# Harvesting Excess Volatility

Guillaume Rabault

*HSBC Global Asset Management*

guillaume.rabault@hsbc.fr

August 2017

**Abstract**

I investigate the volatility pumping intuition in a simple discrete time version of Luenberger[1998]. The literature argues that investors can benefit from volatility by rebalancing positions to constant weights, thereby trading against past returns. I analyze the pay-off of the trading policy induced by rebalancing to understand its return and risk benefits. When returns are i.i.d., the rebalanced portfolio underperforms in expectation its buy-and-hold counterpart so that the trading benefits have to be risk based. I demonstrate analytically that to improve the expected relative return, prices have to exhibit excess volatility. Mean reversion is needed to make volatility pumping a reality for investors.

## 1 Introduction[1]

In portfolio theory, volatility pumping and noise harvesting are terms that evoke the possibility of making profits from randomness in asset prices. They are traditionally used in the context of constant weight portfolios, as opposed to buy-and-hold portfolios. Constant weight (i.e. constant exposure) portfolios trade in a contrarian way. They buy the underperforming assets and sell the outperforming assets. When prices experience cycles, rebalancing harvests profits. An often quoted example comes from Luenberger[1998][2]. Luenberger shows that in a standard model of price dynamics (geometric Brownian motion), a portfolio that rebalances between cash and a risky asset grows even when the

[1]The views expressed here are those of the author alone and not of HSBC Global Asset Management. I thank Paul Bouchey, Michael Dempster and Soumik Pal for feedback on the paper. I am indebted to Fabrizio Coiai for very careful reading and many useful suggestions. I also benefited from discussions with Benoît Bellone, Philippe Declerck and Vis Nayar. All errors are mine.

[2]See Example 15.2 in Luenberger[1998].



Electronic copy available at: https://ssrn.com/abstract=3146917

underlying risky asset does not grow. This mechanism has also been more recently investigated by Dempster[2008] (see also Dempster[2010]), who show that constant weight portfolios composed of assets with i.i.d. prices actually grow, apparently generating wealth out of thin air. This feature is presented as unintuitive even to finance experts. There is an older argument for rebalanced portfolios in Stochastic Portfolio Theory, a theory developed by R.Fernholz and synthesized in Fernholz[2002]. There, the benefits of rebalanced portfolios are linked to a term called the excess growth-rate, which measures the gap between the log-return of a portfolio and the weighted log-returns of the underlying assets. This has been popularized recently by Bouchey[2015].

Although seductive, the idea that one can make money out of pure randomness comes up as suspicious to people acquainted with arbitrage theory. How can one make profits out of unpredictable changes in prices? Two separate mechanisms are usually floated in discussions of volatility pumping. One is a simple diversification argument. Buy-and-hold portfolios can typically lose balance as a result of price fluctuations. This is an entirely standard argument in portfolio theory which recommends diversification to investors. The other argument is that rebalanced portfolios trade in a contrarian way, and therefore have a tendency to buy-low and sell-high, surely a profitable feature. Depending on the context (especially within the context of Luenberger[1998]), this can be a much more contentious argument as recognized for instance by Dempster[1998]. The aim of this note is to clarify the mechanism behind volatility pumping and hopefully dissipate ambiguities.

The main argument of this note originates from the idea that if the contrarian trading process is to explain the benefits of rebalanced portfolios, this has to be confirmed by directly analyzing the relative performance of the rebalanced portfolio versus the buy-and-hold portfolio which is initialized with the same exposures. This is done in a simple cash plus a single risky asset context, in discrete time. I therefore proceed by looking at this very relative performance. Some enlightening analytical results are presented which link the ex-post relative performance to cross-products of returns. The rebalanced portfolio beats the corresponding buy-and-hold portfolio if and only if the price trajectory exhibits reversal. I link this fact to the excess-growth rate of Fernholz[2002] which, apparently, does not involve any temporal covariation of returns, inducing people to believe that no specific assumption regarding temporal correlations are needed to sustain the volatility pumping argument. I show that cross products of returns are subtely hidden in the relative log return of the rebalanced portfolio versus the buy-and-hold portfolio. This purely analytical result is key to the whole argument of this paper.

I then proceed by arguing that what investors need to form a view on the benefits of contrarian trading is a prospective probabilistic view. I start by investigating the case of i.i.d. returns, following the well-known example of Luenberger[1998]. I show that in this context, the expected terminal value of the rebalanced portfolio is strictly lower that the terminal value of the corresponding



Electronic copy available at: https://ssrn.com/abstract=3146917

buy-and-hold portfolio. Contrarian trading loses money on average. Despite this, the rebalanced portfolio almost surely beats the buy-and-hold portfolio asymptotically. Thus almost all trajectories contain a sufficient amount of reversal for the trading policy induced by rebalancing to work in the very long-run (i.e. asymptotically). Indeed a random walk returns to zero from any starting point with probability one. This behavior (recurrence) is not mean reversion however, and the timing ability of rebalancing in this context is completely fortuitous. A comparable situation is provided by the celebrated Saint Petersburg paradox, where doubling down works almost surely asymptotically. There also, one might be tempted to trace this success to some form of mean reversion. Yet there is absolutely no mean reversion in the Saint-Petersburg paradox. The asymptotic dominance of rebalancing over buy-and-hold in Luenberger's case has the same flavour. The strategy can lead to arbitrarily large losses, and is not attractive to an investor who seeks a positive expected excess return. I also show through some Monte-Carlo simulations that the asymptotic result in favour of rebalancing requires very long horizons to materialize in Luenberger's case. In my example, it is clearly visible for investment horizons of 500 years.

I then move to non i.i.d. setups. Relying on the analytical results of the first section, I show that the expected value of the relative pay-off (in logs) of rebalancing is closely related to the variance of the cumulated log returns of the risky asset versus the cumulated variance of the sequential log-returns. The expected log-payoff is very close to being a variance ratio, a classical quantity in financial econometrics. To be more precise, it is close to being a variance difference (as opposed to a ratio). It is well known that the variance difference is zero in the case of i.i.d. returns as the variance of cumulated returns is the sum of the variance of the sequential returns. To improve the prospects of rebalanced portfolios, some mean reversion is needed. The variance ratio test is precisely a mean to detect mean reversion/excess volatility in time series. I illustrate this with simulations. They clearly show that as the persistence of return shocks falls, the rebalanced portfolio starts to meaningfully beat the buy-and-hold portfolio. This concludes the paper.

## 2 Relationship to the literature

There is an abundant applied literature on volatility pumping, noise harvesting and rebalancing (see for instance Bouchey[2014]). As observed by others before (see for instance Hallerbach[2014], Cuthbertson[2015], Chambers[2014], Dempster[2008], Dempster[2010], Quian[2014]) the concepts are not always well delineated and claims are often insufficiently substantiated.

Volatility pumping is often linked to the Stochastic Portfolio Theory of Fernholz[2002], which puts at the center of its analysis the notion of excess growth-rate. However, the excess-growth rate is not itself a financial pay-off. There is no way an investor could harvest the excess growth-rate alone by building a smart portfolio. All portfolios have an excess growth-rate, even the market portfolio which is a buy-and-hold portfolio. The only practical way to benefit from the excess growth-rate is to go long a portfolio with a certain rebalancing frequency and short the corresponding portfolio with a different rebalancing frequency. The bet is then that the long portfolio has a higher excess growth rate than the short one. This is done in Fernholz[2007]. This paper is in the same vein, the starting point being that we need to compare the rebalanced portfolio with the buy-and-hold portfolio to understand the benefits of the trading policy implied by rebalancing[3]. Some simple return arithmetics then reveal the role of time correlations in the outcome. Quian[2014] makes a similar point using specialized examples (footnotes specify when my results have a degree of commonality with his work).

This paper also relates to a very recent literature that analyses excess growth-rates in discrete time, see for instance Pal[2016]. This is helpful in two respects. First it better relates to what is being done in practice. Continuous time is an abstraction. In continuous time, the excess growth-rate is linked to the concept of quadratic variation. In discrete time, there is no need to appeal to quadratic variation and the calculations do not need to be confined to diffusions[4]. The mathematics are more transparent. The difference with Pal[2016] is that I stress that investors need a prospective view of price processes. Trajectory by trajectory analysis is useful, but it then needs to be cast into a probabilistic set-up for investment conclusions to be drawn.

The main analytical result of this paper is Proposition 6 which, as far as I know, is new. For a given rebalancing proportion $\pi$, there is a strictly convex function $\gamma_\pi^\star(\cdot)$, null at zero and positive such that the pay-off of rebalancing versus buy-and-hold, in logs, is the cumulated value $\sum \gamma_\pi^\star(\log(1+r))$ measured on the risky asset return, minus the value $\gamma_\pi^\star(\sum \log(1+r))$, i.e. the value of $\gamma_\pi^\star(\cdot)$ measured on the cumulated return. This puts a lot of structure on the problem because $\gamma_\pi^\star(\cdot)$ acts as $x \mapsto x^2$ would, and the problem boils down to comparing the variability of sequential returns with that of cumulated resturns, with $\gamma_\pi^\star(\cdot)$ instead of $x \mapsto x^2$ embodying the notion of variability. Hallerbach[2014] proposes a related (but different) decomposition. The advantage of Proposition 6 is that due to the positivity and convexity of the excess growth rate function, it lends itself to a more thorough interpretation which reveals the role of time correlations.

# 3 Setup and notation

I will assume that we have one risky asset with price $(p_t)_{t\in\mathbb{N}}$, and a cash account with a zero interest rate. I will also assume that $p_0 = 1$. Rebalancing takes

[3] The same route is taken by Hallerbach[2014] and Quian[2014].

[4] The continuous time case gets closer to the discrete time case if the processes have jumps. Stochastic Portfolio Theory could be generalized to such semimartingales.

place between the two financial instruments. This is similar to the model of Luenberger[1998].

**Remark 1**: This setup can be interpreted differently, along the lines of paragraph 2.3 in Pal[2016], using a suitable choice of numeraire. Indeed, assume that we have two risky assets, with the second one chosen to be the numeraire. With this choice, the value of second risky asset is constant, while $(p_t)_{t\in\mathbb{N}}$ is the relative price of the first risky asset vis-à-vis the second risky asset. All calculations then carry through. I will however stick to the cash plus risky asset interpretation in the text to avoid confusions.

$\square$

The return of the risky asset is given by:

$$R_{t+1} = 1 + r_{t+1} = \frac{p_{t+1}}{p_t},$$

while its log-return is just:

$$\log\left(\frac{p_{t+1}}{p_t}\right) = \log(1 + r_{t+1}).$$

A portfolio is defined by its proportion invested in the risky asset $\boldsymbol{\pi} = (\pi_k)_{k\in\mathbb{N}}$ (I will also use $\boldsymbol{\pi}_t = (\pi_k)_{k=0,\dots,t}$ to denote history up to time $t$). The proportion invested on the cash account is just $(1-\pi_k)_{k\in\mathbb{N}}$. I will concentrate on portfolios which are long in the risky asset and are not leveraged. Therefore, at each date:

$$0 \leq \pi_k \leq 1.$$

*All portfolios are initialized with one dollar at date* 0. Their value at time $t$ is noted $V_{\boldsymbol{\pi}_t,t}$, with the first index reflecting the fact that the value at date $t$ depends on the whole history $\boldsymbol{\pi}_t$ of positions up to that date. The value of a portfolio which rebalances to a fixed proportion $\pi$ has a value of $V_{\pi,t}$ at date $t$. Since its vector of exposure is constant, I index it by the scalar $\pi$ instead of the corresponding vector of proportion.

Its portfolio return satisfies:

$$1 + r_{\pi,t+1} = \frac{V_{\pi,t+1}}{V_{\pi,t}} = (1-\pi) + \pi(1 + r_{t+1}),$$

$$r_{\pi,t+1} = \pi r_{t+1}.$$

The log-return is just:

$$\log\left(\frac{V_{\pi,t+1}}{V_{\pi,t}}\right) = \log(1 + r_{\pi,t+1}) = \log(1 + \pi r_{t+1}).$$

A buy-hold-policy consists in buying a number of shares $n_0$ and keeping it unchanged. Assuming the targeted exposure is $\pi_0$ at inception, the number of shares purchased is (one dollar invested at time 0):

$$n_0 = \frac{\pi_0}{p_0} = \pi_0.$$

As a simplification, I will denote by $W_{\pi,t}$ the value of a buy-and-hold portfolio initialized with exposure $\pi$ ($\pi = \pi_0$). Since the number of shares is held constant, we only need to use the scalar $\pi$ as an index of the value process. Keeping track of the value of a buy-and-hold policy is easy. Indeed:

$$W_{\pi,t} = (1 - \pi) + \pi p_t.$$

It is important to note that the buy-and-hold portfolio has an exposure $\pi_t^{bh}$ which varies with time. I could have written $V_{\boldsymbol{\pi}_t^{bh},t}$ instead of $W_{\pi,t}$. Symmetrically, the rebalanced portfolio associated to proportion $\pi$ holds a number of share $n_t$ that varies with time (see Proposition 2 below).

# 4 The ex-post view of noise harvesting

As an introduction to the benefits of rebalancing, I first describe the mechanism in the context of two periods. I then establish the general relationships that allow to discuss the broader context. I leave probabilistic aspects for later and concentrate on basic return accounting in this section. Both log-returns and standard returns shed light on volatility pumping. I first rely on standard returns and then present the log-return view.

## 4.1 The two period example

Consider a portfolio initialized with exposure $0 < \pi < 1$ in date 0. Assume that in date 1, the risky asset has fallen. The weight on the risky asset has fallen as a result. It is now, before rebalancing:

$$\pi_1^{bh} = \pi \frac{1 + r_1}{1 + \pi r_1}.$$

The quantity of asset that is purchased to restore the exposure $\pi$ is:

$$\pi - \pi_1^{bh} = \frac{(\pi^2 - \pi) r_1}{1 + \pi r_1}.$$

Restoring the weight $\pi$ in date 1 entails redeploying funds towards the risky asset. We have:

$$V_{\pi,2} = (1 + \pi r_1)(1 + \pi r_2) = (1 + \pi r_1)(1 + (\pi_1^{bh} + (\pi - \pi_1^{bh}))r_2) =$$

$$(1+\pi r_1)(1+\pi_1^{bh} r_2) + (\pi^2-\pi) r_1 r_2 = W_{\pi,2} + (\pi^2-\pi) r_1 r_2.$$

Rebalancing thus generates a gain/loss of $(\pi^2-\pi)r_1 r_2$ against buy-and-hold. A gain is experienced as soon as:

$$\mathrm{sgn}(r_1) = -\mathrm{sgn}(r_2),$$

(with a strictly positive gain if and only if $r_1 \neq 0$ and $r_2 \neq 0$), a condition which states that $p_1$ is a local extremum of the price trajectory. This condition is met along a cycle although a cycle is obviously not needed.

I now generalize these results, introducing the needed analytical apparatus.

## 4.2 Compounding rates of returns

Let the function $X$ of $\mathbb{R}^T$ into $\mathbb{R}$ be defined through:

$$X(r_1,\ldots,r_T) = \sum_{i<j\leq T} r_i r_j + \sum_{i<j<k\leq T} r_i r_j r_k + \ldots + r_1 \cdots r_T.$$

Using this, I prove the following result in the appendix:

**Proposition 1**: The relative value of rebalancing versus buy-and-hold is given by:

$$V_{\pi,T} - W_{\pi,T} = X(r_{\pi,1},\ldots,r_{\pi,T}) - \pi X(r_1,\ldots,r_T),$$

and as a second order approximation by:

$$V_{\pi,T} - W_{\pi,T} \approx (\pi^2-\pi) \sum_{i=1}^{T-1} r_i \left( \sum_{j=i+1}^{T} r_j \right).$$

or:

$$V_{\pi,T} - W_{\pi,T} \approx (\pi^2-\pi) \sum_{i=1}^{T-1} \left( \sum_{j=1}^{i} r_j \right) r_{i+1}.$$

□

I note that the condition (arising from the second order approximation):

$$r_i \left( \sum_{j=i+1}^{T} r_j \right) < 0,$$

implies that:

$$\mathrm{sgn}(r_i) = -\mathrm{sgn}(\sum_{j=i+1}^{T} r_j).$$

This says that $p_i$ is a local extremum of the trajectory $(p_{i-1}, p_i, p_T)$ If no such terms is negative, rebalancing cannot beat buy-and-hold.

The condition:

$$\left(\sum_{j=1}^{i} r_j\right) r_{i+1} < 0,$$

implies that:

$$\operatorname{sgn}(\sum_{j=1}^{i} r_j) = \operatorname{sgn}(r_{i+1}).$$

This says that $p_i$ is a local extremum of the trajectory $(p_0, p_i, p_{i+1})$. Again, if no such terms is negative, rebalancing cannot beat buy-and-hold.

We will see after Proposition 3 below that indeed, it is mandatory that the price curves have local extrema in $]0, T[$ for rebalancing to beat buy-and-hold. This will be established without any approximation.

## 4.3 Attribution in level

I give below another expression for the relative performance of the rebalanced portfolio against the buy-and-hold portfolio. It will prove very handy to demonstrate that along monotonic trajectories, buy-and-hold beats rebalancing.

Let $n_t$ be the quantity held by the rebalanced portfolio at date $t$. Given the normalization $p_0 = 1$ as well as the fact that initial wealth is always set at 1, we have $n_0 = \pi$. The buy-and-hold portfolio keeps holding $n_0$ while the rebalanced policy has to adjust the number of share at each date. We have:

**Proposition 2**: The number of shares held by the rebalanced portfolio evolves according to the dynamics:

$$n_{t+1} = n_t \frac{1 + \pi r_{t+1}}{1 + r_{t+1}},\ n_0 = \pi,$$

and we have:

$$\begin{aligned} V_{\pi,t} - W_{\pi,t} &= \sum_{i=0}^{t-1} (n_i - n_0)(p_{i+1} - p_i), \\ &= \sum_{i=0}^{t-1} \pi (V_{\pi,i} - p_i) r_{i+1}. \end{aligned}$$

□

Using this we can easily establish the following result:

**Proposition 3**: If the price trajectory $(p_t)_{0\leq t\leq T}$ is monotonic, buy-and-hold beats rebalancing over the time interval $[0,T]$.

$\square$

As anticipated in the previous section, the price trajectory needs to have local extrema in $[1,T-1]$ for rebalancing to have a chance to beat buy-and-hold.

## 4.4 The excess growth-rate arithmetic

I now introduce the excess growth-rate arithmetic used by Fernholz, see Fernholz[2002]. I start by reviewing log-returns. Log-returns have the advantage of dealing with compounding in an analytically convenient way. As a result, it lends itself to measuring asymptotic growth. A drawback is that it does not easily reveal the role of cross products of returns.

### 4.4.1 From returns to log-return

The strict concavity of the log implies that:

$$\log(1+r_{\pi,t+1}) = \log((1-\pi)+\pi(1+r_{t+1})) \geq (1-\pi)\log(1)+\pi\log(1+r_{t+1}) =$$

$$\pi\log(1+r_{t+1}),$$

with strict inequality if and only if both $0<\pi<1$ and $r_{t+1}\neq 0$. The difference between the two terms in the inequality is called the excess growth-rate. Its noted $\gamma^*_{\pi,r_{t+1}}$ or $\gamma^*_{\pi}(r_{t+1})$. I summarize some useful fact on the excess-growth rate in the following proposition:

**Proposition 4**: Define $h(\cdot)$, $]-1,+\infty[\to\mathbb{R}_+$ through:

$$h(r) = 2\,\frac{r-\log(1+r)}{r^2}.$$

- We have $\log(1+r_{\pi,t+1}) = \pi\log(1+r_{t+1})+\gamma^*_{\pi,r_{t+1}}$, with: $\gamma^*_{\pi,t+1} = \gamma^*_{\pi}(r_{t+1}) = \frac{1}{2}\pi h(r_{t+1})r^2_{t+1} - \frac{1}{2}\pi^2 h(\pi r_{t+1})r^2_{t+1} \geq 0$.
- For $0<\pi<1$, I define $\gamma^\star_{\pi}(\cdot)$ through a reparametrization of $\gamma^*_{\pi}(\cdot)$:

$$\gamma^\star_{\pi}(\log(1+r)) = \gamma^*_{\pi}(e^{\log(1+r)}-1) = \gamma^*_{\pi}(r),$$

- $\gamma^\star_{\pi}(\cdot)$ has domain $\mathbb{R}$, is strictly convex, positive. It verifies $\gamma^\star_{\pi}(\log(1+r))=0$ (respectively $\gamma^*_{\pi}(r)=0$) if and only if $\log(1+r)=0$ (resp. $r=0$).

□

The graph $\gamma^\star_\pi(\cdot)$ is shown in Figure 1. This graph also shows the quadratic function that corresponds to the second order approximation of $\gamma^\star_\pi(\cdot)$ at 0. This quadratic function minorizes $\gamma^\star_\pi(\cdot)$, a feature that will prove useful later (see appendix for details).

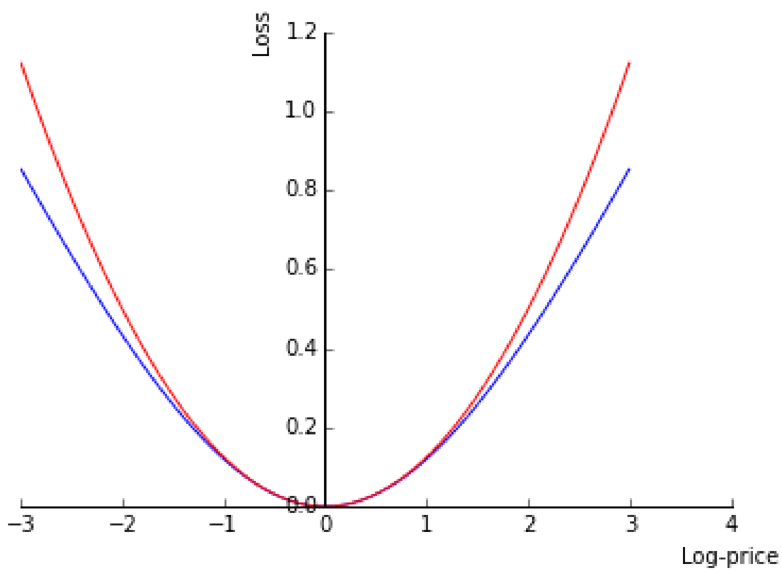


Figure 1: Blue: $\gamma^\star_\pi(x) = \log((1-\pi)+\pi\exp(x)) - \pi x$, Red: $g_\pi(x) = x^2/8$, $\pi = 0.5$

#### 4.4.2 Compounding log-returns

Although interesting, the calculation of the excess growth rate is not bringing much at this stage. In a one period (two dates) context, the key relationship between portfolio returns and asset returns is the arithmetic one. The excess growth-rate should be purely seen as a consequence of choosing log-scales. Where it becomes really handy is when dealing with compounding. We can indeed aggregate the relationship across time. We have for example:

$$\log\left(\frac{V_{\pi,2}}{V_{\pi,0}}\right) = \log(1+r_{\pi,1})+\log(1+r_{\pi,2}) = \pi\log(1+r_1)+\pi\log(1+r_2)+\gamma^*_{\pi,r_1}+\gamma^*_{\pi,r_2},$$

$$= \pi\log(\frac{p_2}{p_0}) + \gamma^*_{\pi,r_1} + \gamma^*_{\pi,r_2}.$$

This can of course be established over any time interval, for a portfolio rebalanced on the fixed exposure $\pi$:

**Proposition 5**: The value of the rebalanced portfolio is given by:

$$\log(V_{\pi,T}) = \pi\log(\frac{p_T}{p_0}) + \Gamma^*_{\pi,T} = \pi\log(\frac{p_T}{p_0}) + \sum_{i=1}^{T}\gamma^*_{\pi,r_i}.$$

$\square$

Along price cycles, we have $p_T = p_0$ and the excess growth-rate measures growth generated by rebalancing trades. In this context, the excess growth-rate is strictly positive as soon as $\pi$ is different from 0 or 1 and for one date $k$ at least, $p_k \neq 1$. Some trajectories will harvest more excess growth-rate than others, but all will harvest a strictly positive growth-rate if the price trajectory is not completely flat. Intuitively also, the excess growth-rate grows with the amount of deviation of the price trajectory and that is probably why volatility is thought to be positive for growth. More on this below.

### 4.4.3 The trading policy induced by rebalancing

The trading policy induced by rebalancing can be isolated by comparing the rebalanced portfolio with exposure $\pi$ and the buy-and-hold portfolio initiated with the same exposure. In section 4.2, we saw how the pay-off decomposes into a sum of cross-products while in section 4.3, we expressed it as a weighted sum of changes in prices. I now look at its expression in the log-log context.

We have[5]:

**Proposition 6**: The relative value of rebalanced versus buy-and-hold is given by:

$$\log\left(\frac{V_{\pi,T}}{W_{\pi,T}}\right) = \sum_{i=1}^{T}\gamma^\star_\pi(\log(1+r_i)) - \gamma^\star_\pi(\sum_{i=1}^{T}\log(1+r_i)).$$

I will call the last term on the right-hand side the divergence term and note it $\nabla_{\pi,T}$:

$$\nabla_{\pi,T} = \gamma^\star_\pi(\sum_{i=1}^{T}\log(1+r_i)).$$

[5] Hallerbach[2014] proposes a decomposition of the ‘rebalancing return’ which is close to that of Proposition 6. The main difference is that I rely on the log-return $\rho$ directly, as opposed to following the tradition of defining the geometric return as $\exp(\rho) - 1$. Because log-returns are additive in time, my decomposition additively disaggregates in the time dimension. My decomposition relies on relative log-returns, whereas his focuses on differences in levels of wealth. Proposition 6 establishes the positivity *and* the convexity of the terms in the decomposition whereas Hallerbach[2014] establishes positivity only. The additional convexity structure achieved in Proposition 6 allows to relate rebalancing gains to excess-volatility.

□

Whereas the excess growth-rate $\Gamma^*_{\pi,T}$ cumulates the values of $\gamma^\star_\pi$ computed on increments $\log(1+r_t)$, $\nabla_{\pi,T}$ is the opposite of $\gamma^\star_\pi$ measured over the whole trajectory (i.e. computed on the cumulated increments $\sum_{t=1}^{T}\log(1+r_t)$).

Thinking heuristically of $\gamma^\star_\pi(\cdot)$ as a quadratic function (which it is not, strictly speaking), the difference:

$$\sum_{i=1}^{T}\gamma^\star_\pi(\log(1+r_i)) - \gamma^\star_\pi(\sum_{i=1}^{T}\log(1+r_i)),$$

appears as a cross product of log-returns. This is the link between the cross-product view introduced in section 4.2 and the excess growth-rate view of the current section. This feature will prove crucial.

### 4.5 Summary

The main take-away points of section 4 are the following:

- the value of the trading process generated by rebalancing is measured by the rebalanced versus buy-and-hold payoff; if this is positive, it means by definition that the trading process buys low and sells high, on average,
- Proposition 3 shows that some local extrema/reversal have to be present in $[1, T-1]$ for rebalanced portfolios to beat buy-and-hold portfolios,
- the cumulated excess growth rate $\Gamma^*$ measures the discrepancy between portfolio log-returns and the weighted sum of asset log-returns; it is positive, and strictly positive if the price trajectory is not entirely flat (under $0 < \pi < 1$),
- the pay-off of the trading process in logs is the sum of the positive excess growth-rate $\Gamma^*_{\pi,T}$ and the divergence term $\nabla_{\pi,T}$ that is necessarily negative:

$$\Gamma^*_{\pi,T} - \nabla_{\pi,T},$$

- it is a sort of cross-product of log-returns.

## 5 The ex-ante view

Investors cannot satisfy themselves with ex-post analysis. They need to take a prospective view. This is why we now turn to a probabilistic set-up. In the probabilistic setup, investors form a view on the process driving asset prices and rationalize their decision through some form of expected utility maximization. I concentrate on the case where returns are i.i.d. (section 5.1) since it is in this context that Luenberger[1998] presents his case for noise harvesting.

In the portfolio optimization framework, the i.i.d case is qualified as a *constant investment opportunities* problem (see for instance Munk[2013] for the distinction between constant investment opportunities and time varying investment opportunities). Indeed, at any point in time, regardless of past prices, the prospective return and risk characteristics are the same. If the investment criterion is also invariant with time, such problems generate constant investment rules: the same exposure is typically chosen at each date, regardless of past prices. Rebalanced policies are thus natural.

The most striking illustration of the optimality of constant weight investment rules is provided by the fact that the growth optimal portfolio $\pi^*$ is itself of this kind. As such, it does not try to time the market. The growth optimal portfolio ends up beating any alternative portfolio with certainty. As a consequence, buy-and-hold portfolios cannot beat the right rebalanced portfolio (i.e. $\pi^*$) in the very long-run. I start by reviewing these facts.

With this in mind, I return to the comparison between the rebalanced portfolio initialized with weight $\pi$ and its buy-and-hold counterpart for arbitrary values of $\pi$ ($0 < \pi < 1$), i.e. the rebalancing profits associated to exposure $\pi$. I show that there is an interval containing $\pi^*$ such that when $\pi$ is in this interval, rebalancing profits are asymptotically positive. Outside of this interval, rebalancing profits are asymptotically negative. This can be understood as follows. The benefit of rebalancing is risk stabilization (see below). However, the exposure of the rebalanced portfolio is fixed at a potentially suboptimal level. In contrast, the exposure of the buy-and-hold portfolio drifts away from the starting point and might get closer to the optimal exposure, giving it a growth advantage. Overall, the sign of the rebalancing profits hinges on the balance of these effects. The advantage attached to rebalancing on exposure $\pi$ diminishes as $\pi$ moves away from $\pi^*$, i.e. as the exposure gets more and more suboptimal. It turns out that in Luenberger's case, the interval of outperformance of the rebalanced portfolio is $]0, 1[$.

The previous results are asymptotic in nature. They are relevant to the very patient investor. Real investors should be more interested in finite distance outcomes and simulations will be presented in the next section. Before doing so, I show analytically that the expected value of the rebalanced portfolio is lower than that of the corresponding buy-and-hold portfolio for any investment horizon. In other words, the rebalancing trading policy has a negative expected return. Rebalancing generates negative expected profits. Indeed, on average, the risky asset grows giving an advantage to buy-and-hold. In addition expected cross correlations of returns are zero. The virtue of rebalancing is that it controls exposure and thus risk.

We are left with apparently contradictory facts. On average, rebalancing loses in Luenberger's case. But almost all trajecories contain enough reversal in the long-run for rebalancing to ultimately beat buy-and-hold. This timing ability is however entirely fortuitous since at no point in time can one spot in real time a low or a high point in the price trajectory in Luenberger's case. In addition, its power is only felt asymptotically. I end this section comparing this result with the Saint Petersburg paradox. Just as in the Saint Peterburg paradox, the buying low selling high interpretation of Luenberger's example is hardly of any relevance for the real investor.

## 5.1 The log-optimal policy

I assume that the risky asset has i.i.d. returns with $r > -1$ almost surely, so that the price remains almost surely strictly positive. I note:

- $E[r] = \mu$, independent of time and assumed strictly positive,
- $V[r] = \nu$, independent of time.

In much more general probabilistic contexts, the financial strategy required to maximize wealth in the long run consists in maximizing at each date $t$:

$$\max_{\pi_t} E_t[\log(1 + \pi_t r_{t+1})],$$

subject to the side constraints, here:

$$0 \leq \pi_t \leq 1.$$

In the i.i.d. context, the conditional expectations $E_t[\log(1 + \pi r_{t+1})]$ boils down to the unconditional expectation $E[\log(1 + \pi r)]$ which is the same for all dates. The growth maximizing investor has no leeway to time the market. He chooses to hold the same exposure $\pi^*$ to risky asset at each date. There is nothing a non prescient investor can do to exploit the variability in returns.

The log-optimal portfolio defined above beats any other adapted portfolio policy sharing the same side constraints. The requirement that the competing policy be adapted merely means the investor has no knowledge of future returns when taking decisions. I spell this out in the following proposition for clarity.

**Proposition 7**: The log-optimal $\pi^*$ maximizes asymptotic growth: it strictly beats any other adapted portfolio strategy in the long run.

□

Since the buy-and-hold policies are adapted, they are beaten by the log-optimal portfolio $\pi^*$. In particular, for large enough $T$,

$$V_{\pi^*,T} > W_{\pi,T},$$

almost surely for any $\pi$ in $[0, 1]$. In particular, the rebalancing profits of portfolio $\pi^*$ are strictly positive (take $\pi = \pi^*$ in the equality above).

**Remark 2**: Suppose we are given two returns $r_1$ and $r_2$ for two consecutive dates, with $r_1 = \mu + \epsilon$ and $r_2 = \mu - \epsilon$, $|\epsilon| > 0$. Simple arithmetics shows that $(1 + r_1)(1 + r_2) < (1 + r)^2$ and is a decreasing function of $|\epsilon|$. Oscillations are detrimental to cumulated returns. Taking logs, this amounts to saying that $\log(1 + \mu + \epsilon) + \log(1 + \mu - \epsilon) < 2 \log(1 + r)$ and is a decreasing function of $|\epsilon|$ which follows from the concavity of the log. The concavity of the log codes for the detrimental effect of oscillations on cumulated returns. In the i.i.d. model, the asymptotic drag created by random fluctuations is known almost surely and is measured by the gap between $E[\log(1 + r)]$ and $\log(1 + E[r])$ .

□

**Remark 3**: The fact that $E[\log(1+r)]$ measures the long term (i.e. asymptotic) growth of an asset implies that for a given mean return, volatility lowers the long-run trajectory of an asset. This fact is called the volatility drag. One consequence is that buying too much of a rising (strictly positive expected return) asset can be detrimental to long term growth. This is easily seen by choosing a portfolio $\pi > \pi^*$. The rebalanced portfolio with exposure $\pi$ underperforms the log-optimal portfolio. Since the risky asset has strictly positive expected return, the log-optimal policy (when measured against portfolio $\pi$) actually sells a 'growing asset' and despite this, it outperforms. The advantage of selling the risky asset (on top of portfolio $\pi$) comes from the fact that risk reduction benefits exceed expected return losses. In the end, achieving optimal portfolio growth requires to properly balance return and risk as coded in the log-utility criterion. See also Remark 2 for related observations.

□

**Remark 4**: We know that:

$$E[\log(1 + r)] = E[r] - \frac{1}{2} E[h(r)r^2],$$

with $h(0) = 1$, asset or portfolio growth is driven by two components: a sequential return component and a sequential risk component (the second term on the right-hand side - the risk iterpretation of this second term is of course clearer in continuous time since it then boils down to cumulated variance). In a portfolio context, boosting growth can be achieved by choosing weights which improve the expected sequential return $E[r]$. In an i.i.d. context, returns are unpredictable and there is no way to boost the expected return by changing weights through time. Timing strategies therefore have no expected sequential return benefits. This is in a nutshell why the trading policy induced by rebalancing cannot be described as enhancing returns. Its benefits have to be risk based.

□

**Remark 5**: Maximizing the expected log-return is thus a legitimate task. What about maximizing the expected excess-growth rate? The excess-growth rate being the log-return of the portfolio minus $\pi$ times the log-return of the asset:

$$E[\gamma^*] = E[\log(1 + \pi r)] - \pi E[\log(1 + r)],$$

it loses the benefits of the log-return criterion unless the term $\pi E[\log(1 + r)]$ actually drops out, in which case it is equivalent. This happens when $E[\log(1 + r)] = 0$. This is Luenberger's case as we will see below.

$\square$

**Remark 6**: Fernholz[2002] (see example 1.3.8 page 21) as well as Luenberger[1998] choose to fix the expected log-return of the risky-asset as opposed to fixing its expected return when they analyze the effect of an increase in volatility. This implies that when they actually change the volatility parameters, they change the expected return of the risky-asset so as to keep its expected log-return unchanged. The observation that volatility is good for growth should therefore be taken with a pinch of salt.

$\square$

## 5.2 Log-normal markets and Luenberger's observation

I now specialize the model further to recover Luenberger's setup. The risky asset is log-normally distributed:

$$\log(1 + r) = \mu - \frac{1}{2}\sigma^2 + \sigma\epsilon,$$

with $\epsilon$ following a centered and standardized normal variable[6] $\mathcal{N}(0, 1)$.

Unfortunately, even in this log-normal case, we cannot compute $E[\log(1 + \pi r)]$ analytically. We can reason qualitatively however. From Proposition 10 in the appendix we know that $E_0[\log(1 + \pi r)] > 0$ on a whole non empty interval that contains $]0, \pi^*]$. Even for $\mu - \frac{1}{2}\sigma^2$ negative, $E_0[\log(1 + \pi^* r)]$ will be strictly positive (as a consequence of $\mu > 0$) and a whole range of rebalanced portfolios will have strictly positive growth rates despite the fact that the risky asset has a negative asymptotic growth rate. This is (an extension of) Luenberger's observation which deals with the case $\mu - \frac{1}{2}\sigma^2 = 0$.

Using a numerical integration technique, I have computed $E[\log(1 + \pi r)]$ for a range of values for $\pi$. Calculations are shown in Figure 2. For the parameter

[6] In this case, we have (for information):

$$\nu^2 = \left(\exp(\sigma^2) - 1\right)\exp(2\mu + \sigma^2).$$

Portfolio returns have a shifted log-normal distribution (affine transformation).

used, they prove very close to what would be deduced from:

$$E_0[\log(1 + \pi r)] \approx \pi\mu - \frac{1}{2}\pi^2\sigma^2.$$

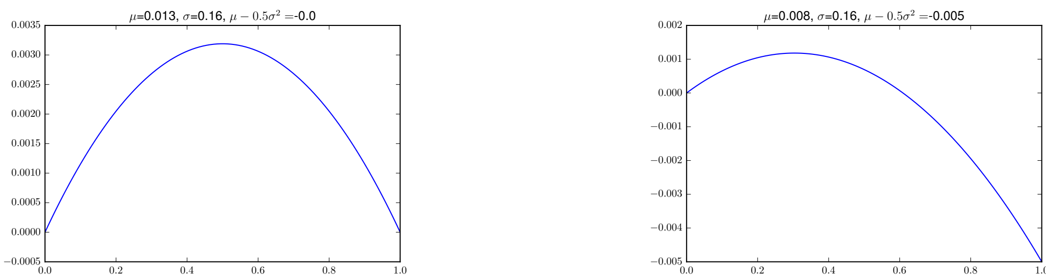


Figure 2: Expected log-return as a function of $\pi$ (Gauss-Hermite quadrature)

**Remark 7**: It should be borne in mind that although the asymptotic growth of the risky asset is zero in Luenberger[1998], we have:

$$E_0[p_t] = \exp(\mu t).$$

The price grows in expectations even in Luenberger's case. Following up on Remark 6, the condition:

$$\mu = \frac{1}{2}\sigma^2,$$

implies that a rise in volatility comes with a rise in the expected return and even a rise in the Sharpe ratio of the risky asset. There is therefore no mistery in the fact that a rise in volatility under the above condition is an improvement in investment opportunities. Clearly however, in the present i.i.d. sense, a rise in volatility does not increase the options of buying low and selling high.

□

Having reviewed the way the risk-return trade-off impacts portolio growth, I now investigate the relative asymptotic performance of rebalanced portfolios versus buy-and-hold portfolios.

## 5.3 Rebalanced portfolios versus buy-and-hold portfolios: asymptotics

I remind the following decomposition (Proposition 5):

$$\log\left(\frac{V_{\pi,T}}{W_{\pi,T}}\right) = \sum_{i=1}^{T}\gamma^*_{\pi,r_i} + \pi\log(p_T) - \log((1-\pi)+\pi p_T).$$

Using this relationship, we can establish the following asymptotic results:

**Proposition 8**: For a given expected return $\mu > 0$ and a given volatility $\sigma$, the rebalancing profits are strictly positive if and only if $\pi$ is in an interval $]\underline{\pi}, \bar{\pi}[$ that contains the log-optimal exposure $\pi^*$. When $\mu - \frac{1}{2}\sigma^2 = 0$ (Luenberger's case), this interval is $]0, 1[$. □

Buy and hold beats rebalancing when the price drifts significantly up or down, offsetting the positive impact of the excess growth-rate. Luenberger's case is a situation where the risky asset price is 'stable' in the long-run because the negative effect of volatility exacly matches the positive expected return (i.e. zero asymptotic growth-rate). This is indeed a feature that is quite detrimental to buy-and-hold. In this case, the relative performance of rebalanced portfolios is driven by the expected excess growth-rate as it also happens to be equal to the expected log-return.

**Remark 8**: In the continuous time version of the present model, the excess growth rate has an analytic form (see Fernholz[2002]), namely:

$$\frac{1}{2}\pi(1-\pi)\sigma^2.$$

In this case the interval of asymptotic outperformance is:

- $]2(\mu - 0.5\sigma^2)/\sigma^2, 1[$ when $\mu - 0.5\sigma^2 > 0$,
- $]0, 1[$ when $\mu - 0.5\sigma^2 = 0$,
- $]0, 2\mu/\sigma^2[$ when $\mu - 0.5\sigma^2 < 0$.

□

## 5.4 The virtue of rebalancing in the constant investment opportunity setup

We have thus seen that when $\mu - 0.5\sigma^2 = 0$, rebalanced portfolios with $0 < \pi < 1$ beat buy-and-hold portfolios asymptotically. I now investigate the mechanism behind this result.

We have[7]:

[7] Proposition 11 can easily be generalized to $N$ assets. The general result is established in Quian[2014] (page 11).

**Proposition 9**: In the i.i.d. case with $\mu > 0$:

$$E_0[V_{\pi,T} - W_{\pi,T}] < 0.$$

$\square$

The most enlightening proof of this uses Proposition 2. When returns are i.i.d. the position induced by rebalancing only depends on past returns. It is therefore independent of the future variation in prices:

$$E_0[(n_t - n_0)(p_{t+1} - p_t)] = E_0[n_t - n_0]E_0[p_{t+1} - p_t] = \pi E_0[V_{\pi,t} - p_t]\mu.$$

However, thanks again to independence:

$$E_0[V_{\pi,t} - p_t] = (1 + \pi\mu)^t - (1 + \mu)^t < 0.$$

Thus the trading policy induced by rebalancing is expected to lose money on average. It is not expected to buy-low and sell-high. Indeed, on average, the risky asset rises and the rebalancing policy actually keeps selling low.

Since rebalancing does not benefit from return enhancement, its growth benefit has to come from risk control (see Remarks 2, 3 and 4). Quian[2014] confirms that in the present two assets context, the variance of the rebalanced portfolio is lower than that of the buy-and-hold portfolio.

The analytical results on ex-post rebalancing profits combined with the asymptotic dominance of rebalancing indicate that almost all trajectories contain a sufficient amount of reversal for the trading policy induced by rebalancing to work (see Proposition 3). It is tempting to conclude that rebalancing works because the risky asset price mean reverts. Yet there is no mean reversion in our context. The log-price follows a random walk which is not a mean reverting process. It is true that a random walk returns to zero from any starting point with probability one. This recurrence is not the result of mean reversion however. In particular, the expected return time to zero is infinite. The asymptotic dominance of rebalancing in this model thus cannot be linked to non fortuitous mean reversion. In the same vein, in the celebrated Saint Petersburg paradox, doubling down works almost surely asymptotically. There also, one might be tempted to trace this success to some form of mean reversion. Yet there is absolutely no mean reversion in the Saint-Petersburg paradox. The asymptotic dominance of rebalancing over buy-and-hold in Luenberger's case has the same flavour as the Saint-Petersburg paradox. It is not an arbitrage opportunity.

I now turn to simulations to investigate finite distance results.

# 6 Simulations

## 6.1 The i.i.d. case

I now use simulations to confirm the results established above and investigate finite distance outcomes. The simulation process consists in drawing trajectories of length $T$ for the log-price, using the log-normal model, and then computing the value of the rebalanced portfolio $\pi$ and its buy-and-hold counterpart.

I concentrate on Luenberger's case as it is the most favourable to rebalancing. I take (see Table 1) $\sigma = 0.16$, $\mu = 0.5\sigma^2$, $T = 5$ (years). The rebalancing frequency is monthly. I choose $\pi = 0.5$, i.e. the rebalanced portfolio is the log-optimal portfolio (see Figure 2).

Table 2 confirms that $E[V_{\pi,T}] \leq E[W_{\pi,T}]$ for different choices of $T$. We have $E[\log(V_{\pi,T}/W_{\pi,T})] \geq 0$. The empirical distribution of $\log(V_{\pi,T}/W_{\pi,T})$ has negative skewness (see Figure 4), which is natural since the divergence contribution is negative or null, unbounded on the downside (see Figure 1). The excess-growth rate is close to being a trend with some moderate variability.

We know from section 5.2.3 (see the proof of Proposition 8) that:

$$\lim_{T \to +\infty} \frac{1}{T} \log(V_{\pi,T}/W_{\pi,T}) = E[\gamma^\star_\pi(\log(1+r))] > 0.$$

From Table 2, $E[\gamma^\star_\pi(\log(1+r))] = 0.0030$, i.e. 30 basis points. We are very far from this for all tested $T$, even 500 years. It is clear that it takes a very long time for asymptotic effects to kick in for the present set of parameters.

**Remark 9 (The quadratic approximation)**: To check these striking results, I explore an almost analytical approximation. Relying on:

$$\gamma^\star_\pi(x) \approx \frac{1}{2}\pi(1-\pi)x^2,$$

I will replace $\gamma^\star_\pi(\cdot)$ by this quadratic function. Given that under Luenberger's assumptions, the log-return follows a normal random-walk, we can deduce that:

$$\frac{1}{T}\sum_{t=1}^{T} \gamma^\star_\pi(\log(1+r_t)),$$

is distributed as $\frac{1}{2}\pi(1-\pi)U$ where $U$ is a random variable with the Gamma distribution $\Gamma(T/2, 2/T)$ (see the appendix for a short reminder on Gamma distributions). Similarly:

$$\frac{1}{T}\gamma^\star_\pi(\sum_{t=1}^{T} \log(1+r_t)),$$

is distributed as $\frac{1}{2}\pi(1-\pi)V$ where $V$ is a random variable with the Gamma distribution $\Gamma(1/2, 2)$. See Figure 3 for an illustration.

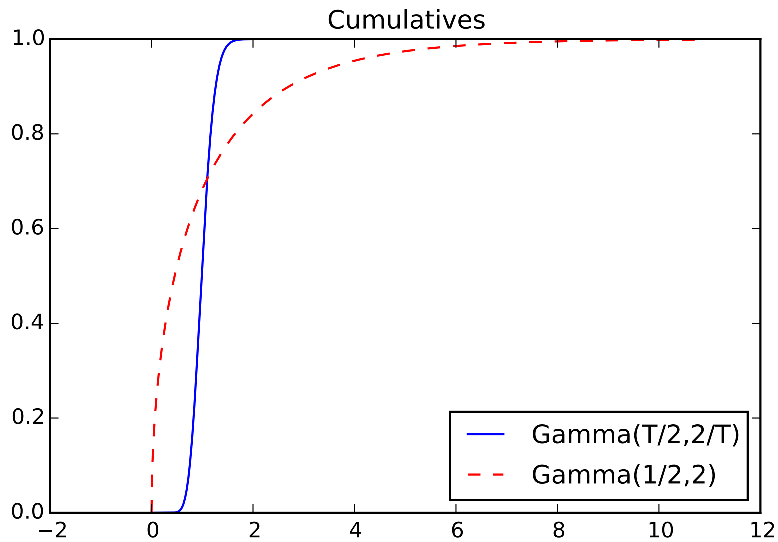


Figure 3: The Cumulatives of $U$ and $V$,

A first observation is that $U$ and $V$ have the same mean, namely 1. This explains that for small $T$, i.e. when the quadratic approximation is accurate, $E[\log(V_{\pi,T}/W_{\pi,T})/T]$ is close to zero. For $T$ large, $\gamma^\star_\pi(\cdot)$ becomes flatter than the quadratic approximation and therefore, $E[\log(V_{\pi,T}/W_{\pi,T})/T]$ becomes positive and converges to its asymptotic value.

A second observation is that the approximation remains very accurate for sequential returns, since they are small. The variability of

$$\frac{1}{T}\sum_{t=1}^{T}\gamma^\star_\pi(\log(1+r_t)),$$

is well represented by $\frac{1}{2}\pi(1-\pi)U$, which is a very peaked random variable for large $T$. We can legitimately expect that it could be replaced by a constant for large $T$. I therefore expect that one could therefore better understand the results assuming:

$$\sum_{t=1}^{T}\gamma^\star_\pi(\log(1+r_t)) - \gamma^\star_\pi(\sum_{t=1}^{T}\log(1+r_t)) \approx \gamma T - \gamma^\star_\pi(\sum_{t=1}^{T}\log(1+r_t)),$$

with $\gamma = E[\gamma^\star_\pi(\log(1+r))]$, that is replacing the sum of excess-growth rates by a linear trend. Under this approximation,

$$\log(V_{\pi,T}/W_{\pi,T}) \approx \gamma T - \gamma^\star_\pi(\sigma\sqrt{T}z_{0,1}),$$

where $z_{0,1}$ is a standardized normal variable. Determining the distribution of $\log(V_{\pi,T}/W_{\pi,T})$ for a given horizon thus boils down to computing the distribution of a known random variable, using the real value of $\gamma^\star_\pi(\cdot)$ this time. Table 3 and 4 show parameters and results which are indeed strikingly close to the simulations in Table 2.

□

**Remark 10**: When cumulated returns remain in a region where the quadratic approximation is correct,

$$E[\frac{1}{T}\sum_{t=1}^{T}\gamma^\star_\pi(\log(1+r_t)) - \gamma^\star_\pi(\sum_{t=1}^{T}\log(1+r_t))],$$

is proportional to:

$$V(\log(1+r_t)) - \frac{1}{T}V(\sum_{t=1}^{T}\log(1+r_t)) = 0.$$

The variance of cumulated returns is $T$ times the variance of returns in the i.i.d. case. In this context, improving the prospects of rebalancing requires to lower the variance of cumulated returns vis-à-vis $T$ times the variance of sequential returns. The difference between the two is closely related variance ratios, a useful tool to detect mean reversion (more on this in the next section).

For the benefits of rebalancing to materialize in Luenberger's case, we need the log-price to spend time in regions where $\gamma^\star_\pi(\cdot)$ is sub-quadratic. This requires time and this is why $T$ needs to be large for asymptotic results to materialize.

□

Thus although in Luenberger's case, rebalancing indeed beats buy-and-hold asymptotically, the speed of convergence is too slow to be of any help in practical situations. I now turn to non i.i.d. cases. They have the potential of making the buying-low selling-high argument legitimate (by inducing a positive expected outperformance), and of leading to exploitable finite distance effects.

## 6.2 Noise harvesting

From Remark 10, we know that to improve the prospects of rebalancing (especially for small to moderate investment horizons), we need to make the divergence term smaller on average than the excess growth-rate term. We need to make the variance of cumulated returns smaller than $T$ times the variance of sequential returns. The ratio between the two is called the variance ratio of a process. In econometrics, a test of mean-reversion is based on it. A time series exhibits mean reversion if sequential shocks are offsets by the future drift of the time series. In this context, short-term volatility, scaled to the investment horizon, overestimates the long-run variability of the time-series.

It is therefore clear that up to the non quadratic nature of $\gamma^\star_\pi(\cdot)$ which makes itself felt only in the very long-run, we need to introduce mean-reversion to improve the prospects of rebalancing[8]. To illustrate the benefits of mean reversion, I run the simulations using an Ornstein-Uhlenbeck process for the price:

$$\log(p_{t+1}) = \rho \log(p_t) + \sigma\epsilon_{t+1},$$

choosing $0 \leq \rho \leq 1$.

### 6.2.1 Simulating rebalancing using an OU process

Up to the mean reversion parameter, I run simulations using the same starting parameters as in the i.i.d. case. The results above can thus be recovered by setting $\rho = 1$. The investment horizon is fixed to five years. Table 5 shows the results and Figure 5 offers some illustrations.

Table 5 shows that the probability of making a loss quickly falls even when mean-reversion remains very low. The probability distribution becomes much more symmetric (Figure 5). Mean reversion makes a striking difference. An interesting observation is that mean-reversion benefits to the excess growth rate. Indeed:

$$\lim_{T\to+\infty} \frac{1}{T}\log(V_{\pi,T}/W_{\pi,T}) = E[\gamma^\star_\pi(\log(1+r))],$$

which is the unconditional variance (almost) of returns. In the Ornstein-Uhlenbeck case, $\sigma^2$ remains the conditional variance of log-returns, but the unconditional variance is mechanically higher.

These simulations cover the case where prices are i.i.d., which is presented as a puzzle in Dempster[2008]. How can a fixed weight allocation to an i.i.d. asset bring financial growth? A buy-and-hold policy of course exhibits no growth. A rebalanced portfolio however trades in a contrarian way. Contrarian trading is highly profitable when prices are i.i.d. because a high realization of the price induces a negative expected return fot the asset. More generally, when the price follows an autoregressive process, we have:

$$E_t[\log(p_{t+1}) - \log(p_t)] = (\rho - 1)\log(p_t),$$

with $\rho - 1 \leq 0$ and $\mathrm{sgn}(E_t[\log(p_{t+1}) - \log(p_t)]) = -\mathrm{sgn}(\log(p_t))$ when $0 \leq \rho < 1$. Using Proposition 2, we see that after a rise in the price, $n_t$ falls below $n_0$ and since the expected return becomes negative, we have $(0 \leq \rho < 1)$:

$$E_t[(n_t - n_0)(p_{t+1} - p_t)] > 0.$$

[8]Other authors have made the point that mean-reversion has the power to enhance rebalancing returns, see for instance Quian[2014] or Chambers[2014].

It is this covariation between positions and prospective returns which generates growth in wealth.

In the non i.i.d. case, investment opportunities are not constant (see Munk[2013]). Optimal strategies entail time-varying exposures. It is known that the optimal growth portfolio remains the myopic log-optimal portfolio in general Markovian contexts. This portfolio takes exposures which are directly proportional to the sequential expected return of the asset. This is not what our trading policy achieves. However, through its contrarian nature, it inherits a similar flavour. See Föllmer[2008] for a thorough investigation of optimal wealth growth in the context of OU price processes.

# 7 Conclusion

The analysis carried out in this paper relies on the observation that rebalancing profits are the difference between positive cumulated sequential excess growth rates and the positive excess growth rate of "cumulated returns" (the divergence term). As an aside, this observation clarifies the link between the sequential excess growth rates and rebalancing profits: the sequential excess growth rate is the upper bound of rebalancing profits. This decomposition also helps describe the sentivity of rebalancing profits to the parameters of the stochastic process of the risky asset. Volatility increases the level of sequential excess growth rates, but it may also increase the magnitude of the expected divergence term. What is beneficial to rebalancing profits is short term volatility that has no lasting effect on prices (due to mean reversion) as it increases the sequential excess growth rate without raising the divergence term. Without mean reversion, rebalancing profits are negative in expectations and positive asymptotically, but asymptotic reality is very elusive.

This paper uses the case of a two asset model where one asset is chosen as the numeraire. It can very likely be extended to the case of an arbitrary number of assets, i.e. to the setup of equity indices for instance. This is a very important extension because there has been a lot of speculation regarding the rebalancing benefits of portfolios with constant (or slowly drifting) weights versus cap-weighted ones. Do markets truely exhibit the right kind of mean reversion for this argument against cap weighting to bear? What is the order of magnitude of this benefit? The framework of the current paper could be used to bring some statistical rigor to this debate.

# 8 References

Bouchey[2016]: P. Bouchey, V. Nemtchinov, A. Paulsen and D.M. Stein, *Volatility Harvesting: Why Does Diversifying and Rebalancing Create Portfolio Growth*, The Journal of Wealth Management, Vol 15 No 2, Fall 2012.

Bouchey[2015]: P. Bouchey, V. Nemtchinov and T.L. Wong, *Volatility Harvesting in Theory and Practice*, The Journal of Wealth Management, Vol 18 No 3, pp 89-100.

Chambers[2014]: D.R. Chambers, J.S. Zdanowitz, *The Limitation of Diversification Return*, The Journal of Portfolio Management, Vol. 40 No 4, pp. 65-76.

Cuthbertson[2015]: K. Cuthbertson, S. Hayley, S. Moston and D. Nitzsche: *Diversification Returns, Rebalancing Returns and Volatility Pumping*, SSRN, 2015.

Dempster[2008]: M.A.H. Dempster, I.V. Evstigneev, and K.R. Schenk-Hoppé,*Volatility Induced Financial Growth*, Quant Finance 8 (2008), no. 1, 1-3.

Dempster[2010]: M.A.H. Dempster, I.V. Evstigneev, and K.R. Schenk-Hoppé, Growing wealth with fixed-mixed strategies, in *The Kelly Capital Growth Investment Criterion: Theory and Practice*, edited by L.C. MacLean, E.O. Thorp and W.T. Ziemba, World Scientific, 2010.

Fernholz[2002]: E.R. Fernholz, *Stochastic Portfolio Theory*, Applications of Mathematics, Springer, 2002.

Fernholz[2007]: E.R. Fernholz and C. Maguire, *The Statistics of Statistical Arbitrage*, Financial Analysts Journal, Vol 63 No 5, pp. 46-52.

Föllmer[2008]: H. Föllmer, W. Schachermayer, *Asymptotic Arbitrage and Large Deviations*, Mathematics and Financial Economics, 1:213-249, Springer.

Hallerbach[2014]: W.G. Hallerbach, *Disentangling Rebalancing Return*, Journal of Asset Management, 15, 301-316, October 2014.

Levy[2016]: H. Levy, *Stochastic Dominance: Investment Decision Making Under Uncertainty*, Springer-Verlag, 2016.

Luenberger[1998]: D.G. Luenberger, *Investment Science*, Oxford University Press, 1998.

Munk[2013]: C. Munk, *Dynamic Asset Allocation*, Working Paper, Copenhagen Business School, 2013.

Pal[2016]: S. Pal and T.l. Wong, *Energy, Entropy and Arbitrage*, Arxiv paper, 2016.

Quian[2014]: E. Quian, *To Rebalance or not to Rebalance: a Statistical Comparison of Terminal Wealth Fixed-Weight abd Buy-and-Hold Portfolios*, Working Paper.

# 9 Appendix

**Proof of Proposition 1**:

For a rebalanced portfolio with exposure $\pi$, we have (using $r_{\pi,i} = \pi r_i$):

$$V_{\pi,T} = \Pi_{i=1}^{T}(1 + r_{\pi,i}) = 1 + \pi \sum_{i=1}^{T} r_i + X(r_{\pi,1}, \ldots, r_{\pi,T}),$$

with:

$$X(r_{\pi,1}, \ldots, r_{\pi,T}) = \sum_{i<j} r_{\pi,i} r_{\pi,j} + \sum_{i<j<k} r_{\pi,i} r_{\pi,j} r_{\pi,k} + \ldots + r_{\pi,1} \cdots r_{\pi,T}.$$

For the risky-asset itself:

$$\Pi_{i=1}^{T}(1 + r_i) = 1 + \sum_{i=1}^{T} r_i + X(r_1, \ldots, r_T),$$

with:

$$X(r_1, \ldots, r_T) = \sum_{i<j} r_i r_j + \sum_{i<j<k} r_i r_j r_k + \ldots + r_1 \cdots r_T.$$

Combining the two relationships we get:

$$V_{\pi,T} = \Pi_{i=1}^{T}(1+r_{\pi,i}) = (1-\pi)+\pi\Pi_{i=1}^{T}(1+r_i)+X(r_{\pi,1}, \ldots, r_{\pi,T})-\pi X(r_1, \ldots, r_T).$$

The quantity $(1 - \pi) + \pi\Pi_{i=1}^{T}(1 + r_i)$ is the value of the buy-and-hold policy initiated with exposure $\pi$, $W_{\pi,T}$. The relative trading gain/loss studied in the two period case can therefore be expressed as:

$$V_{\pi,T} = W_{\pi,T} + X(r_{\pi,1}, \ldots, r_{\pi,T}) - \pi X(r_1, \ldots, r_T).$$

Finally, using $r_{\pi,i} = \pi r_i$:

$$X(r_{\pi,1}, \ldots, r_{\pi,T}) - \pi X(r_1, \ldots, r_T) =$$

$$(\pi^2 - \pi) \sum_{i<j} r_i r_j + (\pi^3 - \pi) \sum_{i<j<k} r_i r_j r_k + \ldots + (\pi^T - \pi) r_1 \cdots r_T.$$

The second order approximations follow easily by rearranging terms.

□

**Proof of Proposition 3**: If the price trajectory is increasing, the returns are positive. The number of shares in the rebalanced portfolio decreases below $n_0$ and:

$$(n_i - n_0)(p_{i+1} - p_i),$$

is negative.

$\square$

**Observations and proof of Proposition 4**: We have:

$$\log(1+r) = r - \frac{1}{2}h(r)r^2,$$

The function $h(\cdot)$ is strictly positive on $]-1,+\infty[$, decreasing, and verifies $h(0)=1$. The function $r - \log(1+r)$ is strictly convex and null at $r=0$.

We have:

$$\gamma_\pi^*(r) = \log(1+\pi r) - \pi\log(1+r),$$

and:

$$\gamma_\pi^\star(x) = \log(1-\pi+\pi\exp(x)) - \pi x.$$

The proposition follows from the Lemma below:

**Lemma 1**: For $0<\pi<1$, the function $\log(1-\pi+\pi\exp(x)) - \pi x$ with domain $\mathbb{R}$ is strictly convex, positive and has its global minimum 0 at $x=0$.

**Proof of Lemma 1**:

We have:

$$\gamma_\pi^{\star\prime}(x) = \frac{\pi e^x}{1-\pi+\pi e^x} - \pi,$$

and:

$$\gamma_\pi^{\star\prime\prime}(x) = \frac{\pi(1-\pi)e^x}{(1-\pi+\pi e^x)^2} \geq 0.$$

The function $\gamma_\pi^{\star\prime\prime}(\cdot)$ is unimodal, with:

$$\lim_{x\to\pm\infty}\gamma_\pi^{\star\prime\prime}(x) = 0.$$

Its maximum is reached at $x^* = \log((1-\pi)/\pi)$ with:

$$\gamma_\pi^{\star\prime\prime}(x^*) = \frac{1}{4}.$$

$\square$

Letting:

$$g_\pi(x) = \frac{1}{8}(x-x^*)^2 + \gamma_\pi^{\star\prime}(x^*)(x-x^*) + \gamma_\pi^\star(x^*),$$

we have:

$$\gamma_\pi^\star(x^*) = g_\pi(x^*),$$
$$\gamma_\pi^{\star\prime}(x^*) = g_\pi'(x^*),$$

$$\gamma_\pi^{\star\prime\prime}(x) \leq g_\pi''(x) = \frac{1}{8}.$$

The function $g_\pi(\cdot)$ is convex, as $\gamma_\pi^\star(\cdot)$. Both functions are tangent to one another at $x^*$ and $g_\pi(\cdot)$ is elsewhere more convex than $\gamma_\pi^\star(\cdot)$. The function $g_\pi(\cdot)$ is thus steeper than $\gamma_\pi^\star(\cdot)$ except at the point of tangency $x^*$. We thus have:

$$\gamma_\pi^\star(x) \leq g_\pi(x).$$

□

**Proof of Proposition 7**:

We have:

$$\frac{1}{T}\sum_{t=0}^{T}\log(1+\pi_t r_{t+1}) =$$

$$\frac{1}{T}\sum_{t=0}^{T}E_t[\log(1+\pi_t r_{t+1})] + \frac{1}{T}\sum_{t=0}^{T}(\log(1+\pi_t r_{t+1}) - E_t[\log(1+\pi_t r_{t+1})])$$

$$= \frac{1}{T}\sum_{t=0}^{T}E_t[\log(1+\pi_t r_{t+1})] + \frac{1}{T}\sum_{t=0}^{T}\varepsilon_{t+1}$$

The process $(\varepsilon_t)$ is a martingale difference:

$$E_t[\varepsilon_{t+1}] = 0.$$

Provided its second moment is bounded, it satisfies the law of large numbers, i.e. almost surely:

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T}\varepsilon_{t+1} = 0.$$

Thus almost surely:

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T}\log(1+\pi_t r_{t+1}) = \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T}E_t[\log(1+\pi_t r_{t+1})].$$

The highest asymptotic portfolio growth rate is obtained by choosing the adapted portfolio policy $(\pi_t)$ that maximizes:

$$E_t[\log(1+\pi_t r_{t+1})],$$

at each date $t$.

□

**Proof of Proposition** 8:

I first establish the following asymptotic results:

$$\lim_{t\to+\infty} \frac{1}{t}\pi \log(p_t) = \pi(\mu - \frac{1}{2}\sigma^2),$$

$$\lim_{t\to+\infty} \frac{1}{t} \log((1-\pi) + \pi p_t) = \mu - \frac{1}{2}\sigma^2,\ \mu - \frac{1}{2}\sigma^2 \geq 0,$$

$$\lim_{t\to+\infty} \frac{1}{t} \log((1-\pi) + \pi p_t) = 0,\ \mu - \frac{1}{2}\sigma^2 \leq 0.$$

I will write $p_t = \exp(x_t)$ so that $x_t$ follows a random-walk with drift $\mu - \frac{1}{2}\sigma^2$ and normally distributed shocks $\sigma\epsilon_t$.

- This is trivial (law of large numbers):

$$\lim_{t\to+\infty} \frac{1}{t}\pi \log(p_t) = \lim_{t\to+\infty} \frac{1}{t}\pi x_t = \pi(\mu - \frac{1}{2}\sigma^2).$$

- For $\mu - \frac{1}{2}\sigma^2 < 0$, we have:

$$\lim_{T\to\infty} x_T = -\infty, \text{ a.s..}$$

  Indeed:

$$\limsup_{T\to+\infty} \frac{\sum_{t=1}^T \epsilon_t}{\sqrt{T\log(\log(T))}} = \sqrt{2},$$

  and:

$$\lim_{T\to+\infty} \frac{\sqrt{T\log(\log(T))}}{T} = 0,$$

  so that in the long run, the negative drift of $x_t$ ($\mu - \frac{1}{2}\sigma^2 < 0$) cannot be offset by positive outcomes of $\sigma\sum_{t=1}^T \epsilon_t$. As a consequence $(x_t)_{t\in\mathbb{N}}$ goes to $-\infty$ with probability one and:

$$\lim_{t\to\infty} p_t = 0, \text{ a.s..}$$

  Thus:

$$\lim_{t\to\infty} \log(1-\pi+\pi p_t) = \log(1-\pi).$$

  We thus get:

$$\lim_{t\to+\infty} \frac{1}{t} \log((1-\pi) + \pi p_t) = 0,\ \mu - \frac{1}{2}\sigma^2 < 0.$$

- For $\mu - \frac{1}{2}\sigma^2 > 0$, just use:

$$\log(1-\pi+\pi\exp(x_t)) = x_t + \log(\pi + 1 - \pi\exp(-x_t)),$$

  and refer to case $\mu - \frac{1}{2}\sigma^2 < 0$.

- For $\mu - \frac{1}{2}\sigma^2 = 0$, use:

$$\log(1-\pi) \leq \log((1-\pi)+\pi p_t) \leq \max(0, \log(p_t)),$$
$$\frac{1}{t}\log(1-\pi) \leq \frac{1}{t}\log((1-\pi)+\pi p_t) \leq \max(0, \frac{1}{t}\log(p_t)),$$

  and then (law of large numbers):

$$\lim_{t\to\infty} \frac{1}{t}\log(p_t) = \lim_{t\to+\infty} \frac{1}{t}x_t = 0, \text{ a.s..}$$

- We also have, from the law of large numbers:

$$\lim_{T\to+\infty} \frac{1}{T}\sum_{t=1}^{T} \gamma_\pi^\star(\log(1+r_t)) = E[\gamma_\pi^\star(\log(1+r))] > 0,$$

  where strict positivity follows from $0 < \pi < 1$.

Assume $\mu - \frac{1}{2}\sigma^2 \geq 0$. The outperformance of rabalancing over buy-and-hold obtains asymptotically if an only if:

$$E[\gamma_\pi^*(\log(1+r))] + \pi(\mu - \frac{1}{2}\sigma^2) - (\mu - \frac{1}{2}\sigma^2) \geq 0.$$

It is easy to check that as a function of $\pi$, $E[\gamma_\pi^*(\log(1+r))]$ is strictly concave (see the proof of Proposition 10) so that the left-hand-side of the above inequality is also concave. This implies that the range of values of $\pi$ for which rebalancing beats buy-and-hold is an open interval. It necessarily contains $\pi^*$.

The same reasoning applies when $\mu - \frac{1}{2}\sigma^2 \leq 0$, the inequality defining the interval of outperformance of rebalancing being:

$$E[\gamma_\pi^*(\log(1+r))] \geq 0.$$

It remains to be shown that the interval of outperformance in Luenberger's case is $]0,1[$, i.e. that:

$$E[\gamma_0^*(\log(1+r))] = 0,$$
$$E[\gamma_1^*(\log(1+r))] = 0,$$
$$E[\gamma_\pi^*(\log(1+r))] > 0, \forall \pi \in ]0,1[.$$

The first two equalities are trivial (the buy-and-hold and the rebalanced policies coincide). We can then compute the derivative:

$$\frac{\partial E[\gamma_\pi^*(\log(1+r))]}{\partial \pi}\Big|_\pi,$$

for $\pi = 0$ and $\pi = 1$. We have:

$$\frac{\partial E[\gamma_\pi^*(\log(1+r))]}{\partial \pi}\Big|_{\pi=0} = E[r - \log(1+r)] > 0,$$

as soon as the distribution of $r$ is not concentrated on $r = 0$. Indeed $r > \log(1+r)$ for $r \neq 0$, $r > -1$.

$$\frac{\partial E[\gamma_\pi^*(\log(1+r))]}{\partial \pi}\Big|_{\pi=1} = E\left[\frac{r}{1+r} - \log(1+r)\right] < 0,$$

as soon as the distribution of $r$ is not concentrated on $r = 0$. This follows from the study of the function $r/(1+r) - \log(1+r)$ which is strictly negative on $]-1, +\infty[\setminus\{0\}$ and null at zero. $\square$

**Proposition 10**: There is a unique optimal solution $\pi^* > 0$ to this strictly concave problem. The set of portfolios for which $E[\log(1 + \pi r)] > 0$ is a non empty interval J that contains $]0, \pi^*]$. In particular, $E[\log(1 + \pi^* r)] > 0$.

$\square$

**Proof of Proposition** 10: Assuming we can interchange integration and derivation, we get that the second derivative of $E[\log(V_\pi)]$ with respect to $\pi$ is strictly positive and therefore the criterion is strictly concave in $\pi$. It is also continuous. As a consequence, the set of weights such that $E[\log(V_\pi)]$ is necessarily an open interval.

Finally the derivative with respect to $\pi$ at $\pi = 0$ is:

$$\frac{\partial E[\log(1 - \pi + \pi p)]}{\partial \pi}|_{\pi=0} = E[p] - 1 = \exp(\mu) - 1 > 0,$$

by the assumption $\mu > 0$.

$\square$

**Gamma distributions**: In the (shape,scale)=$(\alpha, \theta)$ parametrization, the density of $\Gamma(\alpha, \theta)$ is:

$$\frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} \exp(-x/\theta).$$

Its mean is $\alpha\theta$ and its variance is $\alpha\theta^2$. The $\chi_n^2$ distribution (sum of $n$ independent squared standardized normal variables) is $\Gamma(n/2, 2)$. Finally if $Z$ is $\Gamma(\alpha, \theta)$, $cZ$ is $\Gamma(\alpha, c\theta)$ for $c > 0$.

$\square$

| Freq. | $\mu$ | $\sigma$ | $\mu/\sigma$ |
|---|---|---|---|
| 12 | 0.013 | 0.16 | 0.081 |

Table 1: Parameters

| T | $E(V_{\pi,T}-W_{\pi,T})/T$ | $E[\log(V_{\pi,T}/W_{\pi,T})]/T$ | Prob$(\log(V_{\pi,T}/W_{\pi,T})\leq 0)$ | $E[\gamma_\pi^*]$ |
|---|---|---|---|---|
| 5 | -0.0002 | 0 | 32.4 | 0.003 |
| 10 | -0.0002 | 0.0001 | 31.4 | 0.003 |
| 20 | -0.0004 | 0.0002 | 31 | 0.003 |
| 50 | -0.0014 | 0.0004 | 30.1 | 0.003 |
| 100 | -0.0036 | 0.0006 | 29.1 | 0.003 |
| 200 | -0.0156 | 0.0009 | 26.6 | 0.003 |
| 500 | -0.2383 | 0.0014 | 19.8 | 0.003 |

Table 2: Effect of the investment horizon (Luenberger)

| $\sigma$ | $\gamma$ |
|---|---|
| 0.16 | 0.0032 |

Table 3: Parameters (analytic approximation)

| T | Prob$(\log(V_{\pi,T}/W_{\pi,T})\leq 0)$ | $E[\log(V_{\pi,T}/W_{\pi,T})]/T$ |
|---|---|---|
| 5 | 0.32 | 0.0002 |
| 10 | 0.31 | 0.0002 |
| 20 | 0.31 | 0.0003 |
| 50 | 0.3 | 0.0004 |
| 100 | 0.29 | 0.0007 |
| 200 | 0.27 | 0.001 |
| 500 | 0.2 | 0.0014 |

Table 4: Analytic approximation

| $\rho$ | half-life | $E(V_{\pi,T}-W_{\pi,T})/T$ | $E[\log(V_{\pi,T}/W_{\pi,T})]/T$ | Prob($\log(V_{\pi,T}/W_{\pi,T})\leq 0$) | $E[\gamma_\pi^*]$ |
|---:|---:|---:|---:|---:|---:|
| 1 | inf | -0.0002 | 0 | 32.4 | 0.003 |
| 0.994 | 9.2 | 0.0009 | 0.001 | 23 | 0.003 |
| 0.988 | 4.6 | 0.0016 | 0.0016 | 16.4 | 0.003 |
| 0.981 | 3.1 | 0.002 | 0.002 | 11 | 0.003 |
| 0.975 | 2.3 | 0.0022 | 0.0022 | 8.2 | 0.003 |
| 0.969 | 1.8 | 0.0024 | 0.0024 | 5 | 0.003 |
| 0.962 | 1.5 | 0.0025 | 0.0025 | 3.6 | 0.003 |
| 0.956 | 1.3 | 0.0027 | 0.0026 | 2.3 | 0.003 |
| 0.95 | 1.2 | 0.0027 | 0.0027 | 1.6 | 0.003 |
| 0.9 | 0.6 | 0.0031 | 0.0031 | 0.1 | 0.003 |
| 0.7 | 0.2 | 0.0037 | 0.0036 | 0 | 0.004 |
| 0.5 | 0.1 | 0.0042 | 0.0042 | 0 | 0.004 |
| 0.3 | 0.1 | 0.0049 | 0.0048 | 0 | 0.005 |
| 0 | 0.1 | 0.0064 | 0.0063 | 0 | 0.006 |

Table 5: The effect of mean-reversion (T=5)

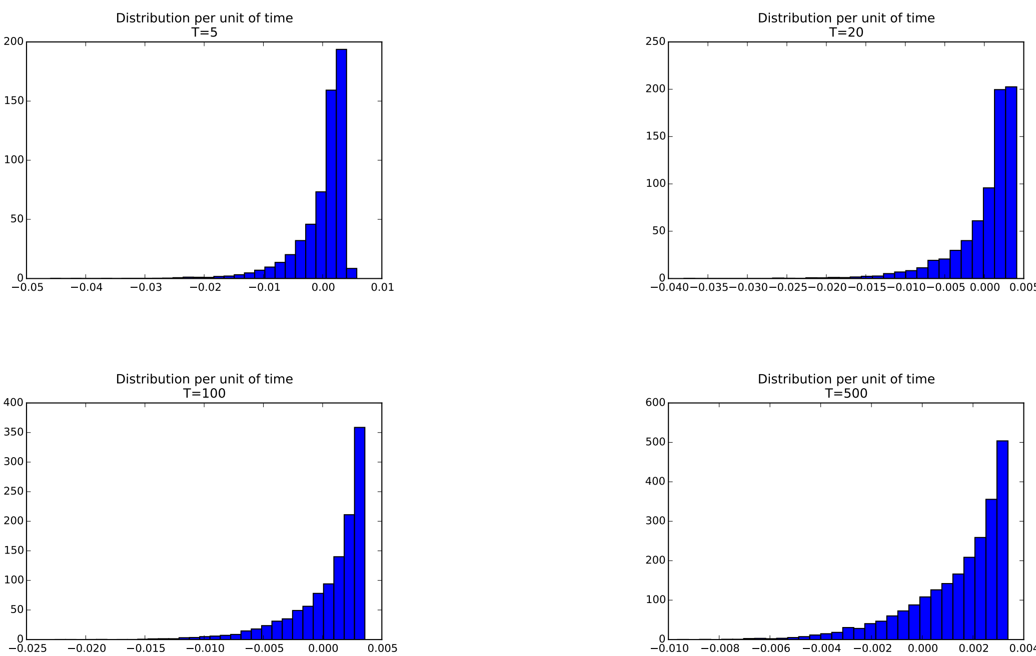


Figure 4: The effect of T on the distribution of $\log(V_{\pi,T}/W_{\pi,T})/T$

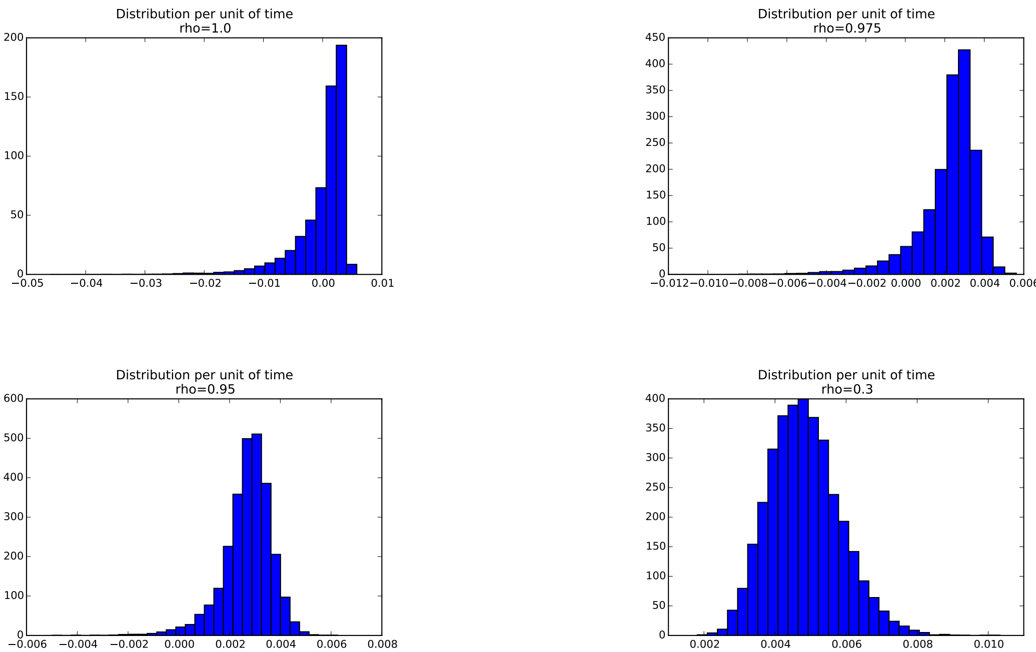


Figure 5: The effect of mean reversion on the distribution of $\log(V_{\pi,T}/W_{\pi,T})/T$ (T=5)