Moment Generating Function (m.g.f.)

$$E(e^{tX})$$

$$M_X(t) = \int e^{tx} f(x)\,dx$$

Q. Find the m.g.f. of the random variable X having the probability density function

$$f(x) = \begin{cases} x & , \quad 0 \le x < 1 \\ 2-x & , \quad 1 \le x < 2 \\ 0 & , \quad \text{otherwise} \end{cases}$$

Find mean and variance of X using m.g.f.

Soln:

$$M_X(t) = E\left[e^{tX}\right]$$

$$= \int_0^1 e^{tx}\, x\, dx + \int_1^2 e^{tx}(2-x)\, dx$$

$$= \left\{ \left[x \int e^{tx}\, dx \right]_0^1 - \int_0^1 \left(\frac{d}{dx} x \int e^{tx}\, dx \right) dx \right\} +$$

$$\left\{ \left[(2-x) \int e^{tx}\, dx \right]_1^2 - \int_1^2 \left(\frac{d}{dx}(2-x) \int e^{tx}\, dx \right) dx \right\}$$

$$= \left[x \left(\frac{e^{tx}}{t} \right) \right]_0^1 - \int_0^1 \frac{e^{tx}}{t}\, dx + \left[(2-x) \cdot \frac{e^{tx}}{t} \right]_1^2 + \int_1^2 \frac{e^{tx}}{t}\, dx$$

$$M_X(t) = \left(\frac{1 \cdot e^t}{t} - 0\right) - \left[\frac{e^{tx}}{t^2}\right]_0^1 + \left[0 - \frac{e^t}{t}\right] + \left[\frac{e^{tx}}{t^2}\right]_1^2$$

$$= \frac{e^t}{t} - \left(\frac{e^t}{t^2} - \frac{1}{t^2}\right) - \frac{e^t}{t} + \left(\frac{e^{2t}}{t^2} - \frac{e^t}{t^2}\right)$$

$$= -\frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{e^t}{t^2}$$

$$= \frac{e^{2t}}{t^2} - \frac{2e^t}{t^2} + \frac{1}{t^2} = \frac{1}{t^2}\left[e^{2t} - 2e^t + 1\right] \quad \text{---} \ 1$$

$$= \frac{(e^t)^2 - 2e^t + 1}{t^2}$$

$$M_X(t) = \frac{(e^t - 1)^2}{t^2} \qquad \text{——— } (1)$$

Expanding $M_X(t)$ using (1)

$$M_X(t) = \frac{1}{t^2}\left[\left(1+2t+\frac{(2t)^2}{2!}+\frac{(2t)^3}{3!}+\frac{(2t)^4}{4!}+\dots\right) - 2\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\dots\right)+1\right]$$

$$= \frac{1}{t^2}\left[t^2+t^3+\frac{7}{12}t^4+\dots\right]$$

$$M_X(t) = 1+t+\frac{7}{12}t^2+\dots$$

Mean $= \mu_1' =$ coefficient of t in $M_X(t) = 1$

$\mu_2' =$ coefficient of $\frac{t^2}{2!}$ in $M_X(t) = 2!\,\frac{7}{12} = \frac{7}{6}$

variance $(\mu_2) = \mu_2' - \mu_1'^2$

$= \frac{7}{6} - (1)^2$

$= \frac{1}{6}$

NORMAL DISTRIBUTION

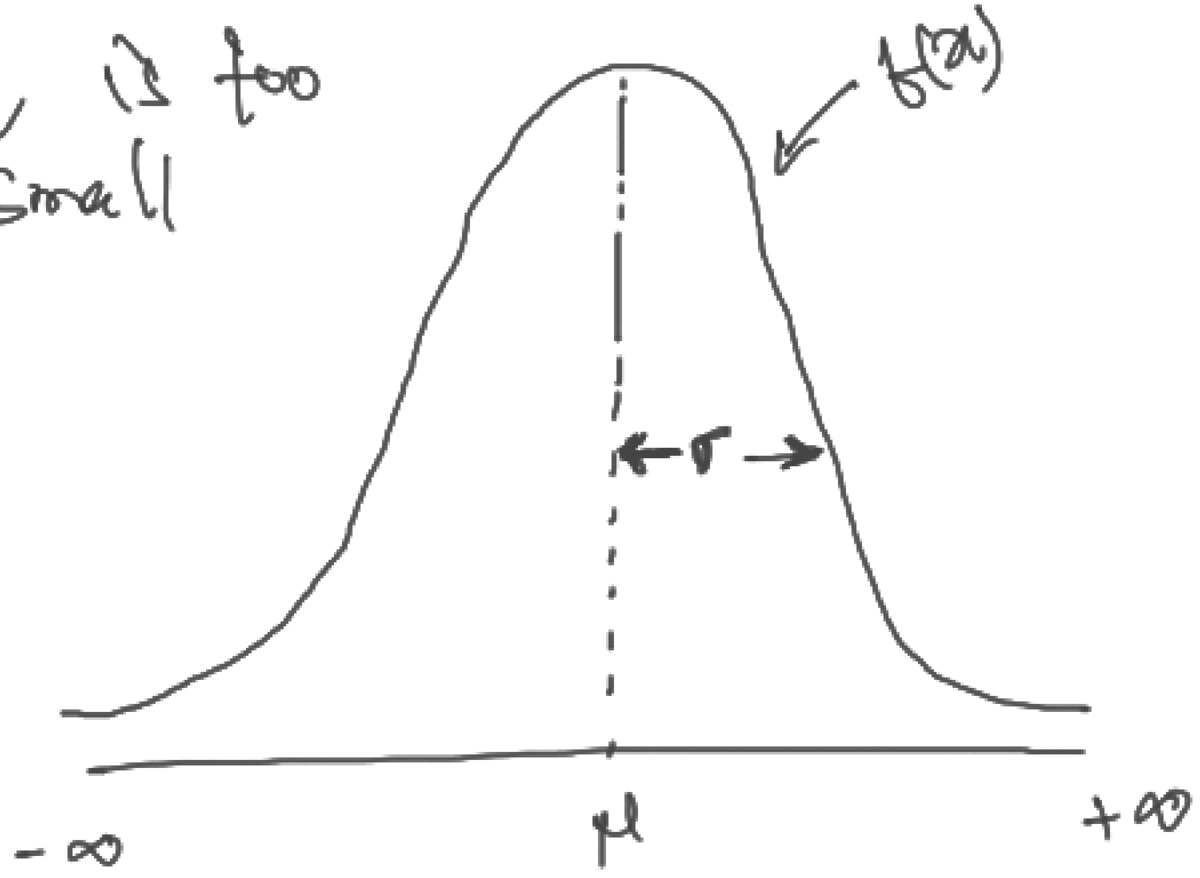
Limiting case of Binomial Distribution

$p \to q$ ← neither p nor q is too small

$n \to \infty$ ← infinitely large

Discovered by De Moivre in 1733

It is continuous R.V. distr.



$X \leftarrow$ continuous R.V.

Then X is said to have normal distribution if its p.d.f. is defined as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad , \quad -\infty < x < \infty$$

$$\sigma > 0$$

Here $\mu, \sigma \leftarrow$ parameters of Normal Distribution

Binomial	Poison	Normal.
1. Discrete R.V.	Discrete R.V.	continuous R.V.
2. 2 parameters (p or q and n)	One parameter (λ)	Two parameters (μ, σ)
3. —	Limiting case of Binomial dist. $p \to 0$, $n \to \infty$	Limiting case binomial distri. $p \to q$, $n \to \infty$
4. $P(r) = {}^nC_r \, p^r q^{n-r}$	$P(r) = \dfrac{e^{-\lambda} \lambda^r}{r!}$	$f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ pdf

B.D	P.D.	N.D
5. mean $= np$	mean $= \lambda$	mean $= \mu$
var $= npq$	var $= \lambda$	var $= \sigma^2$

values det. from formula given at (4)

Q. Prove that the mean and variance of the normal distribution

Mean $= \mu$

Variance $= \sigma^2$

8. For normal distribution

Mean $= \mu$

Median $= \mu$

Mode $= \mu$

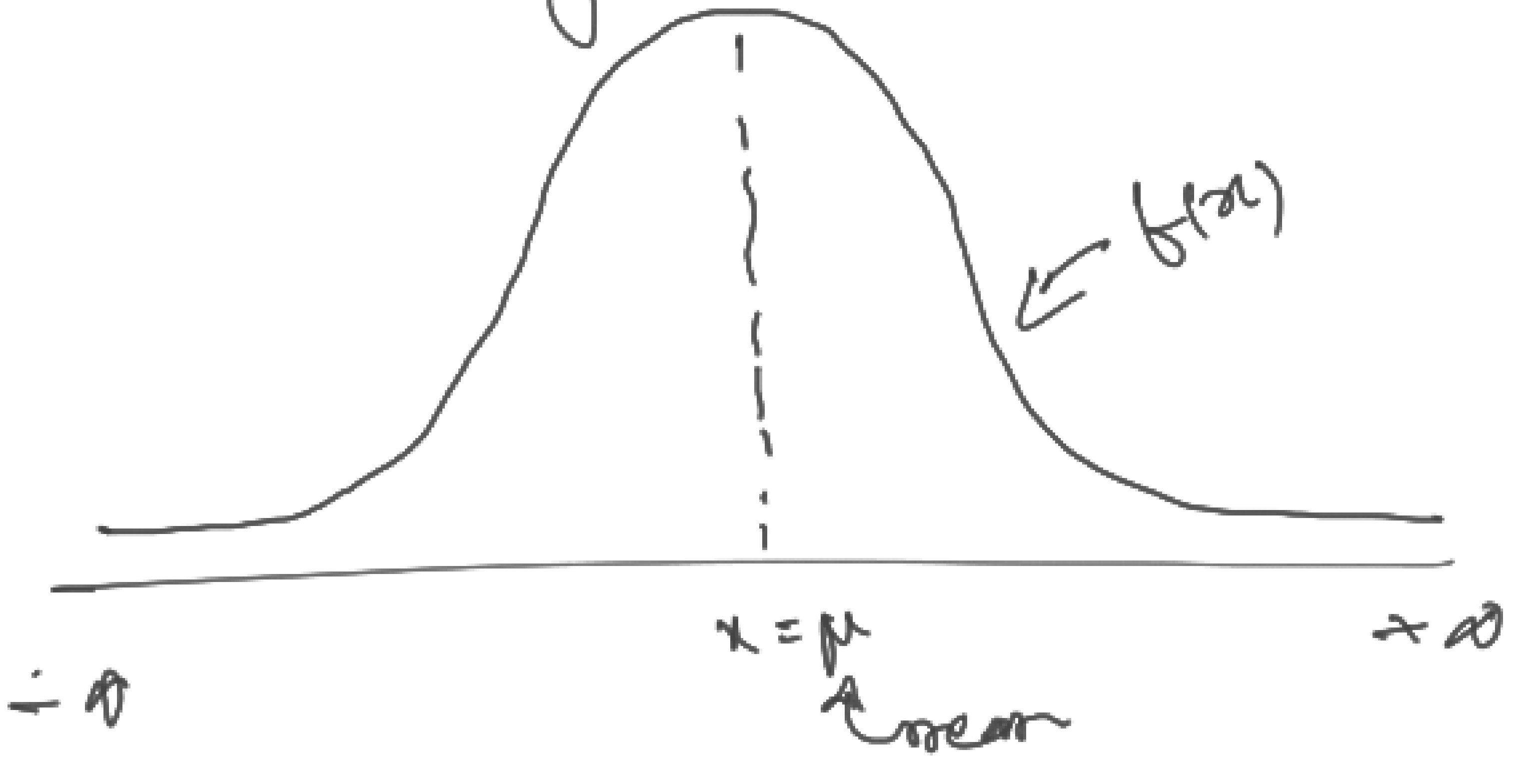
* In case of normal distribution, mean = median = mode.

Properties of Normal Distribution

① The normal probability curve with mean μ and standard deviation σ is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad , \quad -\infty < x < \infty$$

2. The curve is bell-shaped and symmetrical about the line line $x = \mu$



3. Mean, median and mode of the normal distribution coincides i.e. unimodal.

4. $f(x)$ decreases rapidly as x increases.

5. X-axis is an asymtote to the curve.

6. Maximum probability occurs at the point $x = \mu$

and $\max^m$ prob $= \dfrac{1}{\sigma\sqrt{2\pi}}$

7. $M.D.(\bar{x}) = \frac{4}{5}\sigma$ or $M.D.(\mu) = \frac{4}{5}\sigma$

8. The point of inflextion of curve are at $x = \mu \pm \sigma$

(i) Area of the normal curve between $(\mu - \sigma)$ and $(\mu + \sigma)$ is

0.6826 i.e. $P(\mu-\sigma < X < \mu+\sigma) = 0.6826$

(ii) $P(\mu-2\sigma < X < \mu+2\sigma) = 0.9544$

(iii) $P(\mu-3\sigma < X < \mu+3\sigma) = 0.9973$

Area betn — k —