Notes

Hufeng Wen

2024 年 4 月 27 日

前言

本笔记参考的教材为 Strogatz, Steven H 的 nonlinear dynamic and chaos [?]。由于本人水平有限，如果将一些名词翻译成中文，很有可能词不达意，弄巧成拙。所以一些物理名词和一些我看不懂的但感觉很有道理的话，我尽量保留英文，不作翻译。但由于本人是中文母语者，思考的语言还是中文，故笔记大体还是中文。

2024 年 4 月 27 日

目录

第零章　一维情况

首先，一个非常普适的框架就是如下的微分方程组

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1, \ldots, x_n) \\
&\vdots \\
\dot{x}_1 &= f_1(x_1, \ldots, x_n)
\end{aligned} \tag{1}
$$

其中 $\dot{x}_i \equiv dx_i/dt$。

举一个有阻尼的简谐振子，其运动方程为

$$
m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \tag{2}
$$

如果令 $x_1 = x$ $x_2 = \dot{x}$，则有 $\dot{x}_1 = x_2$，并且

$$
\begin{aligned}
\dot{x}_2 = \ddot{x} &= -\frac{b}{m}\dot{x} - \frac{k}{m}x \\
&= -\frac{b}{m}x_2 - \frac{k}{m}x_1
\end{aligned} \tag{3}
$$

这样子这个有阻尼的简谐振子方程就可以化为最开始比较普遍的形式

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{b}{m}x_2 - \frac{k}{m}x_1
\end{aligned} \tag{4}
$$

这样的系统被称为线性的。因为等式右边只有一阶项。

0.1　一维系统

如果只有一个微分方程

$$
\dot{x} = f(x) \tag{5}
$$

那么这样就是一维情况。(one dimensional or first-order systems)。

首先来看一个例子来分析分析如何处理这样的系统。方程如下

$$\dot{x} = sinx \tag{6}$$

这个方程是有解析解的。分离变量即可解得

$$t = -\ln|\csc x + \cot x| + C \tag{7}$$

为了确定常数，假设 $x(t=0) = x_0$。则解为

$$t = \ln\left|\frac{\csc x_0 + \cot x_0}{\csc x + \cot x}\right| \tag{8}$$

我们可以根据下图来形象的理解 x 随时间的变化。我们想象液体沿着 x 轴流动，他的速度满足 $\dot{x} = \sin x$ 。如图所示: $\dot{x} > 0$ 时液体向右流动；$\dot{x} < 0$ 时液体向左流动; $\dot{x} = 0$ 时液体不流动，这样的点就叫做 **fixed points**. 如果流汇聚，那么就是 **stable fixed points**；如果流往两边走则是 **unstable fixed points**。

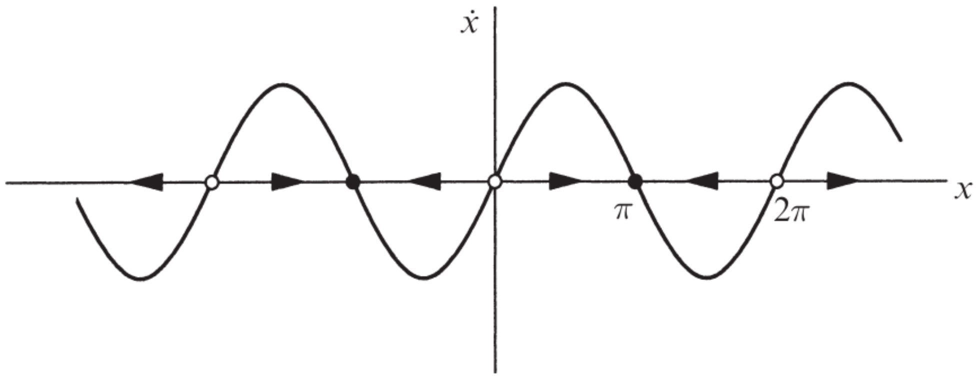

Figure 2.1.1

图 1:

0.2 Fixed Points and Stablility

和之前一样，我们想象一个液体沿着 x 轴流动，速率表达式为 $f(x)$ 。这个想象的液体被称为 phase fluid，x 轴被称为 phase space. $f(x) > 0$ 时流向右，$f(x) < 0$ 时流向左。为了求解 $\dot{x} = f(x)$ 我们放一个想象的点 (known as **phase point**) 在初始位置 x_0，然后观察这个点随流的移动。随着时间的推移，这个点的轨迹是时间的函数，不妨设为 $x(t)$。这个方程称为初始位置为 x_0 的轨迹 (**trajectory**)，也就是微分方程的解嘛。而像上图用来定性描述这个系

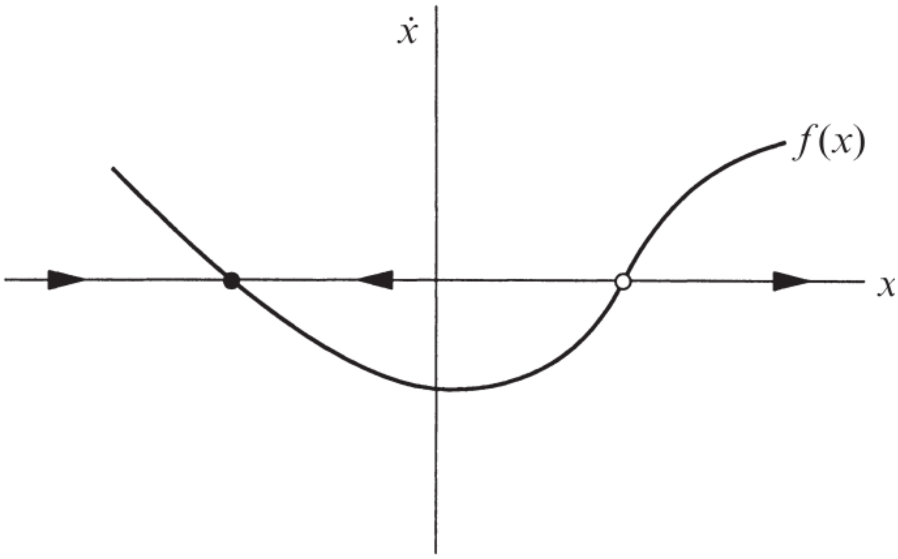

Figure 2.2.1

图 2:

统的图，称为 **phas portratit**。

fixed points 代表了平衡解 (equilibrium solutions)。An equilibrium is defined to be stable if all sufficiently small disturbances away from it damp out in time. Thus stable equilibria are represented geometrically by stable fixed points. Conversely, unstable equilibria, in which disturbances grow in time, are represented by unsatable fixed points.

0.3 Potentials

还有另外一种方式来理解 first-order system $\dot{x} = f(x)$, 这种方式基于物理中势的概念。我们描述一个粒子沿着势垒壁向下滑，the **potential** $V(x)$ 定义为

$$f(x) = -\frac{dV}{dx} \tag{9}$$

利用链式求导法则并联立 $\dot{x} = f(x)$，有

$$\begin{aligned} \frac{dV}{dt} &= \frac{dV}{dx}\frac{dx}{dt} \\ &= -\left(\frac{dV}{dx}\right)^2 \leq 0 \end{aligned} \tag{10}$$

0.4 Bifurcations

The qualitative structure of the flow can changes as parameters are varied. In particular, fixed points can be created or destroyed, or their stability can change. These qualitative changes in the dynamics are called **bifurcations**, and the parameter values at which they occur are called **bifurcations points**.

0.4.1 Pitchfork Bifurcations

Supercritical Pitchfork Bifurcation

The normal form of the supercritical pitchfork bifurcation is

$$\dot{x} = rx - x^3 \tag{11}$$

Note that this equation is **invariant** under the change of variables $x \to -x$. That is,if we replace x by $-x$ and then cancel the resulting minus signs on both sides of the equation and we get 11 back. 当 $r < 0$ 时, 原点是唯一的 fixed point, 并且根据两边的流判断, 它是 stable fixed

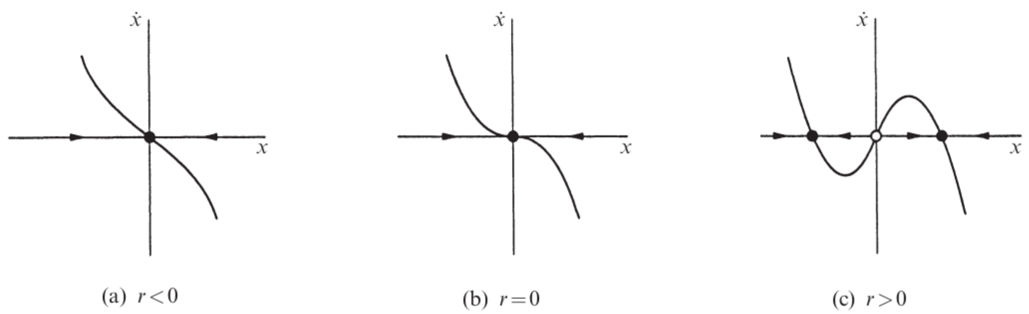

图 3: the vector field for different values of r

point。当 $r = 0$ 时，原点仍然是一个 stable fixed point。但此时 $f'(x)|_{x=1} = 0$。Now solutions no longer decay exponentially fast — instead the decay is a much slower algebraic function of time. This lethargic decay is called **critical slowing down** in the physics literature. 最后，当 $r > 0$ 时，原点变成了一个 unsatable fixed point. 而在原点两边新生成了两个对称的 stable fixed points, 位置在 $x^* - \pm\sqrt{r}$.

这个 bifurcation 为什么叫做 pitchfork bifurcation? 我们可以画一个 bifurcation diagram. 如下 可以看到，这个图的形状就像一个草叉!

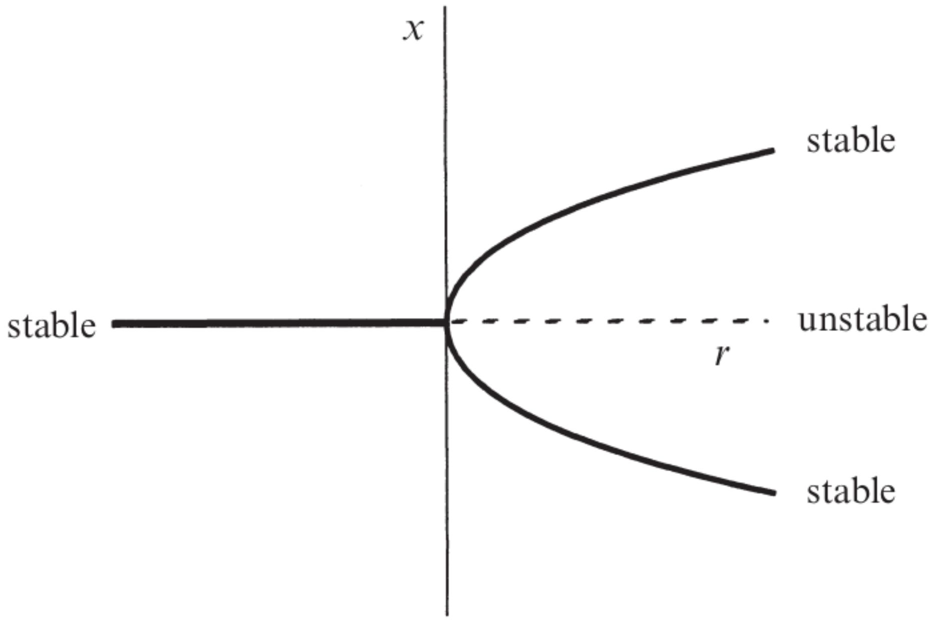

图 4:

Subcritical Pitchfork Bifurcation

In the supercritical case $\dot{x} = rx - x^3$ discussed above, the cubic term is *stabilizing*: is acts as a restoring force that pulls $x(t)$ back toward $x = 0$. If instead the cubic term were *destabilizing*,as in

$$\dot{x} = rx + x^3 \tag{12}$$

then we'd have a **Subcritical** pitchfork bifurcation. Compared to Figure (4), the pitchfork is inverted. the nonzero fixed points $x^* = \pm\sqrt{-r}$ are *unstable*, and exist only *below* the bifurcation $(r < 0)$, which motivates the term "subcritical."

0.5

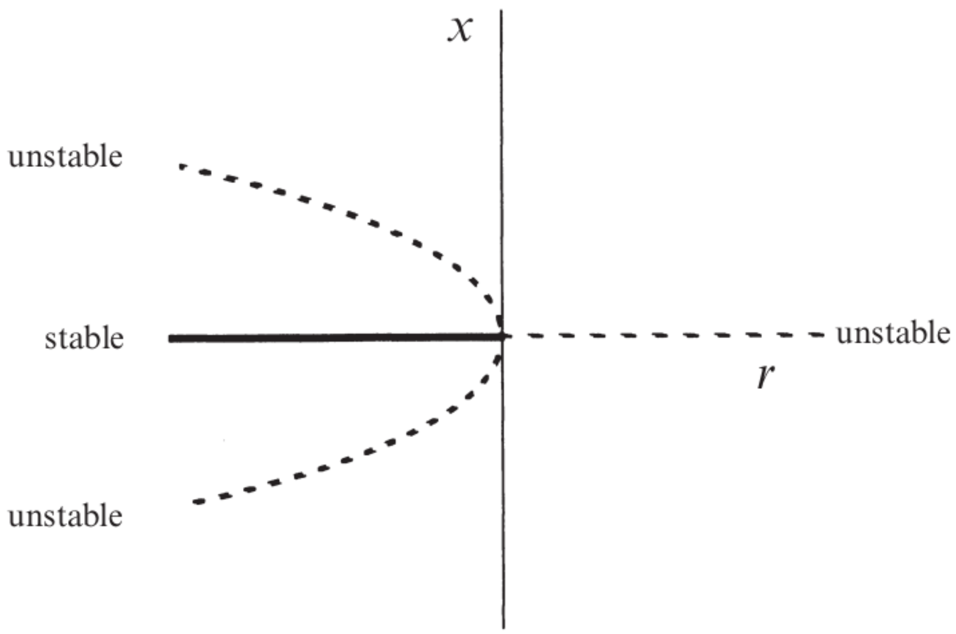

图 5: