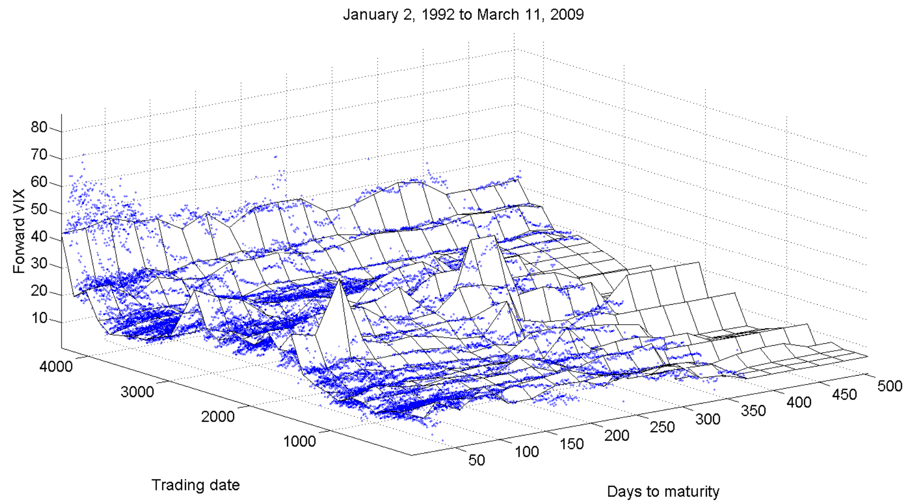


Figure 2 Forward VIX Curve Evolutions. This chart represents the initial *forward* VIX as of the market close dates and days to option maturities over the period from January 2, 1992 to March 11, 2009. The round markers indicate actual *forward* VIX observations.

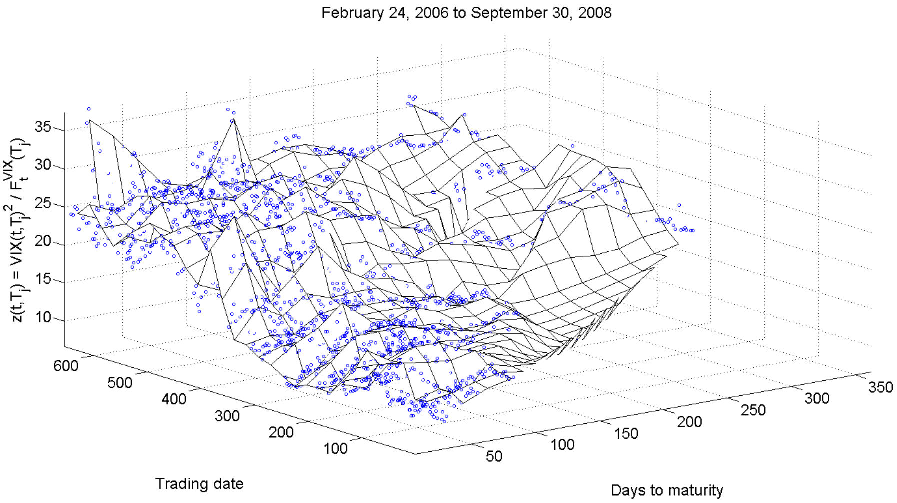


Figure 3 Evolution of *Forward* VIX Squared Normalized by VIX Futures Prices. This chart represents the evolution of the ratio of *forward* VIX squared over VIX futures prices as of the market close dates and days to option maturities over the sample period from February 24, 2006 to September 30, 2008. The round markers indicate actual ratio observations.

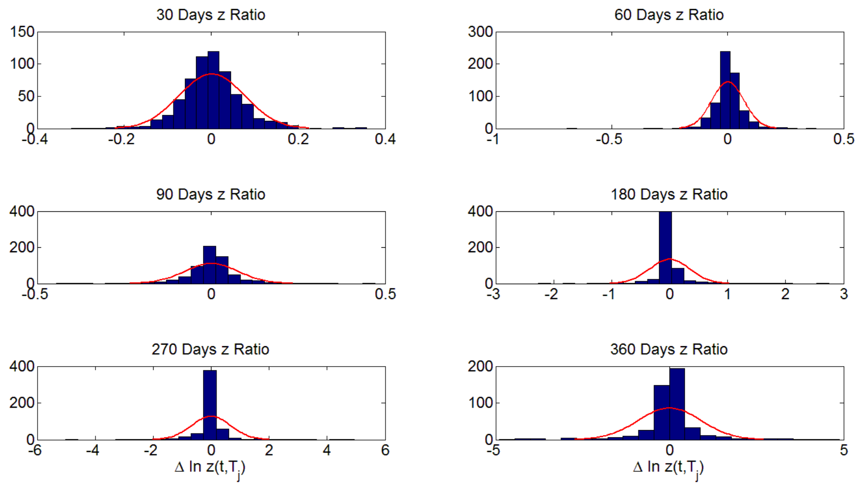


Figure 4 Histogram of Daily Changes in Logarithmic *z* Ratio. This chart represents histograms for changes in a particular maturity logarithmic *z* ratio, defined as *forward* VIX *squared* normalized by VIX futures prices, over the sample period from February 24, 2006 to September 30, 2008.

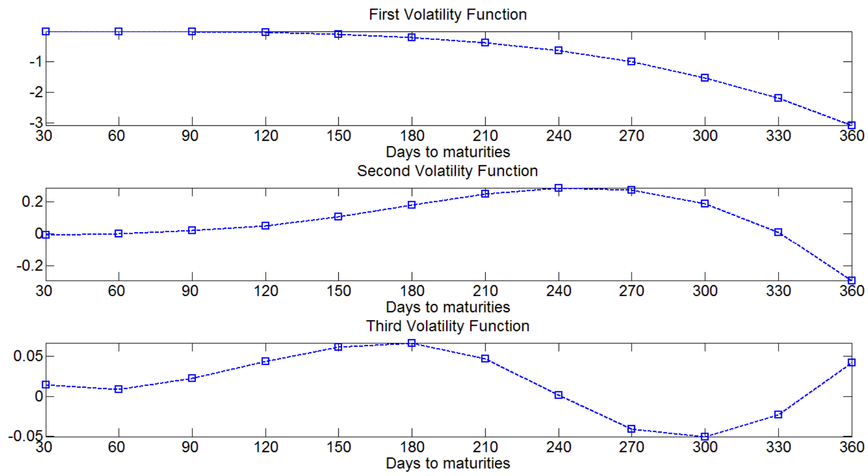


Figure 5 Graphs of the First Three Volatility Functions. The volatility functions are obtained from the principal components of the covariance matrix of the various maturity logarithmic z ratios over the sample period from February 24, 2006 to September 30, 2008.

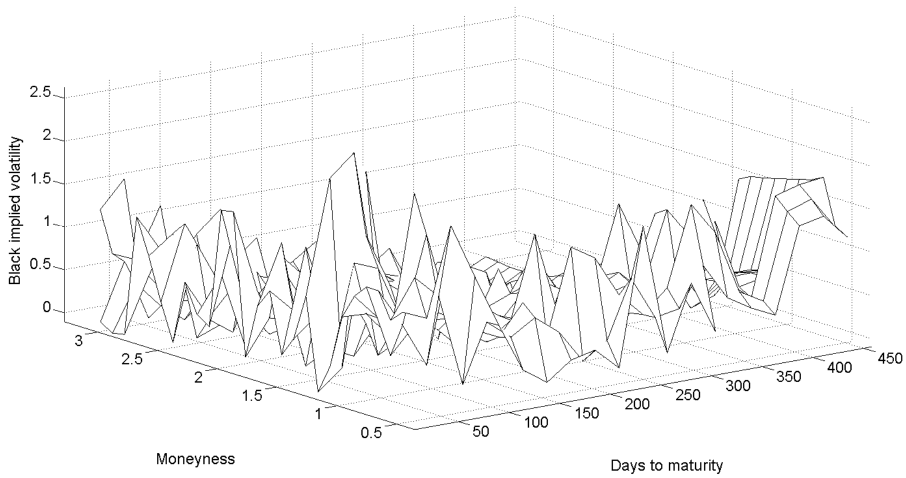
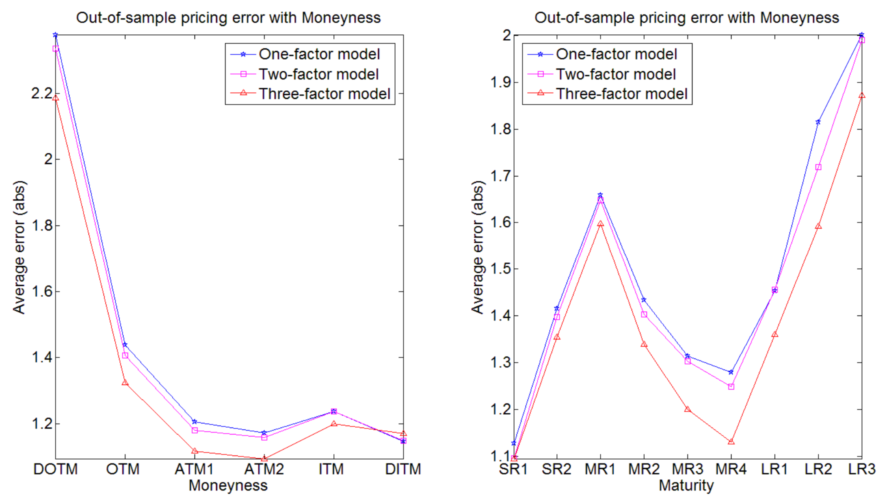


Figure 6 Black's Implied Volatilities. Given the assumption of one-factor constant volatility in Proposition 2, "Black's volatilities" are recovered from market prices of VIX options. Moneyness is defined as the market price of VIX futures divided by the option exercise price. The sample covers the period from February 24, 2006 to September 30, 2008.

Panel A Out-of-sample errors for the same class of models with option moneyness and maturity



Panel B Out-of-sample errors for the volatility group of models with option moneyness and maturity

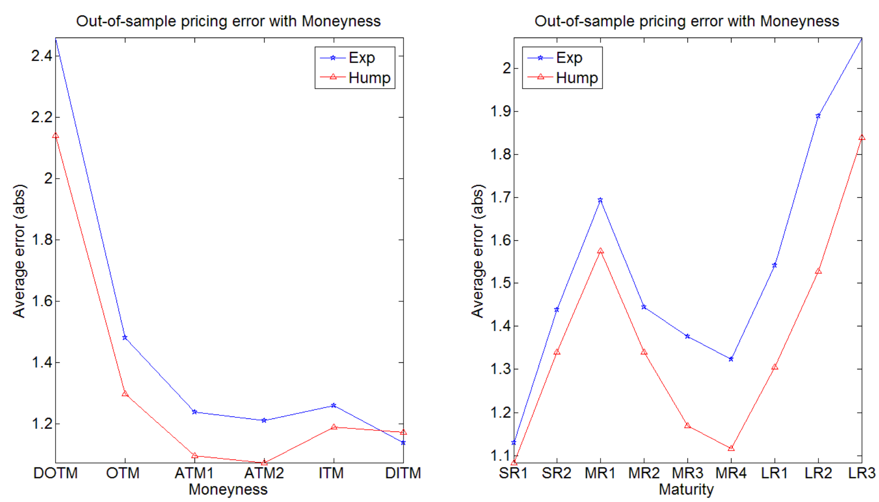


Figure 7 Out-of-Sample Pricing Performance for Factors and Models. In Panel A, the errors for all models with the same factor are added and computed on average across the sample period and then are categorized with moneyness and maturity with eight groups where each group has equal number of observations. In Panel B, the error for the same group of models (one-, two-, and three-factor model) is added and then computed on average based on the procedure in Panel A. Moneyness is defined as the market price of VIX futures divided by the option exercise price. DOTM, OTM,

ATM1, ATM2, ITM and DITM for calls and puts are defined by *Moneyness* <0.70, 0.70–0.85, 0.85–1.00, 1.00–1.15, 1.15–1.30, and >1.30, respectively. SR1, SR2, MR1, MR2, MR3, MR4, LR1, LR2 and LR3 for calls and puts are defined by *days to option's expiry* <30 days, 30–60, 60–90, 90–120, 120–150, 150–180, 180–210, 210–240, and >240, respectively. The out-of-sample data cover the period from March 22, 2006 to September 30, 2008.