Equilibrium in a Dynamic Limit Order Market *

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Abstract

We model a dynamic limit order market as a stochastic sequential game. Since the model is analytically intractable, we provide an algorithm based on Pakes and McGuire (2001) to find a stationary Markov-perfect equilibrium. Given the stationary equilibrium, we generate artificial time series and perform comparative dynamics. We demonstrate that the order flow displays persistence. As we know the data generating process, we can compare transaction prices to the true value of the asset, as well as explicitly determine the welfare gains accruing to investors. Due to the endogeneity of order flow, the midpoint of the quoted prices is not a good proxy for the true value. Further, transaction costs paid by market order submitters are negative on average. The effective spread is negatively correlated with true transaction costs, and largely uncorrelated with changes in investor surplus. As a policy experiment, we consider the effect of a change in tick size, and find that it has a very small positive impact on investor surplus.

1 Introduction

We consider a dynamic pure limit order market in which traders choose between buy and sell orders, and market and limit orders. The endogenous choice of orders implies that many standard intuitions about such markets are reversed. We numerically solve for the equilibrium of the model, and generate time series of trades and quotes. We characterize the equilibrium in terms of traders' strategies, transactions costs of market orders, and welfare accruing to both market and limit orders. A unique feature of our model is that it enables explicit welfare comparison across different policy regimes.

We show that the endogenous choice of orders has important implications for inferences drawn from transactions data. Agents supply liquidity when the reward is high and demand liquidity when it is cheap. On average, market buy (sell) orders are submitted when the ask is below (the bid is above) the consensus value of the asset. As a result, conditional on a trade, the midpoint of the bid ask spread is not a good proxy for the asset's true value. Since market order submitters benefit by "picking off" limit orders, transaction costs for market order submitters are negative on average. Thus, measures that were developed for an intermediated market (such as the effective spread) should be interpreted with caution when liquidity supply is endogenous.

The effective spread, defined as the average transaction price minus the midpoint of the contemporaneous bid and ask quotes, is often used implicitly as a measure of welfare when evaluating policies that affect markets. Using our model, we examine the efficacy of this in a pure limit order market. As all trade is incentive compatible, we find that volume is a better proxy for welfare gains than effective spread. Indeed, effective spread is a poor proxy for welfare as revealed by two policy experiments—a decrease in the tick size, and an increase in the gains to trade. In the first experiment, welfare increases and effective spread decreases. In the second, welfare increases but the effective spread also increases.

In our model, in addition to the common or consensus value of an asset, all agents have a private or liquidity motive for trade.[1] We focus on pure limit order markets (i.e., with no market-making intermediaries), a market form that is gaining prominence. Some exchanges such as Paris, Tokyo, Stockholm, and Vancouver are organized in this way. Other exchanges such as the NYSE or Nasdaq have incorporated limit order books into their market design.

Given the limit order book and common value (which are both publicly observed), agents decide whether to buy or sell (or both), and at what prices. In our model traders arrive sequentially and submit orders to maximize their expected surplus given their private value

[1] Intuitively, the common value represents the true value of the asset (for example, the present value of the future dividend stream), whereas a private value reflects idiosyncratic motives for trade (such as wealth shocks, tax exposures, or hedging needs).

and the current limit order book. Expected surplus for a limit order is computed using beliefs about the order's execution probability and the expected change in the asset's value conditional on the order executing. In equilibrium, order submission strategies generate actual execution probabilities and picking off risk that match traders' beliefs. This picking off risk arises as limit buys execute more often when the value drops and limit sells execute more often when the value increases. In equilibrium a given trader's strategy is a function of only the current book—past traders' actions do not matter, other than in their effect on the current book.

The model is a stochastic dynamic game in which each agent chooses an action only once, upon entry to the market. Since it is analytically intractable, we numerically solve for the equilibrium. Even a numerical solution using traditional techniques is difficult due to the size of the state space. Consider a market with only seven prices (or ticks) and up to twenty buy or sell orders at each price. Suppose the lowest sell is at tick 1. Then, the number of possible books is 21^7. But the lowest sell could be at any of 7 ticks, or there might be no limit sells on the book. Hence the total number of books is 8×21^7. Of course, most of these books never arise when traders play equilibrium strategies. Following Pakes and McGuire (2001), we deal with this curse of dimensionality by obtaining equilibrium values, beliefs, and strategies only on the subset of states in the *recurrent class* of states.

We then characterize it by simulating 500,000 trader arrivals for different values of the key parameters. The equilibrium displays order persistence of the sort documented by Biais, Hillion, and Spatt (1995) for the Paris Bourse. In our model some persistence (such as that of small buy or small sell orders) occurs even when there is no change in the consensus value of the asset, suggesting that other microstructure effects can cause persistence.

Our understanding of the trade-offs involved in submitting limit orders has been enhanced by Cohen, Maier, Schwartz, and Whitcomb (1981), Handa and Schwartz (1996), Chakravarty and Holden (1998), and Kumar and Seppi (1993) who analyze traders' choices between market and limit orders in different environments. Biais, Martimort, and Rochet (1999), Foucault (1999), Glosten (1994), O'Hara and Oldfield (1986), Parlour (1998), Rock (1996), Seppi (1997) and Foucault, Kadan, and Kandel (2002) theoretically analyze prices and trading volumes in markets with limit order books.

Of these papers, Parlour (1998), Foucault (1999), and Foucault, Kadan, and Kandel (2002) are explicitly dynamic. However, these models make restrictive assumptions to obtain analytical solutions. Parlour (1998) assumes a 1-tick market and no volatility in the common value of the asset; Foucault (1999) allows for volatility of the common value of the asset, but truncates the book. Foucault, Kadan, and Kandel (2002) have an interesting interpretation of the cost of immediacy but require limit order submitters to undercut existing orders, as opposed to joining a queue. Ideally, for policy work we would like a model with multiple prices and books of varying thickness.

An interesting empirical literature has shed light on both the characteristics of observed limit order books, and the intuition gleaned from models. In the first category, Biais, Hillion, and Spatt (1995) present an analysis of order flow on the Paris Bourse and document persistence in that order flow. Hamao and Hasbrouck (1993) analyze trades and quotes on the Tokyo exchange.

In the latter category, Sandås (2001) uses data from the Stockholm exchange to develop and test static restrictions implied by Glosten (1994). He strongly rejects the restrictions of the static model, suggesting that a dynamic one is needed to explain both price patterns and orders in a limit order market. Hollifield, Miller, and Sandås (2002) use Swedish data to test a monotonicity condition generated by the equilibrium of a dynamic limit order market. They reject the condition when considering both buy and sell orders, and fail to reject when examining only one side of the market. This provides some support for a dynamic model. Hollifield, Miller, Sandås, and Slive (2002) use a similar technique to investigate the demand and supply of liquidity on the Vancouver exchange, and find that agents indeed supply liquidity when it is dear and consume it when it is cheap.

Our work is complementary to the literature pioneered by Demsetz (1968), Roll (1984), Glosten (1987), and Hasbrouck (1991a, 1991b, 1993) that considers the relationship between quoted spreads, transaction prices, and the true or consensus value of the asset in the presence of an intermediary. We generate artificial data, and thus know the true asset value in our limit order market. We can therefore consider some of the same issues albeit in a different market environment. We comment further on the relationship between this literature and our results in Section 4.

We provide details of our model in Section 2 and our solution technique in Section 3. We present equilibrium characteristics of the book and order floor in Section 4 and then discuss transaction costs and welfare in Section 5. The results of our policy experiments are exhibited in Section 6. Section 7 concludes.

2 Model

We present an infinite horizon version of Parlour (1995). This is a discrete time model of a pure limit order market for an asset. In each period, t, a single trader arrives at the market. The trader at time t is represented by a pair, $\{z_t, \beta_t\}$. Here, $z_t \in \{1, 2, \ldots, \bar{z}\}$ denotes the maximum quantity of shares the trader may trade. The trader may place buy or sell orders for any number up to z_t shares. Thus, the decision to buy or sell is endogenous. Let F_z

denote the distribution of z_t. The trader's private valuation for the asset, β_t, is drawn from a continuous distribution F_β. Both z and β are independently drawn across time, and their distributions are common knowledge. We normalize the mean of β to zero.

The asset's common or consensus value, denoted v_t, is public knowledge at time t. Each period, with probability $\frac{\lambda}{2}$, the consensus value increases by one tick, and with the same probability decreases by one tick. Changes in the consensus value reflect new information about the firm or the economy. The periodic innovations in v_t imply that traders who arrive at $\tau > t$, are better informed than limit order submitters at time t. Thus, this is a model of asymmetric information.

The market place is an open limit order book. The agent in the market at time t can either submit a market order, which trades against outstanding orders in the book, or a limit order at a specified price, which enters the book at that price. There is a finite set of discrete prices, denoted as $\{p^{-(N)}, p^{-(N-1)}, \ldots, p^{-1}, p^0, p^1, \ldots, p^{N-1}, p^N\}$. The distance between any two consecutive prices p^i and p^{i+1} is a constant, d, and we refer to it as "tick size." For convenience, prices are denoted relative to the consensus value v_t, and p^0 is normalized to 0 at each t. An order to buy one share that executes at price p^i requires the buyer to pay $v_t + p^i$. We therefore also refer to the price p^i as "tick i."

Associated with each price $p^i \in \{p^{-(N-1)}, \ldots, p^{N-1}\}$, at each point of time t, is a backlog of outstanding limit orders, ℓ_t^i. We adopt the convention that buy orders are denoted as a positive quantity, and sell orders as a negative one. The limit order book, L_t, is the vector of outstanding orders, so that $L_t = \{\ell_t^i\}_{i=-(N-1)}^{N-1}$. At more extreme prices, a competitive crowd of traders provides an infinite depth of buy orders (at a price p^{-N}) or sell orders (at a price p^N).[2]

The trader who arrives at time t takes an action X_t. X_t is a vector with typical element x_t^i, that denotes the integer number of shares to be traded at price p^i. An action is feasible if $\sum_{i=-N}^{N} |x_t^i| \leq z_t$. A buy (sell) order at price p^i is denoted by $x_t^i > 0$ ($x_t^i < 0$).

Market orders submitted at time t execute in that period. Limit orders submitted at time t execute if a counterparty arrives at some time in the future. Following Hollifield, Miller, and Sandås (2002), in each period, each share in the book is cancelled exogenously with some probability. We assume this probability, δ, is constant and independent across shares. This implies that next period's payoffs are discounted by $(1 - \delta)$. Implicitly, the opportunity cost of submitting a limit order in this asset depends on other asset markets, which are not formally modelled. Changes in these other markets may cause traders to cancel their orders; δ proxies for this.

Agents may submit orders that are in part market orders, and in part limit orders. The

[2] This truncation is a feature of Seppi (1997) and Parlour (1998).

market and limit orders may be at the same or different prices. In addition, she may submit both buy and sell orders. Finally, the agent is allowed to submit no order (i.e., submit an order of 0 shares). An agent may arrive in the market, and decide that, given her type and the current book, she is better off not submitting an order. The decision to trade is thus endogenous with respect to both the quantity and the direction of the trade.

2.1 Evolution of the Limit Order Book

The limit order book at time t, in conjunction with the orders submitted by the trader at time t and the exogenous cancellation rate, generates the book at time $t+1$. We now determine how the book at time $t+1$ evolves from the book at time t for the arbitrary (not necessarily equilibrium) action of the trader at time t, denoted X_t.

At each time t, the following sequence occurs. First, a trader enters and takes an action X_t. Given this action, the cumulative shares listed at price p^i are now $(\ell_t^i + x_t^i)$. This holds regardless of whether x_t^i represents a limit or market order (that is, even when ℓ_t^i and x_t^i have opposite signs). After the orders x_t^i have been submitted (and executed, if they are market orders), each remaining share at price p^i is cancelled with exogenous probability δ.

Figure 1 illustrates the sequence of events with 3 ticks, when there is no change in the consensus value.

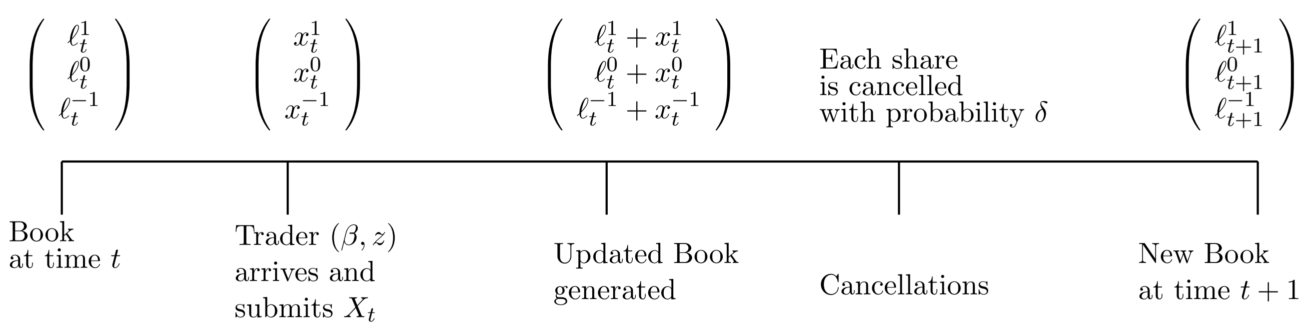

Figure 1: **Evolution of a three tick book**

Now, suppose that at the end of period t, the consensus value of the asset increases from v_t, by one tick. Since all prices are denoted relative to the consensus value, all orders at a price p^i are now listed at price p^{i-1}. That is, such orders are now one tick lower relative to the consensus value. In this process, sell orders at price $p^{-(N-1)}$ will now be listed at p^{-N}, and are automatically crossed off against the crowd willing to buy at that price. Any buy orders that were at $p^{-(N-1)}$ prior to the jump are cancelled.

Similarly, if the consensus value of the asset falls by one tick at the end of period t, all orders previously listed at a price p^i are now listed at a price p^{i+1}, one tick higher relative to the consensus value. Thus, limit orders may execute at a price closer to or further away from the consensus value. In equilibrium this is one of the potential costs of submitting a limit order: orders are more likely to execute if the asset value moves against them.

Limit orders are executed according to time and price priority. Buy orders are accorded priority at higher prices, and sell orders at lower ones. If two or more limit orders are at the same price, time priority is in effect: the one that was submitted first is crossed first. Therefore, an order executes if no other orders have priority, and a trader arrives who is willing to be a counter-party.

Actions of subsequent traders affect the priority of any limit order. Of course, the ultimate change in priority is execution—a counter-party takes the trade. A trader who arrives after an unexecuted limit order can either increase or decrease the price priority. A subsequent trader decreases an existing order's price priority if he submits a competing order closer to the quotes. This moves the unexecuted order further back in the queue. Conversely, a subsequent trader could execute against an order with price priority over the limit order. This moves the limit order toward the front of the queue. Finally, a subsequent order could improve the time priority of the unexecuted order by crossing against orders in the book at the same price, picking off orders with higher time priority. However, it is impossible for a subsequent order to decrease the time priority of a limit order. An agent who submits a limit order is guaranteed a place in the queue at his chosen price. Given that an opposing trade occurs at that price, the agent's order will be executed in sequence.

The per share payoff at time τ to a trader with type β who submits a limit order at time t at price p^i is

$$\begin{cases} p^i - (v_\tau - v_t) - \beta & \text{if he sells the asset at } p^i \text{ at any time } \tau \geq t \\ \beta + (v_\tau - v_t) - p^i & \text{if he buys the asset at } p^i \text{ at any time } \tau \geq t \\ 0 & \text{if the share is cancelled at any time before it is executed} \end{cases} \tag{1}$$

2.2 Transaction Costs and Welfare Measures

The bid and ask prices in the market at time t are defined in the standard fashion. In any period the ask price is the lowest sell price on the book, and the bid price is the highest buy price on the book. Therefore,

Definition 1 *The current bid and ask prices in the market are given by:*

$$\begin{aligned} B_t &= v_t + \max\{\, p^i \mid \ell_t^i > 0 \,\} \\ A_t &= v_t + \min\{\, p^i \mid \ell_t^i < 0 \,\} \end{aligned}$$

The midpoint of the bid and ask prices is $m_t = \frac{A_t+B_t}{2}$.

Next, consider the transaction costs paid at time t by a trader who submits a market buy order of size $\bar{x}$. If the market order is large, it may "walk the book," so that different shares execute at different prices. Suppose that x_t^i shares execute at price p^i, with $\sum_{i=-N}^{N} x_t^i = \bar{x}$. Then, the average execution price for the shares is $P_t(\bar{x}) = v_t + \frac{1}{\bar{x}} \sum_{i=-N}^{N} p^i x_t^i$. The average execution price for a sell order is found analogously.

The average execution price is used to define the total transaction costs paid by a market order submitter.

Definition 2 *The true transaction cost faced by a market order of size $\bar{x}$ at time t is*

$$C_t(\bar{x}) \quad = \quad (P_t(\bar{x}) - v_t)\,\mathrm{sign}(\bar{x}). \tag{2}$$

The effective spread, $S_t(\bar{x})$, faced by a market order of size $\bar{x}$ at time t is

$$S_t(\bar{x}) \quad = \quad (P_t(\bar{x}) - m_t)\,\mathrm{sign}(\bar{x}). \tag{3}$$

In many econometric specifications (see Hasbrouck (2002) for a summary), the execution price is decomposed into the sum of the "efficient price" and microstructure effects. In our model, the efficient price is just the consensus value, v_t. Thus, our transaction cost C_t is simply the microstructure effect times the signed order flow in these specifications.

A commonly used proxy for transaction costs is the effective spread. If a market buy order is small, so that it transacts at the ask and does not go deeper into the book, the effective spread reduces to $(A_t - m_t)$. Similarly, a market sell order that transacts at the bid has an effective spread of $(m_t - B_t)$. Since $S_t(\bar{x}) = ((P_t(\bar{x}) - v_t) \; + \; (v_t - m_t))\mathrm{sign}(\bar{x})$, the effective spread is simply the transaction cost with the midpoint of the quotes as a proxy for the consensus value. If the midpoint of the bid-ask spread equals the consensus value of the asset (that is, $m_t = v_t$), the effective spread is a good proxy for the transactions cost paid by a market order submitter.

Consider a trade that occurs at time t. The consumer surplus accruing to the market order and limit order submitters is a measure of the net change in their welfare. Recall that $\bar{x} > 0$ indicates a market buy order, and $\bar{x} < 0$ a market sell order.

Definition 3 *Consider a trade of $\bar{x}$ shares at t. Then,*
(i) the surplus accruing to the market order submitter is

$$W_t^m \quad = \quad \bar{x}\,(\beta_t + v_t - P_t(\bar{x}))\,,$$

where P_t is the average execution price.
(ii) the surplus accruing to limit order submitters taking the other side of the transaction is

$$W_t^l \;\; = \;\; \bar{x}\left(P_t(\bar{x}) - (v_t + \beta_t^l)\right),$$

where β_t^l is the share-weighted average of the private values of all limit order submitters whose orders trade against the market order at time t.

When $m_t = v_t$, the surplus of a market order submitter can be written in terms of the effective spread. This is the basis for the use of the effective spread to evaluate surplus. However, if $m_t \neq v_t$, this is no longer true.

Proposition 1 *Suppose a trade of size $\bar{x}$ occurs at time t at an effective spread of $S_t(\bar{x})$. If (and only if) $m_t = v_t$,*
(i) the surplus of the market order submitter is $W_t^m = \bar{x}\,\beta_t - |\bar{x}|S_t$.
(ii) the surplus accruing to the limit order submitters who trade at t is $W_t^l = |\bar{x}|\; S_t(\bar{x}) - \bar{x}\;\beta_t^l$.

Proof

(i) The surplus of the market order submitter is $W_t^m = \bar{x}(\beta_t + v_t - P_t(\bar{x}))$. From equation (3), if the market order is a buy order, $P_t(\bar{x}) = m_t + S_t(\bar{x})$, and if it is a sell order, $P_t(\bar{x}) = m_t - S_t(\bar{x})$. Hence, for a buy order,

$$W_t^m \;\; = \;\; \bar{x}\;(\beta_t + v_t - m_t - S_t(\bar{x})).$$

Hence, $W_t^m = \bar{x}(\beta_t - S_t)$ if and only if $m_t = v_t$.

Similarly, for a sell order, $W_t^m = \bar{x}\;(\beta_t + v_t - m_t + S_t(\bar{x})$, and $W_t^m = \bar{x}\;(\beta_t + S_t(\bar{x}))$ if and only if $m_t = v_t$. Putting together the expressions for buy and sell orders, we have $W_t^m = \bar{x}\;\beta_t - |\bar{x}|S_t$ if and only if $m_t = v_t$.
(ii) Next, consider the surplus accruing to the limit order submitters who trade at t. This is $\bar{x}(v_t + P_t(\bar{x}) - \beta^l)$. Similarly to part (i), we obtain $W_t^l = |\bar{x}|\;S_t - \bar{x}\;\beta^l$. ■

We report surplus for both market orders and limit order submitters. For policy purposes, the surplus of limit order submitters should also be considered. Typically, the literature has computed transaction costs for market orders. However, there is no reason why one group of investors should be favored over another. Notice that, even if $m_t \neq v_t$, the aggregate surplus improvement as a result of the trade at t is $\bar{x}(\beta_t - \beta_t^l)$, which is independent of the effective spread, $S_t(\bar{x})$. That is, if one also considers limit order traders in surplus calculations, these transaction costs become irrelevant: in a pure limit order market, these costs are simply transfers between agents. Thus, any measure which determines a cost to one party merely reflects a gain to the counter-party.

We do not have an intermediary: every trade in our model consists of a market order executing against a limit order. In a market with an intermediary market-maker, transaction costs may be an important determinant of retail investor (both market and limit order submitter) surplus. While the intermediary provides a benefit by providing liquidity to market orders, it may also deter limit order submission and thus decrease the surplus of such agents (see Seppi, 1997). As Glosten (1998) observes in this case, one should account for the surplus of all parties in the market.

3 Equilibrium

In section 2 we modelled a limit order market as a stochastic sequential game. We now characterize best responses in this game, discuss the existence of a stationary Markov perfect equilibrium, and present an algorithm for numerically finding such an equilibrium.

3.1 Best Responses

In period t, a trader endowed with type (z_t, β_t) arrives at the market and submits an order X_t specifying the number of shares to buy or sell at each price $\{p^{-N}, \ldots, p^N\}$. He observes the current market conditions, which consist of the current consensus value, v_t, and the current limit order book, L_t. Recall from Section 2 that the trader also knows the (exogenous) order cancellation rate, denoted δ, the probability that v_t will change in any period, denoted λ, and the stationary distributions of types given by F_z and F_β.

Of course, the trader does not know the future sequence of trader types, order cancellations, and changes in consensus value. This sequence determines whether his limit orders execute, as well as the value of any such trades (since v_t may change before execution). Hence, the trader forms beliefs about the probability of execution of an order placed at any price p^i and the change in v_t conditional upon execution at this price.[3]

Let $\mu^e_{t-1}(k, i, L_t, X_t)$ denote the period t trader's belief of the probability of execution of his k^{th} share at price p^i given book L_t and order X_t. Similarly, let $\Delta^v_{t-1}(k, i, L_t, X_t)$ denote his expectation of the net change in the consensus value prior to execution (conditional upon execution). Since the traders are risk-neutral, their expected payoffs depend only on this expectation $\Delta^v_{t-1}(\cdot)$, and not on other features of the underlying distribution of changes

[3] Recall that we normalize prices to be relative to the *current* v_t and therefore shift L_t after a change in the consensus value. Hence, the belief about the "change in v_t conditional upon execution" translates in the algorithm to a belief about "the number of shifts in the book between t and execution."

in consensus value. We refer to $\mu^e_{t-1}(\cdot)$ and $\Delta^v_{t-1}(\cdot)$ together as the beliefs of the agent at time t.

These beliefs are naturally different for market and limit orders. Suppose an agent submits a single buy order at price p^i and time t, so $x^i_t > 0$. Suppose further that the sell depth at price p^i exceeds x^i_t (formally, $\ell^i_t < -x^i_t$). Then, the order is a market order. Since market orders execute immediately, $\mu^e_{t-1}(\cdot) = 1$ and $\Delta^v_{t-1}(\cdot) = 0$. A limit order submitted at t executes only at $(t+1)$ or later. Since it may be cancelled in the interim, $\mu^e_{t-1}(\cdot) < 1$ for any limit order. Similarly, in equilibrium we expect $\Delta^v_{t-1}(\cdot)$ to be positive for limit buy orders, and negative for limit sell orders. That is, a limit order is subject to picking off risk, since future traders are better informed about future v.

Given these beliefs, the risk-neutral trader optimally chooses

$$\begin{aligned} X_t \quad = \quad & \underset{\tilde{X}=(\tilde{x}^{-N},\ldots,\tilde{x}^N)}{\arg\max} \quad \sum_{i=-N}^{N} \sum_{k=1}^{|\tilde{x}^i|} \mu^e_{t-1}(k,i,L_t,\tilde{X})\,(\beta_t + \Delta^v_{t-1}(k,i,L_t,\tilde{X}) - p^i)\,\mathrm{sign}(\tilde{x}^i) \\ & \text{subject to:} \quad \sum_{i=-N}^{N} |\tilde{x}^i| \leq z_t. \end{aligned}$$

A strategy for an agent at time t, therefore, is a mapping $X_t : \mathcal{L} \times [\underline{\beta}, \bar{\beta}] \times \{1, \ldots, \bar{z}\} \to \{-z_t, \ldots, z_t\}^{2N+1}$, where $\mathcal{L}$ is the set of all books. Each agent chooses a strategy to maximize his own payoff, given his beliefs about the execution probabilities, $\mu^e_{t-1}(\cdot)$, and changes in v given execution, $\Delta^v_{t-1}(\cdot)$.

3.2 Existence

In a stationary equilibrium, $\mu^e_t = \mu^e$ and $\Delta^v_t = \Delta^v$ for each t. That is, any two agents facing the same limit order book have the same beliefs about execution probabilities and changes in v conditional on execution. Further, agents' beliefs must be consistent with the actual future course of play. The equilibrium concept we use is Markov perfect equilibrium. The state at any time t depends on the limit order book, L_t.[4] Since time does not enter into the definition of the state, such an equilibrium must be stationary. Thus, we rule out "time of day effects" or equilibria of the form: "Every 333rd period, submit more aggressive orders." The Markov specification requires agents to condition only on the current book, and not on any prior books. In this model it is not restrictive, because the book summarizes the payoff-relevant history of play.

We have a countable state space (since depth at any tick is an integer) and a finite action space. It is well-known that stationary Markov perfect equilibria exist in such models.[5] We

[4]We exclude v_t from the state since L_t shifts as needed to keep prices relative to the current consensus value.

[5]See, for example, Fudenberg and Tirole (1991), page 504, and the references therein.

do not prove uniqueness. However, in keeping with existing literature, we verify that the equilibrium we find appears to be computationally unique. This is done by starting the algorithm at different initial values, and ensuring that it converges to the same equilibrium.

3.3 Solving for Equilibrium

Equilibrium is obtained by finding common beliefs, μ^e and Δ^v, such that when each trader plays his best response, the means of the distributions of realized executions and changes in v conditional on execution indeed match the expected values for these outcomes, as specified by μ^e and Δ^v.

To find this fixed point, we simulate a market session and update beliefs given the simulated outcomes until beliefs converge. We follow Pakes and McGuire (2001), in using a stochastic algorithm to asynchronously update these beliefs. The advantage of this approach is two-fold. Consider the trader's belief for the execution probability of a limit buy for one share at price p^i given the current book, L_t. To update this belief non-stochastically one would integrate over all the possible sequences of future outcomes (of new trader arrivals, order cancellations, and v jumps) that lead to this share either being cancelled or executed. Instead, we simply track whether this share ultimately executes or is cancelled in the market simulation. Upon execution or cancellation, we update the current value of μ^e for the state at which this share was submitted. Updating Δ^v is similar: we keep track of the net changes in v since the order was placed. If the share executes, we average in the net changes to Δ^v. In essence, this approach uses a single draw to perform Monte Carlo evaluation of a complicated integral.

The second advantage of the stochastic algorithm is that beliefs are only updated for states actually visited. Formally, a state is defined by the limit order book L_{t-1}, the action taken by the trader at t, X_t, the price at which a particular share in that order was submitted, p^i, and the number of shares submitted by the trader at that price, k. That is, a state is represented by a (k, i, L, X)-tuple. The fixed point is computed only for the *recurrent class* of states. As discussed in Section 1, the full state space for this game is too large for traditional numerical methods that operate over the entire state space.[6] A natural concern is that false beliefs at points outside the recurrent class may lead players to mistakenly avoid such states. To alleviate this concern, we specify initial beliefs to be overly optimistic: states not in the recurrent class would not be visited even if beliefs for them were correct.

As discussed in Pakes and McGuire (2001), this algorithm may be viewed as a behavioral

[6] To be of practical use, an algorithm must only operate on a set of states that can be stored in the computer's memory (without swapping to "virtual" memory on the hard drive).

description of players learning about the game in "real-time" using the same updating rules as the algorithm. Here, we only use the algorithm to characterize beliefs and behavior in equilibrium. We do not use the model to infer how players "arrive" at the equilibrium.

In more detail, the algorithm works as follows. First, we choose a rule, $\{\mu_0^e(\cdot), \Delta_0^v(\cdot)\}$, for assigning beliefs to states encountered for the first time. As discussed, this rule must be optimistic. The simplest such rule is $\mu_0^e(\cdot) = 1$ and $\Delta_0^v(\cdot) = 0$. A better rule sets $\mu_0^e(\cdot, i, \cdot, \cdot)$ to the probability that a trader, for whom taking the other side of the transaction at price p^i would yield non-negative surplus, arrives before the order is randomly cancelled. To derive this probability, for a limit buy at p^i, note that the probability of surviving τ periods with no such sellers arriving is $(1-\delta)^\tau(1-F_\beta(p^i))^\tau$. Execution at $\tau+1$, therefore, occurs with probability no more than $(1-\delta)^\tau(1-F_\beta(p^i))^\tau(1-\delta)F_\beta(p^i)$. Since execution can occur in any future period,

$$\mu_0^e(\cdot, i, \cdot, \cdot) = \sum_{\tau=0}^{\infty}\Big[(1-\delta)^\tau(1-F_\beta(p^i))^\tau(1-\delta)F_\beta(p^i)\Big] = \frac{(1-\delta)F_\beta(p^i)}{1-(1-\delta)(1-F_\beta(p^i))}.$$

The initial belief rule μ_0^e is similarly derived for limit sells.

Next, we choose an arbitrary initial book, L_0. For simplicity we choose L_0 to be empty. We then set $t = 1$ and iterate over the following steps.

Step 1: Draw the period t trader's (z_t, β_t) and determine the optimal action X_t, given $\mu_{t-1}^e, \Delta_{t-1}^v$.

Step 2: For each market order share, update $\mu_t^e(\cdot)$ and $\Delta_t^v(\cdot)$ for the initial state of the limit order executed by the market order. If the executed limit order was the k^{th} share submitted in period $\tau < t$ at price p^i (relative to v_τ), then

$$\mu_t^e(k, i, L_\tau, X_\tau) = \frac{n}{n+1}\,\mu_{t-1}^e(k, i, L_\tau, X_\tau) + \frac{1}{n+1} \tag{4}$$

$$\Delta_t^v(k, i, L_\tau, X_\tau) = \frac{n}{n+1}\,\Delta_{t-1}^v(k, i, L_\tau, X_\tau) + \frac{v_t - v_\tau}{n+1} \tag{5}$$

where n is the number of shares submitted at state (k, i, L_τ, X_τ) that have either executed or been cancelled between periods 0 and t.[7]

Step 3: Add each limit order share in X_t to the end of the appropriate queue in L_t. At this point the book is $L_t + X_t$.

[7] Since $\Delta_t^v(\cdot)$ refers to the net changes in consensus value conditional on execution, the count used in this updating can alternatively be the number of shares submitted at that state that eventually executed. This leads to faster convergence.

Step 4: Cancel each share in the book with probability δ. Update μ_t^e for the states at which cancelled shares were submitted. The update uses only the first term in equation (4) since the last term has a numerator of zero for cancelled shares (and one for executed shares). Note that Δ_t^v is not updated since it corresponds to changes in v conditional on execution.

Step 5: Determine the consensus value for the next period using the following transition kernel:

$$v_{t+1} = \begin{cases} v_t + 1 & \text{with probability } \lambda/2 \\ v_t & \text{with probability } 1-\lambda \\ v_t - 1 & \text{with probability } \lambda/2. \end{cases}$$

If v changes then shift the book to maintain the normalization of $p^0 = v$ (i.e., to maintain prices being relative to the current consensus value), as discussed in section 2. Consider an increase in v: sell orders at (pre-shift) tick $-(N-1)$ are picked-off by the crowd of buyers at tick $-N$ and buy orders that were at tick $-(N-1)$ are cancelled. The states at which these orders were submitted have beliefs updated in the appropriate manner: executed orders use the update rule in step 2, while cancellations use the update rule in step 4. When v decreases, orders that were at tick $N-1$ are processed similarly.

Step 6: Implicitly set $\mu_t^e = \mu_{t-1}^e$ and $\Delta_t^v = \Delta_{t-1}^v$ for states not updated in steps 2, 4, or 5. Set $t = t+1$, and return to step 1.

As in Pakes and McGuire (2001), we reset n (in step 2) to 1 every 1–10 million periods until beliefs have begun to stabilize. This enables the algorithm to quickly correct for the excessive optimism of initial beliefs at most states.

Since our recurrent class of states is very large, we cannot use the convergence criteria specified by Pakes and McGuire (2001).[8] Instead, we stop the iterative process when beliefs satisfy a probabilistic criterion, similar to the test used by den Haan and Marcet (1994) in another context.

Holding beliefs fixed, we simulate 100 million periods and record the frequency of limit order executions at each state. Consider the execution frequency of the k^{th} share submitted at price p^i as part of order X given book L, $\mu^e(k, i, L, X)$. Under the null hypothesis that beliefs have converged to a fixed point, execution is a binomial process with probability of success μ^e and failure $1 - \mu^e$. By the central limit theorem, the limiting distribution

[8] In our base case, defined in the next section, the recurrent class has about ten million states.

of the empirical execution frequency in each state is approximately normal with mean μ^e and variance $\frac{\mu^e(1-\mu^e)}{N(k,i,L,X)}$, where $N(k,i,L,X)$ denotes how often the state occurred in the 100 million periods. The test statistic standardizes these normal variables and sums their squares. The statistic is χ^2 with degrees of freedom equal to the number of states (i.e., (k,i,L,X)-tuples) used in the summation. We only use states visited at least 100 times to ensure that the central limit approximation is accurate.[9] If the test statistic is less than the 1% critical value, we deem the algorithm to have converged.[10]

This probabilistic test does not involve Δ^v. Therefore, we also check the absolute differences in the realized outcomes (executions and net changes in v_t) and their respective beliefs (μ^e and Δ^v). We find that whenever the chi-squared test is satisfied, the weighted (by visitation frequency) average absolute differences, over the 100 million periods, in both cases are less than 1%.

4 Characterization of Equilibrium: Simulation Results

Once the algorithm has converged, we record 500,000 trader arrivals.[11] We describe characteristics of the book and order flow in a baseline parametrization. To provide intuition for our results, we compare these outcomes with those from a similar model with no asymmetric information (i.e., a constant consensus value).

Differences in outcomes across different parameterizations occur for two reasons. First, as specified in Section 3, an agent's equilibrium actions at time t depend on the state of the limit order book and the consensus value of the asset. Agents in the same state, but across different parametrizations, have different equilibrium strategies. This leads to a difference in transition probabilities between states and hence the frequency of particular states. Second, changes in the consensus value lead to an exogenous transition between states. All the averages we report such as spread frequencies, and submission prices for limit orders incorporate both these effects.

4.1 Numerical parametrization

We do not attempt to calibrate the model. For a baseline benchmark, we chose parameter values that qualitatively capture salient market features while being consistent with com-

[9]For the baseline case, fewer than 1% of the states visited during the 100 million periods are not included in the summation.

[10]As in den Haan and Marcet (1994), the "tolerance" level of this probabilistic stopping criteria is determined by the number of simulated periods used to construct the test. Eventually, as this number approaches infinity, the variance of the execution frequency approaches zero, and even minute discrepancies between execution frequencies and μ^e would lead to a rejection of the null hypothesis.

[11]This sample is large enough that the means we report have very low standard errors; on the order of 10^{-3}.

putational tractability. We experimented with different parameter values, and found the qualitative nature of our results to be robust.

The following parametrization corresponds to our benchmark case.

- There are nine ticks and the tick size is normalized to $d = \$\frac{1}{16}$. The ticks are denoted $\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$. The corresponding price vector relative to the common value is $\{-\frac{1}{4}, -\frac{3}{16}, \cdots, \frac{1}{4}\}$. Traders may submit limit orders at ticks $\{-3, \ldots, 3\}$. At ticks -4 and 4, a trading crowd provides infinite liquidity.

- F_β is a normal distribution with mean 0 and standard deviation 3 ticks, or \$0.1875. Hollifield, Miller, Sandås and Slive (2002) estimate for three stocks on the Vancouver Stock exchange that trades "with a valuation within 2.5% of the average value of the stock account for between 32% and 52% of all traders." This implies a standard deviation of the private value distribution approximately equal to 4.5% of the value of the stock. Given our parametrization, this corresponds to a consensus value of approximately $\frac{3}{16}/.045 = \$4.17$.

  The choice of F_β is not motivated by computational need. The algorithm can handle any distribution for F_β.

- F_z assigns $z \in \{1, 2\}$ with equal probability. That is, each trader has, with equal probability, either one or two units to trade. The potential trade size distribution is difficult to parameterize by casual observation, because order size is endogenous. For the maximal trade size, we choose the lowest number (2) that allows traders to submit multiple orders.

- Each period, each share is cancelled with probability $\delta = 0.04$. If a share is not executed, the expected time before it is cancelled is 25 order arrival periods.[12] If orders arrive every 120 seconds, this parametrization suggests that limit orders stay on the book for about 50 minutes. This is in keeping with the stylized facts presented in Lo, MacKinlay, and Zhang (2002). In a pooled sample of 100 stocks they find that limit orders failing to execute are cancelled on average after 46.92 minutes for buy orders and 34.15 minutes for sell orders.

- We take the innovation to the consensus value to be small, with $\lambda = 0.08$. The probabilities of increases and decreases are the same (0.04); we do not incorporate a trend for the common value. This implies that all aspects of the market are symmetric around the consensus value.

[12]With probability $\delta^k(1-\delta)^{k-1}$, the share will last k periods. Hence, the expected time until cancellation (absent market order executions) is $\delta\{1 + 2(1-\delta) + 3(1-\delta)^2 + 4(1-\delta)^3 + \ldots\} = \frac{1}{\delta}$.

We choose the innovation in the consensus value of the asset to be one tick. That is, each period the asset value can increase or decrease by one tick. As agents' private motives for trade are continuous, traders arrive with varying incentives to undercut the existing book. Further, the variance of the innovation distribution is thus exactly λ.

- For convenience, we report all payoffs in ticks (where one tick represents $\$\frac{1}{16}$).

4.2 The State and Evolution of the Limit Order Book

Since we simulate a symmetric version of the model, we often report the frequencies of buy orders alone. The characteristics of sell orders are exactly similar. 50.0% of the total orders were buy orders, with 21.0% being market buys, and 29.0% limit buys.

Recall that prices are normalized each period to the current consensus value, with p^0 reset to 0. Most orders are submitted at p^0. In Figure 2, we depict the total number of all types of shares submitted at each tick. For convenience, in this figure, we show only orders submitted by traders, and ignore the infinite liquidity supplied by the trading crowd at ticks -4 and 4. Over the 500,000 periods, only 15 shares were executed against the liquidity supplied by the crowd; 5 buys at a tick of $+4$ and 10 sells at -4. A minuscule proportion of traders (just 1 out of 500,000) chose not to submit an order.

While the total number of shares traded is large, on average, the book is thin, suggesting that the market is effective at consummating trades. We present the average buy and sell sides of the book in Figure 3. The average book has a total of 2.82 shares on the buy side, and 2.84 on the sell side. As one might expect, depth is concentrated near the consensus value of the asset, with buy orders more likely to be below it and sell orders more likely to be above it. Further, as expected given the symmetry in the parametrization, the book is symmetric. The number of limit order buys one tick below the consensus value of the asset is equal to the number of limit sells one tick above it.

We next examine persistence in the order flow. Following Biais, Hillion, and Spatt (1995) and the subsequent literature,[13] we start by classifying the types of orders submitted. We present the definitions for buy orders; sell orders are defined analogously.

[13]See, for example, Ahn, Bae, and Chan (2001), Griffiths, Smith, Turnbull, and White (2000) and Ranaldo (2003).

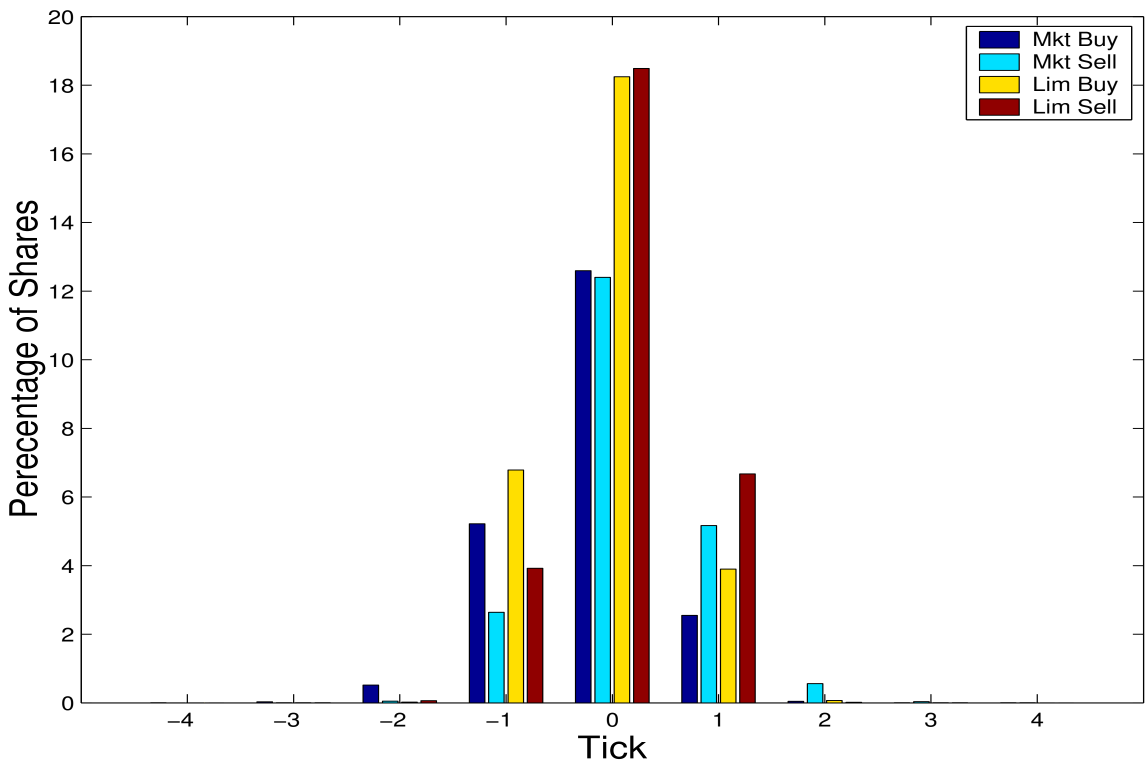

Figure 2: **Total number of shares at different ticks**

Large Buy (LB)	Market buy order that moves the price. This is a market buy for two shares, where the second share is bought at a higher price than the first.
Small Buy (SB)	Market buy order where all shares are bought at the same price.
Aggressive Buy (AB)	Limit buy order at a price that is higher than the current bid.
At the Quote Buy (QB)	Limit buy order at the current bid.
Below the Quote Buy ($< QB$)	Limit buy order at a price lower than the current bid.

Other than large market orders, which necessarily require that two shares be traded, all other order types can involve one or two shares.

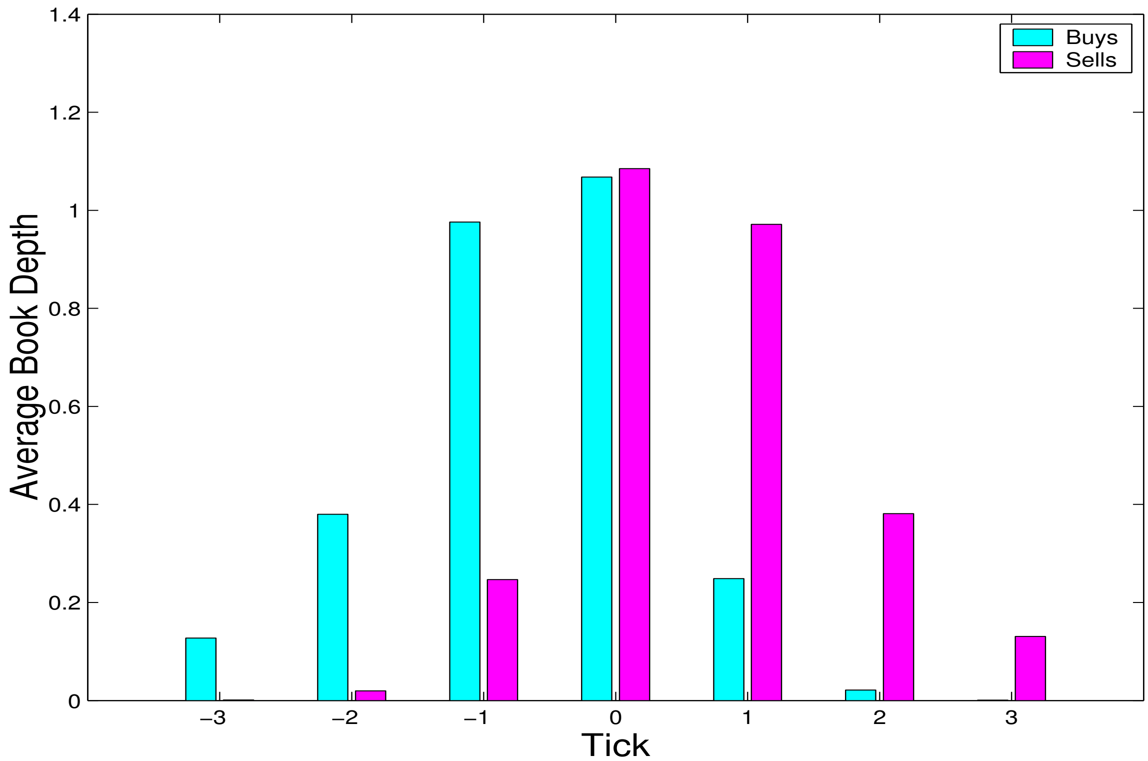

Figure 3: **Average book depth at different ticks**

Biais, Hillion, and Spatt (1995) report on the persistence of orders on the Paris Bourse.[14] They find patterns of trade that are consistent with information effects. For example, a large buy followed by a limit buy above the previous bid "reflects the adjustment in the market expectation to the information content of the trade." Further, they identify a "diagonal effect": the conditional probability of an order following a similar order is typically higher than the unconditional probability of such an order.

To examine persistence, we report the probability of observing an action at time $t+1$, conditional on the action at time t. In our data, a trader with two shares submits them simultaneously. If he submits two different kinds of orders, to determine the transition probabilities, we need to assign one order to be the "first." We use the following rule: if one share is submitted as a market order, it is the first share. When two different limit orders are submitted by the same trader, we randomly assign one order to be the first.

Table 1 reports the transition probabilities for buy orders (again reported as percentages). The model is symmetric, so the conditional probabilities of sell orders are similar.

[14] Hamao and Hasbrouck (1995) document similar persistence on the Tokyo Exchange.

The sum across each row is the conditional probability of a buy order at $t+1$ given the conditioning event in the first column. That is, across each row, the probabilities sum to the conditional frequency of observing a particular kind of buy order at $(t+1)$, given the event at t. Note that the frequency is reported as a percentage of all orders (buy and sell). Hence, the numbers in the total column are around 50% (approximately half the orders are buy orders).

	Market Buys			Limit Buys				Total
Event at t	Large	Small	Total	Agg.	At Quote	$<$ Quote	Total	Buys
Large Buy	0.87	32.38	33.25	21.42	0.14	0.00	21.57	54.82
Small Buy	1.06	23.62	24.69	29.30	6.36	0.56	36.22	60.91
Large Sell	0.00	0.17	0.17	17.98	19.95	8.29	46.22	46.39
Small Sell	0.01	6.30	6.30	13.06	13.30	6.20	32.56	38.86
Agg. Buy	0.03	12.18	12.20	4.62	23.39	7.22	35.23	47.44
Agg. Sell	1.80	35.71	37.51	6.21	6.92	1.82	14.95	52.46
At Quote Buy	0.02	13.59	13.61	7.05	17.76	8.18	33.00	46.61
At Quote Sell	0.03	36.50	36.53	8.07	7.70	1.67	17.44	53.98
$<$ Quote Buy	0.00	3.66	3.66	2.03	23.60	15.01	40.64	44.30
$>$ Quote Sell	3.74	42.21	45.95	4.41	4.70	0.09	9.20	55.16
Overall	0.62	20.39	21.01	12.35	12.24	4.41	29.00	50.01

Table 1: **Conditional frequencies of buy orders at time $t+1$ (column) given the order at time t (row), base case.**

The table reveals some persistence in order submission. For many of the defined events, we do find a diagonal effect. For example, market buys are more likely after market buys than market sells.[15] Further, market orders are frequently followed by aggressive limit orders on the same side of the market. This often happens when traders exhaust the liquidity in the book and become liquidity providers at the same price.

In our model, such patterns might emerge for two reasons. First, the impact of a change in the consensus value, v_t, may induce subsequent traders to take similar actions, until the book has adjusted. For example, following an increase in v_t, sell orders previously on the book are priced “too low.” This should lead to a sequence of buy orders as subsequent traders pick off these limit orders. Second, irrespective of the amount of asymmetric information, actions could be autocorrelated because of persistence in the states.

To determine if information events are the cause of the autocorrelation in the data, we report in Table 2 the transition probabilities for a model with a constant consensus value, and hence no informationally motivated trade.

[15]This also accords with the theoretical results of Parlour (1998) and Foucault (1999), and the empirical findings of Hollifield, Miller, Sandås, and Slive (2002) and Ranaldo (2002).

	Market Buys			Limit Buys				Total
Event at t	Large	Small	Total	Agg.	At Quote	< Quote	Total	Buys
Large Buy	0.00	34.33	34.33	21.12	0.00	0.00	21.12	55.45
Small Buy	0.19	24.87	25.05	23.42	9.78	0.07	33.27	58.32
Large Sell	0.00	0.00	0.00	9.35	32.69	0.87	42.91	42.91
Small Sell	0.00	7.53	7.53	10.33	21.53	2.03	33.90	41.43
Agg. Buy	0.00	10.76	10.76	1.91	30.29	5.57	37.77	48.53
Agg. Sell	1.18	35.56	36.73	3.60	10.48	0.49	14.58	51.31
At Quote Buy	0.00	15.58	15.58	3.19	25.83	1.57	30.59	46.18
At Quote Sell	0.01	33.12	33.13	5.47	14.06	1.27	20.80	53.93
<Quote Buy	0.00	2.29	2.29	0.64	26.45	10.96	38.06	40.35
>Quote Sell	2.50	42.62	45.12	2.06	11.90	0.00	13.96	59.08
Overall	0.19	20.70	20.89	9.17	18.20	1.70	29.07	49.96

Table 2: **Conditional frequencies of buy orders at time $t+1$ (column) given the order at time t (row) with no changes in consensus value**

Even in this scenario, we recover a diagonal effect for some kinds of orders. Small market buys continue to be persistent, and limit buys at the quote are even more persistent when the consensus value is constant. On the other hand, the following events are indicative of information effects—large orders followed by large orders, and limit orders at the quote followed by orders away from the quotes.

4.3 The State and Evolution of the Bid-Ask Spread

In our model, in keeping with standard intuition, a market order is less likely when the bid-ask spread is wide, while limit orders are more likely. In Table 3, we report the unconditional probability (as a percentage) of observing market or limit buy orders given the spread. The results accord with those presented in Foucault, Kadan, and Kandel (2002) and Foucault (1999). When spreads are wide, market orders are more expensive, and thus traders tend to submit aggressive limit orders that narrow the spread. For example, at spreads between 5 and 8 ticks, virtually all orders are aggressive limit orders. When spreads are narrow, traders tend to take liquidity by submitting market orders. The rapid response of traders to profit opportunities ensures that spreads are narrow. Almost half the time, the quoted spread is 1 tick, with a mean of 2.22 ticks.

Glosten (1987) decomposes the quoted spread into order processing and adverse selection components. His market maker framework is not directly applicable to a limit order market. In the latter, quotes are set by (possibly stale) limit orders rather than continuously adjusted by market makers. However, information does play a role in determining the quoted spread in a limit order market: limit orders are placed by traders who are aware that they will

Bid-Ask Spread	Frequency	Market Buys Large	Small	Total	Limit Buys Agg.	At Quote	< Quote	Total	Total Buys
1	48.26	0.51	28.39	28.90	0.00	16.27	4.79	21.07	49.97
2	15.32	0.65	11.94	12.59	25.23	8.13	3.86	37.22	49.82
3	13.06	2.20	22.40	24.60	7.03	8.91	9.63	25.57	50.17
4	17.12	0.01	15.62	15.63	18.82	13.44	2.24	34.50	50.13
5	4.15	0.00	0.63	0.63	48.30	1.08	0.00	49.38	50.01
6	1.03	0.00	0.04	0.04	49.88	0.00	0.00	49.88	49.92
7	0.27	0.00	0.00	0.00	50.45	0.00	0.00	50.45	50.45
8	0.79	0.00	0.00	0.00	50.87	0.00	0.00	50.87	50.87

Table 3: **Frequency (%) of buy orders for different bid-ask spreads.**

execute against market order submitters with superior information. Thus, in equilibrium, the spread distribution is affected by information. Figure 4 demonstrates that, in the absence of asymmetric information, spreads are on average narrower. The mean drops to 1.87 ticks. Thus, volatility of 4% leads to an increase of 18.5% in the average quoted spread.[16]

5 Transaction Costs and Welfare

All our results derive from the fact that order flow is endogenous. In particular, arriving traders take advantage of profit opportunities on the book. This is the other side of the winner's curse or picking off risk faced by limit order submitters: losses to limit order traders accrue as benefits to market order submitters. We first document the winner's curse and the market's response to it. We then examine the implications for the transaction costs paid by a market order, and whether the midpoint of the bid-ask spread is a good proxy for the consensus value.

5.1 The Winner's Curse and Picking off Risk

Limit orders are more likely to be executed after the consensus value moves against the limit order submitter. That is, a limit buy order is more likely to be executed if the asset value moves down and a sell order is more likely to be executed if the consensus value of the asset moves up. Of course, in equilibrium, agents placing limit orders compensate for this. As we have mentioned, an immediate implication is that the market orders that execute against the picked off limit orders represent advantageous trades. How large are these effects?

[16]The effect of volatility shocks on the limit order book has been examined by Ahn, Bae, and Chan (2001), and Coppejans, Domowitz, and Madhavan (2001), and Hasbrouck and Saar (2002).

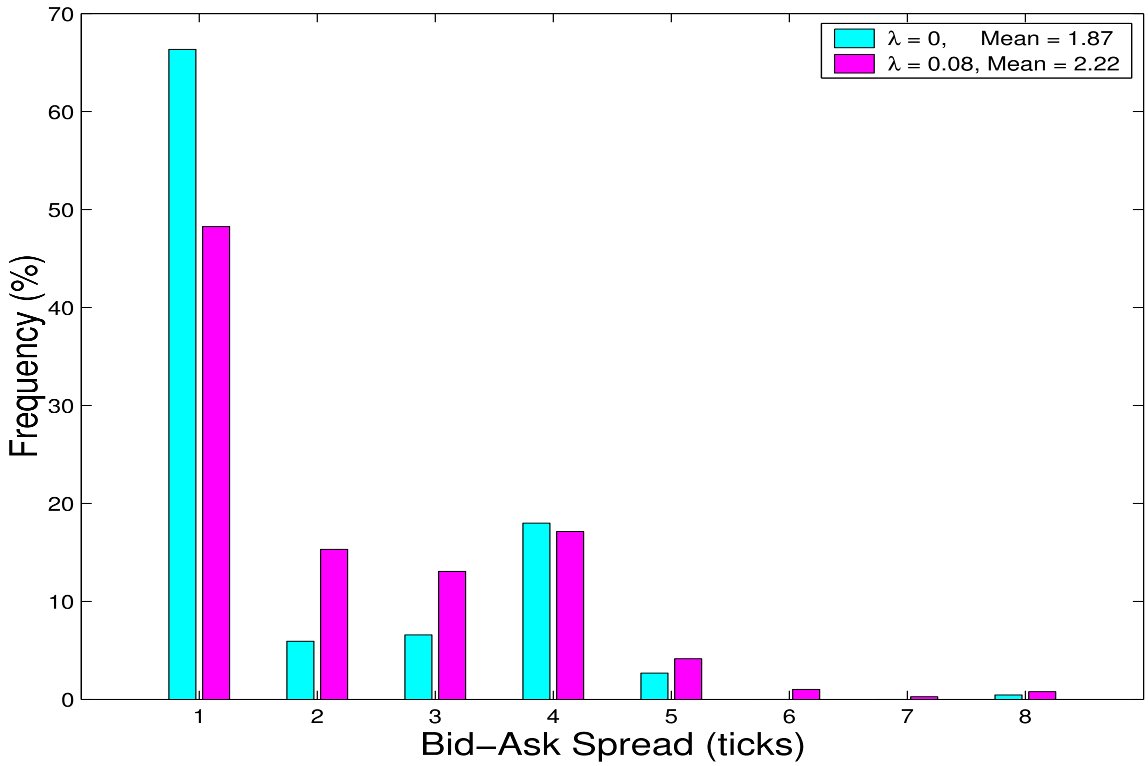

Figure 4: **Frequency distribution of quoted bid-ask spreads**

Our baseline simulation has 40,195 changes in the consensus value, of which 19,979 are increments and 20,216 are decrements. When the consensus value changes, limit orders execute either against incoming market orders or against the trading crowd. In either case, they face traders who now possess an informational advantage about the consensus value, and therefore their execution probability is higher in states with lower payoffs. This, of course, is the realization of the winner's curse. We illustrate this effect in Figure 5.

The horizontal axis of Figure 5 records the number of number of net changes in consensus value before a limit order is executed. The bars represent the proportion of executed limit buy or sell orders, using the left-hand scale. The lines represent the mean welfare change of the limit order submitter on execution, using the right-hand scale.

Limit order submitters optimally place their orders so that, on average, their payoff is still positive after two decrements in consensus value for a buy order and two increments for a sell order.[17] Nonetheless, they are subject to picking off risk: conditional on a limit buy (sell) executing, the average change in the consensus value is -0.17 ($+0.17$) ticks.

[17]This is consistent with the empirical evidence of Nyborg, Rydqvist, and Sundaresan (2002), who find evidence of bidders' compensating for the winner's curse in Swedish Treasury auctions.

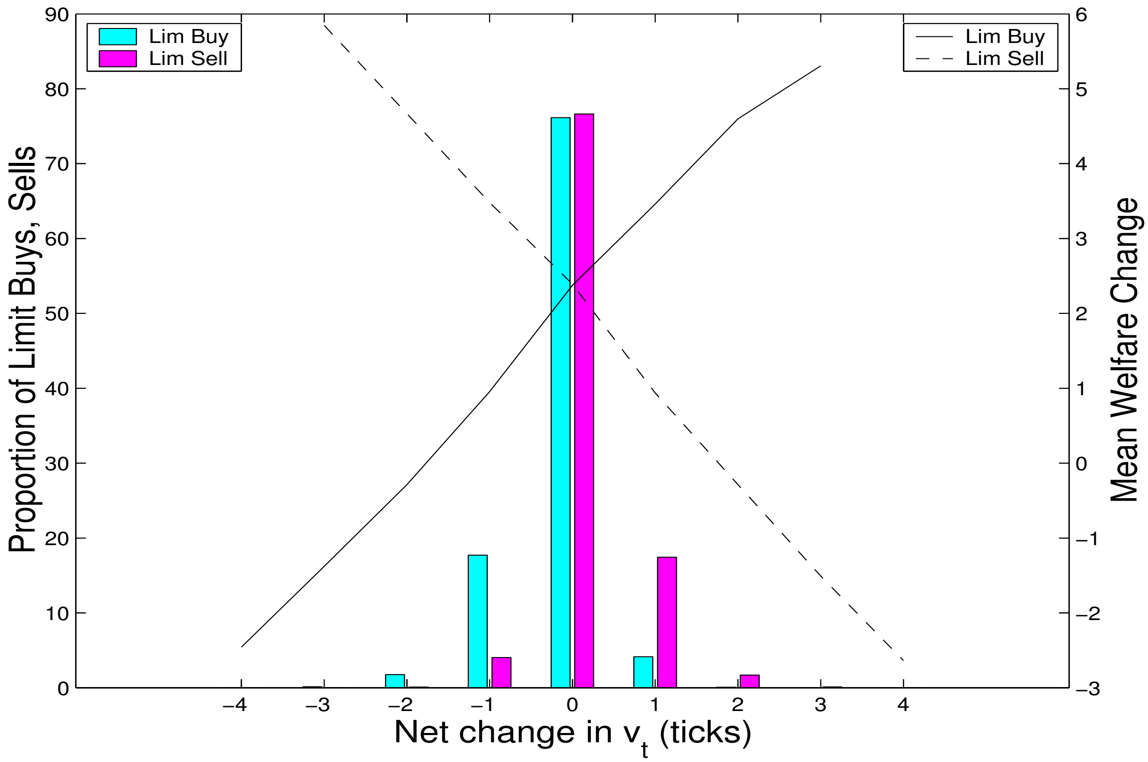

Figure 5: **Number of jumps before a limit order is executed, and trader surplus**

In total, 19.4% of all limit orders experience an adverse change in the consensus value before execution. However, consistent with ex ante optimization, the number of limit order traders who suffer from the "winner's curse" is small—on only about 4.16% of all limit orders does the submitter make a loss relative to his private value. Interestingly, this loss is counterbalanced by some traders (4.18% of all limit orders) who execute after a favorable change in the consensus value.

To determine the equilibrium effect of traders' compensating for the winner's curse, we compare equilibrium behaviors in our baseline case, and a market with no asymmetric information. First, we demonstrate that for a fixed book, agents are more likely to submit conservative orders (away from the consensus value) when the asset volatility is higher. A commonly encountered state is the empty book, which occurs in 0.78% of the periods in the base case. We consider the actions of agents with one share to trade, who enter the market when the book is empty. In Figure 6, we plot their actions against their private values, for the two markets considered.

Agents with $\beta < 0$ submit limit sells, and those with $\beta > 0$ submit limit buys. Further, agents with more extreme private values submit orders at the zero tick, while those with

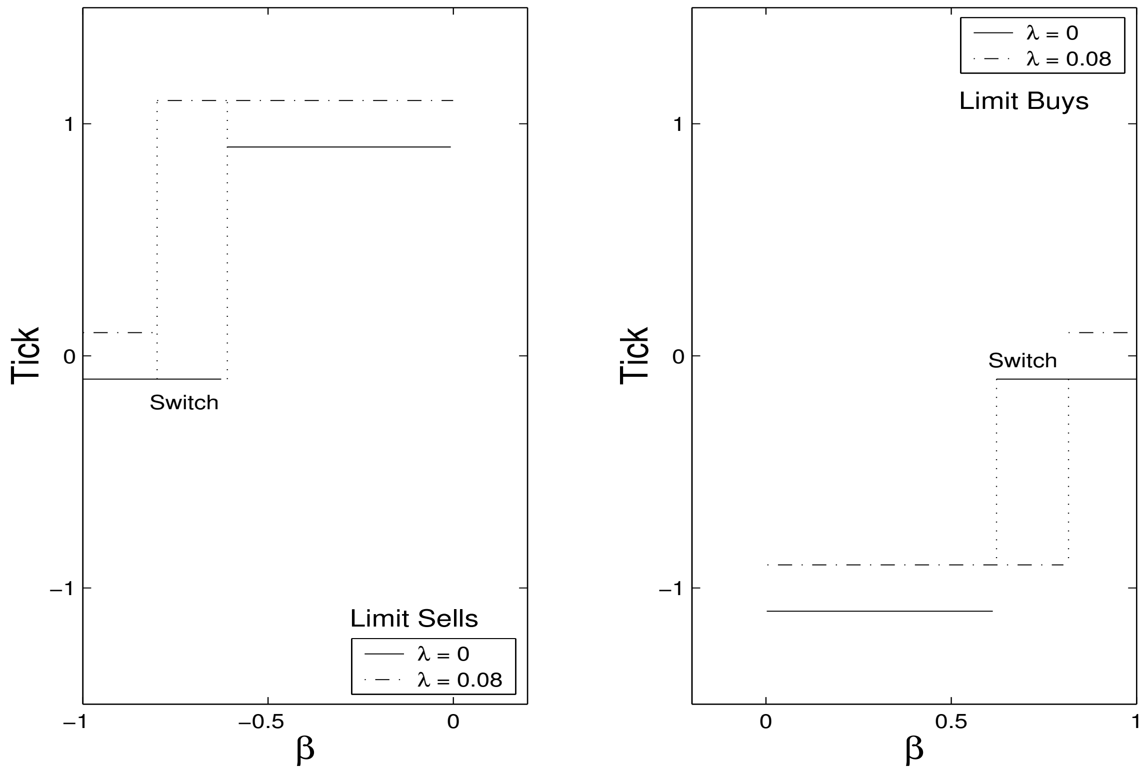

Figure 6: **Actions taken by agents with different** β

values closer to zero submit conservative orders one tick away (limit sells at p^1 or limit buys at p^{-1}). As the figure indicates, the range of agent types who submit conservative orders increases when $\lambda = 0.08$. In the figure, the regions labelled "Switch" consist of agents who submit orders at the zero tick when there is no change in the consensus value, but one tick away when $\lambda = 0.08$. This accords with the intuition that when the asset is more volatile, on average traders submit more conservative orders.

Next, we document differences in average order submission across the two markets. These occur both because agents use different strategies when faced with the same book, and because of equilibrium effects as a result of the different books that emerge. We document three such differences between the markets in Table 4. First, as we have observed, the average spread widens as the volatility in the consensus value increases, confirming the prediction of Foucault (1999). Second, limit orders are submitted at more conservative prices when picking off risk exists. This accounts for the likelihood of an adverse change in the consensus value before execution. Therefore, a limit buy is on average submitted at a price further below an agent's private value; that is, $\beta - p^i$ increases. Third, agents are more willing to accept market buy orders at higher prices, to avoid the winner's curse effect on limit orders. Therefore, $\beta - p^i$ decreases slightly for market buys.

	λ	
Mean (ticks) of	0	0.08
Quoted spread, all orders	1.87	2.22
$(\beta - p^i)$, limit buys	2.06	2.17
$(\beta - p^i)$, market buys	3.00	2.97

Table 4: **Means of quoted spread for all orders, and private value minus submission price for limit buys**

Table 5 shows that, in the absence of an adverse selection component, the book is thicker at the consensus value and one tick below it. However, above the consensus value (for buy orders), the book is thicker if the consensus value is volatile. Further, with no asymmetric information more limit buys are submitted at the consensus value and one tick below it. At all other prices, more orders are submitted if the consensus value is volatile. These average characteristics result from two effects. First, holding the book fixed, to compensate for the winner's curse, a trader fearing asymmetric information is more likely to submit buy orders below the consensus value. Second, spreads are wider when the asset is more volatile. Hence, traders who might submit market orders if spreads were narrow instead submit aggressive limit orders, e.g. limit buy orders above the consensus value. On average we find that above the consensus value more limit buy orders are submitted and the book is thicker.

	Buy Order Depth (no. of shares)		Limit Buy Submission Frequency (%)	
Tick	$\lambda = 0$	$\lambda = 0.08$	$\lambda = 0$	$\lambda = 0.08$
3	0.00	0.00	0.00	0.00
2	0.00	0.02	0.02	0.23
1	0.06	0.25	5.99	13.43
0	1.65	1.07	70.15	62.88
-1	1.41	0.98	23.84	23.38
-2	0.00	0.38	0.00	0.08
-3	0.00	0.13	0.00	0.00
Total	3.12	2.82	100.00	100.00

Table 5: **Buy orders in an average book (left) and frequency of ticks at which limit buys are submitted (right)**

5.2 The Winner's Curse and Market Orders

As we have observed, an immediate implication of the winner's curse faced by limit order submitters is that market order submitters are obtaining bargains. In particular, when a trader arrives at the market he is more likely to take liquidity when it is cheap, or even offered at a subsidy.

How likely are limit orders to execute in adverse circumstances? In Table 6, we examine the frequency of buy order submission based on changes in the consensus value.

Change in v_t	Market Buys			Limit Buys				Total
	Large	Small	Total	Agg.	At Quote	$<$ Quote	Total	Buys
Decrease	0.04	6.57	6.61	10.80	12.22	15.86	38.88	45.49
No Change	0.57	20.25	20.82	12.50	12.55	4.13	29.18	50.00
Increase	2.48	38.01	40.49	9.93	4.13	0.21	14.26	54.75
Overall	0.62	20.39	21.01	12.35	12.24	4.41	29.00	50.01

Table 6: **Frequency of buy orders conditional on changes in the consensus value**

As expected, market buy orders are much more frequent after an increase in the consensus value than after a decrease. This exemplifies picking off risk for limit orders in the book. If the consensus value of the asset increases then last period's ask becomes "too low," offering new traders a profitable opportunity. Of course, last period's bid is also "too low." This leads to fewer limit buys at or below the bid. As shown in the last column, the overall frequency of buy orders increases with an increase in the consensus value, implying that some trader types shift from sell to buy orders in this case.

On average, how much do market order submitters benefit from picking off limit order traders? Such profit opportunities decrease the cost of demanding liquidity. In the simulation, since prices are relative to the consensus value, we have a direct measure of the true transaction costs paid by a market order. A market order executing at tick i pays $v_t + p^i$, corresponding to a transaction cost of p^i for buy orders or $-p^i$ for sell orders. In Table 7, we report statistics on the transaction cost with and without the possibility of changes in the consensus value.

True Transaction Cost	Mean	Std. Dev.
No Change in v_t ($\lambda = 0$)	0.07	0.47
Base Case ($\lambda = 0.08$)	-0.18	0.68

Table 7: **True transaction costs**

In our base case, the true transaction costs when there is picking off risk are *negative*. On average, market orders execute at prices better than the true value of the asset. This happens for two reasons: first, in the presence of asymmetric information, market orders can pick off stale limit orders. Second, in equilibrium, spreads are wider and traders substitute between market orders and aggressive limit orders. Those that do submit market orders do so at a profit. This result is thus consistent with the limit order placement reported in Table 5. Notice, that the standard deviation reported in Table 7 is higher in the presence of asymmetric information: profit opportunities are not always available in the book.[18] Even with no changes in consensus value, the transaction cost is close to zero on average (and negative for some traders). This result re-emphasizes the endogeneity of order submission.

Table 8 shows that the transaction costs paid by market buy orders are increasing in β. This is because traders with low β have a willingness to pay close to the consensus value of the asset. Thus, they only submit market buy orders when transaction costs are negative (that is, the ask is below the consensus value). Only traders with extremely high valuations, those with β above 3, incur positive transaction costs. These traders are so desperate to trade they are willing to do so at a positive cost.

Range of β	No. of Buy Orders	True Transaction Costs (ticks)
-3.0 to -2.0	2	-3.00
-2.0 to -1.0	327	-2.11
-1.0 to 0.0	7,065	-1.11
0.0 to 1.0	21,469	-0.48
1.0 to 2.0	32,705	-0.27
2.0 to 3.0	31,234	-0.14
3.0 to 4.0	25,253	-0.02
4.0 to 5.0	17,466	0.07
5.0 and greater	21,627	0.16

Table 8: **Transaction costs paid by agents with different valuations, market buy orders only**

Traders submit market orders when prices are favorable, and limit orders when they are not. This intuition is appropriate in a limit order market in which it is agents can choose between market and limit orders. If there are substantial costs to flexibility, the transaction costs to market orders are likely to be higher. However, it does suggest that caution be exercised in calculating transaction costs in limit order markets.

[18]Hasbrouck (1993) suggests the standard deviation of difference between the efficient price and transaction price as a measure of market quality. In the context of our model this is just the standard deviation of the transaction cost.

5.3 The Midpoint as a Proxy for the Consensus Value

Given the endogeneity of order flow, is the midpoint of the bid-ask spread a reasonable proxy for the consensus value of the asset? To investigate this, we first examine the difference between the midpoint and the consensus value, $m_t - v_t$, over the 500,000 simulated periods of our base case. Figure 7 displays the frequency distribution of this difference.

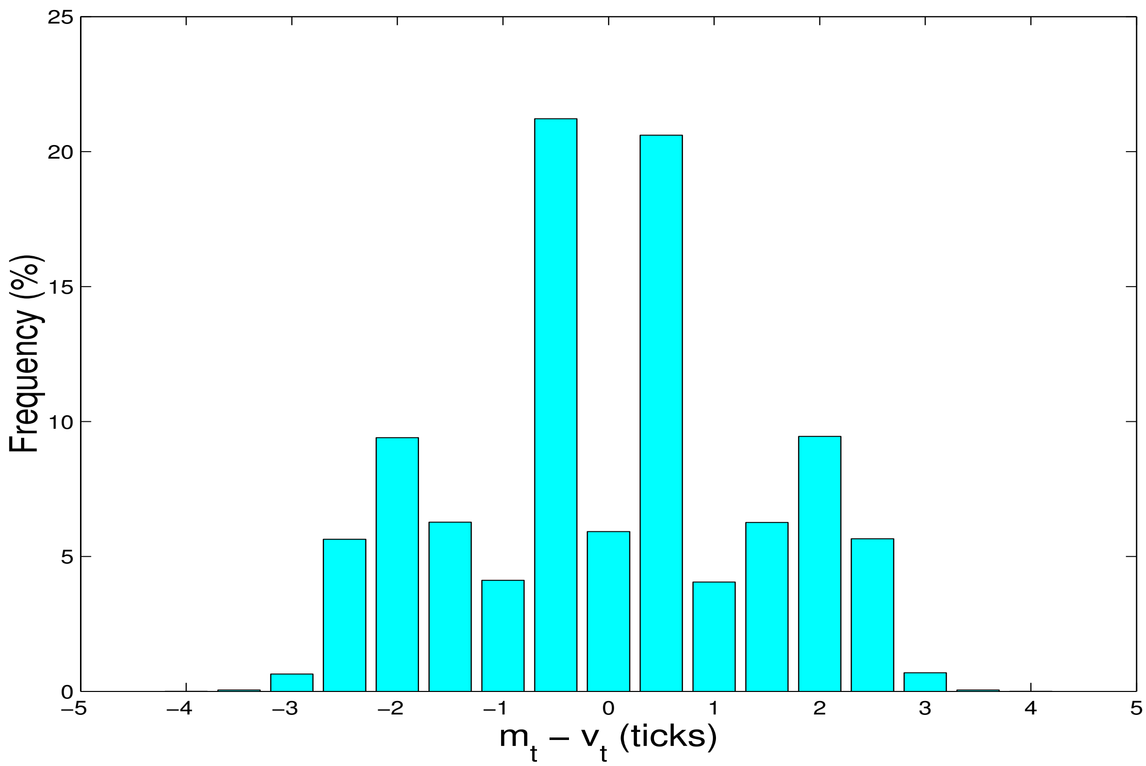

Figure 7: **Histogram of midpoint minus true value**

The mean of this measure, $m_t - v_t$, is 0.003, with a standard deviation of 1.232 ticks. Thus, the midpoint is an unbiased estimator of the consensus value. However, it is frequently incorrect, as shown in the figure. In fact, in about 22% of the periods, the bid-ask spread does not contain the consensus value. This can happen for at least two reasons. First, a trader may optimally submit a limit buy (sell) order above (below) the consensus value if the current ask is "too high" (too low). For example, in Table 5, we show that 13.6% of limit buys are submitted above the consensus value. Second, a change in the consensus value may render the current quotes stale.

In practice, the midpoint is often used to infer the consensus value when a transaction occurs, as in empirical measures of transaction cost. Thus, we next examine the difference between the midpoint and consensus value conditional on a market order being submitted. Since the effective spread is only measured for market orders, this yields a more direct sense of the validity of the condition $m_t = v_t$. Figure 8 plots the distribution of $(m_t - v_t)$ conditional on a market buy or a market sell in that period.

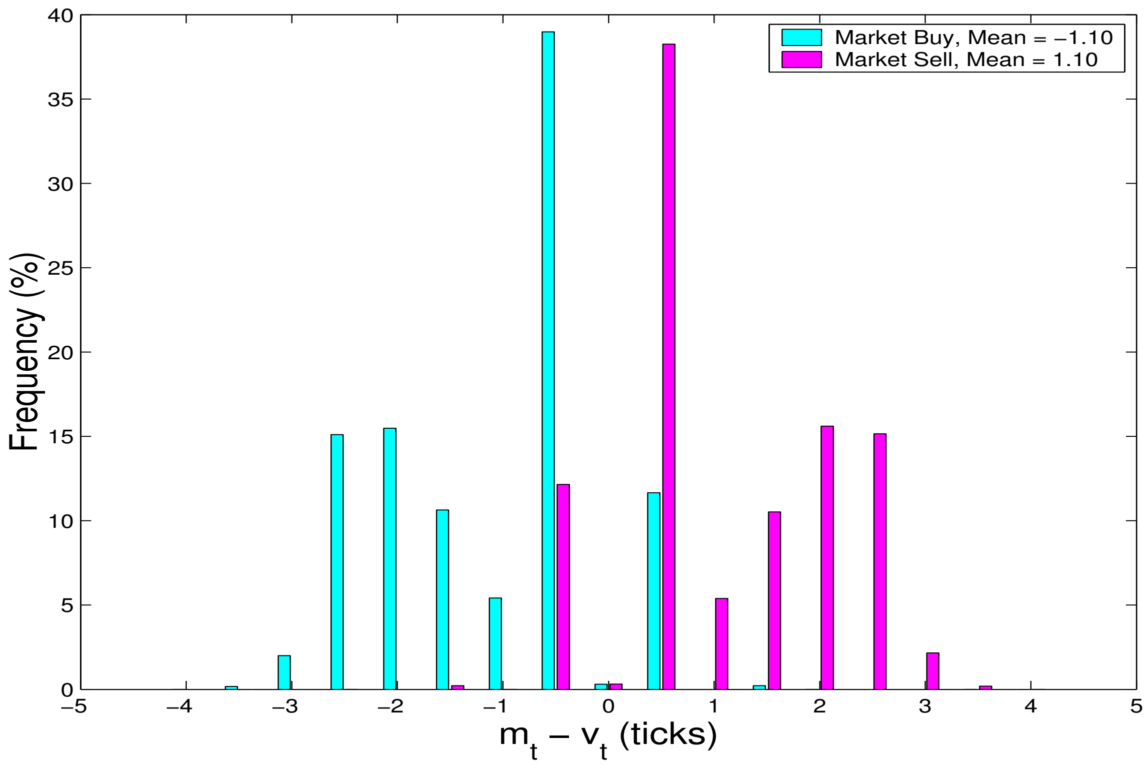

Figure 8: **Histogram of midpoint minus true value, conditional on trade**

¿From the figure, market buy orders are more likely when the midpoint is below the true value of the asset (representing a profitable buy opportunity), and sell orders more likely when $m_t > v_t$. Conditional on observing a market buy (sell), the true value of the asset is on average 1.10 ticks higher (lower) than the midpoint.

To examine the robustness of this result, we checked the corresponding figures for the case when there is no change in consensus value (i.e., $\lambda = 0$). In this case, conditional on a market buy (sell) the true value of the asset is 0.72 ticks higher (lower) than the midpoint of the bid-ask spread. Thus, this result is not solely due to stale limit orders but also because of the endogeneity of orders: market buy orders are more likely when prices are low, and sell orders when prices are high. We conclude that it is important to condition on the transaction in inferring the consensus value from the transaction price.

5.4 Inferences about Surplus and Transaction Costs

We have shown that the midpoint of the bid-ask spread is not a good proxy for the consensus value of the asset. From Proposition 1, we do not expect the effective spread to be a good proxy for either transaction costs or surplus in this situation. In fact, the correlation between true transaction costs and effective spread is -0.23. That is, when transaction costs are low, the effective spread is high. This happens because market orders are more likely when bid and ask quotes do not contain the true value, representing a profitable trading opportunity (that is, a negative transaction cost) on one side of the market. However, by definition, the effective spread is positive in all situations.

Frequently, the effective spread is used as a proxy for the welfare gain of a market order submitter. It would be a perfect proxy if its correlation with the surplus of market order submitters were -1. We next quantify how well effective spread performs as a proxy for market order surplus.

We break our sample into a thousand "trading days," (approximately four business years) each with 500 trader arrivals.[19] For each day, we calculate the volume, average per-share effective spread, total effective spread (which is the average effective spread times the volume on that day), and total surplus garnered by market and limit orders. Table 9 reports the day-to-day correlations of these measures.

	Surplus		
	Market Order	Limit Order	Total
Effective Spread	0.35	0.39	0.46
Average Effective Spread	0.23	0.30	0.32
Volume	0.52	0.43	0.60

Table 9: **Day-to-day correlations of effective spread and surplus**

The correlation between effective spread and surplus (for both market and limit orders) is actually positive. Ceteris paribus, the more desperate a trader is to trader (that is, the higher the β of a buyer), the more willing he is to execute at a worse price. In other words, a transaction consummated at a high effective spread suggests that the surplus obtained by the market order submitter is also high—a problematic finding for the use of effective spread as a surplus measure.

Notice the high correlation between volume and surplus. A higher trade volume must be correlated with higher surplus, since all trades are individually rational. Indeed, as a rule of thumb, volume appears to be a good proxy for surplus.

[19] Alternatively, one could view each subsample as a different stock, under the null hypothesis that trade in each stock is independent and identical. Lehmann and Modest (1994) characterize cross–sectional differences between liquidity provision in stocks on Tokyo.

6 Evaluating Policy Changes Across Different Regimes

Given that effective spread has been used to evaluate market design,[20] we perform two policy experiments and explicitly determine the surplus accruing to traders. Our goal is both to determine if changes in effective spread are a good proxy for changes in surplus across different regimes and to evaluate directly the policy experiments. The two experiments we consider are: (i) changing the tick size, and (ii) changing the standard deviation of the β distribution (i.e., the gains from trade).

6.1 Change in the Tick Size

Besides providing an evaluation of the effective spread, a tick size experiment has policy and market design implications. Both the theoretical and empirical literature are mixed on the effects of a tick size change on surplus. Seppi (1997), in an intermediated market suggests that small traders are better off under a small tick size, while large traders are at a disadvantage. Cordella and Foucault (1999), in examining competing market makers, find that transaction costs are minimized at a non-zero tick size.

Nasdaq and the NYSE, both intermediated markets, have changed their tick size in recent years. Empirical evidence on the effects of these reductions is mixed.[21] In pure limit order markets, there have been a few natural experiments: for example, Toronto moved to decimals in 1996. This change was analyzed by Bacidore(1997) who found that spreads fell but trading volume did not increase. Has the reduction in tick size been a Pareto improvement?

To answer this question, we compare two regimes—one with 9 ticks and one with 5. For computational ease in performing this comparative static, we make a slight modification to the base case. We do this so that the dollar magnitude of changes in the consensus value and the potential gains from trade are the same across the two regimes. In both cases, $\delta = 0.04$, and $\lambda = 0.08$. However, in one case we consider 9 prices in which each change in the consensus value is two ticks, compared to 5 prices in which the corresponding change is 1 tick. Thus, the dollar magnitude of the changes is the same.

The mean and the standard deviation of the β distribution is adjusted so that the same percentage of traders in both cases have valuations more extreme than the trading crowd.

[20] For example, de Jong, Nigman, and Röell (1995) and Venkataraman (2001) use effective spread to measure execution quality of orders on a pure limit order market, the Paris Bourse, with those on an intermediated market (respectively, SEAQ and the NYSE).

[21] The effects have been considered by, among others, Ahn et al. (1998), Bessembinder (1999), Bollen and Whaley (1998), Ronen and Weaver (2001), and Jones and Lipson (2001) who examine the effect on the transaction costs incurred by different parties after the move to "teenies." Goldstein and Kavajecz (2000) and Edwards and Harris (2001) explicitly examine the effect of halving the tick size on liquidity suppliers—the limit order book in the first case and the specialist's ability to "step ahead" in the second.

In particular, with 9 ticks, we use a mean of 0 and a standard deviation of 3 ticks. In the 5 tick case, we have a mean of 0 and a standard deviation of 1.5 ticks. This ensures that, in both cases, the standard deviation of β is $\frac{3}{16}^{th}$ of a dollar. We illustrate the β distributions in Figure 9.

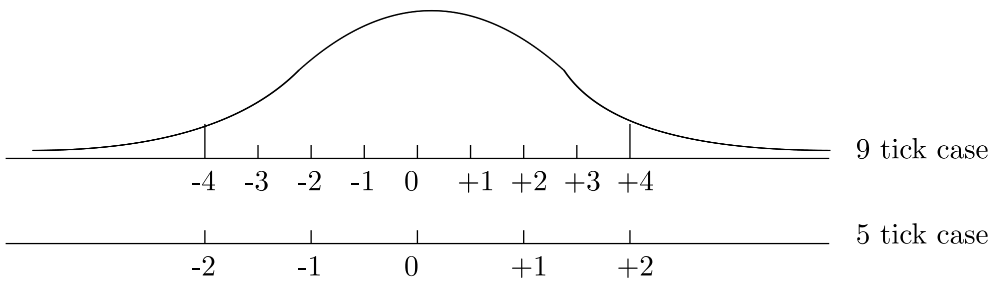

Figure 9: **Relationship of ticks to β distribution**

We report the results of this experiment in Table 10. For ease of comparison, all values are reported relative to the tick-size in the 9-tick model. We report the means of surplus and effective spread per available share and per executed share. For surplus, the mean per available share is the most relevant measure. We define the total number of "available" shares to be the sum over all traders of the maximal quantity an agent may trade; that is, $\sum_{t=1}^{500,000} z_t$. If a policy change results in fewer trades, the mean surplus per available share will fall, while the mean per executed share may rise. For policy prescriptions we should care about forgone trades. For effective spread, the mean across executed shares appears to be the most relevant measure, given its prominence in empirical work.

	Tick Size = $\frac{1}{8}$		Tick Size = $\frac{1}{16}$	
	Mean Per Share		Mean Per Share	
	Available	Executed	Available	Executed
Volume	0.408	1.000	0.420	1.000
Mkt Ord Surplus	1.118	2.736	1.219	2.902
Lim Ord Surplus	0.890	2.178	0.843	2.006
Total Surplus	2.006	4.916	2.062	4.908
Eff Spread	0.526	1.290	0.445	1.059

Table 10: **Results of change in tick size**

Using the 9-tick regime as a base case, effective spread per executed share rises by 18.2% when the tick size is doubled. However, the two regimes have roughly the same volume, and hence surplus. Total surplus per available share falls by 2.7%, and the surplus of market order submitters falls by 8.3%. In other words, a large change in the effective spread can occur despite a relatively small change in surplus. Again, the change in volume is a good proxy for the change in surplus; volume per available share falls by 2.9% as the tick size increases.

These results allow us to reconcile the empirical literature with the theoretical literature. Most of the empirical literature has found that a reduction in tick size leads to a reduction in spreads, and the inference has been drawn (albeit in intermediated markets) that, ceteris paribus, traders are better off. The theoretical literature has suggested that decreases in tick size are not always Pareto improving. Our results suggest that a decrease in effective spread does improve the surplus of market order submitters, but at the expense of limit order submitters. The change in aggregate surplus is negligible. We interpret our result in the light of order endogeneity. In a pure limit order market, the effect of a tick size change must be of second order. If supplying liquidity becomes too expensive, then agents demand liquidity and vice versa. A change that stopped trades from being consummated would affect surplus. Amending the tick size merely perturbs how the gains from trade are split. Any decrease in surplus comes about from limit orders that are cancelled unexecuted.

6.2 Change in the Gains to Trade

Even though the effective spread fails, if the bias is systematic, we can still use it to infer surplus. To see if this is the case, we perform another experiment in which we change the gains to trade for agents. Such a change could occur, for example, if the capital gains tax were reduced or if there were a fall in broker commissions. In our model, gains to trade are larger when traders have more dispersed private valuations. We consider a market in which the standard deviation of the β distribution is smaller—2 ticks instead of 3. Effectively, this implies reducing the gains to trade. All other parameter values are the same as in the base case.

	$\sigma_\beta = 2$ Mean Per Share		$\sigma_\beta = 3$ Mean Per Share	
	Available	Executed	Available	Executed
Volume	0.405	1.000	0.418	1.000
Mkt Ord Surplus	0.776	1.917	1.241	2.968
Lim Ord Surplus	0.593	1.465	0.889	2.126
Total Surplus	1.370	3.382	2.130	5.094
Eff Spread	0.332	0.819	0.382	0.914

Table 11: **Comparison of two β distributions**

As one might expect, if the gains to trade are larger, the surplus from consummated trade is higher. Indeed, there is a 50.6% increase in total surplus per available share, in moving from $\sigma_\beta = 2$ to $\sigma_\beta = 3$ (consistent with the notion that σ represents the gains to trade). However, the effective spread actually increases when the gains to trade increase. Per executed share, the effective spread increases by 15.1%. In this case, as may be expected, the change volume is a poor proxy for the change in surplus (volume per available share increases by 3.2%). A change in the gains to trade leads to an increase in surplus on every trade, and hence to a corresponding increase in surplus even when volume is held constant.

Thus, in our two policy experiments, effective spread goes in the right direction when the tick size changes, but in the wrong direction as the gains to trade change. Further, in the former case, the magnitude of the change in effective spread (18.2%) bears no relationship to the change in surplus (2.7%). We can only conclude that the effective spread can be a very misleading proxy for surplus. Changes in volume are a good proxy for changes in surplus, provided the gains to trade remain approximately the same.

7 Conclusion

The method we introduce opens the door to a class of more realistic models that are closer to existing institutions. The explicit calculation of investor surplus makes it particularly useful for evaluating policy experiments.

In this paper, we use our model to determine the implication of endogenous order submission for the relationships between transaction prices, transaction costs, trader surplus, and some of the commonly used proxies. We find that the midpoint of the quoted spread is an unbiased proxy for the consensus value on average in our symmetric model. However, conditional on a trade occurring it is not. We find that the effective spread is not a good measure of surplus because supply and demand of liquidity are endogenous. Thus, it should not be used to evaluate or motivate policy.

In terms of the model, there are many possible extensions such as including an intermediary, privately informed agents, or competing exchanges. Open questions include: What are reasonable proxies for surplus (to evaluate policy changes), transaction costs (to determine trading strategies), and the consensus value of the asset? Can these be inferred from real data? We hope to answer these questions in future work.

In addition to such market design and policy questions, this method should also be of use to practitioners. In particular, Lo, MacKinlay, and Zhang (2002) report that hypothetical limit order executions are poor proxies for actual ones, suggesting the need for a structural model. We suspect that if practitioners work with a calibrated model of liquidity demand and supply that includes endogenous order flow the predicted estimates of price impacts will be more accurate.

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