Marginal and Conditional Probability Function

Consider a joint distribution of two random variables X and Y then

$$f_X(x) = p_X(x_i) = P(X = x_i) = p_{i1} + p_{i2} + \dots + p_{ij} + \dots + p_{im}$$

$$= \sum_{j=1}^{m} p_{ij}$$

$$= p_i$$

Marginal Probability function of X.

$$f_Y(y) = P_Y(y_j) = P(Y = y_j) = \sum_{i=1}^{n} p_{ij} = p_j$$

$\hookrightarrow$ Marginal Probability function of Y.

<u>Conditional</u> Probability f^n of X when $Y = y_j$ is given

$$f_{X/Y}(x/y) = P(X = x_i / Y = y_j) = \frac{P(X = x_i \cap Y = y_j)}{P(Y = y_j)}$$

$$= \frac{p(x_i, y_j)}{p(y_j)}$$

$$= \frac{p_{ij}}{p_j}$$

Conditional probability f^n of y when $X = x_i$ is given

$$f_{Y/X}(y/x) = P(Y = y_j \mid X = x_i) = \frac{p(x_i, y_j)}{p(x_i)} = \frac{p_{ij}}{p_i}$$

Independent ; $P(X = x_i, Y = y_j) = P(X = x_i)\, P(Y = y_j)$

JOINT DISTRIBUTION FUNCTION

$X, Y \leftarrow$ two R.V.s

Then their joint distribution f^n $F_{X,Y}(x, y)$ is given by

$$F_{X,Y}(x,y) = P(X \le x, Y \le y) \quad \forall\ x, y \in \mathbb{R}$$

where $\sum_x \sum_y f_{X,Y}(x,y) = 1 \leftarrow$ Discrete R.V.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y) = 1 \leftarrow \text{cont. R.V.}$$

Properties

① If $x_1 < x_2$ and $y_1 < y_2$ then (Rectangle Rule)

$$P[x_1 < X \le x_2,\ y_1 < Y \le y_2] = F(x_2, y_2) - F(x_2, y_1) - F(x_1, y_2) + F(x_1, y_1) \geq 0$$

2. (a) $F(-\infty, y) = \lim_{x \to -\infty} F(x, y) = 0 \quad \forall \quad y \in \mathbb{R}$

(b) $F(x, -\infty) = \lim_{y \to -\infty} F(x, y) = 0 \quad \forall \; x \in \mathbb{R}$

(c) $\lim_{\substack{x \to \infty \\ y \to \infty}} F(x, y) = F(\infty, \infty) = 1$

3. $F(x, y)$ is right continuous in each argument i.e.

$$\lim_{h \to 0^+} F(x+h, y) = \lim_{h \to 0^+} F(x, y+h) = F(x, y)$$

4. If the density function $f(x, y)$ is continuous at (x, y) then

$$\frac{\partial^2 F}{\partial x \partial y} = f(x, y).$$

Expectation, Covariance, correlation coefficient

Let us consider (X, Y) as a two dimensional discrete random variable with joint discrete density function $f_{X,Y}(x,y)$.

The expectation of $g(X,Y)$ is denoted by $E[g(X,Y)]$ and defined as

$$E[g(X,Y)] = \sum_x \sum_y g(x,y)\, f_{X,Y}(x,y)$$

Particular Cases

(i) $E[X] = \sum x f_X(x)$

(ii) $E[Y] = \sum y f_Y(y)$

(iii) $E[XY] = \sum_x \sum_y xy\, f_{X,Y}(x,y)$

Covariance : $Cov(X,Y) = E[X - E(X)]\, E[Y - E(Y)] = E[XY] - E[X]E[Y]$

correlation coefficient : $\rho(X,Y) = \dfrac{Cov(X,Y)}{\sqrt{Var\,X}\,\sqrt{Var\,Y}} = \dfrac{Cov(X,Y)}{\sigma_X\,\sigma_Y}$

where, $\sigma_X > 0$, $\sigma_Y > 0$

Note : $-1 \le \rho(X,Y) \le 1$

Conditional Expectation

If (X,Y) are joint discrete random variable then conditional expectation of $g(X, Y)$ given $X = x$ is defined as

$$E[g(X,Y) / X = x] = \sum_j g(x_i, y_j) f_{Y/X}(y_j | x)$$

In Particular,

$$E[Y / X = x] = \sum_j y_j f_{Y/X}(y_j | x) = \sum_j y_j P(Y = y_j | X = x)$$

Q. For the following bivariate probability distribution of X and Y find

(i) $P(X \le 2, Y = 3)$

(ii) $P(X \le 1)$

(iii) $P(Y = 4)$

(iv) $P(Y \le 5)$

(v) $P(X < 2, Y \le 3)$

X \ Y	1	2	3	4	5	6
0	0	0	$1/32$	$2/32$	$2/32$	$3/32$
1	$1/16$	$1/16$	$1/8$	$1/8$	$1/8$	$1/8$
2	$1/32$	$1/32$	$1/64$	$1/64$	0	$2/64$

Sol^n: The marginal distribution is given

$X \downarrow$ \ $Y \rightarrow$	1	2	3	4	5	6	$p_X(x)$
0	0	0	1/32	2/32	2/32	3/32	8/32
1	1/16	1/16	1/8	1/8	1/8	1/8	10/16
2	1/32	1/32	1/64	1/64	0	2/64	8/64
$p_Y(y)$	3/32	3/32	11/64	13/64	6/32	16/64	1

(1) $P(X \leq 2, Y=3) = P(X=0, Y=3) + P(X=1, Y=3) + P(X=2, Y=3)$

$= 1/32 + 1/8 + 1/64$

$= \frac{11}{64}$

(ii) $P(X \leq 1) = P(X=0) + P(X=1)$

$$= \frac{8}{32} + \frac{10}{16}$$

$$= \frac{7}{8}$$

$$P(Y \leq 5) = 1 - P(Y=5)$$
$$= 1 - P(Y=6)$$
$$= 1 - \frac{16}{64} = \frac{48}{64}$$

(iii) $P(Y=4) = \frac{13}{64}$

(iv) $P(Y \leq 5) = P(Y=1) + P(Y=2) + P(Y=3) + P(Y=4) + P(Y=5)$

$= ?$

(v) $P(X<2, Y \leq 3) = P(X=0, Y=1) + P(X=0, Y=2) + P(X=0, Y=3) + P(X=1, Y=1) + P(X=1, Y=2) + P(X=1, Y=3)$

$= ?$