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ORIGINAL PAPER

Optimization of covered call strategies

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Abstract We present a risk-return optimization framework to select strike prices and quantities of call options to sell in a covered call strategy. Covered calls of a general form are considered where call options with different strike prices can be sold simultaneously. Tractable formulations are developed using variance, semivariance, VaR, and CVaR as risk measures. Sample expected return and sample risk are formulated by simulating the price of the underlying asset. We use option market price data to perform the optimization and analyze the structure of optimal covered call portfolios using the S&P 500 as the underlying. The optimal solution is shown to be directly linked to the options' call risk premiums. We find that from a risk-return perspective it is often optimal to simultaneously sell call options of different strike prices for all risk measures considered.

Keywords Portfolio optimization · Call options · Covered call writing

1 Introduction

Covered call or buy-write strategies involve shorting a European call option until maturity while holding a long position in the underlying asset. Asset return beyond the strike price of the shorted call is sacrificed in exchange for gaining the call

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premium. The strategy always provides greater returns than the long position when the option is not exercised. The premium also provides a small buffer in down-side cases, which makes covered calls inherently less risky than the long asset position. For an investor's existing portfolio covered calls are an attractive overlay particularly in bearish conditions.

Typical covered call strategies short calls with a nearby maturity date and marginally out-of-the-money strike price. In recent times literature has emerged exploring strategies with further out-of-the-money strike prices, ranging probabilities of exercise, and farther maturity dates. However, to date the literature has always assumed that the number of shorted calls is equal to the number of units of the asset being held, i.e. that the ratio of calls written to underlying bought or held, the write-buy ratio, is one. Furthermore shorting combinations of calls with different strike prices has not been considered. Shorting combinations of calls or single calls with a write-buy ratio less than one may allow for superior risk-return profiles. The aim of this work is to explore covered call portfolios of this general form and to obtain them using risk-return optimization.

2 Covered call history and recent studies

In 2002, the Chicago Board Options Exchange (CBOE) introduced the CBOE S&P 500 BuyWrite Index. It is designed to track the performance of a covered call strategy where the underlying is the S&P 500 index. When shorted, the calls have approximately a month until maturity and an at-the-money (ATM) strike price. Details about the index can be found in CBOE [3]. Robert Whaley, who was commissioned to design the index, shows in Whaley [11] that from 1988–2001 the index provided nearly the same return as the S&P 500 while having a monthly standard deviation of returns one third lower than that of the S&P 500. Feldman and Roy [5] and Callan Associates [1] repeat Whaley's analysis and draw similar conclusions. While these findings show support for the use of covered calls versus long positions, they only consider ATM call options and a write-buy ratio of one.

McIntyre and Jackson [10] evaluate covered calls for a variety of maturity dates, strike prices, and assets. They contend that covered calls ought to perform poorly from theoretical perspectives but highlight their strong empirical performance. No conclusions or suggestions about optimal strike prices and maturity dates are drawn, and the write-buy ratio is assumed to be one.

Figelman [6] aims to build a framework for evaluating covered calls. He highlights the negative relationship between the covered call return and the call risk premium (CRP). From his analysis Figelman concludes that short-dated options have a lower CRP and are therefore preferable for covered call strategies. Figelman does not make conclusions about optimal strike prices and only considers a write-buy ratio of one.

He et al. [8] test covered call strategies using different maturity dates and strike prices with the S&P 500 as the underlying asset. They obtain their best empirical result from shorting ATM calls with 3 months maturity dates. The portfolio is always fully covered and shorting combinations of options is not explored.

Yang [12] forms a dynamic strategy using technical analysis which holds a covered call during bearish conditions and a put-write during bullish conditions. One month to maturity ATM options are used. The put-write is designed to address the poor performance of typical covered call strategies during bull markets. This is one of the goals we also address though our scope is limited to covered calls.

Hill et al. [9] explore covered calls using the S&P 500 Index and 1-month call options of varying strike prices. They test dynamic strategies where the strike price is set according to the volatility of the underlying asset. They conclude by recommending, based on historical performance, options with less than a 30 % probability of exercise or with a strike price at least 2 % out-of-the-money. Che and Fung [2] perform a similar analysis with Hang Seng Index futures as the underlying. They conclude that the performance of the strategies depends greatly on market conditions, and that for the Hong Kong market the dynamic strategy may provide a benefit in bullish conditions. Though several strike prices are considered, combinations of options are not explored and write-buy ratios are always one.

We note that the current literature has analyzed the effect of various call maturity dates and strike prices. The majority of the published evidence suggests that it is preferable to short call options with 1 month to maturity, though there is no consensus regarding the strike price. Varying the proportion of calls being sold and the possibility of shorting a combination of different calls have not been considered. The focus of this work is to develop an optimization framework that provides combinations of calls with 1 month until maturity for a covered call strategy. This framework is tested by forming covered calls using the S&P 500 Index to explore the risk-return profile and structure of the resulting portfolios. Return variance, semivariance, Value-at-Risk, and Conditional Value-at-Risk are used as risk measures. This framework is new in that it considers selling less options than units of the underlying held, it considers selling combinations of different strike prices, and it selects strike prices and quantities of options to sell based on risk-return optimality.

3 Methodology

Suppose an investor wishes to create a covered call overlay on an existing position. The investor is long n units of an asset with current price S_0. The investor can form a covered call strategy using this asset by shorting any of N_c European call options with maturity date T days away and varying strike prices. Up to n call options in total can be shorted beyond which some calls are not covered. The decision variable p is a vector of write-buy ratios where we short $p_j n$ units of call option j which has strike price k_j and best current market bid C_j. We consider only options which are not in-the-money, i.e. options whose strike prices are above the current asset price S_0. We assume n is large enough that we can ignore the option contract size and treat p as continuous. We assume that proceeds gained from shorting the call option are invested into the risk-free asset with a constant interest rate. An alternative is to put this money against the cost of the underlying asset, but as in Figelman [6] we find that the difference is negligible. We obtain p by solving the following optimization problem:

$$
\begin{aligned}
&\underset{p}{\text{minimize}} && \lambda(risk) - (1-\lambda)\mathbb{E}(r) \\
&\text{subject to} && \sum_{j=1}^{N_c} p_j \leq 1 \\
& && 0 \leq p_j \leq 1, \quad j = 1, \ldots, N_c
\end{aligned}
$$

where:

- $\lambda \in [0, 1]$ represents risk aversion.
- r is the return of the covered call strategy.
- $risk$ is one of: variance of the return, semivariance of the return, Value-at-Risk (VaR), or Conditional Value-at-Risk (CVaR).

For maturity price S_T the return of the covered call is given by:

$$
\begin{aligned}
r &= \frac{nS_T + \eta D - nS_0 + \sum_{j=1}^{N_c} p_j n(C_j e^{r_f T} - \max(S_T - k_j, 0))}{nS_0} \\
&= \frac{S_T + \frac{\eta}{n} D - S_0 + \sum_{j=1}^{N_c} p_j (C_j e^{r_f T} - \max(S_T - k_j, 0))}{S_0}
\end{aligned}
$$

where r_f is the continuously compounded risk-free rate, D is the value of the dividend paid out grown at the risk-free rate, and η is the number of units of the asset held on the dividend record date. Even if they are tractable for some price models, analytical expressions for the expected return and risk of a covered call strategy are highly non-linear and undesirable from an optimization perspective. For examples see Figelman [6] for analytical expressions assuming geometric Brownian motion. We instead simulate the underlying asset price at maturity in scenario i, S_T^i, then formulate the sample returns r_i. The sample returns and expected return can be defined according to the following linear equality constraints, for N simulations:

$$
\begin{aligned}
r_i &= \frac{S_T^i + \frac{\eta}{n} D - S_0 + \sum_{j=1}^{N_c} p_j (C_j e^{r_f T} - \max(S_T^i - k_j, 0))}{S_0}, \quad i = 1, \ldots, N \\
\mathbb{E}(r) &= \frac{1}{N} \sum_{i=1}^{N} r_i
\end{aligned}
$$

We address each of the risk measures separately. The sample variance of the simulated values is given by:

$$
\sigma_r^2 = \frac{1}{N-1} \sum_{i=1}^{N} (r_i - \mathbb{E}(r))^2
$$

This formulation has a quadratic objective, which with our linear constraints can be solved rapidly even for a large number of simulations.

The sample semivariance below the risk-free rate is given by:

$$semivariance = \frac{1}{N}\sum_{i=1}^{N} z_i^2$$
$$-z_i \le r_i - e^{r_f T}, \quad i = 1, \ldots, N$$
$$z_i \ge 0, \quad i = 1, \ldots, N$$

The latter sets of constraints ensure that $z_i = \min(r_i - e^{r_f T}, 0)$, i.e. that positive deviations are set to zero. An investor can easily replace the risk-free rate with their own specified level. As for variance, this leads to a quadratic optimization problem.

Suppose we are interested in optimizing VaR_α. In general this poses a significant challenge as it may lead to a non-convex problem. In our case, since there is only one source of uncertainty which is the maturity price of the underlying asset, the problem is greatly simplified. Suppose we rearrange the index i of the simulated values S_T^i so that the simulated prices at maturity $S_T^1 \cdots S_T^N$ are sorted from smallest to largest. In Appendix A we show that the return is a monotically increasing function of the asset price at maturity S_T. Thus, if the simulated asset values are sorted, i.e. $S_T^1 \le S_T^2 \le \cdots \le S_T^N$, then the sample return values must also be sorted, i.e. $r_1 \le r_2 \le \cdots \le r_N$. Suppose that the number of simulations, N, is chosen such that $(1-\alpha)N$ is an integer. Then we can directly obtain the $(1-\alpha)$ quantile of returns by looking at r_i where the index $i = (1-\alpha)N$. Since the sample VaR_α is the α quantile of sample losses, we must multiply the $(1-\alpha)$ quantile of sample returns by negative one to convert to a loss:

$$\mathrm{VaR}_\alpha = -r_{(1-\alpha)N}$$

Similarly, if we know the sample returns are sorted, $r_1 \le r_2 \le \cdots \le r_N$, then the sample CVaR_α is given by averaging the $(1-\alpha)N$ lowest returns and multiplying by negative one to convert to a loss:

$$\mathrm{CVaR}_\alpha = \frac{-1}{(1-\alpha)N}\sum_{i=1}^{(1-\alpha)N} r_i$$

Numerical examples of the implementation of VaR and CVaR are given in the implementation section.

In Appendix B we show that all constraints for all risk measures are linear, that sample VaR and CVaR have linear objective functions, and that sample variance and semivariance have quadratic objective functions. Additionally, in Appendix C we show that the quadratic objectives are convex, so that minimizing them is tractable. All formulations are thus of the simplest classes of optimization problems, namely quadratic and linear programs. These formulations are rapidly solvable even for large numbers of available call options and large numbers of simulations.

4 Policy structure

To understand the structure of the optimal policy we consider the case of minimal risk aversion, $\lambda = 0$. The objective is then to maximize the expected return. By substituting the simulated return values into the expected return the problem becomes:

$$\underset{\sum p_j \leq 1,\ p \geq 0}{\text{maximize}} \quad \frac{1}{N}\sum_{i=1}^{N} \frac{S_T^i + \frac{\eta}{n}D - S_0 + \sum_{j=1}^{N_c} p_j(C_j e^{r_f T} - \max(S_T^i - k_j, 0))}{S_0}$$

Ignoring constants gives:

$$\underset{\sum p_j \leq 1,\ p \geq 0}{\text{maximize}} \quad \frac{1}{N}\sum_{i=1}^{N} \frac{\sum_{j=1}^{N_c} p_j(C_j e^{r_f T} - \max(S_T^i - k_j, 0))}{S_0}$$

$$\underset{\sum p_j \leq 1,\ p \geq 0}{\text{maximize}} \quad \frac{\sum_{j=1}^{N_c} p_j \left(C_j e^{r_f T} - \frac{1}{N}\sum_{i=1}^{N} \max(S_T^i - k_j, 0)\right)}{S_0}$$

For large N the simulated mean call payoff is approximately equal to the expectation:

$$\underset{\sum p_j \leq 1,\ p \geq 0}{\text{maximize}} \quad \frac{\sum_{j=1}^{N_c} p_j(C_j e^{r_f T} - \mathbb{E}(\max(S_T^i - k_j, 0)))}{S_0}$$

$$\underset{\sum p_j \leq 1,\ p \geq 0}{\text{maximize}} \quad \sum_{j=1}^{N_c} -p_j\ CRP_j$$

The coefficient of p_j, the future value of the call price minus the expected call payoff as a fraction of the asset price, is the negative of the call risk premium (CRP) defined in Figelman [6] for option j. Maximizing the expected return is equivalent to minimizing the CRP of options sold. If according to our simulations the risk premia of all available calls are positive then $p = 0$, selling no options, maximizes the expected return. If there is a call with a negative risk premium then $p_j = 1$ maximizes the expected return where option j has the lowest CRP. In theory, the CRP should always be positive since investors expect to be compensated for risk. However, as noted in Figelman [6], implied volatility has often been higher than realized volatility. This implies that some options had negative risk premiums and were effectively overpriced.

Selling calls produces a liability which reduces up-side returns and provides a premium that increases down-side returns. Call options with high premiums and low strike prices provide the largest down-side benefit and largest up-side liability making them most effective in minimizing risk. Therefore, to minimize variance, semivariance, VaR, or CVaR, it is optimal to set $p_j = 1$ where option j has the lowest strike price of the options available. Thus, if we are not considering options which are in-the-money, a sufficiently risk averse investor always ought to sell the maximum amount of ATM options.

All risk measures lead to the same optimal solution in the case of either minimal or maximal risk aversion. From an analytical perspective it is not clear what the optimal mix should be for intermediate risk aversion settings. However, it is clear that the balance between risk and return is equivalent to the balance between minimizing risk and minimizing the CRP of options sold. Intermediate risk aversion cases are explored experimentally for all four risk measures under consideration.

5 Implementation

We use the S&P 500 index as the underlying asset of our covered call for the period from 1996 to 2014.

To utilize our formulation we must produce simulations of the S&P 500, this requires a model for the price path. Eraker et al. [4] argue that, versus a number of alternatives, their stochastic volatility with correlated jumps (SVCJ) model best explains the price path of the S&P 500 index in their examined time period of 1980–1999. The SVCJ model is similar to vanilla geometric Brownian motion for prices but with a few additional features. Volatility is modelled as a mean reverting stochastic process which has noise which is negatively correlated to noise in the return. There are positive jumps in volatility which are negatively correlated with jumps in returns. This is consistent with market dynamics; in times of stress there are sharp increases in volatility and large drops in price.

Figelman [7] shows that compared to geometric Brownian motion the SVCJ model produces similar expected call payoffs for ATM and near the money options, but may produce significantly different expected call payoffs (and thus CRPs) for further out-of-the-money options. Following Figelman [7] we employ the SVCJ model using the parameter estimates provided for the S&P 500 in Eraker et al. [4]. We adjust the parameter estimate of μ from the SVCJ model so that the expected annual return matches the long run historical average return of the S&P 500 of about 6.5 % annually. We do not adjust the volatility parameters as the expected volatility of 16 % annually is close to the long run historical volatility.

While the SVCJ model holds strong explanatory power, it may not hold strong predictive power beyond the time period for which the parameters were calibrated. We employ the model and parameter estimates from Eraker et al. [4] for the purposes of examining the structure of optimal covered calls. In practice an investor seeking performance should employ a predictive model and should continually update their estimates using newly available information. An investor can readily substitute maturity prices simulated according to their preferred price model into our formulations.

We adjust some parts of our optimization formulation to account for the effectively continuous dividends of the S&P 500. The return of the S&P 500 gross of dividends, S_T^G, was modelled as the return of the S&P 500 index plus a logarithmic return of 7E−5 per day. This is consistent with the approximately 2 % annual return provided by dividends. Where S_T appears outside of the call settlement formula, it is replaced with S_T^G and the dividend D is set to zero. The constraint for return r_i becomes:

$$r_i = \frac{S_T^{Gi} - S_0 + p(C\mathrm{e}^{r_f T} - \max(S_T^i - k, 0))}{S_0}, \quad i = 1, \ldots, N$$

All other constraints and objectives were unchanged and implemented as presented in Sect. 3.

We closely follow the methodology used to construct the CBOE BuyWrite Index with a few noted differences. Call option data was obtained from OptionMetrics which only provides closing prices. Thus, the call premium was taken to be the best closing bid. Lastly, we consider only out-of-the-money options, thus the option with the lowest strike price above the current asset price was taken as the at-the-money option. Each month on the dates where the BuyWrite Index shorts the at-the-money call (the date on which the previous call expires) we instead perform our optimization over all available out-of-the-money (OTM) options to solve for optimal write-buy ratios. Since we are only considering 1 month maturity dates, the 30 days US treasury rate is used as the risk-free rate.

Each formulation used $N = 10{,}000$ simulations of the asset price at maturity. For all risk measures this creates 10,000 variables and equality constraints to model r_i. For semivariance this produces an additional 10,000 variables and inequality constraints to model z_i. Additional variables are required for the write-buy ratios of the available options, usually less than fifty. One additional constraint is required to ensure that the sum of write-buy ratios does not exceed one. Lastly one constraint is required to define the expected return variable. Since the scenario return variables r_i and the expected return variable $\mathbb{E}(r)$ are defined by linear equality constraints, it is possible to eliminate them from the formulation by substituting them with their defining expressions. In practice there is no motivation for doing so since such variables introduce very little overhead from an efficiency perspective. All formulations were implemented and solved using CPLEX.

When optimizing VaR and CVaR we select $\alpha = 95\,\%$. Before optimizing, we rearrange the arbitrary index i of the simulated asset prices at maturity S_T^i so that the prices are arranged in ascending order as described in the methodology section. Since $N = 10{,}000$, $\text{VaR}_{95\%}$ is given by $-r_{500}$. Similarly, $\text{CVaR}_{95\%}$ is given by multiplying the average of the 500 worst cases by negative one, i.e. $\frac{-1}{500}\sum_{i=1}^{500} r_i$.

6 Results

In all tested cases the solution time using CPLEX did not exceed one second. The production of efficient frontiers of 20 to 30 points usually required less than 10 s.

For minimal and maximal risk aversion the optimal solution follows the predicted policy structure. We find that for intermediate risk aversion there is a non-trivial optimal mix involving several options in most months. Example efficient frontiers and optimal mixes for all risk measures can be seen in Fig. 1. The left endpoints of the frontiers correspond to high risk aversion. On the left sides of the mix charts we see that for high risk aversion it is optimal to sell the maximum number of ATM calls. The right endpoints of the frontiers correspond to maximizing the expected return. We see in the mix charts that for low risk aversion or risk-neutrality it was optimal in this example to sell the maximum amount of the 9.4 % OTM option, i.e. the option whose strike price was 9.4 % higher than the asset's price at the time it was sold. This is because the 9.4 % OTM option had a negative simulated CRP value which was lowest among

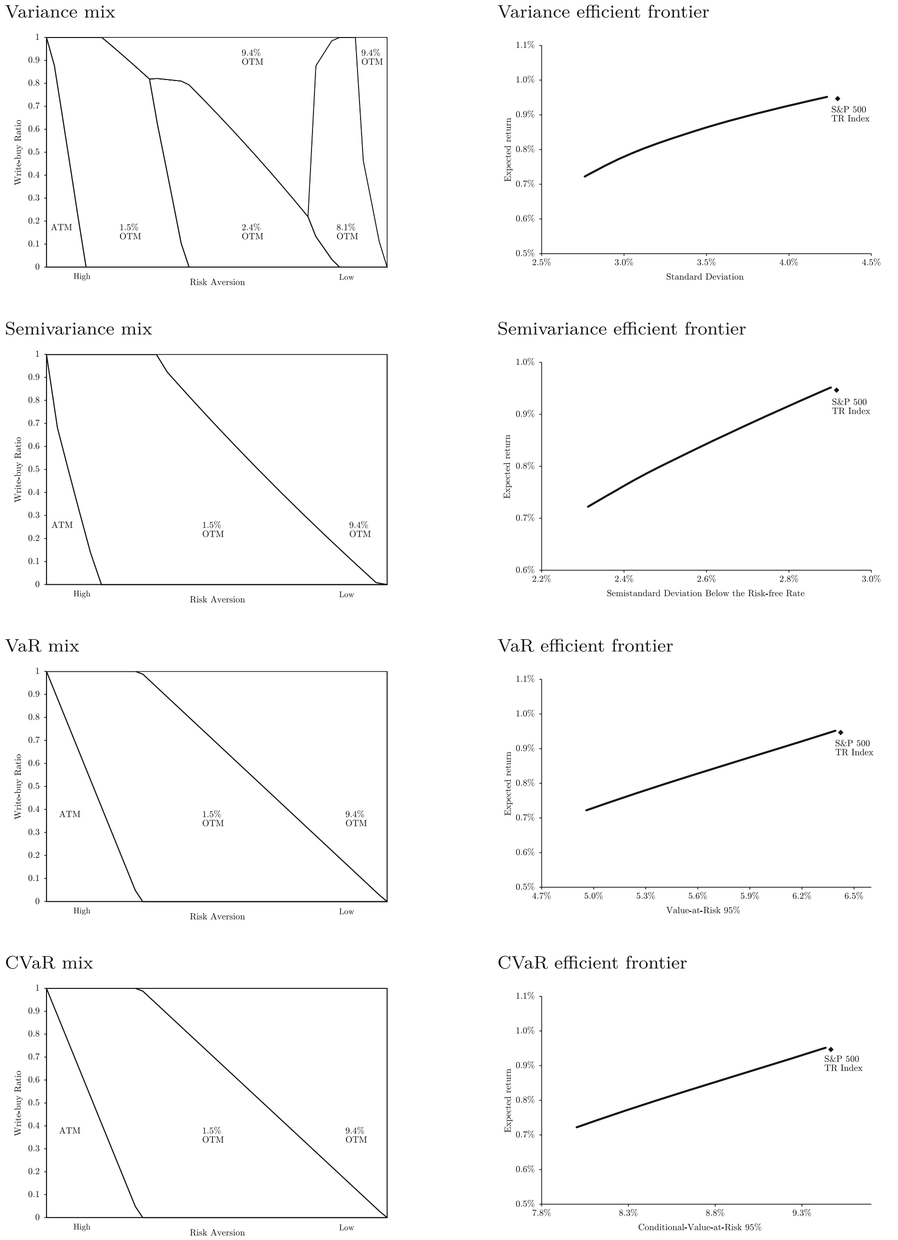

Fig. 1 Sample optimal mixes and frontiers for different risk measures in the same month using the same simulations

all options examined in this particular month. The existence of this slightly negative CRP allows us to achieve a slightly greater expected return than the long position, as seen in the frontiers. When using vanilla geometric Brownian motion to simulate the underlying instead of the SVCJ model, we found portfolio structures similar to Fig. 1.

In the literature there is no consensus on how to best select strike prices of options to sell. Our methodology selects strike prices of options to sell based on risk-return optimality. Intermediate risk aversion values lead us to sell a blend of the ATM, 1.5 % OTM, 2.4 % OTM, 8.1 % OTM, and 9.4 % OTM options. Although there were many other options which could have been sold in this month the optimal mixes contain only these five. In this particular month these options were more efficient from a risk-return perspective than others due to their low CRP values.

Although the current literature has only examined covered calls where a single call option is sold in a one to one ratio to the underlying asset, our results suggest that from a risk-return perspective it is often optimal to sell combinations of call options with different strike prices for intermediate risk aversion. This result is relevant since in practice investors tend to have intermediate risk aversion since maximum return portfolios might have excessively high risk while minimum risk portfolios might provide very poor returns.

We observe that the sample mixes resulting from optimizing semivariance, VaR, and CVaR are all similar, but differ substantially versus variance. This is likely since the first three are all down-side risk measures, while the latter penalizes up-side deviations. The differences in mix observed in Fig. 1 highlight the importance of deciding between variance and a down-side risk measure given that covered call returns are highly asymmetrical.

The optimal mixes for VaR and CVaR in Fig. 1 are identical. This is a general phenomena which stems from our choice of $\alpha = 95$ % and our choice not to consider selling in-the-money options. In the 95th percentile of losses and beyond none of the options under consideration were in the money. Thus, any improvement to the VaR gained by selling an option and not incurring a liability was also an improvement to the CVaR. Conversely, any improvement to the CVaR also caused an improvement to the VaR. Thus, the problems of optimizing VaR and CVaR for $\alpha = 95$ % were identical and led to identical mixes. This would not hold if there were any options that were in-the-money at the α percentile of losses but still out-of-the-money in worse scenarios which are included in the calculation for CVaR. This could occur if a lower value of α were selected, if in-the-money options were considered for sale, or if the price model simply produced scenarios such that the α percentile of losses contained in-the-money options.

7 Conclusion

For all tested risk measures we find that from a risk-return perspective it is often optimal to sell a mix of options with different strike prices. This stands in contrast to current literature which has only examined covered calls with one option shorted with a write-buy ratio of one. While the literature has not reached a consensus on how to select the strike prices of the options to be sold, our methodology selects strike prices and quantities on the basis of risk-return optimality. We have shown that a sufficiently risk averse investor ought to sell the full amount of ATM calls as in the BuyWrite index in order to minimize the risk measures under consideration, while a risk-neutral investor need only concern themself with assessing the CRP of available options in order to maximize expected return. Our methodology can be used to obtain optimal risk-return covered calls in the nontrivial case of intermediate risk aversion. The optimization formulations have few constraints and are thus easy to implement. Since all constraints are linear and the objectives are linear or quadratic they are rapidly solvable for all risk measures considered. While we used the SVCJ model to explore the structure of optimal covered calls, the performance of the methodology could be tested by simulating maturity prices using different predictive price models. In this work we have considered optimizing covered calls as an overlay to an existing position in a single asset. Future work could build upon this framework by simultaneously optimizing covered calls for multiple assets as well as their underlying positions.

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Appendix A

Here we provide proof that the return r is a monotonically increasing function of the asset price at maturity S_T. This permits the convenient formulations for VaR and CVaR seen in the methodology section.

Consider the return of the strategy at maturity as a function of the asset price at maturity:

$$r(S_T) = \frac{1}{S_0}\left(S_T + \frac{\eta}{n}D - S_0 + \sum_{j=1}^{N_c} p_j(C_j e^{r_f T} - \max(S_T - k_j, 0))\right)$$

Now consider the impact of any increase $\delta > 0$ in the asset price at maturity:

$$\begin{aligned}
&r(S_T + \delta) - r(S_T)\\
&\quad = \frac{1}{S_0}\left(S_T + \delta + \frac{\eta}{n}D - S_0 + \sum_{j=1}^{N_c} p_j(C_j e^{r_f T} - \max(S_T + \delta - k_j, 0))\right)\\
&\qquad - \frac{1}{S_0}\left(S_T + \frac{\eta}{n}D - S_0 + \sum_{j=1}^{N_c} p_j(C_j e^{r_f T} - \max(S_T - k_j, 0))\right)\\
&\quad = \frac{1}{S_0}\left(\delta + \sum_{j=1}^{N_c} p_j(\max(S_T - k_j, 0) - \max(S_T + \delta - k_j, 0))\right)
\end{aligned}$$

The lowest possible value of $(\max(S_T - k_j, 0) - \max(S_T + \delta - k_j, 0))$ is $-\delta$ when $S_T \geq k_j$:

$$r(S_T + \delta) - r(S_T) \geq \frac{1}{S_0}\left(\delta + \sum_{j=1}^{N_c} p_j(-\delta)\right)$$

Since we have the constraint $0 \leq \sum p_j \leq 1$, the lowest possible value of $\sum p_j(-\delta)$ is $-\delta$.

$$r(S_T + \delta) - r(S_T) \geq \frac{1}{S_0}(\delta - \delta)$$
$$r(S_T + \delta) \geq r(S_T)$$

Since the above holds for any $\delta > 0$ and any value of S_T, the return is a monotonically increasing function of the asset price at maturity.

Appendix B

Here we show that our VaR and CVaR optimization problems are linear programs and our variance and semivariance optimization problems are quadratic programs. The decision variables are p, r, $\mathbb{E}(r)$, and z; all other symbols represent known constants. Our formulations have the following objectives and constraints:

Variance optimization:

$$\underset{p,r,\mathbb{E}(r)}{\text{minimize}} \quad \lambda\left(\frac{1}{N-1}\sum_{i=1}^{N}(r_i - \mathbb{E}(r))^2\right) - (1-\lambda)\mathbb{E}(r) \tag{1}$$

Semivariance optimization:

$$\underset{p,r,\mathbb{E}(r),z}{\text{minimize}} \quad \lambda\left(\frac{1}{N}\sum_{i=1}^{N} z_i^2\right) - (1-\lambda)\mathbb{E}(r) \tag{2}$$

VaR optimization:

$$\underset{p,r,\mathbb{E}(r)}{\text{minimize}} \quad \lambda\left(-r_{(1-\alpha)N}\right) - (1-\lambda)\mathbb{E}(r) \tag{3}$$

CVaR optimization:

$$\underset{p,r,\mathbb{E}(r)}{\text{minimize}} \quad \lambda\left(\frac{-1}{(1-\alpha)N}\sum_{i=1}^{(1-\alpha)N} r_i\right) - (1-\lambda)\mathbb{E}(r) \tag{4}$$

Common constraints:

$$\sum_{j=1}^{N_c} p_j \leq 1 \tag{5}$$

$$0 \leq p_j \leq 1, \qquad j = 1, \ldots, N_c \tag{6}$$

$$r_i = \frac{S_T^i + \frac{\eta}{n} D - S_0 + \sum_{j=1}^{N_c} p_j (C_j \mathrm{e}^{r_f T} - \max(S_T^i - k_j, 0))}{S_0} \qquad i = 1, \ldots, N \tag{7}$$

$$\mathbb{E}(r) = \frac{1}{N} \sum_{i=1}^{N} r_i \tag{8}$$

Additional constraints for semivariance:

$$-z_i \leq r_i - \mathrm{e}^{r_f T} \qquad i = 1, \ldots, N \tag{9}$$

$$z_i \geq 0 \qquad i = 1, \ldots, N \tag{10}$$

Objectives (1) and (2) can be put into the standard quadratic optimization form: $x^{\mathrm{T}} Q x + c^{\mathrm{T}} x$. For both objectives the vector c contains only zeros with the exception of the coefficient of $\mathbb{E}(r)$ which is $(1 - \lambda)$. Expanding the summation in (1):

$$\begin{aligned}
&\lambda \left(\frac{1}{N-1} \sum_{i=1}^{N} (r_i^2 - 2 r_i \mathbb{E}(r) + \mathbb{E}(r)^2) \right) - (1 - \lambda) \mathbb{E}(r) \\
&= \left(\frac{\lambda}{N-1} \sum_{i=1}^{N} (r_i^2 - 2 r_i \mathbb{E}(r) + \mathbb{E}(r)^2) \right) - (1 - \lambda) \mathbb{E}(r) \\
&= \left(\frac{\lambda N}{N-1} \mathbb{E}(r)^2 + \sum_{i=1}^{N} \frac{\lambda}{N-1} (r_i^2 - 2 r_i \mathbb{E}(r)) \right) - (1 - \lambda) \mathbb{E}(r)
\end{aligned}$$

For objective (1), the matrix Q contains $\lambda N/(N-1)$ in the diagonal corresponding to $\mathbb{E}(r)^2$, $\lambda/(N-1)$ in the diagonals corresponding to r_i^2, $-\lambda/(N-1)$ in the columns and rows corresponding to the cross terms $r_i \mathbb{E}(r)$, and zero elsewhere. The matrix Q for objective (2) contains λ/N in the diagonals corresponding to z_i^2 and zero elsewhere.

Objectives (3) and (4) are in standard linear optimization form: $c^{\mathrm{T}} x$. We assume that α and N are selected so that $(1 - \alpha)N$ is an integer, otherwise this value may not be used as an index for r_i. Noting this, (3) and (4) are evidently linear.

Constraints (5) through (10) can all be put into one of the standard linear forms: $a^{\mathrm{T}} x \leq b$, $a^{\mathrm{T}} x \geq b$, or $a^{\mathrm{T}} x = b$. Constraints (5) and (6) are in linear form. Constraint set (7) can be rearranged into linear form:

$$S_0 r_i + \sum_{j=1}^{N_c} p_j (\max(S_T^i - k_j, 0) - C_j \mathrm{e}^{r_f T}) = S_T^i + \frac{\eta}{n} D - S_0 \quad i = 1, \ldots, N$$

Note that S_T^i is a simulated value which is input to the optimization as a constant, thus the max function is also a constant. Constraint (8) is in linear form if we group the expressions on the left-hand side and equate them to zero. Constraint (9) is in linear form if we move r_i to the left-hand side and note that the exponential term is a constant. Constraint (10) is in linear form.

Since all constraints are linear, the problems of optimizing VaR or CVaR are linear programs, and the problems of optimizing variance of semivariance are quadratic programs. In Appendix C we show that the quadratic terms are convex, so that minimizing them is tractable.

Appendix C

Although our variance and semivariance objectives are quadratic, minimizing a quadratic function is only tractable if it is convex. Here we show that the sample variance and sample semivariance are convex functions.

Consider the sample variance which is a function of the variables r_i and $\mathbb{E}(r)$:

$$\sigma_r^2 = \frac{1}{N-1}\sum_{i=1}^{N}(r_i - \mathbb{E}(r))^2$$

Consider the Hessian of the function $(r_i - \mathbb{E}(r))^2$ for a fixed value of i:

$$\mathrm{H} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

This Hessian has eigenvalues of 4 and 0, and is thus positive semi-definite. Therefore the function $(r_i - \mathbb{E}(r))^2$ is convex for each i. Since the sample variance is a sum of convex functions multiplied by a positive scalar, it is also a convex function.

Similarly, consider the sample semivariance which is a function of z_i:

$$semivariance = \frac{1}{N}\sum_{i=1}^{N} z_i^2$$

For each i the function z_i^2 is convex. Sincle the sample semivariance is a sum of convex functions multiplied by a positive scalar it is also a convex function.

Since the sample variance and semivariance are convex, the objective of minimizing them in a quadratic program is tractable.

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