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Engineering mathematics

Properties of trace of a matrix

1) $tr(\lambda A) = \lambda\, tr(A)$
2) $tr(A+B) = tr(A) + tr(B)$
3) $tr(AB) = tr(BA)$

Properties of transpose of a matrix

1) $(A')' = A$
2) $(A+B)' = A' + B'$
3) $(kA)' = kA'$
4) $(AB)' = B'A'$
5) $(ABC)' = C'B'A'$

Properties of conjugate of a matrix

Properties of conjugate of a matrix

1) $\overline{(\bar{A})} = A$
2) $\overline{(A+B)} = \bar{A} + \bar{B}$
3) $\overline{(kA)} = \bar{k}\bar{A}$
4) $\overline{(AB)} = \bar{A}\bar{B}$
5) $\bar{A} = A\,;if\ it's\ a\ real\ matrix$
6) $\bar{A} = -A\,;if\ it's\ a\ purely\ imaginary\ matrix$

Classification of real matrix

Symmetric matrices ($A^T = A$)
Skew symmetric matrix ($A^T = -A$)
Orthogonal matrix ($A^T = A^{-1}$ or $AA^T = I$)

Classification of complex matrix

Hermitian matrix ($A^\theta = A$)
Skew hermitian matrix ($A^\theta = -A$)
Unitary matrix ($A^\theta = A^{-1}$ or $AA^\theta = I$)

Properties of determinant

1. The value of determinant does not change when row and columns are interchange
   i.e. $|A^T| = |A|$
2. $|AB| = |A||B|$
   $|A^n| = |A|^n$
   $|A^{-1}| = 1/|A|$
3. Using the fact that A.AdjA = |A|I, the following can be proved
   a) $|AdjA| = |A|^{n-1}$
   b) $|Adj(AdjA)| = |(AdjA)|^{n-1} = (|A|^{n-1})^{n-1}$

Inverse of a matrix

The matrix inverse A uniquely exist, iff A is non singular ($|A|\neq 0$) and it is given by

$$A^{-1} = \frac{Adj(A)}{|A|}$$

Adjoint of a square matrix

$$Adj\,(A) = [Cof\,(A)]^T$$

Properties of inverse

1) $AA^{-1} = A^{-1}A = I$
2) A and B are inverse of each other iff AB = BA = I
3) $(AB)^{-1} = B^{-1}A^{-1}$
4) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
5) $(A')^{-1} = (A^{-1})'$
6) $(A^\theta)^{-1} = (A^{-1})^\theta$

Rank of a matrix: a no. r is said to be the rank of a matrix if

a. There is atleast one square sub matrix of A of order r whose determinant is not equal to zero.
b. If the matrix A contains any sub matrix of order (r+1), then the determinant of every such matrix is zero.

Elementary matrix: a matrix obtained from unit matrix by a single elementary transformation is called an elementary matrix.

Eigenvalues and Eigen vectors

- The equation $AX = \lambda X$ is called the eigen value problem.
- The matrix $A-\lambda I$ is called characteristic matrix of A.

Properties of Eigenvalues

1. If $\lambda_1, \lambda_2 \dots\dots\dots \lambda_n$ are the Eigenvalues of A, then $k\lambda_1, k\lambda_2 \dots\dots\dots k\lambda_n$ are eigen values of kA
2. Eigen values of A^{-1} are reciprocals of eigen values of A
3. If $\lambda_1, \lambda_2 \dots\dots\dots \lambda_n$ are the Eigenvalues of A, then $\lambda_1^k, \lambda_2^k \dots\dots\dots \lambda_n^k$ are eigen values of A^k
4. If $\lambda_1, \lambda_2 \dots\dots\dots \lambda_n$ are the Eigenvalues of A, then $|A|/\lambda_1, |A|/\lambda_2 \dots\dots\dots |A|/\lambda_n$ are eigen values of Adj(A).
5. Eigen values of A = eigen values of A^T
6. Maximum distinct no. of eigen values = size of the matrix
7. Sum of the eigen values = trace of A = sum of the diagonal matrix
8. Product of the eigen values = |A|; (at least one eigen value is 0 for A to be singular)
9. In a triangular and diagonal matrix, eigen values are diagonal elements themselves
10. Similar matrix are having same eigen values.
    Two matrices are said to be similar, if there exists a non singular matrix P
    such that $B = P^{-1}AP$.
11. If A and B are two matrix of same order then AB and BA will have same eigen values.

The cayley – Hamilton theorem-

- Every square matrix satisfy its own characteristic equation
- This means that if $C_0\lambda^n + C_1\lambda^{n-1} + \dots\dots\dots + C_n = 0$ then
  $C_0A^n + C_1A^{n-1} + \dots\dots\dots + C_nI = 0$

Similar matrix

- Two matrix A and B are said to be similar, if there exist a non singular matrix P
  Such that $B = P^{-1}AP$

Lagrange's mean value theorem

If a function f(x) is

1. Continuous in [a, b]
2. Differentiable in (a, b) and
3. $f'(c) > 0$ for all x in (a, b) then f(x) is strictly increasing function in [a, b]
   or if $f'(c) < 0$ for all x in (a, b) then f(x) is strictly decreasing function in [a, b]

Derivative of a function

Differentiation rules

1. $(f + g)' = f' + g'$
2. $(f - g)' = f' - g'$
3. $(f.g)' = fg' + f'g$
4. $\left(\frac{f}{g}\right)' = \frac{gf'-fg'}{g^2}$
5. $\frac{d}{dx}(f(g(x))) = \frac{df}{dg}.\frac{dg}{dx}$

Integration by parts

$$\int uvdx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx$$

ILATE rule to be followed to choose as integral function.

Some important formulas

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{(a^2+b^2)}(asinbx - bcosbx)$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{(a^2+b^2)}(acosbx + bsinbx)$$

$$\int_0^{\pi/2}\sin^n x\cos^n x\,dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$$

Where $\Gamma(n+1) = n\Gamma n = n!$

$\Gamma(1) = 1$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Walle's formula

$$\int_0^{\frac{\pi}{2}}\sin^n x = \int_0^{\frac{\pi}{2}}\mathrm{Cos}^n x = \begin{Bmatrix} \frac{(n-1)(n-3)(n-5)}{n(n-2)(n-4)}\dots\dots\frac{2}{3}, when\ n\ is\ odd \\ \frac{(n-1)(n-3)(n-5)}{n(n-2)(n-4)}\dots\dots\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}, when\ n\ is\ even \end{Bmatrix}$$

Linear equation of first order

A differential equation is said to be linear if the dependent variable and its differential coefficients occur only in the first degree and not multiplied together

Leibnitze linear equation

$\frac{dy}{dx} + Py = Q; where\ P\ and\ Q\ are\ arbitary\ functions\ of\ x$

Bernoulli's equation

The equation $\frac{dy}{dx} + Py = Qy^n$; where P and Q are the functions of x

Exact differential equations

A differential equation of the form M(x,y)dx + N(x,y)dy = 0 is said to be exact, if

du = Mdx + Ndy

its solution is therefore u(x,y) = C

Inverse operator

$$f(D)\left\{\frac{1}{f(D)}X\right\} = X$$

Operator $\frac{1}{f(D)}X$ satisfy the equation f(D)y = X and is therefore, its particular integral

f(D) and $\frac{1}{f(D)}$ are inverse operators

Some important conversions

$$\frac{1}{D}X = \int xdy$$

$$\frac{1}{(D-a)}X = e^{ax}\int Xe^{-ax}dx$$

Complex functions

The necessary and sufficient conditions for the derivative of the function w = u(x,y) + iv(x,y) = f(z) to exist for all the values of z in regeion R, are

1. $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ & $\frac{\partial v}{\partial y}$ are continuous function of x and y in R
2. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ **(C-R equation)**

Note: A function is said to be analytic, if Cauchy rieman equation is satisfied

Cauchy's integral formula

$$f(a) = \frac{1}{2\pi i}\int_C\frac{f(z)dz}{z-a}$$

RESIDUE

The coefficient of $(z-a)^{-1}$ in the expression of f(z) around a isolated singularity is called the residue of f(a) at that point

$$a_n = \frac{1}{2\pi i}\int\frac{f(z)}{(z-a)^{n+1}}\,dz$$

$$a_{-1} = Res\,f(a) = \frac{1}{2\pi i}\int_C f(z)dz$$

If f(z) is analytic in a closed curve C except at finite no. of points with in c, Then

$$\int_C f(z)dz = 2\pi i\,X\,(sum\ of\ residue\ at\ the\ singular\ points\ within\ C)$$

Components of vector

If initial point P: (x_1, y_1, z_1) and terminal point Q: (x_2, y_2, z_2)

Then $a_1 = x_2-x_1$, $a_2 = y_2-y_1$, $a_3 = z_2-z_1$ are called the components of a vector PQ

$|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

General properties of vector product

- i x j = k;   j x k = i;   k x i = j
- j x i = -k; k x j = -i;   i x k = -j
- i x i = j x j = k x k =0
- (k**a**) x **b** = k (**a** x **b**) = **a** x (k**b**)
- **a** x (**b** + **c**) = (**a** x **b**) + (**a** x **c**)
- **b** x **a** = **a** x **b** (not commutative but anticommutative)
- **a** x (**b** x **c**) ≠ (**a** x **b**) x **c** (not associative)

Scalar triple product [a b c] = a. (b x c)

Vector triple product

a x (**b** x **c**) = (**a.c**) **b** – (**a.b**) **c**

Gradient of a scalar field

$$grad\,f = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}i + \frac{\partial f}{\partial z}k = \nabla f$$

$$\nabla = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}i + \frac{\partial}{\partial z}k$$

Direction derivatives

$$D_b f = \frac{df}{ds} = \frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z'$$

Divergence of a vector field $(\nabla.V)$

$$Div\,(V) = \nabla.V = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right).(V_1 i + V_2 j + V_3 k)$$

Curl of a vector field $(\nabla X V)$

Line integral

The line integral of a vector function F(r) over a curve is defined by

$$\int_C F(r).dr = \int_a^b F(r(t)).\frac{dr}{dt}dt$$

Line integral independent on path

A line integral with continuous F_1, F_2 and F_3 in a domain D in space is independent of path in D if and only if F = [F_1, F_2, F_3] is the gradient of some function f in D

$$F = grad\,f = \frac{df}{dx}i + \frac{df}{dy}j + \frac{df}{dz}k$$

Green's theorem

$$\iint_R\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dxdy = \oint_C(F_1dx + F_2dy)$$

Triple integrals: divergence theorem of **Gauss**

$$\iiint_T\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)dxdydz = \iint_S(F_1dydz + F_2dzdx + F_3dxdy)$$

Stokes theorem

$$\iint_S\left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)N_1 - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)N_2 + k\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)N_3\right]dudv = \oint_C(F_1dx + F_2dy + F_3dz)$$

Probability & statics

Types of event

1. Complementary event
   In a dice throw, if E = {even no.} = {2,4,6} then E^c = {odd nos} = {1,3,5}
2. Equally likely event
   P(E) = P(F); ex. tossing a coin and probability of getting head or tail =1/2
3. Mutually exclusive events
   If E∩F = φ, i.e. P(E∩F) = 0
4. Collectively exhaustive events
   Two events E and F are collectively exhaustive, if E∪F = S, i.e. together E and F includes all possible outcomes
5. Independent events
   Two events E and F are independent iff
   P(E∩F) = P(E)*P(F)
   Also P(E|F) = P(E∩F)/P(E) = P(F)

Approaches to probability

1. Classical approach
   $$P(E) = \frac{n(E)}{n(S)} = \frac{|E|}{|S|}$$

Rules of probability

- P(A∪B) = P(A)+P(B) – P(A∩B)
  If two mutually exclusive events are there then P(A∩B) = 0
- P(A∩B) = P(A)*P(B/A) = P(B)*P(A/B)
- P(A∩B) is also called the joint probability of A and B
- If A and B are independent events then
  P(A∩B) = P(A)*P(B)
- If A, B and C are three independent events then
  P(A∩B∩C) = P(A)P(B)P(C)

P(E) = P(A∩E) + P(B∩E) = P(A)*P(E|A) + P(B)*P(E|B)

Bays theorem

$$P(A|E) = \frac{P(A\cap E)}{P(E)} = \frac{P(A\cap E)}{P(A\cap E) + P(B\cap E)} = \frac{P(A)*P(E|A)}{P(A)*P(E|A) + P(B)*P(E|B)}$$

Arithmetic mean

$\bar{X} = \frac{\Sigma x}{n}$

Median (central value of the distribution)

Median for the grouped data

1. Identify the median class
2. Calculate median by
   $$Median = L + \left[\frac{\left(\frac{N+1}{2}\right) - (F+1)}{f_m}\right]X\,h$$

Mode: It is the value of the variable which occurs most frequently

Mode of the grouped data

1. Identify the class which has the largest frequency
2. Calculate mode as follows
   $$Mode = L + \left[\frac{f_0 - f_1}{2f_0 - f_1 - f_2}\right]X\,h$$

Mean median and mode relation

1. Mode = 3 Median – 2 Mean (approximate relation)
2. Frequency distribution
   a. Positively skewed distribution
      Mode ≤ Median ≤ Mean
   b. Symmetric distribution
      Mode = Median = Mean
   c. Negatively distribution
      Mean ≤ Median ≤ Mode

Coefficient of variation (CV = σ/μ)

In % CV = $\frac{\sigma}{\mu}X\,100$

Properties of discrete distribution

$\sum P(X) = 1$

E(X) = $\sum X\,P(X)$ is called the expected value or the average value of a random variable

$$V(X) = E(X^2) - (E(X))^2 = \sum X^2\,P(X) - \left[\sum X\,P(X)\right]^2$$

V(X) denotes the variance of the random variable x.

Properties of continuous distribution

$\int_{-\infty}^{\infty}f(x)dx = 1$

$$F(x) = \int_{-\infty}^{x}f(x)dx$$

F(x) is called cumulative distribution function

$E(x) = \int_{-\infty}^{\infty}x\,f(x)dx$

$V(x) = E(X^2) - (E(X))^2 = \int_{-\infty}^{\infty}x^2f(x)dx - \left[\int_{-\infty}^{\infty}xf(x)dx\right]^2$

$P(a < x < b) = \int_a^b x\,f(x)dx$

Properties of expected value and variance

- $E(aX + b) = aE(X) + b$
- $V(aX + b) = a^2E(X) + b$
- $E(aX_1 + bX_2) = aE(X_1) + bE(X_2)$
- $V(aX_1 + bX_2) = a^2V(X_1) + b^2V(X_2) + 2ab\,cov\,(X_1, X_2)$

$COV(X,Y) = E(XY) - E(X)E(Y)$

Binomial distribution (only two outcomes possible, success & failure)

The probability of getting x success from n trials is given by the binomial distribution formula

P(X = x) = ${}^nC_x\,P^x(1-P)^{n-x}$

<u>For binomial distribution</u>

- Mean = E(X) = nP
- Variance = V(X) = nP(1-P)

Poisson's distribution

A random variable X which can take one of the values from 0, 1, 2, 3 ………. is said to be a poisson's random variable with parameter λ if for some λ>0,

P(X = x) = $\frac{e^{-\lambda}\lambda^x}{x!}$

- Mean = E(X) = λ
- Variance = V(X) = λ

λ = average no. of occurrence of event in an observation period Δt. so
λ = αΔt, where α = no. of occurrences of events per unit time

General continuous distribution

For a given **probability density function** f(x) and continuous random variable x

$$P(-\infty \le X \le \infty) = \int_{-\infty}^{\infty}f(x)dx$$

The expected value of x is given by

$$\mu_x = E(X) = \int_{-\infty}^{\infty}Xf(x)dx$$

The cumulative probability function (or **probability distribution function**) F(x)

$$F(X) = P(X \le x) = \int_{-\infty}^{x}f(x)dx$$

Uniform distribution

For discrete uniform distribution

- Mean = E(X) = $\frac{\alpha+\beta}{2}$
- Variance = V(X) = $\frac{(\beta-\alpha)^2}{2}$

Normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty$$

The density function is the bell shaped curve and that is symmetric about μ

For normal distribution

Mean = E(X) = μ and variance = V(X) = σ^2

<u>Standard</u> normal distribution

If we take μ = 0 and σ = 1, then the distribution is standard

Newton raphson method

$$x_{N+1} = x_N - \frac{f(x_N)}{f'(x_N)}$$

Laplace transformations

- $L\{f(t)\} = \int_0^\infty e^{-st}f(t)dt$
- L{f(t)} being a function of s and written also as $\bar{f}(s)$ or as F(s)
- f(t) can also be written as f(t) = $L^{-1}\{\bar{f}(s)\}$

Standard Laplace transformations

1. $L(1) = \frac{1}{s}$ (s>0)
2. $L(t^n) = \frac{n!}{s^{n+1}}$, where n = 0, 1, 2, 3 …….
3. $L(e^{at}) = \frac{1}{s-a}$ (s>a)
4. $L(\sin at) = \frac{a}{s^2+a^2}$ (s<0)
5. $L(\cos at) = \frac{s}{s^2+a^2}$ (s>0)
6. $L(\sinh at) = \frac{a}{s^2-a^2}$ (s > |a|)
7. $L(\cosh at) = \frac{s}{s^2-a^2}$ (s > |a|)

Properties of the laplace transformation

$L\{af(t) + bg(t) - ch(t)\} = a\,L\{f(t)\} + b\,L\{g(t)\} - c\,L\{h(t)\}$

If L{f(t)} = $\bar{f}(s)$ then

- $L\{e^{at}f(t)\} = \bar{f}(s-a)$
- $L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$

Transformation derivative

- $L\{f'(t)\} = s\bar{f}(s) - f(0)$
- If f'(t) and its first (n-1) derivatives be continuous, then
  $L\{f^n(t)\} = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots\dots\dots\dots - f^{n-1}(0)$
- $L\{\int_0^t f(u)du\} = \frac{1}{s}\bar{f}(s)$

Multiplication by t^n

- $L\{t^nf(t)\} = (-1)^n\frac{d^n}{dx^n}[\bar{f}(s)]$

Division by t

- $L\left\{\frac{1}{t}f(t)\right\} = \int_0^\infty\bar{f}(s)\,ds$

Unit step function

- The unit function u(t-a) is defined as
- $u(t-a) = \begin{cases} 0 & for\ t < a \\ 1 & for\ t \ge a \end{cases}$

Transformation of unit function

- $L\{u(t-a)\} = \int_0^\infty e^{-st}u(t-a)dt$