LTL

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Important! This lab is pen-and-paper. I recommend that you submit images of your handwritten solutions. It's also fine to type your solutions, if you're comfortable writing equations in Word or LaTeX.

Linear-time Temporal Logic. A temporal logic is a kind of modal logic where the modality depends on time. Temporal propositions may be true at some times and false at others. The truth or falsity of temporal propositions depends on the current *state* s. For any propositional model M, state s and formula φ, we write $M, s \vDash \varphi$ to mean that formula φ is true in state s in model M.

Kripke structures. The manner in which a state may change at each time-step is described by a *Kripke structure*. A Kripke structure is a labelled transition system; each label is a state, and each arrow represents a single time-step. From now on, a 'model' will refer to a Kripke structure. The truth or falsity of temporal propositions may be defined with respect to a state, a path within the model or the model itself:

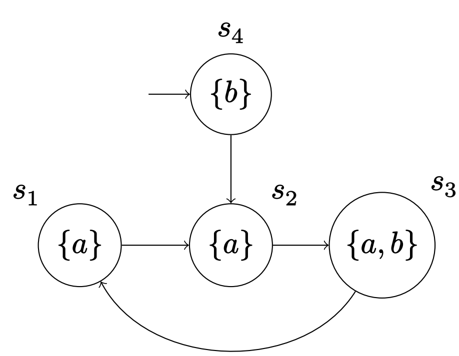
- If a state s in a model M is labelled with an atomic proposition p, then $M, s \vDash p$ is true (by definition). Similarly, if a state s in model M is not labelled with an atomic proposition p, then $M, s \vDash \neg p$. We can extend this to any logical formula φ built with the usual logical operators by looking up their truth tables.
- We can overload our notation for truth to work with paths as well as states. We write $M, \pi \vDash \varphi$ to say that the formula φ is true for a path π in the model M, where $\pi = s_0 s_1 s_2 \ldots$ is a path from state s_0 to s_1 to s_2, and so on.
- We write $M \vDash \varphi$ to say that the formula φ is true for the model M iff $M, \pi \vDash \varphi$ for all paths π which start in the initial state.

Linear-Time Temporal Operators. As presented in lecture notes, the LTL operators Next (X), Future (F), Globally (G) and Until (U) are formally defined as follows.

Let $\pi = s_0 s_1 s_2 \ldots$ be a path, with $\pi^i = s_i s_{i+1} s_{i+2} \ldots$ representing a truncated path. Let AP be a set of atomic propositions and φ an LTL formula. For any model M, we define $M, \pi \models \varphi$ inductively as follows.

- $M, \pi \models p$ for some $p \in AP$ iff p holds in s_0.
- $M, \pi \models \neg\varphi$ iff $\pi \not\models \varphi$.
- $M, \pi \models \varphi \vee \psi$ iff $M, \pi \models \varphi$ or $M, \pi \models \psi$.
- $M, \pi \models \mathsf{X}\,\varphi$ iff $M, \pi^1 \models \varphi$.
- $M, \pi \models \mathsf{G}\,\varphi$ iff for any $i \geq 0$, $M, \pi^i \models \varphi$.
- $M, \pi \models \mathsf{F}\,\varphi$ iff there exists $j \geq 0$, $M, \pi^j \models \varphi$.
- $M, \pi \models (\varphi \,\mathsf{U}\, \psi)$ iff there exists $k \geq 0$ such that $M, \pi^k \models \psi$ and for any j such that $0 \leq j < k$, we have $M, \pi^j \models \varphi$.

Question 1. Consider the following transition system M over the set of atomic propositions $\{a, b\}$:



Indicate for each of the following LTL formulae the set of states for which the following formulae are fulfilled. Explain your solution in each case.

(a) $\mathsf{X}\ a$

(b) $\mathsf{X}\ \mathsf{X}\ \mathsf{X}\ a$

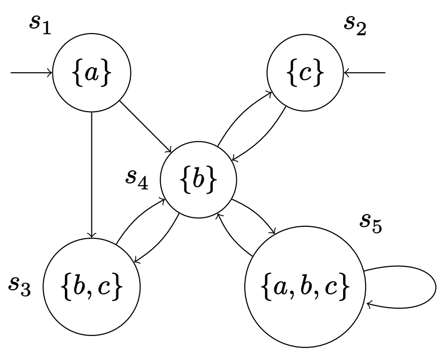
(c) $\mathsf{G}\ b$

(d) $\mathsf{G}\ \mathsf{F}\ a$

(e) $\mathsf{G}\ (b\ \mathsf{U}\ a)$

(f) $\mathsf{F}\ (a\ \mathsf{U}\ b)$

Question 2. Consider the following transition system M over the set of atomic propositions $\{a, b, c\}$:



For each LTL formulae φ_i below, decide if $M \models \varphi_i$ holds. Justify your answer. If $M \not\models \varphi_i$, provide a path π such that $M, \pi \not\models \varphi_i$

(a) $\varphi_1 = \mathsf{F}\ \mathsf{G}\ c$

(b) $\varphi_2 = \mathsf{G}\ \mathsf{F}\ c$

(c) $\varphi_3 = \mathsf{X}\ \neg c \rightarrow \mathsf{X}\ \mathsf{X}\ c$

(d) $\varphi_4 = \mathsf{G}\ a$

(e) $\varphi_5 = (a\ \mathsf{U}\ \mathsf{G}\ (b \vee c))$

Question 3. Specify the following traffic light properties in LTL.

(a) Once red, the light cannot become green immediately.

(b) The light becomes green eventually.

(c) Once red, the light becomes green eventually.

(d) Once red, the light always becomes green eventually after being yellow for some time in-between.

Question 4. Suppose we have a printer device Printer and two users, Peter and Jane. Both users perform several tasks, and every now and then they want to print their results on the printer. Since there is only one printer, only one user can print a job at a time. Suppose we have the following atomic propositions for Peter at our disposal:

- Peter.request—indicates that Peter requests usage of the printer.
- Peter.use—indicates that Peter uses the printer.
- Peter.release—indicates that Peter releases the printer.

Similar predicates are defined for Jane. Specify the following properties in LTL:

(a) *Mutual Exclusion*—only one user at a time can use the printer.
(b) *Finite time of usage*—a user can print only for a finite amount of time.
(c) *Absence of individual starvation*—if a user wants to print something, they are eventually able to do so.
(d) *Absence of blocking*—a user can always request to use the printer.
(e) *Alternating access*—users must strictly alternate in printing.