Copula-based multivariate models with applications to risk management and insurance

Marco Bee

Department of Economics - University of Trento

via Inama 5 - 38100 Trento, Italy

Phone +39-0461-882296, Fax +39-0461-882222

E-mail: marco.bee@economia.unitn.it

ABSTRACT

The purpose of this paper consists in analysing the relevance of dependence concepts in finance, insurance and risk management, exploring how these concepts can be implemented in a statistical model via copula functions and pointing out some difficulties related to this methodology. In particular, we first review the statistical models currently used in the actuarial and financial fields when dealing with loss data; then we show, by means of two risk management applications, that copula-based models are very flexible but sometimes difficult to set up and to estimate; finally we study, by means of a simulation experiment, the properties of the maximum likelihood estimators of the Gaussian and Gumbel copula.

KEY WORDS: Copula, Tail dependence, Loss model, Portfolio model.

JEL Classification: C15, C51, C63.

Mathematics Subject Classification: 62H20, 65C10, 65C60.

1 Introduction

Since the introduction of the mathematical theory of portfolio selection (Markowitz 1952) and of the Capital Asset Pricing Model (CAPM - Sharpe 1964, Lintner 1965 and Mossin 1966), the issue of dependence has always been of fundamental importance to financial economics. For the first time, these models gave a rigorous treatment of the intuitive concept of diversification, which is at the heart of finance, investment science and risk management, showing that it is strictly related to the linear dependence between pairs of securities and possibly between each security and one or more latent factors.

To introduce the problem, we start by recalling the basic version of the CAPM. The CAPM for the return r_i on a risky asset is summarized by the equation

$$r_i - r_f = \beta_i(r_M - r_f) + Z_i, \tag{1}$$

where r_f is the risk-free interest rate, r_M is the return on the market portfolio and $Z \sim N(0,1)$ (Campbell *et al.* 1997). As r_f is non-stochastic, (1) is a one-factor model, where the return on the market portfolio is the common factor; according to financial terminology, r_M conveys all of the *systematic* (or *non diversifiable*) risk. On the other hand, Z_i represents the *specific* (also called *diversifiable*, or *idiosyncratic*) risk, i.e. the risk which is inherent to the i-th asset, and cannot be "explained" by the common factor. From (1) it is easily seen that: (i) $\beta_i = \text{cov}(r_i, r_M)/\text{var}(r_M)$; (ii) $\text{var}(r_i) = \beta^2\text{var}(r_M) + \text{var}(Z_i)$. Thus the parameter β_i measures how much of the i-th asset's risk cannot be diversified away, and this systematic risk is determined by the covariance of the i-th asset's return with the market portfolio.

Besides being, from a theoretical point of view, a cornerstone of modern finance, the CAPM is remarkably similar to the most important portfolio model used in credit risk management. Referring the reader to section 4.1 for details, the so called Vasicek model (Vasicek 1987) is given by

$$r_i = \sqrt{\rho}Y + \sqrt{1-\rho}Z_i,$$

where r_i is the standardized asset return of the firm, the single factor Y represents global economic conditions, ρ is the correlation between the return and the factor and Z_i represents the idiosyncratic risk. The importance of this model has also been emphasized by the regulatory environment, as it is indeed the standard approach enforced to banks and financial institutions by the Basel II Accord for the computation of default probability and required capital. From the statistical point of view, it is not different from the CAPM; we will however see that the standard assumption of normality of Z_i is being abandoned in favor of fat-tailed distributions and, as a consequence, more sophisticated dependence concepts are called for.

A second field of research where non-standard dependence measures are required is the statistical analysis of losses. This is a classical topic in the actuarial field, whose development was mainly spurred by non-life insurance companies, for which "losses" are to be interpreted as "claims". There is therefore a large body of literature concerning the construction and analysis of univariate loss models (see Klugman *et al.* 1998 for a comprehensive account). More recently, increased attention to these techniques has come from the risk management field, because similar techniques are used to model operational losses; multivariate models for the loss frequency distribution have been introduced in the last few years and are mainly based on copulas.

As a consequence of the widespread use of the normal distribution, historically in finance "dependence" has often been identified with "correlation"; for example, in the CAPM model (1) the random variable Z_i is Gaussian, so that zero correlation between r_i and r_M implies independence of the two random variables. However, as different distributional assumptions replace normality, Pearson's linear correlation is no longer an appropriate measure of dependence. This was the source of some misunderstanding and confusion (Embrechts *et al.* 1999), and gave rise to a line of research which has grown impressively in recent years, investigating both financial models and statistical techniques.

As of today, dependence concepts are central to several fields of finance, ranging from the well known equilibrium models of financial markets to insurance and risk management. Here we will concentrate on the role of dependence in the methodologies developed for the analysis of insurance as well as operational and credit losses, and will show that a proper treatment of the non-standard multivariate distributions arising in all of these contexts requires the use of copulas.

The aim of this article is twofold: first, we will show that copulas are a tool which underlies essentially all models of dependence used in the various areas of risk management. Second, we will investigate the estimation and simulation of copulas and will provide examples where the aforementioned models play a key role; we will also point out possible theoretical and computational difficulties, arising in particular when working with large-dimensional data.

The rest of this paper is organized as follows. In section 2 we review the univariate models for loss data; Section 3 summarizes the fundamental mathematical facts about copulas and dependence; In section 4 we give theoretical details on the multivariate models used for risk management purposes; Section 5 analyses estimation and simulation of copulas, with examples based on both real and simulated data. Section 6 discusses the results and concludes.

2 A general framework for univariate loss data

In the classical approach to loss modeling the basic methodology consists of dealing separately with the two sources of randomness: the frequency and the severity of losses. The intuition is that, given a time horizon, the total loss is given by the number of losses and by their magnitude. More formally, the total loss over a predetermined time horizon is given by the random sum

$$S = \sum_{i=1}^{K} W_i, \tag{2}$$

where K is a random variable with a counting distribution and $W_1, \ldots, W_K$ are *iid* continuous positive random variables, also independent from K. The corresponding cdf is

$$\begin{aligned} F_S(w) &= P(S \leq w) \\ &= \sum_{k=0}^{\infty} p_k P(S \leq w | K = k) \\ &= \sum_{k=0}^{\infty} p_k F_W^{*k}(w), \end{aligned}$$

where $p_k = P(K = k)$ and $F_W^{*k}(w)$ is the k-fold convolution of the cdf of W; the distribution of S is commonly known as a compound distribution.

A more general and elegant formulation of this approach requires the use of some concepts from continuous-time stochastic processes. In the latter setup the loss process $S(t),\ 0 \leq t \leq T$ is a compound counting process, represented as

$$S(t) = \sum_{k=1}^{K(t)} W_k, \qquad 0 \leq t \leq T, \tag{3}$$

where $K(t)$ is a counting process and $W_1, \ldots, W_K$ are *iid* continuous r.v.'s. The pair $\{t_k, W_k\}_{k \in \mathbf{Z}}$, where t_k are the points of jump and $\mathbf{Z}$ is the set of integers, is known as a marked point process. If we introduce the hypothesis that $K(t)$ is a Poisson process with parameter ν, (3) is called a Poisson compound process; from the basic properties of the Poisson process (see, for example, Durrett 1996, pag. 145), if we fix $t^* \in [0, T]$, the compound counting process (3) reduces to the compound distribution (2), where the parameter of the Poisson distribution is $\lambda = \nu \cdot t^*$.

The statistical analysis of (2) is not easy. In general, the joint distribution $f_{K,W}(k, w)$ of the two sources of randomness K and W cannot be obtained analytically, and even applying Monte Carlo simulation requires the decomposition of the joint distribution of K and S: $f_{K,S}(k, s) = f_{S|K}(s|k) \cdot f_K(k)$. Conditionally on $K = k$, the model for the aggregated loss is given by the random variable

$$(S|K = k) = \sum_{i=1}^{k} W_i,$$

which may or may not be analytically tractable depending on the distribution of W, but can easily be simulated as long as we can simulate K and W. If $W \sim Logn(\mu, \sigma^2)$, and conditionally on $K \sim P(\lambda)$, this distribution is the convolution of k lognormal r.v.'s, and random number generation from $f_{K,S}(k, s)$ can be accomplished by performing the following steps:

(i) simulate a random number k from the $P(\lambda)$ distribution;

(ii) simulate k random numbers $w_1, \ldots, w_k$ from the $Logn(\mu, \sigma^2)$ distribution and compute $s = \sum_{i=1}^{k} w_i$.

Repeating B times these two steps, we get a random sample of size B from the density $f_{K,S}(k, s)$. This is the approach commonly adopted by banks to compute Capital at Risk

(CaR), which is given by the $(1-\alpha)$-th quantile of the empirical distribution minus its mean.

From the point of view of statistical inference, the parameters of the frequency and of the severity distribution can be estimated separately. However, a further difficulty comes from the fact that loss data are often left-censored or, more frequently, left-truncated; according to whether data are truncated or censored, specific inferential procedures are needed. When using maximum likelihood, for example, the specification of the likelihood function is different in the censored and truncated case, and usually its maximization can only be performed numerically (see Bee 2005 for a detailed analysis).

We already mentioned in the introduction that this approach to the construction of loss models was developed in the non-life insurance field, where it is assumed that insurance losses are not related to underlying financial variables. However, these models are also used in two areas of risk management: operational risk and credit risk. Operational losses are defined as all losses except the financial ones, so that, by definition, the methodology just described is perfectly appropriate. As for credit risk, the family of *reduced-form* models (which share with *structural* models the stage of portfolio credit risk analysis) adopts an approach to the assessment of default probability which does not investigate the financial or macroeconomic causes of default, but only assumes an exogenous process governing the default probability, calibrated to historical or current data. The most important commercial implementation of the reduced-form family is CreditRisk+^{TM}, which is based on the methodology discussed in the present section.

3 Copulas and Dependence Concepts

The concept of copula was introduced long ago (Sklar 1959), but only recently its potential for applications has become clear. A thorough treatment of copulas as well as of their relationship to the different concepts of dependence is given by Joe (1997), to which we refer the reader for theoretical issues; see also Nelsen (1998). A review of applications of copulas to finance can be found in Embrechts *et al.* (2003) and in Cherubini *et al.* (2004).

The copula is a multivariate distribution function whose univariate margins are $U(0,1)$. If $\mathbf{X}$ is a continuous p-variate random vector whose distribution function is $F(x_1,\ldots,x_p;\boldsymbol{\theta})$

with j-th margin $F_j(x_j; \theta_j)$, Sklar's theorem says that there is a unique distribution function $C : [0,1]^p \to [0,1]$ that satisfies

$$F(x_1, \ldots, x_p; \boldsymbol{\theta}) = C[F_1(x_1; \theta_1), \ldots, F_p(x_p; \theta_p); \boldsymbol{\alpha}]. \tag{4}$$

$C(u_1, \ldots, u_p; \boldsymbol{\alpha})$ is called the copula associated to F and $\boldsymbol{\alpha}$ is the vector of its parameters.

It is worth stressing that, given $C : [0,1]^p \to [0,1]$, there is a unique distribution function F such that (4) holds. It follows that, for any choice of the univariate distribution functions $F_j(x)$, using the same copula we obtain a different multivariate distribution.

From the inferential point of view, parameters can be estimated simultaneously with the Full Maximum Likelihood (FML) method; however, for practical applications the method of Inference Functions for Margins (IFM; Joe 1997, sect. 10.1) is more interesting; it is based on the following two steps:

(i) Separate estimation of the parameters of the univariate marginal distributions $F_1(x_1; \lambda_1)$, $\ldots, F_p(x_p; \lambda_p)$.

(ii) Estimation of the parameters of the copula given the estimates obtained in the preceding step:

$$\hat{\boldsymbol{\alpha}} = \arg\max \sum_{i=1}^{N} \log[c(F_1(x_{1i}; \hat{\lambda}_1), \ldots, F_p(x_{pi}; \hat{\lambda}_p)].$$

In particular, IFM is usually much more convenient from a computational point of view, because $p+1$ "simple" optimizations are likely to be more feasible than a single "difficult" optimization. In general, MLE's obtained with FML and IFM are different, and assessing the efficiency of the estimators is not trivial. This problem was studied by Xu (1996; see also Joe 1997, sec. 10.1.2) and the conclusion is that the IFM method is highly efficient compared to the FML method, in the sense that the relative efficiency, as measured by the ratio of the mean squared error of the IFM estimators to the FML estimators, is close to one.

In this paper we will use the Gaussian and the Gumbel copula. Their distribution functions in the bivariate case are respectively given by

$$C(u_1, u_2; \mathbf{R}) = \Phi_{\mathbf{R}}(\Phi^{-1}(u_1), \Phi^{-1}(u_2)),$$
$$C(u_1, u_2; \delta) = e^{-(\bar{u}_1^\delta + \bar{u}_2^\delta)^{1/\delta}},$$

where $\mathbf{R}$ is the correlation matrix, Φ is the standard normal cdf, $\Phi_{\mathbf{R}}$ is the multivariate normal cdf with parameters $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \mathbf{R}$, $\delta \in [1, +\infty)$ and $\bar{u} = -\log(u)$. The corresponding densities are

$$c(u_1, u_2; \mathbf{R}) = \frac{1}{|\mathbf{R}|^{1/2}} \exp\left(-\frac{1}{2}\boldsymbol{\zeta}'(\mathbf{R}^{-1} - \mathbf{I})\boldsymbol{\zeta}\right), \quad \zeta_j = \Phi^{-1}(u_j),$$

$$c(u_1, u_2; \delta) = C(u_1, u_2)(uv)^{-1} \frac{(\bar{u}\bar{v})^{\delta-1}}{(\bar{u}^{\delta} + \bar{v}^{\delta})^{2-1/\delta}} [(\bar{u}^{\delta} + \bar{v}^{\delta})^{1/\delta} + \delta - 1].$$

Contour plots of the Gaussian and Gumbel copulas with Student t_2 marginals are shown in figure 1 for the three values of Kendall's τ which will be used in the applications (see section 5 for details).

Fig. 1a: Gaussian copula, τ=0.3, t_2 marginals

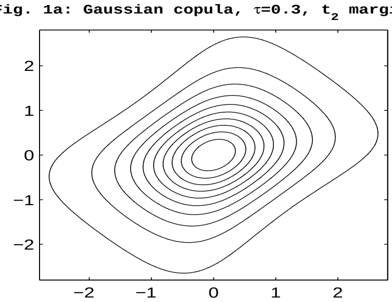


Fig. 1b: Gumbel copula, τ=0.3, t_2 marginals

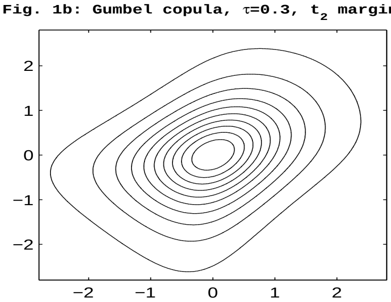


Fig. 1c: Gaussian copula, τ=0.5, t_2 marginals

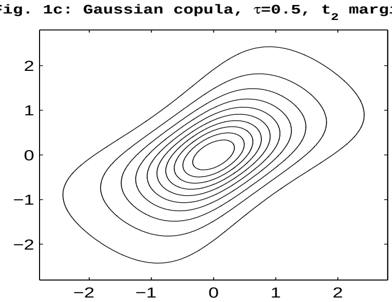


Fig. 1d: Gumbel copula, τ=0.5, t_2 marginals

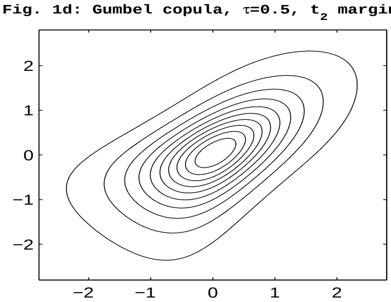


Fig. 1e: Gaussian copula, τ=0.7, t_2 marginals

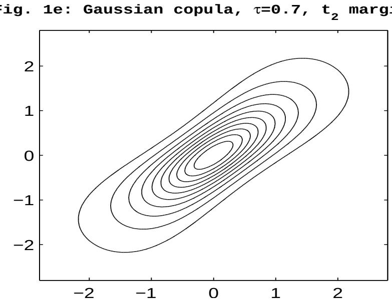
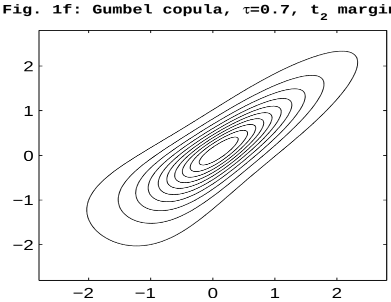


Fig. 1f: Gumbel copula, τ=0.7, t_2 marginals



Concerning the estimation of $\mathbf{R}$ in the Gaussian copula, we can exploit the following simple relationship between the linear correlation coefficient and Kendall's tau (Lindskog *et al.* 2003):

$$\rho = \sin\left(\frac{\pi}{2}\tau\right). \tag{5}$$

For other copulas, the relationship between τ and the parameter of the copula has been derived by Monte Carlo simulation (Joe 1997, tab. 5.1). When data are non-elliptically distributed, computing Kendall's tau for the sample and then estimating, by means of (5), the off-diagonal elements of $\mathbf{R}$, was shown to give a robust and asymptotically efficient estimator (Lindskog 2000), so that we will follow this strategy for the estimation of $\mathbf{R}$. As for the estimation of δ in the Gumbel copula, we use the method of maximum likelihood; MLE's cannot be obtained in closed form, because maximization of the likelihood function has to be performed numerically: here we use the Nelder-Mead simplex algorithm.

Example 1. If we use Poisson marginals and the Gaussian copula, estimation of parameters is straightforward: one has first to estimate the Poisson parameters:

$$\hat{\lambda}_i = \frac{1}{N}\sum_{j=1}^{N} x_{ij} \qquad i = 1, \ldots, p.$$

Then, it is necessary to compute

$$u_i = F(x_i; \hat{\lambda}_i),$$

where $F(x_i; \lambda)$ is the Poisson distribution function with parameter λ. The u_i's can be used to estimate the sample version of τ:

$$\hat{\tau} = \frac{c-d}{c+d},$$

where c and d are respectively the number of concordant and discordant pairs. Finally, by means of the identity (5) we get the off-diagonal elements of $\mathbf{R}$. ∎

As for simulation, it is based on the following steps:

(1) Simulate $\mathbf{u}$ from the chosen copula C;

(2) Compute $y_i = F_i^{-1}(u_i; \hat{\boldsymbol{\theta}}_i)$ for any $i = 1, \ldots, p$, where $F_i(\cdot)$ is the cdf of the i-th marginal distribution;

(3) Repeat steps (1)-(2) a large number of times M.

Example 2. In the case of the Gaussian copula with marginals F_i this procedure specializes as follows:

(1a) Simulate a random vector $\mathbf{x} = (x_1, \ldots, x_p)'$ from the $N_p(\mathbf{0}, \hat{\mathbf{R}})$ distribution;

(1b) Evaluate $u_i = \Phi(x_j)$, for any $i = 1, \ldots, p$;

(2) Compute $y_i = F_i^{-1}(u_i; \hat{\boldsymbol{\theta}}_i)$ for any $i = 1, \ldots, p$;

(3) Repeat steps (1)-(2) a large number of times M. ■

With most copulas, step (1) requires a numerical root finding procedure (Joe 1997, pag. 146), so that it is considerably more involved than example 2; we will come back to this issue in section 5, where we will give the details of the simulation of the Gumbel copula.

The most recent developments in quantitative finance have spurred the interest towards alternative measures of dependence. The need for more sophisticated tools is related to the fact that multivariate models often do not belong to the class of elliptical distributions, which is the only one for which linear correlation has desirable properties. Consequently, other measures of dependence have been considered; while we refer the reader to Joe (1997) for a thorough review of dependence concepts and measures, we focus here on tail dependence, which seems to be particularly relevant to risk management and to the choice of the appropriate copula.

Definition. *Let C be a bivariate copula and $\bar{C}(u,u) = 1 - 2u + C(u,u)$, the so called survival copula. If*

$$\lim_{u \to 1} \bar{C}(u,u)/(1-u) = \lambda_U$$

exists, then C has upper tail dependence if $\lambda_U \in (0,1]$ and no upper tail dependence if $\lambda_U = 0$.

More explicitly, in the bivariate case, if $(U_1, U_2) \sim C$, we have

$$\lambda_U = \lim_{u \to 1} P(U_1 > u | U_2 > u) = \lim_{u \to 1} P(U_2 > u | U_1 > u).$$

The intuition is that, if $\lambda_U > 0$, there is a positive probability that one of U_1, U_2 takes values greater than u given that the other is greater than u for values of u arbitrarily close to 1. Tail dependence has been investigated exhaustively in the last few years (Junker and May 2002, Malevergne and Sornette 2003, Frahm *et al.* 2004); we will see in section 5 that choosing a copula with tail dependence has relevant consequences as concerns the modeling of joint extreme events.

4 Models of dependence in insurance and risk management

Having introduced the tools required to tackle the multivariate problems, we now turn to a description of the models used in insurance and in credit and operational risk management.

4.1 Insurance and operational risk

The usual approach to CaR computation in the multivariate setup assumes independence of losses in different business lines; the model we are going to propose assumes instead that the frequencies of losses among different business lines are correlated. Here the approach based on factor models does not seem to be reasonable, in view of the fact that losses in different business lines are unlikely to be determined by a common risk factor. On the other hand, multivariate Poisson densities are essentially intractable for dimension $p > 2$ (Johnson *et al.* 1997, chap. 37); for this reason it is preferable to use copulas. Formally, this implies that the joint loss frequency distribution is given by

$$F(x_1, \ldots, x_p) = C(F_1(x_1); \ldots; F_p(x_p)),$$

where F_i are $Pois(\lambda_i)$ cdf's. Recently, copula-based models for multivariate count data have been proposed by Embrechts *et al.* (2003), Lindskog and McNeil (2003), Neslehova and Pfeifer (2004) and Chavez-Demoulin *et al.* (2005).

4.2 Credit risk

The basic idea underlying credit risk models is that corporate defaults are likely to depend on a systematic risk factor which can be identified as "the state of the economy": it has indeed be shown empirically that defaults are more frequent in periods of recession and less frequent when the economy expands. This way of reasoning also allows to overcome the problem that historical observations of defaults are not numerous enough to estimate directly a measure of default correlation.

Portfolio credit risk models are a very active research area; here we restrict ourselves to the so called structural models, which are usually based on asset value models, and where the role of copulas is more easily interpretable. As we noticed in section 2, the other important class of models is the reduced-form family; however, we do not make any attempt of giving details about the latter class of models: see Bluhm *et al.* (2002) for a thorough analysis of credit risk models and Gordy (2000) for a comparative analysis; see also Frey and McNeil (2003), who provide further insights into the relationship between structural and reduced-form models.

Structural models, also known as latent variable models, are based on the idea, borrowed from Merton's model (Merton 1974), that a counterparty defaults if, at the end of the period considered for the analysis, the asset value falls below a certain threshold, which in most cases is given by the face value of its liabilities. More formally, consider a portfolio of p counterparties, and a time horizon $[0, T]$, where T is usually equal to one year. Let D_i, $i = 1, \ldots, p$, be the default indicator for obligor i, taking value 1 if at time T the obligor is in default and 0 otherwise.

Let then $\mathbf{r} = (r_1, \ldots, r_p)'$ be the vector of latent variables and $\mathbf{S} = (S_1, \ldots, S_p)'$ be the vector of cut-off levels. In a Merton-type model, the latent variables are interpreted as log-returns of counterparties' asset values: $r_{it} = \log(A_{it}/A_{i0})$, where A_{it} is the asset value process. For notational simplicity, let now $r_i \equiv r_{i,T}$ be the return corresponding to the time horizon T. The thresholds are such that, if r_i falls below S_i at time T, the company defaults. It is therefore clear that the following relation holds:

$$D_i = 1 \Leftrightarrow r_i \leq S_i,$$

so that the time-T *unconditional default probability* of the i-th company is given by

$$\pi_i = P(r_i \leq S_i).$$

It follows that the default indicator variable has a Bernoulli distribution: $D_i \sim Bin(1; \pi_i)$. In the Merton model, which is based on the lognormal hypothesis for the asset value process A_t

and uses an option-theoretic argument to derive the distribution of A_T, this probability can be easily computed by means of the normal cumulative distribution function, as a function of the asset value process A_t, its mean rate of return and volatility, the default threshold and the proportional cash payout rate; see, for example, Duffie and Singleton (2003, pag. 53-54). Finally, the total number of defaults in the portfolio is given by the sum of the default indicators:

$$L = \sum_{i=1}^{p} D_i.$$

Consider now the standard model of the log-returns used in credit risk analysis, the so called Vasicek model (Vasicek 1987):

$$r_i = \sqrt{\rho} Y + \sqrt{1-\rho} Z_i, \qquad i = 1, \ldots, p, \tag{6}$$

where $Y \sim N(0,1)$, $Z_i \sim N(0,1)$, $\text{cov}(Z_i, Z_j) = 0$, when $i \neq j$, $\text{cov}(Z_i, Y) = 0$. It can be verified that $\mathbf{r} \sim N_p(\mathbf{0}, \mathbf{R})$, i.e., $r_i \sim N(0,1)$, $i = 1, \ldots, p$ and $\text{cov}(r_i, r_j) = \text{corr}(r_i, r_j) = \rho$, $i \neq j$. In other words, each r_i is standard normal and any pair (r_i, r_j) has correlation ρ; in multivariate analysis, the matrix $\mathbf{R}$ is known as the *equicorrelation matrix*, and the model (6) (also called *equicorrelation model*) is a repeated measurement model; see, for example, Flury 1997, example 8.4.6.

In our setup, (6) is a one-factor model where the factor Y represents global economic conditions. Under the normality assumption, the unconditional default probability is given by $\pi_i = \Phi(S_i)$, from which we get $S_i = \Phi^{-1}(\pi_i)$; some simple algebraic manipulations show that the *conditional default probability* (conditional on Y), is given by

$$\pi_i(Y) = \Phi\left(\frac{\Phi^{-1}(\pi_i) - \sqrt{\rho} Y}{\sqrt{1-\rho}}\right), \qquad i = 1, \ldots, p.$$

The following result (Bluhm *et al.* 2002, pag. 112) clarifies the importance of copulas in this framework.

Proposition 1. *Given Bernoulli default indicators $D_1, \ldots, D_p$, the distribution of the total number of defaults $L = \sum_{i=1}^{p} D_i$ is uniquely determined by the set of default probabilities $\pi_i = F_i^{-1}(S_i)$ and by the copula C of $r_1, \ldots, r_p$.*

The proof is almost immediate: the distribution of $\mathbf{r}$ is F, where F is a p-variate distribution function. It follows that

$$\begin{aligned} P(D_{i_1} = 1, \ldots, D_{i_k} = 1) = P(r_{i_1} \leq S_{i_1}, \ldots, r_{i_k} \leq S_{i_k}) = \\ F_k(S_{i_1}, \ldots, S_{i_k}) = C_{i_1,\ldots,i_k}(F_{i_1}^{-1}(S_{i_1}), \ldots, F_{i_k}^{-1}(S_{i_k})), \\ = C_{i_1,\ldots,i_k}(p_{i_1}, \ldots, p_{i_k}), \quad \text{with } \{i_1, \ldots, i_k\} \subset \{1, \ldots, p\}, \end{aligned}$$

where $C_{i_1,\ldots,i_k}$ is the k-th dimensional marginal distribution of the copula C.

Another area of credit risk where non standard models of dependence are important is the pricing of credit derivatives. Credit derivatives are instruments that help financial institutions to manage their credit-sensitive investments; more precisely, a credit derivative can be defined as a contingent claim whose payoff depends on the "default / no default" state (or, more generally, on the credit quality) of an underlying security. *Credit Default Swaps* (CDS) are a relevant example, as they are probably the most important credit derivative in the market. A CDS provides insurance against a particular company, called *reference entity*, defaulting on its own debt: by means of such a contract, a counterparty (*buyer*), pays periodically to the other counterparty (*seller*) an amount of money (*CDS spread*) and is protected by the seller in case the reference entity defaults. Here, protection means that if the credit event takes place, the buyer has the right to deliver to the seller a bond issued by the reference entity in exchange for its face value.

Computing risk measures for portfolios of CDS entails the modeling of the joint distributions of CDS spreads; as the univariate distributions show a large excess kurtosis, and have to be modelled by means of non-standard distributions, this usually requires the use of copulas; see Bee (2004) for an application.

The study of dependence is even more important in the case of *basket credit derivatives*; a *basket default swap*, in the standard version, is a single contract that offers protection against the event of the k-th default in a basket of p ($p \geq k$) counterparties. Without entering into details here (see Bluhm *et al.* 2002, sect. 7.3), in this setup it is necessary to model the joint default probabilities (or, more precisely, the *correlated default times*) of k counterparties, a task for which copulas are the most commonly used tool.

5 Applications

In this section we present two applications related to the frameworks introduced in the preceding sections. In both of the following applications we use the Gaussian and the Gumbel copula: whereas the first one is considered a sort of benchmark, because it has no tail dependence and implies a "linear" dependence structure (in the sense that it can be summarized by the Pearson linear correlation coefficient), the latter exhibits tail dependence, which is a very important property in the financial field.

5.1 The actuarial setup

Our purpose here consists in investigating the relevance of loss frequency correlation: in particular, we will show that it is not relevant to CaR computation, whereas it can be important in non-life insurance.

In this example, we will use bivariate operational loss data consisting of losses of two business lines recorded in Banca Intesa in a recent year; for confidentiality reasons, we will just call them "Business Line A" and "Business Line B". Our main interest here consists in identifying the features of the dependence structure; therefore, as concerns the univariate distribution, we assume the appropriateness of the Poisson-lognormal model. However, losses are truncated, a feature which must obviously taken care of for inference purposes. More precisely, we use the maximum likelihood estimation procedure proposed by Bee (2005).

Unfortunately, our data only consist of the total number of losses in one year: while this is enough to estimate the parameter λ of the daily frequency distribution, this does not allow to compute a measure of correlation on a daily horizon. Thus, we decided to simulate data by means of a procedure based on the estimated marginal distributions. To this aim, 10000 bivariate frequencies were simulated from:

(i) a Gaussian copula with $\rho = 0.454$, 0.707 and 0.891, corresponding respectively to $\tau = 0.3$, 0.5 and 0.7 (here we used again the numerical values of table 5.1 in Joe 1997, pag. 146; these equivalences can also be obtained analytically by means of the identity (5));

(ii) a Gumbel copula with $\delta = 1.43$, 2 and 3.33, corresponding respectively to the same values of τ employed with the Gaussian copula, i.e. $\tau = 0.3$, 0.5 and 0.7 (see again table

5.1 in Joe 1997).

In both cases, the marginals are taken to be Poisson with parameters $\lambda_1 = 0.825$ and $\lambda_2 = 2.24$, i.e., the MLE's obtained from the data at hand.

The joint frequency distribution is shown in figure 2 for the Gaussian copula (along with the uncorrelated case) and in figure 3 for the Gumbel copula. It can be seen that, as the correlation increases, the frequencies of the events with an high number of losses for both business lines increase as well. This is confirmed by the results in tables 1 and 2.

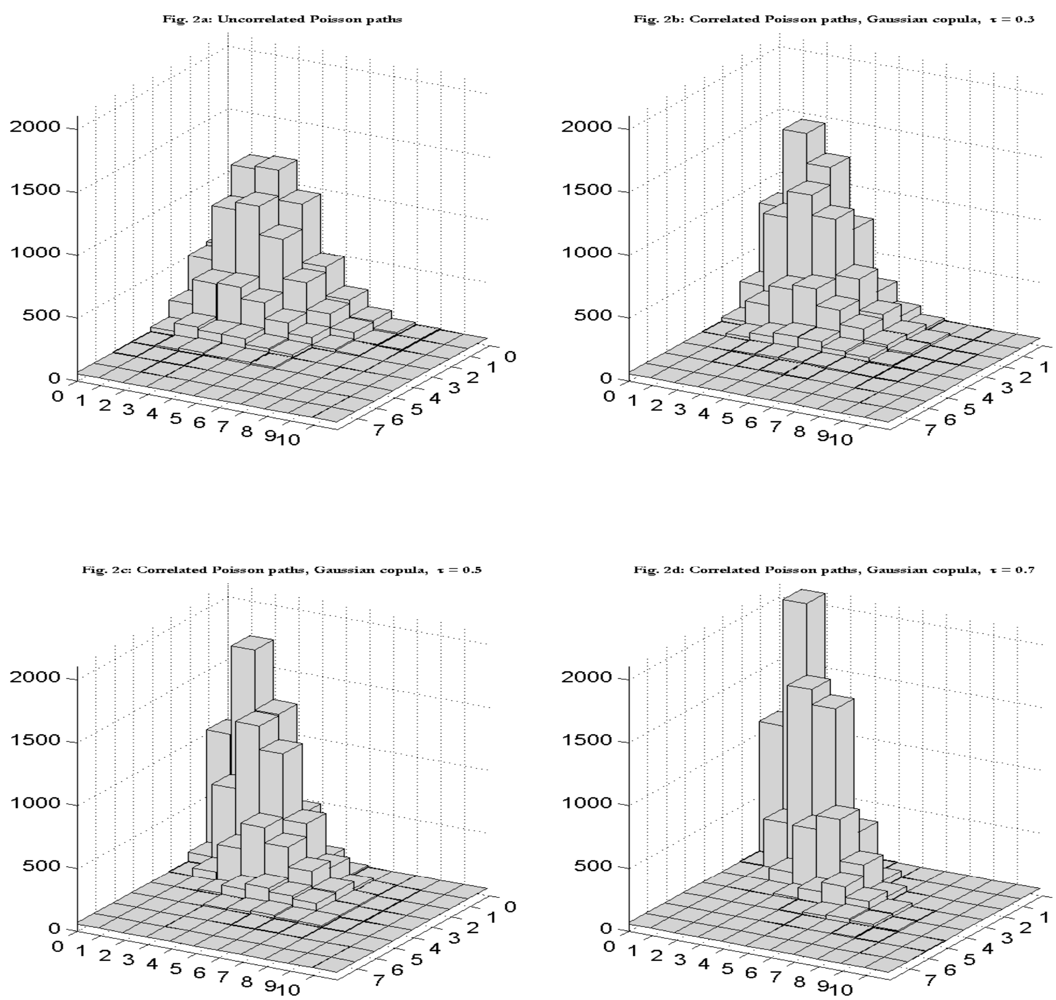


Fig. 3a: Correlated Poisson counts, Gumbel copula, $\tau = 0.3$

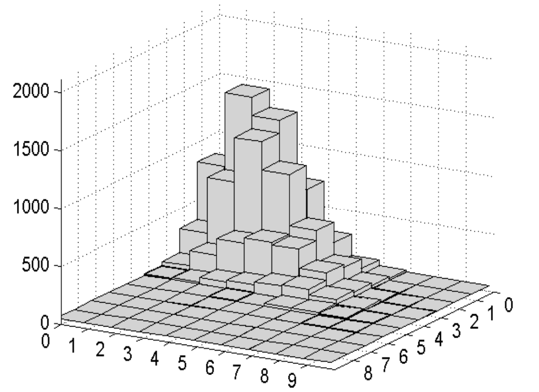


Fig. 3b: Correlated Poisson counts, Gumbel copula, $\tau = 0.5$

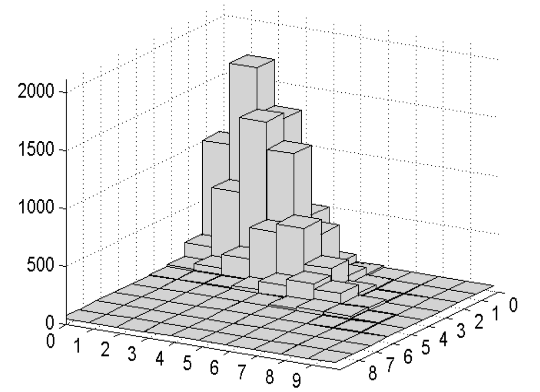


Fig. 3c: Correlated Poisson counts, Gumbel copula, $\tau = 0.7$

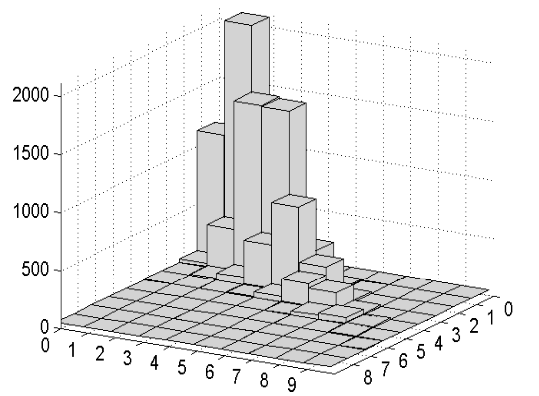


Table 1. Joint probabilities of bivariate Poisson (10000 Monte Carlo replications - Gaussian copula)

	$\tau = 0$	$\tau = 0.3$	$\tau = 0.5$	$\tau = 0.7$
$P(N_1 > 1, N_2 > 1)$	0.1296	0.1771	0.1949	0.201
$P(N_1 > 2, N_2 > 2)$	0.0195	0.0383	0.0511	0.0485
$P(N_1 > 3, N_2 > 3)$	0.0013	0.0065	0.0104	0.0087
$P(N_1 > 4, N_2 > 4)$	0.0001	0.009	0.0018	0.0025
$P(N_1 > 5, N_2 > 5)$	0	0	0.0002	0.0003
$P(N_1 > 6, N_2 > 6)$	0	0	0	0

Table 2. Joint probabilities of bivariate Poisson (10000 Monte Carlo replications - Gumbel copula)

	$\tau = 0.3$	$\tau = 0.5$	$\tau = 0.7$
$P(N_1 > 1, N_2 > 1)$	0.1723	0.1920	0.2022
$P(N_1 > 2, N_2 > 2)$	0.0468	0.0459	0.0507
$P(N_1 > 3, N_2 > 3)$	0.0096	0.0098	0.0091
$P(N_1 > 4, N_2 > 4)$	0.0013	0.0020	0.0019
$P(N_1 > 5, N_2 > 5)$	0.0004	0.0004	0.0005
$P(N_1 > 6, N_2 > 6)$	0	0	0.0002
$P(N_1 > 7, N_2 > 7)$	0	0	0

From the figures and the tables, the difference between the Gaussian and the Gumbel copula can also be appreciated: in the latter case, large loss frequencies tend to take place together more frequently, even in case of weak correlation.

From the practitioner's point of view, this result is of obvious importance whenever the event $\{N_1^{(t)} > n^*, N_2^{(t)} > n^*\}$, i.e. the simultaneous exceedance of some number n^* of losses at a certain date t, is relevant. For example, from tables 1 and 2 we see that, with the Gumbel copula, the probability of having more than 5 losses in both business lines is, for all values of τ, much larger with respect to the Gaussian copula; although these probabilities are quite small, losses associated to these events are usually big, so that they can have a tremendous impact on an insurance company having to face them (see Embrechts *et al.* 2003, section 7.1).

On the other hand, in the LDA approach to operational risk, loss frequency correlation has no influence on CaR computation. To see why, consider the standard way of computing CaR in the uncorrelated case, using the Poisson-lognormal model and a time horizon equal to one year: first, loss frequencies $N_i^{(t)}$, $t = 1, \ldots, T$, with $T = 260$ (assuming 260 working days per year), are simulated from the $Pois(\lambda_i)$ distribution for each business line $i = 1, \ldots, p$, where λ_i is the daily intensity; this is done for every day in the chosen time horizon. Then, for each business line $i = 1, \ldots, p$, we simulate N_i $Logn(\mu_i, \sigma_i^2)$ r.v.'s, where $N_i = \sum_{j=1}^{T} N_i^{(j)} \sim Pois(T\lambda_i)$. These two steps are finally repeated a large number of times B.

When frequencies are correlated, the only change to this procedure concerns the Poisson simulation: instead of simulating separately T Poisson counts for each business line, we simulate joint Poisson counts $\mathbf{N}$, where $\mathbf{N}$ is $(p \times T)$ with $(i,j)-$th element equal to $N_i^{(j)}$; in the next step (the "lognormal" simulation), we sum up Poisson frequencies for each business line, i.e. we compute $N_i = \sum_{j=1}^{T} N_i^{(j)}$, and simulate N_i $Logn(\mu_i, \sigma_i^2)$ r.v.'s. It is now clear that N_i is a Monte Carlo approximation of the i-th marginal of $\mathbf{N}$, which is $Pois(T\lambda_i)$ (no matter of what copula we use to construct the multivariate distribution, the marginals are $Pois(T\lambda_i)$). In conclusion, as long as CaR computation is concerned, in the LDA approach it makes essentially no difference to use correlated or uncorrelated loss frequencies, so that the CaR estimates obtained with correlated frequencies are equal (apart from Monte Carlo sampling errors) to the CaR under the independence hypothesis: see Bee (2005) for the results.

However, from a conceptual point of view, the degree of loss frequency correlation is important: an empirical analysis has been performed by Frachot *et al.* (2004), who show that the correlation is usually low, and therefore there is a wide scope for diversification effects.

Let us now turn to the numerical aspects of the problem. The simulation of the Gaussian copula is easy, because all of the steps described in section 3 are not heavy from a computational point of view: step (1a), i.e. multivariate normal simulation, is based on the fast Choleski decomposition, and steps (1b) and (2) use the univariate normal cdf and its inverse; 10000 Monte Carlo replications take approximately 45 seconds with a Pentium 1.60 GHz, 512 MB RAM.

With the Gumbel copula, Monte Carlo simulation is much more involved. To see where the problem comes from, if C is a bivariate copula, define $C_{2|1}$ to be the conditional distribution function $C_{2|1}(v|u)$. Then, if U and Q are independent $U(0,1)$ random variables, $(U,V) = (U, C_{2|1}^{-1}(Q|U))$ has distribution C (Joe 1997, pag. 146-7). Unfortunately, for most copulas the function $C_{2|1}^{-1}$ does not exist in closed form. In this case, after simulating u, to obtain v we have to solve numerically for v the equation $q = C_{2|1}(v|u)$.

To solve the equation $q = C_{2|1}(v|u)$, in this application we employed the trust-region dogleg algorithm, a variant of the dogleg method described in Powell (1970, chap. 7). However, numerical root-solving turned out to be not straightforward: although the equation always has a real zero in $[0,1]$, in general it has several (real and complex) roots, so that we have to rule out complex zeros and real zeros outside the interval $[0,1]$. When the algorithm finds such a root, we perturbate randomly the starting value and run the root-finding algorithm again until we get a real solution in $[0,1]$; the same strategy is followed when the algorithm converges to a point that is not a root. Not surprisingly, this procedure is rather time-consuming: 10000 Monte Carlo replications with the Gumbel copula take approximately 10 minutes.

5.2 Credit risk

In this section we explore the impact of different copulas on the portfolio loss distribution in the bivariate case. The relevance of the choice of the copula has already been clarified in proposition 1; the same result allows us to investigate numerically the impact of the copula on the resulting loss distribution.

To this aim, we performed a simulation experiment. We set up a portfolio consisting of two loans with probability of default respectively equal to π_1 and π_2. Notice that, as the Gumbel copula has upper tail dependence, we take losses with positive sign and consider counterparty i in default when $r_i > S_i$; thus the thresholds were derived from these PD's as $S_i = \Phi^{-1}(1-\pi_i)$, $i = 1, 2$. The Monte Carlo experiment consists in simulating B standardized returns from the bivariate distribution obtained by means of a copula C and marginals F_i:

$$F(r_1, r_2) = C(F_1(r_1), F_2(r_2)), \tag{7}$$

where C was either the Gaussian or the Gumbel copula and the marginals were either Student t with two degrees of freedom or standard normal. A Monte Carlo estimate of the percentage loss distribution L of the portfolio was then obtained by computing the quantities

$$\begin{aligned}
\hat{p}_0 &= \frac{\#\{r_1 < S_1, r_2 < S_2\}}{B}, \\
\hat{p}_1 &= \frac{\#\{r_1 > S_1, r_2 < S_2\} + \#\{r_1 < S_1, r_2 > S_2\}}{B}, \\
\hat{p}_2 &= \frac{\#\{r_1 > S_1, r_2 > S_2\}}{B},
\end{aligned} \tag{8}$$

where $p_i = P(L = i)$ is the probability of obtaining i defaults in the portfolio, $i = 0, 1, 2$. For both copulas, we ran the simulation for $\tau = 0.3$, 0.5 and 0.7, and for several values of π_1 and

π_2. The results are shown in tables 3, 4 and 5 for the three values of τ; results obtained with Gaussian marginals are similar and are not reported here.

Table 3. Joint default probabilities with t_2-distributed marginals and Gaussian or Gumbel copula (10000 Monte Carlo replications, $\tau = 0.3$)

Default probabilities	Gaussian copula			Gumbel copula		
	$\hat{p}_0$	$\hat{p}_1$	$\hat{p}_2$	$\hat{p}_0$	$\hat{p}_1$	$\hat{p}_2$
$\pi_1 = 0.03, \pi_2 = 0.03$	0.9452	0.048	0.0068	0.9523	0.0355	0.0122
$\pi_1 = 0.02, \pi_2 = 0.02$	0.961	0.0357	0.0033	0.9685	0.0251	0.0064
$\pi_1 = 0.01, \pi_2 = 0.01$	0.9806	0.018	0.0014	0.9841	0.0116	0.0043
$\pi_1 = 0.005, \pi_2 = 0.005$	0.9908	0.009	0.0002	0.9925	0.0054	0.0021
$\pi_1 = 0.003, \pi_2 = 0.003$	0.993	0.0068	0.0002	0.9957	0.0036	0.0007
$\pi_1 = 0.002, \pi_2 = 0.002$	0.9957	0.0042	0.0001	0.997	0.0026	0.0004
$\pi_1 = 0.001, \pi_2 = 0.001$	0.9984	0.0016	0	0.9983	0.0012	0.0005
$\pi_1 = 0.03, \pi_2 = 0.01$	0.9606	0.0371	0.0023	0.9655	0.0277	0.0068
$\pi_1 = 0.01, \pi_2 = 0.005$	0.9866	0.0129	0.0005	0.9889	0.009	0.0021

Table 4. Joint default probabilities with t_2-distributed marginals and Gaussian or Gumbel copula (10000 Monte Carlo replications, $\tau = 0.5$)

Default probabilities	Gaussian copula			Gumbel copula		
	$\hat{p}_0$	$\hat{p}_1$	$\hat{p}_2$	$\hat{p}_0$	$\hat{p}_1$	$\hat{p}_2$
$\pi_1 = 0.03, \pi_2 = 0.03$	0.947	0.0414	0.0116	0.9558	0.0248	0.0194
$\pi_1 = 0.02, \pi_2 = 0.02$	0.9671	0.0256	0.0073	0.9713	0.016	0.0127
$\pi_1 = 0.01, \pi_2 = 0.01$	0.9852	0.0127	0.0021	0.9854	0.0087	0.0059
$\pi_1 = 0.005, \pi_2 = 0.005$	0.9927	0.0061	0.0012	0.9927	0.0041	0.0032
$\pi_1 = 0.003, \pi_2 = 0.003$	0.9943	0.0047	0.001	0.9956	0.003	0.0014
$\pi_1 = 0.002, \pi_2 = 0.002$	0.9961	0.0037	0.0002	0.9963	0.0021	0.0016
$\pi_1 = 0.001, \pi_2 = 0.001$	0.9987	0.0013	0	0.9994	0.0004	0.0002
$\pi_1 = 0.03, \pi_2 = 0.01$	0.9646	0.03	0.0054	0.9706	0.0218	0.0076
$\pi_1 = 0.01, \pi_2 = 0.005$	0.985	0.0129	0.0021	0.9876	0.0089	0.0035

Table 5. Joint default probabilities with t_2-distributed marginals and Gaussian or Gumbel copula (10000 Monte Carlo replications, $\tau = 0.7$)

Default probabilities	Gaussian copula			Gumbel copula		
	$\hat{p}_0$	$\hat{p}_1$	$\hat{p}_2$	$\hat{p}_0$	$\hat{p}_1$	$\hat{p}_2$
$\pi_1 = 0.03, \pi_2 = 0.03$	0.9585	0.0236	0.0179	0.9613	0.0123	0.0264
$\pi_1 = 0.02, \pi_2 = 0.02$	0.9704	0.0167	0.0129	0.9747	0.0097	0.0156
$\pi_1 = 0.01, \pi_2 = 0.01$	0.9843	0.0103	0.0054	0.987	0.0057	0.0073
$\pi_1 = 0.005, \pi_2 = 0.005$	0.9931	0.0041	0.0028	0.9938	0.0017	0.0045
$\pi_1 = 0.003, \pi_2 = 0.003$	0.9956	0.0031	0.0013	0.9965	0.0014	0.0021
$\pi_1 = 0.002, \pi_2 = 0.002$	0.9976	0.0015	0.0009	0.997	0.0008	0.0022
$\pi_1 = 0.001, \pi_2 = 0.001$	0.9979	0.0016	0.0005	0.9988	0.0004	0.0008
$\pi_1 = 0.03, \pi_2 = 0.01$	0.9686	0.0234	0.008	0.9709	0.0192	0.0099
$\pi_1 = 0.01, \pi_2 = 0.005$	0.9885	0.008	0.0035	0.9879	0.007	0.0051

It is worth stressing the impact of a copula like the Gumbel, characterized by tail dependence, on the percentage loss distribution: the probability of joint defaults of both counterparties in the Gumbel case is considerably higher than in the Gaussian case. It follows that, if the joint default event has relevant consequences, as for example with basket credit derivatives, the choice of the copula is crucial. Notice also that tail dependence is a property that only affects the probability of *joint* events; the probability of a single default is indeed often higher in the models based on the Gaussian copula.

MLE's are consitent and asymptotically normal, but if the sample size is small we do not know their properties; to get some insight into this issue, we performed a final simulation. The setup was as follows: for sample sizes $N = 20$, 40, 70, 100, 200 and 400 we simulated bivariate observations from the Gaussian (resp. Gumbel) copula with Student t_2 marginals; the copula parameter was chosen so that $\tau = 0.3$ for both copulas, i.e. $\rho = 0.454$ for the Gaussian copula and $\delta = 1.43$ for the Gumbel copula. For each sample size N, the procedure consists in simulating M bivariate samples with size N, estimating the parameter ρ (resp. δ) and finally computing standard errors and mean squared errors of the empirical distribution. Results are shown in table 6; the same simulation was also run with $\tau = 0.5$ and $\tau = 0.7$; the outcomes are similar and are not displayed here.

Table 6. MSE's and standard errors of the MLE of the parameter of the Gaussian and Gumbel copula (1000 Monte Carlo replications, $\tau = 0.3$, marginals t_2)

	ρ (Gaussian cop.)		δ (Gumbel cop.)	
	MSE	SE	MSE	SE
$N = 20$	0.046	0.213	0.07	0.262
$N = 40$	0.192	0.138	0.034	0.184
$N = 70$	0.011	0.105	0.02	0.139
$N = 100$	0.008	0.089	0.014	0.116
$N = 200$	0.003	0.058	0.006	0.078
$N = 400$	0.002	0.042	0.003	0.056

Overall, the behavior of the estimators is rather good in both cases: although the MSE's are slightly higher in the Gumbel case, they decrease quickly as the sample size increases, and with 100 or more observations they are quite small.

To conclude, it is worth pointing out some serious problems related to copula-based models in this setup. The first one is the extension to the general p-variate ($p > 2$) case, which is absolutely necessary in credit risk portfolio models. This can be easily done with the Gaussian copula, but not with the Gumbel copula. Roughly speaking, what can be said is that the multivariate extension with general correlation structure can be performed within the class of elliptical copulas, but is otherwise intractable, unless one is willing to assume uniform pairwise correlations. The latter assumption is usually too restrictive, so that the actual implementation of credit risk portfolio models seems to be bounded to the use of elliptical copulas.

The second problem has to do with statistical inference: in credit portfolio models, estimating the correlation matrix $\mathbf{R}$ by means of asset correlations is not easy because asset values are unobservable; another approach chooses the correlation matrix $\mathbf{R}$ so as to match the joint one-year probability of default. However, the performance of both procedures is not particularly good (Duffie and Singleton 2003, pag. 240).

In a fully parametric model, similar difficulties arise for the choice and estimation of the marginals. Fermanian and Scaillet (2004) provide some simulation results concerning the impact of misspecified margins on the estimation of the copula parameter; in case of doubts about the true distribution of the marginals, they suggest to use a semiparametric method which estimates the marginals by means of the univariate empirical cdf's and the copula parameter with the maximum likelihood method.

Finally, the selection of the dependence structure is difficult: although some empirical methods have been developed (see Anè and Kharoubi 2003 for an application to market risk), they seem to require far more observations than we usually have in credit risk. The Gaussian copula is usually considered the "standard" model, in the sense that it corresponds to a dependence structure with no tail dependence. Before deciding to use a copula which has tail dependence, like the Gumbel or the Student copula, it would be extremely useful to know whether the data exhibit tail dependence or not. However, as pointed out by Embrechts *et al.* (2003), tail dependence is an asymptotic property and thus it is not easy to assess in advance whether the true underlying dependence structure has tail dependence.

6 Conclusions

In this paper we have examined the statistical tools necessary to understanding and modeling the non-standard dependence concepts which are playing a more and more important role in finance and risk management as the normality assumption loses its centrality. In particular, we have shown that copulas are fundamental to most models used for the measurement of insurance, operational and credit risk.

Although copulas guarantee a high degree of flexibility as concerns the construction of multivariate distributions, their implementation requires some caution: both the choice of copula and its estimation are not straightforward, and different copulas can sometimes produce widely different results; in other words, the so-called model risk is, in this setup, particularly high. It is worth noting that both of these problems become worse when data availability is limited, as is usually the case in credit risk analysis. Finally, with most copulas computational aspects should not be overlooked: simulation of the Gumbel copula requires numerical root-finding, and this is quite time-consuming.

Not surprisingly, research in this field is currently very active, spurred by the existence of unsolved statistical issues, by the needs of new techniques emerging from finance and risk management and, last but not least, by the regulators' pressure towards models that are both sophisticated and transparent. In our opinion, the most important lines of research to be pursued in the future should deal with the following issues: (i) from a statistical point of view, the estimation of the parameters of the copula should be studied more thoroughly: as pointed out by Joe (1997, pag. 297), "Practically the only theory that can be applied is the asymptotic maximum likelihood (ML) theory"; since then, no major improvement seems to have been made in this direction; (ii) more efficient simulation techniques for non-elliptical copulas would be necessary, because currently they are rather slow and unstable; (iii) the analysis of dependence in the actuarial setup is still in its inphancy and some more efforts are probably required to understand how relevant it is to practical applications.

Acknowledgements

I would like to thank Riccardo Cateni (Banca Intesa) for providing the data.

References

Anè T and Kharoubi C (2003) Dependence Structure and Risk Measure. Journal of Business 76: 411-438.

Bee M (2004) Modelling Credit Default Swap Spreads by means of Normal Mixtures and Copulas. Applied Mathematical Finance 11: 125-146.

Bee M (2005) On Maximum Likelihood Estimation of Operational Losses, Discussion paper 2005.3, Department of Economics, University of Trento.

Bluhm C, Overbeck L and Wagner C (2002) An Introduction to Credit Risk Modeling. Chapman & Hall, London.

Campbell J Y, Lo A W and MacKinlay A C (1997) The Econometrics of Financial Markets. Princeton University Press, Princeton.

Chavez-Demoulin V, Embrechts P and Neslehova J (2005) Quantitative Models for Operational Risk: Extremes, Dependence and Aggregation, paper presented at the meeting Implementing an AMA for Operational Risk, Federal Reserve Bank of Boston, May 18-20, 2005.

Cherubini U, Luciano E and Vecchiato W (2004) Copula Methods in Finance, Wiley, New York.

Duffie D and Singleton K J (2003) Credit Risk. Pricing, Measurement and Management, Princeton University Press, Princeton.

Durrett R (1996) Probability: Theory and Examples. Duxbury Press.

Embrechts P, McNeil A and Straumann D (1999) Correlation: Pitfalls and Alternatives, Risk 12: 69-71.

Embrechts P, Lindskog F and McNeil A (2003) Modelling dependence with copulas and applications to Risk Management. In: Rachev S (ed.) Handbook of Heavy Tailed Distributions in Finance. Elsevier, pp. 329-384.

Fermanian J D and Scaillet O (2004) Some Statistical Pitfalls In Copula Modeling For Financial Applications, FAME Research Paper Series rp108, International Center for Financial Asset Management and Engineering.

Flury B (1997) A First Course in Multivariate Statistics. Springer, New York.

Frachot A, Roncalli T and Salomon E (2004) The Correlation Problem in Operational Risk, working paper, Credit Lyonnais.

Frahm G, Junker M and Schmidt R (2004) Estimating the Tail-dependence Coefficient. Properties and Pitfalls, working paper, Center of Advanced European Studies and Research.

Frey R and McNeil A (2003) Dependent Defaults in Models of Portfolio Credit Risk. Journal of Risk 6: 59-92.

Gordy M B (2000) A Comparative Anatomy of Credit Risk Models. Journal of Banking and Finance 24: 119-149.

Joe H (1997) Multivariate Models and Dependence Concepts. Chapman and Hall, London.

Johnson N L, Kotz S and Balakrishnan N (1997) Discrete Multivariate Distributions, Wiley, New York.

Junker M and May A (2002) Measurement of Aggregate Risk with Copulas, working paper, Center of Advanced European Studies and Research.

Klugman S A, Panjer H H and Willmot G E (1998) Loss Models - from Data to Decisions, Wiley, New York.

Lindskog F (2000) Linear Correlation Estimation, working paper, ETH Zürich.

Lindskog F and McNeil A (2003) Common Poisson Shock Models: Applications to Insurance and Credit Risk Modelling. ASTIN Bulletin 33: 209-238.

Lindskog F, McNeil A and Schmock U (2003) A Note on Kendall's tau for Elliptical Distributions. In Bol G, Nakhaeizadeh G, Rachev S, Ridder T, Vollmer K H (eds.) Credit Risk. Measurement, Evaluation and Management. Physica-Verlag, Heidelberg.

Lintner J (1965) The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets. Review of Economics and Statistics 41: 13-37.

Malevergne Y and Sornette D (2002) Tail Dependence of Factor Models, working paper, www.unifr.ch/econophysics.

Markowitz H (1952) Portfolio Selection. Journal of Finance 7: 77-91.

Merton R (1974) On the Pricing of Corporate Debt: The Risk Structure of Interest Rates. Journal of Finance 29: 449-470.

Mossin J (1966) Equilibrium in Capital Asset Market. Econometrica 35: 768-783.

Neslehova J and Pfeifer D (2004) Modeling and Generating Dependent Risk Processes for IRM and DFA. ASTIN Bulletin 34: 333-360.

Powell M J D (1970) A Fortran Subroutine for Solving Systems of Nonlinear Algebraic Equations. In Rabinowitz P (ed.) Numerical Methods for Nonlinear Algebraic Equations.

Sharpe W (1964) Capital Asset Prices: a Theory of Market Equilibrium under Conditions of Risk. Journal of Finance 19: 425-442.

Sklar A (1959) Fonctions de Repartition á n Dimensions et leurs Marges. Publications de l'Institut de Statistique de l'Université de Paris 8: 229-231.

Vasicek O A (1987) Probability of Loss on Loan Portfolio, KMV Corporation.

Xu J J (1996) Statistical Modelling and Inference for Multivariate and Longitudinal Discrete Response Data, unpublished Ph.D. thesis, Department of Statistics, University of British Columbia.