MARKOV CHAIN

Stochastic Process : The collection of the R.V. $\{x_t : t \in T\}$ defined on some probability space (Ω, F, P) is called stochastic process.

Sample space $\to \Omega$

State space $\to S$

(a) T — index set

(b) Values of R.V. x_t takes is called state space. It is denoted by S

Discrete time Markov Chain : A seqn $\{x_n\}_{n \in \mathbb{N}}$ of R.V. with discrete state space is called DTMC if

The Land of Oz is blessed by many things, but not by good weather. They never have two nice days in a row. If they have a nice day, they are just as likely to have snow as rain the next day. If they have snow or rain, they have an even chance of having the same the next day. If there is change from snow or rain, only half of the time is this a change to a nice day.Take the states as R, N, and S for the weather. Also give the transition matrix.

$R \to N \;:\; 1/4$

$R \to R \;:\; 1/2$

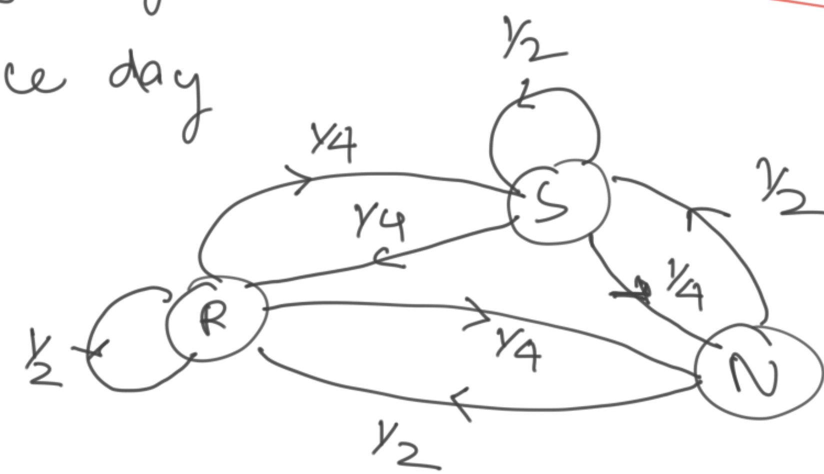
$\boxed{1 - \frac{3}{4} = \frac{1}{4}}$

$R \leftarrow$ rainy day

$S \leftarrow$ snowy day

$N \leftarrow$ nice day

Transition diagram is



Transition matrix

$$P = \begin{array}{c|ccc} & N & R & S \\ \hline N & 0 & 1/2 & 1/2 \\ R & 1/4 & 1/2 & 1/4 \\ S & 1/4 & 1/4 & 1/2 \end{array}$$

In the Dark Ages, Harvard, Dartmouth, and Yale admitted only male students. Assume that, at that time, 80 percent of the sons of Harvard men went to Harvard and the rest went to Yale, 40 percent of the sons of Yale men went to Yale, and the rest split evenly between Harvard and Dartmouth; and of the sons of Dartmouth men, 70 percent went to Dartmouth, 20 percent to Harvard, and 10 percent to Yale. Give the tansition matrix and transition diagram.

H ← Harvard men sons

D ← Dartmouth men sons

Y ← Yale men sons

$H \to H : 80\% = \frac{80}{100} = 0.8$

$H \to D : 0\%$

$H \to Y : 20\% = 0.2$

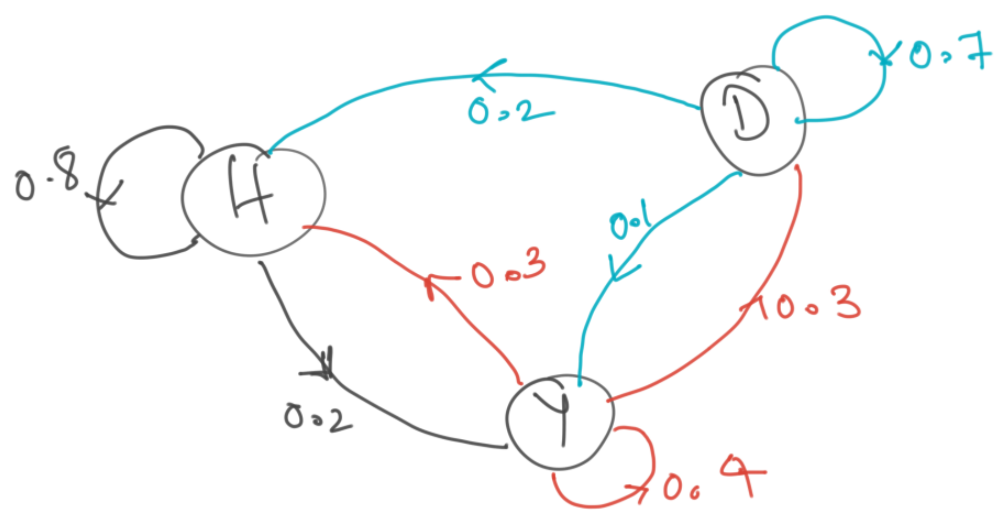
$Y \to Y : 40\% = 0.4$

Rest : 60%

30% 30%

$Y \to H$ $Y \to D$

0.3 0.3



This is the transition diagram

Now,

Transition matrix is given by

$$P = \begin{array}{c} \\ H \\ Y \\ D \end{array} \begin{array}{c} \begin{array}{ccc} H & Y & D \end{array} \\ \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.1 & 0.7 \end{bmatrix} \end{array}$$

$$P(X_{n+1} = j \mid \underset{\text{present}}{X_n = i_n}, \underbrace{X_{n-1} = i_{n-1}, \ldots, X_1 = i_1}_{\text{past}})$$

future

$$= P(X_{n+1} = j \mid X_n = i_n) \quad \forall \; i_n \in S.$$

future present

i.e. future value depends only on the present value and not on the past values.

STATE & STEP

Transition Probability : $P_{ij} = P(X_{n+1} = j \mid X_n = i)$

Probability of going from state i to j in one step.

n

$n+1$ step

$$P_{ij} = P(X_{n+1} = j \mid X_n = i)$$

n

m-step

$n+m$

$$p_{ij}^{(m)} = P(X_{n+m} = j \mid X_n = i)$$

$\hookrightarrow$ prob. of $(i \to j)$ in m steps

Transition Probability Matrix

$$P = P_{ij} = \begin{bmatrix} P_{00} & P_{01} & P_{02} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots \\ - & - & - & \end{bmatrix}$$

$$P_{ij} \geq 0 \quad \forall \ i, j$$

$\sum P_{ij} = 1$ i.e. each row sum = 1

The matrix whose row sum is 1 is called stochastic matrix

" " " row & column sum is 1 is called doubly stochastic

$P_{ij} \leftarrow$ 1 step

2 steps $\longrightarrow$ $P^2_{ij} = P_{ij} \cdot P_{ij}$

3 steps $\longrightarrow$ $P^3_{ij} = P^2_{ij} \cdot P_{ij}$

Q. Given that a person's last cola purchased was a coke there is 90% chance that his next cola will be a coke. If a person's last cola was pepsi there is 80% chance that his next cola will be pepsi. Find the transition matrix and also draw the transition diagram

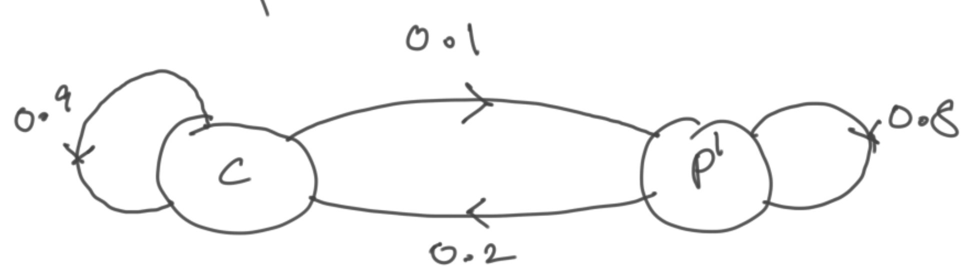
Let, C = person who purchase coke

P' = " " " pepsi

90%

↓

$\frac{90}{100} = 0.9$



$P \to P$

80%

↓

$\frac{80}{100} = 0.8$

This is known as transition diagram

Transition matrix is given by

$$P = \begin{array}{c|cc} & C & P' \\ \hline C & 0.9 & 0.1 \\ P' & 0.2 & 0.8 \end{array}$$

$\leftarrow$ states

$\uparrow$ states

This is called Transition matrix

states $\rightarrow C, P'$

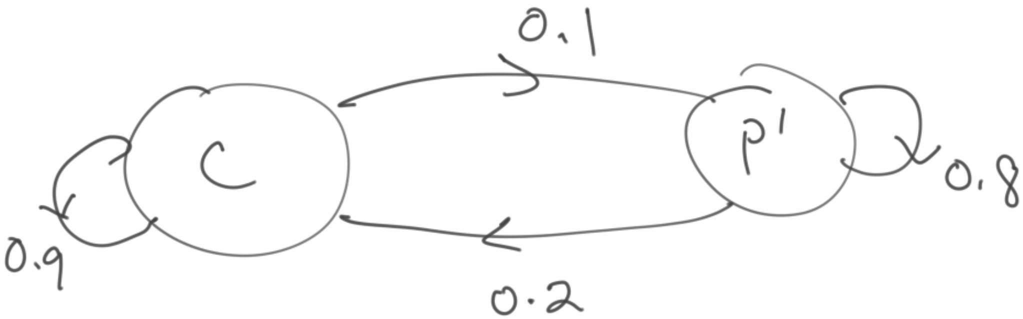
$C \rightarrow C$ in 2 step :

$$C \xrightarrow{0.9} C \xrightarrow{0.9} C \; : \; 0.9 \times 0.9$$

$$C \xrightarrow[0.1]{} p' \xrightarrow[0.2]{} C \; : \; 0.1 \times 0.2$$

1st

$$P^{(2)}_{cp'} = (0.9 \times 0.9) + (0.1 \times 0.2)$$



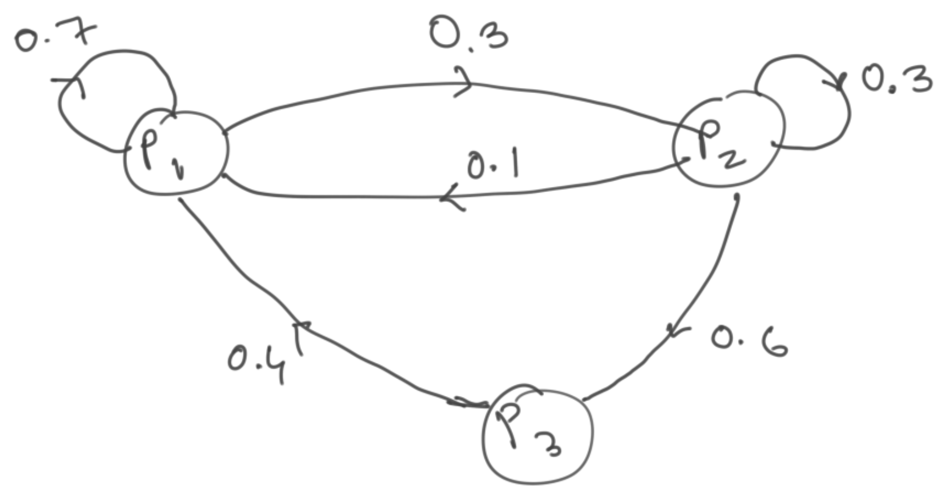
2nd

$$P^{(2)}_{cp'} = P_{cp'} \cdot P_{cp'} = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

$p_{ij}^{(n)}$

$\Downarrow$

prob. of going from i^{th} state to j^{th} state in n steps.



Accessibility : state j is said to be accessible from state i if $P_{ij}^{(n)} > 0$ for some n. (if we can go from i to j in n steps)

Communicating states: Two states i and j are said to be communicating if $i \to j$ and $j \to i$. (if it is possible to go from i to j in n-steps and then back to i in m-steps) (m & n need not be same)

Class of a state i: $C(i) = \{ j \in S : i \leftrightarrow j \}$

= set of all states that communicate with i

Reducible and Irreducible Markov Chain

A Markov Chain is said to be irreducible if every state communicates with each other, i.e. $c(i) = S$ otherwise reducible.

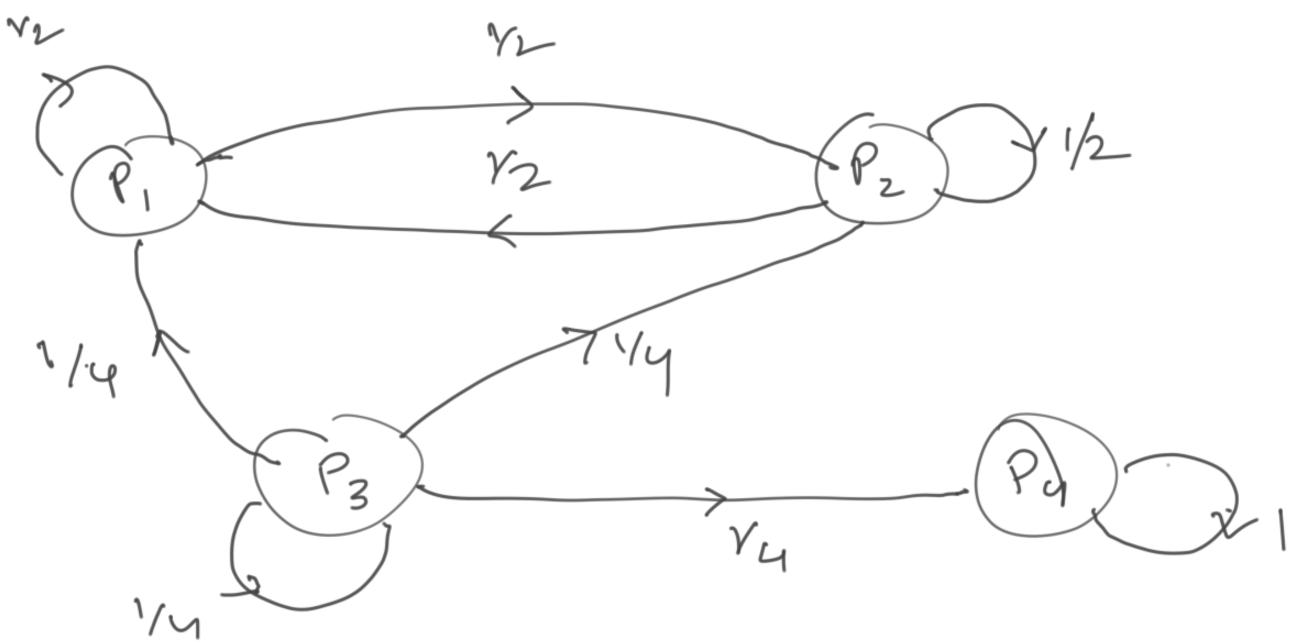
Closed communicating class

A class is said to be closed communicating class if the exit from a class is not possible. It consists of recurrent states only.

States
- Recurrent : Returning to a state is possible
- Transient : Returning to a state is not possible.

$P_1 \to P_1$

self loop



Class of state P_1

$$C(P_1) = \{P_1, P_2\}$$

$$C(P_2) = \{P_1, P_2\} = C(P_1)$$

$$C(P_3) = \{P_3\}$$

$$C(P_4) = \{P_4\}$$

$$C(P_1) = C(P_2), \quad C(P_3), \quad C(P_4)$$

states $= \cancel{2}$ 4, class $= 3$

$\Rightarrow$ chain is reducible

$P_1 \to P_2$, $P_2 \to P_1$

$\Rightarrow P_1 \leftrightarrow P_2$

$P_3 \to P_1$, $P_3 \to P_2$, $P_3 \to P_4$

but cannot travel back

$P_3 \leftrightarrow P_3$

states $- P_1, P_2, P_3, P_4$