Q. Let X and Y have joint p.d.f.

Y \ X	-1	0	1
0	b	2b	b
1	3b	2b	b
2	2b	b	2b

Find marginal distribution of X and Y. Also find conditional distribution of X given Y = 1.

Sol^n : Marginal Distribution table will be given by

$Y \backslash X$	-1	0	1	$p_Y(y)$
0	b	$2b$	b	$4b$
1	$3b$	$2b$	b	$6b$
2	$2b$	b	$2b$	$5b$
$p_X(x)$	$6b$	$5b$	$4b$	$15b$

Marginal Distribution of X is

$P(X=-1) = 6b$, $P(X=0) = 5b$, $P(X=1) = 4b$

Marginal Distribution of Y is

$P(Y=0) = 4b$, $P(Y=1) = 6b$, $P(Y=2) = 5b$

Conditional distribution of X when $Y=1$

$$P(X=x / Y=1) = \frac{P(X=x \cap Y=1)}{P(Y=1)} \begin{cases} \dfrac{P(X=-1 \cap Y=1)}{P(Y=1)} \\[2ex] \dfrac{P(X=0 \cap Y=1)}{P(Y=1)} \\[2ex] \dfrac{P(X=1 \cap Y=1)}{P(Y=1)} \end{cases}$$

$$P(X=x \mid Y=1) = \begin{cases} \frac{3b}{6b} & \text{when } X=-1,\ Y=1 \\ \frac{2b}{6b} & \text{when } X=0,\ Y=1 \\ \frac{b}{6b} & \text{when } X=1,\ Y=1 \end{cases}$$

$$= \begin{cases} \frac{1}{2}, & \text{when } X=-1,\ Y=1 \\ \frac{1}{3}, & \text{when } X=0,\ Y=1 \\ \frac{1}{6}, & \text{when } X=1,\ Y=1 \end{cases}$$

Alternative

$P(X=-1 \mid Y=1) = \frac{1}{2}$, $P(X=0 \mid Y=1) = \frac{1}{3}$, $P(X=1, Y=1) = \frac{1}{6}$

Q. The joint probability distribution of X and Y is given in the following table

X \ Y	1	3	9
2	$1/8$	$1/24$	$1/12$
4	$1/4$	$1/4$	0
6	$1/8$	$1/24$	$1/12$

ⓐ find the marginal probability distribution of Y

(b) Find the conditional distribution of Y given $X = 4$

(c) Find covariance of X and Y.

(d) Are X and Y independent?

Soln: Marginal distribution table is given by

X \ Y	1	3	9	$f_X(x)$
2	$1/8$	$1/24$	$1/12$	$6/24$
4	$1/4$	$1/4$	0	$2/4$
6	$1/8$	$1/24$	$1/12$	$6/24$
$f_Y(y)$	$4/8$	$8/24$	$2/12$	1

(a) Marginal Probability Distribution of Y is

$$P(Y=1) = \frac{4}{8} = \frac{1}{2}$$

$$P(Y=3) = \frac{8}{24} = \frac{1}{3}$$

$$P(Y=9) = \frac{2}{12} = \frac{1}{6}$$

i.e. $$P(Y=y) = \begin{cases} \frac{1}{2}, & y=1 \\ \frac{1}{3}, & y=3 \\ \frac{1}{6}, & y=9 \end{cases}$$

(b) The conditional distribution of Y given $X=4$ is

$$P(Y=y \mid X=4) = \frac{P(Y=y \cap X=4)}{P(X=4)}$$

Now,

$$P(Y=1 \mid X=4) = \frac{P(Y=1 \cap X=4)}{P(X=4)} = \frac{1/4}{2/4} = \frac{1}{2}$$

$$P(Y=3 \mid X=4) = \frac{P(Y=3 \cap X=4)}{P(X=4)} = \frac{1/4}{2/4} = \frac{1}{2}$$

$$P(Y=9 \mid X=4) = \frac{P(Y=9 \cap X=4)}{P(X=4)} = \frac{0}{2/4} = 0$$

i.e. $P(Y=y \mid X=4) = \begin{cases} \frac{1}{2}, & y=1, x=4 \\ \frac{1}{2}, & y=3, x=4 \\ 0, & y=9, x=4 \end{cases}$

(c) $\operatorname{cov}(X,Y) = E[XY] - E[X]\,E[Y] \longrightarrow (1)$

Now, $E[X] = \sum x f_X(x) = \left[\left(2\times\frac{6}{24}\right) + \left(4\times\frac{2}{4}\right) + \left(6\times\frac{6}{24}\right)\right]$

$= 4$

$$E[Y] = \sum y f_Y(y) = \left[\left(1 \times \frac{4}{8}\right) + \left(3 \times \frac{8}{24}\right) + \left(9 \times \frac{2}{12}\right) \right]$$

$$= 3$$

$$E[XY] = \sum xy f_{X,Y}(x,y)$$

$$= \left[\left(2 \times 1 \times \frac{1}{8}\right) + \left(2 \times 3 \times \frac{1}{24}\right) + \left(2 \times 9 \times \frac{1}{12}\right) \right] + \left[\left(4 \times 1 \times \frac{1}{4}\right) + \left(4 \times 3 \times \frac{1}{4}\right) + (4 \times 9 \times 0) \right]$$

$$+ \left[\left(6 \times 1 \times \frac{1}{8}\right) + \left(6 \times 3 \times \frac{1}{24}\right) + \left(6 \times 9 \times \frac{1}{12}\right) \right]$$

$$= 12$$

from ①,

$$\operatorname{cov}(X, Y) = E[XY] - E[X]E[Y] = 12 - (4 \times 3) = 12 - 12 = 0$$

$f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$ ← if true then indp.

Check

(d) $f_{X,Y}(2,1) = ?$

$f_X(2) = ?$

$f_Y(1) = ?$

Check if $f_{X,Y}(2,1) = f_X(2)\, f_Y(1)$ or not

and also check for the rest.