Poisson Distribution

Limiting case of Binomial Distribution

S.D. Poisson

(French Mathematician)

$n \to \infty$ (very large)

$p \to 0$ (very small)

Parameter $\to \lambda$

$\boxed{\lambda = np}$ *

Formula

$$P(x) = \frac{e^{-\lambda} \lambda^{x}}{x!}, \quad x = 0, 1, 2, \ldots$$

Conditions

1. R.V. must be discrete
2. There are two alternatives = failure & success;

3. It is applicable when n is very large and p is very small but mean np finite.

$$np = \lambda$$

parameter

Mean (μ) $= \lambda$

Variance (σ^2) $= \lambda$

S.D. $= \sqrt{\sigma^2} = \sigma = \sqrt{\lambda}$

Constants of Poisson Distribution

$\mu_1 = 0$, $\mu_2 = \lambda$, $\mu_3 = \lambda$, $\mu_4 = \lambda + 3\lambda^2$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{1}{\lambda}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = 3 + \frac{1}{\lambda}$$

Recurrence Formula

$$p(x+1) = \frac{\lambda}{x+1}\, p(x), \quad x = 0, 1, 2, \ldots$$

Suppose a book of 585 pages contains 43 typographical errors. If these errors are randomly distributed throughout the book, what is the probability that 10 pages, selected at random will be free from errors?

Here, $p = \frac{43}{585} = 0.0735$, $n = 10$

As p is very small so we are going to use Poisson Distribution

Now,

$$\lambda = np = 0.735$$

Let, X be the no. of errores in 10 pages

$$\therefore \quad P(X=0) = \frac{e^{-\lambda}\lambda^{0}}{0!} = \frac{e^{-0.735}(0.735)^{0}}{0!}$$

$$= e^{-0.735} = 0.4795$$

Average number of accidents on any day on a highway is 1.8. Determine the probability that the number of accidents are :

(i) atleast one

(ii) atmost one

$(e^{-1.8} = 0.16529)$

Given,

Mean, $\lambda = 1.8$

By Poisson Distribution,

$$P(X=x) = \frac{e^{-\lambda} \lambda^{x}}{x!}$$ where x denotes no. of accidents

(1) Prob. that no. of accidents is atleast one

$= P(X \geq 1) = 1 - P(X<1) = 1 - P(X=0)$

$$P(X \geq 1) = 1 - \frac{e^{-1.8}(1.8)^0}{0!}$$

$$= 1 - \frac{e^{-1.8}(1)}{1}$$

$$= 1 - e^{-1.8}$$

$$= 1 - 0.16529$$

$$= 0.8347$$

(ii) prob. that no. of accidents atmost one

$= P(X \leq 1)$

$= P(X=0) + P(X=1)$

$$= \frac{e^{-1.8}(1.8)^0}{0!} + \frac{e^{-1.8}(1.8)^1}{1!}$$

$= 0.4628$

If the variance of the Poisson distribution is 2, find the probabilities for $r = 1, 2, 3, 4$ from the recurrence relⁿ. Also find $P(x \geq 4)$ $[e^{-2} = 0.1353]$

Here,

Variance, $\lambda = 2$

$$P(r) = \frac{e^{-\lambda} \lambda^{r}}{r!} = \frac{e^{-2}\, 2^{r}}{r!} \quad \text{--- (1)}$$

Recurrence relⁿ is given by

$$P(r+1) = \frac{\lambda}{r+1} P(r) \quad \text{--- (1)}$$

From (1),

$$P(0) = \frac{e^{-2}\, 2^{0}}{0!} = \frac{e^{-2}(1)}{1}$$

$$= e^{-2}$$

$$= 0.1353$$

$$P(1) = \frac{\lambda}{0+1} P(0)$$

$$= \frac{2}{1} (0.1353)$$

$$= 0.2706$$

$$P(2) = \frac{\lambda}{1+1} P(1) = \frac{2}{2} (0.2706)$$

$$= 0.2706$$

$$P(3) = \frac{\lambda}{2+1} P(2)$$

$$= \frac{2}{3} (0.2706)$$

$$= 0.1804$$

$$P(4) = \frac{\lambda}{3+1} P(3) = \frac{2}{4} (0.1804)$$

$$= 0.0902$$

Again,

$$P(x \geq 4) = 1 - P(x < 4)$$

$$= 1 - [P(0) + P(1) + P(2) + P(3)]$$

$$= 0.1431$$

If a random variable X follows a Poisson distribution s.t. $P(X=2) = 9P(X=4) + 90P(X=6)$ find mean and variance of X

By Poisson distribution,

$$P(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \qquad \text{———} \; (1)$$

Six coins are tossed 6400 times. Using the Poisson distribution, determine the approximate probability of getting six heads x times.

Probability of getting a head with one coin $= \frac{1}{2}$

$\therefore$ The probability of getting 6 heads with 6 coins $= \left(\frac{1}{2}\right)^6$

$$\therefore \quad p = \frac{1}{64}$$

Now, mean, $\lambda = np = 6400 \times \frac{1}{64} = 100$

Approximate probability of getting 6 heads x times $= \dfrac{\lambda^x e^{-\lambda}}{x!}$ (using Poisson Dist.)

$$= \frac{100^x e^{-100}}{x!}$$

If the probability of a bad reaction from a certain injection is 0.001, determine the chance that out of 2000 individuals more than two will get a bad reaction.

Here,

Probability of a bad reaction, $p = 0.001$

$n = 2000$

Let x denotes the no. of bad reaction

$\therefore$ Probability of getting more than 2 bad reactions

$= P(X > 2)$

$= 1 - P(X \leq 2)$

$= 1 - (P(X=0) + P(X=1) + P(X=2))$

$= 1 - P(X=0) - P(X=1) - P(X=2)$

$P(X=r)$

$= \dfrac{\lambda^{r} e^{-\lambda}}{r!}$

$\lambda = np$

Now,

$$\lambda = np = 2000 \times 0.001 = 2$$

$$P(X>2) = 1 - \frac{e^{-\lambda}\lambda^0}{0!} - \frac{e^{-\lambda}\lambda^1}{1!} - \frac{e^{-\lambda}\lambda^2}{2!}$$

$$= 1 - \frac{e^{-2}}{1} - \frac{e^{-2}(2)}{1} - \frac{e^{-2}(2)^2}{2}$$

$$= 1 - e^{-2}(1+2+2)$$

$$= 0.32$$

Suppose that x has a Poisson distribution. If

$P(X=2) = \frac{2}{3} P(X=1)$. Find (i) $P(X=0)$ (ii) $P(X=3)$

Let, λ be the mean of the distribution

Now,

$$P(X=2) = \frac{2}{3} P(X=1)$$

$$\Rightarrow \frac{e^{-\lambda}\lambda^2}{2!} = \frac{2}{3} \cdot \frac{e^{-\lambda}\lambda^1}{1!}$$

Recurrence Formula

$\therefore \lambda = \frac{4}{3}$

① $P(X=0) = \frac{e^{-\lambda}\lambda^0}{0!} = \frac{e^{-4/3}(4/3)^0}{1} = e^{-4/3}$

② $P(X=3) = \frac{e^{-\lambda}\lambda^3}{3!} = \frac{e^{-4/3}(4/3)^3}{6}$

$$P(X=2) = P(2) = \frac{e^{-\lambda} \lambda^2}{2!}$$

$$P(X=4) = P(4) = \frac{e^{-\lambda} \lambda^4}{4!}$$

$$P(X=6) = P(6) = \frac{e^{-\lambda} \lambda^6}{6!}$$

Given,

$$P(X=2) = 9\,P(X=4) + 90\,P(X=6)$$

$$\frac{e^{-\lambda}\lambda^2}{2!} = 9\,\frac{e^{-\lambda}\lambda^4}{4!} + 90\,\frac{e^{-\lambda}\lambda^6}{6!}$$

$$\Rightarrow \frac{\lambda^2}{2} = \frac{9\lambda^4}{4.3.2} + 90\,\frac{\lambda^6}{6.5.4.3.2}$$

$$\Rightarrow \lambda^2 = \frac{3\lambda^4}{4} + \frac{\lambda^6}{4}$$

$$\Rightarrow 4\lambda^2 = 3\lambda^4 + \lambda^6$$

$$4 = 3\lambda^2 + \lambda^4$$

$$\Rightarrow \lambda^4 + 3\lambda^2 - 4 = 0$$

$$\Rightarrow \lambda^4 + 4\lambda^2 - \lambda^2 - 4 = 0$$

$$\Rightarrow \lambda^2(\lambda^2 + 4) - 1(\lambda^2 + 4) = 0$$

$$\Rightarrow (\lambda^2 + 4)(\lambda^2 - 1) = 0 \quad \Rightarrow \lambda = 1$$

$\lambda^2 + 4 = 0$

$\Rightarrow \lambda^2 = -4$

Not possible

$\therefore$ Mean $= \lambda$

$= 1$

Variance $= \lambda$

$= 1$

Fit a Poisson distribution on the following

x	0	1	2	3	4
f	192	100	24	3	1

Total no. of trials = 320

$$\left(e^{-0.5} = 0.6065\right)$$

$$\text{Mean, } \lambda = \frac{(0\times192)+(1\times100)+(2\times24)+(3\times3)+(4\times1)}{192+100+24+3+1}$$

$$\Rightarrow \lambda = \frac{161}{320} = 0.5 \text{ (approx)}$$

$$P(0) = \frac{e^{-\lambda}\lambda^0}{0!} = \frac{e^{-0.5}(0.5)^0}{1} = e^{-0.5}$$

$$= 0.6065$$

$$\Rightarrow f = 320 \times 0.6065 = 194 \text{ (approx)}$$

$$P(1) = \frac{e^{-\lambda}\lambda^1}{1!} = e^{-0.5}(0.5) = 0.30325$$

$$\Rightarrow f = 320 \times 0.30325 = 97 \text{ (approx)}$$

$$P(2) = \frac{e^{-\lambda}\lambda^2}{2!} = \frac{e^{-0.5}(0.5)^2}{2!} = 0.07581$$

$$\Rightarrow f = 320 \times 0.07581 = 24 \text{ (approx)}$$

$$P(3) = \frac{e^{-\lambda}\lambda^3}{3!} = \frac{e^{-0.5}(0.5)^3}{3!} = 0.0126$$

$$\Rightarrow f = 320 \times 0.0126 = 4 \text{ (approx)}$$

$$P(4) = \frac{e^{-\lambda}\lambda^4}{4!} = \frac{e^{-0.5}(0.5)^4}{4!} = 0.0016$$

$$\Rightarrow f = 320 \times 0.0016 = 0.512 = 1 \text{ (approx)}$$

$\therefore$ Approximate value by Poisson Distribution is

x	0	1	2	3	4
f	194	97	24	4	1