# Option Pricing with Quadratic Volatility: a Revisit

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**Abstract**

This paper considers the pricing of European options on assets that follow a stochastic differential equation with a quadratic volatility term. We correct several errors in the existing literature, extend the pricing formulas to arbitrary root configurations, and list alternative representations of option pricing formulas to improve computational performance. Our exposition is based entirely on probabilistic arguments, adding a fresh perspective and new intuition to the existing PDE-dominated literature on the subject. Our main tools are martingale methods and shift of probability measure; the fact that the underlying process is typically a strict local martingale is carefully considered throughout the paper.

## 1 Introduction

Many authors (e.g., Blacher [5], Ingersoll [10], Lipton [13], and Zuhlsdorff [18], to name a few) have suggested derivative pricing models where financial variables (e.g., foreign exchange rates, equity prices, or forward interest rates) following diffusion processes with quadratic volatility. Consider therefore the fundamental problem of pricing European put and call options on an asset that satisfies a stochastic differential equation (SDE) of the type

$$dx(t) = \left(\alpha + \beta x(t) + \gamma x(t)^2\right) dW(t), \quad x(0) = x_0, \tag{1}$$

where $\gamma \neq 0$. Rady [14] and Ingersoll [10], among several others, consider the bounded case where the function $A(x) = \alpha + \beta x + \gamma x^2$ has two real roots that straddle the initial value $x_0$. [1] outlines a general strategy that allows for a transformation of the pricing PDE for (1) into the heat equation; this strategy is used in Lipton [13] to compute a call option pricing formula for the case where $A(x)$ has two negative roots and an absorbing barrier[1] has been inserted at $x = 0$. Other root-configurations have been considered in Zuhlsdorff [17], but several of the given option pricing results contain errors.

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[1] [13] does not explicitly state that the origin is absorbing, but the form of the given option pricing solution indicates that this must be the case.

In this paper, we carefully analyze the characteristics of the process (1), paying particular attention to the circumstances under which $x(t)$ fails to be a martingale. This analysis, in turn, serves as the starting point for a probabilistic derivation of put and call option pricing expressions for all non-trivial[2] root-configurations of $A(x)$. In doing so, we take care to appropriately incorporate into the pricing expressions the strict local martingale property of (1), thereby avoiding the issues that plague existing result in the literature. We also discuss how to modify results if a range truncation through absorbing boundaries is desired. Our analysis contributes new intuition to SDEs with quadratic volatility and lists many new formulas for option pricing.

## 2 One or Two Real Roots

We consider a semi-martingale asset process $x(t)$ adapted to a filtration generated by a scalar Brownian motion. For simplicity, we also assume that interest rates are zero, an assumption that can easily be relaxed by the usual numeraire-deflation of the asset. From results in Delbaen and Schachermayer [7], the absence of a *free lunch with vanishing risk* (FLVR) is equivalent to the existence of a "risk-neutral" probability measure $\mathbb{Q}$ in which $x(t)$ is a local martingale. Without going into detail, we recall that FLVR is a slight strengthening of the usual definition of arbitrage from admissible[3] trading strategies; see [7] for the complete account.

We proceed to fix a scalar Brownian motion $W$ on the probability space $(\Omega, \mathscr{F}, \mathbb{Q})$, with the standard filtration $(\mathscr{F}_t)_{t\geq 0}$ generated by $W$. We assume that $x(t)$ satisfies the $\mathbb{Q}$-SDE (1); clearly $x(t)$ is a local martingale. We first consider the case where $A(x) = \alpha + \beta x + \gamma x^2$ has two real roots $l$ and $u$, $l < u$, lying to the left of $x_0$. Without loss of generality, we may then consider the normalized process

$$dx(t) = \frac{(x(t)-u)(x(t)-l)}{u-l} dW(t), \quad x_0 > u > l, \tag{2}$$

where $W(t)$ is a Brownian motion in the risk-neutral probability measure $\mathbb{Q}$.

Let $p(t)$ and $c(t)$ denote the time $t$ fair market prices of European put and call options, respectively. Assuming that the option strike is $K > u$ and the option maturity is $T$, by definition of puts and calls we have the terminal payout conditions

$$\begin{aligned} p(T) &= (K - x(T))^+ , \\ c(T) &= (x(T) - K)^+ . \end{aligned}$$

A problem of fundamental interest in this paper is to establish time 0 prices $p(0)$ and $c(0)$.

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[2]We skip the simple case dealt with in Rady [14] and Ingersoll [10].

[3]While our treatment here is generally informal, the notion of "arbitrage" used in this paper is one that precludes the use of doubling strategies. More precisely, we require that all trading strategies be *admissible*, in the sense that trading gains are not allowed to go below some large (but finite) negative floor.

**Remark 1** *Instead of (2), in applications we often need to consider*

$$dx(t) = \sigma(t) \times \frac{(x(t)-u)(x(t)-l)}{u-l} dW(t), \quad x_0 > u > l,$$

*where $\sigma(t)$ is a bounded deterministic function of time. By the usual rules for time-change of Brownian motion, computation of $p(0)$ and $c(0)$ for this process proceeds by replacing, in the pricing formulas for the case (1), the maturity $T$ with the integral*

$$\int_0^T \sigma(s)^2 ds.$$

We start by listing a few straightforward lemmas.

**Lemma 1** *The range for $x(t)$ in (2) is $x(t) \in (u,\infty)$. In particular, the process for $x(t)$ does not explode in measure $\mathbb{Q}$.*

**Proof:** That $x(t)$ cannot go below $u$ is obvious; further, Feller's boundary criteria (e.g. Karlin and Taylor [12], Chapter 15.6) establishes that $u$ is not accessible when $x_0 > u$. As $x(t)$ is described by a time-homogenous one-dimensional SDE, it cannot explode (Karatzas and Shreve [11], p. 332). ∎

**Lemma 2** *The process (2) is a strict supermartingale in measure $\mathbb{Q}$.*

**Proof:** In Appendix A.1. ∎

**Lemma 3** *Suppose that $x(t)$ is a local martingale (2) in measure $\mathbb{Q}$; let $\mathrm{E}(\cdot)$ denote expectation in measure $\mathbb{Q}$. Then the no-arbitrage prices at time 0 for the put and call are*

$$\begin{aligned} p(0) &= \mathrm{E}(p(T)), \\ c(0) &= p(0) + x_0 - K > \mathrm{E}(c(T)). \end{aligned}$$

**Proof:** A slight adaptation of the proof of Proposition 6.I in [8] shows that the put price $p(t)$ can be replicated by an admissible trading strategy, the value process of which is a local martingale in $\mathbb{Q}$. In the absence of arbitrage, $p(t)$ must therefore be a local martingale in measure $\mathbb{Q}$. As the put price is here bounded between 0 and $K-u$, it follows elementarily that, in fact, $p(t)$ must be a true $\mathbb{Q}$-martingale. The expression for $p(0)$ follows. By standard put-call parity, we must also have $c(0) = p(0) + x_0 - K > \mathrm{E}(p(T) + x(T) - K) = \mathrm{E}(c(T))$, where the inequality follows from Lemma 2. ∎

**Remark 2** *If we start from the assumption that there is no FLVR in the market, then the results for $p(0)$ and $c(0)$ also hold.*

We emphasize the non-standard result $c(0) > \mathrm{E}(c(T))$ which is a consequence of the strict local martingale property of $x(t)$. The inequality holds for arbitrarily large strikes; indeed (and rather counter-intuitively) $\lim_{K\to\infty} c(0) = x_0 - \mathrm{E}(x(T)) > 0$.

### 2.1 Option Pricing Formulas

The results in Lemma 3 suggest that we focus our attention on the pricing of put options. For this purpose, let us note the useful equality

$$x - K = \frac{(x-u)(K-l) - (K-u)(x-l)}{u-l} \tag{3}$$

which allows us to write[4]

$$\begin{aligned} p(T) &= \frac{1}{u-l}\left((K-u)(x(T)-l) - (x(T)-u)(K-l)\right)^{+} \\ &= \frac{(K-u)(x(T)-l)}{u-l} 1_{(K-u)(x(T)-l)-(x(T)-u)(K-l)>0} \\ &\quad - \frac{(x(T)-u)(K-l)}{u-l} 1_{(K-u)(x(T)-l)-(x(T)-u)(K-l)>0} \\ &\equiv p_1(T) - p_2(T). \end{aligned} \tag{4}$$

The payouts $p_1$ and $p_2$ have identical structure, so it suffices to focus our attention on pricing one of them, e.g. $p_1$.

From Lemma 3, we have $p_1(0) = \mathrm{E}(p_1(T))$, which we rewrite as

$$p_1(0) = \frac{(K-u)}{u-l} \mathrm{E}\left((x(T)-l) 1_{(x(T)-u)/(x(T)-l)<(K-u)/(K-l)}\right). \tag{5}$$

At this point our first instinct would be to perform a measure shift that eliminates that factor $x(T) - l$ in the expectation, i.e. we would like to introduce a new measure $\mathbb{P}$ such that $\mathbb{P}(B) = \frac{1}{x_0 - l}\mathrm{E}((x(T)-l)B)$, for any $\mathscr{F}_T$-measurable event $B$. We recall, however, that $x(t)$ (and therefore $x(t) - l$) is not a martingale in $\mathbb{Q}$, so such a measure shift cannot be performed outright. To get around this, we follow the localization argument in Sin [16] and stop the process $x$ at a finite level. Specifically, let us define a process $x^{(n)}(t)$ as

$$x^{(n)}(t) = x(t \wedge \tau_n)$$

where $\tau_n$ is a stopping time,

$$\tau_n = \inf(t : x(t) - u = n).$$

The process for $y^{(n)}(t) \equiv x^{(n)}(t) - l$ satisfies (up to $\tau_n$)

$$dy^{(n)}(t) = y^{(n)}(t) \frac{x^{(n)}(t) - u}{u-l} dW(t), \quad y^{(n)}(0) = x_0 - l.$$

As $x^{(n)}(t) - u \leq n$ is bounded at or below $n$ for all $t$, it follows that $y^{(n)}(t)$ is a valid $\mathbb{Q}$-martingale, so we can define a measure $\mathbb{P}^n$ by $\mathbb{P}^n(B) = \frac{1}{x_0 - l}\mathrm{E}\left(\left(x^{(n)}(T) - l\right) B\right)$ for any $\mathscr{F}_T$-measurable event $B$. Let $\mathrm{E}^{(n)}$ denote expectation in measure $\mathbb{P}^n$.

[4] $1_B$ is the indicator function for the set $B$.

**Lemma 4** *Set*

$$Y^{(n)}(T) = \frac{x^{(n)}(T) - u}{x^{(n)}(T) - l} = \frac{x^{(n)}(T) - u}{y^{(n)}(T)}.$$

*Then*

$$\mathrm{E}\left(y^{(n)}(T) 1_{Y^{(n)}(T) < (K-u)/(K-l)} 1_{\tau_n > T}\right) = (x_0 - l)\, \mathrm{E}^{(n)}\left(1_{Y^{(n)}(T) < (K-u)/(K-l)} 1_{\tau_n > T}\right)$$

*where $Y^{(n)}(T)$ satisfies, up to time $\tau_n$,*

$$dY^{(n)}(t) = Y^{(n)}(t) dW^{(n)}(t), \quad Y^{(n)}(0) = \frac{x_0 - u}{x_0 - l} < 1,$$

*with $W^{(n)}$ being a $\mathbb{P}^n$-Brownian Motion.*

**Proof:** By Girsanov's Theorem applied to the valid change of measure from $\mathbb{Q}$ to $\mathbb{P}^n$. ∎

This lemma leads to the following.

**Proposition 1** *Let*

$$dY(t) = Y(t) dW(t), \quad Y(0) = \frac{x_0 - u}{x_0 - l} < 1,$$

*be geometric Brownian motion in $\mathbb{Q}$. Define $\tau = \inf(t : Y(t) = 1)$, and let $K > u$. Then $p_1(0)$ in (5) is given by*

$$p_1(0) = \frac{(K-u)(x_0 - l)}{u - l} \mathrm{E}\left(1_{Y(T) < (K-u)/(K-l)} 1_{\tau > T}\right). \tag{6}$$

*Stated explicitly,*

$$p_1(0) = K_1 \Phi\left(\frac{-\ln(X_1/K_1) + \frac{1}{2}T}{\sqrt{T}}\right) - X_2 \Phi\left(\frac{\ln(X_2/K_2) + \frac{1}{2}T}{\sqrt{T}}\right), \tag{7}$$

*with $\Phi$ being the cumulative Gaussian distribution function, and*

$$\begin{aligned} K_1 &= \frac{(K-u)(x_0 - l)}{u - l}, \quad X_1 = \frac{(x_0 - u)(K - l)}{u - l}, \\ K_2 &= \frac{(K-l)(x_0 - l)}{u - l}, \quad X_2 = \frac{(x_0 - u)(K - u)}{u - l}. \end{aligned}$$

**Proof:** In Appendix A.2. ∎

Following similar steps leads to an expression for $p_2(0)$, which in turn leads to the following result for $p(0) = p_1(0) - p_2(0)$.

**Proposition 2** *Let $K_i, X_i$, $i = 1, 2$, be given as in Proposition 1. Assuming $K > u$, the put price $p(0)$ for the model (2) has the explicit representation*

$$p(0) = K_1 \Phi\left(-d_-^{(1)}\right) - X_2 \Phi\left(d_+^{(2)}\right) - X_1 \Phi\left(-d_+^{(1)}\right) + K_2 \Phi\left(d_-^{(2)}\right),$$

$$d_\pm^{(i)} = \frac{\ln\left(X_i/K_i\right) \pm \frac{1}{2}T}{\sqrt{T}}, \quad i = 1, 2.$$

An application of put-call parity then immediately gives the call price.

**Corollary 1** *The call price for the model (2) is*

$$c(0) = x_0 - K + p(0),$$

*with $p(0)$ given in Proposition 2.*

**Remark 3** *Proposition 2 corrects an erroneous result*[5] *in Zuhlsdorff [17].*

## 2.2 Extensions

### 2.2.1 Roots to the Right of $x_0$

Now, let the roots $l, u$, $l < u$, both be to the *right* of $x_0$,

$$dx(t) = \frac{(u - x(t))(l - x(t))}{u - l} dW(t), \quad x_0 < l < u. \tag{8}$$

Define the process $z(t) = l + u - x(t)$, such that, by Ito's Lemma,

$$dz(t) = \frac{(z(t) - u)(z(t) - l)}{u - l} dW(t), \quad z(0) = l + u - x_0 > u. \tag{9}$$

As $x(t) < l$ for all $t$, the call option payout

$$c(T) = (x(T) - K)^+, \quad K < l,$$

is now bounded; written in terms of $z(T)$ it becomes

$$c(T) = (K_z - z(T))^+, \quad K_z \equiv l + u - K, \tag{10}$$

where $K_z > u$. We recognize (10) as being of the form (2), with the call payout (10) being equivalent to a put payout on $z(T)$. Proposition 2 then immediately gives us a pricing result for the call option $c(0)$.

[5] The formulas in [17] for option pricing with absorption at zero are also incorrect; see Section 2.2.2 for the correct expressions.

**Lemma 5** *Assume $K < l$, and define*

$$\begin{aligned} K_1 &= \frac{(l-K)(u-x_0)}{u-l}, \quad X_1 = \frac{(l-x_0)(u-K)}{u-l}, \\ K_2 &= \frac{(u-K)(u-x_0)}{u-l}, \quad X_2 = \frac{(l-x_0)(l-K)}{u-l}. \end{aligned}$$

*For the process (8), the call option price is*

$$c(0) = K_1 \Phi\left(-d_-^{(1)}\right) - X_2 \Phi\left(d_+^{(2)}\right) - X_1 \Phi\left(-d_+^{(1)}\right) + K_2 \Phi\left(d_-^{(2)}\right),$$

$$d_\pm^{(i)} = \frac{\ln(X_i/K_i) \pm \frac{1}{2}T}{\sqrt{T}}, \quad i = 1, 2.$$

The put option price follows by put-call parity, as in Corollary 1.

**Remark 4** *For the process (8), it is clear from Lemmas 2 and 3 that i) $x(t)$ is a strict submartingale; ii) $p(0) > \mathrm{E}(p(T))$.*

### 2.2.2 Absorption at Zero

If $x(t)$ is supposed to model a non-negative asset price, in some cases it may be desirable to insert an absorbing boundary at $x = 0$. In a probabilistic framework, this is generally straightforward. To demonstrate, let the process for $x$ be as in (2), with both roots to the left of $x_0$. Also assume that $l < u < 0$, such that the unrestricted process $x(t)$ may go below zero. Finally, set

$$dY_l(t) = Y_l(t)dW(t), \quad Y_l(0) = \frac{x_0 - u}{x_0 - l} < 1, \tag{11}$$

$$dY_u(t) = Y_u(t)dW(t), \quad Y_u(0) = \frac{x_0 - l}{x_0 - u} > 1. \tag{12}$$

**Lemma 6** *Let $p_1(0)$ and $p_2(0)$ be as defined in (4), and assume that $x(t)$ satisfies (2), with $l < u < 0$ and an absorbing barrier at zero. Define*

$$\begin{aligned} \tau_l &= \inf(t : Y_l(t) = 1 \text{ or } Y_l(t) = u/l); \\ \tau_u &= \inf(t : Y_u(t) = 1 \text{ or } Y_u(t) = l/u). \end{aligned}$$

*Then*

$$\begin{aligned} p_1(0) &= \frac{(K-u)(x_0-l)}{u-l} \mathrm{E}\left(1_{Y_l(T \wedge \tau_l) < (K-u)/(K-l)}\right), \\ p_2(0) &= \frac{(K-l)(x_0-u)}{u-l} \mathrm{E}\left(1_{Y_u(T \wedge \tau_u) > (K-l)/(K-u)}\right). \end{aligned}$$

**Proof:** An obvious extension of the argument in Appendix A.2, to insert an absorbing barrier in $x = 0$, in addition to the absorbing barrier at $x = \infty$. ∎

Computation of $p_1(0)$ and $p_2(0)$ can be done by classical means, using known expressions for the density of Brownian motion in the presence of two absorbing boundaries; for the relevant results see, e.g., Cox and Miller [6], Chapter 5, and Bhattacharya and Waymire [4], Chapter 7.2. We notice that two different representations of the density are possible, either a Fourier sine-series or a series obtained by the method of images. Propositions 3 and 4 explore both possibilities.

**Proposition 3 (Method of Images)** *For the process (1), assume that $l < u < 0 < x_0$ and insert an absorbing boundary at $x = 0$. With $K > 0$, define*

$$F^{\pm}(x,z) = \Phi\left(\frac{x - z \pm \frac{1}{2}T}{\sqrt{T}}\right), \quad k = \ln\left(\frac{(K-u)(x_0-l)}{(K-l)\,(x_0-u)}\right),$$
$$z_U = \ln\left(\frac{x_0 - l}{x_0 - u}\right), \quad z_L = z_U - \ln(l/u).$$

*The put price is*

$$p(0) = \frac{(K-u)\,(x_0-l)}{u-l}\left\{e_1^+(k) + e_2^+\right\} - \frac{(K-l)\,(x_0-u)}{u-l}\left\{e_1^-(k) + e_2^-\right\},$$

*where, with $z_n' = 2n\,(z_U - z_L)$ and $z_n'' = 2z_U - z_n'$,*

$$e_1^{\pm}(k) = \sum_{n=-\infty}^{\infty}\left(e^{\mp\frac{1}{2}z_n'}(F^{\pm}(k,z_n') - F^{\pm}(z_L,z_n')) - e^{\mp\frac{1}{2}z_n''}(F^{\pm}(k,z_n'') - F^{\pm}(z_L,z_n''))\right),$$

$$e_2^{\pm} = \psi^{\pm} \mp \frac{e_1^{\pm}(z_U)}{D^{\pm}} \pm \frac{\left(\frac{x_0-u}{x_0-l}\right)^{\pm 1} e_1^{\mp}(z_U)}{D^{\pm}},$$

*and*

$$\psi^+ = \frac{l}{l - x_0}, \quad \psi^- = \frac{u}{u - x_0}, \quad D^+ = 1 - u/l, \quad D^- = l/u - 1.$$

**Proof:** In Appendix A.3. ∎

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**Proposition 4 (Fourier Series)** *For the process (1), assume that $l < u < 0 < x_0$ and insert an absorbing boundary at $x = 0$. Define*

$$\lambda_n = \frac{1}{2}\left(\frac{1}{4} + \frac{n^2\pi^2}{(z_U - z_L)^2}\right), \quad a_n = \frac{n\pi z_L}{z_U - z_L}, \quad k_n = \frac{n\pi\,(k - z_L)}{z_U - z_L},$$

*and let $k$, $z_U$, $z_L$, and $\psi^{\pm}$ be as in Proposition 3. Then the put price is*

$$p(0) = \frac{(K-u)(x_0-l)}{u-l}\left\{e_1^+ + e_2^+\right\} - \frac{(K-l)(x_0-u)}{u-l}\left\{e_1^- + e_2^-\right\},$$

*where*

$$e_1^{\pm} = \frac{1}{z_U - z_L}\sum_{n=1}^{\infty}\sin(a_n)\frac{e^{-\lambda_n T}}{\lambda_n}\left[\frac{a_n}{z_L}\left(e^{\mp\frac{1}{2}k}\cos(k_n) - e^{\mp\frac{1}{2}z_L}\right) \pm \frac{1}{2}e^{\mp\frac{1}{2}k}\sin(k_n)\right],$$

$$e_2^{\pm} = \psi^{\pm} + \frac{e^{\mp\frac{1}{2}z_L}}{(z_U - z_L)^2}\sum_{n=1}^{\infty}\frac{e^{-\lambda_n T}}{\lambda_n} n\pi\sin(a_n).$$

**Proof:** In Appendix A.4. ∎

**Remark 5** *Lipton [13] uses classical PDE methods to list an alternative (but equivalent) form for the Fourier-series in Proposition 4. Note that the series in [13] has a typo: the constant $\xi$ should be $\theta$.*

For the case where $u/l$ is large relative to the variance of $\log Y_l(T)$, the representation in Proposition 3 will typically require substantially fewer terms to converge than will the Fourier series representation in 4 . On the other hand, the latter will be more convenient for the case where $u/l$ is small, i.e. when the roots are close together or the option maturity is large. An intelligent implementation will branch between the two solutions, in the manner discussed in, say, Andersen [2].

**Remark 6** *Insertion of an absorbing boundary at a non-zero level above $u$ is a trivial extension of the results above, as is the insertion of an additional absorbing boundary* above $x_0$ *(which would effectively replace the one in $\infty$). See also Section 3.*

We leave to the reader the case where zero is an absorbing boundary and the two real roots are to the right of $x_0$.

### *2.2.3 A single real Root*

Consider now the case where there is only a single root, i.e. $l = u$. Let us assume that $x_0 > u$; the case $x_0 < u$ can be solved by the symmetry arguments in Section 2.2.1. We write

$$dx(t) = (x(t) - u)^2\, dW(t), \quad x_0 > u, \tag{13}$$

and note that the range of $x(t)$ is $(u, \infty)$. It follows from the proof of Lemma 2 that $x(t)$ remains a strict supermartingale.

If we make the variable transformation $y(t) = x(t) - u$, then

$$dy(t) = y(t)^2 dW(t), \quad y(0) > 0, \tag{14}$$

with the put option payout being

$$p(T) = (K - u - y(T))^+, \quad K > u.$$

$y(t)$ is a constant elasticity of variance (CEV) process with a power of 2 and, as such, $p(0)$ can, in principle, be computed from the general CEV option pricing formulas in Schroder [15] (see also Andersen and Andreasen [3]). However, these involve infinite series of chi-square distributions and are impractical for the special case of (14). Instead, the simple formula below should be used.

**Proposition 5** *For the process (13), the put option price is*

$$p(0) = (x_0 - u)(K - u)\sqrt{T}\left\{d_+\Phi(d_+) + \phi(d_+) - d_-\Phi(d_-) - \phi(d_-)\right\},$$

*where* $\phi(x) = (2\pi)^{-1/2} e^{-x^2/2}$ *is the Gaussian density, and*

$$d_\pm = \frac{\pm\frac{1}{x_0-u} - \frac{1}{K-u}}{\sqrt{T}}.$$

**Proof:** Let us write

$$\begin{aligned} p(0) &= \mathrm{E}\left((K-u)\,1_{x(T)-u<K-u}\right) - \mathrm{E}\left((x(T)-u)\,1_{x(T)-u<K-u}\right) \\ &= \mathrm{E}\left((K-u)\,1_{1/(x(T)-u)>1/(K-u)}\right) - \mathrm{E}\left((x(T)-u)\,1_{1/(x(T)-u)>1/(K-u)}\right). \end{aligned}$$

We notice that the process $(x(t)-u)^{-1}$ satisifies

$$d\left((x(t)-u)^{-1}\right) = -dW(t) + (x(t)-u)\,dt.$$

Let therefore $dz(t) = -dW(t)$, with $z(0) = (x_0 - u)^{-1} > 0$; and set $\tau = \inf(t : z(t) = 0)$. Following the route of arguments that lead to Proposition 1, we can show that

$$\begin{aligned} \mathrm{E}\left((K-u)\,1_{1/\{x(T)-u\}>1/(K-u)}\right) &= z(0)^{-1}(K-u)\,\mathrm{E}\left(z(T)1_{z(T)>1/(K-u)}1_{\tau>T}\right), \\ \mathrm{E}\left((x(T)-u)\,1_{1/\{x(T)-u\}>1/(K-u)}\right) &= z(0)^{-1}\mathrm{E}\left(1_{z(T)>1/(K-u)}1_{\tau>T}\right). \end{aligned}$$

Standard results for Brownian motion with an absorbing barrier (Cox and Miller [6], Chapter 5) allows us to easily evaluate these expressions in closed form. ∎

**Remark 7** *An alternative proof for Proposition 5 observes that for the process*

$$dx(t) = (x(t)-u)(x(t)-l)\,dW(t), \quad l < u < x_0,$$

*the put price can be computed from the result in Proposition 2, after a time-change, from* $T$ *to* $T(u-l)^2$*; see Remark 1. Taking the limit of the put price as* $l \uparrow u$ *then establishes the result. For a pure PDE proof of the result in Proposition 5, see Zuhlsdorff [17].*

We leave the case where the lone real root lies to the right of $x_0$ to the reader.

#### 2.2.4 A single real Root: Absorption at Zero

Assume that $x(t)$ satisfies (13) with $u < 0$ and assume now that an absorbing barrier has been inserted at the origin, ensuring that $x(t)$ never goes negative. We can easily show the following result.

**Lemma 7** *Assume that $x(t)$ satisfies (2), with $u < 0$ and an absorbing barrier at zero. Let*

$$dz(t) = -dW(t), \quad z(0) = \frac{1}{x_0 - u} > 0,$$

*and define*

$$\tau = \inf\left(t : z(t) = 0 \text{ or } z(t) = -1/u\right).$$

*Then the put price is given by*

$$p(0) = (x_0 - u)(K-u)\,\mathrm{E}\left(z(T\wedge\tau)1_{z(T\wedge\tau)>1/(K-u)}\right) - (x_0 - u)\,\mathrm{E}\left(1_{z(T\wedge\tau)>1/(K-u)}\right). \tag{15}$$

**Proof:** A simple extension of the argument in the proof of Proposition 5, to insert an absorbing barrier at $x = 0$. ∎

Evaluation of the two expectations in (15) is straightforward, and can proceed along the lines of the proofs of Propositions 3 and 4. In the interest of brevity, we omit the results, since they can be found by simply taking the limits $u \uparrow l$ in Propositions 3 and 4; see Remark 7.

When $u$ is close to zero – that is when the range $z \in [-1/u, 0]$ is large – a series-solution based on the method of images will require less terms to converge than a sine-solution. The opposite holds when $u$ is far away from zero.

**Remark 8** *In Zuhlsdorff [17], the expression for the single-root case with absorption is incorrect.*

## 3 No Real Roots

We now consider the case where the polynomial $A(x)$ in (1) has no real roots. After suitable normalization[6], our $\mathbb{Q}$-SDE has the form

$$\begin{aligned} dx(t) &= b\left(1 + \left(\frac{x(t)-a}{b}\right)^2\right)dW(t) \qquad (16)\\ &= \frac{1}{b}\left((x(t)-a)^2 + b^2\right)dW(t) \\ &= \frac{1}{b}(x(t)-c_+)(x(t)-c_-)\,dW(t) \end{aligned}$$

[6] Our normalization follows that of Zuhlsdorff [17].

where $c_\pm$ are two complex-valued roots,

$$c_\pm = a \pm ib, \quad b > 0, \tag{17}$$

with $i$ being the imaginary unit, $i^2 = -1$.

Without further restrictions, the range for $x(t)$ is now the entire real line. Following the argument in the proof for Lemma 2, it can be demonstrated that $x(t)$ is a strict local martingale, but, in the absence of lower and upper bounds on $x$, we cannot characterize $x$ as either a supermartingale or a submartingale. Absence of FLVR dictates that put and call prices be local martingales in measure $\mathbb{Q}$; as neither the put nor the call have bounded payouts for the case of (16), neither can be argued to be martingales in measure $\mathbb{Q}$. (We can, however, argue that both are supermartingales, as they are local martingales bounded from below at zero).

To get firmer ground under our feet, we proceed to introduce explicit bounds on the process $x(t)$, through the insertion[7] of *absorbing boundaries* $L$ and $U$, with $L < x_0 < U$. $x(t)$ is thus a bounded local martingale, and hence a martingale. The same argument applies to put and call prices, wherefore we have the following Lemma.

**Lemma 8** *Assume that the process $x(t)$ is equipped with absorbing boundaries $L$ and $U$. Define*

$$\tau = \inf\left(t : x(t) = L \text{ or } x(t) = U\right).$$

*Then, with $L < K < U$,*

$$\begin{aligned} c(0) &= \mathrm{E}\left((x(T \wedge \tau) - K)^+\right), \\ p(0) &= \mathrm{E}\left((K - x(T \wedge \tau))^+\right). \end{aligned}$$

Looking at the form of the diffusion term in (16) suggests, as in previous sections, to focus on the (complex-valued) ratio

$$Y(t) = \frac{x(t) - c_+}{x(t) - c_-} \tag{18}$$

as well as its logarithm. To gain some intuition, the following result is useful.

**Lemma 9** *Let $Y(t)$ be as given in (18), and set*[8] $R(t) = \frac{1}{2}\ln\left(-Y(t)\right)$, $Z(t) = \mathrm{Im}\left(R(t)\right)$. *Define $\tau$ as in Lemma 8. Then, for $t < \tau$,*

$$\begin{aligned} dY(t) &= i2Y(t)\left(dW(t) - \frac{1}{b}\left(x(t) - a - ib\right)dt\right), \\ dR(t) &= i\left(dW(t) - \frac{1}{b}\left(x(t) - a\right)dt\right), \\ dZ(t) &= dW(t) - \frac{1}{b}\left(x(t) - a\right)dt = dW(t) - \tan Z(t)dt. \end{aligned}$$

[7] As should be obvious from previous results, it actually suffices to insert a single absorbing boundary to make *either* the put or the call payout bounded. For generality, we use two boundaries, with the understanding that one of them may, in fact, be set at either $\infty$ or $-\infty$. Indeed, a natural configuration of our bounds $U$ and $L$ is to have $U = \infty$ and $L = 0$.

[8] We use $\mathrm{Im}(\cdot)$ to denote the imaginary part of a complex number.

**Proof:** The dynamics for $Y$ and $R$ follows from Ito's Lemma, as does the first equation for $dZ(t)$. To show the second equality for $dZ(t)$, we only need to notice that, by the definition of $Z(t)$,

$$Z(t) = \arctan\left(\frac{x(t)-a}{b}\right)$$

a result that is evident from the basic relation $\arctan(x) = \frac{1}{2}i\left(\ln(1-ix) - \ln(1+ix)\right)$. ∎

The quantity $Z(t)$ in Lemma 9 is of particular interest, as i) it is monotonic in $x(t)$; and ii) it can be reduced to a Brownian motion by a change of measure. Acting on ii), we proceed to introduce a probability measure $\mathbb{P}$ in which $d\tilde{W} = dW(t) - \tan Z(t)dt$ defines a Brownian motion. To characterize the measure $\mathbb{P}$, let $\tilde{\mathrm{E}}(\cdot)$ denote expectation in measure $\mathbb{P}$ and introduce the $\mathbb{P}$-martingale

$$\eta(t) = \tilde{\mathrm{E}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathscr{F}_t\right).$$

For $t < \tau$, Girsanov's Theorem shows that then, in measure $\mathbb{P}$,

$$\begin{aligned} d\eta(t)/\eta(t) &= -\tan Z(t)d\tilde{W}(t), \quad \eta(0) = 1, && (19)\\ dZ(t) &= d\tilde{W}(t), \quad Z(0) = \arctan\left(\frac{x_0-a}{b}\right). && (20) \end{aligned}$$

From this, we can derive the following result.

**Proposition 6** *Define* $Z_L = \arctan(\frac{L-a}{b})$, $Z_U = \arctan(\frac{U-a}{b})$, *where* $Z_L, Z_U \in (-\pi/2, \pi/2)$. *Set*

$$\tau = \inf\left(t : Z(t) = Z_L \text{ or } Z(t) = Z_U\right),$$

*where* $Z(t)$ *is a Brownian motion in* $\mathbb{P}$, *started at level* $\arctan\left(\frac{x_0-a}{b}\right)$. *With* $\tilde{T} \equiv T \wedge \tau$ *we have, for* $K \in (L, U)$,

$$p(0) = b\sqrt{1+\left(\frac{x_0-a}{b}\right)^2}\,\tilde{\mathrm{E}}\left(e^{\frac{1}{2}\tilde{T}}\left(\tilde{K}\cos Z(\tilde{T}) - \sin Z(\tilde{T})\right)^+\right), \quad \tilde{K} = \frac{K-a}{b}.$$

**Proof:** In Appendix A.5. ∎

Writing $1 = 1_{\tau>T} + 1_{\{\tau\le T, Z(\tau)=Z_L\}} + 1_{\{\tau\le T, Z(\tau)=Z_U\}}$ allows us to decompose the result in Proposition 6 as

$$\begin{aligned} p(0) &= \sqrt{bA(x_0)}e^{\frac{1}{2}T}\tilde{\mathrm{E}}\left(\left(\tilde{K}\cos Z(T) - \sin Z(T)\right)^+ 1_{\tau>T}\right)\\ &\quad + \sqrt{bA(x_0)}\left(\tilde{K}\cos Z_L - \sin Z_L\right)\tilde{\mathrm{E}}\left(e^{\frac{1}{2}\tau}1_{\{\tau\le T, Z(\tau)=Z_L\}}\right)\\ &= \sqrt{bA(x_0)}e^{\frac{1}{2}T}\tilde{\mathrm{E}}\left(\left(\tilde{K}\cos Z(T) - \sin Z(T)\right)^+ 1_{\tau>T}\right)\\ &\quad + \sqrt{\frac{A(x_0)}{A(L)}}(K-L)\,\tilde{\mathrm{E}}\left(e^{\frac{1}{2}\tau}1_{\{\tau\le T, Z(\tau)=Z_L\}}\right), && (21) \end{aligned}$$

where $A(x) = b[1 + ((x-a)/b)^2]$. We have used the fact that the payout of the put is zero whenever $z(\tau) = z_U$ (since we assume that $L < K < U$). The expectations involved in the expression above can be computed analytically, using results similar to those used to prove Propositions 3 and 4. Again, we will have at least two representations, either as a sine-series or as a series based on the method of images. The sine-series result is listed in Proposition 7 below.

**Proposition 7 (Fourier Series)** *Consider the model (16) with absorbing barriers at L and U. Let $Z_L$, $Z_U$, and $\tilde{K}$ be as in Proposition 6, and define*

$$Z_0 = \arctan\left(\frac{x_0 - a}{b}\right), \quad \alpha_n = \frac{n^2\pi^2}{2\left(Z_U - Z_L\right)^2}, \quad a_n = \frac{n\pi\left(Z_L - Z_0\right)}{Z_U - Z_L}.$$

*Then $p(0)$ is given by (21), with*

$$\tilde{\mathrm{E}}\left(\left(\tilde{K}\cos Z(T) - \sin Z(T)\right)^+ 1_{\tau > T}\right) = \frac{2}{Z_U - Z_L}\sum_{n=1}^{\infty} e^{-\alpha_n T}\sin\left(-a_n\right)\left(\tilde{K}I_n^{(c)} - I_n^{(s)}\right),$$

$$\tilde{\mathrm{E}}\left(e^{\frac{1}{2}\tau}1_{\{\tau \le T, Z(\tau) = Z_L\}}\right) = \frac{\sin(Z_U - Z_0)}{\sin\left(Z_U - Z_L\right)} - \frac{\sum_{n=1}^{\infty} n\pi \sin\left(-a_n\right)\frac{e^{-(\alpha_n - \frac{1}{2})T}}{\alpha_n - \frac{1}{2}}}{(Z_U - Z_L)^2}.$$

*Here, the terms $I_n^{(c)}$ and $I_n^{(s)}$ are given in closed form in (29) and (28) in Appendix A.6.*

**Proof:** In Appendix A.6. ∎

Zuhlsdorff [17] lists an alternative representation of the Fourier sine series in Proposition 7. As written, the series in [17] suffers from overflow issues[9] and, additionally, will typically require the computation of many 100's of terms (the author lists 200-300 terms as an average number). In contrast, the series representation above will, on average, converge with 5-10 terms or less.

Application of the method of images here does not lead to a closed form solution (or so we believe), but the put price can still be computed by one-dimensional numerical integration. In cases where the sine series in Proposition 7 is slow to converge ($L$ and $U$ far apart, small value of $T$), the method of images result may still be worthwhile pursuing. We list it below.

**Proposition 8 (Method of Images)** *Consider the model (16) with absorbing barriers at L and U. Let $Z_L$, $Z_U$, and $\tilde{K}$ be as in Proposition 6, and define*

$$Z_0 = \arctan\left(\frac{x_0 - a}{b}\right), \quad z_n' = 2n\left(Z_U - Z_L\right), \quad z_n'' = 2\left(Z_U - Z_0\right) - z_n'.$$

---

[9] The series in [17] involves rapidly growing terms of the form $\exp\left(\left(const \cdot n^2\pi^2 - 1\right)T/2\right)$ where $const > 0$. We should also note that there are several typos in the result in [17].

*Also, set*

$$I_c(y) = \frac{1}{\sqrt{2\pi T}} \int_{z_L}^{k} \exp\left\{ -\frac{(z-y)^2}{2T} \right\} \cos(z+Z_0)\, dz, \tag{22}$$

$$I_s(y) = \frac{1}{\sqrt{2\pi T}} \int_{z_L}^{k} \exp\left\{ -\frac{(z-y)^2}{2T} \right\} \sin(z+Z_0)\, dz. \tag{23}$$

*Then* $p(0)$ *is given by* (21)*, with*

$$\tilde{\mathrm{E}}\left( \left(\tilde{K}\cos Z(T) - \sin Z(T)\right)^{+} 1_{\tau > T} \right) = \sum_{n=-\infty}^{n=\infty} \left( \tilde{K}\left(I_c(z_n') - I_c(z_n'')\right) - \left(I_s(z_n') - I_s(z_n'')\right) \right),$$

$$\tilde{\mathrm{E}}\left( e^{\frac{1}{2}\tau} 1_{\{\tau \le T, Z(\tau) = Z_L\}} \right) = \int_0^T \varphi(t) e^{\frac{1}{2}t} dt,$$

*where*

$$\varphi(t) = \frac{z_U t^{-3/2}}{2(z_U - z_L)} \sum_{n=-\infty}^{n=\infty} \left\{ (z_U - z_n')\, \phi\left(\frac{z_U - z_n'}{\sqrt{t}}\right) - (z_L - z_n')\, \phi\left(\frac{z_L - z_n'}{\sqrt{t}}\right) \right.$$
$$\left. - (z_U - z_n'')\, \phi\left(\frac{z_U - z_n''}{\sqrt{t}}\right) + (z_L - z_n'')\, \phi\left(\frac{z_L - z_n''}{\sqrt{t}}\right) \right\}.$$

**Proof:** In Appendix A.7. ∎

**Remark 9** *While the topic is somewhat outside the scope of this paper, we note that several computational tricks can be used to optimize the computation of the integrals in Proposition 8. For instance, using Euler's formulas for* $\sin(\cdot)$ *and* $\cos(\cdot)$*, the integrals* $I_c$ *and* $I_s$ *in* (22)*-*(23) *can be rewritten in terms of the* complex error function*, allowing for quick computation using well-known expansion series.*

## 4 Parameterization & Numerical Example

In practical applications, the quadratic volatility model may be parameterized through to the intuitive form

$$dx(t) = \sigma \cdot \left( qx(t) + (1-q)x_0 + \frac{1}{2}s\frac{(x(t)-x_0)^2}{x_0} \right) dW(t), \tag{24}$$

where $\sigma > 0$ is a proxy for at-the-money volatility level[10], $q$ is a volatility slope or "skew" parameter, and $s$ is a measure of the convexity of the quadratic volatility function. The

[10] Notice that the volatility function reduces to $\sigma_0 x_0$ whenever $x(t) = x_0$. As discussed in Remark 1, we can easily allow $\sigma$ to be a function of time.

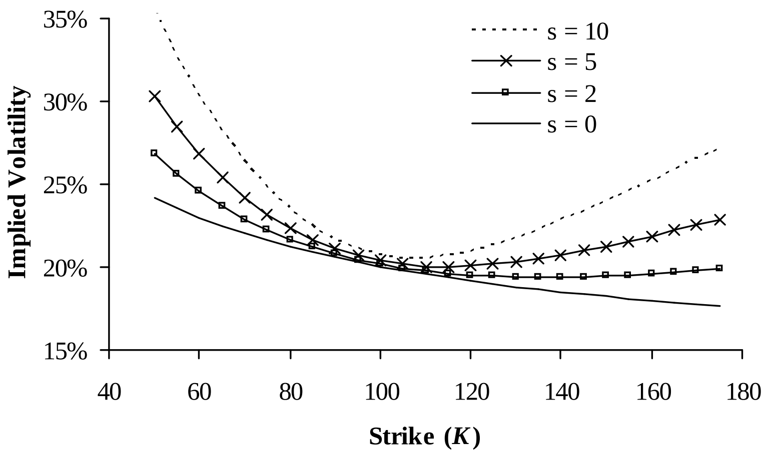
three new parameters $\sigma, q, s$ maps to the parameters $\alpha, \beta, \gamma$ in (1) in obvious fashion. The roots of the quadratic function in (24) are

$$\left(s - q \pm \sqrt{q^2 - 2s}\right) \frac{x_0}{s},$$

and there will be two real roots if $q^2 > 2s$; one real root if $q^2 = 2s$; and no real roots if $q^2 < 2s$.

Figure 1 below shows an example of the types of implied Black-Scholes volatility[11] smile that can be produced by the model (24). In computing the smile, we used the option formulas in this paper with appropriate scaling on the maturity $T$, as outlined in Remark 1.

Figure 1: Implied Volatility Smile for Quadratic Model




**Notes:** The model setup was as in (24, with $\sigma = 20\%$, $q = 0.5$, and $x_0 = 100$. The option maturity is $T = 1$ and the convexity parameter $s$ varies as indicated in the graph. We assumed absorption in zero, the effect of which is minimal for the strike range in the figure.

## 5 Conclusion

As should be obvious at this point, call and put option pricing in the quadratic volatility model is a rather delicate problem that scrapes against the limits of no-arbitrage theory. We have here provided a careful analysis which, we hope, clarifies some confusion in the existing literature and solves the problem once and for all. Our presentation has, on the whole, been focused on theoretical issues, but the numerous pricing formulas listed in the paper should be of use to practitioners that are interested in quick calibration of quadratic volatility models to quoted put and call prices. Due to their tractability, quadratic volatility

[11] See any finance textbook (e.g. [9]) for the definition of "implied" Black-Scholes volatility.

models may serve as a convenient back-bone to more complicated models, such as the "universal" local-stochastic volatility model in Lipton [13] and others.

As a final comment, we note that we would expect practitioners to find it convenient in numerical work to "regularize" the quadratic volatility model to something like

$$dx(t) = \max\left(\sigma_{\min} x(t), \min\left(\sigma_{\max} x(t), A\left(x(t)\right)\right)\right) dW(t), \tag{25}$$

in effect stitching linear tails on to the quadratic form $A(x)$. See Andersen and Andreasen [3] for similar ideas in a CEV setting. Option computations for such a model would necessarily require numerical methods, but if $\sigma_{\min}$ and $\sigma_{\max}$ are small and large, respectively, the formulas in this paper appear to give an excellent approximation of European put and call prices for the model (25). A detailed examination of the quality of this approximation is left for future work.

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# A Appendix: Proofs

## A.1 Proof of Lemma 2

Set $y(t) = x(t) - u$, such that

$$dy(t) = \frac{y(t)(y(t)+u-l)}{u-l}dW(t) = y(t)v(t)dW(t), \quad v(t) \equiv \frac{y(t)+u-l}{u-l}.$$

The process for $v(t)$ is, by Ito's Lemma,

$$dv(t) = \frac{1}{u-l}y(t)v(t)\,dW(t) = (v(t)-1)\,v(t)dW(t).$$

According to the arguments of Sin [16], $y(t)$ – and therefore $x(t) = y(t) + u$ – will be a strict local martingale provided that the "augmented" $v(t)$ process

$$d\hat{v}(t) = (\hat{v}(t)-1)\,\hat{v}(t)^2dt + (\hat{v}(t)-1)\,\hat{v}(t)dW(t)$$

explodes in finite time. Application of the standard Feller boundary criteria for SDEs (e.g. Karlin and Taylor [12], Chapter 15.6) shows that $\infty$ is accessible by $\hat{v}(t)$, proving that $x(t)$ is a strict local martingale. As a strict local martingale bounded from below is a strict supermartingale, the lemma follows. ∎

### A.2 Proof of Proposition 1

Adopting the notation of Lemma 4, the fact that $x(t)$ does not explode in $\mathbb{Q}$ (see Lemma 1) shows that

$$p_1(T) = \lim_{n\to\infty} y^{(n)}(T) 1_{Y^{(n)}(T)<(K-u)/(K-l)} 1_{\tau_n>T}.$$

By dominated convergence, we then have, from Lemma 4

$$\begin{aligned} p_1(0) &= \mathrm{E}\left(p_1(T)\right) = \lim_{n\to\infty} \mathrm{E}\left(y^{(n)}(T) 1_{Y^{(n)}(T)<(K-u)/(K-l)} 1_{\tau_n>T}\right) \\ &= (x_0 - l) \lim_{n\to\infty} \mathrm{E}^{(n)}\left(1_{Y^{(n)}(T)<(K-u)/(K-l)} 1_{\tau_n>T}\right) \end{aligned}$$

The event $x^{(n)}(t) - u = n$ translates to $Y^{(n)}(T) = \frac{n}{n+u-l}$, the limit of which is 1 for $n \to \infty$. As $Y^{(n)}(t)$ is a Geometric Brownian motion in measure $\mathbb{P}^n$ up to the hitting time $\tau_n$, the result (6) follows.

To prove (7), we write $Z(t) = \ln Y(t)$, such that

$$dZ(t) = -\frac{1}{2}dt + dW(t), \quad Z(0) = \ln\left(Y(0)\right) < 0.$$

The absorbing barrier at $Y_l = 1$ becomes an absorbing barrier at the origin of $Z$. The expectation in the expression

$$p_1(0) = \frac{(K-u)(x_0-l)}{u-l} \mathrm{E}\left(1_{Z(T)<\ln k} 1_{\tau>T}\right), \quad k \equiv (K-u)/(K-l),$$

can be evaluated from standard methods for absorbed Brownian motion with drift; see Cox and Miller [6], Chapter 5. The result is

$$\mathrm{E}\left(1_{Z(T)<\ln k} 1_{\tau>T}\right) = \Phi\left(\frac{\ln k - Z(0) + \frac{1}{2}T}{\sqrt{T}}\right) - e^{Z(0)}\Phi\left(\frac{\ln k + Z(0) + \frac{1}{2}T}{\sqrt{T}}\right).$$

A few simplifications lead to (7). ∎

### A.3 Proof of Proposition 3

Let us focus on evaluation of $p_1(0)$. For this, define $z_l(t) = \ln\left(Y_l(t)\right) - \ln\left(Y_l(0)\right)$, such that

$$\begin{aligned} p_1(0) &= \frac{(K-u)(x_0-l)}{u-l} \mathrm{E}\left(1_{Y_l(T\wedge\tau_l)<(K-u)/(K-l)}\right) \\ &= \frac{(K-u)(x_0-l)}{u-l} \mathrm{E}\left(1_{z_l(T\wedge\tau_l)<k}\right), \quad k = \ln\left(\frac{K-u}{K-l}\right) - \ln\left(\frac{x_0-u}{x_0-l}\right). \end{aligned}$$

Clearly $z_l(t)$ is a drifting Brownian motion, started at zero:

$$dz_l(t) = -\frac{1}{2}dt + dW(t), \quad z_l(0) = 0.$$

If $Y_l$ hits 1, $z_l$ hits $z_U = -\ln(Y_l(0))$; if $Y_l$ hits $u/l$, $z_l$ hits $z_L = \ln(u/l) + z_U$. Note that $1 = 1_{\tau_l > T} + 1_{\{\tau_l \le T, z_l(\tau) = z_L\}} + 1_{\{\tau_l \le T, z_l(\tau) = z_U\}}$, such that

$$\begin{aligned} p_1(0) &= \frac{(K-u)(x_0-l)}{u-l}\left\{\mathrm{E}\left(1_{z_l(T\wedge\tau_l)<k}1_{\tau_l>T}\right) + \mathrm{E}\left(1_{\{\tau_l\le T, z_l(\tau)=z_L\}}\right)\right\} \\ &\equiv \frac{(K-u)(x_0-l)}{u-l}\left\{e_1^+(k) + e_2^+\right\}, \end{aligned} \tag{26}$$

where we have used the fact that $1_{z_l(T\wedge\tau_l)<k} = 0$ whenever $1_{\{\tau_l\le T, z_l(\tau)=z_U\}} = 1$. The two expectations $e_1^+(k)$ and $e_2^+$ can be found by standard means. First, from the interior density given in Cox and Miller [6], p. 223,

$$\begin{aligned} e_1^+(k) &= \mathrm{E}\left(1_{z_l(T\wedge\tau_l)<k}1_{\tau_l>T}\right) \\ &= \int_{z_L}^{k} \frac{1}{\sqrt{2\pi T}} \sum_{n=-\infty}^{\infty} \left(\exp\left\{-\frac{z_n'}{2} - \frac{\left(z - z_n' + \frac{1}{2}T\right)^2}{2T}\right\} - \exp\left\{-\frac{z_n''}{2} - \frac{\left(z - z_n'' + \frac{1}{2}T\right)^2}{2T}\right\}\right) dz, \end{aligned}$$

where $z_n' = 2n(z_U - z_L)$, $z_n'' = 2z_U - z_n''$. Evaluating this integral leads to the expression for $e_1^+(k)$ given in Proposition 3.

The expectation $e_2^+$ is the probability of $z$ hitting barrier $z_L$ i) before time $T$, and ii) before $z_U$ is hit. To compute this probability, we first compute $\psi(x)$, the outright probability that $z_L$ will be hit before $z_U$, conditional on $z(0) = x$. From standard results (e.g. [12], Chapter 15.3) we have, for $x \in (z_L, z_U)$,

$$\psi(x) = \frac{\int_x^{z_U} \exp(z - z_L)\,dz}{\int_{z_L}^{z_U} \exp(z - z_L)\,dz} = \frac{1 - \exp(x - z_U)}{1 - \exp(z_L - z_U)}.$$

According to Bhattacharya and Waymire [4], pp. 406-407, we can now compute $e_2^+$ as

$$e_2^+ = \psi(0) - \int_{z_L}^{z_U} \frac{\psi(z)}{\sqrt{2\pi T}} \sum_{n=-\infty}^{\infty} \left(\exp\left\{-\frac{z_n'}{2} - \frac{\left(z - z_n' + \frac{1}{2}T\right)^2}{2T}\right\} - \exp\left\{-\frac{z_n''}{2} - \frac{\left(z - z_n'' + \frac{1}{2}T\right)^2}{2T}\right\}\right) dz$$

Computing the integral leads to the expression for $e_2^+$ given in the proposition.

As for $p_2(0)$, we have

$$p_2(0) = \frac{(K-l)(x_0-u)}{u-l}\mathrm{E}\left(1_{Y_u(T\wedge\tau_u)>(K-l)/(K-u)}\right) = \frac{(K-l)(x_0-u)}{u-l}\mathrm{E}\left(1_{z_u(T\wedge\tau_u)\le k}\right),$$

where $k$ is as defined earlier and $z_u(t) = -\ln(Y_u(T\wedge\tau_u)) + \ln(Y_u(0))$; $z_u(t)$ is a Brownian motion with drift $+\frac{1}{2}$ (rather than $-\frac{1}{2}$, as above) and starting point $z_u(0) = 0$. The expectation $\mathrm{E}\left(1_{z_u(T\wedge\tau_u)\le k}\right)$ can consequently be computed easily from the expressions above (essentially by changing sign on all terms that involve $\frac{1}{2}T$). We omit the details. ∎

### A.4 Proof of Proposition 4

The proof proceeds as for Proposition 3 (see Appendix A.3 above), but now we use Fourier series representations for survival and absorption probabilities. Adopting the notation of Appendix A.3 everywhere, we first notice, from Cox and Miller [6], p. 223,

$$\begin{aligned} e_1^+ &= \mathrm{E}\left(1_{z_l(T\wedge\tau_l)<k} 1_{\tau_l>T}\right) \\ &= \frac{2}{z_U-z_L}\int_{z_L}^{k} e^{-\frac{1}{2}z}\sum_{n=1}^{\infty} e^{-\lambda_n T}\sin\left(n\pi\frac{-z_L}{z_U-z_L}\right)\sin\left(n\pi\frac{z-z_L}{z_U-z_L}\right)dz \\ &= \sum_{n=1}^{\infty}\frac{2\sin\left(n\pi\frac{-z_L}{z_U-z_L}\right)e^{-\lambda_n T}}{z_U-z_L}\int_{z_L}^{k} e^{-\frac{1}{2}z}\sin\left(n\pi\frac{z-z_L}{z_U-z_L}\right)dz, \end{aligned}$$

where

$$\lambda_n = \frac{1}{2}\left(\frac{1}{4}+\frac{n^2\pi^2}{(z_U-z_L)^2}\right). \tag{27}$$

Setting $a_n = \frac{n\pi z_L}{z_U-z_L}$ and $k_n = \frac{n\pi(k-z_L)}{z_U-z_L}$ it is easy to demonstrate that

$$\int_{z_L}^{k} e^{-\frac{1}{2}z}\sin\left(n\pi\frac{z-z_L}{z_U-z_L}\right)dz = \frac{a_n}{2z_L\lambda_n}\left(e^{-\frac{1}{2}z_L}-e^{-\frac{1}{2}k}\cos(k_n)\right)-\frac{e^{-\frac{1}{2}k}}{4\lambda_n}\sin(k_n),$$

So,

$$e_1^+ = \frac{1}{z_U-z_L}\sum_{n=1}^{\infty}\sin(-a_n)e^{-\lambda_n T}\left[\frac{a_n}{\lambda_n z_L}\left(e^{-\frac{1}{2}z_L}-e^{-\frac{1}{2}k}\cos(k_n)\right)-\frac{e^{-\frac{1}{2}k}}{2\lambda_n}\sin(k_n)\right].$$

As for the lower-boundary absorption probability $e_2^+$, using the same technique as in Appendix A.3,

$$\begin{aligned} e_2^+ &= \mathrm{E}\left(1_{\{\tau_l\le T, z_l(\tau)=z_L\}}\right) \\ &= \psi(0)-\int_{z_L}^{z_U}\frac{\psi(z)}{\sqrt{2\pi T}}\sum_{n=-\infty}^{\infty}\frac{2\sin(-a_n)e^{-\lambda_n T}}{z_U-z_L}\int_{z_L}^{z_U} e^{-\frac{1}{2}z}\sin\left(n\pi\frac{z-z_L}{z_U-z_L}\right)dz, \end{aligned}$$

where $\psi(x) = (1-\exp(x-z_U))/(1-\exp(z_L-z_U))$. Evaluating the integral gives us

$$e_2^+ = \psi^+ + \frac{e^{-\frac{1}{2}z_L}}{(z_U-z_L)^2}\sum_{n=1}^{\infty}\frac{e^{-\lambda_n T}}{\lambda_n}n\pi\sin(a_n),$$

with $\psi^+$ as given in the Proposition. Computation of $e_1^-$ and $e_2^-$ proceeds the same way, after a shift of drift of $z_l$, from $-\frac{1}{2}$ to $+\frac{1}{2}$. ∎

### A.5 Proof of Proposition 6

The form of (19) suggests that, for $t \le \tau$,

$$\eta(t) = q(t)\cos(Z(t))$$

for some deterministic function $q$. Applying Ito's Lemma, we get

$$\begin{aligned} d\eta(t) &= -q(t)\sin(Z(t))\,d\tilde{W}(t) - \frac{1}{2}q(t)\cos(Z(t))\,dt + \frac{dq(t)}{dt}\cos(Z(t))\,dt \\ &= -\eta(t)\tan(Z(t))\,d\tilde{W}(t) + \cos(Z(t))\left(\frac{dq(t)}{dt} - \frac{1}{2}q(t)\right)dt. \end{aligned}$$

From (19), it follows that the function $q(t)$ must satisfy

$$\frac{dq(t)}{dt} = \frac{1}{2}q(t), \quad q(0)\cos(Z(0)) = 1,$$

i.e.

$$q(t) = q(0)e^{\frac{1}{2}t}, \quad q(0) = \frac{1}{\cos\left(\arctan\left(\frac{x_0-a}{b}\right)\right)} = \sqrt{1+\left(\frac{x_0-a}{b}\right)^2}.$$

By Lemma 8 and the usual rules of measure change

$$\begin{aligned} p(0) &= \mathrm{E}\left(\left(K - x(\tilde{T})\right)^+\right) = \tilde{\mathrm{E}}\left(\eta(\tilde{T})\left(K - x(\tilde{T})\right)^+\right) \\ &= q(0)\tilde{\mathrm{E}}\left(e^{\frac{1}{2}\tilde{T}}\cos Z(\tilde{T})\left(K - a - b\tan Z(\tilde{T})\right)^+\right) \\ &= q(0)b\tilde{\mathrm{E}}\left(e^{\frac{1}{2}\tilde{T}}\left(\tilde{K}\cos Z(\tilde{T}) - \sin Z(\tilde{T})\right)^+\right), \quad \tilde{K} = \frac{K-a}{b}. \end{aligned}$$

∎

### A.6 Proof of Proposition 7

Let us first turn to the computation of $e_1 = \tilde{\mathrm{E}}\left((\tilde{K}\cos Z(T) - \sin Z(T))^+ 1_{\tau > T}\right)$, where we recall that $\tau$ is the first time that the Brownian motion $Z(T)$ hits either $Z_U$ or $Z_L$. We define $Z_0 = \arctan\left(\frac{x_0-a}{b}\right)$, such that $z(t) = Z(t) - Z_0$ is a regular Brownian motion started at zero. We first notice that $\tilde{K}\cos Z(T) - \sin Z(T) > 0 \Leftrightarrow z(T) < \arctan(\tilde{K}) - Z_0 \equiv k$, such that, from results similar to those in Appendix A.4,

$$e_1 = \int_{z_L}^{k} n(z; z_L, z_U, T)\left(\tilde{K}\cos(z + Z_0) - \sin(z + Z_0)\right)dz,$$

where $z_L = Z_L - Z_0$, $z_U = Z_U - Z_0$, and

$$n(z; z_L, z_U, T) = \frac{2}{z_U - z_L}\sum_{n=1}^{\infty} e^{-\alpha_n T}\sin\left(n\pi\frac{-z_L}{z_U - z_L}\right)\sin\left(n\pi\frac{z - z_L}{z_U - z_L}\right),$$

with

$$\alpha_n = \frac{n^2\pi^2}{2\left(z_U - z_L\right)^2}.$$

It follows that

$$e_1 = \frac{2}{z_U - z_L}\sum_{n=1}^{\infty} e^{-\alpha_n T}\sin\left(-a_n\right)\left(\tilde{K}I_n^{(c)} - I_n^{(s)}\right),$$

where, with $a_n = n\pi\frac{z_L}{z_U - z_L}$,

$$\begin{aligned} I_n^{(c)} &= \int_{z_L}^{k} \sin\left(a_n\frac{z}{z_L} - a_n\right)\cos\left(z + Z_0\right)dz, \\ I_n^{(s)} &= \int_{z_L}^{k} \sin\left(a_n\frac{z}{z_L} - a_n\right)\sin\left(z + Z_0\right)dz. \end{aligned}$$

Defining $b_n^{\pm} = 1 \pm a_n/z_L$ and $c_n^{\pm} = b_n^{\pm}k \mp a_n + Z_0$ it is easily shown that

$$I_n^{(c)} = \frac{z_L}{2}\left[\frac{\cos\left(c_n^-\right)}{z_L - a_n} - \frac{\cos\left(c_n^+\right)}{z_L + a_n} + \frac{2\cos\left(Z_L\right)a_n}{a_n^2 - z_L^2}\right], \tag{28}$$

where we have used that $b_n^{\pm}z_L = z_L \pm a_n = Z_L - Z_0 \pm a_n$. Similarly, we get

$$I_n^{(s)} = \frac{z_L}{2}\left[\frac{\sin\left(c_n^-\right)}{z_L - a_n} - \frac{\sin\left(c_n^+\right)}{z_L + a_n} + \frac{2\sin\left(Z_L\right)a_n}{a_n^2 - z_L^2}\right]. \tag{29}$$

As an aside, notice that

$$\begin{aligned} \cos\left(Z_L\right) &= \cos\left(\arctan\left(\frac{L-a}{b}\right)\right) = \frac{1}{\sqrt{A(L)/b}}, \\ \sin\left(Z_L\right) &= \sin\left(\arctan\left(\frac{L-a}{b}\right)\right) = \frac{L-a}{\sqrt{bA(L)}}. \end{aligned}$$

Having now computed an explicit expression for $e_1$, we turn to the computation of $e_2 = \tilde{\mathrm{E}}\left(e^{\frac{1}{2}\tau}1_{\{\tau\le T, Z(\tau)=Z_L\}}\right)$. First, we establish that

$$1 - \tilde{\mathrm{E}}\left(1_{\{\tau\le T\}}\right) = \int_{z_L}^{z_U} n\left(z; z_L, z_U, T\right)dz.$$

As $z$ is a drift-free Brownian motion started at zero, a symmetry argument shows that

$$\tilde{\mathrm{E}}\left(1_{\{\tau\le T, Z(\tau)=Z_L\}}\right) = \frac{z_U}{z_U - z_L}\tilde{\mathrm{E}}\left(1_{\{\tau\le T\}}\right).$$

The (defect) density of the random time $\tau_L$ at which $z$ gets absorbed at $z_L$ is therefore (employing somewhat loose notation)

$$\begin{aligned} \varphi(t) &\equiv \mathbb{P}\left(\tau_L \in [t, t+dt]\right)/dt = -\frac{z_U}{z_U - z_L}\int_{z_L}^{z_U}\frac{\partial}{\partial t}n\left(z; z_L, z_U, t\right)dz \\ &= \frac{1}{\left(z_U - z_L\right)^2}\sum_{n=1}^{\infty} e^{-\alpha_n t}n\pi\sin\left(-a_n\right), \end{aligned}$$

Consequently,

$$
\begin{aligned}
e_2 &= \tilde{\mathrm{E}}\left(e^{\frac{1}{2}\tau}1_{\{\tau\le T, Z(\tau)=Z_L\}}\right) = \int_0^T \varphi(t)e^{\frac{1}{2}t}dt \\
&= \frac{1}{(z_U-z_L)^2}\sum_{n=1}^{\infty} n\pi \sin(-a_n)\int_0^T e^{-(\alpha_n-\frac{1}{2})t}dt \\
&= \frac{1}{(z_U-z_L)^2}\sum_{n=1}^{\infty} n\pi \sin(-a_n)\frac{1-e^{-(\alpha_n-\frac{1}{2})T}}{\alpha_n-\frac{1}{2}} \\
&= \frac{\sin(z_U)}{\sin(z_U-z_L)} - \frac{1}{(z_U-z_L)^2}\sum_{n=1}^{\infty} n\pi \sin(-a_n)\frac{e^{-(\alpha_n-\frac{1}{2})T}}{\alpha_n-\frac{1}{2}}.
\end{aligned}
$$

∎

### A.7 Proof of Proposition 8

We borrow all notation from Appendix A.6 above. As before, we have

$$
e_1 = \int_{z_L}^{k} n(z; z_L, z_U, T)\left(\tilde{K}\cos(z+Z_0) - \sin(z+Z_0)\right)dz,
$$

where we now use a method of images representation for $n(z; z_L, z_U, T)$ :

$$
n(z; z_L, z_U, T) = \frac{1}{\sqrt{2\pi T}}\sum_{n=-\infty}^{n=\infty}\left[\exp\left\{-\frac{(z-z_n')^2}{2T}\right\} - \exp\left\{-\frac{(z-z_n'')^2}{2T}\right\}\right],
$$

with $z_n' = 2n(z_U - z_L)$, $z_n'' = 2z_U - z_n'$. Defining $I_c(\cdot)$ and $I_s(\cdot)$ as in (22)-(23) we get

$$
e_1 = \sum_{n=-\infty}^{n=\infty}\left(\tilde{K}\left(I_c(z_n') - I_c(z_n'')\right) - \left(I_s(z_n') - I_s(z_n'')\right)\right).
$$

As for the computation of $e_2 = \tilde{\mathrm{E}}\left(e^{\frac{1}{2}\tau}1_{\{\tau\le T, Z(\tau)=Z_L\}}\right)$, we first note that

$$
\begin{aligned}
1 - \tilde{\mathrm{E}}\left(1_{\{\tau\le T\}}\right) &= \int_{z_L}^{z_U} n(z; z_L, z_U, T)\,dz \\
&= \sum_{n=-\infty}^{n=\infty}\left\{\Phi\left(\frac{z_U - z_n'}{\sqrt{T}}\right) - \Phi\left(\frac{z_L - z_n'}{\sqrt{T}}\right) - \Phi\left(\frac{z_U - z_n''}{\sqrt{T}}\right) + \Phi\left(\frac{z_L - z_n''}{\sqrt{T}}\right)\right\}.
\end{aligned}
$$

Following the steps in Appendix A.6, we get

$$
\varphi(t) \equiv \mathbb{P}\left(\tau_L \in [t, t+dt]\right)/dt = -\frac{z_U}{z_U - z_L}\int_{z_L}^{z_U}\frac{\partial}{\partial t}n(z; z_L, z_U, t)\,dz
$$

the explicit evaluation of which leads to the the result for $\varphi$ given in the proposition. Finally, from the definition of $e_2$,

$$e_2 = \int_0^T \varphi(t) e^{\frac{1}{2}t} dt.$$

∎