A Revised Closed-Form Solution for Bond Convexity

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Since duration works best for only small changes of the interest rate, convexity helps to improve the estimation accuracy. Traditionally, convexity is displayed with a formula, which depends on the number of outstanding payments. Closed-form solutions do exist, typically building on the work of Chua (1984). Based on the slightly different approach of Kruschwitz and Schöbel (1986a), who also developed a closed-form equation for the duration, I derive a shorter closed-form solution for convexity than previously known.

1 Introduction

Generally, financial instruments are merely a series of cash-flows, promoting the *time value of money* and the *present value* formulae to essential concepts in finance. In this framework, straight bonds are securitized loans, paying regular coupons to the bond holder and redeeming the notional at maturity. Even if we ignore credit risk, i.e. a decline of the rating or even a default, there is still market risk attached to bonds, since interest rates are subject to change (see Malkiel (1962) for a sensitivity analysis of bond prices with respect to interest rate risk).

The *duration* of a bond, as introduced by Macaulay (1938) and Hicks (1939), is a widely known risk measure for bond price sensitivity due to changes of the interest rate. It works best for flat yield curves and rather small changes of the interest rate. For larger movements of the interest rate, Redington (1952) has shown by a Taylor series expansion, that *convexity* helps to improve the accuracy of the sensitivity estimate.

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The Macaulay duration of a straight bond is typically defined as a time dependent equation, i.e. the formula has to be adapted to the number of the outstanding cash flows

$$D_{Mac} = \frac{\sum\limits_{t=1}^{T} (t \cdot C_t \cdot q^{-t}) + T \cdot N \cdot q^{-T}}{\sum\limits_{t=1}^{T} (C_t \cdot q^{-t}) + N \cdot q^{-T}}$$

with C as coupon payment, q as discount factor $(1+i)$, i as interest rate, t as index for a given point in time, T as maturity, and N as nominal value. For different reasons, like transferring formulas into software code, closed-form equations are desirable. Chua (1984; 1985; 1988) has derived a closed-form solution for the duration in a series of articles. In German language, Kruschwitz and Schöbel (1986a; 1986b) have also derived a closed-form duration formula for broken periods.

Whereas most authors use Chua's terminology to derive a closed-form formula for convexity, I use the slightly different equation from Kruschwitz and Schöbel (1986a) to derive an even shorter closed-form solution than hitherto known.

2 Closed-Form Solution for Convexity

Traditionally, the formula for convexity $\mathbb{C}$ is shown as

$$\mathbb{C} = q^{-2} \cdot \frac{\sum\limits_{t=1}^{T} \left((t+1) \cdot t \cdot C_t \cdot q^{-t}\right) + T \cdot N \cdot q^{-T}}{\sum\limits_{t=1}^{T} (C_t \cdot q^{-t}) + N \cdot q^{-T}} \tag{1}$$

which means the formula has to be adapted to the number of payments. To derive a closed-form solution, we start with the general formula for the dirty price of a straight bond, which is defined as

$$PV = \sum_{t=1}^{T} \left(C \cdot q^{-t}\right) + N \cdot q^{-T} \tag{2}$$

with PV as present value of the bond, C as coupon payment, q as discounting factor $(1+i)$, N as the notional, and T as maturity. Using a geometric series, Equation (2) can be reformulated to

$$PV = C \cdot \frac{q^T - 1}{i \cdot q^T} + \frac{N}{q^T}. \tag{3}$$

The closed-form equation (3) can be further adapted to broken periods

$$PV = q^a \cdot \left(C \cdot \frac{q^n - 1}{i \cdot q^n} + \frac{N}{q^n}\right) \tag{4}$$

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with n as the number of outstanding coupon payments and a as the time that has passed since the last coupon payment.

In their paper, Kruschwitz and Schöbel (1986a) derived Equation (4) to get a closed-form solution for the Macaulay duration. Their first derivative is given in their equation (9) on p. 552, and evolves to:

$$\frac{dPV}{di} = q^a \cdot \left(\frac{a \cdot C}{i \cdot q} - \frac{C}{i^2} + \frac{C}{i^2 \cdot q^n} + \frac{(n-a) \cdot C}{i \cdot q^{n+1}} - \frac{(n-a) \cdot N}{q^{n+1}} \right). \tag{5}$$

Based on their result, I use the following expression, which is more convenient to find the second derivative:

$$\frac{dPV}{di} = \frac{C \cdot a \cdot q^{a-1}}{i} - \frac{C \cdot q^a}{i^2} - \frac{C \cdot (a-n) \cdot q^{a-n-1}}{i} + \frac{C \cdot q^{a-n}}{i^2} + N \cdot (a-n) \cdot q^{a-n-1}. \tag{6}$$

The second derivative thus is given as

$$\begin{aligned} \frac{d^2PV}{di^2} = {} & \frac{C \cdot a \cdot (a-1) \cdot q^{a-2} \cdot i - C \cdot a \cdot q^{a-1}}{i^2} - \frac{C \cdot a \cdot q^{a-1} \cdot i^2 - C \cdot q^a \cdot 2i}{i^4} - ... \\ & ... \frac{C \cdot (a-n) \cdot (a-n-1) \cdot q^{a-n-2} \cdot i - C \cdot (a-n) \cdot q^{a-n-1}}{i^2} + ... \\ & ... \frac{C \cdot (a-n) \cdot q^{a-n-1} \cdot i^2 - C \cdot q^{a-n} \cdot 2i}{i^4} + N \cdot (a-n) \cdot (a-n-1) \cdot q^{a-n-2}. \end{aligned}$$

This expression can be simplified to

$$\begin{aligned} \frac{d^2PV}{di^2} = q^a \cdot \Bigg[C \cdot \Bigg(& \frac{a \cdot (a-1) - (a-n) \cdot (a-n-1) \cdot q^{-n}}{i \cdot q^2} + ... \\ & ... + \frac{-a + (a-n) \cdot q^{-n}}{0.5 \cdot i^2 \cdot q} + \frac{1 - q^{-n}}{0.5 \cdot i^3} \Bigg) + N \cdot (a-n) \cdot (a-n-1) \cdot q^{-n-2} \Bigg]. \end{aligned} \tag{7}$$

Since convexity is defined as (Tuckman and Serrat, 2012, p.132)

$$\mathbb{C} = \frac{d^2PV}{PV(i)} \cdot \frac{1}{di^2} \tag{8}$$

we have to divide Equation (7) by the present value from formula (4), which delivers a closed-form solution for the convexity

$$\mathbb{C} = \frac{C \cdot \left(\frac{a \cdot (a-1) - (a-n) \cdot (a-n-1) \cdot q^{-n}}{i \cdot q^2} + \frac{-a + (a-n) \cdot q^{-n}}{0.5 \cdot i^2 \cdot q} + \frac{1 - q^{-n}}{0.5 \cdot i^3} \right) + \frac{N \cdot (a-n) \cdot (a-n-1) \cdot q^{-n}}{q^2}}{C \cdot \frac{q^n - 1}{i \cdot q^n} + \frac{N}{q^n}}. \tag{9}$$

Further simplification and substitution evolves to

$$\mathbb{C} = \frac{C \cdot \left(\frac{a \cdot (a-1)}{q^2} - \gamma + \frac{-a+(a-n) \cdot q^{-n}}{0.5 \cdot i \cdot q} + \frac{1-q^{-n}}{0.5 \cdot i^2} \right)}{i \cdot PV} + \gamma \tag{10}$$

$$\text{with} \quad \gamma = (a-n) \cdot (a-n-1) \cdot q^{-2}$$

and PV from Equation (3), adjusted to non-broken periods, i.e. T as the number of outstanding coupons n. To the best of my knowledge, equation (10) is shorter than the formulas previously known, especially those of Brooks (1989), Cole and Young (1995), Blake and Orszag (1996), and Kuipers (2006).

3 Numeric Example for Convexity

As an example, we use a straight bond with a maturity of 7 years and 4 months, a notional of $ 1,000, a coupon rate of 4.2 %, and a market rate of 5 %. By applying

Table 1: Cash-flow table

Time	0	0.33	1.33	2.33	3.33	4.33	5.33	6.33	7.33
CF_t	- 1,000	42	42	42	42	42	42	42	1,042
$CF_t \cdot q^{-t}$	- 1,000	41	39	37	36	34	32	31	729
PV_t	**979.65**	1,029	1,080	1,134	1,191	1,250	1,313	1,378	1,447
$CF_t \cdot q^{-t} \cdot t$		13.77	52.47	87.45	118.99	147.32	172.68	195.29	5,342.95
$CF_t \cdot q^{-t} \cdot t \cdot (t+1)$		18.37	122.44	291.52	515.61	785.69	1,093.63	1,432.14	44,524.57

Formula (1), we need to add the elements of the last row and divide this sum by $ 979.65, the present value of the bond, and by the squared market rate. As a result, we get 45.1678 as approximation for the convexity.

Using the closed-form solution from Equation (10), we obtain the same result:

$$\begin{aligned} \mathbb{C} &= \frac{42 \cdot (-0.201562 - 55.429579 - 214.482107 + 258.528510)}{47.414715} + 55.429579 \\ \mathbb{C} &= 45.1678. \end{aligned} \tag{11}$$

4 Summary

Based on the work of Kruschwitz and Schöbel (1986a) and Kruschwitz and Schöbel (1986b), who have derived a closed-form equation for the Macaulay duration, I have derived a revised closed-form solution for convexity. Although there are closed-form equations available, and Brooks' (1989) approach is rather close to mine, I need significantly less calculation steps, i.e. variables in the equation. Hence, the approach derived in this paper is significantly shorter than previously published formulas, which allows for application with lower effort and reduced error-proneness.

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