Analysis and Comparison of Leveraged ETFs and CPPI-type Leveraged Strategy

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Abstract

Among leveraged funds, leveraged ETFs are designed to achieve multiple exposure (e.g., twice) to some financial index returns, on a daily basis. In this paper, we derive an analytical expression for the value process of a leveraged ETF. We analyze it as a convex constant allocation portfolio strategy and examine its properties. In the continuous-time framework, we prove an equivalence result stating that a CPPI with a floor proportional to the portfolio value itself is also a constant allocation portfolio strategy. Next, we focus on another CPPI type strategy with a specific variable leverage. This type of Leveraged CPPI portfolio is not fully equivalent to a Leveraged ETF because the leverage is reduced in falling markets as well as bounded from above. We derive a quasi explicit expression for the value of such Leveraged CPPI. Then, we compare Leveraged ETFs and Leveraged CPPI by means of Monte Carlo simulations. Finally, we present an empirical study and comparison of the Leveraged CPPI (illustrated by the SGAM Leveraged CAC 40 ETF) and of the Leveraged ETF (issued by Lyxor), to highlight differences in their behavior and performance.

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1 Introduction

Exchange-traded funds (ETFs), also known as trackers in Europe, are similar to mutual funds but their shares can be traded at any time on an exchange. The first ETFs have been designed to track an index and, as such, have been passive funds. Since the mid 2000's, actively managed ETFs have appeared on the market, with leveraged and inverse ETFs designed to achieve multiple exposure (positive or negative, e.g., expositions equal to 2x or -2x the benchmark ETF) to index returns, on a daily basis. The first leveraged and inverse ETFs were launched between June 2006 and June 2009 in the U.S. More than 150 leveraged and inverse ETFs are currently traded in the U.S, covering a broad range of equity, sector, international, fixed-income, commodity and currency markets. In France, a leveraged ETF which differed slightly from the US model, was introduced on the market by SGAM AI[1] in October 2005.

Recently, there has been some controversy surrounding leveraged ETFs in the U.S. market, focused mainly on the performance results delivered by these products over extended periods of time. The most frequent criticism has been that leveraged ETFs do not perform as they should or as they are claimed to. The Financial Industry Regulatory Authority (FINRA) published in June 2009 a regulatory notice containing the following : "While such products may be useful in some sophisticated trading strategies, they are highly complex financial instruments that are typically designed to achieve their stated objectives on a daily basis. Due to the effects of compounding, their performance over longer periods of time can differ significantly from their stated daily objective. Therefore, inverse and leveraged ETFs that are reset daily typically are unsuitable for retail investors who plan to hold them for longer than one trading session, particularly in volatile markets." For the moment, this controversy has not crossed the Atlantic.

In Europe, NYSE Euronext publishes leveraged indices[2] (which belong to the family of strategy indices) on several national European equity indices (CAC 40, AEX, BEL20, PSI20) which are designed to be the underlying indices for ETFs. It gives the following definition of the double leverage index: "The leverage index tracks the performance of a strategy which doubles exposure to an underlying index with the support of a short-term financing" on a daily basis. For instance, on the French market, on February 10, 2010, ComStage (a subsidiary of Commerzbank) issued a

[1] SGAM stands for "Société Générale Asset Management". AI means "Alternative Investment".
[2] It also publishes short, double short, triple short and triple leverage indices.

leveraged ETF on the CAC 40 that is aimed at replicating the ETF CAC 40 Leverage index, an index launched on December 21, 2007 by Euronext. It is worth noting that in France, unlike the U.S., some of the first of these leveraged funds have been managed with a slightly modified Constant Proportion Portfolio Insurance (CPPI) type strategy[3]. Examples include SGAM ETF Leveraged CAC 40 and SGAM ETF Bear CAC 40 launched on October 19, 2005 by SGAM Alternative Investment[4]. The first of these funds is not strictly speaking a leveraged ETF, since the leverage is at most 200% but can be less, depending on market circumstances and also on the expectations of the portfolio manager[5]. In what follows, we analyze these funds in more details. Avellaneda and Jian Zhang (2009), Jarrow (2010) and Giese (2010), have recently studied these financial products mainly on a theoretical basis. Their studies only explore the US case and none of them draws parallels with CPPI-type strategies, nor considers products like that issued by SGAM AI. More recently, Charupat and Miu (2011) conducts an empirical analysis of leveraged ETFs based on Canadian financial market data, examining the pricing efficiency and the tracking errors of some ETFs.

This paper is organized as follows. Section 2 presents an introductory example which sets the background of the paper and illustrates differences between standard LETF and Leveraged CPPI strategy. In Section 3, we set the theoretical modelling and provide an analytical expression for the value process of a leveraged ETF. We analyze this latter one as a constant allocation portfolio strategy and then examine its properties (subsection 3.1). We establish a CPPI equivalence result in continuous-time (subsection 3.2). In Section 4, we compare Leveraged ETFs and specific Leveraged CPPI strategies[6]. First, we determine a quasi-explicit expression of the Leveraged CPPI value, taking account of the discrete-time variations of the leverage level. Then, we conduct comparisons by means of Monte Carlo simulations. Finally, we present an empirical comparison of the SGAM ETF Leveraged CAC 40 (which is a Leveraged CPPI type strategy) and of a CAC 40 LETF issued by Lyxor, to highlight differences in their behavior and performance.

[3] To the best of our knowledge, leveraged funds have not been analyzed as CPPI portfolio in the US.

[4] After a reorganization of the asset management activity of Société Générale, these funds are now managed by LYXOR since the 1st of September 2009.

[5] See the "prospectus complet" of the SGAM ETF Leveraged CAC 40, certified by the "Autorité des marchés financiers" (AMF).

[6] This latter one is illustrated by the SGAM Leveraged CAC 40 strategy.

2 Introductory example and motivations

There is a widely held misconception of what exactly leveraged ETFs provide: returns of 2x or 3x the annual return of the underlying market index? To date, leveraged ETFs have been designed to deliver daily returns that are a positive or negative multiple of the daily return of the underlying index. We illustrate this on the following example which considers three baseline scenarii: an upward trending market (+5% per day), a downward trending market (−5% per day) and a volatile market where a +5% increase is followed by a −5% decrease[7]. While unrealistic, this data set is chosen for expositional purposes. However, note that the comments madefrom this example do not depend on these specific parameter values.

Recall briefly the mechanism of a CPPI strategy[8]. This method uses a simplified strategy to allocate assets dynamically over time. The investor starts by setting a floor equal to the lowest acceptable value of the portfolio. Then, he computes the cushion as the excess of the portfolio value over the floor and determines the amount invested in the risky asset by multiplying the cushion by a predetermined multiple, denoted m. The total amount invested in the risky asset is known as the exposure. The remaining funds are invested in the riskless asset. For the Leveraged CPPI, the floor is proportional to the initial portfolio value. For the Leverage ETF strategy, the exposure is proportional to the portfolio value at any time. The proportion L corresponds to the leverage as soon as we set L higher than 1[9].

For the first two scenarii, the volatility of the daily return is equal to zero, whereas for the last scenario the daily volatility is 5%[10]. The first column contains the underlying index price, the next two columns show the leveraged (2x) ETF and the fund whose performance is twice that of the index over the whole period (static leverage). Columns (4) and (7) show a leveraged CPPI (LCPPI) portfolio which is designed to have maximum exposure to the risky asset of 200% of the total portfolio value. We also report the value of the exposure (columns (6) and (9)) if it had not been bounded above. The portfolios differ in the parameters used in the setting of the portfolio insurance strategy: LCPPI (resp. LCPPI 1) has a constant floor of 50% (resp. 75%) of the portfolio value at inception and a multiple of 4 (resp. 8). The initial exposure to the risky asset of these

[7] For expositional simplicity and w.l.o.g., we assume here that interest rate is equal to zero.
[8] See Appendix A for more details about CPPI and leveraged CPPI type strategies in discrete-time.
[9] See Appendix B where the static leverage case is discussed.
[10] As long as we consider an infinite sequence of returns.

two portfolios is set at 200%. Then a maximal bound of 200% is set on the risky asset exposure. However, note that these funds can reach less than 200% risky asset exposure during their lifetime. The difference between these two funds is that the one with the multiple m of 8 is more sensitive to the fluctuations of the risky asset (its payoff is more convex in the value of the underlying index[11]) than the one with $m = 4$. Note also that the gap risk of this fund is higher. Table 1 reports data for this example.

Table 1: Effect of Return Volatility and Compounding on 2× leveraged and CPPI Funds

S	Daily Leveraged	Static Leveraged	LCPPI	Exposure bounded	Exposure no bound	LCPPI 1	Exposure bounded	Exposure no bound
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Upward trending market								
100,00	**100,00**	**100,00**	**100,00**	200%	200%	**100,00**	200%	200%
105,00	**110,00**		**110,00**	200%	218%	**110,00**	200%	255%
110,25	**121,00**		**121,00**	200%	235%	**121,00**	200%	304%
115,76	**133,10**	**131,53**	**133,10**	200%	250%	**133,10**	200%	349%
Downward trending market								
100,00	**100,00**	**100,00**	**100,00**	200%	200%	**100,00**	200%	200%
95,00	**90,00**		**90,00**	178%	178%	**90,00**	133%	133%
90,25	**81,00**		**82,00**	156%	156%	**84,00**	86%	86%
85,74	**72,90**	**71,48**	**75,60**	135%	135%	**80,40**	54%	54%
Volatile Market								
100,00	**100,00**	**100,00**	**100,00**	200%	200%	**100,00**	200%	200%
105,00	**110,00**		**110,00**	200%	218%	**110,00**	200%	255%
99,75	**99,00**		**99,00**	198%	198%	**99,00**	194%	194%
104,74	**108,90**	**109,48**	**108,80**	200%	216%	**108,60**	200%	248%

We compare first the performance of Leveraged ETF and Period Leveraged funds. In a trending market (upward or downward), the Leveraged ETF takes advantage of the daily compounding of the leveraged index return and exhibits a higher three-day return ($33,10\%$ in an upward market and $-27,10\%$ in a downward market) than the return of the static leveraged fund ($31,53\%$ and $-28,52\%$). Thus, in both cases and as we would expect, the leveraged ETF performs better than the static leveraged fund. Note also that, in both cases, the return volatility of the underlying index is nil.

In a volatile market situation, the leveraged ETF return may be less than the return of the static leveraged fund, whether the underlying index experiences an increase (as in Table 1) or a decrease.

[11] See below Section (3.1) for a formal statement of this property.

The longer the buy-and-hold period, the stronger this effect, as we will see below. Volatility in financial markets has reached a very high level since the fall of 2008, which explains why investors may have very bad experiences with such funds.

In an upward trending market, both LCPPI and LCPPI 1 funds have the same performance as the Daily Leveraged fund due to the bound on their exposure. But, in a downward trending market, these two funds take advantage of their trend-following feature by gradually cutting their exposure to the risky asset. Thus, they dampen the effect of the fall and exhibit better performance than the Daily Leveraged fund. LCPPI 1, with a higher multiple ($m = 8$) than LCPPI ($m = 4$), amplifies this effect. In the end, CPPI type strategies behaves rather poorly in volatile markets without trends.

3 The Leveraged portfolio value

In this section, we prove that the leveraged portfolio strategy belongs to the class of the constant allocation portfolio strategies. Next, we establish under which assumptions a leveraged portfolio becomes a CPPI type portfolio.

3.1 The value of the leveraged and bear (*i.e.* inverse leveraged) strategy as a constant allocation (constant-mix) portfolio strategy

A constant allocation (also named "constant mix") portfolio strategy with two asset classes is a dynamically rebalanced strategy that aims to maintain exposures to both asset classes proportional to the portfolio value at any time.

3.1.1 The financial market

We adopt a simple continuous-time model where the stock index price[12] dynamics is given by the following stochastic process :

$$dS_t = S_t[\mu_t dt + \sigma_t dW_t], \tag{1}$$

[12] In the context of the present article, the stock index is the ETF. Additionally, we assume that all price processes are dividend-adjusted.

which implies:

$$S_t = S_0 \exp\left[\int_0^t (\mu_s - \frac{1}{2}\sigma_s^2)ds + \int_0^t \sigma_s dW_s\right], \tag{2}$$

where $(W_t)_t$ is a standard Brownian motion with respect to a given filtration $(\mathcal{F}_t)_t$, S_0 is the initial stock index price and $\mu(.)$ and $\sigma(.)$ are respectively the drift and volatility functions of the stock index price. These two processes are assumed to be adapted to the filtration $(\mathcal{F}_t)_t$ and satisfy usual assumptions. In this framework, we can consider, for instance, stochastic volatility models, as in Hull and White (1987) and Heston (1993). Note that when $\mu(.)$ and $\sigma(.)$ are constant, we recover the geometric Brownian motion (GBM hereafter) as a particular case.

The riskless asset price at time t is denoted by B_t:

$$B_t = B_0 \exp[rt], \tag{3}$$

where r is the instantaneous riskless interest rate, which is assumed to be constant.

3.1.2 Value process

The time period considered is $[0, T]$ and the strategies are self-financing. We consider the constant allocation portfolio strategy in which the constant proportion L of initial wealth V_0 is invested in the risky asset while the remainder proportion $(1 - L)$ is allocated to the riskless asset. Thus, the initial portfolio value is given by:

$$\begin{aligned} V_0 &= LV_0 + (1-L)V_0, & (4) \\ &= q_0^S S_0 + q_0^B B_0, & (5) \end{aligned}$$

where $q_0^S = \frac{LV_0}{S_0}$ and $q_0^B = \frac{(1-L)V_0}{B_0}$ are the initial numbers of shares in the risky and riskless asset respectively. The portfolio is continuously adjusted so as to maintain the exposures to each asset proportional to the portfolio value.

Proposition 1 *The value of a constant allocation (constant-mix) self-financing portfolio strategy* *at any time is given by:*

$$V_t = V_0 \exp\left[rt + \int_0^t \left(L\left(\mu_s - r\right) - L^2\frac{\sigma_s^2}{2}\right)ds + L\int_0^t \sigma_s dW_s\right]. \quad (6)$$

This expression can be rewritten as:

$$V_t = V_0\left(\frac{S_t}{S_0}\right)^L \exp\left[r\left(1-L\right)t + \frac{1}{2}L\left(1-L\right)\int_0^t \sigma_s^2 ds\right]. \quad (7)$$

Proof. The instantaneous variation of the portfolio value V_t in time interval dt is given by:

$$dV_t = q_t^S dS_t + q_t^B dB_t = V_t\left[L\frac{dS_t}{S_t} + (1-L)\frac{dB_t}{B_t}\right]. \quad (8)$$

A straightforward application of Itô's formula allows us to write the solution of this stochastic differential equation (SDE) as in Relation (6).

Additionally, by using the relation:

$$S_t^L = S_0^L \exp\left[L\int_0^t \left(\mu_s - \frac{1}{2}\sigma_s^2\right)ds + L\int_0^t \sigma_s dW_s\right],$$

we deduce that:

$$\exp[L\int_0^t \sigma_s dW_s] = \left(\frac{S_t}{S_0}\right)^L \exp\left[-L\int_0^t \left(\mu_s - \frac{1}{2}{\sigma_s}^2\right)ds\right].$$

Substituting this expression of $\exp[L\int_0^t \sigma_s dW_s]$ into the expression for V_t leads to Relation (7).

■

From Relation (7), we can see that, depending on the value of L, the portfolio value V_t is a convex or concave function of the stock index price S_t:

- If $0 < L < 1$, V_t is strictly concave.

Usually, constant-mix strategies are conceived for proportion L between zero and one. This is why these strategies are usually referred to as concave strategies. This strategy performs well under relatively flat but volatile market conditions. Moreover, it capitalizes on price reversals.

- If $L < 0$ or $L > 1$, V_t is strictly convex.

This case can be reduced without loss of generality to the case $L > 1$, the one that concerns us when considering leveraged funds[13]. For instance, if L is set at 2, the portfolio holdings in the stock index are twice the portfolio's value. The leveraged part is financed by borrowing V_t at the riskless rate.

Note also that with the index price dynamics given in (1), the LETF value is always above zero. But, as shown in Appendix B, this property is no longer true for the static case, for which continuous-time rebalancing is not allowed. Such difference between the discrete-time and continuous-time cases is well-known for the CPPI strategies.

3.1.3 Some statistical properties

Consider the first four moments of the leveraged ETF return. If $\mu(.)$ is deterministic, the first two moments of the leveraged ETF value are given by:

$$\begin{cases} \mathbb{E}[\frac{V_t - V_0}{V_0}] & = & \exp\left[rt + L\int_0^t (\mu_s - r)ds\right] - 1, \\ Var[\frac{V_t - V_0}{V_0}] & = & \exp\left[2rt + 2L\int_0^t (\mu_s - r)ds\right]\mathbb{E}\left[\exp\left(L^2\int_0^t \sigma_s^2 ds\right) - 1\right]. \end{cases}$$

Therefore, for $\mu > r$, both expectation and variance are increasing functions of the leverage parameter L. This implies that no such leveraged ETF dominates another with respect to the mean-variance criterion.

In Table 2, the first four moments of the return of these leveraged ETFs are displayed. They illustrate the strong effect of leverage on these statistics.

Table 2: First Four Moments of V_t/V_0-1 for various L (t=1)

	$L=1$	$L=2$	$L=3$	$L=5$
Expectation (%)	8.33	13.88	19.72	32.31
Volatility (%)	21.88	47.44	78.81	173.4
Skewness	0.614	1.322	2.26	6.185
Kurtosis	3.678	6.26	13.27	113.94

Remark 2 *In the GBM case (μ and σ constant), the probability distribution of the portfolio value*

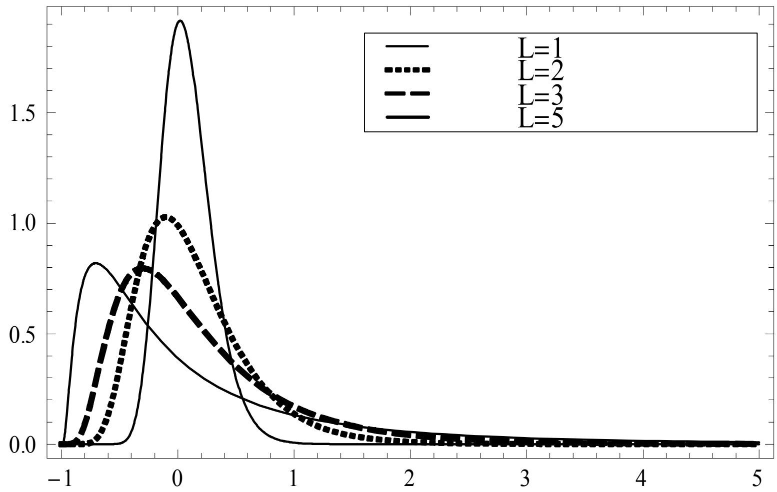
[13] The case, $L < 0$, a bear or inverse leveraged strategy, can be treated similarly.

is the Lognormal distribution:

$$\frac{V_t}{V_0} \sim Ln\mathcal{N}\left((r + L(\mu - r) - L^2\frac{\sigma^2}{2})t, L\sigma\sqrt{t}\right).$$

Figure 1 displays the probability density function (pdf) of the return on the leveraged ETF for different values of the leverage parameter L in the GBM case. The following values are used for the financial market parameters: $\mu = 8\%$, $r = 3\%$ and $\sigma = 20\%$.

Figure 1: Pdf of $\left(\frac{V_t}{V_0} - 1\right)$ for various L $(t = 1)$



From Equation (6), the expected continuous time growth rate over the period $[0, t]$ is equal to $(r + L\frac{1}{t}\int_0^t (\mu_s - r)\, ds - \frac{1}{2}L^2\frac{1}{t}\int_0^t \sigma_s^2 ds)$. Apart from time, it has three components[14]:

- The riskless rate, r;
- The temporal mean of the leveraged excess return of S over r, $\frac{1}{t}\int_0^t (\mu_s - r)\, ds$;
- A variance term,$-\frac{1}{2}L^2\frac{1}{t}\int_0^t \sigma_s^2 ds$, that reduces the expected Logreturn irrespective of the sign of L. The term is proportional to the temporal mean of the realized variance $\frac{1}{t}\int_0^t \sigma_s^2 ds$ of the ETF.

[14] Alternatively, the first two components can be organized in the following way:

- Leveraged return on the index, $L\mu$,
- Financing cost, $r(1 - L) < 0$ for $L > 1$.

The volatility correction is the same.

The variance correction term can even be such that the expected logreturn becomes negative. It is maximized for the value of the leverage given by:

$$L^* = \frac{\int_0^t (\mu_s - r)\, ds}{\int_0^t \sigma_s^2 ds}, \tag{9}$$

where it reaches the value $r + \frac{1}{2}\frac{\int_0^t (\mu_s - r) ds}{\int_0^t \sigma_s^2 ds}$. Relation (9) shows that the optimal leverage is stochastic and corresponds to a Sharpe-type ratio. For the GBM case, this optimal level is indeed constant: for instance, if $\mu = 8\%$, $r = 3\%$ and $\sigma = 20\%$, we obtain: $L^* = 1.25$.

It is possible to give another but equivalent interpretation of the value process, based on relation (7). The growth return of the LETF over the period $[0, t]$ is equal to the underlying index return compounded L times reduced by the interest paid on $[0, t]$, $\exp[r\,(1 - L)\, t]$, and by the effect of the ETF's volatility on $[0, t]$, $\exp[\frac{1}{2} L\,(1 - L) \int_0^t \sigma_s^2 ds]$.

As already shown by Bertrand and Prigent (2005, 2011) for the CPPI case, the portfolio value given by Relation (7) is inversely related to the volatility of the risky asset. In other words, the Vega of the leveraged ETF is negative. Note that it is the realized volatility that matters here, and not the expected (or implied) volatility, as is the case for option prices.

3.1.4 Portfolio return as function of the risky asset return

From Relation (7), we deduce:

$$\ln\left(\frac{V_t}{V_0}\right) = L \ln\left(\frac{S_t}{S_0}\right) + \left[r\,(1 - L)\, t + \frac{1}{2} L\,(1 - L) \int_0^t \sigma_s^2 ds\right]. \tag{10}$$

Therefore, in continuous-time, it is the portfolio logreturn which is equal to the risky asset logreturn multiplied by the leverage parameter L to which a correction term is added (this is due to the compound rates). The impact of this latter term has to be emphasized. Indeed, the portfolio return is an increasing function of the risky asset return but the correction term is negative for values of the leverage coefficient L higher than one. Then, it is interesting to evaluate the probability that the stock index will experience an increase at the same time as the leveraged fund decreases. This is an event that is probably difficult to accept for an investor in such a fund.

Proposition 3 *For the GBM case, the probability that the stock index increases while the leveraged* *fund decreases is given by:*

$$\Phi\left(\frac{-a/L-\left(\mu-\frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right)-\Phi\left(\frac{-\left(\mu-\frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right), \tag{11}$$

where Φ denotes the cumulative distribution function (cdf) of the standard Gaussian distribution.

Proof. The probability of this joint event can be calculated by first considering Relation (7):

$$Y = X^L \exp[a], \tag{12}$$

with $Y = \frac{V_t}{V_0}$, $X = \frac{S_t}{S_0}$, $a = r(1-L)t + \frac{1}{2}L(1-L)\sigma^2 t$.

Note that $a < 0$ as soon as $L > 1$. We are looking for the value $\mathbb{P}[(X > 1) \cap (Y < 1)]$. First, we observe that

$$Y < 1 \Leftrightarrow X^L \exp[a] < 1, \tag{13}$$

$$\Leftrightarrow X < \exp[-a/L](> 1). \tag{14}$$

Thus, we must evaluate the probability that $1 < X < \exp[-a/L]$, which is equivalent to: $0 < Ln(X) < -a/L$.

Therefore, we deduce:

$$\mathbb{P}\left[0 < Ln(X) < -a/L\right] = \Phi\left(\frac{-a/L-\left(\mu-\frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right)-\Phi\left(\frac{-\left(\mu-\frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right).$$

■

We can gain some intuition on this phenomenon by closely looking at the expression (12). On the one hand and in the event that the return on the risk asset is positive, the first term in the right-hand side of equation (12) tells us that the return on the fund is greater than that of the risky asset because of the leverage effect. On the other hand, the second term is less than one. Thus, the product of these two terms might be less than the return of the risky asset. This is even more likely that the price S_t is low, that the volatility, the leverage, the riskless rate and the time are high. A situation where the terminal price may be low and the volatility is high, is typically what financial markets have experienced recently.

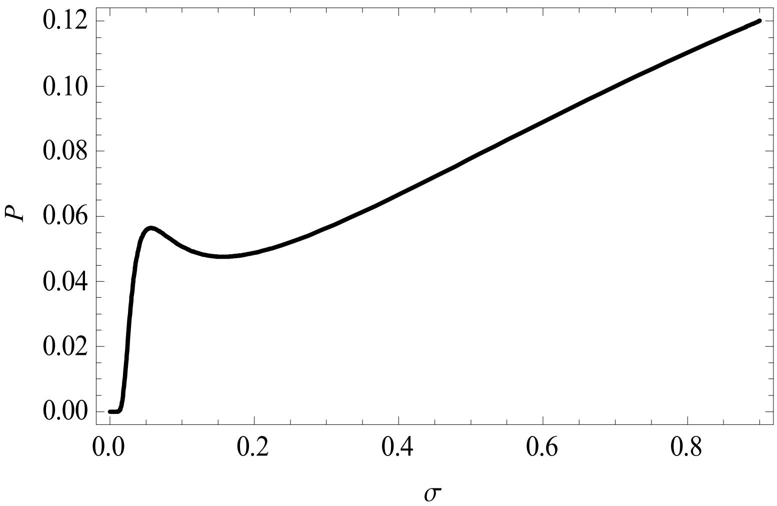
We consider the following parameter values to assess the magnitude of this probability:

$$\mu = 8\%, \sigma = 20\%, r = 3\%, t = 0.5, L = 2.$$

Therefore, the probability is equal to 4.9%, and if the management period is set at one year, this probability becomes 6.82%.

This feature is not negligible and makes a difference to investors. Moreover, the sensitivity of this probability to each of these parameters needs to be analyzed.

Figure 2: Probability as a function of volatility.



As shown in Figure 2, the behavior of the probability as a function of the volatility of the stock index is rather complicated, since it also depends on the value of the other parameters. For small values of t (typically less than one year), it is first increasing, then reaches a local maximum, then becomes decreasing again to reach a local minimum before finally becoming increasing again. There is no direct intuition for the shape of this curve which arises from the way volatility enters expression (11). Nevertheless, for values of the volatility between 5% and 40%, the probability is between 5.57% and 6.67%, meaning that the sensitivity is small.

Figure 3 displays the probability as a function of leverage. The probability is increasing in the leverage because the ratio $(-a/L)$ is also increasing in the leverage. Thus, we are computing a probability of the same random variable on a wider interval. Moreover, it shows a high degree of sensitivity, since it rises from 4.5% to 14.5% when the leverage goes from 2 to 5.

Figure 3: Probability as a function of leverage

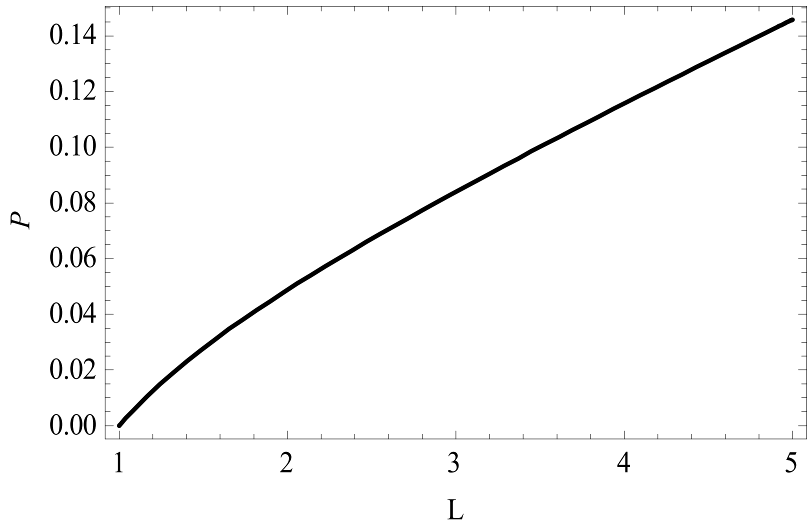
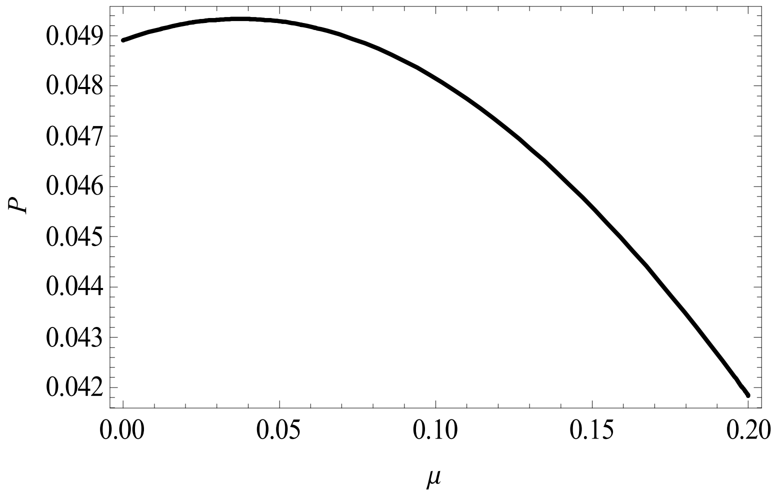


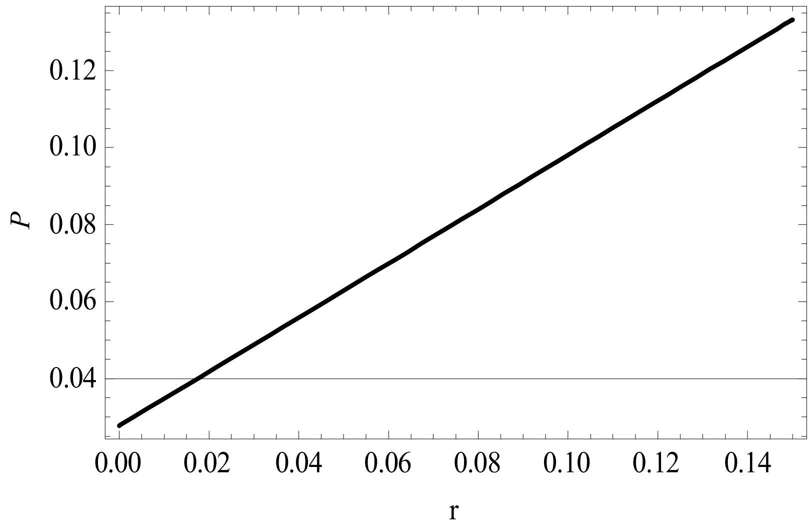
Figure 4: Probability as a function of expected return.



As a function of the expected return of the underlying index (see Figure 4), the probability is first slightly increasing and then decreasing. Note nevertheless that the magnitude of the effect is small. For example, when the expected return of the risky asset is between 5% and 15%, the probability is not very sensitive to it and lies between 4.93% and 4.55%. This arises because expected return has an opposite effect on both parts of the right-hand side of expression (11).

As shown in Figure 5, the probability is a quasi-linear and increasing function of the riskless interest rate. The effect of the riskless rate on the probability is stronger than that of the expected return. For r equal to 2%, the probability is equal to 4.18% and for r equal to 6%, it is equal to 6.99%. This is because interest rate enters Formula (11) only in the first cumulative distribution

Figure 5: Probability as a function of riskless rate



function.

Figure 6: Probability as a function of time.

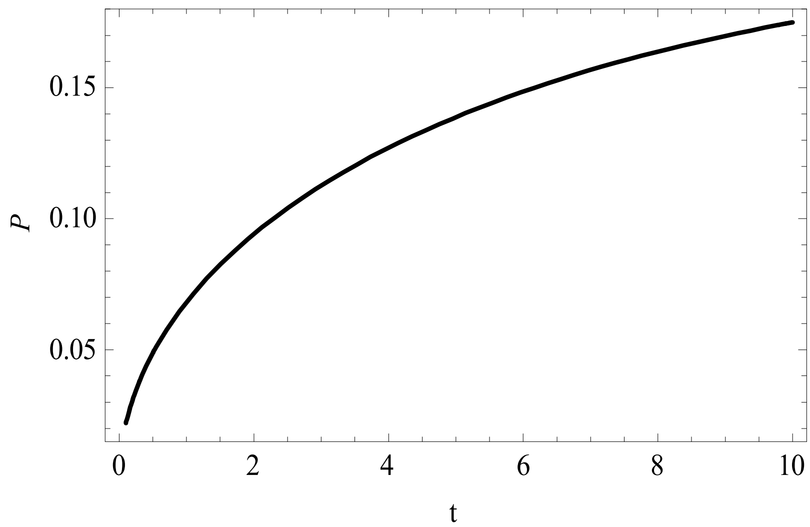


Figure 6 displays the probability as a function of time and shows that time also has a positive, although concave effect, on this probability. The value of the strategy being convex, it resembles the constant proportion portfolio insurance (CPPI) strategy with the usual multiple values. This point will be addressed in the following section.

3.2 CPPI equivalence

In this section, we establish that, in continuous-time, a CPPI portfolio with a floor proportional to the value of the portfolio itself is a constant allocation portfolio strategy.

First, we recall some basic features of CPPI strategy. Portfolio insurance is designed to give the investor the ability to limit downside risk while allowing some participation in upside markets. Such methods allow investors to recover, at maturity, a given percentage of their initial capital, in particular in falling markets. The CPPI method consists of managing a dynamic portfolio so that its value is above a floor F at any time t. The value of the floor gives the dynamically insured amount. It is assumed to evolve according to:

$$dF_t = F_t r dt$$

Obviously, the initial floor F_0 is less than the initial portfolio value V_0^{CPPI}. This difference $V_0^{CPPI} - F_0$ is called the cushion, denoted by C_0. Its value C_t at any time t in $[0, T]$ is given by :

$$C_t = V_t^{CPPI} - F_t$$

Denote by e_t the exposure, which is the total amount invested in the risky asset. The standard CPPI method consists in letting $e_t = mC_t$ where m is a constant called the multiple. The interesting case is when $m > 1$, that is, when the payoff function is convex. Thus, here the constant proportion is between the amount invested in the risky asset and the cushion, whereas in constant-mix strategies, this constant proportion is maintained between the amount invested in the risky asset and the portfolio value.

The value of this portfolio V_t^{CPPI} at any time t in the period $[0, T]$ is given by[15]:

$$\begin{aligned} V_t^{CPPI} &= F_0.e^{rt} + C_0 exp[rt + m\int_0^t \left(\mu_s - r - \frac{m^2\sigma_s^2}{2}\right) ds + m\int_0^t \sigma dW_s], \\ &= F_0.e^{rt} + C_0 \left(\frac{S_t}{S_0}\right)^m exp[r(1-m)t - (m^2 - m)\int_0^t \frac{\sigma_s^2}{2} ds]. \end{aligned}$$

[15] This formula was proved by Black and Perold (1989) and has been extended to the stochastic volatility case by Bertand and Prigent (2005).

Thus, the CPPI method is parametrized by the initial floor F_0 and the multiple m.

It is usually believed that a CPPI with a zero floor is a necessary and sufficient condition for this strategy to be considered as a constant-mix strategy. Nevertheless, we show in what follows that, at least formally, it is not necessary for the floor to be zero.

Proposition 4 *A CPPI portfolio with a multiple m and with a floor proportional to the portfolio value at any time t in the period $[0,T]$ (proportion equal to ξ)is a constant-mix portfolio strategy with leverage coefficient L equal to:*

$$L = m(1-\xi). \tag{15}$$

Proof. A floor proportional to the portfolio value at any time t in the period $[0,T]$ is written:

$$F_t = \xi V_t^{CPPI},\ 0 \leq \xi < 1.$$

Then, the cushion is written:

$$e_t = m\left(V_t^{CPPI} - F_t\right) = m(1-\xi)V_t^{CPPI}.$$

The evolution at time t of the CPPI portfolio value is given by

$$dV_t^{CPPI} = (1 - m(1-\gamma))V_t^{CPPI}\frac{dB_t}{B_t} + m(1-\xi)V_t^{CPPI}\frac{dS_t}{S_t}.$$

It can be checked that setting $L = m(1-\xi)$ leads to the equation (8) which describes the evolution of the constant allocation portfolio value. ∎

Leveraged CPPI portfolios at inception are obtained when $m > 1/(1-\xi)$ (*i.e.* $e_0 > V_0^{CPPI}$).

Thus, the critical feature that makes a constant proportion strategy becoming a constant allocation strategy is that the floor is proportional to the portfolio value at any time. This means that, contrary to the usual CPPI, there is no longer any downside protection because the floor is now stochastic and can therefore reach zero[16].

[16] We could have set the value of the floor such that: $F_t = \xi V_0^{CPPI} \equiv F_0$, $0 < \xi < 1$. In this case, we would obtain a constant-mix-type strategy in which the part of the wealth invested in the risky asset is no longer proportional to the whole wealth at each time, but rather to $\left(V_t^{CPPI} - F_0\right)$.

4 Comparison of Leveraged ETF and Leveraged CPPI Portfolio

In this section, we first provide a detailed analysis of the value process of the leveraged CPPI portfolio, illustrated by the SGAM Leveraged CAC 40 strategy. Even if we assume that this strategy is rebalanced in continuous time, some features of this product prevent a fully explicit formula for the value of the leveraged CPPI portfolio from being obtained. This is because the floor is adjusted at the beginning of each month and then held constant during the month. We conduct a Monte Carlo simulation in discrete time in order to compare LETF and leveraged CPPI portfolio.

4.1 Leveraged CPPI

The Leveraged CPPI fund is very similar to the CPPI funds introduced in section 1. It is designed to have maximum exposure to the risky asset of 200%. However, this fund may reach less than 200% risky asset exposure during its lifetime. In this respect, it differs from the standard LETF, for which the leverage is constant through time. It also has a monthly floor of 50% of the start-of-the-month portfolio value, which remains constant during the rest of the month. The value of the multiple at the inception of the fund is set at 4.6 and subsequently allowed to vary between 3.00 and 5.50. It is readjusted at the beginning of each quarter. This point is not addressed here because it is a tactical asset allocation decision. For expositional purposes, we will consider a constant multiple equal to 4 which provides an initial exposure of 200%. The exposure is assumed to be upper bounded: it is capped with leverage coefficient $\lambda = 2$ (see proposition below).

Although no insurance is provided for the whole period of management and, as a consequence, the entire amount invested may be lost, insurance is obtained within each month.

Consider the sequence $(t_n)_n$ of times at which the floor is determined by the relation[17]:

$$F_{t_n} = \xi V_{t_n} \text{ with } 0 < \xi < 1$$

Usually, t_n is the first day of the $n-th$ month of the portfolio management period. Assume that portfolio is rebalanced in discrete time during each period $[t_n, t_{n+1}[$. Let $t_{n,l} = t_n + l.h_n$ the rebalancing times during $[t_n, t_{n+1}[$, where $l \in \left\{0, ..., \frac{t_{n+1}-t_n}{h_n}\right\}$ and h_n denotes the rebalancing frequency (typically, h_n corresponds to one day).

[17] The parameter $\xi = 50\%$ for instance.

Then, taking account of all previous assumptions about leveraged CPPI leads to:

Proposition 5 *The exposure of the leveraged CPPI is given by:*

$$e_{t_{n,l}} = Max\left[Min\left[m \times (V_{t_{n,l}} - F_{t_n}); \lambda V_{t_{n,l}}\right]; 0\right],$$

where m is the multiple and λ denotes the leverage coefficient.

The leveraged CPPI portfolio value satisfies:

$$V_{t_{n,l}} = Max\left[e_{t_{n,l-1}} \times \left(\frac{S_{t_{n,l}}}{S_{t_{n,l-1}}}\right) + \left(V_{t_{n,l-1}} - e_{t_{n,l-1}}\right) \times [1 + rh_n]; F_{t_n}\right], \tag{16}$$

where the parameter λ is the maximum exposure allowed for the fund[18]*.*

Relation (16) does not allow an explicit solution but can be numerically illustrated by using Monte Carlo simulations (see next subsection). However, under some additional mild assumptions, we can provide a quasi-explicit solution (see 6).

4.2 Simulations and comparisons of Leveraged ETF and Leveraged CPPI Portfolios

In this section, we implement Monte Carlo simulations to compare both strategies. A discretized geometric Brownian motion is simulated on a daily basis over a period of 240 days[19]. We use the same values of market parameters as previously and 500,000 paths for the underlying stock index are drawn. Then, a 2x LETF portfolio strategy is simulated. We consider three Leveraged CPPI strategies described in Table 3:

Table 3: Leveraged CPPI Strategies

	ξ	m
LCPPI	50%	4
LCPPI 1	75%	8
LCPPI 2	90%	20

[18] In the case of the SGAM product, it was set at 200%.
[19] Here a stylized year is assumed to contain twelve months of 20 days.

All three strategies are designed to provide an initial exposure to the risky asset of 200%. Although the last case is unrealistic, the value of the multiple being much too high, especially for a stock index, it is used for expositional purposes. These strategies are also simulated along the 500,000 paths of the stock index.

Table 4 shows the first four moments corresponding to the four strategies.

Table 4: First four Moments of LETF and Leveraged CPPI Strategies

	LETF	LCPPI	LCPPI 1	LCPPI 2
Expectation (%)	13.12	12.90	12.40	11.02
Volatility (%)	42.97	42.00	40.25	36.23
Skewness	1.19	1.21	1.24	1.30
Kurtosis	5.64	5.73	5.88	6.18

The first two portfolio strategies come out as very close. Thus, the LCPPI strategy is essentially equivalent to a 2x LETF with a little more downside protection, as will be highlighted in Figure (7). The LCPPI 1 and LCPPI 2 portfolios exhibit less expected returns, less volatility, more positive asymmetry and more extreme risk than the first two strategies.

Figure (7) represents the cumulative distribution function (CDF) of the four portfolio strategies. It highlights the downside protection that is provided by the Leveraged CPPI strategies. This effect is all the more marked as the values of the multiple m and of the floor parameter ξ are high.

For instance, the probability that the return on the LETF is less than -20.64% is 22.5%, whereas the same probability for the LCPPI 2 fund is 18.5%. The LETF curve crosses the LCPPI curve at a return level of 1.815%, the LCPPI 1 curve at a return level of -0.279% and the LCPPI 2 curve at a return level of -2.977%.

Another point needing to be addressed is the case where the value of the multiple of the LCPPI strategy with $\xi = 50\%$ is such that the expected returns on the LETF and on the LCPPI fund are equal. In our example, this value of the multiple denoted by m^* is approximately 4.4. We thus obtain the moments for the two strategies that appear in Table 5.

In this case, the LCPPI strategy is equivalent to the LETF. Actually, it is slightly more dominant if we consider only the first three moments.

Figure 7: CDF of the four portfolio strategies

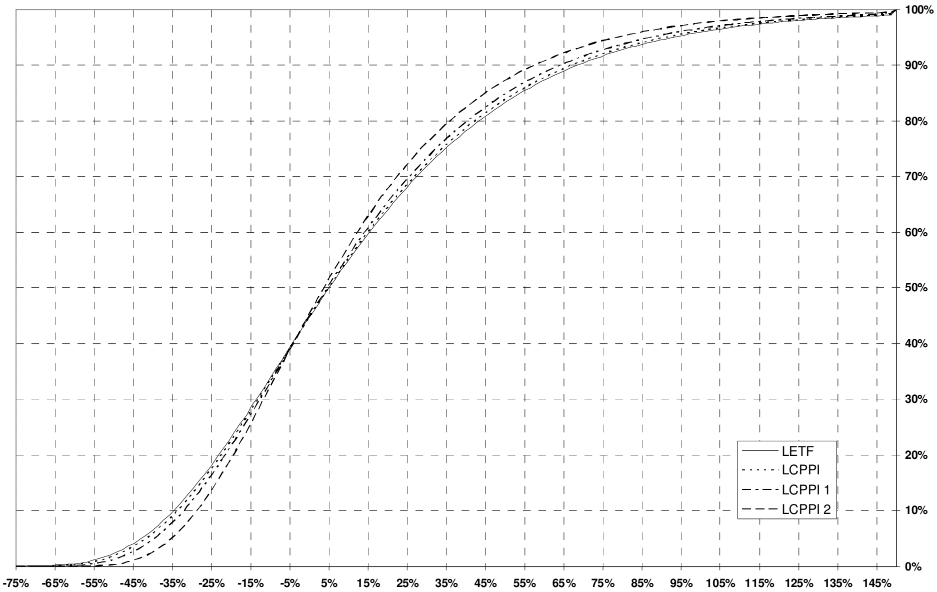


Table 5: First four Moments of LETF and SGAM Strategy for m*

	LETF	LCPPI
Expectation (%)	13.12	13.12
Volatility (%)	42.97	42.83
Skewness	1.18	1.19
Kurtosis	5.60	5.63

4.3 Pricing efficiency and tracking error of two CAC 40 LETFs

This section examines the pricing efficiency and tracking errors of two French leveraged ETFs on the CAC 40 stock index[20]: the SGAM ETF Leveraged CAC 40 managed under a leveraged CPPI portfolio strategy and a standard 2x leveraged ETF issued by LYXOR. As recalled by Charupat and Miu (2011): "Pricing efficiency measures how close the fund's market price is to its NAV (*i.e.* Net Asset Value), while tracking errors measure how well the fund's NAV reflects the underlying benchmark's returns".

4.3.1 Sample description

As of November 2010, there were 50 leveraged funds listed on NextTrack, Euronext's dedicated product segment for trackers and structured funds. More precisely, these consisted of 15 leveraged

[20]The CAC 40 index is the main benchmark for the Paris equity market The index contains 40 stocks selected among the top 100 market capitalizations and the most active stocks listed on Euronext Paris.

funds and 35 short or bear funds. Note that the oldest of these ETFs was issued at the beginning of 2007. The SGAM product was issued on October 19, 2005 and managed as a CPPI portfolio with exposure to the CAC 40 index which is variable up to a maximum 200%. The CPPI method makes it possible to optimize exposure to the CAC 40 index, with a 200% cap, while limiting the maximum monthly loss to 50% of the previous month's last NAV. The multiple can be adjusted at the beginning of each quarter allowing for tactical portfolio decision. This ETF was absorbed by the LYXOR LETF CAC 40 fund on July 2, 2010. This latter fund was first listed on June 13, 2008 and was designed to replicate the CAC 40 Leverage Index, a strategy index provided by NYSE Euronext[21]. This index combines an investment aiming to replicate a long position on the CAC 40 index with double exposure and a borrowing at the EONIA[22] interest rate. We study these two ETFs on a daily basis both on their common time interval (June 13, 2008 - July 2, 2010) and on their whole available sample (SGAM fund: October 19, 2005 - July 7, 2010; Lyxor fund: June 13, 2008 - October 8, 2010).

Leveraged ETF's are traded in a primary market open to Authorized Participants (AP) for the creation and redemption of shares directly from the fund, and a secondary market where ETF shares can be traded in the same way as shares are bought and sold on an ordinary stock exchange. As usual with leveraged ETFs, the creation/redemption process involves in-cash transactions at the NAV. Creation/redemption order submissions are allowed each trading day between 10 a.m. and 5 p.m. and calculated at the NAV, which is determined at the end of the day. Thus, there may be a time interval during which there is a price risk. Note that previously, the timing of creation/redemption order submissions was different, requiring orders to be sent before noon. Thus, the process has now become more efficient.

4.3.2 Pricing efficiency

In this section, we study the pricing efficiency of these two LETFs. Pricing efficiency should be achieved through the creation and redemption process for LETFs, which allows arbitrage opportunities to be exploited if the LETF prices diverge from the fund's NAV. Thus, LETF shares should not be priced at significant premiums or discounts from the NAV of the underlying portfolio.

[21] See http://www.euronext.com/fic/000/060/711/607112.pdf
[22] Euro Overnight Index Average.

Previous research provides evidence that U.S. ETFs are efficiently priced, since only small and non-persistent deviations from the NAV are observed (e.g., Ackert and Tian, 2000, 2008; Elton, Gruber, Comer and Li, 2002). In addition, mispricing for other-country ETFs is more significant and more persistent (e.g., Engle and Sarkar, 2006; Ackert and Tian, 2008). In an empirical study of the pricing efficiency of Canadian LETFs by Charupat and Miu (2011), the authors conclude that leveraged ETFs are efficiently priced but that, compared to traditional ETFs, the price deviations of LETFs have greater volatility. Moreover, they find that, on average, leveraged ETFs trade at a "discount or a slight premium to their NAVs"[23].

Pricing efficiency is measured by the deviation of end-of-day prices of leveraged ETFs from their NAVs. To be more precise, it is defined as:

$$PE_{i,t} = \frac{P_{i,t} - NAV_{i,t}}{NAV_{i,t}} \tag{17}$$

where $P_{i,t}$ is the closing price of a leveraged ETF i on day t, $NAV_{i,t}$ is the net asset value of the fund on the same day. If $PE_{i,t}$ is positive (resp. negative), the fund trades at a premium (resp. discount). Table 6 reports the results[24] for these two ETFs.

Table 6: Pricing Efficiency

	SGAM		LYXOR	
	whole sample (N=1096)	common sample (N=520)	whole sample (N=579)	common sample (N=509)
Average (%)	0.00629	-0.00777	-0.01545	-0.01450
Std. Dev. (%)	0.36076	0.42706	0.46343	0.49275
5th Percentile	-0.3775	-0.6229	-0.5744	-0.6301
95th Percentile	0.5361	0.8201	0.6169	0.7025
Skewness	1.96527	2.44753	2.2386	2.11377
Kurtosis	23.5592	23.1312	33.9999	30.2563
1st-Order Autocorrel.	0.141**	0.174**	0.055	0.053

The Lyxor LETF exhibits negative average price deviation, in line with what is found by Charupat and Miu (2011) for the Canadian market. Of the four LETFs they examine, three trade at a

[23] Ackert and Tian (2008) find that for traditional ETFs, "the average price deviation is more often a premium, as opposed to a discount, which may result from the redemption and creation process." This arises because it is less costly for an AP to redeem a fund that trades at a discount than to purchase additional creation units from the ETF.

[24] ** denotes significance at the 1 level.

discount: -0.024%, -0.0321% and -0.0320%. Here, the Lyxor LETF price deviation is roughly two times less than that of the Canadian LETFs. Although it is difficult to make comparisons across markets in different countries, the reason may lie in the now later cut-off time on the NextTrack market, which leads to greater efficiency. Depending on the sample period considered, the SGAM fund experiences an average price deviation that is slightly positive or negative and which is two times less in absolute value than the Lyxor one. Although the average price deviations are small, large discounts or premiums may occur, as shown by their standard deviation (from 0.36% to 0.49%) as well as by their 90% inter-percentile range (from 0.91% to 1.4%). Positive skewness and excess kurtosis show clear departure from the normal distribution[25]. With respect to the observed auto-correlations, even though some are statistically significant, we conclude, as in Charupat and Miu (2011) that, given their small magnitude, they are not economically significant.

Thus, with respect to their pricing efficiency, we find that both funds with their distinct management techniques are similar.

4.3.3 Tracking-error

Tracking-error is a measure of how far the return on a portfolio deviates from the return on its benchmark index. Most of the time it is defined as the standard deviation of the differences between the return on the portfolio and the return on the benchmark. It is also known as the relative risk of the portfolio with respect to its benchmark and is often used in portfolio optimization contexts (e.g., Bertrand, 2010). In our study, the tracking-error is calculated on the difference between the return on the NAV of the ETF without dividends and management fee and the return on the CAC 40 Leverage Index. Table 7 reports our results.

Table 7: Tracking Error

	SGAM	LYXOR
	whole sample (N=1010)	whole sample (N=572)
Average (%)	-0.00259	-0.000472
Std. Dev. (%)	0.7731	0.1377

The Lyxor LETF has a low daily average tracking error, meaning that on average the fund is able to accurately track its benchmark. This result is in line with findings by Harper *et al* (2006).

[25] which is confirmed by a Jarque-Bera test not reported here.

Indeed, their study reports that the average monthly tracking errors for US and international ETF is equal to -0.016%, to be compared with a monthly average tracking error for the Lyxor LETF equal to -0.0104%. The volatility of the tracking error is quite acceptable, with a monthly figure of 0.646% (annual figure of 2.178%). This is higher than the result obtained by the previous authors, but our data sample does contain a turbulent financial periods. Over the whole management period, this fund slightly underperforms, by -0.58%, the CAC 40 Leverage Index. The under-performance is equal to 0.25% per year.

As for the SGAM LETF, it has an average annualized tracking error of -0.647%. Over the whole management period, it outperforms, by 3.078%, the CAC 40 Leverage Index[26] (i.e. an outperformance of 0.653% per year). Thus, it has been able to yield excess performance. Recall that the SGAM fund is not designed to replicate the CAC 40 Leverage Index and therefore shows a daily tracking error 5.5 times higher than that of the Lyxor LETF. Its tracking-error volatility is also quite high. However on average, the SGAM LETF has been able to outperform the Lyxor LETF.

Results on tracking error confirms the distinct empirical behavior of these two funds arising from their different objective and feature.

5 Concluding remarks

In this paper, we show that a leveraged ETF can be analyzed as a convex constant allocation portfolio strategy. We analyze some of its main properties. In particular, we illustrate how the stock index can experience an increase while at the same time the leveraged fund decreases, calculating the explicit probability of such an event for the GBM case. We also establish that, in continuous-time, a CPPI portfolio with a floor proportional to the value of the portfolio itself is a convex constant allocation portfolio strategy. Taking a leveraged ETF issued by SGAM and managed under a slightly modified CPPI strategy that we call Leveraged CPPI, we propose a quasi explicit expression for its value. Comparisons based on Monte Carlo simulations allow us to point out the similarities in behavior of a standard leveraged ETF and of a Leveraged CPPI strategy. An empirical study of

[26] Note that there is an outperformance on the whole period while we find an under-performance on average. This comes from the fact that the tracking error is computed as an arithmetic average of a difference of a simple return whereas the return over the whole period would be obtained as a geometric average.

the SGAM fund and of a standard CAC 40 LETF issued by Lyxor allows us to highlight differences in their behavior and performance.

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Appendix A. CPPI in discrete-time

Suppose that the investor trades in discrete-time. Denote by X_k the arithmetical return of the risky asset between times t_{k-1} and t_k. We have:

$$X_k = \frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}}. \tag{18}$$

Denote by V_k the portfolio value at time t_k.

Standard CPPI

The guarantee constraint is to keep the portfolio value V_k above the floor F_k whose dynamics is given by:

$$F_{k+1} = F\left(1 + r_k\right), \tag{19}$$

where the riskless asset on the time period $[t_k, t_{k+1}]$ is denoted by r_k. The exposure e_k (amount invested on the risky asset S_k) is equal to mC_k, where the cushion value C_k is equal to the difference $V_k - F_k$. The remaining amount $(V_k - e_k)$ is invested on the riskless asset. Therefore, the dynamic evolution of the portfolio value is given by:

$$V_{k+1} = V_k + e_k X_{k+1} + (V_k - e_k) r_{k+1}, \tag{20}$$

where the cushion value is defined by:

$$C_{k+1} = C_k \left[1 + m\ X_{k+1} + (1-m) r_{k+1}\right], \tag{21}$$

which implies:

$$C_k = C_0 \prod_{1 \le l \le k} \left[1 + m\ X_l + (1-m) r_l\right]. \tag{22}$$

Leveraged CPPI

In the introductory example, a modified version of the standard CPPI method, a Leveraged CPPI (LCPPI), is used. This method has a floor that is constant over time that is $F_k = F_0 = \xi V_0$, $\forall k$. The parameter ξ corresponds to the initial proportion of the portfolio value that is designed to be guaranteed. However, contrary to usual portfolio insurance, the initial exposure to the risky asset is significantly higher than the portfolio value (200% for example), such that a leveraged portfolio is obtained. Then a maximal bound equal to the initial exposure, 200% for example, is set on the risky asset exposure. This bound named the leverage is denoted by λ. Therefore, the exposure is now given by:

$$e_k = Min\left[m \times (V_k - \xi V_0); \lambda V_0\right] \tag{23}$$

Note that, for the first subperiod, if we set $\lambda \geq m(1-\xi)$, we have:

$$e_0 = m \times (V_0 - \xi V_0) = LV_0, \tag{24}$$

with

$$L = m(1-\xi).$$

Thus, in that case, the LETF and the LCPPI are identical. For instance, we can take:

1) $m = 4, \xi = 1/2$ and $\lambda = 2$. This corresponds to LCPPI fund in Section 2.

2) $m = 8, \xi = 3/4$ and $\lambda = 2$. This corresponds to LCPPI 1 fund in Section 2.

They differ from the second subperiod since for the LETF the amount is proportional to portfolio value at time 1 ($e_1 = LV_1$) while for the LCPPI the amount is still partly based on the portfolio value at time 0 ($e_1 = Min\left[m \times (V_1 - \xi V_0); \lambda V_1\right]$). All these features are illustrated in Table 1.

Appendix B. Static leveraged strategy

The static leveraged portfolio strategy is a buy-and-hold strategy that aims to reproduce the leveraged performance of the stock index over a given period without continuous-time rebalancing (for practical purposes, daily). This can be considered as a benchmark for the sake of comparison. Let us denote V_t^{SL} the value process of this strategy where L is still the leverage coefficient of the initial portfolio value. This initial portfolio value is split into two amounts:

$$V_0^{SL} = LV_0^{SL} + (1-L)V_0^{SL}, \tag{25}$$

where LV_0^{SL} is the amount (or "exposure") invested on the risky asset S and $(1-L)V_0^{SL}$ is the amount invested on the riskless asset B.

Therefore, at any time t, the static leveraged portfolio value is given by:

$$\frac{V_t^{SL}}{V_0^{SL}} = L\left(\frac{S_t}{S_0}\right) - (L-1)e^{rt}, \tag{26}$$

which implies (for short horizon):

$$\frac{V_t^{SL} - V_0^{SL}}{V_0^{SL}} \simeq L\left(\frac{S_t - S_0}{S_0}\right). \tag{27}$$

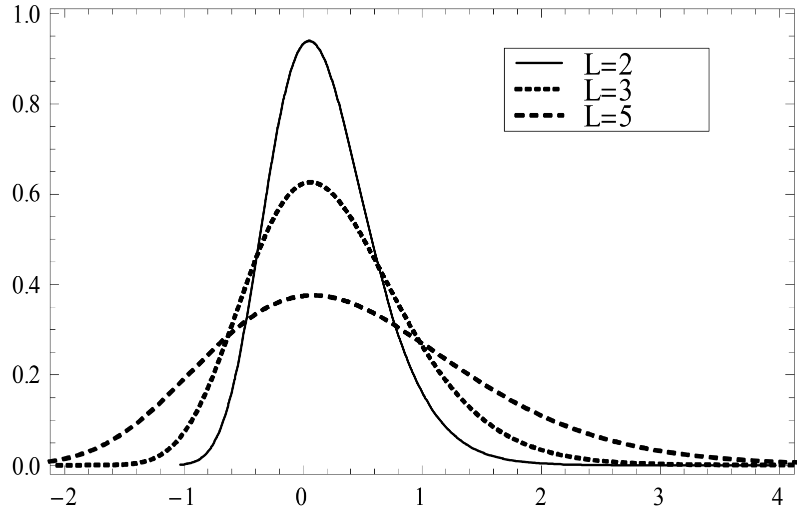
In that case, the (arithmetic) rate of return of portfolio value is approximately equal to the rate of risky asset return multiplied by the leverage coefficient L.

More precisely, under assumption (2) on the risky asset dynamics, we deduce:

$$\frac{V_t^{SL}}{V_0^{SL}} = L\exp\left[\int_0^t \left(\mu_s - \frac{1}{2}\sigma_s^2\right) ds + \int_0^t \sigma_s dW_s\right] - (L-1)e^{rt}. \tag{28}$$

Thus, for the GBM case, the random variable $X = \frac{V_t^{SL}}{V_0^{SL}} + (L-1)e^{rt}$ has a Lognormal distribution $Ln\mathcal{N}\left(Log\left(L\right) + (\mu - \frac{1}{2}\sigma^2)t, \sigma\sqrt{t}\right)$. The probability density function of $\frac{V_t^{SL}}{V_0^{SL}} - 1$ is plotted in Figure 8 for three different levels of leverage ($L = 2$, 3 and 5):

Figure 8: Pdf of $\left(\frac{V_t^{SL}}{V_0^{SL}} - 1\right)$ $(t = 0.5)$



6 Appendix C. Quasi-explicit solution for Leveraged CPPI fund

Assume now that the portfolio manager trades in continuous-time during each period $[t_n, t_{n+1}[$. Suppose also that the CPPI strategy is not capped. Then, the portfolio value can be determined by induction as follows:

Recall that the standard CPPI strategy, with a deterministic floor F_t, induces a portfolio value defined by:

$$V_t^{CPPI} = F_0.e^{rt} + \alpha_t. \left(\frac{S_t}{S_0}\right)^m, \tag{29}$$

$$\text{where} \quad \alpha_t = C_0 \exp\left[\beta_t\right] \quad \text{and} \quad \beta_t = r(1-m)t - (m^2 - m)\int_0^t \frac{\sigma_s^2}{2} ds.$$

Consequently, applying relation (29) for each management period $[t_n, t_{n+1}[$, we can determine the portfolio value of the leveraged CPPI:

Denote for all $s < t$,

$$\alpha_{s,t} = C_0 \exp\left[\beta_{s,t}\right] \quad \text{and} \quad \beta_{s,t} = r(1-m)(t-s) - (m^2 - m)\int_s^t \frac{\sigma_u^2}{2} du.$$

We have:

$$\begin{aligned}
\text{For } t &\in [0, t_1[,\ V_t = F_0 e^{rt} + \alpha_t \left(\frac{S_t}{S_0}\right)^m, \\
\text{For } t &\in [t_1, t_2[,\ V_t = \xi V_{t_1} e^{r(t-t_1)} + \alpha_{t_1,t} \left(\frac{S_t}{S_{t_1}}\right)^m, \\
\text{with } \alpha_{t_1,t} &= (1-\xi) V_{t_1} \exp\left[\beta_{t_1,t}\right]. \\
\text{For } t &\in [t_n, t_{n+1}[,\ V_t = \xi V_{t_n} e^{r(t-t_n)} + \alpha_{t_n,t} \left(\frac{S_t}{S_{t_n}}\right)^m, \\
\text{with } \alpha_{t_n,t} &= (1-\xi) V_{t_n} \exp\left[\beta_{t_n,t}\right].
\end{aligned}$$

Then, we deduce: If the leveraged CPPI is continuously rebalanced and not capped, then its value is given by: for any $t \in [t_n, t_{n+1}[$,

$$\begin{aligned}
V_t &= V_0 \left[\prod_{1 \le k \le n} \left(\xi e^{r(t_k - t_{k-1})} + (1-\xi) e^{\beta_{t_{k-1},t_k}} \left(\frac{S_{t_k}}{S_{t_{k-1}}}\right)^m\right)\right] \\
&\quad \times \left(\xi e^{r(t-t_n)} + (1-\xi) \exp\left[\beta_{t,t_n}\right] \left(\frac{S_t}{S_{t_n}}\right)^m\right).
\end{aligned} \tag{30}$$

Consider for instance the geometric Brownian case. Assume also to simplify that durations $(t_{n+1} - t_n)$ are equal to a constant δ and that portfolio maturity T is equal to t_N. Then, the ratio $\frac{S_{t_k}^m}{S_{t_{k-1}}^m}$ is given by:

$$\left(\frac{S_{t_k}}{S_{t_{k-1}}}\right)^m = \exp\left(m\left[(\mu - 1/2\sigma^2)\delta + \sigma\left(W_{t_k} - W_{t_{k-1}}\right)\right]\right).$$

i) The portfolio value V_T has a quasi-explicit form:

$$V_T = V_0 \left[\prod_{1 \leq n \leq N} \left(\xi e^{r\delta} + (1-\xi)e^{\beta_\delta} \frac{S_{t_n}^m}{S_{t_{n-1}}^m}\right)\right]. \tag{31}$$

Note that the rate of return on the period $[t_n, t_{n+1}[$ is given by:

$$(1-\xi)\exp\left[\beta_\delta\right] \frac{S_{t_n}^m - S_{t_{n-1}}^m}{S_{t_{n-1}}^m} + \xi e^{\beta_\delta} - 1.$$

ii) The probability distribution of portfolio return can be determined as follows: The random variables $\left(\frac{S_{t_n}^m}{S_{t_{n-1}}^m}\right)$ are *i.i.d.* with Lognormal distribution. Thus, the logreturn of the portfolio $\ln(V_T/V_0)$ is the sum of N *i.i.d.* random variables. Their common probability distribution is an affine transformation of the Lognormal distribution with parameters $\left(m\left[(\mu - 1/2\sigma^2)\delta;\ m\sigma\sqrt{\delta}\right]\right)$. Thus, the pdf of the logreturn of the portfolio $\ln(V_T/V_0)$ is given by[27]:

$$f_{\ln(V_T/V_0)}(x) = *_{1 \leq n \leq N}\ g_n(x), \tag{32}$$

where $g_n(x)$ is given by:

$$g_n(x) = \frac{1}{\sqrt{2\pi}} \times \frac{1}{m\sigma\sqrt{\delta}} \exp\left[-\frac{\left(\ln\left[\frac{x-\xi}{1-\xi}\right] - m(\mu - 1/2\sigma^2)\delta\right)^2}{2m^2\sigma^2\delta}\right] \mathbb{I}_{x>\xi}.$$

[27] The symbol $*$ denotes the convolution product.