μ_X = mean wrt X

MOMENTS OF RANDOM VARIABLE

The moments of a random variable or its distribution are expected values of powers or related functions of the random variable.

General –

The rth moment of X is $\mu'_r = \sum x^r P(X = x) = E[X^r]$

The rth central moment of X is $\mu_r = E(X - \mu_X)^r$

In particular –

1st moment – $\mu'_1 = \sum x P(X = x) = E[X]$ i.e. 1st moment = mean

2nd central moment – $\mu_2 = E(X - \mu_X)^2 = \sigma^2$ i.e. 2nd central moment = variance

Relation betn Moment & central Moment

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = \mu_2' - \mu_1'^2$$

$$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3 = \mu_3' - 3\mu_1'\mu_2' + 2\mu_1'^3$$

$$\mu_4 = \mu_4' + 6\mu_1'^2\mu_2' - 4\mu_1'\mu_2' - 3\mu_1'^4$$

Q. Let X be a R.V. have pmf

$$P_X(x) = \begin{cases} 1/2 , & x = 1 \\ 1/3 , & x = 2 \\ 1/6 , & x = 3 \\ 0 ; & \text{otherwise} \end{cases}$$

find 3rd moment of X.

Soln: Third moment of $X = E[X^3]$

$$= \sum x^3 p(x)$$

$$= \left(1^3 \cdot \frac{1}{2}\right) + \left(2^3 \cdot \frac{1}{3}\right) + \left(3^3 \cdot \frac{1}{6}\right)$$

$$= 7.67$$

Q. Let X be a discrete random variable with probability mass function (pmf)

$$P_X(x) = \begin{cases} 3/4 & , \quad x = 1 \\ 1/4 & , \quad x = 2 \\ 0 & , \quad \text{otherwise} \end{cases}$$

Find the third central moment of X.

<u>Soln:</u> Third central moment, $\mu_3 = E(X - \mu_X)^3$ — (1)

Now, $\mu_X = E[X] = \sum x p(x) = \left(1 \cdot \frac{3}{4}\right) + \left(2 \cdot \frac{1}{4}\right) = \frac{5}{4}$

$$\therefore \quad \mu_3 = E(X-\mu_X)^3 = \sum (x-\mu_X)^3 p(x)$$

$$= \sum \left(x - \frac{5}{4}\right)^3 p(x)$$

$$= \left[\left(1-\frac{5}{4}\right)^3 \cdot \left(\frac{3}{4}\right)\right] + \left[\left(2-\frac{5}{4}\right)^3 \cdot \left(\frac{1}{4}\right)\right]$$

$$= \frac{3}{32}$$

MOMENT GENERATING FUNCTION (m.g.f.)

The moment generating function (m.g.f.) of a random variable X having the probability function $f(x)$ is given by

$$M_x(t) = E(e^{tx}) = \sum_x e^{tx} f(x) \longrightarrow \text{Discrete R.V.}$$

where t = real constant

$$\boxed{M_x(t) = \int e^{tx} f(x)\,dx \quad \hookrightarrow \text{continuous R.V.}}$$

$$M_x(t) = E(e^{tx}) = E\left[1 + tx + \frac{t^2x^2}{2!} + \dots + \frac{t^r x^r}{r!} + \dots\right]$$

$$= 1 + tE[x] + \frac{t^2}{2!}E[x^2] + \dots + \frac{t^r}{r!}E[x^r] + \dots$$

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \dots}$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \dots + \frac{t^r}{r!}\mu_r' + \dots \quad (*)$$

coefficient of $\frac{t^r}{r!}$ in the expansion $(*)$ is μ_r' i.e. rth moment of X about origin

$\therefore$ It generates moments

$\therefore$ It is called m.g.f.

$$\mu_r' = \frac{d^r}{dt^r}\left[M_x(t)\right]$$

m.g.f. of X about the point $x = a$ is

$$M_x(t) \text{ (about } x = a) = E\left[e^{t(x-a)}\right]$$

m.g.f of X about mean

$$M_x(t) \text{ (about mean)} = E\left[e^{t(x-\bar{x})}\right] \text{ or } E\left[e^{t(x-\mu)}\right]$$

Properties

① $M_{cx}(t) = M_x(ct)$ where c = constant

② The moment generating function of the sum of a number of independent random variables is equal to the product of their respective moment generating function i.e.

if $x_1, x_2, \ldots, x_n$ are independent random variables then the moment generating function of their sum $x_1 + x_2 + \ldots + x_n$ is given by

$$M_{X_1+X_2+\dots+X_n}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \dots M_{X_n}(t).$$

EFFECT OF CHANGE OF ORIGIN AND SCALE ON M.G.F.

X be a R.V. with $M_X(t) = E[e^{tX}]$

let $U = \dfrac{X-a}{h}$ where a and h are constants

$\Rightarrow X = a + hU$

$$M_U(t) = e^{-at/h} M_X\left(\frac{t}{h}\right) \quad \text{or} \quad M_X(t) = e^{at} M_U(th)$$