

IEOR E4718

Topics in Derivatives Pricing: An Introduction to the Volatility Smile

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Aim of the Course

This isn't a course about mathematics, calculus, differential equations or stochastic calculus, though it does use all of them. Much of the time the approach is going to be mathematical, but not extremely rigorous. I want to develop intuition about models, not just methods of solution. No assumptions behind financial models are genuinely true, and no financial models are really correct, so it's very important to understand what you're doing and why.

This is a course about several themes:

1. Understanding the practical use of the Black-Scholes-Merton model. There's more to it than just the equation and it's solution.
2. The theoretical and practical limitations of the model.
3. The extensions of the model to accommodate/explain the volatility smile.
4. Understanding the consequences of these extensions. It's easy to make up new models but we want to understand whether they are realistic and what they lead to.

Preamble to the Course

I assume you've all learned about the Black-Scholes-Merton model itself. According to Prof. Steve Ross of MIT, one of the inventors of the binomial options pricing model, risk-neutral valuation and arbitrage pricing theory, "... options pricing is the most successful theory not only in finance, but in all of economics." And it is indeed.

Academics in finance tend to think of options valuations as a solved problem, of little academic interest anymore. But for those of you who end up working as practitioners – on options trading desks in equities, fixed income, currencies or commodities, as risk monitors or risk managers or controllers or model auditors – you'll find that it isn't really a solved problem at all. Financial markets disrespect the Black-Scholes-Merton results even while they use its language. Most academics who haven't lived or died by models don't see this clearly, but practitioners and traders who are responsible for coming up with the prices for which they are willing to trade securities, especially exotic illiquid securities, grapple with this every day. They have to figure out how to amend the models results to cope with the real world.

In this class we're going to talk a lot about the Black-Scholes model and its discontents. In one sense the Black-Scholes model is a total miracle: it lets you value, in a totally rational way, securities that before its existence had no clear value. In the Platonic world of Black-Scholes-Merton – a world with normal returns, geometric Brownian motion, infinite liquidity, continuous hedging and no transactions costs – it provides a method of synthesizing an option and it works perfectly.

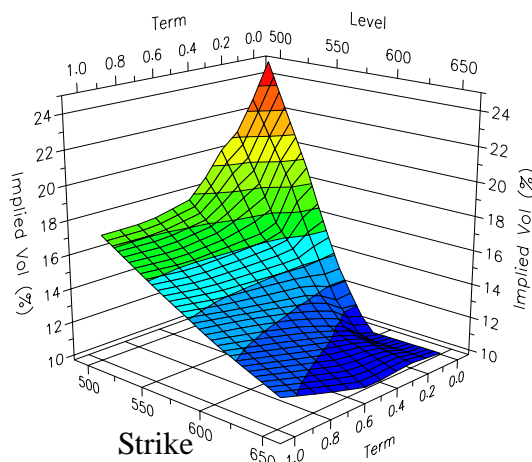
It's a masterpiece of engineering in a world that doesn't quite exist, because markets don't obey all of its assumptions. Some are violated approximately, and some more dramatically. The assumptions that you can hedge continuously, at zero transaction cost, are approximations we can adjust for, and we'll illustrate that. Skilled traders and quants do this with a mix of skill and intuition every day. For example, you can practically adjust for transactions costs by adding some dollars to your price, or some volatility points to the Black-Scholes formula. In that sense the model is pretty robust -- you can perturb it from its Platonic view of the world to take account, approximately, of the in-reality less modelable aspects of that world.

And the Black-Scholes model is so rational, and has such a strong grip on everyone's imagination that even people who don't totally believe in it use it to quote prices they are willing to trade at. When you deal with models, quoting is always a problem. For example, the US dollar is the standard currency for quoting gold prices, at least for the present, and so when you quote gold prices in dollars rather than in Euros or Swiss francs you are seeing something idiosyncratic, especially if you are not a dollar-based investor. The gold price might look approximately constant in Euros, but variable in US dollars if the dollar is falling. Similarly bond traders quote their bond prices in terms of yields to maturity, the constant instantaneous forward rate at which you must discount a bond's payoff to get its price. Once upon a time a constant yield to maturity was the best way to model bond prices; nowadays, people have more sophisticated bond pricing models based off forward rates and even embedded optionality. Yet, even though you may not believe that rates will be constant in the future, yield to maturity is still a convenient metric for quoting, though not necessarily a good model. In the same way, markets use the Black-Scholes formula to quote options prices, even though the model has its flaws.

But there are fundamental problems with the model. For example, stocks don't really follow geometric Brownian motion. They can jump, their distributions have fat tails, and some people even believe that their variance is infinite rather than finite, in which case all bets are off. This is a big issue, one we *won't* tackle much here. What we will focus on especially is the problem of the volatility smile.

Prior to the stock market crash of October 1987, the Black-Scholes implied volatility of equity options varied little with strike, though it did vary with expiration. Since that crash, the behavior of implied volatility in equity index markets has changed: market participants now think of implied volatility as a two-dimensional surface whose level at any time is a function of strike and expiration. This surface, with combined term and strike structure, is called the volatility smile, or sometimes the volatility skew. Here's a vintage smile surface from ten years ago; I'll show more recent ones in class.

FIGURE 1.1. The implied volatility surface for S&P 500 index options as a function of strike level and term to expiration on September 27, 1995.



This so-called volatility smile, initially a feature of equity index options markets only, has now become a feature of not just equity index options markets (CAC, DAX, Nikkei, S&P, etc.) but also single-stock options markets, interest-rate options markets, currency options markets, credit derivatives markets, and almost any other volatility market. New markets typically begin with traders using the canonical Black-Scholes model with no smile; then, as they gain experience with the idiosyncrasies of their particular markets and its movements, the implied volatility smile structure starts to develop.

The industry-standard Black-Scholes model alone cannot account for this structure, and so, though options prices are still quoted by means of their Black-Scholes implied volatilities, trading desks at hedge funds and investment banks now use more complex smile-consistent models to value and hedge their options.

What's the right replacement for Newton's laws? Well, special relativity and quantum mechanics, I suppose, maybe string theory eventually, or maybe not. But Newton's laws are still useful, and much better for most practical uses; you just have to know the limitations. What's the right replacement model for Black-Scholes? Think a little about how you would determine this. Or, it might be even better to ask, is there a right replacement model for Black-Scholes? That's in part what this course is about, and there isn't an easy answer.

This course will describe the smile structure of implied volatilities and the way that structure contradicts the classic Black-Scholes model. We'll then consider some of the sorts of models that can account for the smile. Black-Scholes tells you that you can value an option because you can hedge away its risk. If Black-Scholes isn't right, then you don't know how to hedge the risk of options. Smile models are critical to the correct hedging of ordinary options and even more crucial for valuing the exotic options and structured products that are very popular because they are custom-tailored for clients and generate higher margins. Because they are custom-tailored, these products are relatively illiquid, and, like custom-tailored clothes, can't easily be resold. What are they

therefore worth? If you can't get a mark from another dealer, then only a model can tell you, and therefore their values are marked by model, and subject to *model risk*, or more accurately, to *model uncertainty*.

Quantitative strategists and financial engineers on derivatives trading desks and within the firm-wide risk departments at investment banks have to worry about which is the best model to use. There are many important issues of model choice, model validity and model testing that are of practical concern. Quants and controllers must worry about the marks of positions, to what extent they are model-dependent, what the effect a change in model has on the P&L of the firm. These are weighty issues that involve many people in the front and back office, and in I.T. groups. For example, it is common to value an exotic option with a very fancy slow model when you first think about the deal, and then mark it again daily with a less accurate and less sophisticated off-the-shelf model, because the slow model may use Monte Carlo simulation and take too much time to run. The question of model uncertainty also generates interesting and relevant questions about how to determine your profit and how to pay your traders for profits that depend on models and are therefore uncertain.

I'll try to approach the models with a mixture of theory and pragmatism. I don't like simply writing down formulas without proofs, though I'll do that occasionally. Knowing the formulas like a table of integrals isn't enough when you're working in the field, because many of the derivative products you have to deal with, and their markets, violate the assumptions behind the simple formulas. So while you should know the standard models, and know how to derive and play around with them, you should aim to learn how to build your own models and understand what they lead to. You need to develop intuition about the models, so that you can know when your calculations are giving you the right answer or when you've made a mathematical or computational or programming mistake.

So, I'll put a lot of effort into deriving simple or approximate proofs of the key model formulas and ideas. Often these proofs and formulas may not be the best way to implement a model for rapid and accurate computational use, but they can be good for understanding ideas. My aim is to develop these models logically, to get a feel for the phenomena to be explained, and to estimate the effects of the models. I will also bring in two or three practitioners or traders from derivatives desks on Wall Street or at hedge funds to give talks about their parts of the business as it relates to options pricing, options trading and the volatility smile.

References

I don't require you to buy any textbook for the course, and in fact until recently there were almost no books devoted to the smile. But I can recommend the following additional material.

- *The Volatility Surface: A Practitioner's Guide* by Jim Gatheral, Wiley 2006.
This book probably contains material most relevant to what I'm doing, though it approaches it somewhat differently.
- *Semiparametric Modeling of Implied Volatility* by Matthias R. Fengler, Springer, 2005

Some other useful books:

- *Volatility and Correlation: The Perfect Hedger and the Fox* by Riccardo Rebonato, Wiley, 2004. This is a very comprehensive book, over 800 pages long, full of information, but on the verbose side.
- *Derivatives in Financial Markets with Stochastic Volatility* by JP Fouque, G. Papanicolaou and R. Sircar, Cambridge U. Press. This book is devoted to a particular perturbative treatment of rapidly mean-reverting stochastic volatility models. It has a very good introductory chapter, then gets pretty technical.
- *Option Valuation under Stochastic Volatility* by Alan Lewis, Finance Press.
Good book, not easy to read -- uses Fourier transforms extensively.
- There is also a qualitative chapter (Chapter 14) on the smile in my book, *My Life as a Quant: Reflections on Physics and Finance*. I will post an electronic version of that chapter on Courseworks.

Some more useful books of a more general nature are listed below. All of them have sections on the volatility smile. The list isn't comprehensive; there are so many others I don't know about or haven't mentioned:

- *Paul Wilmott on Quantitative Finance*, Wiley, by Paul Wilmott (who else?) is a very good general book on options theory. He's not afraid to tell you what he thinks is important and what isn't, which is valuable. (Actually, he's not afraid to tell you what he thinks in general.) This is a good book in which to look up topics -- you don't have to read it cover to cover, but can dip in. And it's always sensible.
- *Financial Derivatives: Pricing, Applications, and Mathematics*, Cambridge University Press 2004, by Jamil Baz and George Chacko.
A very nice book that tries to make things simple and clear rather than complicated.
- *The Concepts and Practice of Mathematical Finance*, Cambridge University Press 2004, by Mark S. Joshi.
Good book, very good on static hedging.
- *Black-Scholes and Beyond* by Neil Chriss, McGraw Hill. Good on local volatility models, follows the Derman-Kani papers closely.
- *Option Theory* by Peter James, Wiley.
Written in a physicist's style, it is straightforward and has a section on local volatility models and the Fokker-Planck equation.

- *Principles of Financial Engineering* by Salih N. Neftci, Elsevier Academic Press
This book really takes an engineering approach. It focuses on how to use the little we know about the behavior of stocks, bonds and other assets to create the payoffs we want with a minimum of theory. Good common sense.
- *Options, Futures and Other Derivatives*, Prentice Hall, by John Hull. The standard comprehensive teaching book.

In terms of journals to read, look for example at

- Risk Magazine
- Wilmott Magazine
- Journal of Derivatives
- www.ssrn.com has many papers in the FEN (Financial Economics Network) section, and most of the latest papers get posted there before publication.
- There are a bunch of papers, some quite old, on volatility and on local volatility models on my web site, www.ederman.com.

I'll give further references during the course. But mostly I will rely on my class lectures, which I will post on Courseworks.

Contacting me

You can email me at ed2114@columbia.edu.

Grades

20% of the final grade will depend on homework, 30% on the midterm and 50% on the final examination. The worst homework grade will be discarded.

Course Outline

This is roughly what I would like to cover in the course, but we will have to play it by ear and see exactly what we can cover in each class.

- **The Principles of Valuation**
 - Aim of the course.
 - A quick look at the smile.
 - Viewpoints: dealers vs. retail clients.
 - The principles of quantitative finance
 - Static hedging, Dynamic hedging, Utility-based
 - The theory of dynamic hedging.
- **Option Valuation: Realities and Myths**
 - The theory of dynamic replication
 - Option replication
 - The Black-Scholes equation
 - P&L (profit and loss) of options trading
 - The difficulties of dynamic hedging; which hedge ratio to use
 - The approximations and assumptions involved
 - Simulations of discrete hedging
 - Reserves for illiquid securities
- **Introduction to the Implied Volatility Smile**
 - The smile in various markets
 - The difficulties the smile presents for trading desks and for theorists
 - Pricing and hedging
 - Different kinds of volatility
 - Parametrizing options prices: delta, strike and their relationship
 - Estimating the effects of the smile on delta and on exotic options
 - Reasons for a smile
 - No-riskless-arbitrage bounds on the size of the smile
 - Fitting the smile
 - Some simple models and a look at their smiles
- **Implied Distributions Extracted from the Smile**
 - Arrow-Debreu state prices
 - Breeden Litzenberger formula
 - Black-Scholes implied density and its use
 - Static replication of path-independent exotic options with vanilla options
- **Static Hedging**
 - Static replication of path-dependent exotic options with vanilla options

- **Extending Black-Scholes beyond constant-volatility lognormal stock price evolution**
 - Binomial trees
 - Time-dependent deterministic rates
 - Time-dependent deterministic volatility
 - Calibration to rates and volatility
 - Changes of numeraire to simplify problems
 - Alternative stochastic processes that could account for the smile
- **Local Volatility Models/ Implied Trees**
 - Binomial local volatility trees
 - Difficulties encountered
 - Trinomial local volatility trees
- **Fitting Implied Binomial Trees to the Volatility Smile**
 - Dupire equation
 - Fokker-Planck/ forward Kolmogorov equation.
 - Calibration of implied binomial trees
 - How to build an implied tree from options prices.
 - The relation between local and implied volatilities
- **The Consequences of Local Volatility Models**
 - The local volatility surface
 - The relationship between local and implied volatility
 - Estimating the deltas of vanilla options in the presence of the smile
 - Estimating the values of exotic options
 - Static hedging of barrier options
 - Some specific local volatility models: displaced diffusion, CEV, mixed distributions
- **Model classification**
 - Empirical behavior of implied volatility with time and market level
 - Sticky strike, sticky delta, sticky implied tree
- **Stochastic Volatility Models**
 - Are they reasonable, and if so, when?
 - Mean reversion of volatility
 - The PDE for option value under stochastic volatility
 - The mixing formula for option value under stochastic volatility
 - Estimating the smile in stochastic volatility models
 - Simulations of the smile in these models
 - The relationship between local and stochastic volatility

- **Jump-Diffusion Models**
Are they reasonable, and if so, when?
Poisson jumps
The Merton jump-diffusion model and its solution
Estimating the smile in jump-diffusion models
Simulations
- **Other Models**
Vega matching
...
- **Some Guest Speakers**