Do Leveraged and Inverse ETFs Converge to Zero?

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New leveraged exchange-traded funds (ETFs) such as the Proshares Ultra S&P 500 (SSO) and the more aggressive Direxion Large Cap Bull 3x Shares (BGU), which aim to deliver two and three times the daily return of their respective indices, seemingly offer an attractive way for investors to take concentrated bets on a particular sector. However, as documented elsewhere (Mackintosh [2008]; Trainor and Baryla [2008]), the longer-term performance of these funds can be undesirable. Specifically, observed returns for 2x leveraged ETFs over longer time periods tend to be less than two times the return of the underlying index. These previous studies have demonstrated through Monte Carlo simulation that the geometric nature of returns compounding results in highly skewed logarithmic distributions of ending portfolio values. These studies also suggest that continuous rebalancing to maintain constant leverage—what Trainor and Baryla call the "constant leverage trap"—may play a role in the poor longer-term performance of these ETFs.

Rebalancing raises red flags, because in order to maintain an ETF's 2x leverage, the fund must buy more of the underlying index after it rises and sell more after it falls. To illustrate, assume that on day one an investor (maybe a leveraged ETF portfolio manager) has an account worth $1,000 and institutes 2x leverage by buying $2,000 worth of an index. If on the next day the index increases in price by 10%, the investor's account will be worth $1,200 and the index holdings will be worth $2,200, leaving him with 1.83x leverage. To maintain 2x leverage, the investor must now purchase an additional $200 of the index at the now higher price. If markets do not trend in one direction or the other, this continuous buying high and selling low will tend to decay the ETF's value.

In this article, we hope to accomplish three things that we think will clarify the long-term leveraged ETF's underperformance. First, we would like to address the question of how much leverage is too much. More specifically, we use the idea of a growth-optimized portfolio to determine the conditions under which ETFs can be expected to perform well and when they can be expected to converge to zero over long time horizons. Second, we clarify the role of rebalancing in the poor performance of leveraged ETFs. Finally, we suggest how our understanding of the performance drivers of leveraged ETFs can help us construct an arguably more attractive ETF.

OPTIMAL GROWTH THEORY: LESSONS FOR BLACKJACK AND THE MARKET

The performance of levered ETFs can be understood through the lens of optimal portfolio growth theory. Optimal portfolio growth theory stemmed from Kelly's [1956] article on optimal betting strategies. Kelly came up with the insight that if a gambler picks a strategy that maximizes the expected log return of each successive bet, that strategy will outperform all other strategies over the long run. The maximization of log returns then became known as the Kelly criterion for portfolio growth. It didn't take long before these ideas were applied to investments (Latane [1959]; Hakansson [1971a, 1971b]). As chronicled in a popular press book by Poundstone [2005], the debate concerning the merits of growth-optimal investing continued into the late 1970s, but application of these ideas to investment management never fully carried over into the mainstream.

Standard Markowitz [1952] portfolio theory characterizes portfolios by their expected raw (not log) return and standard deviation of returns. One might ask: How can a portfolio with high expected return have low expected log return or growth? The answer lies in the geometric nature of portfolio returns. The portfolio value at some future time is the product, not the sum, of each intermediate period's return. As such, investors achieve high-growth-rate portfolios by maximizing the geometric average of a series of returns rather than their arithmetic average. To put it in the context of a familiar example, if in one period a portfolio achieves 100% return and in the next it achieves –50% return, the portfolio value ends where it started. Some simple calculations will show that in this example the mean return per time period is 25%, but the mean log return (growth) is 0%.

Under the assumption that stock prices follow geometric Brownian motion, the relationship between expected growth and expected return can be derived through the application of stochastic calculus, more specifically Ito's Lemma (the same Ito's Lemma that has been applied with great success in the valuation of derivatives). If over some short time period a portfolio has an expected return of μ and standard deviation σ, the expected growth g over that time period is approximately:

$$g = \mu - \frac{\sigma^2}{2} \tag{1}$$

This highlights a significant difference between the portfolio growth perspective and the risk-return framework of standard portfolio theory. Under standard Markowitz mean-variance portfolio construction, there is no maximum return. Theoretically, we can achieve infinite return levels by continuing to apply higher and higher levels of leverage. This is not the case with portfolio growth. As we assume higher and higher levels of leverage, it naturally takes a toll on a portfolio's level of expected growth.

Should we care more about growth or returns? The answer of course depends on the investor's goals, but we can say the following about continuously rebalanced portfolios with negative expected growth rates (Thorpe [1969]): A portfolio with a negative expected growth rate will converge to zero with near certainty over long time periods. Consequently, the median (but not necessarily the mean) account value of a negative growth rate portfolio will converge to zero over time.

The above suggests that a leveraged ETF with a positive expected raw return but negative expected growth, increasingly resembles a lottery ticket over time. As time passes, the chances of the lottery player ending up with zero approach certainty, but the payoff if he wins continues to increase to ensure that the lottery itself has favorable odds.

Further, we can highlight the following characteristics of a growth-optimized portfolio: that is, one whose portfolio weights are selected to maximize log return, proofs of which can be found in Breiman [1961]:

- At any time in the future, the growth-optimized portfolio is more likely to have a value in excess of any other portfolio. More formally, if X_{optimal} is the value of the growth optimal portfolio at some future time, and X_{other} is the value of any other portfolio at that time, then $E(X_{\text{other}}/X_{\text{optimal}}) < 1$.
- The expected length of time it takes for a growth-optimized portfolio to achieve a high wealth target is lower than the expected length of time for any other portfolio.

As forcefully argued by Samuelson [1971] and Merton and Samuelson [1974], these characteristics alone do not guarantee the superiority of a growth-optimized portfolio over other candidate portfolios, but they do provide a framework for explaining the performance of different leveraging strategies that ETFs have used.

THE GROWTH CHARACTERISTICS OF LEVERAGED, INVERSE AND DE-LEVERED ETFs

A portfolio manager designing a leveraged (or inverse) ETF has the task of choosing portfolio weights for two assets: the risky asset and the risk-free asset. For a

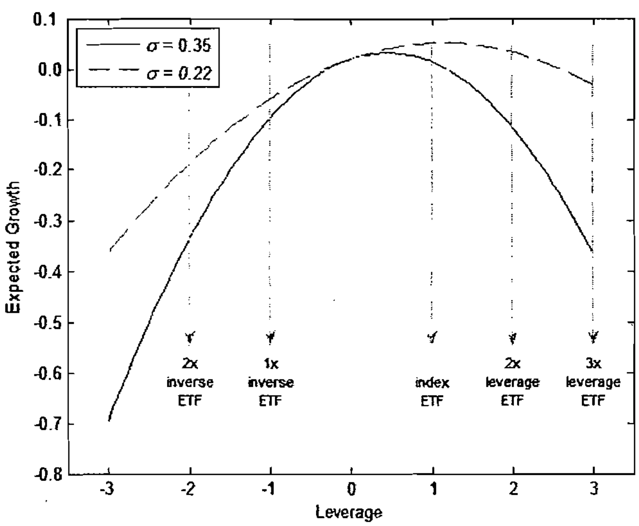
2x leveraged ETF, the weights are a constant 200% in the risky asset and −100% in the risk-free asset. If w_{risky} is the proportion of the portfolio invested in the risky asset, the growth of the ETF according to Equation (1) becomes:

$$g = \mu(w_{\text{risky}}) + r(1 - w_{\text{risky}}) - \frac{(w_{\text{risky}}\sigma)^2}{2} \qquad (2)$$

where μ is the rate of return the investor expects from the risky asset, r is the borrowing and lending rate, and σ is the standard deviation of the risky asset's log returns. The above formula assumes that the portfolio proportions are held constant, which is approximately true for levered ETFs since they rebalance daily. For illustrative purposes, assume that the expected annualized rate of return for the risky asset μ is 7.5% and the borrowing and lending rate for implementing leveraged long and short positions r is 2%. Further assume that the standard deviation of the risky asset's log return is 35%, a recently observed level of the CBOE volatility index (VIX). For these assumptions, the expected growth rate for leverage levels (w_{risky}) ranging from −3x to 3x is plotted in Exhibit 1. Consistent with the criticism leveled at them, we find that leveraged and inverse ETFs deliver negative expected growth rates. This implies that given these assumptions and enough time, the value of an investment in leveraged and inverse ETFs will eventually converge to zero with near certainty.

Exhibit 1

Expected Growth for ETFs with Different Leverage Levels



By taking the derivative of Equation (2) with respect to w_{risky}, we can show that the portfolio with the optimal growth rate is:

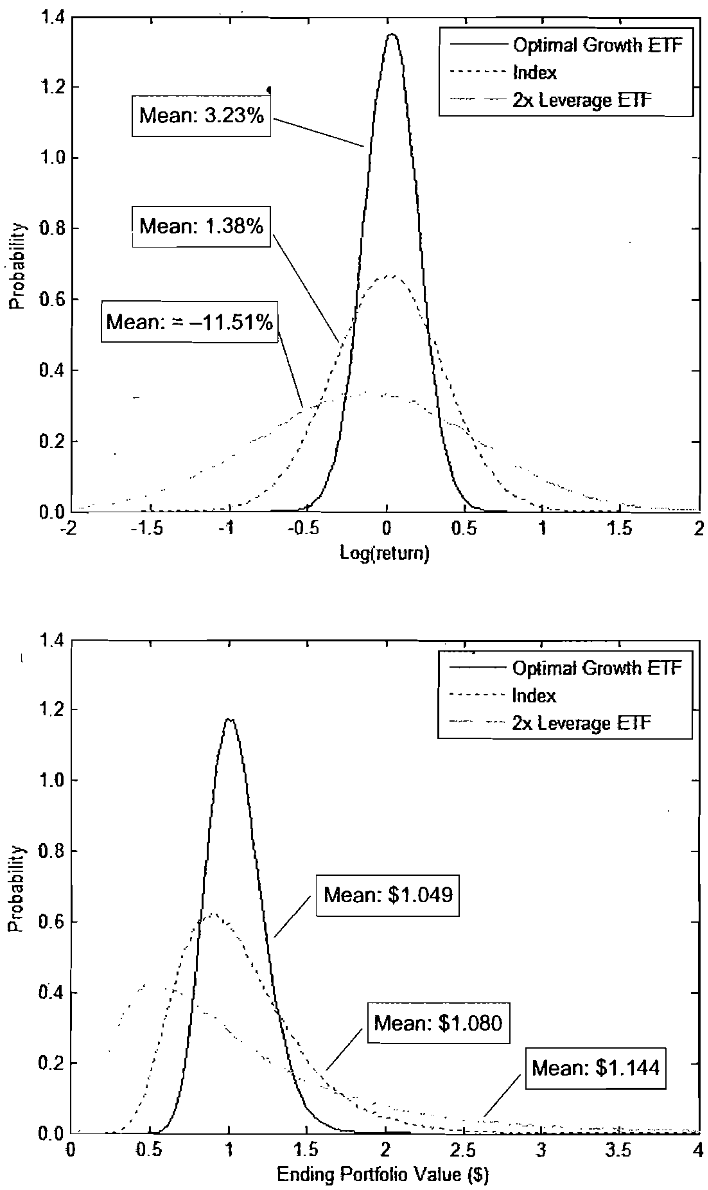
$$w^*_{\text{risky}} = \frac{\mu - r}{\sigma^2} \qquad (3)$$

Using Equation (3), we find that in our illustrative example, the value of w_{risky} that maximizes expected growth is approximately 0.45. So under these assumptions, the optimal growth ETF is actually de-levered.

Under more normal market conditions—that is, if σ were lower—we could expect optimal growth with a larger weight in the risky asset. For example, if we retain our return estimates from above and use a standard deviation of 22%, which is approximately the average closing level of the VIX index from January 2, 1999 to May 8, 2009, we find the optimal w_{risky} is 1.14. Therefore, the growth-optimized ETF could be leveraged. Normal market conditions also suggest a more benign outlook for the performance of 2x leveraged ETFs. From Equation (2), setting σ equal to 22%, we find that 2x and 3x levered ETFs produce expected growth rates of 3.3% and −3.3%, respectively. We conclude that in such a situation, an investor in a 2x leveraged ETF might not be doomed to eventual ruin, but funds invested in a 3x ETF will almost certainly approach a value of zero over time.

We construct model portfolios and perform a Monte Carlo simulation to compare the return performance of leveraged and short ETFs to their non-leveraged counterparts. Retaining our above assumptions (μ = 7.5%, r = 2%, σ = 35%), we then convert these return and standard deviation assumptions into equivalent daily returns and standard deviations, based on a 250-trading-day year. We assume that an investor places $1 in a market index, a levered ETF, and an optimally designed de-levered ETF according to Equation (3) and holds these investments for a time horizon of one year. Then we simulate the one-year performance of each portfolio using 300,000 trials, assuming no fees or transaction costs. The outcome of each of these trials allows us to estimate a probability distribution of log returns (growth) and the distribution of ending portfolio values (Exhibit 2). We find that the expected raw return of the levered ETF is the highest simply because the investor borrows at 2% to invest at 7.5%; however, the distribution of return outcomes is arguably unattractive. The investor in a levered ETF

EXHIBIT 2
Simulated Distribution of Log Returns and Ending Portfolio Values for Different Leverage Levels



achieves negative expected growth and also has the lowest median portfolio return.

For the growth-optimized ETF, performance appears more promising. In addition to the greatest expected growth rate, it also has the highest median return. Note that the levered ETF investor has a higher expected portfolio value, but paradoxically due to the skewed distribution of returns, the de-levered investor has a higher probability of ending the year with more money. If we performed the simulation over a longer time horizon, we would see that the levered portfolio's distribution of log returns would continue to migrate to the left and its distribution of ending portfolio values would continue to become more skewed with a high probability of minimal value.

HOW IMPORTANT IS REBALANCING FREQUENCY FOR ETF GROWTH?

In theory, the expected growth of a portfolio that is leveraged and not often rebalanced is negative infinity. This is because with a lognormal distribution of returns, the use of leverage will lead to some scenarios, however improbable, of complete ruin (a zero or negative portfolio value) and the logarithm of zero is negative infinity. For this reason, we emphasize that a policy of not rebalancing a leveraged ETF would be disastrous for long-term expected growth. Based on growth alone, rebalancing is unquestionably good.

However, expected growth does not tell the whole story. Unlike leveraged ETFs with negative expected growth rates, the levered portfolio that is not rebalanced need not converge to zero with near certainty. To illustrate, we retain our prior return and standard deviation assumptions and simulate the five-year portfolio value of a 2x levered ETF and the value of a 2x leveraged portfolio that does not rebalance. This exercise is not completely straightforward, since the 2x levered portfolio that does not rebalance will face a number of scenarios where portfolio value goes negative. To eliminate the possibility of negative account values, we assume that the investor must always meet a 30% margin requirement and de-leverages to meet this threshold when necessary. We plot the 20th, 50th, and 80th percentiles for portfolio values in Exhibit 3.

As indicated by the median and upper and lower percentiles, we can see that the distribution of returns differs between the rebalanced and not-rebalanced portfolios. Consistent with its negative expected growth, the levered ETF has a median return that drops fairly quickly towards zero. Further, after about three years, the 80th percentile line also begins its decent toward zero. Over even longer time horizons, every percentile (except the 100th) of the ETF's value will eventually converge to zero. This is not to say that rebalancing is always bad. Rebalancing a portfolio with positive expected growth will enhance median returns over time.

Exhibit 3

Distribution of Portfolio Values for a 2x Leveraged ETF and a 2x Leveraged Portfolio that Does Not Rebalance

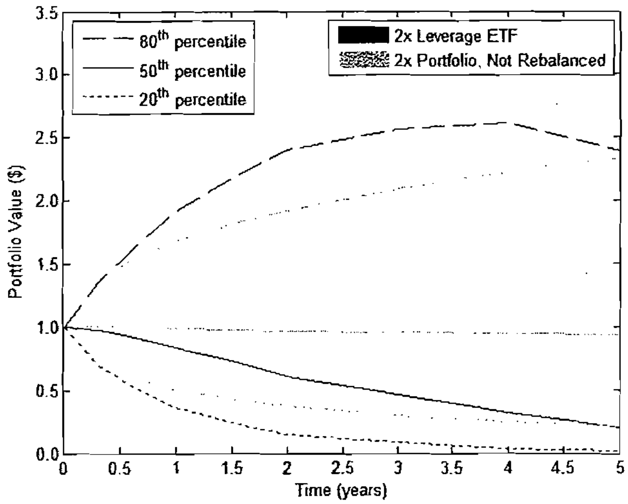
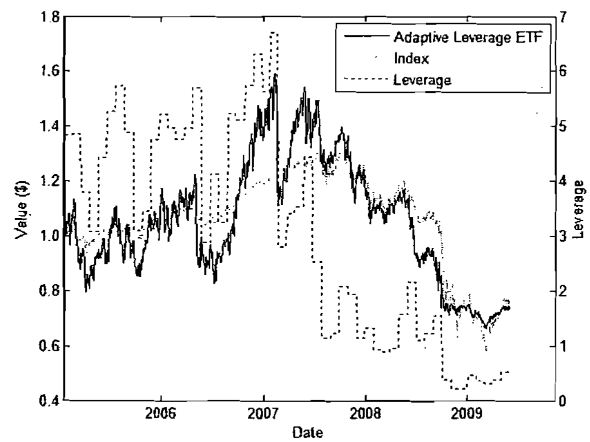


Exhibit 4

Hypothetical Performance of an Adaptive Leverage ETF Based on the S&P from May 2005 to May 2009



Note: The adaptive leverage ETF adjusts leverage each month based on prevailing option implied volatilities.

THE CASE FOR AN ADAPTIVE LEVERAGE ETF

We used the concept of portfolio growth to shed light on the performance of levered ETFs. The problem with levered ETFs is not that they have low expected returns, since these funds can expect a return as high as or even higher than the underlying index. Rather, the problem is with the expected growth or log returns. We showed that under reasonable assumptions, highly levered ETFs (3x and inverse ETFs) can be expected to have negative expected growth and therefore are likely to converge to zero over longer time horizons. When based on high-volatility indexes, 2x leveraged ETFs can also be expected to decay to zero; however, under moderate market conditions, these ETFs should avoid the fate of their more highly leveraged counterparts. We also saw that daily rebalancing is a necessary condition for negative-growth ETFs to converge to zero.

Since negative expected growth tends to hurt ETF performance, we pose the question: Might an ETF aimed at maximizing expected growth work well? The ability to use just a few inputs (Equation (3)) to calculate weights for a growth-optimized portfolio (expected asset return, standard deviation, and risk-free rate) suggests the possibility of updating leverage dynamically as market conditions change. The risk-free borrowing and lending rates are easily observable. A fund could use as its expected asset return either historical rates of return or returns based on an equilibrium model such as CAPM. For the standard deviation of returns, the fund could use implied volatilities from traded options.

To illustrate how an adaptive leverage ETF would work, we test its performance on the S&P 500 using available data from Bloomberg. We assume that the adaptive leverage ETF will alter its target leverage each month based on the implied volatilities for one-month at-the-money options and a constant market risk premium ($\mu - r$) of 5.5%. To simplify, we assume the fund pays no fees or transaction costs and must fund leveraged positions at the prevailing one-month LIBOR rate. The performance of this hypothetical adaptive leverage fund as well as that of the S&P 500 for the period from May 2005 through May 2009 is shown in Exhibit 4. We can see that from 2005 through the early part of 2007, the fund used significant leverage. After volatilities increased in 2007, the fund de-levered significantly. During this time, the adaptive leverage fund would have performed about as well as the S&P 500—not a terrible performance, considering the amount of leverage assumed in 2005–2007 and the fact that a 2x leveraged ETF would have fared much worse.

This example is meant to be illustrative, since much more data would be needed to perform a definitive test of an adaptive leverage ETF's desirability. However, we find appealing the fact that such a fund would tend to de-lever in high-volatility times and thus help ETF investors avoid repeated buying high and selling low, which come with continuous rebalancing. Under more stable market conditions, the fund could take on more leverage to earn the spread between the risky asset and the risk-free rate. This type of strategy should bolster long-run growth. Whether investors would buy in is another question entirely.

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