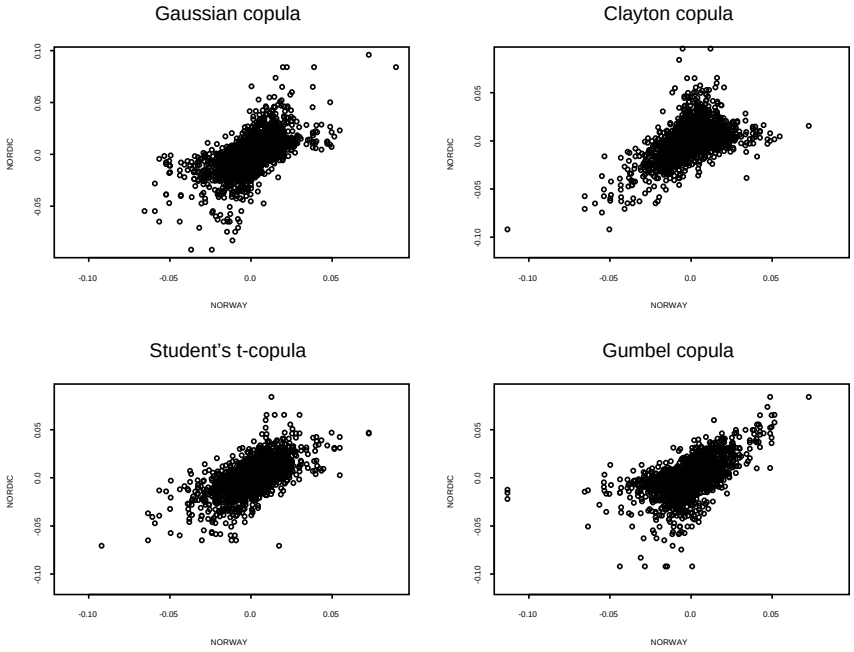
Modelling the dependence structure of financial assets: A survey of four copulas



SAMBA/22/04

Kjersti Aas

December 2004

Norwegian Computing Center
APPLIED RESEARCH AND DEVELOPMENT

NR Note

Title: Modelling the dependence structure of financial assets: A survey of four copulas

Date: December 2004
Year: 2006
Note no.: SAMBA/22/04

Author: Kjersti Aas

Abstract:

Understanding and quantifying dependence is at the core of all modelling efforts in financial econometrics. The linear correlation coefficient, which is the far most used measure to test dependence in the financial community and also elsewhere, is only a measure of linear dependence. This means that it is a meaningful measure of dependence if asset returns are well represented by an elliptical distribution. Outside the world of elliptical distributions, however, using the linear correlation coefficient as a measure of dependence may lead to misleading conclusions. Hence, alternative methods for capturing co-dependency should be considered. One class of alternatives are copula-based dependence measures. In this survey we consider two parametric families of copulas; the copulas of normal mixture distributions and Archimedean copulas.

Keywords: Copulas, Gaussian, Student's t, Gumbel, Clayton, dependence measures, estimation, simulation

Target group:

Availability: Open

Project: GB-BFF

Project no.: 220195

Research field: Finance, insurance and power market

No. of pages: 19

Norwegian Computing Center
Gaustadalléen 23, P.O. Box 114 Blindern, NO-0314 Oslo, Norway
Telephone: +47 2285 2500, telefax: +47 2269 7660, http://www.nr.no
Copyright © 2006 by Norwegian Computing Center, Oslo, Norway
All rights reserved. Printed in Norway.

Contents

1 Introduction **1**

2 Definition **1**

3 Examples of copulas **2**
3.1 Implicit copulas 2
3.2 Explicit copulas 3

4 Copula-based dependence measures **3**
4.1 Kendall's tau and Spearman's rho 4
4.2 Tail dependence 5

5 Estimating copula parameters **6**
5.1 Implicit copulas 7
5.1.1 Gaussian copula 8
5.1.2 Student's t-copula with ν degrees of freedom 10
5.2 Explicit copulas 12
5.2.1 Clayton copula 12
5.2.2 Gumbel copula 12
5.3 How to choose copulas empirically? 13

6 Simulating from copulas **14**
6.1 Implicit copulas 14
6.1.1 Gaussian copula 14
6.1.2 Student's t-copula 14
6.2 Explicit copulas 14
6.2.1 Clayton copula 15
6.2.2 Gumbel copula 15
6.3 Example 16

References **16**

1 Introduction

Understanding and quantifying dependence is the core of all modelling efforts in financial econometrics. The linear correlation coefficient, which is by far the most used measure to test dependence in the financial community (and also elsewhere), is not a measure of general, but only of *linear* dependence. If asset returns are well represented by an elliptical distribution, such as the multivariate Gaussian or the multivariate Student's t, their dependence structure is linear. Hence, the linear correlation coefficient is a meaningful measure of dependence. Outside the world of elliptical distributions, however, the use of the linear correlation coefficient as a measure of dependence may induce misleading conclusions. In financial markets, there is often a non-linear dependence between returns. Thus alternative methods for capturing co-dependency should be considered, such as copula-based ones. Copulas are used to combine marginal distributions into multivariate distributions.

The concept of copulas was introduced by Sklar (1959), and has for a long time been recognized as a powerful tool for modelling dependence between random variables. The use of copula theory in financial applications is a relatively new (introduced by Embrechts et al. (1999)) and fast-growing field. From a practical point of view, the advantage of the copula-based approach to modelling is that appropriate marginal distributions for the components of a multivariate system can be selected freely, and then linked through a suitable copula. That is, copula functions allow one to model the dependence structure independently of the marginal distributions. Any multivariate distribution function can serve as a copula.

As an example of how copulas may be successfully used, consider the modelling of the joint distribution of a stock market index and an exchange rate. The Student's t-distribution has been found to provide a reasonable fit to the univariate distributions of daily stock market as well as of exchange rate returns. Hence, it might seem natural to model the joint distribution with a bivariate Student's t-distribution. However, the standard bivariate Student's t-distribution has the restriction that both marginal distributions must have the same tail heaviness, while the distributions of daily stock market and exchange rate returns don't. Decomposing the multivariate distribution into marginal distributions and a copula, allows for the construction of better models of the individual variables than would be possible if only explicit multivariate distributions were considered.

We first give the formal definition of a copula in Section 2. In Section 3 we present the functional forms of four different copulas, and in Section 4 we describe dependence measures that are based on copulas. Section 5 treats the problem of estimating the parameters of the four different copulas, and finally algorithms for simulating from these copulas are given in Section 6.

2 Definition

The definition of a d-dimensional copula is a multivariate distribution, C, with uniformly distributed marginals $U(0,1)$ on [0,1]. Sklar's theorem states that every multivariate distribution F with marginals F_1, F_2,...,F_d can be written as

$$F(x_1, \ldots, x_d) = C(F_1(x_1), F_2(x_2), ..., F_d(x_d)) \tag{1}$$

for some copula C. If we have a random vector $\boldsymbol{X} = (X_1, \ldots, X_d)$ the copula of their joint distribution function may be extracted from Equation 1:

$$C(u_1, \ldots, u_d) = F(F_1^{-1}(u_1), F_2^{-1}(u_2), ..., F_d^{-1}(u_d)), \tag{2}$$

where the F_i^{-1}'s are the quantile functions of the marginals.

3 Examples of copulas

In this section, we present the functional forms of four of the most commonly used copulas. For the sake of simplicity we only consider the bivariate versions of the copulas here. In Section 5 extensions to the multivariate case are treated. It is common to represent a bivariate copula by its distribution function

$$C(u, v) = P(U \leq u, V \leq v) = \int_{-\infty}^{u} \int_{-\infty}^{v} c(s, t)\, ds\, dt, \tag{3}$$

where $c(s, t)$ is the density of the copula. In this survey we concentrate on two parametric families of copulas; the copulas of normal mixture distributions and Archimedean copulas. The first are so-called *implicit* copulas, for which the double integral at the right-hand side of Equation 3 is implied by a well-known bivariate distribution function, while the latter are *explicit* copulas, for which this integral has a simple closed form.

3.1 Implicit copulas

Implicit copulas do not have a simple closed form, but are implied by well-known multivariate distribution functions. In this survey we will consider two implicit copulas; the Gaussian and the Student's t-copula.

Gaussian copula The Gaussian copula is given by

$$C_\rho(u, v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\,\pi\,(1-\rho^2)^{1/2}} \exp\left\{-\frac{x^2 - 2\,\rho\,x\,y + y^2}{2\,(1-\rho^2)}\right\} dx\, dy, \tag{4}$$

where ρ is the parameter of the copula, and $\Phi^{-1}(\cdot)$ is the inverse of the standard univariate Gaussian distribution function. Although it probably not is obvious to the reader that the double integral in Equation 4 is equal to the double integral in Equation 3, it can be shown that this is the case.

Student's t-copula The Student's t-copula allows for joint fat tails and an increased probability of joint extreme events compared with the Gaussian copula. This copula can be written as

$$C_{\rho,\nu}(u, v) = \int_{-\infty}^{t_\nu^{-1}(u)} \int_{-\infty}^{t_\nu^{-1}(v)} \frac{1}{2\,\pi\,(1-\rho^2)^{1/2}} \left\{1 + \frac{x^2 - 2\,\rho\,x\,y + y^2}{\nu\,(1-\rho^2)}\right\}^{-(\nu+2)/2} ds\, dt, \tag{5}$$

where ρ and ν are the parameters of the copula, and t_ν^{-1} is the inverse of the standard univariate student-t-distribution with ν degrees of freedom, expectation 0 and variance $\frac{\nu}{\nu-2}$.

The Student's t-dependence structure introduces an additional parameter compared with the Gaussian copula, namely the degrees of freedom ν. Increasing the value of ν decreases the tendency to exhibit extreme co-movements. As will be shown in Section 4.2, the Student's t-dependence structure supports joint extreme movements regardless of the marginal behaviour of the individual assets. This is even the case when the assets have light-tailed distributions.

3.2 Explicit copulas

There are also a number of copulas which are not derived from multivariate distribution functions, but do have simple closed forms. In this survey we will consider two explicit copulas; the Clayton and Gumbel copulas. Both are so-called Archimedian copulas.

Clayton copula The Student's t-copula allows for joint extreme events, but not for asymmetries. If one believes in the asymmetries in equity return dependence structures reported by for instance Longin and Solnik (2001) and Ang and Chen (2000), the Student's t-copula may also be too restrictive to provide a reasonable fit. Then, the Clayton copula, which is an asymmetric copula, exhibiting greater dependence in the negative tail than in the positive, might be a better choice. This copula is given by

$$C_\delta(u, v) = (u^{-\delta} + v^{-\delta} - 1)^{-1/\delta},$$

where $0 < \delta < \infty$ is a parameter controlling the dependence. Perfect dependence is obtained if $\delta \to \infty$, while $\delta \to 0$ implies independence.

Gumbel copula The Gumbel copula is also an asymmetric copula, but it is exhibiting greater dependence in the positive tail than in the negative. This copula is given by

$$C_\delta(u, v) = \exp\left(-[(-\log u)^\delta + (-\log v)^\delta]^{1/\delta}\right),$$

where $\delta \geq 1$ is a parameter controlling the dependence. Perfect dependence is obtained if $\delta \to \infty$, while $\delta = 1$ implies independence.

4 Copula-based dependence measures

Since the copula of a multivariate distribution describes its dependence structure, it might be appropriate to use measures of dependence which are copula-based. The bivariate concordance measures *Kendall's tau* and *Spearman's rho*, as well as the coefficient of *tail dependence*, can, as opposed to the linear correlation coefficient, be expressed in terms of the underlying copula alone. In Section 4.1 we describe Kendall's tau and Spearman's rho, while we in Section 4.2 discuss the concept of tail dependence.

4.1 Kendall's tau and Spearman's rho

Kendall's tau of two variables X and Y is

$$\rho_\tau(X,Y) = 4\int_0^1\int_0^1 C(u,v)\,dC(u,v) - 1,$$

where $C(u,v)$ is the copula of the bivariate distribution function of X and Y. For the Gaussian and Student's t-copulas and also all other elliptical copulas, the relationship between the linear correlation coefficient and Kendall's tau is given by

$$\text{cor}(X,Y) = \sin\left(\frac{\pi}{2}\,\rho_\tau\right), \tag{6}$$

where *cor* is the linear correlation coefficient. Schweizer and Wolff (1981) established that for Archimedean copulas, Kendall's tau can be related to the dependence parameter. For the Clayton copula it is is given by

$$\rho_\tau(X,Y) = \frac{\delta}{\delta+2},$$

and for the Gumbel copula it is

$$\rho_\tau(X,Y) = 1 - \frac{1}{\delta}.$$

For the Gaussian and Student's t-copulas, Kendall's tau must be estimated empirically. See Section 5.1.2 for a description.

Spearman's rho of two variables X and Y is given by

$$\rho_S(X,Y) = 12\int_0^1\int_0^1 C(u,v)\,du\,dv - 3,$$

where $C(u,v)$ is the copula of the bivariate distribution function of X and Y. Let X and Y have distribution functions F and G, respectively. Then, we have the following relationship between Spearman's rho and the linear correlation coefficient

$$\rho_S(X,Y) = \text{cor}(F(X),F(Y)).$$

For the Gaussian and Student's t-copulas, we have that the relationship between the linear correlation coefficient and Spearman's rho is

$$\text{cor}(X,Y) = 2\sin\left(\frac{\pi}{6}\,\rho_S\right).$$

Both $\rho_\tau(X,Y)$ and $\rho_S(X,Y)$ may be considered as measures of the degree of monotonic dependence between X and Y, whereas linear correlation measures the degree of linear dependence only. Moreover, these measures are invariant under monotone transformations, while the linear correlation generally isn't. Hence, according to Embrechts et al. (1999) it is slightly better to use these measures than the linear correlation coefficient. In their opinion, however, one should choose a model for the dependence structure that reflects more detailed knowledge of the risk management problem at hand instead of summarizing dependence with a "dangerous single number like (linear or rank) correlation". One such measure is *tail dependence*, which is the issue of the following section.

4.2 Tail dependence

There is a saying in finance that in times of stress, correlations will increase. Bivariate tail dependence measures the amount of dependence in the upper and lower quadrant tail of a bivariate distribution. This is of great interest for the risk manager trying to guard against concurrent bad events. In this section we discuss the coefficient of tail dependence, which was first introduced in the financial context by Embrechts et al. (2001).

Let $X \sim F_X$ and $Y \sim F_Y$. By definition, the upper tail dependence coefficient is

$$\lambda_u(X,Y) = \lim_{\alpha \to 1} P(Y > F_Y^{-1}(\alpha) | X > F_X^{-1}(\alpha)),$$

and quantifies the probability to observe a large Y, assuming that X is large. Analogously, the coefficient of lower tail dependence is

$$\lambda_l(X,Y) = \lim_{\alpha \to 0} P(Y \leq F_Y^{-1}(\alpha) | X \leq F_X^{-1}(\alpha)).$$

These measures are independent of the marginal distributions of the asset returns. Moreover, they are invariant under strictly increasing transformations of X and Y.

For elliptical distributions, $\lambda_u(X,Y) = \lambda_l(X,Y)$. If $\lambda_u(X,Y) > 0$, large events tend to occur simultaneously. On the contrary, when $\lambda_u(X,Y) = 0$, the distribution has no tail dependence, and the variables X and Y are said to be asymptotically independent. It is important to note that while independence of X and Y implies $\lambda_u(X,Y) = \lambda_l(X,Y) = 0$, the converse is not true in general. That is, $\lambda_u(X,Y) = \lambda_l(X,Y) = 0$ does not necessarily imply that X and Y are statistically independent. Thus asymptotic independence should be considered as the "weakest dependence which can be quantified by the coefficient of tail dependence" (Ledford and Tawn, 1998).

Gaussian copula: For the Gaussian copula, the coefficients of lower tail and upper tail dependence are

$$\lambda_l(X,Y) = \lambda_u(X,Y) = 2 \lim_{x \to -\infty} \Phi\left(x \frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}\right) = 0.$$

This means, that regardless of high correlation ρ we choose, if we go far enough into the tail, extreme events appear to occur independently in X and Y.

The Student's t-copula: For the Student's t-copula, the coefficients of lower and upper tail dependence are

$$\lambda_l(X,Y) = \lambda_u(X,Y) = 2\, t_{\nu+1}\left(-\sqrt{\nu+1}\sqrt{\frac{1-\rho}{1+\rho}}\right),$$

where $t_{\nu+1}$ denotes the distribution function of a univariate Student's t-distribution with $\nu+1$ degrees of freedom. The stronger the linear correlation ρ and the lower the degrees of freedom ν, the stronger is the tail dependence. Surprisingly perhaps, the Student's t-copula gives asymptotic dependence in the tail, even when ρ is negative (> -1) and zero.

The Clayton copula: The Clayton copula is lower tail dependent. That is, the coefficient of the upper tail dependence $\lambda_u(X, Y) = 0$, and the coefficient of the lower tail dependence is

$$\lambda_l(X, Y) = 2^{-1/\delta}.$$

The Gumbel copula: The Gumbel copula is upper tail dependent. That is, the coefficient of the lower tail dependence $\lambda_l(X, Y) = 0$ and the coefficient of the upper tail dependence is

$$\lambda_u(X, Y) = 2 - 2^{1/\delta}.$$

The coefficient of tail dependence seems to provide a useful measure of the extreme dependence between two random variables. However, a difficult problem remains unsolved, namely to estimate the tail dependence for an empirical data set. One alternative is to use a parametric approach. Longin and Solnik (2001) for instance, choose to model dependence via the Gumbel copula and determine the associated tail dependence. The problem with this method is that the choice of the copula amounts to choose a priori whether or not the data presents tail dependence.

Another alternative is to estimate the coefficient of tail dependence empirically. Unfortunately, this is a strenuous task, and to our knowledge, there is not yet any reliable estimator. A direct estimation of the conditional probability $P(Y > F_Y^{-1}(\alpha)|X > F_X^{-1}(\alpha))$, which should tend to λ when $\alpha \to 1$ is impossible in practice, due to the curse of dimensionality and drastic decrease of the number of realisations as α becomes close to one. Hence, a fully non-parametric approach is not reliable.

5 Estimating copula parameters

There are mainly two ways of estimating the parameters of a copula; a fully parametric method or a semi-parametric method. The fully parametric method may again be divided into two sub-approaches. The first, is performed by maximizing the likelihood function to all parameters simultaneously, both copula and margin parameters. As the scale of the problem increases, this method gets computationally very demanding. Hence, a simpler, but less accurate approach, that has been termed the inference functions for margins (IFM) method (Joe, 1997), have been proposed. In this approach the parameters of the margins are first estimated, and then each parametric margin is plugged into the copula likelihood, and this full likelihood is maximized with respect to the copula parameters.

The success of the fully-parametric method obviously depends upon finding appropriate parametric models for the margins, which may not always be so straightforward if they show evidence of heavy tails and/or skewness. Hence, it would be better to have a procedure that avoids marginal risk as much as possible. Several authors, e.g. Demarta and McNeil (2004) and Romano (2002), have therefore proposed a semi-parametric approach, for which one do not have any parametric assumptions for the margins. Instead, the univariate empirical cumulative distribution functions are plugged into the likelihood. The estimation procedure can be described as follows:

Assume that we want to determine the copula for the iid sample $\boldsymbol{Y}_1, .., \boldsymbol{Y}_n$, where $\boldsymbol{Y}_i = (Y_{i,1}, ..., Y_{i,d})'$. Assume further that the empirical marginal distribution functions are denoted $F_1, ..., F_d$. Hence, a pseudo-sample $\boldsymbol{U}_1, .., \boldsymbol{U}_n$ from the copula may be constructed by

$$\boldsymbol{U}_i = (U_{i,1}, ..., U_{i,d})' = (F_1(Y_{i,1}, ..., F_d(Y_{i,1}))'.$$

The upper panel of Figure 1 shows a plot of Norwegian (TOTX) vs. Nordic (MSCI Nordic) geometric stock index returns from the period 01.01.1988-28.08.2002. The lower panel shows the same returns after they have been transformed into uniform variates using the empirical distribution function. Having obtained the pseudo-samples, we can use maximum likelihood to estimate the copula parameters. The estimates are obtained by maximizing

$$\log L(\boldsymbol{\Theta}, \boldsymbol{U}_1, .., \boldsymbol{U}_n) = \sum_{i=1}^{n} \log c(\boldsymbol{U}_i; \boldsymbol{\Theta}), \tag{7}$$

where $\boldsymbol{\Theta}$ denotes the set of copula parameters, and $c(\boldsymbol{u}; \boldsymbol{\Theta})$ denotes the density of the copula. In Genest et al. (1995) it is shown that the estimator $\boldsymbol{\Theta}$ resulting from (7) is consistent and has asymptotically normal distribution under regularity conditions similar to those of maximum likelihood theory. The estimator is shown to be semiparametrically efficient in the case of independence and the normal copula. Moreover, Genest et al. (1995) have shown that the efficiency condition is not fulfilled for the Gumbel-Morgenstern family, but otherwise the efficiency of the estimator is not yet completely clarified.

In Section 5.1 we describe parameter estimation for the two implicit copulas, while the estimation for the explicit copulas is treated in Section 5.2.

5.1 Implicit copulas

For the implicit copula of an absolutely continuous joint distribution function F with strictly continuous marginal distribution functions $F_1, \ldots, F_d$, the copula density is given by

$$c(\boldsymbol{u}) = \frac{f(F_1^{-1}(u_1), \ldots, F_1^{-1}(u_d))}{f_1(F_1^{-1}(u_1)) \cdots f_d(F_d^{-1}(u_d))}, \tag{8}$$

where f is the density of the joint distribution, $f_1, \ldots, f_d$ are the marginal densities, and $F_1^{-1}, \ldots, F_d^{-1}$ are the ordinary inverses of the marginal distribution functions. Using this technique, we can calculate the density of the Gaussian and the Student's t-copulas. Figure 2 shows the density of the bivariate Gaussian distribution with correlation 0.7 and standard normal marginals together with the density of the bivariate Gaussian copula with parameter $\rho = 0.7$. In Sections 5.1.1 and 5.1.2 we will give the densities of the multivariate Gaussian and Student's t-copulas and show how the parameters of these copulas can be estimated.

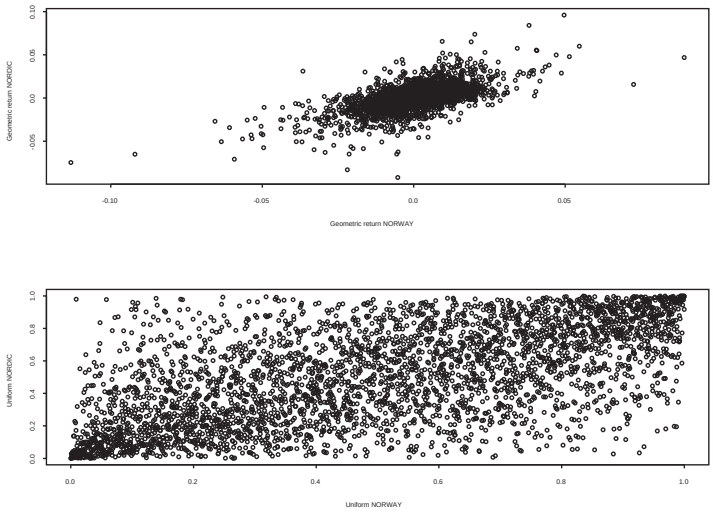


Figure 1: *Upper panel: Norwegian (TOTX) vs. Nordic (MSCI Nordic) geometric stock index returns from the period 01.01.1988-28.08.2002. Lower panel: The same returns after they have been transformed into uniform variates.*

5.1.1 Gaussian copula

A stochastic d-vector $\mathbf{X}$ is multivariate Gaussian distributed with expectation vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ if its density is given by

$$f(\mathbf{x}) = \frac{1}{(2\,\pi)^{d/2}\,|\boldsymbol{\Sigma}|^{1/2}} e^{-(1/2)(\mathbf{x}-\boldsymbol{\mu})'\,\boldsymbol{\Sigma}^{-1}\,(\mathbf{x}-\boldsymbol{\mu})}.$$

The copula is based on the standardized distribution, in which $\boldsymbol{\mu}$ is the null-vector, and the covariance matrix $\boldsymbol{\Sigma}$ is equal to the correlation matrix R. As previously mentioned, the density of an implicit copula equals the density of the corresponding multivariate distribution divided by the product of the densities of its marginals. That is, the density of the d-dimensional Gaussian copula is given by

$$c(\boldsymbol{u}) \;=\; \frac{\frac{1}{(2\,\pi)^{d/2}\,|\boldsymbol{R}|^{1/2}} e^{-1/2\,\mathbf{x}'\,\boldsymbol{R}^{-1}\,\mathbf{x}}}{\prod_{j=1}^{d} \frac{1}{(2\,\pi)^{1/2}} e^{-1/2\,x_j^2}} = \frac{e^{-1/2\,\mathbf{x}'\,\boldsymbol{R}^{-1}\,\mathbf{x}}}{|\boldsymbol{R}|^{1/2}\, e^{-1/2\,\sum_{j=1}^{d} x_j^2}}$$

where $\mathbf{x} = (\Phi^{-1}(u_1), ..., \Phi^{-1}(u_d))$.

The Gaussian copula has one parameter, the correlation matrix $\boldsymbol{R}$. The maximum likelihood estimator for $\boldsymbol{R}$ can be obtained by

$$\hat{\boldsymbol{R}} = \arg\max_{\boldsymbol{R}\in\mathcal{P}} \sum_{i=1}^{n} \log c(\boldsymbol{U}_i; \boldsymbol{R}), \tag{9}$$

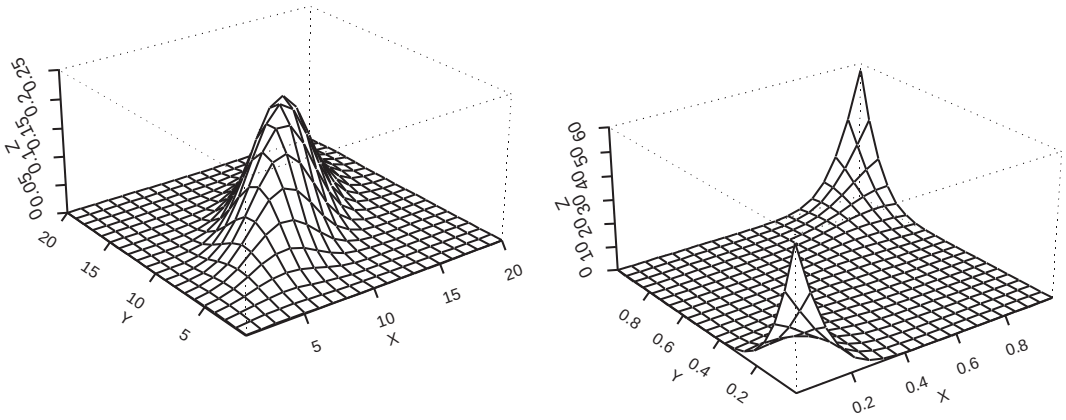


Figure 2: *Left: The density of the bivariate Gaussian distribution with correlation 0.7 and standard normal marginals. Right: The density of the bivariate Gaussian copula with parameter* $\rho = 0.7$.

where the pseudo-samples $\boldsymbol{U}_i$ are generated using either the IFM or CML method described above, and $\mathcal{P}$ denotes the set of all possible linear correlation matrices. To find the maximum, one has to search over the set of unrestricted lower-triangular matrices with ones on the diagonal. According to McNeil et al. (2005), this search is feasible in low dimensions, but very slow in high dimensions.

An approximate solution to maximization can be obtained by instead of maximizing over the set of all possible linear correlation matrices as in Equation 9, we maximize over the set of all covariance matrices. This problem has an analytical solution, which is the maximum likelihood estimator of the covariance matrix $\boldsymbol{\Sigma}$ for a $N(\mathbf{0}, \boldsymbol{\Sigma})$ distribution:

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{N} \sum_{i=1}^{n} (\Phi^{-1}(U_{i,1}), ..., \Phi^{-1}(U_{i,d}))' (\Phi^{-1}(U_{i,1}), ..., \Phi^{-1}(U_{i,d})).$$

This matrix is likely to be close to being a correlation matrix, but it does not necessarily have ones at the diagonal. However, the corresponding correlation matrix $\hat{\boldsymbol{R}}$ can easily be obtained by

$$\hat{\boldsymbol{R}} = \boldsymbol{\Delta}^{-1} \hat{\boldsymbol{\Sigma}} \boldsymbol{\Delta}^{-1},$$

where $\boldsymbol{\Delta}$ is a diagonal matrix containing the diagonal elements of $\hat{\boldsymbol{\Sigma}}$.

5.1.2 Student's t-copula with ν degrees of freedom

A stochastic d-vector $\mathbf{X}$ is multivariate Student's t-distributed with ν degrees of freedom, expectation $\boldsymbol{\mu}$ and scale matrix $\mathbf{S}$ if its density function is given by

$$f(\mathbf{x}) = \frac{\Gamma[(\nu + d)/2]}{(\pi \nu)^{d/2} \Gamma(\nu/2) |\mathbf{S}|^{1/2}} \left(1 + \frac{(\mathbf{x} - \boldsymbol{\mu})' \mathbf{S}^{-1} (\mathbf{x} - \boldsymbol{\mu})}{\nu}\right)^{-\frac{(\nu+d)}{2}},$$

The covariance matrix of the Student's t-distribution is given by

$$\mathrm{Cov}(\mathbf{X}) = \frac{\nu}{\nu - 2} \mathbf{S}.$$

The copula is based on the standardized distribution, in which $\boldsymbol{\mu}$ is the null-vector, and the scale matrix $\mathbf{S}$ is equal to the correlation matrix R. The density of the t-copula is given by (Bouyé et al., 2000):

$$c(\mathbf{u}) = \frac{\Gamma(\frac{\nu+d}{2}) \Gamma(\frac{\nu}{2})^{d-1} \left(1 + \frac{\mathbf{x}' \mathbf{R}^{-1} \mathbf{x}}{\nu}\right)^{-\frac{(\nu+d)}{2}}}{|\mathbf{R}|^{1/2} \Gamma(\frac{\nu+1}{2})^{d} \prod_{j=1}^{d} (1 + \frac{x_j^2}{\nu})^{-\frac{\nu+1}{2}}}, \tag{10}$$

where $\mathbf{x} = (t_\nu^{-1}(u_1), ..., t_\nu^{-1}(u_d))$.

Unlike for the Gaussian copula, estimation of the Student's t-copula parameters requires numerical optimisation of the log-likelihood function. Ideally, the likelihood function should be maximized with respect to the degrees of freedom ν and the correlation matrix $\boldsymbol{R}$

simultaneously. This maximization is generally very involved, and a naive numerical search is likely to fail because of the high dimensionality (the number of parameters is $1 + d(d-1)/2$) of the parameter space. A simpler approach, used by Mashal and Zeevi (2002) and Demarta and McNeil (2004), is a two-stage procedure in which $\boldsymbol{R}$ is estimated first using Kendall's tau, and then the pseudo-likelihood function is maximized with respect to ν.

The first step can be summarized as follows. First, an empirical estimate $\hat{\rho}_\tau(Y_j, Y_k)$ of Kendall's tau is constructed for each bivariate margin, using the original data vectors $\boldsymbol{Y}_1, .., \boldsymbol{Y}_n$:

$$\hat{\rho}_\tau(Y_j, Y_k) = \begin{pmatrix} n \\ 2 \end{pmatrix}^{-1} \sum_{1 \le i \le l \le n} \text{sign}(Y_{i,j} - Y_{l,j})(Y_{i,k} - Y_{l,k}). \tag{11}$$

Note, that we could equally well have computed Kendall's tau from the uniform variables $\boldsymbol{U}_i = (F_1(Y_{i,1}, ..., F_d(Y_{i,1}))^{'}$, where the F_j's are marginal distribution functions, or from the Student's t distributed variables $\boldsymbol{X}_i = (t_\nu^{-1}(U_{i,1}), ..., t_\nu^{-1}(U_{i,d}))$ with an arbitrary ν, since Kendall's tau is invariant under strictly increasing transformations of the random variables.

The next step is to use the relationship between Kendall's tau and the linear correlation coefficient for an elliptical distribution given in Equation 6, to obtain the linear correlation of the bivariate pairs. That is,

$$\hat{\boldsymbol{R}}_{ij} = \sin\left(\frac{\pi}{2}\,\hat{\rho}_\tau\right).$$

There is no guarantee that this componentwise transformation of the empirical Kendall's taus will result in a positive definite correlation matrix. However, the matrix may be adjusted using a procedure such as the method of Rousseeuw and Molenberghs (1993) or that of Rebonato and Jäckel (1999).

The easiest way to estimate the remaining parameter ν is by maximum likelihood with the correlation matrix $\hat{\boldsymbol{R}}$ held fixed (Mashal and Zeevi, 2002). That is, perform a numerical search for $\hat{\nu}$:

$$\hat{\nu} = \text{argmax}_\nu \left[\sum_{i=1}^{n} \log(c(\boldsymbol{U}_i, \nu, \hat{\boldsymbol{R}}))\right].$$

Here $c(\cdot)$ is the density of the Student's t-copula given by Equation 10, and the pseudo-samples $\boldsymbol{U}_i$ are generated using either the IFM or CML method described above. Note that for each step in the numerical search for ν we have to first transform the pseudo-samples $\boldsymbol{U}_i$ to $\boldsymbol{X}_i = (t_\nu^{-1}(U_{i,1}), ..., t_\nu^{-1}(U_{i,d}))$, where ν is the current degrees of freedom.

Bouyé et al. (2000) (p. 41-42) do the opposite and propose an algorithm to estimate $\boldsymbol{R}$, when ν is known (they do not say anything about how ν is determined). The authors show that the maximum likelihood estimate of the correlation matrix then is given by

$$\hat{\boldsymbol{R}}_{m+1} = \frac{1}{n}\left(\frac{\nu+2}{\nu}\right) \sum_{i=1}^{n} \frac{\boldsymbol{X}_i\,\boldsymbol{X}_i^{'}}{1 + \frac{1}{\nu}\,\boldsymbol{X}_i^{'}\,\hat{\boldsymbol{R}}_m^{-1}\,\boldsymbol{X}_i}, \tag{12}$$

where $\mathbf{X}_i = (t_\nu^{-1}(U_{i,1}), ..., t_\nu^{-1}(U_{i,d}))$ and the pseudo-samples $\boldsymbol{U}_i$ are generated using either the IFM or CML method described above. Equation 12 is iterated until convergence is

obtained. Bouyé et al. (2000) use the maximum likelihood estimate of the correlation matrix for the Gaussian copula as the starting value.

According to Mashal and Zeevi (2002), this procedure is computationally intensive and suffers from numerical stability problems arising from the inversion of close to singular matrices. Moreover, it still remains to find an appropriate value for ν. They therefore state that the first method given in this section should be used.

5.2 Explicit copulas

5.2.1 Clayton copula

It is not obvious how to define the d-dimensional Clayton copula, but one valid choice (see McNeil et al. (2005) for the necessary conditions for a valid copula) is given by

$$C(\boldsymbol{u}) = \left(\sum_{j=1}^{d} u_j^{-\delta} - d + 1 \right)^{-1/\delta}.$$

The density of this copula is given by

$$c(\boldsymbol{u}) = \frac{\delta^d C(\boldsymbol{u})}{\delta u_1 \delta u_2 \dots \delta u_d} = \delta^d \frac{\Gamma(\frac{1}{\delta} + d)}{\Gamma(\frac{1}{\delta})} \left(\prod_{j=1}^{d} u_j^{-\delta-1} \right) \left[\sum_{j=1}^{d} u_j^{-\delta} - d + 1 \right]^{-1/\delta - d}.$$

The parameter δ is generally found by numerical maximization of the log-likelihood in Equation 7. In the bivariate case, however, a simpler approach can be used, in which the relationship between Kendall's tau and the copula parameter δ is utilised. The density of the bivariate Clayton copula is given by (Venter, 2001)

$$c(u, v) = \frac{\delta^2 C(u, v)}{\delta u \delta v} = (1 + \delta)(uv)^{-1-\delta} \left(u^{-\delta} + v^{-\delta} - 1 \right)^{-1/\delta - 2},$$

and the parameter δ can be estimated using the following procedure:

- Compute an empirical estimate of Kendall's tau $\hat{\rho}_\tau$ (see for instance Equation 11).
- Use the relationship between Kendall's tau and δ for the Clayton copula given in Section 4.1 to obtain an estimate for δ:

$$\hat{\delta} = \frac{2\,\hat{\rho}_\tau}{1 - \hat{\rho}_\tau}.$$

5.2.2 Gumbel copula

Like for the Clayton copula, it is not obvious how to define d-dimensional version of the Gumbel copula. One valid choice is given by

$$C(\boldsymbol{u}) = \exp\left[\left(-\sum_{j=1}^{d} (-\log u_j)^{\delta} \right)^{1/\delta} \right].$$

According to McNeil et al. (2005) it is straightforward to derive the density of this copula, at least if a computer algebra system such as *Mathematica* is used. We limit ourselves to the bivariate density (Venter, 2001) here

$$\begin{aligned} c(u,v) &= \frac{\delta^2 C(u,v)}{\delta u \delta v} \\ &= C(u,v)\,(u\,v)^{-1}((-\log u)^{\delta} + (-\log v)^{\delta})^{-2+2/\delta}(\log u \log v)^{\delta-1} \\ &\times \left\{1 + (\delta-1)((-\log u)^{\delta} + (-\log v)^{\delta})^{-1/\delta}\right\}. \end{aligned}$$

The parameter δ may be estimated using maximum likelihood, see Bouyé et al. (2000) (p. 56), but like for the Clayton copula, a simpler approach is to use the relationship between Kendall's tau and the copula parameter:

- Compute an empirical estimate of Kendall's tau $\hat{\rho}_\tau$ (see for instance Equation 11).
- Use the relationship between Kendall's tau and δ for the Gumbel copula given in Section 4.1 to obtain an estimate for δ:

$$\hat{\delta} = \frac{1}{1-\hat{\rho}_\tau}.$$

5.3 How to choose copulas empirically?

One of the main issues with copulas is to choose the copula that provides the best fit for the data set at hand. According to Blum et al. (2002), giving an answer to this question is essentially as difficult as estimating the joint distribution in the first place. The choice among different copulas can be done via so-called goodness-of-fit (GOF) tests (Fermanian). However, while there in the 1-D case are a lot of well-known distribution-independent GOF statistics available, e.g. Kolmogorov-Smirnov and Anderson-Darling, it is more difficult to build distribution-independent GOF tests in the multi-dimensional framework.

Alternatively, one may choose the copula that minimizes the distance to the *empirical copula* of the data (Romano, 2002). The empirical copula is given by

$$C_e(u_1,..,u_d) = \frac{1}{n}\sum_{i=1}^{n}\prod_{j=1}^{d} I(U_{i,j} < u_j),$$

where $0 \leq u_j \leq 1$, and $\boldsymbol{U}_1,....,\boldsymbol{U}_n$ are pseudo-samples obtained by transforming the d-dimensional empirical returns $\boldsymbol{Y}_1,..,\boldsymbol{Y}_n$ using the empirical distribution function. The best copula C can be chosen as the one that minimizes the distance

$$d(C,C_e) = \sqrt{\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}\left(C\left(\frac{i_1}{n},\ldots,\frac{i_d}{n}\right) - C_e\left(\frac{i_1}{n},\ldots,\frac{i_d}{n}\right)\right)^2}.$$

Finally, a more informal test involves comparing the plot of the estimated parametric copula with the one of a copula function estimated non-parametrically via kernel methods (Fermanian and Scaillet, 2002).

6 Simulating from copulas

In Section 6.1 we give algorithms for simulating from the two implicit copulas, while the explicit copulas are treated in Section 6.2.

6.1 Implicit copulas

The implicit copulas are particularly easy to simulate. If we can simulate a vector $\boldsymbol{X}$ with the distribution function F, we can obtain a simulation from the copula by transforming each component of $\boldsymbol{X}$ by its marginal distribution function, i.e. $\boldsymbol{U} = (U_1, \dots, U_d) = (F_1(X_1), ..., F_d(X_d))$. In this section we give the simulation algorithms for the Gaussian and Student's t-copulas.

6.1.1 Gaussian copula

An easy algorithm for random variate generation from the d-dimensional Gaussian copula with correlation matrix $\boldsymbol{R}$ is given by (Embrechts et al., 2003)

- Simulate $\boldsymbol{X} \sim N_d(\boldsymbol{0}, \boldsymbol{R})$
- Set $\boldsymbol{U} = (\Phi(X_1), ..., \Phi(X_d))$

6.1.2 Student's t-copula

An easy algorithm for random variate generation from the d-dimensional Student's t-copula with correlation matrix $\boldsymbol{R}$ and ν degrees of freedom is given by (Embrechts et al., 2003)

- Simulate $\boldsymbol{X} \sim t_d(\nu, \boldsymbol{0}, \boldsymbol{R})$
- Set $\boldsymbol{U} = (t_\nu(X_1), ..., t_\nu(X_d))$

Here $t_d(\nu, \boldsymbol{0}, \boldsymbol{R})$ is the multivariate Student's t-distribution with expectation $\boldsymbol{0}$, scale matrix $\boldsymbol{R}$ and ν degrees of freedom (see Section 5.1.2 for the density), and t_ν denotes the distribution function of a standard univariate Student's t-distribution.

6.2 Explicit copulas

The explicit copulas present slightly more challenging simulation problems. Before we explain how to simulate from the Clayton and Gumbel copulas we have to give some definitions. The d-dimensional Archimedian copulas may be written as

$$C(u_1, \cdots, u_d) = \phi^{-1}(\phi(u_1) + \cdots + \phi(u_d)), \tag{13}$$

where ϕ is a decreasing function known as the *generator* of the copula and ϕ^{-1} denotes the inverse of the generator (Frees and Valdez, 1998; McNeil et al., 2005). If ϕ in addition equals the inverse of the Laplace transform of a distribution function G on $\mathcal{R}^+$ satisfying

$G(0) = 0$[1], the following algorithm can be used for simulating from the copula (Marshall and Olkin, 1988):

- Simulate a variate X with distribution function G such that the Laplace transform of G is the inverse of the generator.
- Simulate d independent variates $V_1, \ldots, V_d$.
- Return $\boldsymbol{U} = (\phi^{-1}(-\log(V_1)/X), \ldots, \phi^{-1}(-\log(V_d)/X))$.

Both the Clayton and the Gumbel copula can be simulated using this procedure.

6.2.1 Clayton copula

For the Clayton copula we have that

$$\phi(t) = (t^{-\delta} - 1) \text{ and } \phi^{-1}(t) = (t+1)^{-1/\delta}.$$

The inverse of the generator is equal to the Laplace transform of a Gamma variate $X \sim Ga(1/\delta, 1)$ (see e.g. Frees and Valdez (1998)). Hence, the simulation algorithm becomes

- Simulate a gamma variate $X \sim Ga(1/\delta, 1)$.
- Simulate d independent standard uniforms $V_1, \ldots, V_d$.
- Return $\boldsymbol{U} = \left((1 - \frac{\log V_1}{X})^{-1/\delta}, \ldots, (1 - \frac{\log V_d}{X})^{-1/\delta}\right)$.

6.2.2 Gumbel copula

For the Gumbel copula we have that

$$\phi(t) = (-\log(t))^{\delta} \text{ and } \phi^{-1}(t) = \exp(-t^{1/\delta}).$$

The inverse of the generator is equal to the Laplace transform of a positive stable variate $X \sim St(1/\delta, 1, \gamma, 0)$, where $\gamma = (\cos(\frac{\pi}{2\delta}))^{\delta}$ and $\delta > 1$ (see e.g. Frees and Valdez (1998)). Hence, the simulation algorithm becomes

- Simulate a positive stable variate $X \sim St(1/\delta, 1, \gamma, 0)$.
- Simulate d independent standard uniforms $V_1, \ldots, V_d$.
- Return $\boldsymbol{U} = \left(\exp - \left(\left(-\frac{\log V_1}{X}\right)^{1/\delta}\right), \ldots, \exp\left(-\left(-\frac{\log V_d}{X}\right)^{1/\delta}\right)\right)$.

[1]The Laplace transform of a distribution function G on $\mathcal{R}^+$ satisfying $G(0) = 0$ is

$$\hat{G}(t) = \int_0^{\infty} e^{-tx} dG(x), \ t \geq 0.$$

Since the positive stable distribution is not as known as the other distributions mentioned in this survey, an algorithm for simulating a stable random variate $X \sim St(\alpha, \beta, \gamma, \delta)$ is enclosed below. We use the parameterisation and simulation algorithm proposed by Nolan (2005):

- Simulate a uniform variable $\Theta \sim U(-\frac{\pi}{2}, \frac{\pi}{2})$
- Simulate an exponentially distributed variable W with mean 1 independently of Θ.
- Set $\theta_0 = \arctan(\beta \tan(\pi\alpha/2))/\alpha$.
- Compute $Z \sim St(\alpha, \beta, 1, 0)$:

$$Z = \begin{cases} \frac{\sin\alpha(\theta_0+\Theta)}{(\cos\alpha\,\theta_0\,\cos\Theta)^{1/\alpha}} \left[\frac{\cos(\alpha\,\theta_0+(\alpha-1)\Theta)}{W}\right]^{(1-\alpha)/\alpha} & \alpha \neq 1 \\ \frac{2}{\pi}\left[(\frac{\pi}{2}+\beta\Theta)\tan\Theta - \beta\log\left(\frac{\frac{\pi}{2}W\cos\Theta}{\frac{\pi}{2}+\beta\Theta}\right)\right] & \alpha = 1 \end{cases}$$

- Compute $X \sim St(\alpha, \beta, \gamma, \delta)$:

$$X = \begin{cases} \gamma\,Z + \delta & \alpha \neq 1 \\ \gamma\,Z + (\delta + \beta\frac{2}{\pi}\gamma\log(\gamma)) & \alpha = 1 \end{cases}$$

6.3 Example

The issue of the last section of this survey is to illustrate how the choice of copula can affect the joint distribution of two assets. In the upper left corner of Figure 3 we show the Norwegian vs. Nordic geometric returns from Figure 1. We have fitted a Gaussian, a Clayton and a Student's t-copula to this data set (since the data obviously does not exhibit greater dependence in the positive tail than in the negative, we have not fitted the Gumbel copula) using the methods described in Section 5. The parameter ρ of the Gaussian distribution was estimated to be 0.64, the parameter δ of the Clayton copula was estimated to be 1.14, and the parameters ρ and ν of the Student's t-copula were estimated to be 0.64 and 7, respectively. The three remaining panels of Figure 3 show simulations from the estimated copulas generated using the algorithms given in Sections 6.1 and 6.2. The marginal distributions are the same in all three panels; the empirical distribution functions of the Norwegian and Nordic returns. Hence, the different appearance in each panel is explained on the basis of the choice of copula alone.

Acknowledgements: The author would like to thank Dr. Alexander J. McNeil at ETH Zurich for answering my questions and giving me access to the draft of the book *Quatitative Risk Management: Concepts, Techniques and Tools*.

References

A. Ang and J. Chen. Asymmetric correlations of equity portfolios. Technical report, Stanford, 2000.

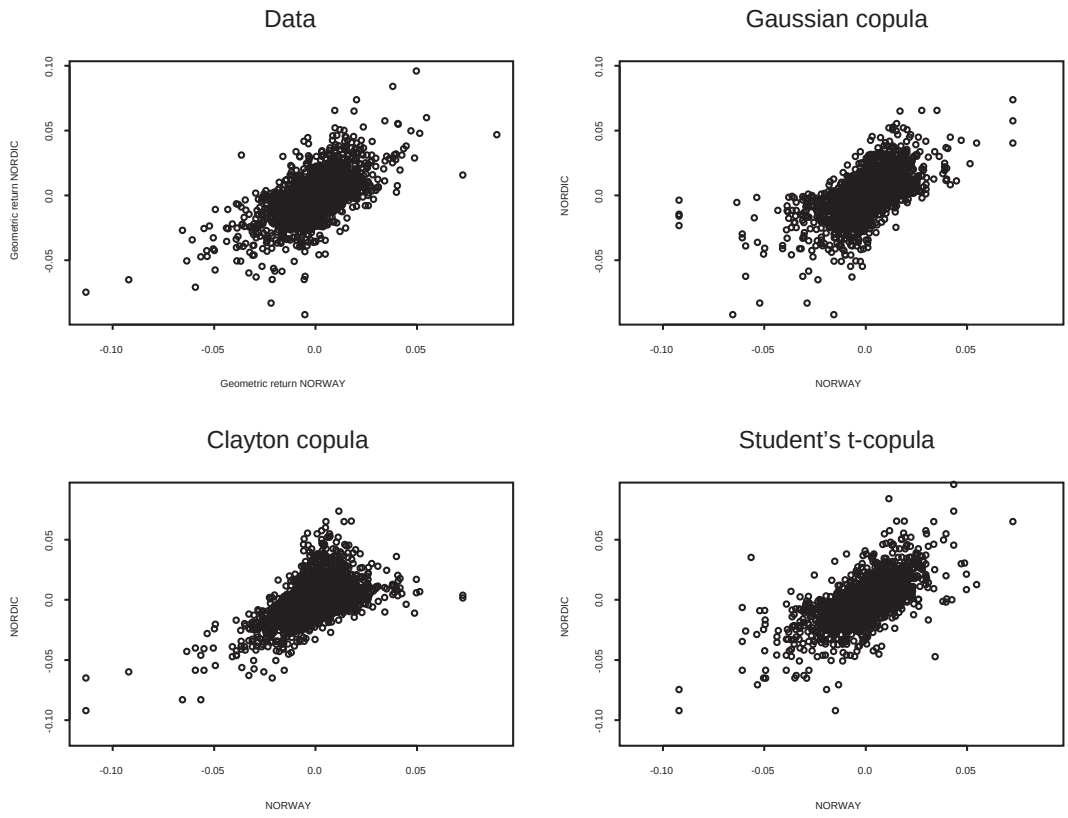


Figure 3: *Upper left corner: Norwegian and Nordic geometric returns from Figure 1. Upper right corner: Gaussian copula fitted to the returns. Lower left corner: Clayton copula fitted to the returns. Lower right corner: Student's t-copula fitted to the returns. In all three panels the marginals are the empirical distribution functions of the Norwegian and Nordic returns, respectively.*

P. Blum, A. Dias, and P. Embrechts. The art of dependence modelling: the latest advances in correlation analysis. In *Alternative Risk Strategies*. Risk Books, London, 2002.

E. Bouyé, V. Durrleman, A. Nikeghbali, G. Riboulet, and T. Roncalli. Copulas for finance - a reading guide and some applications. Technical report, Crédit Lyonnais, 2000.

S. Demarta and A. J. McNeil. The t copula and related copulas. Technical report, ETH Zurich, 2004.

P. Embrechts, F. Lindskog, and AJ. McNeil. Modelling dependence with copulas and applications to risk management. In S.T.Rachev, editor, *Handbook of Heavy Tailed Distributions in Finance*. Elsevier, North-Holland, 2003.

P. Embrechts, A. J. McNeil, and D. Straumann. Correlation: Pitfalls and alternatives. *Risk*, 12:69–71., 1999.

P. Embrechts, A. J. McNeil, and D. Straumann. Correlation and dependency in risk management: Properties and pitfalls. In *Value at Risk and Beyond*. Cambridge University Press, 2001.

J-D. Fermanian. Goodness-of-fit tests for copulas. *To appear in Journal of Multivariate analysis*.

J-D. Fermanian and O. Scaillet. Nonparametric estimation of copulas for time series. *Journal of Risk*, 4:25–54, 2002.

E. W. Frees and E. A. Valdez. Understanding relationships using copulas. *North American Actuarial Journal*, 2(1):1–25, 1998.

C. Genest, K. Ghoudi, and L. Rivest. A semi-parametric estimation procedure of dependence parameters in multivariate families of distributions. *Biometrika*, 82:543–552, 1995.

H. Joe. *Multivariate Models and Dependence Concepts*. Chapman & Hall, London, 1997.

A. W. Ledford and J. A. Tawn. Concomitant tail behavior for extremes. *Advances in Applied Probability*, 30:197–215, 1998.

F. Longin and B. Solnik. Extreme correlation of international equity markets. *The Journal of Finance*, 56(2):649–676, 2001.

A. W. Marshall and I. Olkin. Families of multivariate distributions. *The Journal of the American Statistical Association*, 83:834–841, 1988.

R. Mashal and A. Zeevi. Beyond correlation: Extreme co-movements between financial assets. Technical report, Columbia University, 2002.

A. J. McNeil, R. Frey, and P. Embrechts. *Quantitative Risk Management: Concepts, Techniques and Tools*. Forthcoming, 2005.

J. P. Nolan. *Stable Distributions: Models for Heavy Tailed Data.* Forthcoming, 2005.

R. Rebonato and P. Jäckel. The most general methodology to create a valid correlation matrix for risk management and option pricing purposes. Technical report, Quantitative Research Centre of the NatWest Group, 1999.

C. Romano. Calibrating and simulating copula functions: An application to the italian stock market, 2002. Working paper n. 12, CIDEM.

P. Rousseeuw and G. Molenberghs. Transformation of non positive semidefinite correlation matrices. *Comm. Statist. Theory Methods*, 22:965–984, 1993.

B. Schweizer and E. F. Wolff. On non parametric measures of dependence for random variables. *The annals of statistics*, 9:879–885, 1981.

A. Sklar. Fonctions dé repartition á n dimensions et leurs marges. *Publ. Inst. Stat. Univ. Paris*, 8:229–231, 1959.

G. G. Venter. Tails of copulas. In *Proceedings ASTIN Washington, USA*, pages 68–113, 2001.