VIX FUTURES

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VIX futures are exchange-traded contracts on a future volatility index (VIX) level derived from a basket of S&P 500 (SPX) stock index options. The authors posit a stochastic variance model of VIX time evolution, and develop an expression for VIX futures. Free parameters are estimated from market data over the past few years. It is found that the model with parameters estimated from the whole period from 1990 to 2005 overprices the futures contracts by 16–44%. But the discrepancy is dramatically reduced to 2–12% if the parameters are estimated from the most recent one-year period. © 2006 Wiley Periodicals, Inc. Jrl Fut Mark 26:521–531, 2006

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INTRODUCTION

The VIX futures is the first-ever listed futures contract on the Chicago Board Options Exchange (CBOE) Volatility Index (VIX). Its trading began on the CBOE Futures Exchange (CFE) on March 26, 2004.[1]

The VIX Index is a key measure of market expectations of near-term volatility conveyed by S&P 500 (SPX) stock index option prices. There are two versions of the volatility index, an old one and a new one. The old VIX, renamed VXO in 2003, was first introduced by the CBOE in 1993. It represents the implied volatility of a hypothetical 30-calendar-day at-the-money S&P 100 index (OEX) options. In September 2003, the CBOE began disseminating a new VIX index, which is computed based on the prices of a portfolio of 30-calendar-day out-of-the-money SPX calls and puts with weights being inversely proportional to the squared strike price. The VIX squared is equal to the 30-calendar-day variance swap rate. Since its introduction in 1993, the VIX (VXO) has quickly become the benchmark for the stock market volatility and is often referred to as the "investor fear gauge."

Since its inception, the VIX futures market has been steadily growing. To establish an intuition on the market size, we looked at the data on the CBOE Web site. For example, the open interest of VIX futures was 9240 on February 28, 2005. This corresponds to a market value of 112 million USD.[2] The trading volume was 332, which corresponds to 4 million USD.

The realized variance can be replicated by a portfolio of options (Carr & Madan, 1998; Demeterfi, Derman, Kamal, & Zou, 1999); therefore, the variance swap rate can be determined relatively by the current values of the replicating options. Researchers are looking for possible ways to replicate realized volatility (Carr & Lee, 2003), but the results available so far are very limited and only applicable to zero correlation between stock return and volatility. Because the VIX cannot be replicated by a portfolio of the SPX options and it is not a traded asset, one cannot use the no-arbitrage principle to obtain a simple relationship between VIX futures and the VIX as that of stock futures and stock price. Pricing VIX futures becomes an issue.

Carr and Wu (2004) present a lower bound and a upper bound of the VIX futures price. By applying Jensen's inequality they show that the current VIX futures (with maturity T) price is between the forward

[1]Some key features of the VIX futures are attached in the Appendix.
[2]On February 28, 2005, VIX = 12.08, VXB = 10 × VIX = 120.8. The VIX futures contract size is 100 × VXB = 12,080, therefore the total market value is 12.08 × 10 × 100 × 9240 = 111,619,200.

volatility swap rate and the forward variance swap rate over the period of $(T, T + 30/365)$.

In this article, the dynamics of the new and evolving volatility market are explored in terms of a simple and intuitively appealing model. To begin, a Heston-type of stochastic variance model was chosen; its tractability provides an intuitive linkage between the model and the market. Our model for the VIX index and VIX futures is presented in the next section. Then the model parameters are estimated by using the historical VIX data and the VIX futures pricing formula is tested against the market data. A conclusion is presented in the last section.

THEORY

VIX Model

In statistical measure, suppose the SPX, denoted as S_t, is modeled by following diffusion process with stochastic instantaneous variance, V_t (Heston, 1993),

$$dS_t = \mu S_t\,dt + \sqrt{V_t}S_t\,dB_t^S \tag{1}$$

$$dV_t = \kappa(\theta - V_t)\,dt + \sigma_V\sqrt{V_t}\,dB_t^V \tag{2}$$

where μ is the expected return of investing in the SPX, θ is long-run mean level of the instantaneous variance, κ is the mean-reverting speed of the variance, σ_V measures the variance of the variance. Two standard Brownian motions, B_t^S and B_t^V, describe the random noises in asset return and variance. They are correlated with a correlation coefficient, ρ, which is assumed to be a constant.

With the change of probability measure, we are able to describe the SPX in a risk-neutral measure by following

$$dS_t = rS_t\,dt + \sqrt{V_t}S_t\,d\widetilde{B}_t^S \tag{3}$$

$$dV_t = [\kappa\theta - (\kappa + \lambda)V_t]\,dt + \sigma_V\sqrt{V_t}\,d\widetilde{B}_t^V \tag{4}$$

where the two new standard Brownian motions, $\widetilde{B}_t^S$ and $\widetilde{B}_t^V$, are correlated with constant correlation coefficient, ρ, as well. The new parameter, λ, measures volatility risk premium.

Because the VIX squared is defined to be the variance swap rate,[3] we are able to evaluate it by computing the conditional expectation in

[3]The definition of the VIX and the methodology of computing it from options price are clearly described in the white paper of the CBOE (2003).

the risk-neutral measure

$$VIX_t^2 = E_t^Q\left[\frac{1}{\tau_0}\int_t^{t+\tau_0} V_s\,ds\right] \tag{5}$$

where τ_0 is 30 calendar days by definition. From Equation (4), we have

$$E_t^Q[V_s] = \frac{\kappa\theta}{\kappa + \lambda} + \left(V_t - \frac{\kappa\theta}{\kappa + \lambda}\right) e^{-(\kappa+\lambda)(s-t)} \tag{6}$$

Substituting this equation into Equation (5) gives following result.

Proposition 1: *The VIX squared is a linear function of the instantaneous variance*

$$VIX_t^2 = A + BV_t \tag{7}$$

where A and B are functions of structural parameters, given by

$$A = \frac{\kappa\theta}{\kappa + \lambda}\left[1 - \frac{1 - e^{-(\kappa+\lambda)\tau_0}}{(\kappa + \lambda)\tau_0}\right], \qquad B = \frac{1 - e^{-(\kappa+\lambda)\tau_0}}{(\kappa + \lambda)\tau_0} \tag{8}$$

with $\tau_0 = 30/365$.

Because the process of the VIX squared is observable, one can use the proposition to back out the process of instantaneous variance.

VIX Futures

In the risk-neutral measure, the square root process of instantaneous variance in Equation (4) determines the transition probability density (Cox, Ingersoll, & Ross, 1985)

$$f^Q(V_T|V_t) = c\, e^{-u-v}\left(\frac{v}{u}\right)^{q/2} I_q(2\sqrt{uv}) \tag{9}$$

where

$$c = \frac{2(\kappa + \lambda)}{\sigma_V^2(1 - e^{-(\kappa+\lambda)(T-t)})}, \quad u = cV_t e^{-(\kappa+\lambda)(T-t)}, \quad v = cV_T, \quad q = \frac{2\kappa\theta}{\sigma_V^2} - 1$$

and $I_q(\cdot)$ is the modified Bessel function of the first kind of order q. The distribution function is the noncentral chi-square, $\chi^2(2v;\, 2q + 2,\, 2u)$, with $2q + 2$ degrees of freedom and parameter of noncentrality $2u$ proportional to the current variance, V_t.

By using risk-neutral valuation formula, we have the following result.

Proposition 2: *The price of VIX futures with maturity T is given by following formula*

$$F_t = E_t^Q(VIX_T) = E_t^Q(\sqrt{A + BV_T}) = \int_0^{+\infty} \sqrt{A + BV_T}\, f^Q(V_T \mid V_t)\, dV_T \quad (10)$$

where A and B are given by Equation (8), and $f^Q(V_T \mid V_t)$ is given by Equation (9).

Because the price of VIX futures is a function of the instantaneous variance, V_t, which is linked to VIX_t by proposition 1, then the VIX futures price is a function of VIX_t as well, i.e.,

$$F_t = F_t(VIX_t, T - t; \kappa, \theta, \sigma_V, \lambda)$$

One is unable to describe the function analytically because it is unlikely that the integration (10) can be carried out in a closed-form.

DATA, MODEL ESTIMATION, AND TESTING

The VIX data was downloaded from the Web site of the CBOE to estimate the VIX model. The VIX time series is shown in Figure 1. The daily VIX futures price was retrieved from the same Web site to test the futures price formula.

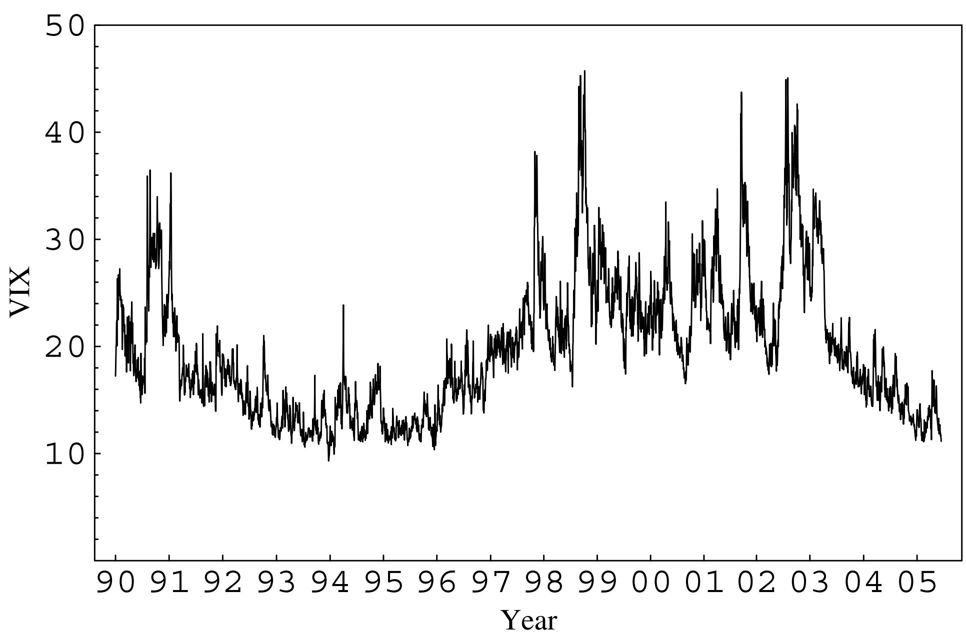

FIGURE 1
The closing level of the S&P 500 Volatility Index (VIX). The sample period is January 2, 1990–June 16, 2005. Data from the Chicago Board Options Exchange. Retrieved June 17, 2005, from http://www.cboe.com

Estimating the VIX Model

Given the physical process of the instantaneous variance in Equation (2), one may have the following conditional probability density function

$$f^P(V_T|V_t) = c\, e^{-u-v}\left(\frac{v}{u}\right)^{q/2} I_q(2\sqrt{uv}) \tag{11}$$

where

$$c = \frac{2\kappa}{\sigma_V^2(1 - e^{-\kappa(T-t)})}, \quad u = cV_t e^{-\kappa(T-t)}, \quad v = cV_T, \quad q = \frac{2\kappa\theta}{\sigma_V^2} - 1$$

With proposition 1, one may have the conditional density function for VIX_t^2 as follows

$$f^P(VIX_t^2|\,VIX_{t-\Delta t}^2) = \frac{1}{B} f^P(V_t|\,V_{t-\Delta t}) \tag{12}$$

The likelihood function is then given by

$$L(\kappa, \theta, \sigma_V, \lambda) = \sum_{t=1}^{N} \ln[f^P(VIX_t^2|VIX_{t-\Delta t}^2)] \tag{13}$$

Maximizing the likelihood function by using the historical VIX data gives us values of the parameters.

The VIX index data is available from January 2, 1990. We use the data up to March 1, 2005, because we will test our pricing theory for the VIX futures contracts by using their market prices on March 1, 2005. We have done the estimation exercises for three different periods: the whole period from January 2, 1990 to March 1, 2005, the high-VIX-level period from January 2, 1998 to December 31, 2003 as illustrated in Figure 1, and the most recent one-year period from March 1, 2004 to March 1, 2005. The results are presented in Table I.

We have two observations: First, the standard deviation of the estimate for λ is very large, which means that the current estimate of λ is not accurate. We have also done some additional exercises to observe the sensitivity of other estimates by changing the controlled value of λ. We find that our estimation results are not sensitive to the value of λ. Second, the long-term mean level of the variance is changing over time. It is very high from January 2, 1998 to December 31, 2003 with a value of 0.0837, which corresponds to a volatility of 29%. Yet it is relatively

TABLE I

The Values of Parameters Estimated From the Historical VIX Data Over Three Different Periods: The Whole Period, High-Level Period, and Most Recent One-Year Period

κ	θ	σ_V	λ
The whole period: January 2, 1990–March 1, 2005			
5.8313	0.0415	0.4900	−0.5594
(0.6533)	(0.0143)	(0.0950)	(10.6045)
The high level period: January 2, 1998–December 31, 2003			
11.7360	0.0837	0.8936	10.0000
(1.5564)	(0.0346)	(0.1961)	(16.1485)
The most recent one-year period: March 1, 2004–March 1, 2005			
18.4971	0.0284	0.5819	10.0000
(5.0625)	(0.0211)	(0.2241)	(32.4239)

Note. The estimations of the last two periods are done with a control of $\lambda \leq 10$. The numbers in parentheses are standard deviations of the estimations.

low from March 1, 2004 to March 1, 2005, with a value of 0.0284, which corresponds to a volatility of 17%.

Because the current estimation does not give a reliable λ, we choose to use the value, $\lambda = -0.8716$, estimated by Shu and Zhang (2004) with the SPX options market data from 1995 to 1999 and keep it fixed. We have done a new estimation exercise by controlling for the fixed value of λ. The results of the new estimation are presented in Table II.

With the estimated parameters, one may use proposition 1 to establish the linear relationship between VIX_t^2 and instantaneous variance, V_t.

Testing Futures Price Formula

On March 1, 2005, there were four kinds of VIX futures traded in the CFE: VIX/H5, VIX/K5, VIX/Q5, and VIX/X5. They represent four different maturities: March 16, May 18, August 17, and November 16 with time to maturities being 15, 78, 169, and 260 days, respectively. The VIX level was 12.04.

In Table III, the model price A is computed by using proposition 2 with the parameters, $(\kappa, \theta, \sigma_V, \lambda) = (5.7895, 0.0414, 0.4868, -0.8716)$, which is estimated from the historical VIX data for the whole period from January 2, 1990 to March 1, 2005. The model price is 16% larger than the market price for the March futures. It is 44% larger for the

TABLE II

The Values of Parameters Estimated From the Historical VIX Data Over Three Different Periods: The Whole Period, High-Level Period, and Most Recent One-Year Period With a Control of $\lambda = -0.8716$

κ	θ	σ_V	λ
The whole period: January 2, 1990–March 1, 2005			
5.7895	0.0414	0.4868	−0.8716
(0.6534)	(0.0143)	(0.0939)	(10.4989)
The high level period: January 2, 1998–December 31, 2003			
13.0468	0.0611	0.7707	−0.8716
(1.5777)	(0.0222)	(0.1488)	(12.5145)
The most recent one-year period: March 1, 2004–March 1, 2005			
19.8741	0.0217	0.5073	−0.8716
(5.1268)	(0.0147)	(0.1809)	(26.1812)

Note. The numbers in parentheses are standard deviations of the estimations.

TABLE III

The Prices of VIX Futures With Four Different Maturities on March 1, 2005

Maturity	*VIX/H5* *March 16*	*VIX/K5* *May 18*	*VIX/Q5* *August 17*	*VIX/X5* *November 16*	*RMSE*
$T - t$ (days)	15	78	169	260	
Market price	122.0	131.6	137.8	146.4	
Model price A	141.8	185.6	204.9	210.2	54.5
Model price B	136.9	148.9	149.3	149.3	12.8
Model price C	123.3	135.0	141.7	143.6	3.0

Note. Model price A is computed with parameters, $(\kappa, \theta, \sigma_V, \lambda) = (5.7895, 0.0414, 0.4868, -0.8716)$, which are estimated from the VIX data for the whole period from January 2, 1990 to March 1, 2005. Model price B is computed with parameters, $(\kappa, \theta, \sigma_V, \lambda) = (19.8741, 0.0217, 0.5073, -0.8716)$, which are estimated from the VIX data for the most recent one-year period from March 1, 2004 to March 1, 2005. Model price C is computed with parameters, $(\kappa\theta, \kappa + \lambda, \sigma_V) = (0.1177, 4.8899, 0.4851)$, which are calibrated from the market prices of VIX futures on March 1, 2005. RMSE is the root of mean squared error between model and market prices.

November futures. The model price B is computed with the parameters, $(\kappa, \theta, \sigma_V, \lambda) = (19.8741, 0.0217, 0.5073, -0.8716)$, which is estimated from the historical VIX data for the recent one-year period from March 1, 2004 to March 1, 2005. The discrepancy of the November futures price has been reduced to 2%, but that of the March futures price is 12%, still quite large.

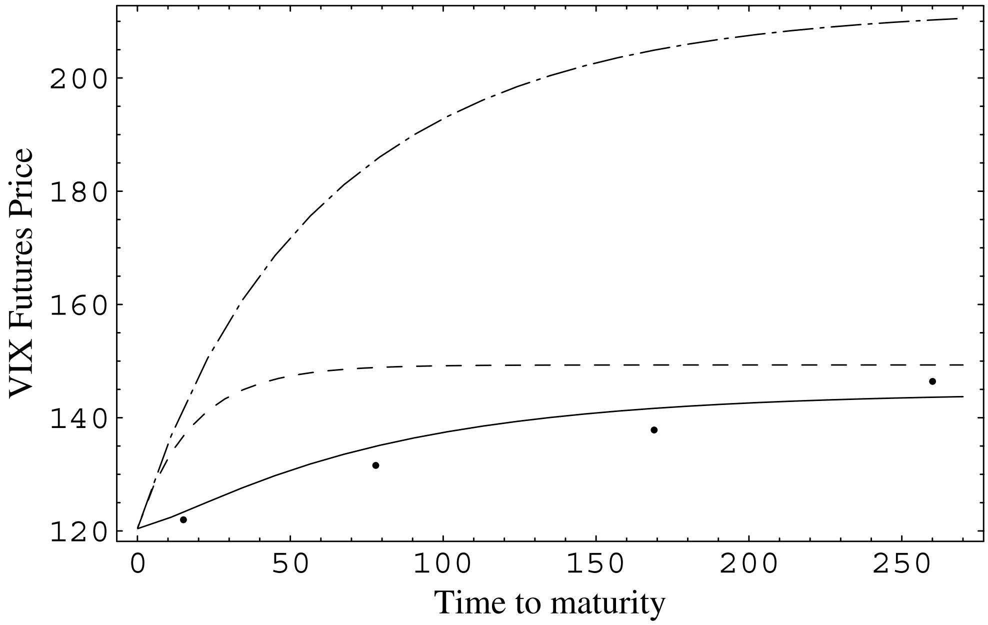

FIGURE 2

The VIX futures price as a function of time to maturity (days) on March 1, 2005. The dots are market prices. The dash-doted line is model price computed with parameters, $(\kappa, \theta, \sigma_V, \lambda) = (5.7895, 0.0414, 0.4868, -0.8716)$, which are estimated from the VIX data for the whole period from January 2, 1990 to March 1, 2005. The dashed line is model price computed with parameters, $(\kappa, \theta, \sigma_V, \lambda) = (19.8741, 0.0217, 0.5073, -0.8716)$, which are estimated from the VIX data for the most recent one-year period from March 1, 2004 to March 1, 2005. The solid line is model price computed with parameters, $(\kappa\theta, \kappa + \lambda, \sigma_V) = (0.1177, 4.8899, 0.4851)$, which are calibrated from the market prices of VIX futures on the day.

We now study the volatility parameters implied in the VIX futures prices. By numerically solving the following minimization problem

$$\min_{(\kappa,\theta,\lambda,\sigma_V)} [F_t^{model}(\kappa, \theta, \lambda, \sigma_V) - F_t^{market}]^2$$

we obtain an optimal set of parameters

$$(\kappa\theta, \kappa + \lambda, \sigma_V) = (0.1177, 4.8899, 0.4851)$$

The VIX futures price computed with this set of parameters is listed in Table I as model price C. One may notice that we are unable to determine the market price of risk, λ, from VIX futures price, because the first three parameters enter into the VIX futures formula as two identities, $\kappa\theta$ and $\kappa + \lambda$. If we take $\lambda = -0.8716$, then $\kappa = 5.7615$, and $\theta = 0.0204$. VIX futures price as a function of time to maturity is shown in Figure 2 graphically.

The exercise shows that the futures price is very sensitive to the long-term mean, and the value estimated from the historical VIX data for the overall period from January 2, 1990 to March 1, 2004 ($\theta = 0.0414$) is too large for the purpose of pricing VIX futures on March 1, 2005. The value estimated from the historical VIX data for the recent one-year period from March 1, 2004 to March 1, 2005 ($\theta = 0.0217$) is closer to the long term mean implied by the VIX futures market prices ($\theta_{\text{Implied}} = 0.0204$). It is therefore a better candidate for VIX futures pricing.

Our estimation of the volatility of volatility parameter, $\sigma_V = 0.5073$, is also very close to that implied by the VIX futures market, $\sigma_{V\text{Implied}} = 0.4851$. But the mean reverting speed parameter, $\kappa = 19.8741$, is much larger than that implied by the VIX futures market, $\kappa_{\text{Implied}} = 5.7615$. This indicates that the VIX futures traders believe that the volatility reverts to the long-term mean much slower than what one-year historical VIX data tells us.

CONCLUSION

We have developed a simple and intuitively appealing model to price the VIX futures. By using the historical time series of the VIX data from January 2, 1990 to March 1, 2005, we determine the structural parameters in the variance process with the maximum likelihood method. We use the market price of volatility risk estimated by Shu and Zhang (2004) by using the SPX option market data from 1995 to 1999. We then price the VIX futures with different maturities and compare the model prices with the market prices on one particular day, March 1, 2005.

The results show that the model with the parameters estimated from the whole period overprices all four futures contracts by 16% for March futures and 44% for November futures. We find that the long-term mean level of the variance is crucial to the futures price in the model. By using the parameters estimated from the recent one-year period, we are able to reduce the discrepancy between the model price and the market price from 16% to 12% for the March futures and from 44% to 2% for the November futures. Our research suggests that the most recent VIX data should be used to estimate the volatility structural parameters in the VIX futures-pricing model.

APPENDIX

Some Key Features of the VIX Futures

The VIX futures has following key features:

1. CBOE Volatility Index (VIX) Futures ticker symbol: VX.
2. Trading hours: 8:30 A.M. to 3:15 P.M. CST (Central Standard Time).

3. Underlying value: The Increased-Value VIX = 10 times the index value, disseminated by the CBOE, through Options Price Reporting Authority (OPRA), under the symbol "VXB," and through the CFE under the symbol "VBI."
4. Contract size: $100 times VXB. For example, with a VIX value of $16.50, the VXB would be 165 and the contract size would be $16,500.
5. Minimum tick size: 10 cents. Therefore, minimum value change will be in $10 intervals.
6. Contract months: Initially, May, June, August, and November. Thereafter, two near-term and two additional months on the February quarterly cycle.
7. Last trading date: Usually the Tuesday prior to the third Friday of the month.
8. Settlement date: Usually the Wednesday prior to the third Friday of the month.
9. VIX Futures are settled in cash.

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