High Frequency Market Making: Optimal Quoting*

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Abstract

We propose a model of market making where a strategic high frequency trader exploits his speed and informational advantages to place quotes that interact with the orders of low frequency traders. We characterize the optimal market making policy of the high frequency trader analytically. Our model shows that higher speed translates into higher profits through a more aggressive quoting policy. The optimal policy is consistent with empirically documented features of high frequency trading such as order cancellations and predatory trading.

Keywords: High Frequency Trading, Market Making, Liquidity, Stochastic Optimal Control.

1. Introduction

This paper focuses on a specific type of high frequency trading, market making. High frequency traders (HFTs) have to a large extent become the de facto market makers, or providers of liquidity to the market. To better understand how HFTs can be expected to behave in that capacity, and what influences their quoting, we model a fully dynamically optimizing high frequency market maker in the classical inventory control framework of Amihud and Mendelson (1980) and Ho and Stoll (1981) for "traditional" market makers (see also Avellaneda and Stoikov (2008), Guilbaud and Pham (2013), Guéant et al. (2013), Cartea et al. (2014) and Hendershott and Menkveld (2014)).

In the classical setting, the informational advantages between providers and consumers of liquidity are split: while the market maker has access to the order book, generating microstructure information about the order flow, his trading counterparties could potentially be better informed about the fundamental value of the asset, generating adverse selection risk for the market maker

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(see Glosten and Milgrom (1985)). We depart from the classical inventory literature by modeling the optimization problem to account for the features that are new to high frequency market making: first, we add speed as a variable to the problem; second, instead of splitting the advantages between traders and market makers, we turn the tables by endowing HFTs with both speed and informational advantages. Both elements are essential to capture the new world of market making. The fact that the HFT "holds all the cards" in our model matches the current situation where HFTs are both faster and typically better informed than their trading counterparties. The informational advantage of the HFT in our model is microstructure-driven, rather than pertaining to the fundamental value of the asset. The HFT is still exposed to adverse selection risk in our model, but it is adverse selection arising from speed asymmetries instead of informational asymmetries.

Our modeling choices are informed by some of the main empirical stylized facts that are known about HFTs (see e.g., Brunetti et al. (2011)), particularly as they relate to market making. They include the facts that only a small number of HFT firms exist relative to the mass of low frequency traders (LFTs); HFTs are recognizable by their high frequency of quoting updates, small size on each quote, use of inventory controls as their primary risk mitigation strategy and unwillingness to take directional bets. They also tend to place many limit orders, with only few actually leading to an execution; and they seem to exploit order flow information and generate trading signals on a very short time scale, including potential order anticipation strategies.[1]

The main contribution of the paper is to set up and solve a tractable yet fairly realistic model of liquidity provision by a HFT based on the outlined empirical facts and characterize his optimal quoting policy as a function of his speed when trading against many uninformed LFTs with different propensity to trade, and different willingness to wait to get the best price. In the model, the bid-ask spread is endogenously determined as a result of the equilibrium between the HFT's optimal quoting strategy and the incoming LFT orders, while the HFT utilize informative signals about the order flow to optimally determine his level of quote aggressiveness.

Technically, we use multiple, staggered, Poisson processes to represent the arrival of the various elements of the model: market orders by LFTs, technological latency of the HFT, informational signals to the HFT, and jumps in the asset's fundamental price. This modeling device keeps the analysis of the dynamic optimization problem facing the HFT tractable and flexible, since merging the different Poisson processes on a common time clock results in a single Poisson process with

[1] There is a growing empirical literature analyzing the relationship between institutional trading costs and HFTs' activities. For example, Hirschey (2011) finds empirical evidence that HFTs may increase the trading cost of non-HFTs by trading ahead. Korajczyk and Murphy (2016) observe that HFT liquidity provision is significantly lower for large order trades executed by institutional investors. Kervel and Menkveld (2015) study the cost dynamics of institutional large orders by studying the impact of the HFTs. Trading costs of informed investors increase dramatically when they start to trade in the same direction. Sağlam (2016) provides robust signals that can quantify the predictability of institutional order flow and find increased execution costs for predictable trades.

arrival rate equal to the sum of the individual processes' arrival rates. We solve for the optimal quoting policy for the HFT and show that it can be explicitly characterized by a series of asymmetric thresholds. Finally, we analyze different market environments by varying the parameters of the model.

In economic terms, the HFT in our model is subject to three trade-offs: first, a conflict between inventory concerns and spread capture, which is the same as in the classical inventory control literature; second, a new speed-dependent trade-off between earning the spread by posting quotes on both sides vs. the adverse selection risk due to speed differentials or simply randomness of order arrivals; third, a new information-dependent trade-off between the extensive and intensive margins, as the HFT processes more volume with more aggressive prices but earns a higher profit on each transaction when the spread is higher.

The model provides several new insights relative to the classical literature on low frequency market making with inventory costs. First, thanks to the introduction to speed as a determinant, we show that when the HFT is faster, he will provide more aggressive quotes, i.e., he will quote within wider ranges of inventory limits and at lower spreads on average. We illustrate that wider inventory limits are not purely driven by the lower risk of leaving stale quotes. Managing inventory risk efficiently via increased ability in quote monitoring also incentivizes the HFT to quote more. This finding is consistent with the robust empirical evidence that high-frequency trading activity and increase in their speed is positively correlated with overall liquidity proxies (see e.g., Hendershott et al. (2011), Brogaard et al. (2014) and Brogaard et al. (2015)).

Second, due both to time-varying signals and asymmetry in inventory limits, the HFT may find it optimal to cancel his existing limit orders endogenously. Thus, our model provides a mechanism for the rapid cancellation rates widely observed in empirical data. Baruch and Glosten (2013) provide another channel for cancellation rates based on pursuing mixed strategies in a trading game.

Third, we show how the HFT may engage in predatory quoting strategies, or price discrimination, against impatient liquidity consumers by exploiting his order anticipation skills, modifying the spread between his quotes in the process. Empirically, this prediction of the model matches the observed increase in trading costs when HFTs pursue order anticipation strategies and decrease their liquidity provision to large institutional order flow as documented by Korajczyk and Murphy (2016) and Kervel and Menkveld (2015).

Finally, the optimal quoting policy in our model provides a new motive for quoting through the use of order direction signal, i.e., the ability to imperfectly predict the sign of the incoming market order. In the presence of large adverse selection costs due to stale quotes, this signal can be utilized to hedge against being picked off by arbitrageurs. If this risk is small, the HFT will utilize the signal to increase the aggressiveness of the quote.

In a companion paper, Aït-Sahalia and Sağlam (2016), we use our model to complement these findings by exploring how price volatility affects the HFT's optimal quoting policy, the impact of different proposed HFT regulations, as well as the effects of competition for liquidity provision. Overall, these extensions imply that our model offers a flexible framework for HFTs' quoting behavior and can be utilized to examine the impact of ongoing changes in the microstructure of financial markets from a market making perspective.

The paper is organized as follows. Section 2 sets up the model. Section 3 starts by deriving the optimal quoting strategy of the HFT in a simple version of the model without price volatility, before solving the full model in Section 4. In Section 5, we discuss the economic insights gained from the optimal market making policy. Finally, Section 6 concludes. Proofs and technical results are in the Appendix.

2. The Model

2.1. HFT's Quotes and Endogenous Bid-Ask Spread

The HFT and a large number of uninformed LFTs are trading a single asset in an electronic limit order book. The HFT acts exclusively as a market maker, employing only limit orders to buy and sell. The quantity of each order, market or limit, is fixed at 1 share or lot. Small volume on each trade matches what is observed empirically in markets that are popular with HFTs, such as the S&P500 eMini futures. Generally speaking, the quantity exchanged in each transaction has been going down over time (see, e.g., Angel et al. (2015)).

We assume that the HFT can place limit orders at four discrete price levels around the fundamental value of the asset, X_t. The tick size in the market is $2C$ for some constant $C > 0$ (e.g., \$0.01). The minimum price at which the HFT is willing to sell the asset is $X_t + C$ with $C > 0$ while the minimum price at which he is willing to buy the asset is $X_t - C$. We will refer to these prices as the best ask and the best bid prices. The HFT can also choose to quote at price levels one tick away from the best ones, $X_t + 3C$ and $X_t - 3C$, and transactions will take place at these levels in the absence of quotes at the best bid and/or ask prices. We will refer to these prices as the second-best ask and the second-best bid prices.

These quoting decisions define the set of available actions to the HFT. Formally, $\ell_t^a = 1$ (resp. $\ell_t^b = 1$) will imply that the HFT has an active quote at the best ask (resp. bid) at time t, and similarly, $\ell_t^a = 2$ (resp. $\ell_t^b = 2$) will imply that the HFT has an active quote at the second-best ask

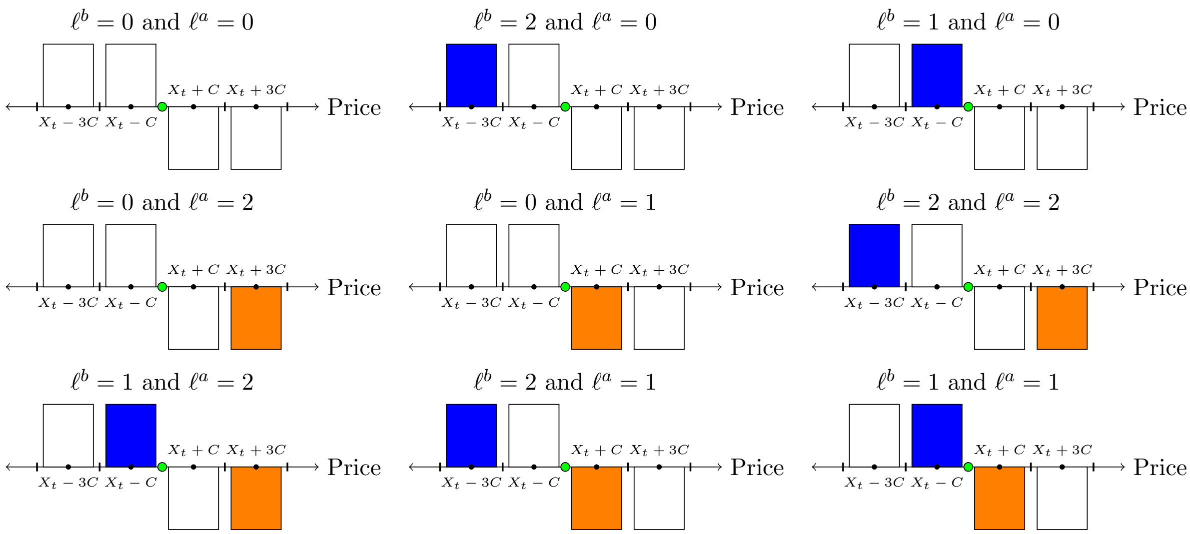

Fig. 1. HFT's possible quotes.

(resp. bid) at time t. Finally, $\ell_t^a = 0$ (resp. $\ell_t^b = 0$) means that the HFT either chooses not to quote to sell (resp. buy) for inventory considerations or his most recent active order has been filled by a market order but the HFT has not yet submitted a new order due to the technological constraints. Figure 1 illustrates the possible quoting decisions by the HFT. A solid bar in the figure indicates that the HFT is quoting at that price level; there are nine possible quoting decisions.

The HFT is a monopolistic liquidity provider at tick levels $X_t \pm C$ and $X_t \pm 3C$.[2] We assume that the limit order book contains exogenous depth supplied by competing liquidity providers at tick levels $X_t + 5C$ and/or $X_t - 5C$. This assumption is needed only to define a valid spread measure should the HFT completely withdraw from providing liquidity ($\ell_t^b = 0$ or $\ell_t^a = 0$).

The bid-ask spread will be endogenously determined as a result of the equilibrium between the HFT's optimal quoting strategy and the incoming LFT orders, to be described in Section 2.3. The spread between the active quotes at the ask and the bid can be $2C$, $4C$, $6C$, $8C$ and $10C$, depending

[2] In Aït-Sahalia and Sağlam (2016), we extend the model to a duopoly situation in which there is an additional nonstrategic liquidity provider quoting at the available bid and ask prices, competing with the HFT. The main implications of the model also hold in that duopolistic market.

upon the quoting decisions by the HFT:

$$
\begin{array}{cc}
\text{HFT Quoting Decisions} & \text{Resulting Bid-Ask Spread} \\
\ell_t^a = 1 \text{ and } \ell_t^b = 1 & 2C \\
\ell_t^a = 1 \text{ and } \ell_t^b = 2 & 4C \\
\ell_t^a = 2 \text{ and } \ell_t^b = 1 & 4C \\
\ell_t^a = 1 \text{ and } \ell_t^b = 0 & 6C \\
\ell_t^a = 0 \text{ and } \ell_t^b = 1 & 6C \\
\ell_t^a = 2 \text{ and } \ell_t^b = 2 & 6C \\
\ell_t^a = 2 \text{ and } \ell_t^b = 0 & 8C \\
\ell_t^a = 0 \text{ and } \ell_t^b = 2 & 8C \\
\ell_t^a = 0 \text{ and } \ell_t^b = 0 & 10C
\end{array}
\tag{2.1}
$$

2.2. *Dynamics of the Underlying Asset Price and Volatility*

We assume that the asset's fundamental price is subject to exogenous variability in the form of pure compound Poisson jumps:

$$X_t = X_0 + \sum_{i=1}^{N_t^\sigma} Z_i, \tag{2.2}$$

where N_t^σ is a Poisson process with arrival rate σ counting the number of price jumps up to time t and Z_i is the stochastic magnitude of the ith jump with distribution

$$Z_i \sim \begin{cases} J & \text{with probability } 1/2, \\ -J & \text{with probability } 1/2. \end{cases} \tag{2.3}$$

J takes values kC, where k is a positive integer. X_0 is a given multiple of C.

With this specification, the fundamental value of the asset is a martingale. The variance of the fundamental value changes is proportional to σ. Although it is in principle a misnomer to call σ "volatility" in light of the differences between discontinuous and continuous components of a semimartingale, in the absence of a continuous price variation, jumps are the only source of price volatility in the model.

2.3. *LFT Orders and Equilibrium*

The HFT acts as an intermediary who executes incoming LFTs' orders through his own inventory. LFTs are exclusively liquidity takers. Their orders come with a price condition (like limit orders) combined with an immediacy condition (like market orders): they must either be immediately executed if the price condition is met or cancelled if it is not ("immediate-or-cancel" orders). Consider for instance, an LFT order to buy the asset at price $X+C$ or lower. If an HFT quote to sell is present at tick level $X+C$, the order will get immediately filled, and the HFT inventory will decrease by one share. If the HFT is only quoting at that instant to sell at $X+3C$, or not at all on that side of the book, an LFT order to buy at $X+C$ or lower will not get executed and will be returned to the LFT as being cancelled; an LFT order to buy at $X+3C$ or lower would get executed, however.

The demand for liquidity is as follows. LFTs' orders to buy and sell arrive to the market as Poisson processes, in the form of a function $\lambda^B(X,Y,\sigma)$ to buy the asset and a function $\lambda^S(X,Y,\sigma)$ to sell the asset (representing the arrival rate or quantity expected, per unit of time) that depend upon both the fundamental value of the asset X, the transaction price $Y = X \pm kC$ (where k is an odd integer) quoted by the HFT and potentially the asset volatility as well. The micro foundations for this Poissonian assumption are similar to those in Garman (1976), with the addition of the fundamental value and asset price volatility: if a large number of agents demand and supply the asset, each acting independently in the timing of their orders, none being large, and none being able to generate an infinite number of orders per unit of time, then as the number of agents grow the aggregate demand and supply will converge to Poisson processes. It is natural to assume that $\partial\lambda^B/\partial X \geq 0$ and $\partial\lambda^B/\partial Y \leq 0$, and vice versa for λ^S. For simplicity, we assume that λ^B and λ^S depend on (X,Y) through $Y-X$ only, so the demand and supply functions become $\lambda^B(Y-X,\sigma)$ and $\lambda^S(Y-X,\sigma)$.

For the market to be in equilibrium, we require that $\lambda^B(0,\sigma) = \lambda^S(0,\sigma)$. Recognizing the discreteness of the possible prices, we also require that in equilibrium

$$\begin{cases} \lambda^B(-C,\sigma) = \lambda^S(C,\sigma) \equiv \lambda_A + \lambda_P + \lambda_I \\ \lambda^B(C,\sigma) = \lambda^S(-C,\sigma) \equiv \lambda_P + \lambda_I \\ \lambda^B(3C,\sigma) = \lambda^S(-3C,\sigma) \equiv \lambda_I \end{cases} \tag{2.4}$$

with λ_I, λ_P and λ_A parameters capturing the slope of the demand and supply curves near X. When Y is further away from X, the demand and supply curves satisfy for simplicity:

$$\begin{cases} \lambda^B(-kC,\sigma) = \lambda^S(kC,\sigma) = \lambda_A + \lambda_P + \lambda_I \text{ for } k \geq 3 \\ \lambda^B(kC,\sigma) = \lambda^S(-kC,\sigma) = 0 \text{ for } k \geq 5 \end{cases} \tag{2.5}$$

Note that λ_I, λ_P and λ_A as defined above are potentially functions of σ, but we leave this dependence implicit.

LFTs arrive with equal probability on both sides of the market, to buy or sell the asset, as expressed by the equality between λ^B and λ^S at the relevant tick levels in (2.4-2.5). This equilibrium condition is necessary for the HFT not to be in a position to accumulate inventory over time, which would otherwise force him to stop quoting on one side of the market due to inventory aversion; this would happen if some imbalance existed between the arrival rates of the buyers and sellers. This notion of equilibrium is statistical; it holds on average over time but at any given instant a trader may or may not arrive, and do so on either side of the market, with the HFT's inventory absorbing the imbalance whenever the HFT is quoting and a transaction takes place against an incoming market order.

Due to the facts that fundamental and transaction prices live on a discrete grid determined by the tick level, the only values of the functions λ^B and λ^S that matter are those at the discrete values of $Y - X$ given in (2.4-2.5). Therefore, the arrival of the LFT orders can be interpreted as emanating from three types of LFTs, patient, impatient and arbitrageurs. Patient LFTs submit orders only at the best available bid or ask price i.e., at $X_t + C$ or $X_t - C$, and their order is only executed if there is an existing limit order (by the HFT) at these prices. By contrast, impatient LFTs are also willing to trade at the second best available quotes when that is all that is available, i.e., sell at $X_t - 3C$ and buy at $X_t + 3C$. Arbitrageurs are only willing to trade if they can buy the asset at a cheaper price than its fundamental value or sell the asset at a higher price than its fundamental value.

Patient, impatient and arbitrageurs arrive at random times according to Poisson processes with respective arrival rates λ_P, λ_I and λ_A; the sum of their arrivals produces the aggregate demand for liquidity functions λ^B and λ^S faced by the HFT. In practice, if λ_A is sufficiently high, then arbitrageurs, although formally classified as LFTs in the model, act more like high frequency traders themselves who may be referred to as high frequency "bandits" (Menkveld (2016)), "toxic arbitrageurs" (Foucault et al. (2016)) or "back-runners" (Yang and Zhu (2015)): with high probability, someone will quickly take liquidity from the HFT by buying at a price Y below X and selling at a price Y above X.

2.4. *Price Volatility, Stale Quotes and Speed-Induced Adverse Selection*

We assume that the HFT can only alter his quotes at random times specified by a Poisson distribution with arrival rate μ. Although the HFT is in principle in charge of the timing of his quoting decisions, this technologically-induced randomness is consistent with how communication networks at high frequencies operate: it accounts for the randomness in the packet transit times between the HFT and the exchange, the order in which messages are processed by the exchange, etc. Many technological investments by HFTs are directed at increasing μ : for instance, tower-based microwave technology between New York and Chicago or more prosaically colocation within an exchange aim to reduce the time it takes an order sent by the algorithm to be received by the matching engine of the trading platform. A higher value of μ allows the HFT to revise his quotes more frequently, and thus μ can be interpreted as the speed of the HFT, or $1/\mu$ as a measure of his latency.

Consistent with the notion that the HFT trades on the basis of short-lived market microstructure information (see Section 2.5 below), as opposed to fundamental information regarding the true value of the asset, we assume that the HFT has no informational advantage regarding the exogenous jumps in the asset fundamental price, X_t, that were described in Section 2.2. So, when a price jump occurs, the HFT is as surprised as everyone else in the market and is stuck with his existing bid and ask quotes until the arrival of the next decision event (μ) at which point he can peg his limit order to the new mid-price.

The HFT's quotes can become stale after a price jump. For example, consider a specific instance at which the HFT is quoting at the best prices around both sides of the market. If the fundamental price of the security jumps up by $J = 4C$, the HFT's earlier quote at the pre-jump $X_t + C$ becomes stale and leads to an arbitrage opportunity for LFTs. Patient, impatient and as well as arbitrageurs would certainly like to buy the asset at the post-jump $X_t - 3C$. If they submit an order during this stale-quote period, a trade occurs, and the HFT would lose $3C$ to the the arbitrageur (or the LFT) that submitted the order. Note that the HFT's quote to buy the asset at the post-jump $X_t - 5C$ is not attractive to any LFT, patient, impatient or arbitrageur, due to the demand functions in (2.4)-(2.5), and thus no trade would take place at that price and so the potential loss from staleness is not compensated by a potential gain.

The HFT is by definition fast relative to any single LFT. However the arrival rates of the LFT orders $\lambda = (\lambda_P, \lambda_I, \lambda_A)$ result from the aggregation of many such LFTs, so we do not necessarily expect μ to be very large relative to λ. In any event, the model does not require assumptions about their relative values. If the HFT is much faster than the arbitrageurs and other LFTs (μ high relative to λ), he is unlikely to get caught with stale quotes by an incoming order, since a $\mu-$event is likely to have occurred before a $\lambda-$event, but should an arbitrageur's order nevertheless arrive in the interim, it will get executed against the HFT's prevailing pre-jump quotes, creating a loss for the HFT. This creates adverse selection risk for the HFT, although one which is due to relative speed or simpliy randomness of order arrivals rather than an informational advantage about the fundamental value by the LFTs (unlike the situation in Glosten and Milgrom (1985)). Stale quotes are attractive not only to patient and impatient LFTs but also to arbitrageurs, who are likely to be high frequency traders themselves. Since the latter would be fast traders, it is plausible for λ_A to be high (equivalently, the demand to buy and sell functions can be very steep) and thus the risk of being "picked off" can be substantial for the HFT.

2.5. HFT's Informational Advantage

We endow the HFT with a microstructure-driven informational advantage by letting him make quoting decisions based on two independent signals which are respectively informative about the direction of the next incoming market order (buy or sell), and about the type of the trader submitting the market order (patient or impatient). The signals are not predictive about arbitrageurs orders as they will only trade with the HFT if he is stuck with a stale quote due to a price jump, a situation which the HFT will never voluntarily face.

The motivation behind the existence of these signals is the empirical evidence that HFTs are able to identify and exploit fleeting opportunities arising from the "plumbing" of the trading process, derived for instance from collocating their trading engine near the exchange matching engine, obtaining and exploiting pricing information faster than other traders, extracting information from the current state of the limit order book, such as real time order book imbalances, observing the arrival of orders on one exchange before they hit other exchanges, running a securities information processor (SIP) that is faster than the publicly-available one, observing recent trading patterns that may be predictive of the direction of the future orders, etc: see e.g., Hirschey (2011) and Sağlam (2016).

2.5.1 First Signal: Direction of the Order Flow

The first signal is an i.i.d. Bernoulli random variable, $S^{\text{dir}} \in \{B, S\}$, imperfectly predicting the direction of the next incoming market order: B corresponds to a LFT order to buy (i.e., on the ask side of the quotes from the perspective of the HFT) and S refers to a LFT order to sell (on the bid side of the HFT's quotes). The accuracy level of the signal is $1/2 \leq p < 1$. If $p = 1/2$, the signal is uninformative. If $M^{\text{dir}} \in \{B, S\}$ denotes the direction of the next LFT order, and S^{dir} denote the most recent directional signal received, then by definition

$$\mathbb{P}(S^{\text{dir}} = B | M^{\text{dir}} = B) = p \quad \text{and} \quad \mathbb{P}(S^{\text{dir}} = S | M^{\text{dir}} = B) = (1 - p). \tag{2.6}$$

Given that buy and sell LFT orders are equally likely, the unconditional probabilities for $\mathbb{P}\left(S^{\text{dir}} = B\right)$ and $\mathbb{P}\left(S^{\text{dir}} = S\right)$ are

$$\mathbb{P}\left(S^{\text{dir}} = B\right) = \mathbb{P}(M^{\text{dir}} = B)\mathbb{P}(S^{\text{dir}} = B | M^{\text{dir}} = B) + \mathbb{P}(M^{\text{dir}} = S)\mathbb{P}(S^{\text{dir}} = B | M^{\text{dir}} = S) = 0.5$$

and similarly $\mathbb{P}\left(S^{\text{dir}} = S\right) = 0.5$. The conditional probabilities, $\mathbb{P}(M^{\text{dir}} = B|S^{\text{dir}} = B)$ and $\mathbb{P}(M^{\text{dir}} = S|S^{\text{dir}} = B)$, using Bayes' rule, behave as expected:

$$\mathbb{P}(M^{\text{dir}} = B|S^{\text{dir}} = B) = \frac{\mathbb{P}(M^{\text{dir}} = B)\mathbb{P}(S^{\text{dir}} = B|M^{\text{dir}} = B)}{\mathbb{P}\left(S^{\text{dir}} = B\right)} = \frac{0.5p}{0.5p + 0.5(1-p)} = p.$$
$$\mathbb{P}(M^{\text{dir}} = S|S^{\text{dir}} = B) = \frac{\mathbb{P}(M^{\text{dir}} = S)\mathbb{P}(S^{\text{dir}} = B|M^{\text{dir}} = S)}{\mathbb{P}\left(S^{\text{dir}} = B\right)} = \frac{0.5(1-p)}{0.5p + 0.5(1-p)} = 1-p.$$

The conditional probabilities for $S^{\text{dir}} = S$ are obtained by symmetry.

Signals arrive to the HFT as a Poisson process with rate θ. In order to avoid introducing any autocorrelation in the order flow from the signal due to (2.6), we assume that each previous signal is cancelled by either the arrival of a new signal (on a Poisson scale θ) or by the arrival of a LFT order (on a Poisson scale λ). It is natural to assume that each transaction cancels the previous signal and replaces it with a fresh one since the fact that the transaction occurred will often itself be informative. Both events lead to the replacement of the previous signal by a new signal, drawn from its unconditional distribution.

2.5.2 Second Signal: Type of Trader

We assume that the HFT is able, imperfectly again, to predict the type of LFT behind the next incoming order. This introduces the potential for price discrimination by the HFT: if his signal indicates that the next incoming order is more likely to come from an impatient investor, the HFT could potentially widen his quotes to benefit from the impatient LFTs willing to buy at a higher price and sell at a lower one. Institutional investors often criticize HFTs for providing "phantom liquidity", arguing that quotes at the best bid and ask suddenly disappear when they try to act on them, leaving them with quotes away from the best bid and ask prices. Quote widening by the HFT may not necessarily happen, though, as the HFT trades off the potential for a higher gain with the lower probability of a trade, since the signal could be incorrect and a patient LFT may materialize, as well as the HFT's inventory concerns.

Like the first, this signal is an i.i.d. Bernoulli random variable, $S^{\text{type}} \in \{P, I\}$, with P and I refer to patient and impatient LFTs respectively. The accuracy level of the second signal is $1/2 \leq q < 1$. As before, if $q = 1/2$, the second signal becomes uninformative. Letting $M^{\text{type}} \in \{P, I\}$ denote the type of the next LFT, we have

$$\mathbb{P}(S^{\text{type}} = P|M^{\text{type}} = P) = q \quad \text{and} \quad \mathbb{P}(S^{\text{type}} = I|M^{\text{type}} = P) = (1-q). \tag{2.7}$$

From 2.3, the unconditional distributions of patient and impatient market orders are given by

$$\mathbb{P}(M^{\text{type}} = P) = \frac{\lambda_P}{\lambda_P + \lambda_I} \text{ and } \mathbb{P}(M^{\text{type}} = I) = \frac{\lambda_I}{\lambda_P + \lambda_I}. \tag{2.8}$$

We can then compute the unconditional probabilities $\mathbb{P}\left(S^{\text{type}} = P\right)$ and $\mathbb{P}\left(S^{\text{type}} = I\right)$:

$$\begin{aligned}\mathbb{P}\left(S^{\text{type}} = P\right) &= \mathbb{P}(M^{\text{type}} = P)\mathbb{P}(S^{\text{type}} = P|M^{\text{type}} = P) + \mathbb{P}(M^{\text{type}} = I)\mathbb{P}(S^{\text{type}} = P|M^{\text{type}} = I) \\ &= \frac{q\lambda_P + (1-q)\lambda_I}{\lambda_P + \lambda_I}\end{aligned}$$

Similarly, we have $\mathbb{P}\left(S^{\text{type}} = I\right) = \frac{q\lambda_I + (1-q)(\lambda_P)}{\lambda_P + \lambda_I}$. We can then compute the conditional probabilities, $\mathbb{P}(M^{\text{type}} = P|S^{\text{type}} = P)$ and $\mathbb{P}(M^{\text{type}} = I|S^{\text{type}} = P)$ using Bayes' rule:

$$\begin{aligned}\mathbb{P}(M^{\text{type}} = P|S^{\text{type}} = P) &= \frac{\mathbb{P}(M^{\text{type}} = P)\mathbb{P}(S^{\text{type}} = P|M^{\text{type}} = P)}{\mathbb{P}\left(S^{\text{type}} = P\right)} = \frac{q\lambda_P}{q\lambda_P + (1-q)\lambda_I}. \\ \mathbb{P}(M^{\text{type}} = I|S^{\text{type}} = P) &= \frac{\mathbb{P}(M^{\text{type}} = I)\mathbb{P}(S^{\text{type}} = P|M^{\text{type}} = I)}{\mathbb{P}\left(S^{\text{type}} = P\right)} = \frac{(1-q)\lambda_I}{q\lambda_P + (1-q)\lambda_I}.\end{aligned}$$

The conditional probabilities for $S^{\text{type}} = I$ are obtained symmetrically. We assume again that each previous signal is cancelled by either the arrival of a new signal or by the arrival of a LFT order, so that each order arrival is preceded by a minimum of one new signal, in order to preclude any correlation between the type of incoming LFTs.

We can aggregate the conditional probabilities based on both signals $s = \left(S^{\text{dir}}, S^{\text{type}}\right)$ by independence of the two signals. For example,

$$\begin{aligned}\mathbb{P}(M^{\text{dir}} = S, M^{\text{type}} = P|S^{\text{dir}} = S, S^{\text{type}} = P) &= \mathbb{P}(M^{\text{dir}} = S|S^{\text{dir}} = S)\mathbb{P}(M^{\text{type}} = P|S^{\text{type}} = P) \\ &= \frac{pq\lambda_P}{q\lambda_P + (1-q)\lambda_I}.\end{aligned} \tag{2.9}$$

2.5.3 Two Notions of Speed

Recall that the HFT is able to make quoting decisions upon the arrival of a Poisson process with parameter μ. He makes his quoting decisions on the basis of the most recent signals received before the arrival of his quote decision time. Once his quotes are in place, it is possible for a new set of signals to arrive before the next incoming market order, drawn from the signals' unconditional distribution and the new set of signals may be different from the existing one. If the HFT cannot update his quotes (no arrival of his quote decision time in between) then he will get stuck quoting of the basis of an out-of-date signal when the next market order arrives. The new order will always be in accordance with the latest signal, whether the HFT was able to act upon it or not. Thus a

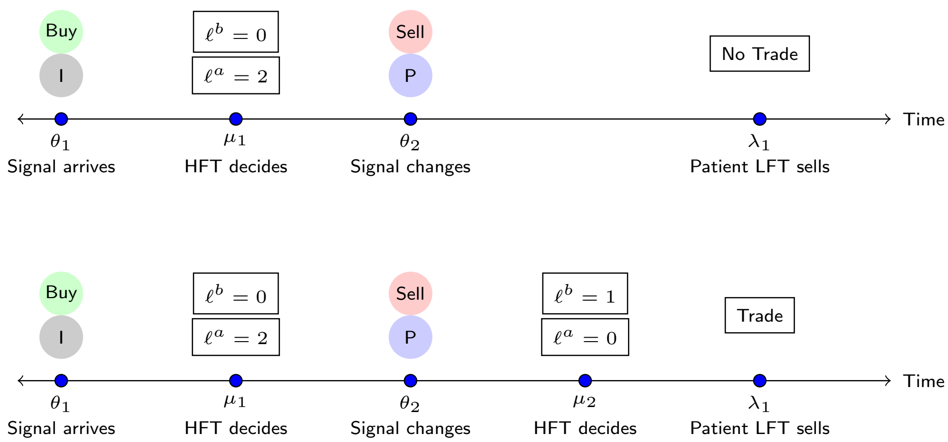

Fig. 2. Potential sequence of signal and order arrivals and HFT's corresponding actions.

high arrival rate of the signals (θ) relative to the HFT decision speed (μ) is a risk factor from the HFT's perspective.

The two parameters μ and θ represent two different notions of speed that need not be tied: while μ is the speed at which the HFT reacts to market events (as determined by his technological level), θ is the speed at which the market evolves (as determined by the rate at which incoming orders change and he receives information regarding them). There is a potential conflict between the two from the HFT's perspective.

Figure 2 illustrates the quoting decisions of a relatively slower and faster HFTs respectively in response to two sequences of signal and order arrivals. In the upper sequence, the relatively slower HFT gets stuck with a quote determined on the basis of an out-of-date signal. Of course, the HFT internalizes the possibility of this happening when he computes his optimal quoting strategy. In the lower sequence, a relatively faster HFT is able to update his quotes in response to the latest signal and take advantage of it before the next LFT order arrives.

2.6. HFT Inventory Aversion and Objective Function

The HFT's position in the asset is denoted by x_t, starting with $x_0 = 0$. This position can be positive or negative. In the case of a stock, this means that we impose no restrictions on short selling, while in the case of a futures contract a positive (resp. negative) value of x_t denotes a long (resp. short) net position in the contract. We assume that the HFT is risk-neutral, but penalizes itself for holding excess inventory at a rate of $\Gamma|x_t|$ where Γ is a constant parameter of inventory aversion. In practice, limiting or penalizing inventory is one of the primary sources of risk mitigation by HFTs and is often what is coded into the actual algorithms that perform market making. Inventory aversion is also the reason why, even without competition and no jumps in the fundamental value, so no possibility of adverse selection, a monopolistic HFT may not systematically quote on both sides of the market and attempt to systematically capture every spread.

The HFT's objective is to maximize his expected discounted rewards earned from transacting against the incoming order flow from LFTs, which earns him the bid-ask spread, minus the amount of price jump if the quote is stale, and the potential penalty costs from holding an inventory. The discount rate $D > 0$ is assumed to be constant. Let π denote any feasible policy that chooses ℓ^b and ℓ^a at decision times, T_k^q, and T_i^a be the ith sell order submitted by LFT type $y_i^a \in \{P, I\}$ and T_j^b is the jth market buy order submitted by the LFT type $y_j^b \in \{P, I\}$ where i,j and k are positive integers. To track the most recent decision time by the HFT before the arrival of market orders, define

$$\begin{cases} \tau_i = \max\left\{k : T_k^q < T_i^a\right\} \\ \tau_j = \max\left\{k : T_k^q < T_j^b\right\}. \end{cases} \tag{2.10}$$

There are three potential outcomes in terms of the HFT's reward function when an LFT submits a buy or sell order. By symmetry, we focus on the bid side of the HFT's quotes, i.e., the ith sell order. The first case refers to a zero payoff due to no trade. If the HFT is not quoting at the bid side ($\ell_{\tau_i}^b = 0$) or the fundamental price has a positive jump after the most recent decision time ($X_{T_i^a} - X_{\tau_i} > 3C$), there will be no trade. If a patient LFT sends the ith sell order ($y_i^a = P$) and the HFT's quote is at the second-best bid ($\ell_{\tau_i}^b = 2$), then there is again no trade.

In the second case, the HFT gains C by trading with either a patient or an impatient LFT ($y_i^a \in \{P, I\}$) when he is quoting at the best bid ($\ell_{\tau_i}^b = 1$) and the fundamental price has not changed since the HFT's most recent decision time ($X_{T_i^a} = X_{\tau_i}$). If there has been a negative jump during this period, the HFT would lose the jump amount mJ despite his spread gain where $X_{T_i^a} - X_{\tau_i} = -mJ$ with $m = 0, 1, 2 \ldots$.

In the third case, the HFT gains $3C$ if the order is submitted by an impatient LFT ($y_i^a = I$) when he is quoting at the second-best bid ($\ell_{\tau_i}^b = 2$) and the fundamental price has not changed since the HFT's most recent decision time ($X_{T_i^a} = X_{\tau_i}$). If there has been a negative jump during this period, the HFT would again lose the jump amount mJ ($X_{T_i^a} - X_{\tau_i} = -mJ$).

We can summarize these cases with the following HFT gains function from LFTs' sell orders:

$$G^-(\ell, y, X_T, X) = \begin{cases} C - mJ & \text{if } \ell = 1, X_T - X = -mJ \\ 3C - mJ & \text{if } \ell = 2, y = I, X_T - X = -mJ \\ 0 & \text{otherwise.} \end{cases} \tag{2.11}$$

Similarly, we can obtain the symmetric function to accommodate the gains from LFTs' buy orders:

$$G^{+}(\ell, y, X_T, X) = \begin{cases} C - mJ & \text{if } \ell = 1, X_T - X = mJ \\ 3C - mJ & \text{if } \ell = 2, y = I, X_T - X = mJ \\ 0 & \text{otherwise.} \end{cases} \tag{2.12}$$

Putting it all together, the HFT has the following objective function:

$$\max_{\pi} \mathbb{E}^{\pi} \left[\sum_{i=1}^{\infty} e^{-DT_i^a} G^{-}(\ell^b_{\tau_i}, y^a_i, X_{T_i^a}, X_{\tau_i}) + \sum_{j=1}^{\infty} e^{-DT_j^b} G^{+}(\ell^a_{\tau_j}, y^b_j, X_{T_j^b}, X_{\tau_j}) \right.$$
$$\left. - \Gamma \int_0^{\infty} e^{-Dt} |x_t| dt \right] \tag{2.13}$$

The first (resp. second) term is the HFT's gain from an incoming sell (resp. buy) order crossed against his existing limit order net of adverse selection costs due to stale quotes. The last term captures the HFT's inventory penalty costs.

2.7. *Limitations of the Model*

To keep the model tractable, a number of elements are left out and simplifying assumptions are made.

First, this paper is only about high frequency market making, not every possible trading strategy that a HFT might employ: in particular, the HFT in our model does not make a strategic choice between limit and market orders, but employs limit orders only. We refer to it as a "HFT" rather than a "HFMM" for ease of exposition, but this paper is only about "HFMM". In Rosu (2009), traders dynamically choose between limit and market orders; however they are classical in the sense that there is no speed differential, unlike the issue we are focusing on here.

Second, the HFT does not place orders larger than for one contract, so the quantity is not a strategic variable chosen by the HFT. This implies that the relevant notion of market depth at a given price tick in our model is the probability that a quote will be placed by the HFT at that price tick rather than the number of shares available.

Third, HFTs' limit orders are always placed at the best bid and ask price or at the next prices one tick away from the best ones, rather than at every possible price level. Extending the model to allow for quotes at levels further removed from X needlessly complicates the model and provides no clear further insights. But it means that we exclude order placement strategies known as "quote stuffing" that place large numbers of quotes far away from the best prices to falsely give the impression to other traders of an incoming imbalance, presumably without the intent of ever executing these orders. This sort of strategy is arguably not part of market making. It also means that we preclude the use by HFTs of the now-banned "stub quotes", which are place-holding quotes far from the current market price, employed by market makers to post quotes without any desire to trade, but which may become relevant in a flash crash event for instance. Nevertheless, in the model, the HFT decision to quote at the best bid and ask prices vs. quoting at one level removed from them is strategic, changes the spread faced by LFTs, and consequently the spread is determined endogenously in the model.

Fourth, we do not allow for parameter uncertainty: the HFT knows the parameters of the model, including the arrival rates of the various types of LFTs. Cartea et al. (2016) study market making strategies that are robust to model misspecification.

Finally, we do not model explicitly any rebate that might be provided by the exchange to market makers (or payment for order flow in the form of a rebate to liquidity takers). But from the perspective of the HFT, a rebate to liquidity makers can be easily incorporated into the G^- and G^+ functions.

3. Optimal Market Making in a Simplified Model

The key advantage of the model's Poisson-based setup is that we can convert the existing continuous-time model to a discrete-time equivalent one in terms of the arrival times of the LFTs' orders, signals and the HFT's decision time. This effectively merges the different Poisson clocks into a single chronologically-ordered one (with arrival rate equal to the sum of the individual ones), and then performing a time change from the (random) Poisson clock to the corresponding discrete-time event clock.

To demonstrate, we first analyze a simplified version of the model where the fundamental price is constant ($\sigma = 0$), and the arrival rate of impatient LFTs is zero ($\lambda_I = 0$). In this case, the second signal about the likely type of LFT submitting the next order becomes irrelevant, and the only signal will be $s = S^{\text{dir}}$, predicting the likely direction of the next order submitted by LFTs. In the next section, we build on the results of this simplified analysis and solve the full model.

We describe the main elements of the solution method and leave the technical details to the Appendix, where we also prove the results stated in this section. The state space after the discrete-time transformation is (x, s, l, e) where $x \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ denotes the current holdings or inventory of the HFT. The second state variable is the order direction signal, $s \in \{B, S\}$. The third state variable, $l \in \{00, 10, 01, 11\}$, denotes the quotes of the HFT currently in the limit order book. $l = 10$ denotes that the HFT is quoting at the best available bid price and is not quoting at the best ask price, and similarly for the other three possible values. In the simplified model, there is no point for the HFT to quote away from the best bid and ask prices in the absence of impatient LFTs.

Recall that the HFT can only make quoting decisions after μ-events in the merged Poisson clock. In the remaining Poisson arrivals, the HFT does not have the ability to change his existing quotes but can only maintain his quotes initiated at the most recent decision time. The fourth state variable, $e \in \{0, 1\}$, is therefore a binary state variable denoting when $e = 1$ that the discrete date in the merged clock corresponds to a $\mu-$event , in which case the HFT can revise his quotes; whereas $e = 0$ refers to the arrival of either a LFT order (λ) or a signal event (θ), in which cases the HFT cannot revise his quotes in this state.

The action taken by the HFT at each state is whether to quote a limit order or not at the best bid and the best ask, i.e., $d^b(x, s, l, e) \in \{0, 1\}$ and $d^a(x, s, l, e) \in \{0, 1\}$. When $e = 0$, the corresponding action taken by the HFT is determined by l so we can actually consider these states as fake decision epochs that force the HFT to continue with existing active quotes. For example, if $l = 10$, then $d^{b*}(x, s, 10, 0) = 1$ and $d^{a*}(x, s, 10, 0) = 0$. Thus, the relevant states for the HFT's decision making are the ones when $e = 1$ and in this case the existing active quotes given by l are no longer binding. For this purpose, we will suppress the last two states when we refer to the optimal market-making policy writing $\ell^{b*}(x, s) = d^{b*}(x, s, l, 1)$ and $\ell^{a*}(x, s) = d^{a*}(x, s, l, 1)$.

3.1. *Optimal Market Making Policy by the HFT*

In the simplified model, the optimal policy of the HFT reflects only the economic trade-off between quoting in order to capture the spread as often as possible and the risk of quoting too much and increasing his inventory. We let $V(x, s, l, e)$ be the value function of the HFT at state (x, s, l, e). Since the model is symmetric around the bid and ask side of the market, we can first eliminate s from the state space:

Lemma 1.

$$V(-x, S, l, e) = \begin{cases} V(x, B, l, e) & \textit{when } l \in \{00, 11\}, \\ V(x, B, 01, e) & \textit{when } l = 10, \\ V(x, B, 10, e) & \textit{when } l = 01. \end{cases}$$

Second, the value function at a decision time is independent of the existing quotes of the HFT, as at this state $(e = 1)$, the HFT can revise his quotes without any restriction. Formally, we have the following result:

Lemma 2.

$$V(x,B,00,1) = V(x,B,01,1) = V(x,B,10,1) = V(x,B,11,1).$$

We show in Section A in the Appendix how to express the transition probabilities of the system, and the HFT's reward function, as a function of the HFT's actions. Using these results, we can concisely state the Hamilton-Jacobi-Bellman optimality equations for the HFT's value functions:

Proposition 1. *Let $v(x,l) \equiv V(x,S,l,0)$ and $h(x) \equiv V(x,S,00,1)$. Then, $v(x,l)$ and $h(x)$ jointly satisfy*

$$\begin{aligned}
v(x,00) &= -\gamma|x| + \delta\left(\frac{\mu}{r}h(x) + \frac{\theta+\lambda}{2r}v(x,00) + \frac{\theta+\lambda}{2r}v(-x,00)\right) \\
v(x,10) &= -\gamma|x| + \delta\Big(\frac{\mu}{r}h(x) + \frac{\theta+(1-p)\lambda}{2r}v(x,10) + \frac{\theta+(1-p)\lambda}{2r}v(-x,01) + \frac{p\lambda}{2r}v(x+1,00) \\
&\quad + \frac{p\lambda}{2r}v(-x-1,00) + \frac{pc\lambda}{r}\Big) \\
v(x,11) &= -\gamma|x| + \delta\Big(\frac{\mu}{r}h(x) + \frac{\theta}{2r}(v(x,11)+v(-x,11)) + \frac{p\lambda}{2r}v(x+1,01) + \frac{p\lambda}{2r}v(-x-1,10) \\
&\quad + \frac{(1-p)\lambda}{2r}v(x-1,10) + \frac{(1-p)\lambda}{2r}v(-x+1,01) + \frac{c\lambda}{r}\Big) \\
v(x,01) &= -\gamma|x| + \delta\Big(\frac{\mu}{r}h(x) + \frac{\theta+p\lambda}{2r}v(x,01) + \frac{\theta+p\lambda}{2r}v(-x,10) + \frac{(1-p)\lambda}{2r}v(x-1,00) \\
&\quad + \frac{(1-p)\lambda}{2r}v(-x+1,00) + \frac{(1-p)c\lambda}{r}\Big)
\end{aligned} \tag{3.1}$$

where $h(x) = \max\{v(x,00), v(x,01), v(x,10), v(x,11)\}$ and

$$\lambda \equiv 2\lambda_P, \quad r \equiv \lambda+\mu+\theta, \quad \delta \equiv \frac{r}{r+D}, \quad c \equiv \delta C \quad \textit{and} \quad \gamma \equiv \frac{\Gamma}{r+D}. \tag{3.2}$$

Proposition 1 shows that the HFT aims to choose the optimal action when the state is at $e = 1$ by maximizing over all possible quoting actions and the corresponding value is stored in $h(x)$. On the other hand, $v(x,l)$ computes the expected one-step reward resulting from possible transitions determined by the active quotes in state l.

We can now characterize the optimal quoting policy of the HFT. The following result shows that it is based on (endogenously determined) thresholds:

Theorem 1. *The optimal quoting policy π^* of the HFT consists in quoting at the best bid and the* *best ask according to a threshold policy, i.e., there exist $L^* < 0 < U^*$ such that*

$$\ell^{b*}(x,S) = \begin{cases} 1 & \text{when } x < U^* \\ 0 & \text{when } x \geq U^* \end{cases} \qquad \ell^{a*}(x,B) = \begin{cases} 1 & \text{when } x > -U^* \\ 0 & \text{when } x \leq -U^* \end{cases}$$

$$\ell^{a*}(x,S) = \begin{cases} 1 & \text{when } x > L^* \\ 0 & \text{when } x \leq L^* \end{cases} \qquad \ell^{b*}(x,B) = \begin{cases} 1 & \text{when } x < -L^* \\ 0 & \text{when } x \geq -L^* \end{cases}$$

The limits L^* and U^* are functions of the model parameters, but not of the state.

We can interpret the result of Theorem 1 as follows. U^* and $-U^*$ are the limits that matter for quoting in the direction of the anticipated sign of the next LFT order to arrive. Suppose that the HFT holds a positive inventory $x > 0$. If he receives the signal $s = S$, he is going to act upon it by placing or keeping a limit order to buy, i.e., on the bid side of the book ($\ell^b = 1$), as long as his current long inventory is not already too high ($x < U^*$); if $x \geq U^*$ the HFT will not quote to buy and risk increasing his already positive inventory even further. Symmetrically, if he receives the signal B, he will quote to sell ($\ell^a = 1$) as long as his inventory is not already too negative ($x > -U^*$).

The HFT wishes to capture the spread, and his signal is imperfect, so the HFT may often quote on the side of the book that is opposite to what the signal predicts. For example, even if he receives the signal $s = S$, he may place or keep a limit order of the ask side of the book, that is, offer to sell from his inventory. L^* and $-L^*$ are the limits that matter for quoting in that case. If the HFT receives the signal $s = S$, he is going to quote to sell ($\ell^a = 1$) as long as his current long inventory is not already too negative ($x > L^*$). Symmetrically, if he receives the signal B, he will quote to buy ($\ell^b = 1$) as long as his inventory is not too positive ($x < -L^*$).

These limits are of course such that the HFT will always quote to buy when his inventory is negative, and quote to sell when his inventory is positive, irrespective of his signal. The limits only bind for quoting in the direction that would increase his inventory further.

A final remark is that the HFT's inventory will be contained in the inventory region $[-N, N]$ where $N \equiv \max(|L^*|, |U^*|)$ since at the widest inventory levels, the HFT will have either buy or sell quotes.

3.2. Analytical Computation of L^* and U^*

We can use the structure of the optimal market making policy to derive a system of equations that will let us solve the threshold limits and the corresponding value functions. The following proposition provides a sufficient condition for threshold limits L and U to be optimal using the structure of the optimal policy:

Proposition 2. *The limits L and U are optimal if L satisfies $v(L,10) > v(L,11)$ and $v(L+1,10) \leq v(L+1,11)$, and U satisfies $v(U,01) > v(U,11)$ and $v(U-1,01) \leq v(U-1,11)$, and the value functions solve the system of equations (3.1) where the function $h(x)$ is replaced by the function*

$$m(x) = \mathbf{1}(x \leq L)v(x,10) + \mathbf{1}(L < x < U)v(x,11) + \mathbf{1}(x \geq U)v(x,01). \tag{3.3}$$

We propose a simple and efficient algorithm to solve for the optimal thresholds in the form of Algorithm 1 in Section A.5 in the Appendix.

4. Optimal Market Making in the Complete Model

We now solve the complete model as described in Section 2, adding back price volatility, impatient traders and a signal about trader type. Compared to the simplified model, the optimal policy of the HFT now reflects two additional trade-offs: quoting at the aggressive price level and transacting with every incoming LFTs vs. attempting to price-discriminate by quoting at a price one tick removed from the best one and trading only with impatient LFTs, but earning more from these rarer trades; and quoting in a volatile environment to earn the spread vs. running the adverse selection risk of stale quotes.

4.1. Action and State Space

In its full generality, the state space in our model can be represented by (x, s, l, e, j) where x is as before, and s is the most recent signal with now $s = \left(S^{\text{dir}}, S^{\text{type}}\right) \in \{BP, SP, BI, SI\}$. The third state variable, l, denotes the active quotes of the HFT in the limit order book at the best prices or one tick away from them, with $l \in \{00, 10, 20, 01, 02, 11, 22, 12, 21\}$. For example, $l = 12$ denotes that HFT is quoting at the best price level on the bid side and at the second level on the ask side. The fourth state variable, $e \in \{0, 1\}$, is as before. The fifth and last state variable, j, keeps track of the jumps realized in the asset fundamental value since the HFT's last quoting action, with $j \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

Using the symmetry result in Lemma 1, we can again reduce our signal state by only focusing on $s \in \{SP, SI\}$ as the remaining cases will be given by symmetric states. The action taken by the HFT at each state is whether to quote at the second-best or best available prices, i.e., $d^b(x, s, l, e, j) \in \{0, 1, 2\}$ and $d^a(x, s, l, e, j) \in \{0, 1, 2\}$. When $e = 0$, the corresponding action taken by the HFT is determined by l so we can, as in the simplified model, consider these states as fake decision epochs that force the HFT to continue with existing active quotes. For example, if $l = 10$, then $d^{b*}(x, s, 10, 0, j) = 1$ and $d^{a*}(x, s, 10, 0, j) = 0$. Thus, the relevant state for the HFT is when $e = 1$ and in this case the existing active quotes given by l is no longer binding. At this state, j also reverts to 0 as the HFT can now peg his quotes around the new fundamental price. For this purpose, we will suppress the last three states when we refer to the optimal market-making policy using $\ell^{b*}(x, s) = d^{b*}(x, s, l, 1, 0)$ and $\ell^{a*}(x, s) = d^{a*}(x, s, l, 1, 0)$.

4.2. Optimal Market Making

We show in the Appendix how to express the transition probabilities of the system, and the HFT's reward function, as a function of his actions. We again use two value functions to reflect the quoting decision times of the HFT. Let $V(x, s, l, e, j)$ denote the value function of the HFT. Using Lemma 2, we can again suppress some of the states in the case of $e = 1$ with $h(x, s) = V(x, s, 00, 1, 0)$, as the HFT can revise his quotes at this time. Similarly, let $v(x, s, l, j)$ be the value function in the no-decision event types, i.e., a LFT order, signal or volatility event arrives (i.e., when $e = 0$).

We can now characterize the optimal quoting policy of the HFT, showing that, as in the simplified model, it is based on thresholds but now with additional levels that dictate when the HFT should not quote ($\ell = 0$), quote at the best prices ($\ell = 1$) or quote at the second best prices ($\ell = 2$).

Theorem 2. *The optimal quoting policy π^* of the HFT consists in quoting according to a threshold policy, i.e., there exist U_P^0, U_P^1, U_I^0, U_I^1 and L_P^0, L_P^1, L_I^0, L_I^1 which are functions of the model parameters, but not of the state, such that*

$$\ell^{b*}(x, SP) = \begin{cases} 1 & x \leq U_P^1, \\ 2 & U_P^1 < x < U_P^0, \\ 0 & x \geq U_P^0. \end{cases} \quad \ell^{a*}(x, BP) = \begin{cases} 1 & x \geq -U_P^1, \\ 2 & -U_P^1 > x > -U_P^0, \\ 0 & x \leq -U_P^0. \end{cases}$$

$$\ell^{b*}(x, SI) = \begin{cases} 1 & x \leq U_I^1, \\ 2 & U_I^1 < x < U_I^0, \\ 0 & x \geq U_I^0. \end{cases} \quad \ell^{a*}(x, BI) = \begin{cases} 1 & x \geq -U_I^1, \\ 2 & -U_I^1 > x > -U_I^0, \\ 0 & x \leq -U_I^0. \end{cases}$$

$$\ell^{b*}(x, BP) = \begin{cases} 1 & x \leq L_P^1, \\ 2 & L_P^1 < x < L_P^0, \\ 0 & x \geq L_P^0. \end{cases} \quad \ell^{a*}(x, SP) = \begin{cases} 1 & x \geq -L_P^1, \\ 2 & -L_P^1 > x > -L_P^0, \\ 0 & x \leq -L_P^0. \end{cases}$$

$$\ell^{b*}(x, BI) = \begin{cases} 1 & x \le L_I^1, \\ 2 & L_I^1 < x < L_I^0, \\ 0 & x \ge L_I^0. \end{cases} \quad \ell^{a*}(x, SI) = \begin{cases} 1 & x \ge -L_I^1, \\ 2 & -L_I^1 > x > -L_I^0, \\ 0 & x \le -L_I^0. \end{cases}$$

The optimal quoting policy in the complete model can interpreted similarly as in the simplified model with obvious nuances. The limits U_P^0 and U_I^0 are the relevant limits for the HFT to stop quoting for inventory considerations when the direction signal is aligned with the direction of quoting. The limits L_P^0 and L_I^0 are the relevant limits for the HFT to stop quoting for inventory considerations when the direction signal is opposite to the direction of quoting. These limits now depend on the anticipated type of the LFT (patient or impatient) submitting the order. Since the inventory limits may be different in the presence of informative signals, the main implication of this quoting policy is endogenous cancellation of existing limit orders.

The second main difference is the additional set of limits, U_P^1, U_I^1, L_P^1 and U_I^1, that lets the HFT to decide between quoting at the best bid or at the second best bid. These limits are driven by the economic consideration based on the likelihood of trade given the accuracy of the signals and the relative ratio of patient LFTs to impatient LFTs, and the additional gain from transacting away from the best prices with an impatient LFT. The presence of the volatility and the arrival rate of arbitrageurs (which increases the risk caused by price jumps) also affect the optimal thresholds.

Beside its advantages in interpretation, this monotonic structure in the optimal policy can be used to develop a structured algorithm to solve the MDP (e.g., monotone policy iteration). Such an algorithm can reduce the computational effort significantly in model instances with large number of inventory states.

4.3. Comparative statics

We now provide a comparative statics analysis of the model, examining the sensitivity of the HFT's objective value to the model's parameters.

Proposition 3. *$h(0, s)$ is increasing in λ_I, λ_P, μ, p, q, C and decreasing in λ_A, θ, Γ, δ, and σ.*

This result shows the role that each parameter of the model plays in determining the overall value achieved by the HFT. The HFT achieves higher value when: he is faster ($\mu \uparrow$) since he is more likely to be able replenish his quotes between the arrival of successive LFT orders, thereby transacting with two LFT orders in close succession, more likely to be able to take advantage of the latest signal, and less likely to be caught with stale quotes when the asset value is volatile; he receives more LFT orders that he benefits from ($\lambda_I, \lambda_P \uparrow$) since he will capture the spread more often over time; his signal(s) are more accurate ($p, q \uparrow$) since he can price-discriminate more effectively and his inventory risk is lower, allowing him to adjust his quotes asymmetrically when close to his inventory limits; the tick size is higher ($C \uparrow$) since that is what he gains on each transaction. Conversely, the HFT achieves lower value when: more arbitrageurs arrive ($\lambda_A \uparrow$) and/or more fundamental price volatility occur ($\sigma \uparrow$) since both increase his risk of being picked off by arbitrageurs who take advantage of his stale quotes; his signals become more frequent ($\theta \uparrow$) since a signal update without the ability to update his quotes ($\mu-$event) is detrimental to the HFT as it makes it more likely that he will continue quoting on the basis of an outdated signal; he is more inventory averse ($\Gamma \uparrow$) since this is from his perspective a pure penalty with no benefits; his discount rate is higher ($\delta \uparrow$) since it lowers the present value of his rewards, i.e., his objective function.

5. Analysis of the Optimal Market Making Policy

The key insight of the existing models of market making with inventory holding costs (see e.g., Amihud and Mendelson (1980) and Ho and Stoll (1981)) is the inverse relationship between the willingness to quote and existing inventory. That is, the market maker posts less aggressive bid (ask) price when he has a large long (short) position in the asset. The resulting optimal market making policy in our model is also aligned with this feature but, thanks to the introduction of speed as a key determinant, provides further novel insights which are not captured in the classical market making literature.

First, our model implies that when the HFT is faster, he will provide more aggressive quotes as he can manage his inventory risk better by tracking the signals and price jumps. Second, due to time-varying signals and asymmetry in limits, the HFT may cancel his existing limit orders endogenously. Third, the HFT will engage in predatory quoting strategies by offering worse prices when his signal points to incoming impatient LFTs. Finally, the HFT will be more aggressive in quoting in the opposite direction of the signal in the presence of adverse selection, i.e., his willingness to sell (buy) the asset is higher when the signal points to an incoming market-sell (market-buy) order.

To illustrate, we will examine these features in a realistic calibration of the model: the HFT will be able to make decisions every 100 milliseconds which implies that $\mu = 600$ per minute; $C = \$0.005$ which makes the tick-size and the minimum spread to be \$0.01; the arrival rate of impatient LFTs on each side of the market is set to be $\lambda_I = 7.5$ per minute while the arrival rate of patient LFTs is given $\lambda_P = 22.5$ per minute which then implies that the total arrival rate of LFTs on both sides of the market is 1 order per second; the arrival rate of arbitrageurs is given by $\lambda_A = 300$ per minute on each side, which implies the same rate of arrivals as the HFT's decision events in aggregate. We set $D = 10^{-6}$ per minute so that the corresponding annualized discount rate is roughly 10% per year. For the accuracy of the signals, we set $p = 0.7$ and $q = 0.6$, i.e., the signal will predict the correct sign of the next market order with 70% chance and the type of the LFT submitting the next order with 60% chance. The signal will be subject to change $\theta = 30$ times per minute on average. The fundamental price will be subject to a jump occurring 10 times per minute on average, i.e., $\sigma = 10$. Each jump will be in the amount \$0.04 in either positive or negative direction with equal probability. Given that $\mu \gg \sigma$, we use a truncated state space for the number of jumps since the HFT's last quoting action with letting $j \in \{-1, 0, 1\}$. Finally, $\Gamma = 0.05$ so that HFT is paying \$0.05 per minute for each non-zero inventory he is holding.

With these parameters, the quoting limits are computed as $U_P^1 = 0$, $U_P^0 = 4$, $U_I^1 = -1$, $U_I^0 = 5$, and $L_P^1 = 0$, $L_P^0 = 2$, $L_I^1 = -1$, $L_I^0 = 4$. The HFT optimal quoting policy is given in (5.1). When the optimal policy consists in quoting $\ell^{b*} = 0$ and $\ell^{a*} = 1$, we write 01 below:

Signal (s) / Inventory (x)	-5	-4	-3	-2	-1	0	1	2	3	4	5
SP	10	10	10	10	12	11	21	21	21	01	01
SI	10	10	12	12	12	22	21	21	21	21	01
BP	10	10	12	12	12	11	21	01	01	01	01
BI	10	12	12	12	12	22	21	21	21	01	01

(5.1)

The quote aggressiveness of the optimal market making policy at the bid side is captured by U_P^0 when $s = SP$, U_I^0 when $s = SI$, L_P^0 when $s = BP$, and L_I^0 when $s = BI$. Our first novel insight suggests that these limits will increase as the HFT gets faster. Figure 3 illustrate this key implication of the model as we increase the speed of the HFT from $\mu = 50$ to $\mu = 600$. This finding suggests that the HFT quotes at wider inventory limits when his speed increases which implies higher liquidity provision, and is a desirable feature from the perspective of liquidity demanders since it implies a higher probability that their trading needs will be fulfilled.[3]

It is worthwhile to emphasize that the increase in quote aggressiveness is not purely due to lower risk of leaving stale quotes (see e.g., Foucault et al. (2003) and Hoffmann (2014)). Even in the absence of adverse selection costs due to volatility, the HFT will quote at the wider inventory limits as he can manage his inventory more efficiently by monitoring and replenishing his quotes more frequently. Figure 4 illustrates this insight when we plot the speed of the HFT against optimal thresholds while keeping the price volatility, σ, at zero.

[3] Aït-Sahalia and Sağlam (2016) verify this implication by considering various liquidity measures in a long-run equilibrium setting.

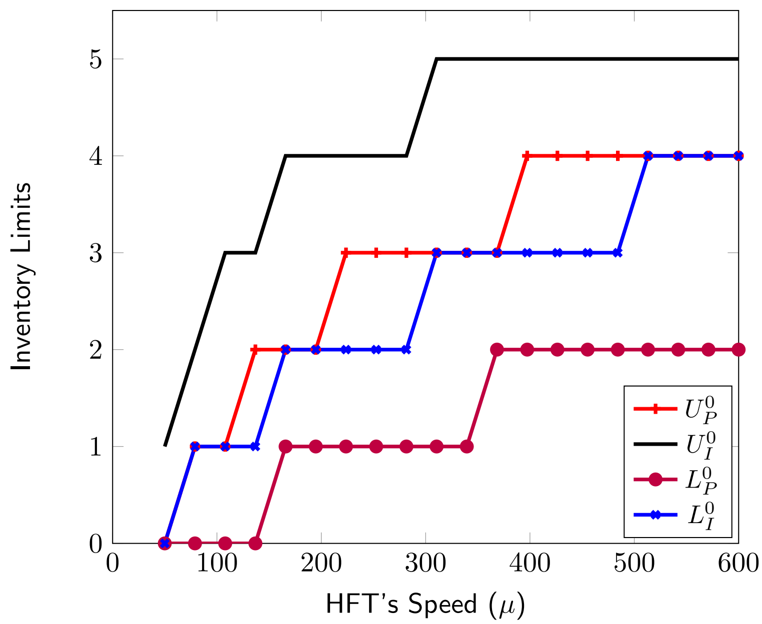

Fig. 3. HFT's inventory limits widen as he gets faster in the presence of stale quote risk.

The optimal market making policy with asymmetric inventory limits proposes another plausible mechanism for the observed cancellation of limit orders by HFTs in the empirical data. The optimal policy expressed in (5.1) illustrates this new cancellation motive that is absent in the classical market making literature. For example, when the signal is $s = SI$ and the HFT has a long inventory of 2 lots, he has a limit order to buy the asset at $X_t - 3C$. If the signal gets revised to $s = BP$, then the HFT will cancel this limit order.

The HFT's order anticipation ability also gives rise to predatory quoting motive as illustrated in the optimal policy in (5.1). Specifically, when the HFT has a neutral position with a latest observed signal of $s = SP$ or BP, he is quoting on both sides of the market at the best bid and ask but when the signal points to the potential arrival of an impatient LFT, the HFT starts quoting at a tick away from the best prices. This type of predatory trading activity by the HFTs has been empirically documented. For example, Korajczyk and Murphy (2016) and Kervel and Menkveld (2015) observe that institutional investors pay higher trading costs as the HFTs drop liquidity provision during their large-order execution. These findings are broadly consistent with our model implications as we can draw a parallel between large-order executions by institutional investors and the impatient LFTs.

Finally, the optimal quoting policy in our model provides a new motive for quoting through the use of order direction signal. In the presence of adverse selection due to stale quotes, the HFT utilizes the order direction signal to determine his quote aggressiveness. For example, the optimal

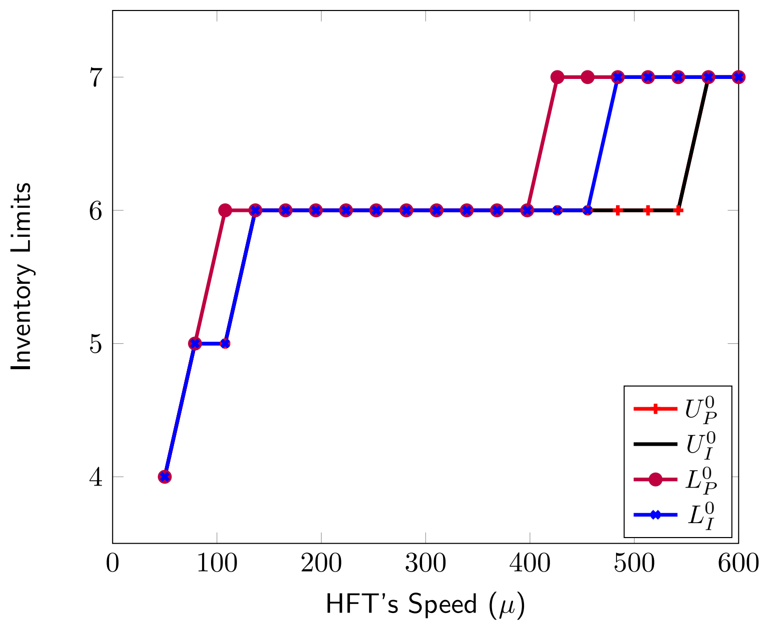

Fig. 4. HFT's inventory limits widen as he gets faster in the absence of stale quote risk, i.e., $\sigma = 0$.

quoting policy in (5.1) illustrates that when the signal points to an incoming market-sell order that will transact with HFT's buy order, the HFT is less aggressive to sell the asset to hedge himself against a potential stale quote. Specifically, in the case of $s = SP$, he stops quoting at the ask side when $x = -L_P^0 = -2$ whereas in the case of $s = BP$, he keeps quoting at the ask side till $x = -U_P^0 = -4$, highlighting the increase in aggressiveness. However, if the stale quote risk is not substantially high (e.g., low volatility environment or no arbitrageur activity), the HFT can use this signal to increase the aggressiveness of the quote in the same direction of the signal.

6. Conclusions

We propose a theoretical model of high frequency market making in a setting designed to match the current market environment where liquidity is primarily provided by HFTs who are both faster and better informed than the rest of the market participants. We superpose different Poisson processes running on different time clocks to represent the arrival of different elements of market information and orders, resulting in a tractable and flexible framework where the optimal market making strategy of the HFT and the equilibrium between the HFT quotes and incoming LFTs' orders is fully characterized. We characterize analytically the optimal quoting policy of the HFT, and show the influence of the economically-relevant parameters of the model. In a companion paper, Aït-Sahalia and Sağlam (2016), we analyze the implications of the model for the HFT's liquidity provision in different market environments, analyzing in particular the effect of volatility, competition with other market makers and the role of potential market regulations on the HFT's optimal quoting behavior.

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Appendix for "High Frequency Market Making: Optimal Quoting"
Technical Results and Proofs

A. Discrete-time Embedding in the Simplified Model

We start by recalling the definition of a discounted infinite horizon Markov Decision Process (MDP), before showing that our continuous-time HFT optimization problem can be represented as such. A MDP is defined by a 4-tuple, $(I, A_i, \mathbb{P}(.|i,a), \mathbb{R}(.|i,a))$, in which I is the state space, A_i is the action space, i.e., the set of possible actions that a decision maker can take when the state is $i \in I$, $\mathbb{P}(.|i,a)$ is the probability transition matrix determining the state of the system in the next decision time, and finally $\mathbb{R}(.|i,a)$ is the reward matrix, specifying the reward obtained using action a when the state is in i. The HFT seeks a quoting policy that maximizes the expected discounted reward

$$v(i) = \max_{\pi} \mathbb{E}\left[\sum_{t=0}^{\infty} \alpha^t \mathbb{R}(i_{t+1}|i_t, \pi(i_t))|i_0 = i\right], \tag{A.1}$$

where α is the discount rate. An admissible stationary policy π maps each state $i \in I$ to an action in A_i. Under mild technical conditions, we can guarantee the existence of optimal stationary policies (see Puterman (1994)). Conditioning on the first transition from i to i', we obtain the Hamilton-Jacobi-Bellman optimality equation

$$\begin{aligned}
v(i) &= \max_{\pi}\left\{\sum_{i'} \mathbb{P}(i'|i,\pi(i))\left(\mathbb{R}(i'|i,\pi(i)) + \alpha\mathbb{E}\left[\sum_{t=1}^{\infty} \alpha^{t-1}\mathbb{R}(i_{t+1}|i_t,\pi(i_t))|i_1 = i'\right]\right)\right\} \\
&= \max_{\pi}\left\{\sum_{i'} \mathbb{P}(i'|i,\pi(i))\left(\mathbb{R}(i'|i,\pi(i)) + \alpha\mathbb{E}\left[\sum_{k=0}^{\infty} \alpha^{k}\mathbb{R}(i_{k+1}|i_k,\pi(i_k))|i_k = i'\right]\right)\right\} \\
&= \max_{a \in A_i}\left\{\sum_{i'} \mathbb{P}(i'|i,a)\left(\mathbb{R}(i'|i,a) + \alpha v(i')\right)\right\}.
\end{aligned} \tag{A.2}$$

A.1. Transition Probabilities

We now calculate the transition probabilities at each state of the HFT in the simplified model. First, note that the state transitions occur at a rate of $\mu + \lambda + \theta$ where $\lambda \equiv 2\lambda_P$ and the transition rate is the same for all states and actions. Let $\mathbb{P}((x', s', l', e')|(x, s, l, e), (\ell^b, \ell^a))$ be the probability of reaching state (x', s', l', e') when the system is in state (x, s, l, e) and the trader takes the actions of ℓ^b and ℓ^a. First, we define

$$\mathsf{tr}(s) = \begin{cases} p & \text{if } s = S, \\ 1-p & \text{if } s = B. \end{cases}$$

Suppose that the current state of the HFT is (x, s, l, e). Let $r = (\lambda + \mu + \theta)$. We provide the transition probabilities with respect to four possible actions that can be employed by the HFT at decision epochs with $\ell^b \in \{0, 1\}$ and $\ell^a \in \{0, 1\}$.

First, if the HFT does not quote in either side of the market, the inventory level in the next state cannot change. If a new decision event arrives before the arrival of a market order or a signal, the state remains the same. Otherwise, we know that either a market order or a signal arrived before HFT makes a new quoting decision. Formally, we obtain

$$\mathbb{P}\left((x', s', l', e')|(x, s, l, 1), (0, 0)\right) = \begin{cases} \frac{(\lambda+\theta)}{2r} & \text{if } x = x', s' \in \{B, S\}, e' = 0, l' = 00, \\ \frac{\mu}{r} & \text{if } x = x', s = s', e' = 1, l' = 00, \\ 0 & \text{otherwise.} \end{cases}$$

When the HFT takes the action $(1, 0)$, he may increase his inventory by trading with the incoming market-sell order submitted by a patient LFT, which occurs with probability $\frac{\mathsf{tr}(s)\lambda}{r}$.

$$\mathbb{P}\left((x', s', l', e')|(x, s, l, 1), (1, 0)\right) = \begin{cases} \frac{(\lambda(1-\mathsf{tr}(s))+\theta)}{2r} & \text{if } x = x', s' \in \{B, S\}, e' = 0, l' = 10 \\ \frac{\lambda \mathsf{tr}(s)}{2r} & \text{if } x + 1 = x', s' \in \{B, S\}, e' = 0, l' = 00 \\ \frac{\mu}{r} & \text{if } x = x', e' = 1, s = s', l' = 10 \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, if the HFT takes the action $(0, 1)$, he may increase his inventory by trading with the incoming market-sell order submitted by a patient LFT, which occurs with probability $\frac{\mathsf{tr}(s)\lambda}{r}$. Similarly, he may decrease his inventory by only trading with the incoming market-buy order submitted by a patient LFT, which occurs with probability $\frac{(1-\mathsf{tr}(s))\lambda}{r}$.

$$\mathbb{P}\left((x', s', l', e')|(x, s, l, 1), (1, 1)\right) = \begin{cases} \frac{\theta}{2r} & \text{if } x = x', s' \in \{B, S\}, e' = 0, l' = 11 \\ \frac{\lambda \mathsf{tr}(s)}{2r} & \text{if } x + 1 = x', s' \in \{B, S\}, e' = 0, l' = 01 \\ \frac{\lambda(1-\mathsf{tr}(s))}{2r} & \text{if } x - 1 = x', s' \in \{B, S\}, e' = 0, l' = 10 \\ \frac{\mu}{r} & \text{if } x = x', e' = 1, s = s', l' = 11 \\ 0 & \text{otherwise.} \end{cases}$$

If the system is observed at the arrival time of a market order or signal event, that is $e = 0$, the HFT cannot revise his quotes. We can accommodate these states in our model using fake decisions that merely sets the action to the existing quotes tracked by $l = (l^b, l^a)$. In this case,

$$\mathbb{P}\left((x', s', l', e')|(x, s, l, 0), (\ell^b, \ell^a)\right) = \begin{cases} \mathbb{P}\left((x', s', l', e')|(x, s, l, 1), (\ell^b, \ell^a)\right) & \text{if } \ell^b = l^b, \ell^a = l^a \\ 0 & \text{otherwise.} \end{cases}$$

A.2. HFT's Reward Function

Let $\mathbb{R}\left((x', s', l', e')|(x, s, l, e), (\ell^b, \ell^a)\right)$ be the total reward achieved by the HFT when the system is in state (x, s, l, e), the HFT chooses quoting actions ℓ^b and ℓ^a and the system reaches the state (x', s', l', e'). In the simplified model, the HFT will be able to make C from trade events and will not lose anything in the absence of price jumps.

We would like to write the HFT's objective in (2.13) in the form of an MDP objective function as in (A.1). We first introduce the following notation. Let t_k be the time of the kth state transition due to a decision, signal or market order arrival (by convention $t_0 = 0$) and let η_k be the length of this cycle, i.e., $\eta_k = t_k - t_{k-1}$. We start with the positive reward terms in $G^-(.)$ and $G^+(.)$ that measure the spreads earned by the HFT when there is a trade. We can track this sum of the discounted rewards in our MDP framework with

$$\sum_{k=1}^{\infty} e^{-Dt_k}\mathbb{R}^+\left((x'_{t_k}, s'_{t_k}, l'_{t_k}, e'_{t_k})|(x_{t_{k-1}}, s_{t_{k-1}}, l_{t_{k-1}}, e_{t_{k-1}}), (\ell^b_{t_{k-1}}, \ell^a_{t_{k-1}})\right),$$

where

$$\mathbb{R}^+\left((x', s', l', e')|(x, s, l, e), (\ell^b, \ell^a)\right) = C\mathbb{1}\left(x + 1 = x'\right)\mathbb{1}\left(\ell^b = 1\right)$$

We can take the expectation of the HFT's discounted earnings using the independence of each cycle length, η_i, which is an exponentially distributed random variable with mean $1/r$:

$$\begin{aligned}
&\mathbb{E}\left[\textstyle\sum_{k=1}^{\infty} e^{-Dt_k}\mathbb{R}^+\left((x'_{t_k}, s'_{t_k}, b'_{t_k}, e'_{t_k})|(x_{t_{k-1}}, s_{t_{k-1}}, b_{t_{k-1}}, e_{t_{k-1}}), (\ell^b_{t_{k-1}}, \ell^a_{t_{k-1}})\right)\right] \\
&= \textstyle\sum_{k=1}^{\infty} \mathbb{E}\left[e^{-D\sum_{i=1}^{k}\eta_i}\right]\mathbb{E}\left[\mathbb{R}^+\left((x'_{t_k}, s'_{t_k}, b'_{t_k}, e'_{t_k})|(x_{t_{k-1}}, s_{t_{k-1}}, b_{t_{k-1}}, e_{t_{k-1}}), (\ell^b_{t_{k-1}}, \ell^a_{t_{k-1}})\right)\right] \\
&= \textstyle\sum_{k=1}^{\infty} \mathbb{E}\left[e^{-D\eta_1}\right]^k\mathbb{E}\left[\mathbb{R}^+\left((x'_{t_k}, s'_{t_k}, b'_{t_k}, e'_{t_k})|(x_{t_{k-1}}, s_{t_{k-1}}, b_{t_{k-1}}, e_{t_{k-1}}), (\ell^b_{t_{k-1}}, \ell^a_{t_{k-1}})\right)\right] \\
&= \textstyle\sum_{k=1}^{\infty} \left(\int_0^{\infty} re^{-(r+D)t}dt\right)^k\mathbb{E}\left[\mathbb{R}^+\left((x'_{t_k}, s'_{t_k}, b'_{t_k}, e'_{t_k})|(x_{t_{k-1}}, s_{t_{k-1}}, b_{t_{k-1}}, e_{t_{k-1}}), (\ell^b_{t_{k-1}}, \ell^a_{t_{k-1}})\right)\right] \\
&= \textstyle\sum_{k=1}^{\infty} \left(\frac{r}{r+D}\right)^k\mathbb{E}\left[\mathbb{R}^+\left((x'_{t_k}, s'_{t_k}, b'_{t_k}, e'_{t_k})|(x_{t_{k-1}}, s_{t_{k-1}}, b_{t_{k-1}}, e_{t_{k-1}}), (\ell^b_{t_{k-1}}, \ell^a_{t_{k-1}})\right)\right] \\
&= \delta\textstyle\sum_{k=0}^{\infty} \delta^k\mathbb{E}\left[\mathbb{R}^+\left((x'_{t_{k+1}}, s'_{t_{k+1}}, b'_{t_{k+1}}, e'_{t_{k+1}})|(x_{t_k}, s_{t_k}, b_{t_k}, e_{t_k}), (\ell^b_{t_k}, \ell^a_{t_k})\right)\right].
\end{aligned}$$

where δ is the "adjusted discount factor," defined in (3.2). Inventory costs in the third term of

(2.13) can be simplified as

$$
\begin{aligned}
\mathbb{E}\left[\Gamma \int_0^\infty e^{-Dt}|x_t|dt\right] &= \Gamma \sum_{k=0}^\infty \mathbb{E}\left[\int_{t_k}^{t_{k+1}} e^{-Dt}|x_t|dt\right] \\
&= \Gamma \sum_{k=0}^\infty \mathbb{E}\left[\int_{t_k}^{t_{k+1}} e^{-Dt}dt\right] \mathbb{E}\left[|x_{t_k}|\right] \\
&= \frac{\Gamma}{D} \sum_{k=0}^\infty \mathbb{E}\left[e^{-Dt_k}\right]\left(1-\mathbb{E}\left[e^{-D\eta_{k+1}}\right]\right) \mathbb{E}\left[|x_{t_k}|\right] \\
&= \frac{\Gamma}{D} \sum_{k=0}^\infty \delta^k \left(\frac{D}{\lambda+\mu+D}\right) \mathbb{E}\left[|x_{t_k}|\right] \\
&= \frac{\Gamma}{r+D} \sum_{k=0}^\infty \left(\frac{r}{r+D}\right)^k \mathbb{E}\left[|x_{t_k}|\right].
\end{aligned}
$$

Let $\mathbb{R}\left((x', s', b', e')|(x, s, l, e), (\ell^b, \ell^a)\right)$ be the probability of reaching state (x', s', l', e') when the system is in state (x, s, l, e) and the trader takes the actions of ℓ^b and ℓ^a. We are now ready to define the total reward matrix. Let

$$
\mathbb{R}\left((x', s', l', e')|(x, s, l, e), (\ell^b, \ell^a)\right) = c\mathbb{1}\left(x+1=x'\right)\mathbb{1}\left(\ell^b=1\right) + c\mathbb{1}\left(x-1=x'\right)\mathbb{1}\left(\ell^a=1\right) - \gamma|x| \tag{A.3}
$$

where c and γ, defined in (3.2) are the "adjusted spread" and "adjusted inventory aversion" parameters for the discrete-time formulation. Then, the HFT maximizes

$$
V(x, s, l, e) = \max_\pi \mathbb{E}^\pi\left[\sum_{k=0}^\infty \delta^k \mathbb{R}\left((x'_{t_{k+1}}, s'_{t_{k+1}}, b'_{t_{k+1}}, e'_{t_{k+1}})|(x_{t_k}, s_{t_k}, b_{t_k}, e_{t_k}), (\ell^b_{t_k}, \ell^a_{t_k})\right)\right], \tag{A.4}
$$

starting from his initial state, (x, s, l, e), which is in the requisite MDP form.

A.3. HFT's Value Function

We have now transformed our continuous-time problem into an equivalent discrete-time MDP. Using the Hamilton-Jacobi-Bellman optimality equations, $V(x, s, l, e)$ in (A.4) can be computed by solving the following set of equations:

$$
\begin{aligned}
V(x, s, l, e) = \max_{\ell^b,\ \ell^a} \Bigg\{ \sum_{(x', s', l', e')} & \mathbb{P}\left((x', s', l', e')|(x, s, l, e), (\ell^b, \ell^a)\right) \Big[\mathbb{R}\left((x', s', l', e')|(x, s, l, e), (\ell^b, \ell^a)\right) \\
& + \delta V(x', s', l', e')\Big]\Bigg\}.
\end{aligned} \tag{A.5}
$$

By substituting the expressions for $\mathbb{P}\left((x', s', l', e')|(x, s, l, e), (\ell^b, \ell^a)\right)$ and $\mathbb{R}\left((x', s', l', e')|(x, s, l, e), (\ell^b, \ell^a)\right)$, we can obtain the implicit equations for the value functions of each state.

Proof. Proof of Proposition 1Since the model is symmetric around the bid and ask side of the market, we can first eliminate s from our state space. We have that:

$$V(-x,S,l,e) = \begin{cases} V(x,B,l,e) & \text{when } l \in \{00,11\}, \\ V(x,B,01,e) & \text{when } l = 10, \\ V(x,B,10,e) & \text{when } l = 01. \end{cases}$$

Using this result, we let $v(x,l) \equiv V(x,S,l,0)$ and $h(x) \equiv V(x,S,l,1)$ for ease in notation and obtain the following set of equations for the value functions:

$$\begin{aligned}
v(x,00) &= -\gamma|x| + \delta\left(\frac{\mu}{r}h(x,00) + \frac{\theta+\lambda}{2r}v(x,00) + \frac{\theta+\lambda}{2r}v(-x,00)\right) \\
v(x,10) &= -\gamma|x| + \delta\Big(\frac{\mu}{r}h(x,10) + \frac{\theta+(1-p)\lambda}{2r}v(x,10) + \frac{\theta+(1-p)\lambda}{2r}v(-x,01) + \frac{p\lambda}{2r}v(x+1,00) \\
&\quad + \frac{p\lambda}{2r}v(-x-1,00) + \frac{pc\lambda}{r}\Big) \\
v(x,01) &= -\gamma|x| + \delta\Big(\frac{\mu}{r}h(x,01) + \frac{\theta+p\lambda}{2r}v(x,01) + \frac{\theta+p\lambda}{2r}v(-x,10) + \frac{(1-p)\lambda}{2r}v(x-1,00) \\
&\quad + \frac{(1-p)\lambda}{2r}v(-x+1,00) + \frac{(1-p)c\lambda}{r}\Big) \\
v(x,11) &= -\gamma|x| + \delta\Big(\frac{\mu}{r}h(x,11) + \frac{\theta}{2r}(v(x,11)+v(-x,11)) + \frac{p\lambda}{2r}v(x+1,01) + \frac{p\lambda}{2r}v(-x-1,10) \\
&\quad + \frac{(1-p)\lambda}{2r}v(x-1,10) + \frac{(1-p)\lambda}{2r}v(-x+1,01) + \frac{c\lambda}{r}\Big)
\end{aligned}$$

and

$$h(x) = \max\{v(x,00), v(x,01), v(x,10), v(x,11)\}.$$

□

A.4. *Proof of Theorem 1*

First, we prove some auxiliary results.

Lemma 3. *If $x \le -\frac{c}{\gamma(1-\delta)}$ ($x \ge \frac{c}{\gamma(1-\delta)}$) then $\ell^{b*} = 1$ and $\ell^{a*} = 0$ ($\ell^{b*} = 0$ and $\ell^{a*} = 1$)*

Proof. Proof of Lemma 3 We know that the discounted expected cost between decision epochs is $\gamma|x|$. We know that the maximum discounted revenue from earning spreads is less than $\frac{c}{1-\delta}$. Thus, you would not quote to sell (buy) if $x \le -\frac{c}{\gamma(1-\delta)}$ ($x \ge \frac{c}{\gamma(1-\delta)}$). □

We establish by induction that $v(x,l)$ is concave in x, $v(x,11) - v(x,01)$, $v(x,10) - v(x,00)$ is non-increasing in x and $v(x,10) - v(x,11)$, $v(x,01) - v(x,00)$ is non-decreasing in x.

We use induction on the steps of the dynamic programming operator. We focus on $v(x,l)$ for simplicity. Let $v^{(0)}(x,l)=0$ for all x and l. Then, in the base case, we obtain

$$\begin{aligned}
v^{(1)}(x,00) &= -\gamma|x|,\\
v^{(1)}(x,10) &= -\gamma|x| + \frac{\delta p\lambda c}{r},\\
v^{(1)}(x,01) &= -\gamma|x| + \frac{\delta(1-p)\lambda c}{r},\\
v^{(1)}(x,11) &= -\gamma|x| + \frac{\delta\lambda c}{r},
\end{aligned}$$

which shows that $v^{(1)}(x,l)$ is concave. Assume that $v^{(n)}(x,l)$ satisfies the induction hypothesis. Then, $v^{(n+1)}(x,l)$ satisfies the following set of equations:

$$\begin{aligned}
v^{(n+1)}(x,00) &= -\gamma|x| + \delta\left(\tfrac{\mu}{r}h^{(n)}(x) + \tfrac{\theta+\lambda}{2r}v^{(n)}(x,00) + \tfrac{\theta+\lambda}{2r}v^{(n)}(-x,00)\right)\\
v^{(n+1)}(x,10) &= -\gamma|x| + \delta\Big(\tfrac{\mu}{r}h^{(n)}(x) + \tfrac{\theta+(1-p)\lambda}{2r}v^{(n)}(x,10) + \tfrac{\theta+(1-p)\lambda}{2r}v^{(n)}(-x,01) + \tfrac{p\lambda}{2r}v^{(n)}(x+1,00)\\
&\quad + \tfrac{p\lambda}{2r}v^{(n)}(-x-1,00) + \tfrac{pc\lambda}{r}\Big)\\
v^{(n+1)}(x,01) &= -\gamma|x| + \delta\Big(\tfrac{\mu}{r}h^{(n)}(x) + \tfrac{\theta+p\lambda}{2r}v^{(n)}(x,01) + \tfrac{\theta+p\lambda}{2r}v^{(n)}(-x,10) + \tfrac{(1-p)\lambda}{2r}v^{(n)}(x-1,00)\\
&\quad + \tfrac{(1-p)\lambda}{2r}v^{(n)}(-x-1,00) + \tfrac{(1-p)c\lambda}{r}\Big)\\
v^{(n+1)}(x,11) &= -\gamma|x| + \delta\Big(\tfrac{\mu}{r}h^{(n)}(x) + \tfrac{\theta}{2r}(v^{(n)}(x,11) + v^{(n)}(-x,11)) + \tfrac{p\lambda}{2r}v^{(n)}(x+1,01)\\
&\quad + \tfrac{p\lambda}{2r}v^{(n)}(-x-1,10) + \tfrac{(1-p)\lambda}{2r}v^{(n)}(x-1,10) + \tfrac{(1-p)\lambda}{2r}v^{(n)}(-x+1,01) + \tfrac{c\lambda}{r}\Big)
\end{aligned}$$

Using this set of equations, $v^{(n+1)}(x,11)-v^{(n+1)}(x,01)$ is non-increasing as it is a positive sum of non-increasing functions:

$$\begin{aligned}
v^{(n+1)}(x,11) - v^{(n+1)}(x,01) &= \tfrac{\theta\delta}{2r}\left(v^{(n)}(x,11) - v^{(n)}(x,01)\right) + \tfrac{\theta\delta}{2r}\left(z^{(n)}(x,11) - z^{(n)}(x,01)\right)\\
&\quad + \tfrac{p\lambda\delta}{2r}\left(v^{(n)}(x+1,01) - v^{(n)}(x,01)\right) + \tfrac{p\lambda\delta}{2r}\left(z^{(n)}(x+1,01) - z^{(n)}(x,01)\right)\\
&\quad + \tfrac{(1-p)\lambda\delta}{2r}\left(v^{(n)}(x-1,10) - v^{(n)}(x-1,00)\right) + \tfrac{(1-p)\lambda\delta}{2r}\left(z^{(n)}(x-1,10) - z^{(n)}(x-1,00)\right) + \tfrac{pc\lambda\delta}{r},
\end{aligned}$$

where $z(x,l)\equiv v(-x,\mathsf{sym}(l))$. Similarly, $v^{(n+1)}(x,10)-v^{(n+1)}(x,00)$ is also non-increasing:

$$\begin{aligned}
v^{(n+1)}(x,10) - v^{(n+1)}(x,00) &= \tfrac{(\theta+(1-p)\lambda)\delta}{2r}\left(v^{(n)}(x,10) - v^{(n)}(x,00)\right) + \tfrac{(\theta+(1-p)\lambda)\delta}{2r}\left(z^{(n)}(x,10) - z^{(n)}(x,00)\right)\\
&\quad + \tfrac{p\lambda\delta}{2r}\left(v^{(n)}(x+1,00) - v^{(n)}(x,00)\right) + \tfrac{p\lambda\delta}{2r}\left(z^{(n)}(x+1,00) - z^{(n)}(x,00)\right) + \tfrac{pc\lambda\delta}{r}.
\end{aligned}$$

Using the same reasoning, $v^{(n+1)}(x,11)-v^{(n+1)}(x,10)$ and $v^{(n+1)}(x,01)-v^{(n+1)}(x,00)$ are nondecreasing in x as they are positive sum of non-decreasing functions:

$$\begin{aligned}
v^{(n+1)}(x,11) - v^{(n+1)}(x,10) &= \tfrac{\theta\delta}{2r}\left(v^{(n)}(x,11) - v^{(n)}(x,10)\right) + \tfrac{\theta\delta}{2r}\left(z^{(n)}(x,11) - z^{(n)}(x,10)\right) \\
&\quad + \tfrac{p\lambda\delta}{2r}\left(v^{(n)}(x+1,01) - v^{(n)}(x+1,00)\right) + \tfrac{p\lambda\delta}{2r}\left(z^{(n)}(x+1,01) - z^{(n)}(x+1,00)\right) \\
&\quad + \tfrac{(1-p)\lambda\delta}{2r}\left(v^{(n)}(x-1,10) - v^{(n)}(x,00)\right) + \tfrac{(1-p)\lambda\delta}{2r}\left(z^{(n)}(x-1,10) - z^{(n)}(x,00)\right) + \tfrac{(1-p)c\lambda\delta}{r}, \\
v^{(n+1)}(x,01) - v^{(n+1)}(x,00) &= \tfrac{(\theta+p\lambda)\delta}{2r}\left(v^{(n)}(x,01) - v^{(n)}(x,00)\right) + \tfrac{(\theta+p\lambda)\delta}{2r}\left(z^{(n)}(x,01) - z^{(n)}(x,00)\right) \\
&\quad + \tfrac{(1-p)\lambda\delta}{2r}\left(v^{(n)}(x-1,00) - v^{(n)}(x,00)\right) + \tfrac{(1-p)\lambda\delta}{2r}\left(z^{(n)}(x-1,00) - z^{(n)}(x,00)\right) + \tfrac{(1-p)c\lambda\delta}{r}.
\end{aligned}$$

Finally, $v^{(n+1)}(x,l)$ is concave for all l if $h^{(n)}(x)$ is concave. Let L and U satisfy

$$L \triangleq \max\{x : v(x,11) < v(x,10)\} \text{ and } U \triangleq \min\{x : v(x,11) < v(x,01)\}.$$

We now show that for all inventory regions $h^{(n)}(x)$ is concave. For $x \le L-1$,

$$h^{(n)}(x) - h^{(n)}(x+1) = v^{(n)}(x,10) - v^{(n)}(x+1,10),$$

which is nondecreasing as $v^{(n)}(x,10)$ is concave. For $x \ge U$,

$$h^{(n)}(x) - h^{(n)}(x+1) = v^{(n)}(x,01) - v^{(n)}(x+1,01),$$

which is nondecreasing as $v^{(n)}(x,01)$ is concave. If $x \ge L$ and $x+1 < U$,

$$\begin{aligned}
h^{(n)}(x) - h^{(n)}(x+1) &= v^{(n)}(x,10) - v^{(n)}(x+1,11) \\
&= (v^{(n)}(x,10) - v^{(n)}(x,11)) + (v^{(n)}(x,11) - v^{(n)}(x+1,11)),
\end{aligned}$$

which is again nondecreasing as it is the sum of nondecreasing functions. If $x \ge L$ and $x+1 = U$,

$$\begin{aligned}
&h^{(n)}(x) - h^{(n)}(x+1) = v^{(n)}(x,10) - v^{(n)}(x+1,01) \\
&= (v^{(n)}(x,10) - v^{(n)}(x+1,10)) + (v^{(n)}(x+1,10) - v^{(n)}(x+1,11)) \\
&+ (v^{(n)}(x+1,11) - v^{(n)}(x+1,01)),
\end{aligned}$$

which is again nondecreasing as it is the sum of nondecreasing functions. If $x > L$ and $x+1 < U$,

$$h^{(n)}(x) - h^{(n)}(x+1) = v^{(n)}(x,11) - v^{(n)}(x+1,11),$$

which is nondecreasing as $v^{(n)}(x,11)$ is concave. Finally, if $x > L$ and $x+1 = U$,

$$\begin{aligned} h^{(n)}(x) - h^{(n)}(x+1) &= v^{(n)}(x,11) - v^{(n)}(x+1,01) \\ &= (v^{(n)}(x,11) - v^{(n)}(x+1,11)) + (v^{(n)}(x+1,11) - v^{(n)}(x+1,01)), \end{aligned}$$

which is sum of nondecreasing functions.

A.5. *Computation of the HFT's Threshold Quoting Policy*

We proved that the optimal market making policy involves thresholds. In this section, we exploit this solution structure and provide an efficient algorithm to solve for the threshold limits L^* and U^*, and the value functions v. Using the linear system of equations in Proposition 1 and the optimality conditions in Proposition 2, we obtain the following algorithm to compute the optimal limits.

Algorithm 1: Efficient algorithm to compute L^* and U^*.

```
Output: L* and U*
Initialize L = −1, K = c/(γ(1−δ)) and flag = 0;
while flag = 0 do
    U ⟵ 1;
    while U ≤ K do
        Solve for v(L,l), v(L+1,l), ..., v(U,l) using the linear system in Proposition 1;
        if v(L,10) > v(L,11), v(L+1,10) ≤ v(L,11), v(U,01) > v(U,11),
         v(U−1,01) ≤ v(U−1,11) then
            flag ⟵ 1, L* ⟵ L and U* ⟵ U ;
            break ;
        U ⟵ U + 1;
    L ⟵ L − 1;
```

B. Analysis of the Complete Model

B.1. *Transition Probabilities*

We now calculate the transition probabilities at each state of the HFT. First, note that the state transitions occur at a rate of $\mu+\lambda+\theta+\sigma$ where $\lambda \equiv 2(\lambda_P+\lambda_I+\lambda_A)$ and using uniformization we can have the same transition rate for all states and actions. Let $\mathbb{P}((x',s',l',e',j')|(x,s,l,e,j),(\ell^b,\ell^a))$ be the probability of reaching state (x',s',l',e',j') when the system is in state (x,s,l,e,j) and the trader takes the actions of ℓ^b and ℓ^a. First, we define our auxiliary variables.

Let $\mathsf{pr}(s)$ denote the unconditional probability of receiving signal $s = \left(S^{\text{dir}}, S^{\text{type}}\right)$ right after the arrival of a market order or a signal.

$$\begin{aligned}
\mathsf{pr}(s) &= \sum_{i\in\{P,I\}} \sum_{j\in\{B,S\}} \mathbb{P}(M^{\text{type}} = i, M^{\text{dir}} = j)\mathbb{P}(s|M^{\text{type}} = i, M^{\text{dir}} = j) \\
&= \sum_{i\in\{P,I\}} \sum_{j\in\{B,S\}} (0.5/\lambda)\left(p\mathbb{1}\{s_1 = i\} + (1-p)\mathbb{1}\{s_1 \neq i\}\right)\left(q\mathbb{1}\{s_2 = j\} + (1-q)\mathbb{1}\{s_2 \neq j\}\right) \\
&\qquad\qquad \left(\lambda^I \mathbb{1}\{i = I\} + \lambda^P \mathbb{1}\{i = P\}\right) \\
&= \begin{cases} 0.5p\lambda^I/\lambda + 0.5(1-p)\lambda^P/\lambda & \text{if } s_1 = I, \\ 0.5(1-p)\lambda^I/\lambda + 0.5p\lambda^P/\lambda & \text{if } s_1 = P. \end{cases}
\end{aligned}$$

Let $\mathsf{m}_s^{s'}$ denote the conditional probability of receiving a market order with type s' (e.g., buy order submitted by an impatient LFT will be denoted by $s' = IB$) when the last signal appeared is s.

$$\begin{aligned}
\mathsf{m}_s^{s'} &= \frac{\mathbb{P}(M^{\text{type}} = s'_1, M^{\text{dir}} = s'_2)\mathbb{P}(S^{\text{type}} = s_1, S^{\text{dir}} = s_2|M^{\text{type}} = s'_1, M^{\text{dir}} = s'_2)}{\sum_{i\in\{P,I\}}\sum_{j\in\{B,S\}} \mathbb{P}(M^{\text{type}} = i, M^{\text{dir}} = j)\mathbb{P}(S^{\text{type}} = s_1, S^{\text{dir}} = s_2|M^{\text{type}} = i, M^{\text{dir}} = j)} \\
&= \left(p\mathbb{1}\{s_1 = s'_1\} + (1-p)\mathbb{1}\{s_1 \neq s'_1\}\right)\left(q\mathbb{1}\{s_2 = s'_2\} + (1-q)\mathbb{1}\{s_2 \neq s'_2\}\right) \\
&\left(\lambda^I \mathbb{1}\{s'_1 = I\} + \lambda^P \mathbb{1}\{s'_1 = P\}\right)/(\lambda\mathsf{pr}(s))
\end{aligned}$$

Suppose that the current state of the HFT is (x, s, l, e, j). Let $r = (\lambda + \mu + \theta + \sigma)$. First, we provide the transition probabilities with respect to each action taken at decision epochs.

If the HFT does not quote in either side of the market, the inventory level cannot change. If a new decision event arrives before the arrival of an LFT order, a signal or jump, the state for tracking jumps, j, reverts to zero. Since the arrival of arbitrageurs will not change the state of the system, we also have the additional self-transition at the rate of $2\lambda_A/r$ to uniformize the model.

Formally, we obtain

$$\mathbb{P}\left((x',s',l',e',j')|(x,s,b,1,j),(0,0)\right)=\begin{cases}\frac{(\lambda^{PI}+\theta)\mathsf{pr}(s')}{r} & \text{if } x=x', e'=0, l'=00\\ \frac{\sigma}{2r} & \text{if } x=x', e'=0, l'=00, j'=j+1\\ \frac{\sigma}{2r} & \text{if } x=x', e'=0, l'=00, j'=j-1\\ \frac{\mu}{r} & \text{if } x=x', e'=1, s=s', l'=00, j'=0\\ \frac{2\lambda_A}{r} & \text{if } x=x', e'=e, s=s', l'=l, j'=j\\ 0 & \text{otherwise,}\end{cases}$$

where $\lambda^{PI}\equiv 2(\lambda_P+\lambda_I)$.

We now analyze the trade scenarios for one-sided quoting. We are going to state the transition probabilities in which the HFT takes the action $(1,0)$. The remaining one-sided HFT actions are also very similar. When the HFT's action is $(1,0)$ he may increase his inventory by trading with the incoming market-sell order submitted by a patient or an impatient LFT, which occurs with probability $m_s^{IS}+m_s^{PS}$. We also need to account for the existence of stale quotes. Since the HFT is quoting at the bid side in this case, the stale quotes can only appear if $j<0$. Formally, we have the following transition probabilities:

$$\mathbb{P}\left((x',s',l',e',j')|(x,s,b,1,j),(1,0)\right)=\begin{cases}\frac{(\lambda^{\mathsf{sell}}+\theta)\mathsf{pr}(s')}{r} & \text{if } x=x', e'=0, l'=10, j\le 0\\ \frac{(\lambda_P+\lambda_I+\theta)\mathsf{pr}(s')}{r} & \text{if } x=x', e'=0, l'=10, j>0\\ \frac{\lambda^{\mathsf{buy}}\mathsf{pr}(s')}{r} & \text{if } x+1=x', e'=0, l'=00, j=0\\ \frac{\left(\lambda_A+\lambda^{\mathsf{buy}}\right)\mathsf{pr}(s')}{r} & \text{if } x+1=x', e'=0, l'=00, j<0\\ \frac{\sigma}{2r} & \text{if } x=x', e'=0, l'=10, j'=j+1\\ \frac{\sigma}{2r} & \text{if } x=x', e'=0, l'=10, j'=j-1\\ \frac{\mu}{r} & \text{if } x=x', e'=1, s=s', l'=10, j'=0\\ \frac{2\lambda_A}{r} & \text{if } x=x', e'=e, s=s', l'=l, j'=j\ge 0\\ \frac{\lambda_A}{r} & \text{if } x=x', e'=e, s=s', l'=l, j'=j<0\\ 0 & \text{otherwise,}\end{cases}$$

where $\lambda^{\mathsf{buy}}=\lambda^{PI}(m_s^{IS}+m_s^{PS})$ and $\lambda^{\mathsf{sell}}=\lambda^{PI}(m_s^{IB}+m_s^{PB})$ denotes the corresponding intensity for HFT's buy and sell trade in absence of any jumps, respectively.

We now analyze the trade scenarios for both-sided quoting. We are going to state the transition probabilities in which the HFT takes the action $(2,2)$. The remaining both-sided HFT actions are also very similar. When the HFT's action is $(2,2)$ he may increase his inventory by trading with the incoming market-sell order submitted by an impatient LFT, which occurs with probability m_s^{IS}. He may decrease his inventory by trading with the incoming market-buy order submitted by a patient LFT, which occurs with probability m_s^{IB}. We also need to account for the existence of stale quotes. Since the HFT is quoting at both sides of the market, the stale quotes will emerge when $j \neq 0$. Formally, we have the following transition probabilities:

$$\mathbb{P}\left((x', s', l', e', j')|(x, s, b, 1, j), (2, 2)\right) = \begin{cases} \frac{(\lambda_P+\theta)\mathsf{pr}(s')}{r} & \text{if } x = x', e' = 0, l' = 22, j = 0 \\ \frac{(\lambda^{\mathsf{buy}}+\theta)\mathsf{pr}(s')}{r} & \text{if } x = x', e' = 0, l' = 22, j > 0 \\ \frac{(\lambda^{\mathsf{sell}}+\theta)\mathsf{pr}(s')}{r} & \text{if } x = x', e' = 0, l' = 22, j < 0 \\ \frac{(\lambda_P+\lambda_I)m_s^{IS}\mathsf{pr}(s')}{r} & \text{if } x+1 = x', e' = 0, l' = 02, j = 0 \\ \frac{\left(\lambda_A+\lambda^{\mathsf{buy}}\right)\mathsf{pr}(s')}{r} & \text{if } x+1 = x', e' = 0, l' = 02, j < 0 \\ \frac{(\lambda_P+\lambda_I)m_s^{IB}\mathsf{pr}(s')}{r} & \text{if } x-1 = x', e' = 0, l' = 20, j = 0 \\ \frac{\left(\lambda_A+\lambda^{\mathsf{sell}}\right)\mathsf{pr}(s')}{r} & \text{if } x-1 = x', e' = 0, l' = 20, j < 0 \\ \frac{\sigma}{2r} & \text{if } x = x', e' = 0, l' = 22, j' = j+1 \\ \frac{\sigma}{2r} & \text{if } x = x', e' = 0, l' = 22, j' = j-1 \\ \frac{\mu}{r} & \text{if } x = x', e' = 1, s = s', l' = 22, j' = 0 \\ \frac{2\lambda_A}{r} & \text{if } x = x', e' = e, s = s', l' = l, j' = j = 0 \\ \frac{\lambda_A}{r} & \text{if } x = x', e' = e, s = s', l' = l, j' = j \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

If the system is observed at the arrival time of a market order or signal event, that is $e = 0$, the HFT cannot revise his quotes. We can accommodate these states in our model using fake decisions that merely sets the action to the existing quotes. In this case,

$$\mathbb{P}\Big((x', s', l', e', j')|(x, s, b, 0, j), (\ell^b, \ell^a)\Big) = \begin{cases} \mathbb{P}\left((x', s', l', e', j')|(x, s, b, 1, j), (\ell^b, \ell^a)\right) & \text{if } \ell^b = b_1, \ell^a = b_2 \\ 0 & \text{otherwise.} \end{cases}$$

B.2. HFT's Reward Function

Let $\mathbb{R}\left((x', s', l', e', j')|(x, s, l, e, j), (\ell^b, \ell^a)\right)$ be the probability of reaching state (x', s', l', e', j') when the system is in state (x, s, l, e, j) and the trader takes the actions of ℓ^b and ℓ^a. We can define the total reward matrix using a similar analysis as in Section A.2. The main difference in the complete model is the possibility of earning $3C$ and losing the jump amount to the LFT in the presence of a stale quote. Formally, let

$$\mathbb{R}\left((x',s',l',e',j')|(x,s,l,e,j),(\ell^b,\ell^a)\right) = \mathbb{1}\left(x+1=x'\right)\left((c-\phi j^-)\mathbb{1}\left(\ell^b=1\right)+(3c-\phi j^-)\mathbb{1}\left(\ell^b=2\right)\right)$$
$$\mathbb{1}\left(x-1=x'\right)\left((c-\phi j^+)\mathbb{1}\left(\ell^a=1\right)+(3c-\phi j^+)\mathbb{1}\left(\ell^a=2\right)\right)-\gamma|x|,$$

where $j^+ \equiv \max(j,0)$, $j^- \equiv \max(-j,0)$ and c, ϕ and γ are defined as the "adjusted spread," "adjusted jump size" and "adjusted inventory aversion" parameters for the discrete-time formulation of the complete model and given by

$$\lambda \equiv 2(\lambda_P+\lambda_I+\lambda_A), \quad r \equiv \lambda+\mu+\theta+\sigma, \quad \delta \equiv \frac{r}{r+D}, \quad c \equiv \delta C, \quad \phi \equiv \delta J \ \text{ and } \ \gamma \equiv \frac{\Gamma}{r+D}. \tag{B.1}$$

Using this definition, the HFT maximizes

$$V(x,s,l,e,j) = \max_{\pi} \mathbb{E}^{\pi}\left[\sum_{k=0}^{\infty} \delta^k \mathbb{R}\left((x'_{t_{k+1}}, s'_{t_{k+1}}, b'_{t_{k+1}}, e'_{t_{k+1}}, j'_{t_{k+1}})|(x_{t_k}, s_{t_k}, b_{t_k}, e_{t_k}, j_{t_k}), (\ell^b_{t_k}, \ell^a_{t_k})\right)\right], \tag{B.2}$$

starting from his initial state, (x,s,l,e,j), which is in the requisite MDP form.

B.3. HFT's Value Function

We have now transformed our continuous-time problem into an equivalent discrete-time MDP. We can now solve for $V(x,s,l,e,j)$ in (B.2) using the discrete-time HJB equations:

$$V(x,s,l,e,j) = \max_{\ell^b,\ \ell^a}\Bigg\{\sum_{(x',s',l',e',j')} \mathbb{P}\left((x',s',l',e',j')|(x,s,l,e,j),(\ell^b,\ell^a)\right)\Big[\mathbb{R}\left((x',s',l',e',j')|(x,s,l,e,j),(\ell^b,\ell^a)\right)$$
$$+\delta V(x',s',l',e',j')\Big]\Bigg\}. \tag{B.3}$$

Since the model is symmetric around the bid and ask side of the market, we can first eliminate the order direction signal from our state space. We have the following reduction for each $s_1 \in \{P, I\}$.

$$V(-x, s_1S, l, e, j) = \begin{cases} V(x, s_1B, l, e, -j) & \text{when } l \in \{00, 11, 22\}, \\ V(x, s_1B, 01, e, -j) & \text{when } l = 10, \\ V(x, s_1B, 10, e, -j) & \text{when } l = 01, \\ V(x, s_1B, 21, e, -j) & \text{when } l = 12, \\ V(x, s_1B, 12, e, -j) & \text{when } l = 21, \\ V(x, s_1B, 02, e, -j) & \text{when } l = 20, \\ V(x, s_1B, 20, e, -j) & \text{when } l = 02. \end{cases}$$

Using this result, we let $v(x, P, l, j) \equiv V(x, PS, l, 0, j)$, $z(x, P, l, j) \equiv V(x, PB, l, 0, j)$, $v(x, I, l, j) \equiv V(x, IS, l, 0, j)$ and $h(x, P) \equiv V(x, PS, l, 1, 0)$, $h(x, I) \equiv V(x, IS, l, 1, 0)$ for ease in notation.

By substituting the expressions for $\mathbb{P}$ and $\mathbb{R}$, we can obtain the implicit equations for the value functions of each state. In the following proposition, we state the HJB equations for our value function. We provide value functions for the bid side of the market by excluding the symmetric states.

Proposition 4. *$v(x, P, 00, j)$ satisfies the following set of equations. Since there is no active quote, there is no risk of stale quotes.*

$$\begin{aligned} v(x, P, 00, j) = -\gamma|x| + \delta\Big\{ &\frac{\mu}{r}h(x, P) + \frac{\lambda^{PI} + \theta}{r}\Big(\mathsf{pr}(PS)v(x, P, 10, j) + \mathsf{pr}(IS)v(x, I, 10, j) + \mathsf{pr}(PB)z(x, P, 10, j) \\ &+ \mathsf{pr}(IB)z(x, I, 10, j)\Big) + \frac{\sigma}{2r}\Big(v(x, P, 10, j-1) + v(x, P, 10, j+1)\Big) \\ &+ \frac{2\lambda_A}{r}v(x, P, 10, j)\Big\} \end{aligned}$$

$v(x, P, 10, j)$ satisfies the following set of equations. If $j < 0$, the HFT is subject to the risk of leaving stale quotes:

$$\begin{aligned} v(x, P, 10, j) = -\gamma|x| + \delta\Big\{ &\frac{\mu}{r}h(x, P) + \frac{\lambda^{\mathsf{sell}} + \theta}{r}\Big(\mathsf{pr}(PS)v(x, P, 10, j) + \mathsf{pr}(IS)v(x, I, 10, j) + \mathsf{pr}(PB)z(x, P, 10, j) \\ &+ \mathsf{pr}(IB)z(x, I, 10, j)\Big) + \frac{\lambda^{\mathsf{buy}} + \lambda_A}{r}\Big(c + j\phi + \mathsf{pr}(PS)v(x+1, P, 00, j) + \mathsf{pr}(IS)v(x+1, I, 00, j) \\ &+ \mathsf{pr}(PB)z(x+1, P, 00, j) + \mathsf{pr}(IB)z(x+1, I, 00, j)\Big) + \frac{\sigma}{2r}\Big(v(x, P, 10, j-1) + v(x, P, 10, j+1)\Big) \\ &+ \frac{\lambda_A}{r}v(x, P, 10, j)\Big\} \end{aligned}$$

For $j = 0$, the HFT does not suffer from stale quotes:

$$v(x,P,10,0) = -\gamma|x| + \delta\Big\{\frac{\mu}{r}h(x,P) + \frac{\lambda^{\mathsf{sell}}+\theta}{r}\Big(\mathsf{pr}(PS)v(x,P,10,0) + \mathsf{pr}(IS)v(x,I,10,0)$$
$$+ \mathsf{pr}(PB)z(x,P,10,0) + \mathsf{pr}(IB)z(x,I,10,0)\Big)$$
$$+ \frac{\lambda^{\mathsf{buy}}}{r}\Big(c + \mathsf{pr}(PS)v(x+1,P,00,0) + \mathsf{pr}(IS)v(x+1,I,00,0)$$
$$+ \mathsf{pr}(PB)z(x+1,P,00,0) + \mathsf{pr}(IB)z(x+1,I,00,0)\Big) + \frac{\sigma}{2r}\Big(v(x,P,10,-1) + v(x,P,10,1)\Big)$$
$$+ \frac{2\lambda_A}{r}v(x,P,10,0)\Big\}$$

If $j > 0$, the HFT's bid quote is not attractive to any LFT so there is no possibility of a trade:

$$v(x,P,10,j) = -\gamma|x| + \delta\Big\{\frac{\mu}{r}h(x,P) + \frac{\lambda^{PI}+\theta}{r}\Big(\mathsf{pr}(PS)v(x,P,00,j) + \mathsf{pr}(IS)v(x,I,00,j) + \mathsf{pr}(PB)z(x,P,00,j)$$
$$+ \mathsf{pr}(IB)z(x,I,00,j)\Big) + \frac{\sigma}{2r}\Big(v(x,P,10,j+1) + v(x,P,00,j-1)\Big)$$
$$+ \frac{2\lambda_A}{r}v(x,P,00,j)\Big\}$$

$v(x,P,20,j)$ satisfies the following set of equations. If $j < 0$, the HFT is again subject to the risk of leaving stale quotes at the bid side:

$$v(x,P,20,j) = -\gamma|x| + \delta\Big\{\frac{\mu}{r}h(x,P) + \frac{\lambda^{\mathsf{sell}}+\theta}{r}\Big(\mathsf{pr}(PS)v(x,P,20,j) + \mathsf{pr}(IS)v(x,I,20,j) + \mathsf{pr}(PB)z(x,P,20,j)$$
$$+ \mathsf{pr}(IB)z(x,I,20,j)\Big) + \frac{\lambda^{\mathsf{buy}}+\lambda_A}{r}\Big(c + j\phi + \mathsf{pr}(PS)v(x+1,P,00,j) + \mathsf{pr}(IS)v(x+1,I,00,j)$$
$$+ \mathsf{pr}(PB)z(x+1,P,00,j) + \mathsf{pr}(IB)z(x+1,I,00,j)\Big) + \frac{\sigma}{2r}\Big(v(x,P,20,j-1) + v(x,P,20,j+1)\Big)$$
$$+ \frac{\lambda_A}{r}v(x,P,20,j)\Big\}$$

For $j = 0$, the HFT does not suffer from stale quotes:

$$v(x,P,20,0) = -\gamma|x| + \delta\Big\{\frac{\mu}{r}h(x,P) + \frac{\lambda^{PI}(1-m_s^{IS})+\theta}{r}\Big(\mathsf{pr}(PS)v(x,P,20,0) + \mathsf{pr}(IS)v(x,I,20,0)$$
$$+ \mathsf{pr}(PB)z(x,P,20,0) + \mathsf{pr}(IB)z(x,I,20,0)\Big)$$
$$+ \frac{\lambda^{PI}m_s^{IS}}{r}\Big(c + \mathsf{pr}(PS)v(x+1,P,00,0) + \mathsf{pr}(IS)v(x+1,I,00,0)$$
$$+ \mathsf{pr}(PB)z(x+1,P,00,0) + \mathsf{pr}(IB)z(x+1,I,00,0)\Big) + \frac{\sigma}{2r}\Big(v(x,P,20,-1) + v(x,P,20,1)\Big)$$
$$+ \frac{2\lambda_A}{r}v(x,P,20,0)\Big\}$$

If $j > 0$, the HFT's bid quote is not attractive to any LFT so there is no possibility of a trade:

$$v(x,P,20,j) = -\gamma|x| + \delta\Big\{\frac{\mu}{r}h(x,P) + \frac{\lambda^{PI}+\theta}{r}\Big(\mathsf{pr}(PS)v(x,P,20,j) + \mathsf{pr}(IS)v(x,I,20,j) + \mathsf{pr}(PB)z(x,P,20,j)$$
$$+ \mathsf{pr}(IB)z(x,I,20,j)\Big) + \frac{\sigma}{2r}\Big(v(x,P,20,j+1) + v(x,P,20,j-1)\Big)$$
$$+ \frac{2\lambda_A}{r}v(x,P,20,j)\Big\}$$

$v(x,P,11,j)$ satisfies the following set of equations. In this case, the HFT is subject to the risk of

leaving stale quotes at the both sides of the market. If $j < 0$,

$$\begin{aligned} v(x,P,11,j) = -\gamma|x| + \delta\Big\{ & \frac{\mu}{r}h(x,P) + \frac{\lambda^{\mathsf{sell}}+\theta}{r}\Big(\mathsf{pr}(PS)v(x,P,11,j) + \mathsf{pr}(IS)v(x,I,11,j) + \mathsf{pr}(PB)z(x,P,11,j) \\ & + \mathsf{pr}(IB)z(x,I,11,j)\Big) + \frac{\lambda^{\mathsf{buy}}+\lambda_A}{r}\Big(c + j\phi + \mathsf{pr}(PS)v(x+1,P,01,j) + \mathsf{pr}(IS)v(x+1,I,01,j) \\ & + \mathsf{pr}(PB)z(x+1,P,01,j) + \mathsf{pr}(IB)z(x+1,I,01,j)\Big) + \frac{\sigma}{2r}\Big(v(x,P,11,j-1) + v(x,P,11,j+1)\Big) \\ & + \frac{\lambda_A}{r}v(x,P,11,j)\Big\} \end{aligned}$$

For $j = 0$, the HFT does not suffer from stale quotes:

$$\begin{aligned} v(x,P,11,0) = -\gamma|x| + \delta\Big\{ & \frac{\mu}{r}h(x,P) + \frac{\theta}{r}\Big(\mathsf{pr}(PS)v(x,P,11,0) + \mathsf{pr}(IS)v(x,I,11,0) + \mathsf{pr}(PB)z(x,P,11,0) \\ & + \mathsf{pr}(IB)z(x,I,11,0)\Big) + \frac{\lambda^{\mathsf{buy}}}{r}\Big(c + \mathsf{pr}(PS)v(x+1,P,01,0) + \mathsf{pr}(IS)v(x+1,I,01,0) \\ & + \mathsf{pr}(PB)z(x+1,P,01,0) + \mathsf{pr}(IB)z(x+1,I,01,0)\Big) + \frac{\sigma}{2r}\Big(v(x,P,11,-1) + v(x,P,11,1)\Big) \\ & + \frac{2\lambda_A}{r}v(x,P,11,0) + \frac{\lambda^{\mathsf{sell}}}{r}\Big(c + \mathsf{pr}(PS)v(x-1,P,10,0) + \mathsf{pr}(IS)v(x-1,I,10,0) \\ & + \mathsf{pr}(PB)z(x-1,P,10,0) + \mathsf{pr}(IB)z(x-1,I,10,0)\Big)\Big\} \end{aligned}$$

If $j > 0$,

$$\begin{aligned} v(x,P,11,j) = -\gamma|x| + \delta\Big\{ & \frac{\mu}{r}h(x,P) + \frac{\lambda^{\mathsf{buy}}+\theta}{r}\Big(\mathsf{pr}(PS)v(x,P,11,j) + \mathsf{pr}(IS)v(x,I,11,j) + \mathsf{pr}(PB)z(x,P,11,j) \\ & + \mathsf{pr}(IB)z(x,I,11,j)\Big) + \frac{\lambda^{\mathsf{sell}}+\lambda_A}{r}\Big(c - j\phi + \mathsf{pr}(PS)v(x-1,P,10,0) + \mathsf{pr}(IS)v(x-1,I,10,0) \\ & + \mathsf{pr}(PB)z(x-1,P,10,0) + \mathsf{pr}(IB)z(x-1,I,10,0)\Big) + \frac{\sigma}{2r}\Big(v(x,P,11,j+1) + v(x,P,11,j-1)\Big) \\ & + \frac{\lambda_A}{r}v(x,P,11,j)\Big\} \end{aligned}$$

$v(x,P,22,j)$ satisfies the following set of equations. In this case, the HFT is subject to the risk of leaving stale quotes at the both sides of the market. If $j < 0$,

$$\begin{aligned} v(x,P,22,j) = -\gamma|x| + \delta\Big\{ & \frac{\mu}{r}h(x,P) + \frac{\lambda^{\mathsf{sell}}+\theta}{r}\Big(\mathsf{pr}(PS)v(x,P,22,j) + \mathsf{pr}(IS)v(x,I,22,j) + \mathsf{pr}(PB)z(x,P,22,j) \\ & + \mathsf{pr}(IB)z(x,I,22,j)\Big) + \frac{\lambda^{\mathsf{buy}}+\lambda_A}{r}\Big(c + j\phi + \mathsf{pr}(PS)v(x+1,P,02,j) + \mathsf{pr}(IS)v(x+1,I,02,j) \\ & + \mathsf{pr}(PB)z(x+1,P,02,j) + \mathsf{pr}(IB)z(x+1,I,02,j)\Big) + \frac{\sigma}{2r}\Big(v(x,P,22,j-1) + v(x,P,22,j+1)\Big) \\ & + \frac{\lambda_A}{r}v(x,P,22,j)\Big\} \end{aligned}$$

For $j=0$, the HFT does not suffer from stale quotes:

$$v(x,P,22,0)=-\gamma|x|+\delta\Big\{\frac{\mu}{r}h(x,P)+\frac{\lambda^{PI}(1-m_s^{IB}-m_s^{IS})+\theta}{r}\Big(\mathsf{pr}(PS)v(x,P,22,0)+\mathsf{pr}(IS)v(x,I,22,0)+\mathsf{pr}(PB)z(x,P,22,0)$$
$$+\mathsf{pr}(IB)z(x,I,22,0)\Big)+\frac{\lambda^{PI}m_s^{IS}}{r}\Big(c+\mathsf{pr}(PS)v(x+1,P,02,0)+\mathsf{pr}(IS)v(x+1,I,02,0)$$
$$+\mathsf{pr}(PB)z(x+1,P,02,0)+\mathsf{pr}(IB)z(x+1,I,02,0)\Big)+\frac{\sigma}{2r}\Big(v(x,P,22,-1)+v(x,P,22,1)\Big)$$
$$+\frac{2\lambda_A}{r}v(x,P,22,0)+\frac{\lambda^{PI}m_s^{IB}}{r}\Big(c+\mathsf{pr}(PS)v(x-1,P,20,0)+\mathsf{pr}(IS)v(x-1,I,20,0)$$
$$+\mathsf{pr}(PB)z(x-1,P,20,0)+\mathsf{pr}(IB)z(x-1,I,20,0)\Big)\Big\}$$

If $j>0$,

$$v(x,P,22,j)=-\gamma|x|+\delta\Big\{\frac{\mu}{r}h(x,P)+\frac{\lambda^{\mathsf{buy}}+\theta}{r}\Big(\mathsf{pr}(PS)v(x,P,22,j)+\mathsf{pr}(IS)v(x,I,22,j)+\mathsf{pr}(PB)z(x,P,22,j)$$
$$+\mathsf{pr}(IB)z(x,I,22,j)\Big)+\frac{\lambda^{\mathsf{sell}}+\lambda_A}{r}\Big(c-j\phi\mathsf{pr}(PS)v(x-1,P,20,0)+\mathsf{pr}(IS)v(x-1,I,20,0)$$
$$+\mathsf{pr}(PB)z(x-1,P,20,0)+\mathsf{pr}(IB)z(x-1,I,20,0)\Big)+\frac{\sigma}{2r}\Big(v(x,P,22,j+1)+v(x,P,22,j-1)\Big)$$
$$+\frac{\lambda_A}{r}v(x,P,22,j)\Big\}$$

$v(x,P,21,j)$ *satisfies the following set of equations. In this case, the HFT is subject to the risk of leaving stale quotes at the both sides of the market. If $j<0$,*

$$v(x,P,21,j)=-\gamma|x|+\delta\Big\{\frac{\mu}{r}h(x,P)+\frac{\lambda^{\mathsf{sell}}+\theta}{r}\Big(\mathsf{pr}(PS)v(x,P,21,j)+\mathsf{pr}(IS)v(x,I,21,j)+\mathsf{pr}(PB)z(x,P,21,j)$$
$$+\mathsf{pr}(IB)z(x,I,21,j)\Big)+\frac{\lambda^{\mathsf{buy}}+\lambda_A}{r}\Big(c+j\phi+\mathsf{pr}(PS)v(x+1,P,01,j)+\mathsf{pr}(IS)v(x+1,I,01,j)$$
$$+\mathsf{pr}(PB)z(x+1,P,01,j)+\mathsf{pr}(IB)z(x+1,I,01,j)\Big)+\frac{\sigma}{2r}\Big(v(x,P,21,j-1)+v(x,P,21,j+1)\Big)$$
$$+\frac{\lambda_A}{r}v(x,P,21,j)\Big\}$$

For $j=0$, the HFT does not suffer from stale quotes:

$$v(x,P,21,0)=-\gamma|x|+\delta\Big\{\frac{\mu}{r}h(x,P)+\frac{\lambda^{PI}(m_s^{PS})+\theta}{r}\Big(\mathsf{pr}(PS)v(x,P,21,0)+\mathsf{pr}(IS)v(x,I,21,0)+\mathsf{pr}(PB)z(x,P,21,0)$$
$$+\mathsf{pr}(IB)z(x,I,21,0)\Big)+\frac{\lambda^{PI}m_s^{IS}}{r}\Big(c+\mathsf{pr}(PS)v(x+1,P,01,0)+\mathsf{pr}(IS)v(x+1,I,01,0)$$
$$+\mathsf{pr}(PB)z(x+1,P,01,0)+\mathsf{pr}(IB)z(x+1,I,01,0)\Big)+\frac{\sigma}{2r}\Big(v(x,P,21,-1)+v(x,P,21,1)\Big)$$
$$+\frac{2\lambda_A}{r}v(x,P,21,0)+\frac{\lambda^{\mathsf{sell}}}{r}\Big(c+\mathsf{pr}(PS)v(x-1,P,20,0)+\mathsf{pr}(IS)v(x-1,I,20,0)$$
$$+\mathsf{pr}(PB)z(x-1,P,20,0)+\mathsf{pr}(IB)z(x-1,I,20,0)\Big)\Big\}$$

If $j > 0$,

$$v(x,P,21,j) = -\gamma|x| + \delta\Big\{\frac{\mu}{r}h(x,P) + \frac{\lambda^{\mathsf{buy}}+\theta}{r}\Big(\mathsf{pr}(PS)v(x,P,21,j) + \mathsf{pr}(IS)v(x,I,21,j) + \mathsf{pr}(PB)z(x,P,21,j) + \mathsf{pr}(IB)z(x,I,21,j)\Big) + \frac{\lambda^{\mathsf{sell}}+\lambda_A}{r}\Big(c - j\phi\mathsf{pr}(PS)v(x-1,P,20,0) + \mathsf{pr}(IS)v(x-1,I,20,0) + \mathsf{pr}(PB)z(x-1,P,20,0) + \mathsf{pr}(IB)z(x-1,I,20,0)\Big) + \frac{\sigma}{2r}\Big(v(x,P,21,j+1) + v(x,P,21,j-1)\Big) + \frac{\lambda_A}{r}v(x,P,21,j)\Big\}$$

Finally, as in the simplified model, HFT's value function at the decision epoch is given by

$$h(x,P) = \max\Big\{v(x,P,00,0), v(x,P,10,0), v(x,P,01,0), v(x,P,11,0), v(x,P,20,0), v(x,P,02,0), v(x,P,22,0), v(x,P,12,0), v(x,P,21,0)\Big\}$$

Proposition 4 shows that the HFT aims to choose the optimal action when the state is in $e = 1$ by maximizing over all possible quoting actions and the corresponding value is stored in $h(x, s)$. On the other hand, $v(x, s, l, j)$ computes the expected one-step reward resulting from possible transitions determined by the active quotes in state l.

B.4. Proof of Theorem 2

The proof is very similar to that of Theorem 1. Using the value iteration algorithm, we first establish by induction that value functions are concave in x. We also need to show that as inventory gets larger (smaller), less quoting at the bid (ask) side will be more and more attractive. Formally, we need to show that $v(x,s,0a,j) - v(x,s,2a,j)$, $v(x,s,2a,j) - v(x,s,1a,j)$ will be nondecreasing in x for any fixed policy $a \in \{0,1,2\}$ at the ask side and similarly, $v(x,s,b2,j) - v(x,s,b0,j)$, $v(x,s,b1,j) - v(x,s,b2,j)$ will be nondecreasing in x for any fixed policy $b \in \{0,1,2\}$ at the bid side. We will refer to this condition as the "nondecreasing property."

Let $v^{(0)}(x,s,l,j) = 0$ for all (x,s,l,j). Then, in the base case, all value functions will include $\gamma|x|$ and an appropriate constant term. Therefore, the base case will satisfy the concavity and non-decreasing property. Assume that $v^{(n)}(x,s,l,j)$ satisfies the induction hypothesis. Then, we first illustrate that the nondecreasing property holds using $v^{(n+1)}(x,P,20,j) - v^{(n+1)}(x,P,10,j)$. If $j \neq 0$,

$$v^{(n+1)}(x,P,20,j) - v^{(n+1)}(x,P,10,j) = \delta\Big\{\frac{\lambda^{\mathsf{sell}}+\theta}{r}\Big(\mathsf{pr}(PS)\Big(v^{(n)}(x,P,20,j) - v^{(n)}(x,P,10,j)\Big) + \mathsf{pr}(IS)\Big(v^{(n)}(x,I,20,j) - v^{(n)}(x,I,10,j)\Big) + \mathsf{pr}(PB)\Big(z^{(n)}(x,I,20,j) - z^{(n)}(x,I,10,j)\Big) + \mathsf{pr}(IB)\Big(z^{(n)}(x,I,20,j) - z^{(n)}(x,I,10,j)\Big)\Big) + \frac{\sigma}{2r}\Big(v^{(n)}(x,P,20,j-1) - v^{(n)}(x,P,10,j-1)\Big) + \frac{\sigma}{2r}\Big(v^{(n)}(x,P,20,j+1) - v^{(n)}(x,P,10,j+1)\Big) + \frac{\lambda_A}{r}\Big(v^{(n)}(x,P,20,j) + v^{(n)}(x,P,10,j)\Big)\Big\}$$

is also nondecreasing in x as each term satisfies the non-decreasing property via the induction hypothesis. If $j = 0$, we have the following additional terms:

$$\frac{\lambda^{PI} m_s^{PS}}{r} \Big(\mathsf{pr}(PS) \left(v^{(n)}(x, P, 20, j) - v^{(n)}(x+1, P, 00, j) \right) + \mathsf{pr}(IS) \left(v^{(n)}(x, I, 20, j) - v^{(n)}(x+1, I, 00, j) \right) \\ + \mathsf{pr}(PB) \left(z^{(n)}(x, I, 20, j) - z^{(n)}(x+1, I, 00, j) \right) + \mathsf{pr}(IB) \left(z^{(n)}(x, I, 20, j) - z^{(n)}(x+1, I, 00, j) \right) \Big)$$

which equals to

$$\frac{\lambda^{PI} m_s^{PS}}{r} \Big(\mathsf{pr}(PS) \left(v^{(n)}(x, P, 20, j) - v^{(n)}(x, P, 00, j) \right) + \mathsf{pr}(IS) \left(v^{(n)}(x, I, 20, j) - v^{(n)}(x, I, 00, j) \right) \\ + \mathsf{pr}(PB) \left(z^{(n)}(x, I, 20, j) - z^{(n)}(x, I, 00, j) \right) + \mathsf{pr}(IB) \left(z^{(n)}(x, I, 20, j) - z^{(n)}(x, I, 00, j) \right) \\ + \mathsf{pr}(PS) \left(v^{(n)}(x, P, 00, j) - v^{(n)}(x+1, P, 00, j) \right) + \mathsf{pr}(IS) \left(v^{(n)}(x, I, 00, j) - v^{(n)}(x+1, I, 00, j) \right) \\ + \mathsf{pr}(PB) \left(z^{(n)}(x, I, 00, j) - z^{(n)}(x+1, I, 00, j) \right) + \mathsf{pr}(IB) \left(z^{(n)}(x, I, 00, j) - z^{(n)}(x+1, I, 00, j) \right) \Big)$$

which is also nondecreasing in x as the first four terms satisfies the non-decreasing property via the induction hypothesis, and the last four terms satisfy the concavity in x via the induction hypothesis.

B.5. *Proof of Proposition 3*

Suppose that we increase μ by $\Delta\mu$. Assume that π^* is the optimal quoting policy when the arrival rate of decision epochs is μ. Using the similar thinning argument, if the HFT does not quote at the new decision times due to $\Delta\mu$, he is going to get the same objective value as in the original model. However, he can quote to buy when his inventory is negative and quote to sell when his inventory is positive and thus he can achieve a higher value.

The sensitivity with respect to λ_I and λ_P is very similar. We prove for λ_I. Suppose that we increase λ_I by $\Delta\lambda_I$. Assume that π^* is the optimal quoting policy when the arrival rate of impatient orders is λ_I. Consider any sample path under this model. For any sample path, we can construct a model with an arrival rate of impatient orders of $\lambda_I + \Delta\lambda_I$ in which via thinning Poisson processes, we can obtain the original sequence of orders plus the new arrivals due to $\Delta\lambda_I$. Now at these new arrivals, there is a positive probability that the HFT will reduce his inventory and achieve higher value.

Suppose that we decrease θ by $\Delta\theta$. Assume that π^* is the optimal quoting policy when the arrival rate of signal is θ. In the new model π^* is still feasible, and the probability of receiving an order as predicted by the signal is higher and thus the HFT's value increases as long as the signals are informative. The relationship with respect to p and q are similar as by construction increasing the accuracy of the signals will increase the probability of receiving the order that the signal predicts at the decision epoch. Thus, the value of the HFT will be nondecreasing in p and q.

The relationship with respect to C and Γ can be obtained using induction on the steps of the value iteration algorithm. We use the simplified model for ease in notation. Let $v^{(0)}(x,l) = 0$ for all x and l. Then, in the base case, we obtain

$$\begin{aligned}
v^{(1)}(x,00) &= -\gamma|x|,\\
v^{(1)}(x,10) &= -\gamma|x| + \frac{\delta p\lambda c}{r},\\
v^{(1)}(x,01) &= -\gamma|x| + \frac{\delta(1-p)\lambda c}{r},\\
v^{(1)}(x,11) &= -\gamma|x| + \frac{\delta\lambda c}{r},
\end{aligned}$$

which shows that $v^{(1)}(x,l)$ is increasing in C and Γ as $c \equiv C\delta$ and $\gamma \equiv \Gamma/(r+D)$. Assume that $v^{(n)}(x,l)$ satisfies the induction hypothesis. Then, $v^{(n+1)}(x,l)$ satisfies the following set of equations:

$$\begin{aligned}
v^{(n+1)}(x,00) &= -\gamma|x| + \delta\left(\tfrac{\mu}{r}h^{(n)}(x) + \tfrac{\theta+\lambda}{2r}v^{(n)}(x,00) + \tfrac{\theta+\lambda}{2r}v^{(n)}(-x,00)\right)\\
v^{(n+1)}(x,10) &= -\gamma|x| + \delta\Big(\tfrac{\mu}{r}h^{(n)}(x) + \tfrac{\theta+(1-p)\lambda}{2r}v^{(n)}(x,10) + \tfrac{\theta+(1-p)\lambda}{2r}v^{(n)}(-x,01) + \tfrac{p\lambda}{2r}v^{(n)}(x+1,00)\\
&\quad + \tfrac{p\lambda}{2r}v^{(n)}(-x-1,00) + \tfrac{pc\lambda}{r}\Big)\\
v^{(n+1)}(x,01) &= -\gamma|x| + \delta\Big(\tfrac{\mu}{r}h^{(n)}(x) + \tfrac{\theta+p\lambda}{2r}v^{(n)}(x,01) + \tfrac{\theta+p\lambda}{2r}v^{(n)}(-x,10) + \tfrac{(1-p)\lambda}{2r}v^{(n)}(x-1,00)\\
&\quad + \tfrac{(1-p)\lambda}{2r}v^{(n)}(-x-1,00) + \tfrac{(1-p)c\lambda}{r}\Big)\\
v^{(n+1)}(x,11) &= -\gamma|x| + \delta\Big(\tfrac{\mu}{r}h^{(n)}(x) + \tfrac{\theta}{2r}(v^{(n)}(x,11) + v^{(n)}(-x,11)) + \tfrac{p\lambda}{2r}v^{(n)}(x+1,01)\\
&\quad + \tfrac{p\lambda}{2r}v^{(n)}(-x-1,10) + \tfrac{(1-p)\lambda}{2r}v^{(n)}(x-1,10) + \tfrac{(1-p)\lambda}{2r}v^{(n)}(-x+1,01) + \tfrac{c\lambda}{r}\Big)
\end{aligned}$$

We observe that each v is increasing in C and decreasing in Γ as the positive sum of these functions preserve the property.

The objective value is decreasing in λ_A and J as under any fixed policy these two parameters control the expected loss of the HFT in the presence of stale quotes. The value is also decreasing in σ as the probability of HFT's stale quote increases with σ. Finally, the value is also decreasing in D as the expected reward from each period is discounted more under any fixed policy.