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CW

Q. Derive the expressions for the trigonometric fourier series coefficients.

Ans:- Let us consider a sinusoidal wave,
$x(t) = A \sin \omega_0 t$ with period $T = \frac{2\pi}{\omega_0}$

The sum of two sinusoids is periodic provided that their frequencies are integral multiples of a fundamental frequency, ω_0.

We can show that a signal $x(t)$, a sum of sine & cosine functions whose functions are integral multiples of ω_0, is a periodic signal.

Let, the signal $x(t)$ be,

$$x(t) = a_0 + a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t + \dots + a_k \cos k\omega_0 t$$
$$+ b_1 \sin \omega_0 t + b_2 \sin 2\omega_0 t + \dots + b_k \sin k\omega_0 t$$

$$\Rightarrow x(t) = a_0 + \sum_{n=1}^{k} a_n \cos \omega_0 n t + b_n \sin \omega_0 n t$$

Where,

$a_0, a_1, a_2, \dots, a_k$

& $b_0, b_1, b_2, \dots, b_k$ are consts.

& ω_0 is the fundamental frequency.

Now,

$$x(t+T) = a_0 + \sum_{n=1}^{k} a_n \cos \omega_0 n (t+T) + b_n \sin \omega_0 n (t+T)$$
$$= a_0 + \sum_{n=1}^{k} a_n \cos \omega_0 n \left(t + \frac{2\pi}{\omega_0}\right) + b_n \sin \omega_0 n \left(t + \frac{2\pi}{\omega_0}\right)$$
$$= a_0 + \sum_{n=1}^{k} a_n \cos(\omega_0 n t + 2\pi n) + b_n \sin(\omega_0 n t + 2\pi n)$$
$$= a_0 + \sum_{n=1}^{k} a_n \cos \omega_0 n t + b_n \sin \omega_0 n t$$
$$= x(t)$$

This proves that s/g $x(t)$, which is summation of sine & cosine functions of freq., $0, \omega_0, 2\omega_0, \ldots, k\omega_0$ is a periodic s/g with period T.

If $k \to \infty$ in the expn for $x(t)$, we obtain the Fourier series representation of any periodic s/g $x(t)$.

$$x(t) = \sum_{n=0}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

$$\Rightarrow \boxed{x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t} \quad \text{--- (1)}$$

where, $a_n, b_n \to$ constants

$a_0 \to$ also called dc component

$a_1 \cos \omega_0 t + b_1 \sin \omega_0 t \longrightarrow 1^{st}$ harmonic

$a_2 \cos 2\omega_0 t + b_2 \sin 2\omega_0 t \longrightarrow 2^{nd}$ harmonic

--- --- so on

$\left.\begin{matrix} a_0, a_1, \ldots a_n \\ b_0, b_1, \ldots b_n \end{matrix}\right\}$ fourier coefficients

Integrating (1) on both sides $\int_{t_0}^{t_0+T}$ (over 1 period)

$$\int_{t_0}^{t_0+T} x(t)dt = a_0 \int_{t_0}^{t_0+T} dt + \int_{t_0}^{t_0+T} \left\{\left[\sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \cos n\omega_0 t\right] dt\right\}$$

$$= a_0 T + \sum_{n=1}^{n} a_n \int_{t_0}^{t_0+T} \cos n\omega_0 t\, dt \nearrow 0 + \sum_{n=1}^{\infty} b_n \int_{t_0} \sin n\omega_0 t\, dt \nearrow 0$$

[$\because$ net areas of sinusoides are zero over complete periods for any non-zero 'n' & any time t_0]

$$\therefore \int_{t_0}^{t_0+T} x(t)\,dt = a_0 T$$

$$\Rightarrow \boxed{a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t)\, dt}$$

$$\therefore \boxed{a_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos n\omega_0 t\, dt}$$

$$\therefore \boxed{b_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \sin n\omega_0 t\, dt}$$

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