Liouvillian Matrix in real and image basis

Ding Yi
yi.ding@ubbcluj.ro

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Introduction

Liouvillian equation which can help us to understand the transporting trait is a crucial tool in quantum system. Still for me, it was difficult to re-write the equations to the matrix form. Here I give an example to show the path from equations to matrix form. Since the single quantum dot system is too easy to re-write, we start from double quantum dot system with follwoing equations [1],

$$\begin{aligned}
\dot{\sigma}_{aa} &= -\Gamma_L\sigma_{aa} + \Gamma_R\sigma_{cc}, &(1)\\
\dot{\sigma}_{bb} &= \Gamma_L\sigma_{aa} + \Gamma_R\sigma_{dd} + i\Omega_0(\sigma_{bc} - \sigma_{cb}), &(2)\\
\dot{\sigma}_{cc} &= -\Gamma_R\sigma_{cc} - \Gamma_L\sigma_{cc} - i\Omega_0(\sigma_{bc} - \sigma_{cb}), &(3)\\
\dot{\sigma}_{dd} &= -\Gamma_R\sigma_{dd} + \Gamma_L\sigma_{cc}, &(4)\\
\dot{\sigma}_{bc} &= i(E_2 - E_1)\sigma_{bc} + i\Omega_0(\sigma_{bb} - \sigma_{cc}) - \frac{1}{2}(\Gamma_L + \Gamma_R)\sigma_{bc}. &(5)
\end{aligned}$$

Here, $\sigma_{aa}, \sigma_{bb}, \sigma_{bc}, \sigma_{cc}, \sigma_{dd}, \sigma_{bc}, \sigma_{cb}$ represent the empty state, dot1 occupied, dot2 occupied state, double occupied state and coherent states between dot1 and dot2, respectively. Γ_L and Γ_R are the tunneling amplitudes between source to dot and dot to drain electrode, respectively. $\dot{\sigma}_{cb}$ is the conjugated transpose counterpart of $\dot{\sigma}_{bc}$ which is $\dot{\sigma}^*_{bc}$.

$$\dot{\sigma}_{cb} = \dot{\sigma_{bc}}^* = -i(E_2 - E_1)\sigma_{cb} + i\Omega_0(\sigma_{cc} - \sigma_{bb}) - \frac{1}{2}(\Gamma_L + \Gamma_R)\sigma_{cb}. \tag{6}$$

For convenience, some abbriviations are asked ($\Delta E \equiv E_2 - E_1$, $\Gamma \equiv \Gamma_L + \Gamma_R$). And here, we ignore the double occupied states by employing the big bias voltage. The new Liouvillian equations read as

$$\begin{aligned}
\dot{\sigma}_{aa} &= -\Gamma_L\sigma_{aa} + \Gamma_R\sigma_{cc}, &(7)\\
\dot{\sigma}_{bb} &= \Gamma_L\sigma_{aa} + i\Omega_0(\sigma_{bc} - \sigma_{cb}), &(8)\\
\dot{\sigma}_{cc} &= -\Gamma_R\sigma_{cc} - i\Omega_0(\sigma_{bc} - \sigma_{cb}), &(9)\\
\dot{\sigma}_{bc} &= i\Delta E\sigma_{bc} + i\Omega_0(\sigma_{bb} - \sigma_{cc}) - \frac{1}{2}\Gamma\sigma_{bc} &(10)\\
\dot{\sigma}_{cb} &= -i\Delta E\sigma_{cb} + i\Omega_0(\sigma_{cc} - \sigma_{bb}) - \frac{1}{2}\Gamma\sigma_{cb}. &(11)
\end{aligned}$$

Make it as a matrix with the basis of $(\sigma_{aa}, \sigma_{bb}, \sigma_{bc}, \sigma_{cc}, \sigma_{bc}, \sigma_{cb})^{\mathrm{T}}$

$$\begin{pmatrix} \dot{\sigma}_{aa} \\ \dot{\sigma}_{bb} \\ \dot{\sigma}_{cc} \\ \dot{\sigma}_{bc} \\ \dot{\sigma}_{cb} \end{pmatrix} = \begin{pmatrix} -\Gamma_L & 0 & \Gamma_R & 0 & 0 \\ \Gamma_L & 0 & 0 & iV_0 & -iV_0 \\ 0 & 0 & -\Gamma_R & -iV_0 & iV_0 \\ 0 & iV_0 & -iV_0 & i\Delta E - \frac{\Gamma}{2} & 0 \\ 0 & -iV_0 & iV_0 & 0 & -i\Delta E - \frac{\Gamma}{2} \end{pmatrix} \begin{pmatrix} \sigma_{aa} \\ \sigma_{bb} \\ \sigma_{cc} \\ \sigma_{bc} \\ \sigma_{cb} \end{pmatrix}. \tag{12}$$

In fact, we care more about the real and image part of coherent state instead of its conjugated transpose.

$$\begin{aligned}
\sigma_{bc} &= \Re\sigma_{bc} + i\Im\sigma_{bc}, \\
\sigma_{cb} &= \Re\sigma_{bc} - i\Im\sigma_{bc}, \\
\Re\sigma_{bc} &= \frac{\sigma_{bc} + \sigma_{cb}}{2}, \\
\Im\sigma_{bc} &= \frac{\sigma_{bc} - \sigma_{cb}}{2i}.
\end{aligned} \tag{13}$$

Here are two ways to transform the Eq.(12) to real and image basis. Acutually, they are same way with different begining.

Method 1 of $\Re\sigma_{bc}$ and $\Im\sigma_{bc}$

$$\begin{aligned}\Re\dot{\sigma}_{bc} &= \frac{\dot{\sigma_{bc}} + \dot{\sigma_{cb}}}{2} = \frac{1}{2}(i\Delta E\sigma_{bc} + i\Omega_0(\sigma_{bb} - \sigma_{cc}) - \frac{1}{2}\Gamma\sigma_{bc} - i\Delta E\sigma_{cb} + i\Omega_0(\sigma_{cc} - \sigma_{bb}) - \frac{1}{2}\Gamma\sigma_{cb}) \\ &= \frac{1}{2}(i\Delta E(\sigma_{bc} - \sigma_{cb}) - \frac{1}{2}\Gamma(\sigma_{bc} + \sigma_{cb})) \\ &= \frac{1}{2}(i\Delta E \times 2 \times i\Im\sigma_{bc} - \frac{1}{2}\Gamma \times 2\Re\sigma_{bc}) \\ &= -\Delta E\Im\sigma_{bc} - \frac{1}{2}\Gamma\Re\sigma_{bc} \\ \\ \Im\dot{\sigma}_{bc} &= \frac{\dot{\sigma_{bc}} - \dot{\sigma_{cb}}}{2i} = \frac{1}{2i}(i\Delta E\sigma_{bc} + i\Omega_0(\sigma_{bb} - \sigma_{cc}) - \frac{1}{2}\Gamma\sigma_{bc} - (-i\Delta E\sigma_{cb} + i\Omega_0(\sigma_{cc} - \sigma_{bb}) - \frac{1}{2}\Gamma\sigma_{cb})) \\ &= \frac{1}{2i}(i\Delta E(\sigma_{bc} + \sigma_{cb}) - \frac{1}{2}\Gamma(\sigma_{bc} - \sigma_{cb}) + 2i\Omega_0(\sigma_{bb} - \sigma_{cc})) \\ &= \frac{1}{2i}(i\Delta E \times 2\Re\sigma_{bc} - \frac{1}{2}\Gamma \times 2 \times i\Im\sigma_{bc} + 2i\Omega_0(\sigma_{bb} - \sigma_{cc})) \\ &= \Delta E\Re\sigma_{bc} - \frac{1}{2}\Gamma\Im\sigma_{bc} + \Omega_0(\sigma_{bb} - \sigma_{cc})\end{aligned} \tag{14}$$

Method 2 of $\Re\sigma_{bc}$ and $\Im\sigma_{bc}$

$$\begin{aligned}&\dot{\sigma_{bc}} = \Re\dot{\sigma}_{bc} + i\Im\dot{\sigma}_{bc} \\ &i\Delta E\sigma_{bc} + i\Omega_0(\sigma_{bb} - \sigma_{cc}) - \frac{1}{2}\Gamma\sigma_{bc} = \Re\dot{\sigma}_{bc} + i\Im\dot{\sigma}_{bc} \\ &i\Delta E[\Re\sigma_{bc} + i\Im\sigma_{bc}] + i\Omega_0(\sigma_{bb} - \sigma_{cc}) - \frac{1}{2}\Gamma[\Re\sigma_{bc} + i\Im\sigma_{bc}] = \Re\dot{\sigma}_{bc} + i\Im\dot{\sigma}_{bc} \\ &-\Delta E\Im\sigma_{bc} - \frac{1}{2}\Gamma\Re\sigma_{bc} + i\Delta E\Re\sigma_{bc} + i\Omega_0(\sigma_{bb} - \sigma_{cc}) - \frac{1}{2}\Gamma i\Im\sigma_{bc} = \Re\dot{\sigma}_{bc} + i\Im\dot{\sigma}_{bc} \\ \\ &\Rightarrow \Re\dot{\sigma}_{bc} = -\Delta E\Im\sigma_{bc} - \frac{1}{2}\Gamma\Re\sigma_{bc} \\ &\Rightarrow \Im\dot{\sigma}_{bc} = \Delta E\Re\sigma_{bc} - \frac{1}{2}\Gamma\Im\sigma_{bc} + \Omega_0(\sigma_{bb} - \sigma_{cc})\end{aligned} \tag{15}$$

σ_{bb} and σ_{cc}

$$\begin{aligned}\dot{\sigma_{bb}} &= \Gamma_L\sigma_{aa} + i\Omega_0(\sigma_{bc} - \sigma_{cb}) \\ \dot{\sigma_{cc}} &= -\Gamma_R\sigma_{cc} - i\Omega_0(\sigma_{bc} - \sigma_{cb}) \\ \\ \Rightarrow \dot{\sigma_{bb}} &= \Gamma_L\sigma_{aa} + i\Omega_0 \times 2i\Im\sigma_{bc} = \Gamma_L\sigma_{aa} - 2\Omega_0\Im\sigma_{bc} \\ \Rightarrow \dot{\sigma_{cc}} &= -\Gamma_R\sigma_{cc} - i\Omega_0 \times 2i\Im\sigma_{bc} = -\Gamma_R\sigma_{cc} + 2\Omega_0\Im\sigma_{bc}\end{aligned} \tag{16}$$

At last, let's make it as a Liouvillian matrix with the basis of $(\sigma_{aa}, \sigma_{bb}, \sigma_{bc}, \sigma_{cc}, \Re\sigma_{bc}, \Im\sigma_{bc})^{\mathrm{T}}$ and keep the others items unchanged,

$$\begin{pmatrix} \dot{\sigma}_{aa} \\ \dot{\sigma}_{bb} \\ \dot{\sigma}_{cc} \\ \Re(\sigma_{bc}) \\ \Im(\sigma_{bc}) \end{pmatrix} = \begin{pmatrix} -\Gamma_L & 0 & \Gamma_R & 0 & 0 \\ \Gamma_L & 0 & 0 & 0 & -2\Omega_0 \\ 0 & 0 & -\Gamma_R & 0 & 2\Omega_0 \\ 0 & 0 & 0 & -\frac{\Gamma}{2} & -\Delta E \\ 0 & \Omega_0 & -\Omega_0 & \Delta E & -\frac{\Gamma}{2} \end{pmatrix} \begin{pmatrix} \sigma_{aa} \\ \sigma_{bb} \\ \sigma_{cc} \\ \Re(\sigma_{bc}) \\ \Im(\sigma_{bc}) \end{pmatrix} \tag{17}$$

References

[1] S. A. Gurvitz and Ya S. Prager, Physical Review B 53 (23), 15932 (1996).