JOINT CONTINUOUS DENSITY FUNCTION

A two dimensional R.V. (X, Y) is said to be continuous iff $\exists$ a f^n $f_{X,Y}(x,y) \geq 0$ s.t.

$$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\,du\,dv$$

The f^n $f_{X,Y}(x,y)$ or $f(x,y)$ is called Joint Probability Density f^n.

Properties

(i) $f(x,y) \geq 0 \quad \forall \ (x,y) \in \mathbb{R}^2$

(ii) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

Note: $F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$

Marginal and Conditional Density Function

$X, Y \leftarrow$ Joint continuous R.V.

$f_{X,Y}(x,y) \leftarrow$ p.d.f having

Then,

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$ $\leftarrow$ marginal probability density f^n of X

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$ $\leftarrow$ marginal probability density f^n of Y

Conditional probability f^n of Y given $X=x$ is

$$f_{Y/X}(y/x) = \frac{f_{X,Y}(x,y)}{f_X(x)} \quad \text{if } f_X(x) > 0$$

Conditional Probability f^n of X given $Y=y$ is

$$f_{X/Y}(x/y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \quad \text{if } f_Y(y) > 0$$

Conditional Cumulative Distributive

$f_{XY}(x,y)$ this one

$f_{X,Y}(x,y)$ in general

$X, Y \leftarrow$ Joint Continuous R.V.

with

$f_{X,Y}(x,y) \leftarrow$ pdf

Then conditional cumulative distribution of Y when $X = x$

$$F_{Y/X}(y/x) = \int_{-\infty}^{y} f_{Y/X}(t/x)\, dt$$

conditional cumulative distribution of X when $Y = y$

$$F_{X/Y}(x/y) = \int_{-\infty}^{x} f_{X/Y}(t/y)\, dt$$

Note: if X and Y independent then

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y).$$

Expectation

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f(x,y)\, dx\, dy$$

In Particular,

$$E[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\, dx\, dy$$

$$E[Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f(x,y)\, dx\, dy$$

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$$

Moment Generating Function

$$M_{X,Y}(t_1, t_2) = E\left[e^{t_1 X + t_2 Y}\right]$$

Q. Find k so that $f(x,y) = kxy$, $1 \le x \le y \le 2$ will be a joint probability density fn.

Soln: $\because$ $f(x,y)$ is a joint probability density fn

$\therefore$ $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1.$$

$$\Rightarrow \int_{1}^{2}\int_{x}^{2} kxy\,dy\,dx = 1$$

$$\Rightarrow k\int_{1}^{2} x\left(\int_{x}^{2} y\,dy\right)dx = 1$$

$1 \le x \le y \le 2$

$1 \le x \le y$

$1 \le y \le 2$

or

$1 \le x \le 2$

$x \le y \le 2$

$$\int_{x=1}^{x=2}\int_{y=x}^{y=2}$$

$$\Rightarrow k \int_1^2 x \left[\frac{y^2}{2}\right]_x^2 dx = 1$$

$$\Rightarrow k \int_1^2 x \left(2 - \frac{x^2}{2}\right) dx = 1$$

$$\Rightarrow k \int_1^2 \left(2x - \frac{x^3}{2}\right) dx = 1$$

$$\Rightarrow k \left\{ \left[\frac{2x^2}{2}\right]_1^2 - \frac{1}{2}\left[\frac{x^4}{4}\right]_1^2 \right\} = 1$$

$$\Rightarrow k \left\{ [4-1] - \frac{1}{8}[16-1] \right\} = 1$$

$$\Rightarrow \quad \frac{9k}{8} = 1$$

$$\Rightarrow \quad K = \frac{8}{9}$$

Q. Find K so that $f(x, y) = K(x+y)$, $0 < x < 1$ and $0 < y < 1$, is a joint probability density fn.

Ans. $K = 1$

Q. The joint p.d.f. of (X, Y) is given by

$$f(x,y) = \begin{cases} 2, & 0 < x < 1, \quad 0 < y < x \\ 0, & \text{elsewhere} \end{cases}$$

ⓐ find marginal density f^n of X and Y

ⓑ find the conditional density f^n of Y given $X = x$ and that of X given $Y = y$

ⓒ Are X and Y independent ?

Sol^n: Given,

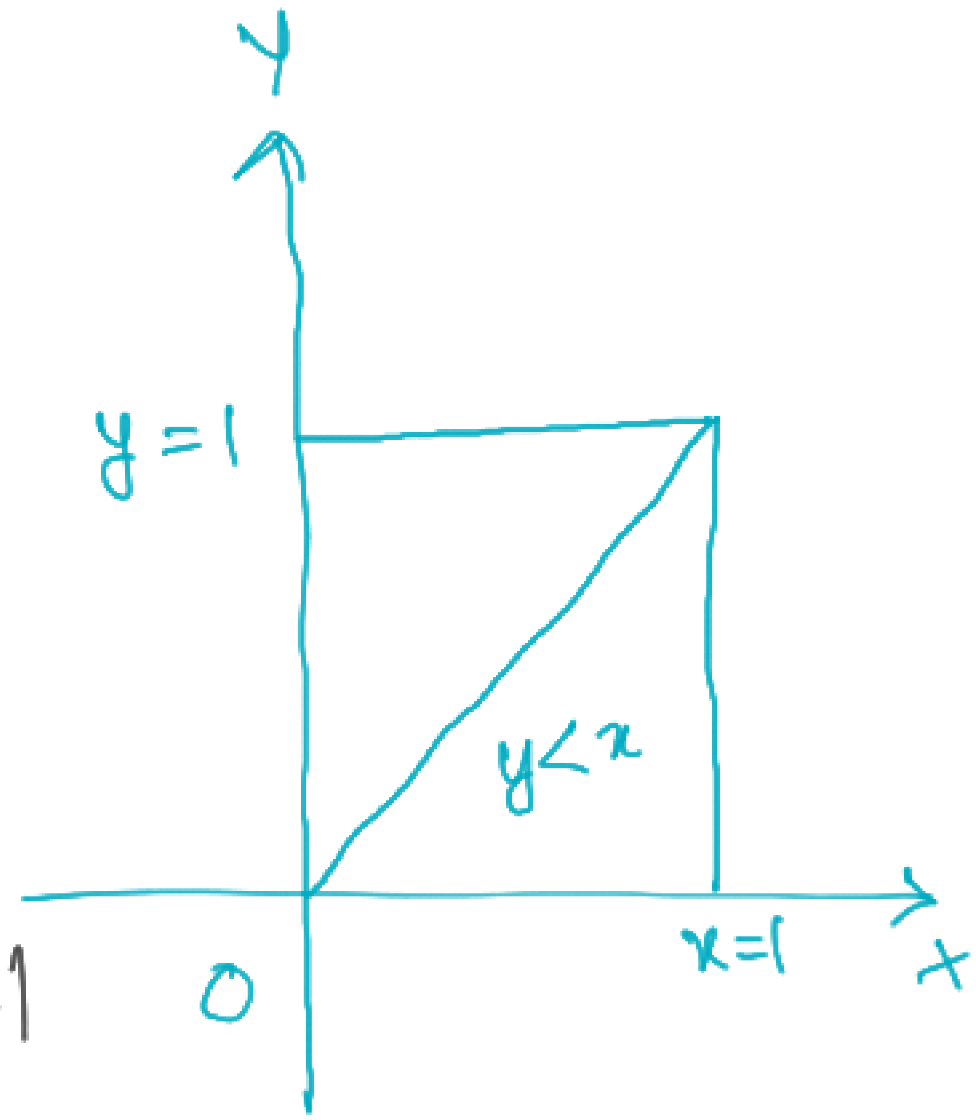
$$f(x,y) = \begin{cases} 2 & , \ 0<x<1, \ 0<y<x \\ 0 & , \ \text{elsewhere.} \end{cases}$$

(i) Marginal density f^n of X is

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_0^x 2\,dy \qquad 0<x<1$$

$$= \left[2x\right]_0^x = 2x \quad , 0<x<1$$



$$f_X(x) = \begin{cases} 2x , & 0 < x < 1 \\ 0 , & \text{elsewhere} \end{cases}$$

marginal density f^n of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

$$= \int_{y}^{1} 2\, dx , \quad 0 < y < 1$$

$$= 2 \left[x \right]_{y}^{1} , \quad 0 < y < 1$$

$= 2(1-y) \quad , \quad 0 < y < 1$

$$f_Y(x) = \begin{cases} 2(1-y) \quad , & 0 < y < 1 \\ 0 \quad , & \text{elsewhere} \end{cases}$$

(ii) Conditional density f^n of Y when $X = x$ is

$$f_{Y/X}(y/x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{f(x,y)}{f_X(x)}$$

$$= \begin{cases} \frac{2}{2x} \quad , & 0 < x < 1 \\ 0 \quad , & \text{elsewhere} \end{cases}$$

$$f_{Y/X}(y|x) = \begin{cases} \frac{1}{x}, & 0<x<1 \\ 0, & \text{else} \end{cases}$$

Conditional density fn of X given Y = y

$$f_{X/Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

$$= \begin{cases} \frac{2}{2(1-y)}, & 0<y<1 \\ 0, & \text{elsewhere.} \end{cases}$$

we know that,

$$f_X(x)\, f_Y(y) = f(x, y) \quad \text{for independent}$$

Now,

$$f_X(x)\, f_Y(y) = 2x \cdot 2(1-y) = 4x(1-y)$$

$$f(x, y) = 2$$

$$\Rightarrow f_X(x)\, f_Y(y) \neq f(x, y)$$

$\Rightarrow$ Not independent