FOURIER SERIES (FS)

Fourier Series Representation :-

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}.}$$

This is the FS representation for fourier s/g $x(t)$.

$$x(t) = x(t+T)$$ → fundamental period of the s/g.

Fundamental frequency, $\omega_0 = \frac{2\pi}{T} = 2\pi f$.

we will consider the complex s/g, $x(t) = e^{j\omega_0 t}$, to represent the FS because then we can also solve s/. simple s/g using the complex s/g.

Fundamental period of $e^{j\omega_0 t}$ is T i.e. $e^{j\omega_0 t}$ keeps on repeating for every 'T', i.e., the value of exponential s/g may ~~be~~ change i.e. the value can be $e^{j\omega_0 t}$, $e^{j2\omega_0 t}$, $e^{j3\omega_0 t}$ .... etc.

So if we want to represent the set of exponentials, then,

$$\phi_k = \{ e^{j\omega_0 t}, e^{j2\omega_0 t}, e^{j3\omega_0 t} \dots e^{jk\omega_0 t} \}_{k=-\infty \text{ to } \infty}$$

If we want to represent the linear combination of the set, then,

$$\phi_k(t) = \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t}$$

If we want to represent this linear combination in the form of s/g then,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

where, a_k is the coefficient of approximation.

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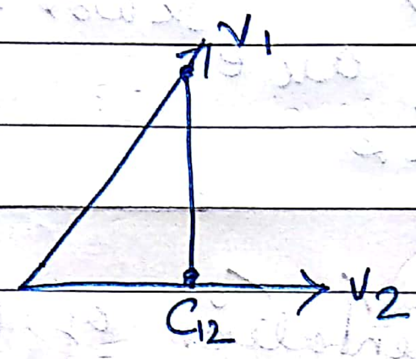
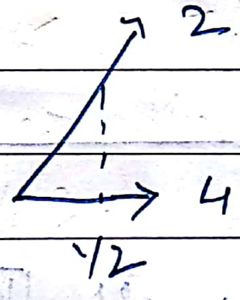
* <u>Vector Analogy</u>

If you want to represent any vector v_1 in terms of v_2, then draw a perpendicular bisector & you need to trace the component from v_1 to v_2.



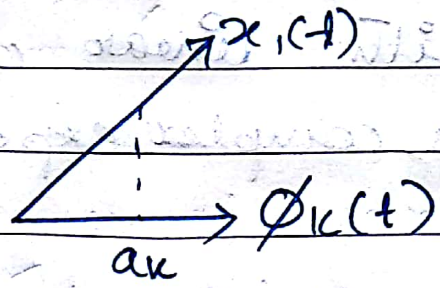
If you want to write v_1 in terms of v_2 then $v_1 = C_{12} v_2$.

* If you want to write 2 in terms of 4 then



$2 = \frac{1}{2} \times 4 = 2$

Here



Represent $x(t)$ in terms of set $\phi_k(t)$

$\therefore x(t) = a_k \phi_k(t)$.

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$$x(t) = a_k \phi_k(t)$$

$$\Rightarrow x(t) = a_k \sum_{k=-\infty}^{\infty} \phi\, e^{jk\omega_0 t}$$

$$\Rightarrow \boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}}$$, This is the

FS representation of the s/g $x(t)$.

Replace 'k' by 'n',

$$\boxed{x(n) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}}$$

$\therefore$ fourier series is the representation of a s/g with linear combination of continuous complex exponential set.

Deriving the coefficient :-

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

We are going to derive the formula for a_k.

Multiply $e^{-jn\omega_0 t}$ on both sides.

Multiply $e^{-jn\omega_0 t}$ on both sides.

$$\therefore x(t)\, e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t} \cdot e^{-jn\omega_0 t}.$$

$$\Rightarrow x(t)\, e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k\, e^{j(k-n)\omega_0 t}$$

Integrating $\int_0^T$ both sides,

Integrating $\int_0^T$ both sides,

$$\Rightarrow \int_0^T x(t)\, e^{-jn\omega_0 t} = \int_0^T \sum_{k=-\infty}^{\infty} a_k\, e^{j(k-n)\omega_0 t}$$

$$\Rightarrow \int_0^T x(t)\, e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t}$$

RHS using Euler's formula,

$$\int_0^T e^{j(k-n)\omega_0 t} dt = \int_0^T \cos(k-n)\omega_0 t \, dt + j\int_0^T \sin(k-n)\omega_0 t \, dt$$

$$\int_0^T e^{j(k-n)\omega_0 t} dt = \begin{cases} T, & \text{when } k=n \\ 0, & \text{when } k \neq n. \end{cases}$$

* If $k=n$, $k-n=0$

$$\cos 0 = 1$$

$$\sin 0 = 0$$

$$\therefore \int_0^T \cos(k-n)\omega_0 t \, dt$$

$$= (t_0)_0^T = T.$$

If you take diff. values of k and n you will get 0.

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At one point of k, we are getting a value i.e. at $k=n$. So, we will consider that.

$$\therefore \int_0^T x(t)e^{-jn\omega_0 t} = a_n T$$

$\rightarrow k=n$.

$$\boxed{\therefore a_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt}$$

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* Representation of Fourier Series in Exponential form

Last class we got,

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}} \quad \text{—— (1)}$$

where $\omega_0 t = \frac{2\pi}{T}$

$$a_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt.$$

or if we integrate over one time period $\int_{t_0}^{t_0+T}$

$$\boxed{a_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, e^{-jn\omega_0 t}\, dt} \quad \text{—— (2)}$$

$$\therefore \quad \boxed{a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, dt.} \quad \text{—— (3)}$$

* Representation in Trigonometric form :—

let us consider a sinusoidal wave, with

$x(t) = A\sin\omega_0 t.$ with period $T = \frac{2\pi}{\omega_0}$

The sum of two ~~periodics~~ sinusoids is periodic provided that their frequencies are integral multiples of a fundamental freq, ω_0.

We can ... function ... periodi...

We can show that a signal $x(t)$, a sum of sine & cosine functions whose functions are integral multiples of ω_0, is a periodic signal.

Let the s/g $x(t)$ be

$$x(t) = a_0 + a_1 \cos\omega_0 t + a_2 \cos 2\omega_0 t + \cdots + a_K \cos K\omega_0 t$$
$$+ b_1 \sin\omega_0 t + b_2 \sin 2\omega_0 t + \cdots + b_K \sin K\omega_0 t$$

$$\Rightarrow x(t) = a_0 + \sum_{n=1}^{K} a_n \cancel{\cos n\omega_0 t} \cos\omega_0 n t$$
$$+ \sum_{n=1} b_n \sin\omega_0 n t .$$

where, $a_0, a_1, a_2 \ldots a_K$

& $b_0, b_1, b_2 \ldots b_K$ are constants

& ω_0 is the fundamental frequency.

Now,

$$x(t+T)$$

Now,

$$x(t+T) = a_0 + \sum_{n=1}^{k} a_n \cos \omega_0 n (t+T) + b_n \sin \omega_0 n (t+T)$$

$$= a_0 + \sum_{n=1}^{k} a_n \cos \omega_0 n \left(t + \frac{2\pi}{\omega_0}\right) + b_n \sin \omega_0 n \left(t + \frac{2\pi}{\omega_0}\right)$$

$$= a_0 + \sum_{n=1}^{k} a_n \cos (\omega_0 n t + 2\pi n) + b_n \sin (\omega_0 n t + 2\pi n)$$

$$= a_0 + \sum_{n=1}^{k} a_n \cos \omega_0 n t + b_n \sin \omega_0 n t$$

$$= x(t).$$

This proves that s/g $x(t)$, which is summation of sine & cosine functions of freq. $0, \omega_0, 2\omega_0 \ldots k\omega_0$ is a periodic s/g with period T.

If $k \to \infty$ in the exp^n for $x(t)$, we obtain the Fourier series representation of any periodic s/g $x(t)$

This proves that s/g $x(t)$, which is summation of sine & cosine functions of freq. $0, \omega_0, 2\omega_0 \ldots k\omega_0$ is a periodic s/g with period T.

If $k \to \infty$ in the expⁿ for $x(t)$, we obtain the Fourier series representation of any periodic s/g $x(t)$

$$x(t) = \sum_{n=0}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t .$$

$$\boxed{x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t} \quad \text{— (4)}$$

$a_n, b_n \rightarrow$ constants.

$a_0 \rightarrow$ also called dc component.

$a_1 \cos \omega_0 t + b_1 \sin \omega_0 t \rightarrow$ 1st harmonic

$a_2 \cos 2\omega_0 t + b_2 \sin 2\omega_0 t \rightarrow$ 2nd harmonic.

.... so on.

* Evaluation of Fourier coefficients

$$\left.\begin{matrix} a_0, & a_1 & \cdots & a_n \\ b_0, & b_1 & \cdots & b_n \end{matrix}\right\} \text{ Fourier coefficients.}$$

Now $x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$.

Integrating both sides $\int_{t_0}^{t_0+T}$ (over 1 period)

$$\Rightarrow \int_{t_0}^{t_0+T} x(t)\,dt = a_0 \int_{t_0}^{t_0+T} dt + \int_{t_0}^{t_0+T} \left\{ \left[\sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \cos n\omega_0 t \right] dt \right\}$$

$$= a_0 T + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T} \cancel{\cos n\omega_0 t \, dt}^{\,0} + \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T} \cancel{\sin n\omega_0 t \, dt}^{\,0}$$

[$\because$ net areas of sinusoides are zero over complete periods for any non-zero integer n & any time t_0].

$$\therefore \int_{t_0}^{t_0+T} x(t)\,dt = a_0 T$$

$$\therefore \boxed{a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) \, dt} \quad \text{--- (5)}$$

$$\boxed{a_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos n\omega_0 t \, dt} \quad \text{--- (6)}$$

$$\boxed{b_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \sin n\omega_0 t \, dt} \quad \text{--- (7)}$$

[To find a_n, multiply the eq^n ④

by $\cos m\omega_0 t$ & integrate over 1 period.

To find b_n, multiply the eq^n ④

~~wt~~ by $\sin m\omega_0 t$ & integrate over 1 period.

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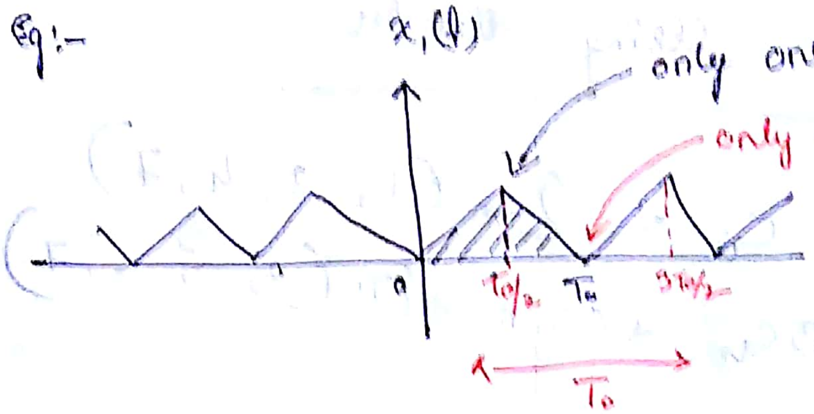
German mathematician Dirichlet

Dirichlets Conditions

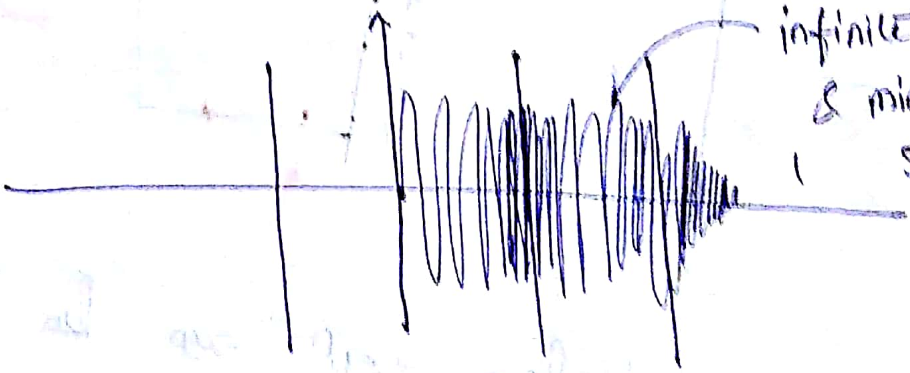
(Conditions for Existance of Fourier Series)

Conditions 1 s/g should have finite no. of maxima & minima over the range of time period.

Eg:-



over a range of a time period T_0 we have only ~~only~~ one max. and one min.



FS not exist.

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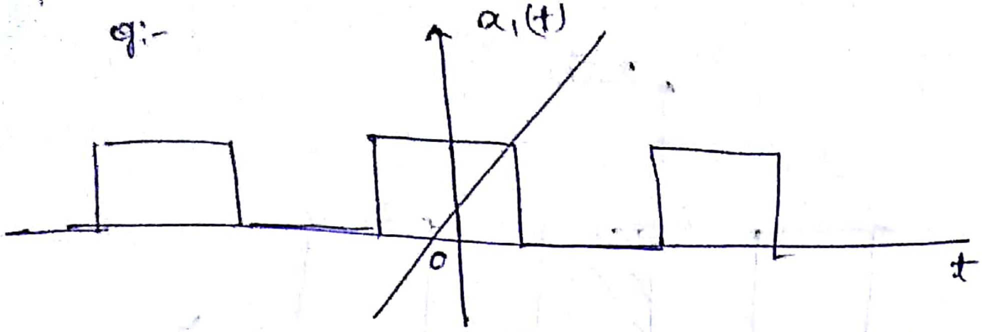
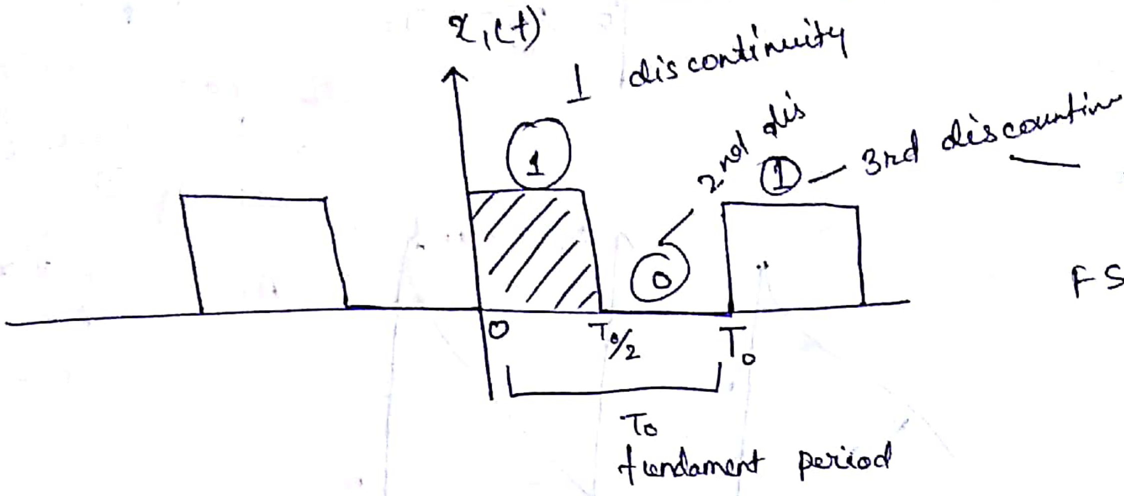
Condition 2

S/g should have finite no. of discontinuities over the range of time period.

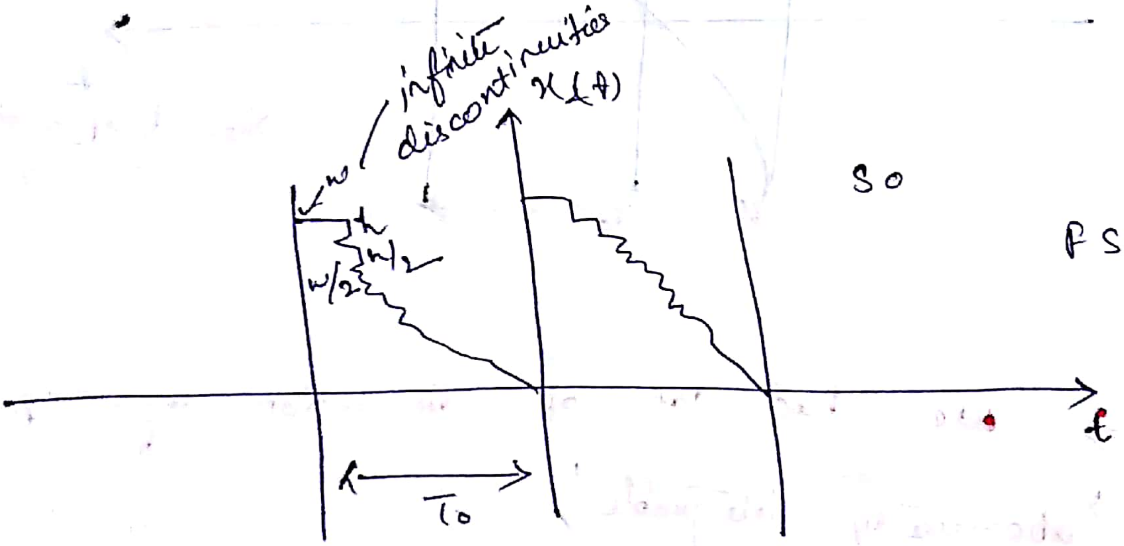
eg:-





finite.

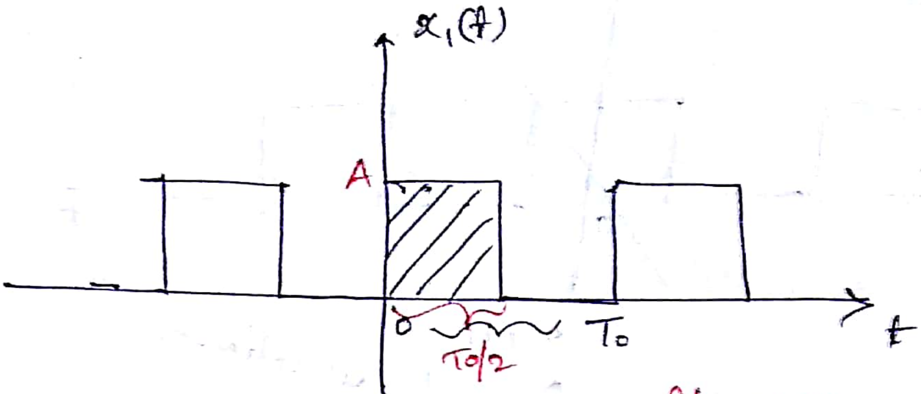
FS exist.



So

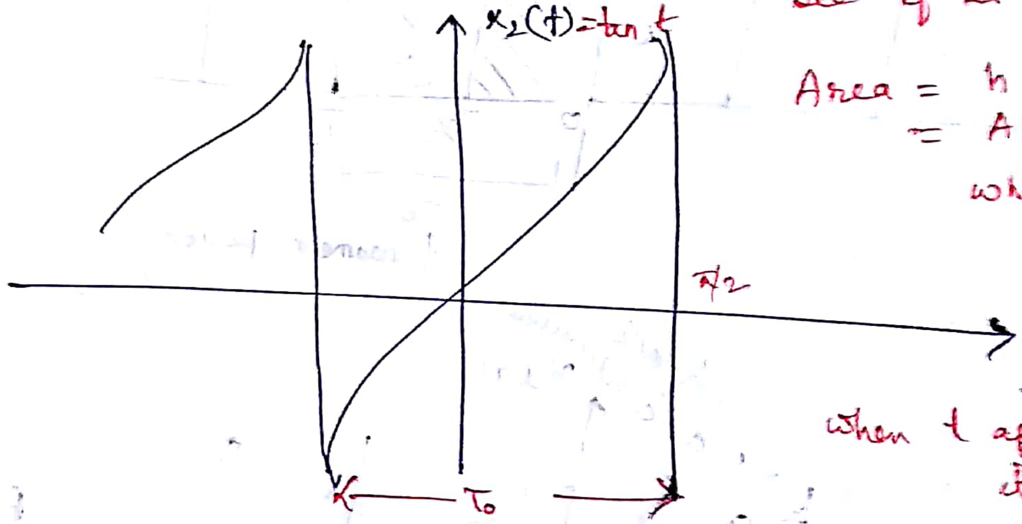
FS does not exist

Condition 3   S/g should be absolutely integrable over the range of Time period.



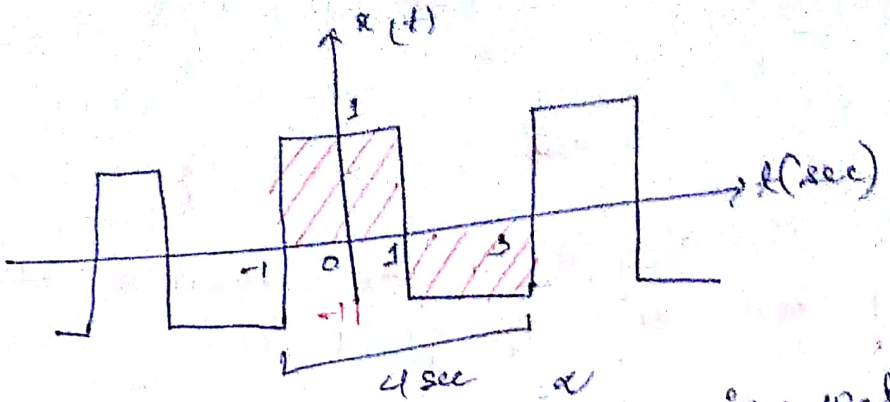
If you integrate it for 1 time period then see if it is finite.

Area = h × b
= A × $T_0/2$
which is finite



when t approaches $\pi/2$ it gives infinity

Ex:-



$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

a_0 : The given gr s/g is symmetrical about Time axis

i.e. the area in 1 Time period is = 0

The positive area will cancel out the negative area

$\therefore$ When you divide the total area by total Time period then a it is 0.

$\therefore \boxed{a_0 = 0.}$

b_n

Again, $x(-t) = x(t)$

$\therefore$ It is even s/g

When there is even s/g no need to calculate b_n as there will be no sine terms.

a_n

$$\therefore x(t) = \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$$

$$a_n = \frac{2}{T_0} \int_{T_0} x(t) \cos n\omega_0 t \, dt$$

=

$T_0 = 4$ sec

$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{4} = \frac{\pi}{2} \text{ rad/sec.}$$

$$\therefore a_n = \frac{2}{4}\int_{-1}^{3} x(t) \cdot \cos \frac{n\pi}{2} t \, dt$$

$$= \frac{1}{2}\left[\int_{-1}^{1} (1) \cos \frac{n\pi}{2} t \, dt + \int_{1}^{3} (-1) \cdot \cos \frac{n\pi}{2} t \, dt \right]$$

$$= \frac{1}{2}\left[\int_{-1}^{1} \cos \frac{n\pi}{2} t \, dt - \int_{1}^{3} \cos \frac{n\pi}{2} t \, dt \right]$$

Assume $\frac{n\pi}{2} t = \theta$

$\frac{n\pi}{2} dt = d\theta$

$\Rightarrow dt = \frac{2}{n\pi} d\theta$

when $t = -1$

$\Rightarrow \theta = -\frac{n\pi}{2}$

$t = 1, \ \theta = \frac{n\pi}{2}$

$t = 3 = \theta = \frac{3n\pi}{2}$

$$= \frac{1}{2}\left[\int_{-n\pi/2}^{n\pi/2} \cos \theta \, \frac{2}{n\pi} d\theta - \int_{n\pi/2}^{3\pi/2} \cos \theta \, \frac{2}{n\pi} d\theta \right]$$

$$= \frac{1}{\cancel{2}} \times \frac{\cancel{2}}{n\pi} \left[\int_{-n\pi/2}^{n\pi/2} \cos \theta \, d\theta - \int_{n\pi/2}^{3n\pi/2} \cos \theta \, d\theta \right]$$

$$= \frac{1}{\cancel{2}} \times \frac{\cancel{2}}{n\pi} \left[(\sin \theta)_{-n\pi/2}^{n\pi/2} - (\sin \theta)_{n\pi/2}^{3n\pi/2} \right]$$

$$= \frac{1}{n\pi}\left[\sin n\pi/2 - \sin\left(-\frac{n\pi}{2}\right) - \sin 3n\pi/2 + \sin(n\pi/2) \right]$$

$= \frac{1}{n\pi} [$

S | A

T | TC

Case I $\quad n = \text{even},$

$$a_n = \frac{1}{n\pi}\left[\sin\frac{n\pi}{2} + \cancel{\sin\frac{n\pi}{2}} - \cancel{\sin\frac{n\pi}{2}} + \sin\frac{n\pi}{2}\right]$$

$$a_n = \frac{2}{n\pi}\sin\frac{n\pi}{2}.$$

$\boxed{a_n = 0}$ when n is even. (put values of n)

Case II $\quad n = \text{odd}$

Case a $\quad n = 1, 5, 9, 13, \ldots$

a_n Case b $\quad n = 3, 7, 11, 15, \ldots$

Case a $\quad a_n = \frac{1}{n\pi}\left[\sin\frac{n\pi}{2} + \sin\frac{n\pi}{2} + \sin\frac{n\pi}{2} + \sin\frac{n\pi}{2}\right]$

$\sin\frac{1\pi}{2}$ or $\sin 5\pi/2 = +1$

$$= \frac{4}{n\pi}\sin\frac{n\pi}{2}$$

Case b $\quad a_n = -\frac{4}{n\pi}$

but $\sin 3\pi/2 \quad \sin 7\pi/2 = -1$

Parseval's Power Theorem

fourier coefficient

$$x(t) \rightleftharpoons C_n$$

& the time period $= T_0$

then Average Power

**

$$P_{x(t)} = \sum_{n=-\infty}^{\infty} |C_n|^2$$

Using this theorem we can calculate the power of the signal if we know its Complex exponential Fourier coeff.

Proof

$$x(t) = \sum_{n=-\infty}^{\infty} C_n . e^{jn\omega_0 t}$$

Taking conjugates on both sides,

$$x^*(t) = \sum_{n=-\infty}^{\infty} C_n^* . e^{-jn\omega_0 t}$$

Now,

$$x(t)\, x^*(t) = |x(t)|^2$$

$z = a+ib$

$z^* = a-ib$

$|z| = \sqrt{a^2+b^2}$

$|z|^2 = a^2+b^2$

$z . z^* = a^2 - i^2 b^2$

$= a^2+b^2$

$= |z|^2$

Again

$$P_{x(t)} = \frac{1}{T}\int_0^T |x(t)|^2\, dt$$

$$= \frac{1}{T}\int_0^T x(t) . x^*(t)\, dt$$

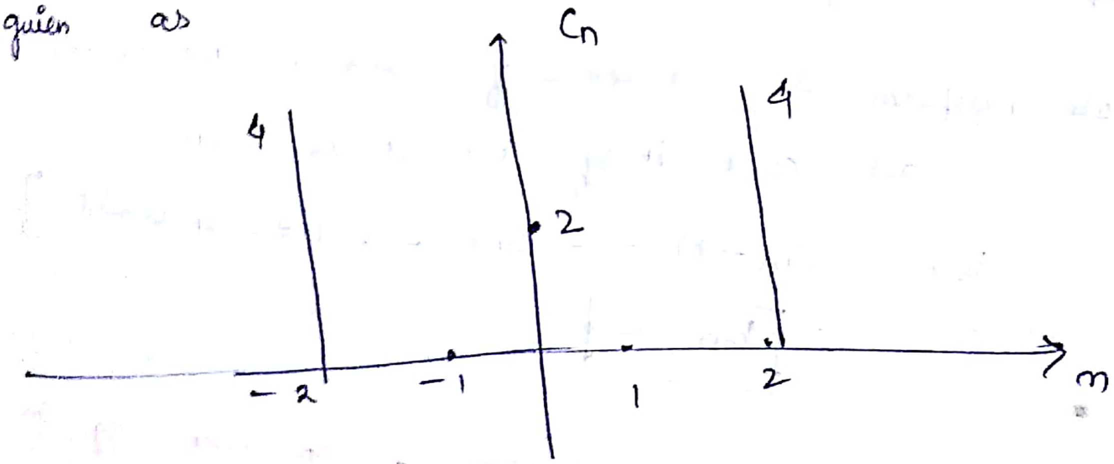
$$= \frac{1}{T}\int_0^T x(t) \sum_{n=-\infty}^{\infty} C_n^* e^{-jn\omega_0 t}\, dt$$

$$= \sum_{n=-\alpha}^{\alpha} C_n^* \frac{1}{T_0} \underbrace{\int_0^{T_0} x(t)\, e^{-jn\omega_0 t}\, dt}_{C_n}$$

$$= \sum_{n=-\alpha}^{\alpha} C_n^* . C_n$$

$$= \sum_{n=-\alpha}^{\alpha} |C_n|^2$$

Q. Find the average power of s/g $x(t)$, when C_n is given as



Sol^n

$$P_{x(t)} = \sum_{n=-\alpha}^{\alpha} |C_n|^2$$

$-\alpha$ to -3, $C_n = 0$

$C_{-2} = 4$

$\underline{\underline{\text{Sol}^n}}$

$$P_x(t) = \sum_{n=-\alpha}^{\alpha} |C_n|^2$$

$-\alpha$ to -3, $C_n = 0$

$C_{-2} = 4$

$C_{-1} = 0$

$C_0 = 2$

$C_1 = 0$

$C_2 = 4$

3 to $+\alpha$, $C_n = 0$

$$= |C_{-2}|^2 + |C_0|^2 + |C_2|^2$$

$$= |4|^2 + |2|^2 + |4|^2$$

$$= 16 + 4 + 16$$

$$= 36 \text{ Watts}$$

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Symmetricities in Fourier Series (Chan

① Even Symmetry :-

$$x(t) = a_0 + \underbrace{\sum_{n=-\infty}^{\infty} a_n \cos n\omega_0 t}_{\text{cosine terms.}} + \underbrace{\cancel{\sum_{n=-\infty}^{\infty} b_n \sin n\omega_0 t}}_{\text{sine terms}} \rightarrow 0$$

$a_0 \downarrow$ dc value

∴ $x(t) \rightarrow$ even means F.S. Expansion will have harmonics of even s/g.

[when we perform time reversal of $\cos t$, i.e $\cos(-t)$ we get $\cos t$ itself i.e. it is even

but $\sin(-t) = -\sin t \rightarrow$ i.e. it is odd]

∴ $\boxed{b_n = 0}$

∴ The Fourier series expansion of an even s/g does not contain sine terms. $\Rightarrow$ $b_n = 0$.

② Odd Symmetry :-

$x(t) =$ odd in nature.

F.S. expansion won't have even terms.

$\Rightarrow \boxed{a_n = 0}$

also avg. values $= 0$ i.e. $\boxed{a_0 = 0}$

& $\boxed{b_n \neq 0}$

& $\vdots$.

(3) Half Wave Symmetry :- (HWS)

Fourier series expansion of HWS s/g contains only "odd harmonics."

$$x(t) = -x(t + T_0/2)$$

Condition for HWS

If there is s/g $x(t)$ & $x(t) = -x(t + T_0/2)$ half time period

& we perform time shifting by half time period & reversal & after performing it if

$$\underset{\downarrow \; C_{n_1}}{x(t)} = -x\underset{\downarrow \; C_{n_2}}{(t + T/2)}$$

∵ the two s/g are same

So their fourier coefficients are also same.

Now we have to find C_{n_2} in terms of C_{n_1}.

$$x(t) \rightleftharpoons C_{n_1}$$

$$x(t - t_0) \rightleftharpoons C_{n_1} e^{-jn\omega_0 t_0}$$

$t_0 = -T/2$ to get $T/2$

$$x(t + T/2) \rightleftharpoons C_{n_1} e^{jn\omega_0 T/2}$$

$$-x(t + T_0/2) \rightleftharpoons -C_{n_1} e^{jn\omega_0 T/2} = C_{n_2}$$

Now, $C_{n_1} = C_{n_2}$

$$\cancel{C_{n_1}} = -\cancel{C_{n_1}} e^{jn\omega_0 T/2} \qquad \omega_0 = \frac{2\pi}{T}$$

$$1 = -e^{jn\omega_0 T/2} \qquad \frac{\omega_0 T}{2} = \pi$$

$$1 = -e^{jn\pi}$$

$$\Rightarrow 1 + e^{jn\pi} = 0 \qquad \Rightarrow 1 + (-1)^n = 0$$

$$\Rightarrow 1 + e^{(j\pi)n} = 0$$

$\Rightarrow n$ is an odd integer

freq is $n\omega_0$ and it is odd.

$\Rightarrow$ only odd harmonics.

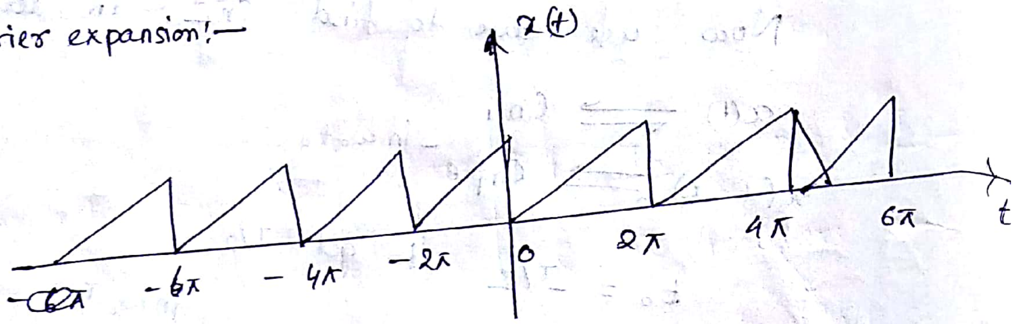
Odd + HWS

Even + HWS

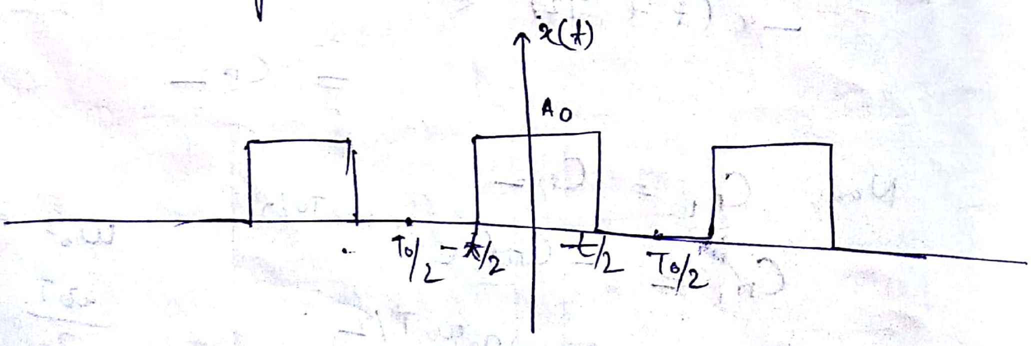
Hidden Symmetry.

Questions :—

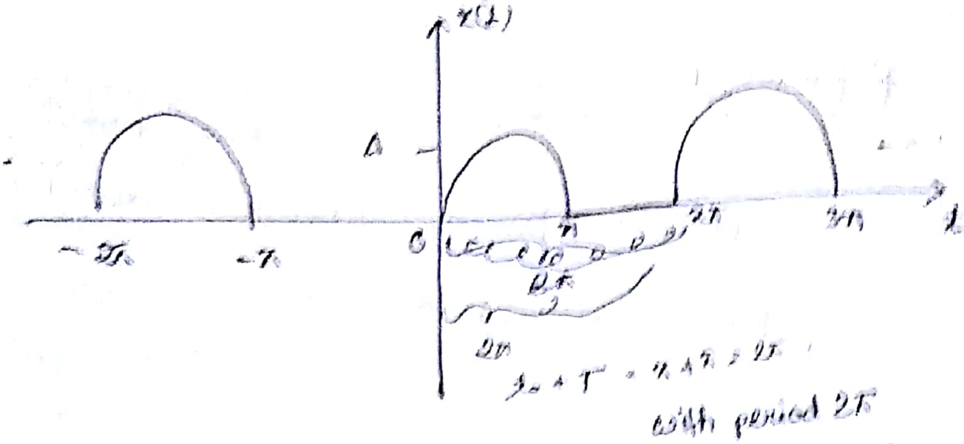
① Find the fourier expansion :—



② Find C_n for the s/g x(t) :—



Q Find the Fourier Series expansion of the half wave rectified sine wave —



Solⁿ :- The periodic waveform, shown in the figure is half of a sine wave with period 2π.

$$\omega = \frac{2\pi}{T}$$

$$x(t) = \begin{cases} A \sin \omega t = A \sin \frac{2\pi}{2\pi} t = A \sin t & , 0 \le t \le \pi \\ 0 & , \pi \le t \le 2\pi \end{cases}$$

[Now the fundamental period $= 2\pi$

" freq. of $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$

Let, $t_0 = 0$, $t_0 + T = T = 2\pi$.]

$$a_0 = \frac{1}{T}\int_0^T x(t)\, dt = \frac{1}{2\pi}\int_0^{2\pi} x(t)\, dt = \frac{1}{2\pi}\int_0^{\pi} A \sin t\, dt$$

$$= \frac{A}{2\pi}\left[-\cos t\right]_0^{\pi}$$

$$= \frac{A}{2\pi}\left[(-\cos\pi - \cos 0)\right]$$

$$= \frac{2A}{2\pi} = \frac{A}{\pi}$$

$$\therefore \boxed{a_0 = \frac{A}{\pi}}$$

a_n

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\omega_0 t\, dt$$

$$= \frac{2}{2\pi}\int_0^{2\pi} x(t)\cos nt\, dt \qquad \because \omega_0 = 1$$

$$= \frac{1}{\pi}\int_0^{\pi} A\sin t\cos nt\, dt$$

$$= \frac{A}{\pi}\int_0^{\pi} \sin t\cos nt\, dt$$

$$= \frac{A}{2\pi}\int_0^{\pi} \sin(1+n)t + \sin(1-n)t$$

$$= \frac{A}{2\pi}\left[-\frac{\cos(1+n)t}{1+n} - \frac{\cos(1-n)t}{1-n}\right]_0^{\pi}$$

$$= -\frac{A}{2\pi}\left[\frac{\cos(1+n)\pi - \cos 0}{1+n} + \frac{\cos(1-n)\pi - \cos 0}{1-n}\right]$$

$$= -\frac{A}{2\pi}\left\{\left[\frac{(-1)^{n+1}-1}{1+n}\right] + \frac{(-1)^{n-1}-1}{1-n}\right\}$$

For odd n, $a_n = -\frac{A}{2\pi}\left[\frac{1-1}{1+n} + \frac{1-1}{1-n}\right] = 0$

For even n, $a_n = \frac{-A}{2\pi}\left[\frac{-1-1}{1+n} + \frac{-1-1}{1-n}\right]$

$$= \frac{A}{2\pi}\left[\frac{2}{n+1} - \frac{2}{n-1}\right]$$

$$= -\frac{2A}{\pi(n^2-1)}$$

$\therefore$ $a_n = \frac{-2A}{\pi(n^2-1)}$

$$b_n = \frac{2}{T}\int_0^T x(t) \sin n\omega_0 t \, dt$$

$$= \frac{2}{2\pi}\int_0^{2\pi} x(t) \sin nt \, dt$$

$$= \frac{1}{\pi}\int_0^{\pi} A \sin t \sin nt \, dt = \frac{A}{\pi}\int_0^{\pi} \sin t \sin nt \, dt$$

~~n=1~~ $\frac{A}{\pi}\int_0^{\pi} \sin t \sin t \, dt$

n=1 $\frac{A}{\pi}\int_0^{\pi} \sin^2 t \, dt$

$$= \frac{A}{2\pi}\int_0^{\pi} (1-\cos 2t) \, dt$$

$$= \frac{A}{2\pi}\left[t - \frac{\sin 2t}{2}\right]_0^{\pi}$$

$$= \frac{A}{2\pi}\left[\left(\pi - \frac{\sin 2\pi}{2}\right) - 0 - \frac{\sin}{2}\right.$$

π

BSS

Q.1. Find The Time domain signal whose Fourier series coefficient is given by,

$$C_n = j\delta(n-1) - j\delta(n+1) + \delta(n-3) + \delta(n+3), \quad \omega_0 = \pi.$$

⇒ We have,

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

BSS

Q.1. Find the time domain signal whose Fourier series coefficient is given by,

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⇒ We have,

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$\Rightarrow x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\pi t}$$

$$= je^{j\pi t} - je^{-j\pi t} + e^{j3\pi t} + e^{-j3\pi t}$$

$$= 2\cos 3\pi t - 2\sin \pi t$$

Q2. Find the time domain s/g corresponding to

$$C_n = \left(-\frac{1}{2}\right)^{|n|} \; ; \quad \omega_0 = 1$$

Q2. Find the time domain s/g corresponding to

$$C_n = \left(-\tfrac{1}{2}\right)^{|n|} \; ; \quad \omega_0 = 1$$

Sol^n:-

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \, e^{jn\omega_0 t}$$

$$\therefore \; x(t) = \sum_{n=-\infty}^{\infty} \left(-\tfrac{1}{2}\right)^{|n|} e^{jnt}$$

$$= \sum_{n=-\infty}^{-1} \left(-\tfrac{1}{2}\right)^{-n} e^{jnt} + \sum_{n=0}^{\infty} \left(-\tfrac{1}{2}\right)^{n} e^{jnt}$$

$$= \sum_{n=1}^{\infty} \left(-\tfrac{1}{2}\right)^{n} e^{-jnt} + \sum_{n=0}^{\infty} \left(-\tfrac{1}{2}\right)^{n} e^{jnt}$$

$$= \sum_{n=0}^{\infty} \left(-\tfrac{1}{2}\right)^{n} e^{-jnt} - \left[\left(-\tfrac{1}{2}\right)^{n} e^{-jnt}\right]_{n=0} + \sum_{n=0}^{\infty} \left(-\tfrac{1}{2}\right)^{n} e^{jnt}$$

$$= \frac{-(1/2)e^{-jt}}{1+(1/2)e^{-jt}} + \frac{1}{1+(1/2)e^{jt}}$$

$$= \frac{3/4}{(5/4) + \cos t}$$

Q3. Find the complex exponential Fourier series representation of the following signals:-

(a) $x(t) = 4\cos 2\omega_0 t$

(b) $x(t) = \cos^2 t$

(c) $x(t) = \sin\left(2t + \frac{\pi}{4}\right)$

Table 1: **Properties of the Continuous-Time Fourier Series**

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$$

Property	**Periodic Signal**	**Fourier Series Coefficients**
	$\left.\begin{array}{r} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time-Shifting	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency-Shifting	$e^{jM\omega_0 t} = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	$x^*(t)$	a^*_{-k}
Time Reversal	$x(-t)$	a_{-k}
Time Scaling	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution	$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation	$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration	$\int_{-\infty}^{t} x(t)dt$ (finite-valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x(t)$ real and even	a_k real and even
Real and Odd Signals	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Table 2: **Properties of the Discrete-Time Fourier Series**

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk(2\pi/N)n}$$

Property	Periodic signal	Fourier series coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time shift	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shift	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m] & \text{if } n \text{ is a multiple of } m \\ 0 & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \left(\begin{array}{l}\text{viewed as} \\ \text{periodic with} \\ \text{period } mN\end{array}\right)$
Periodic Convolution	$\sum_{r=\langle N \rangle} x[r] y[n-r]$	$N a_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N \rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)}) a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \left(\begin{array}{l}\text{finite-valued and} \\ \text{periodic only if } a_0 = 0\end{array}\right)$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right) a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$ $x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$

Table 3: **Properties of the Continuous-Time Fourier Transform**

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Property	**Aperiodic Signal**	**Fourier transform**
	$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
Time-shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
Frequency-shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time-Reversal	$x(-t)$	$X(-j\omega)$
Time- and Frequency-Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}X(j\omega) * Y(j\omega)$
Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \Re e\{X(j\omega)\} = \Re e\{X(-j\omega)\} \\ \Im m\{X(j\omega)\} = -\Im m\{X(-j\omega)\} \\ \|X(j\omega)\| = \|X(-j\omega)\| \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ $[x(t)$ real$]$ $x_o(t) = \mathcal{O}d\{x(t)\}$ $[x(t)$ real$]$	$\Re e\{X(j\omega)\}$ $j\Im m\{X(j\omega)\}$

Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

Table 4: **Basic Continuous-Time Fourier Transform Pairs**

Signal	**Fourier transform**	**Fourier series coefficients (if periodic)**
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0, \quad \text{otherwise}$
$\cos\omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0, \quad \text{otherwise}$
$\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0, \quad \text{otherwise}$
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1, \quad a_k = 0,\ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t+T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2\sin\omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t),$ $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

Table 5: **Properties of the Discrete-Time Fourier Transform**

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

Property	**Aperiodic Signal**	**Fourier transform**
	$x[n]$ $y[n]$	$\left.\begin{matrix} X(e^{j\omega}) \\ Y(e^{j\omega}) \end{matrix}\right\}$ Periodic with period 2π
Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
Time-Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
Frequency-Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
Time Reversal	$x[-n]$	$X(e^{-j\omega})$
Time Expansions	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega - 2\pi k)$
Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \Re e\{X(e^{j\omega})\} = \Re e\{X(e^{-j\omega})\} \\ \Im m\{X(e^{j\omega})\} = -\Im m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
Symmetry for Real, Even Signals	$x[n]$ real and even	$X(e^{j\omega})$ real and even
Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$ $x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$\Re e\{X(e^{j\omega})\}$ $j\Im m\{X(e^{j\omega})\}$

Parseval's Relation for Aperiodic Signals

$$\sum_{n=-\infty}^{+\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$$

Table 6: **Basic Discrete-Time Fourier Transform Pairs**

Signal	**Fourier transform**	**Fourier series coefficients (if periodic)**
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \ldots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \leq N_1 \\ 0, & N_1 < \lvert n\rvert \leq N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \quad k \neq 0, \pm N, \pm 2N, \ldots$ $a_k = \frac{2N_1 + 1}{N}, \quad k = 0, \pm N, \pm 2N, \ldots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n] \begin{cases} 1, & \lvert n\rvert \leq N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \leq \lvert\omega\rvert \leq W \\ 0, & W < \lvert\omega\rvert \leq \pi \end{cases}$ $X(\omega)$periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi \delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n+1)a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
$\frac{(n+r-1)!}{n!(r-1)!} a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—

Table 7: **Properties of the Laplace Transform**

Property	**Signal**	**Transform**	**ROC**
	$x(t)$	$X(s)$	R
	$x_1(t)$	$X_1(s)$	R_1
	$x_2(t)$	$X_2(s)$	R_2
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R [i.e., s is in the ROC if $(s - s_0)$ is in R]
Time scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{s}{a}\right)$	"Scaled" ROC (i.e., s is in the ROC if (s/a) is in R)
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\Re e\{s\} > 0\}$

Initial- and Final Value Theorems

If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \to \infty$, then

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

Table 8: **Laplace Transforms of Elementary Functions**

Signal	**Transform**	**ROC**
1. $\delta(t)$	1	All s
2. $u(t)$	$\frac{1}{s}$	$\Re e\{s\} > 0$
3. $-u(-t)$	$\frac{1}{s}$	$\Re e\{s\} < 0$
4. $\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$
5. $-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\Re e\{s\} < 0$
6. $e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} > -\Re e\{\alpha\}$
7. $-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} < -\Re e\{\alpha\}$
8. $\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} > -\Re e\{\alpha\}$
9. $-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} < -\Re e\{\alpha\}$
10. $\delta(t-T)$	e^{-sT}	All s
11. $[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
12. $[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
13. $[e^{-\alpha t}\cos\omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\Re e\{\alpha\}$
14. $[e^{-\alpha t}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\Re e\{\alpha\}$
15. $u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16. $u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$

Table 9: **Properties of the z-Transform**

Property	Sequence	Transform	ROC
	$x[n]$	$X(z)$	R
	$x_1[n]$	$X_1(z)$	R_1
	$x_2[n]$	$X_2(z)$	R_2
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2
Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R except for the possible addition or deletion of the origin
Scaling in the z-Domain	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R
	$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$z_0 R$
	$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e., $\|a\|R =$ the set of points $\{\|a\|z\}$ for z in R)
Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1} where z is in R)
Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$ where z is in R)
Conjugation	$x^*[n]$	$X^*(z^*)$	R
Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of R_1 and R_2
First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $\|z\| > 0$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1-z^{-1}}X(z)$	At least the intersection of R and $\|z\| > 1$
Differentiation in the z-Domain	$nx[n]$	$-z\frac{dX(z)}{dz}$	R

Initial Value Theorem
If $x[n] = 0$ for $n < 0$, then
$x[0] = \lim_{z\to\infty} X(z)$

Table 10: **Some Common z-Transform Pairs**

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $\alpha^n u[n]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
6. $-\alpha^n u[-n-1]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
7. $n\alpha^n u[n]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
8. $-n\alpha^n u[-n-1]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
9. $[\cos\omega_0 n]u[n]$	$\frac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\frac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\frac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\frac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

8) Find the complex exponential Fourier Series representation of the following signals -

a) $x(t) = 4\cos 2\omega_0 t$

Sol Comparing with $x(t) = A\cos\omega_0 t$, we get -

Fundamental angular frequency, $\omega = 2\omega_0$

$$\therefore x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}$$ [Complex Fourier series ~~exp~~ representation]

Now,

$$x(t) = 4\cos 2\omega_0 t = \sum_{n=-\infty}^{\infty} a_n e^{jn2\omega_0 t}$$

$$\therefore 4\cos 2\omega_0 t = 2\left[\cos 2\omega_0 t + j\sin 2\omega_0 t + \cos 2\omega_0 t - j\sin 2\omega_0 t\right]$$

$$= 2\left[e^{j2\omega_0 t} + e^{-j2\omega_0 t}\right]$$

$$\Rightarrow x(t) = 2e^{j2\omega_0 t} + 2e^{-j2\omega_0 t} = \sum_{n=\infty}^{\infty} a_n e^{jn2\omega_0 t}$$

$\therefore$ The complex fourier coefficients for $4\cos 2\omega_0 t$ are -

$a_{-1} = 2$ and $a_1 = 2$, $a_n = 0$ $|n| \neq 1$.

$\cos\theta = \dfrac{e^{j\theta}+e^{-j\theta}}{2}$

$\sin\theta = \dfrac{e^{j\theta}-e^{-j\theta}}{2\theta}$

b) $x(t) = \cos^2 t$

$$\Rightarrow x(t) = \frac{1+\cos 2t}{2} = \frac{1}{2} + \frac{\cos 2t}{2}$$

Here, fundamental angular frequency of $\cos 2t$ is $\omega_0 = 2$.

$$\therefore x(t) = \cos^2 t = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} a_n e^{jn2t}$$

By Euler's formula :-

$$x(t) = \cos^2 t$$

$$\Rightarrow x(t) = \left(\frac{e^{jt}+e^{-jt}}{2}\right)^2$$

$$= \frac{e^{2jt}+e^{-2jt}+2e^{2jt}\cdot e^{-2jt}}{4}$$

$$\Rightarrow x(t) = \frac{1}{2} + \frac{1}{4}e^{2jt} + \frac{1}{4}e^{-2jt} = \sum_{n=-\infty}^{\infty} a_n e^{jn2t}.$$

The complex Fourier coefficients of $\cos^2 t$ are -

$a_0 = \frac{1}{2}$, $a_{-1} = \frac{1}{4}$, $a_1 = \frac{1}{4}$, and $a_n = 0$ for $|n| \neq 1$.

6) $x(t) = \sin(2t + \pi/4)$

Comparing with $\sin(\omega_0 t + \theta)$ we get -

Fundamental angular frequency, $\omega_0 = 2$.

$\therefore x(t) = \sin\left(2t + \frac{\pi}{4}\right) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} a_n e^{jn2t}$

By Euler's formula -

$x(t) = \sin(2t + \pi/4)$

$\Rightarrow x(t) = \dfrac{e^{j(2t+\pi/4)} - e^{-j(2t+\pi/4)}}{2j} = -\frac{1}{2j} e^{-j\pi/4} \cdot e^{-j2t} + \frac{1}{2j} e^{j\pi/4}$

$\Rightarrow x(t) = \sum_{n=-\infty}^{\infty} a_n e^{j2nt}$

$\therefore$ The complex fourier coefficient for $\sin(2t + \pi/4)$ are -

$a_{-1} = -\frac{1}{2j} e^{-j\pi/4} = -\frac{1}{2j}\left[\cos \pi/4 - j \sin \pi/4\right]$

$= -\frac{1}{2j}\left(\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right)$

$\Rightarrow a_{-1} = -\frac{1}{2j}\left[\frac{1-j}{\sqrt{2}}\right] = \frac{\sqrt{2}}{4j}(j-1)$

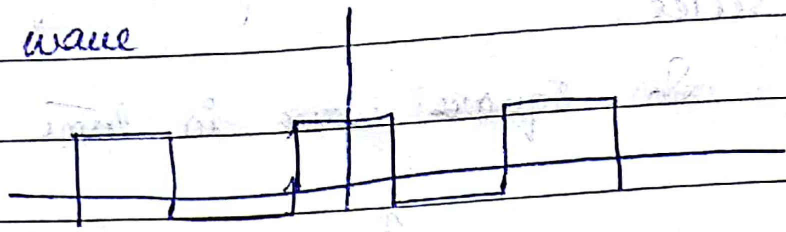
$a_1 = \frac{1}{2j} e^{j\pi/4} = \frac{1}{2j}\left[\cos \pi/4 + j \sin \pi/4\right] = \frac{1}{2j}\left[\frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}\right]$

$\Rightarrow a_1 = \frac{1}{2j}\left[\frac{1+j}{\sqrt{2}}\right] = \frac{\sqrt{2}}{4j}(j+1)$

$a_n = 0$, for $|n| \neq 1$.

FOURIER TRANSFORM

Take a square wave

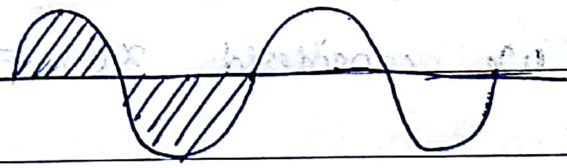


$$x(t) = \sum_{n=-\infty}^{\infty} |C_n| \, e^{jn\omega_0 t}$$

$$|C_n| = \begin{cases} \dfrac{2A}{\pi n}, & n = \text{odd.} \\ 0, & n = \text{even.} \end{cases}$$

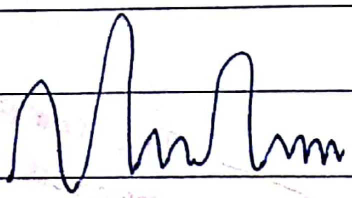
Frequency spectrum

Time period :— Duration of time ~~is~~ of one cycle in a repeating unit.



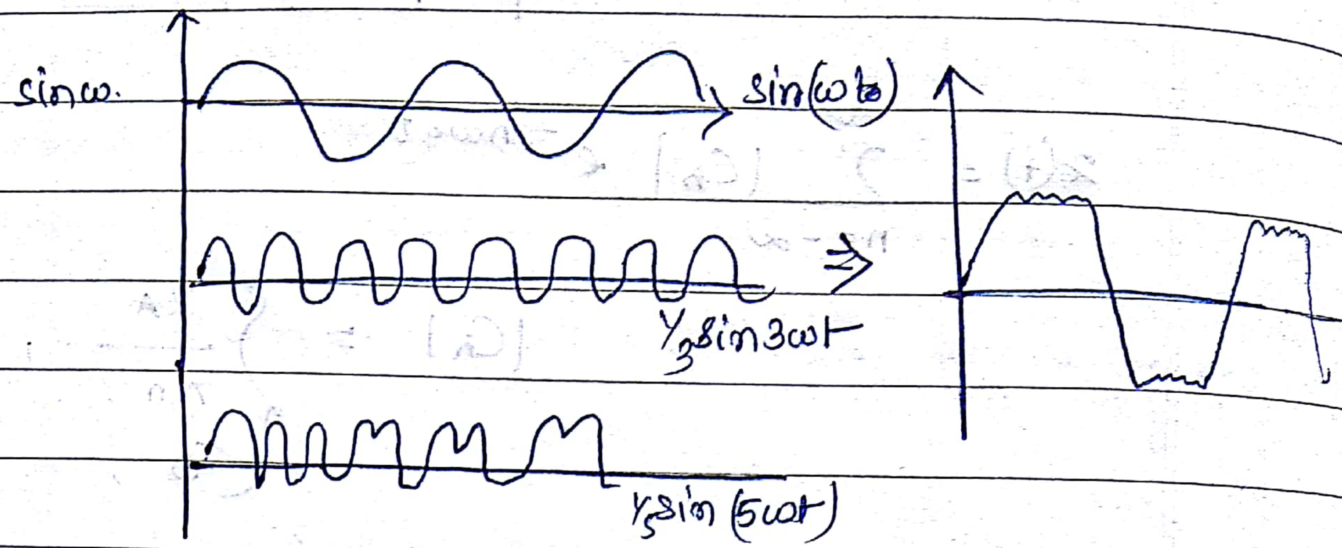
Frequency :— No. of cycles per unit time.

$$F = \frac{1}{T}$$



→ this s/g does not have time period but it has a frequency range.

So square wave in terms of sine →

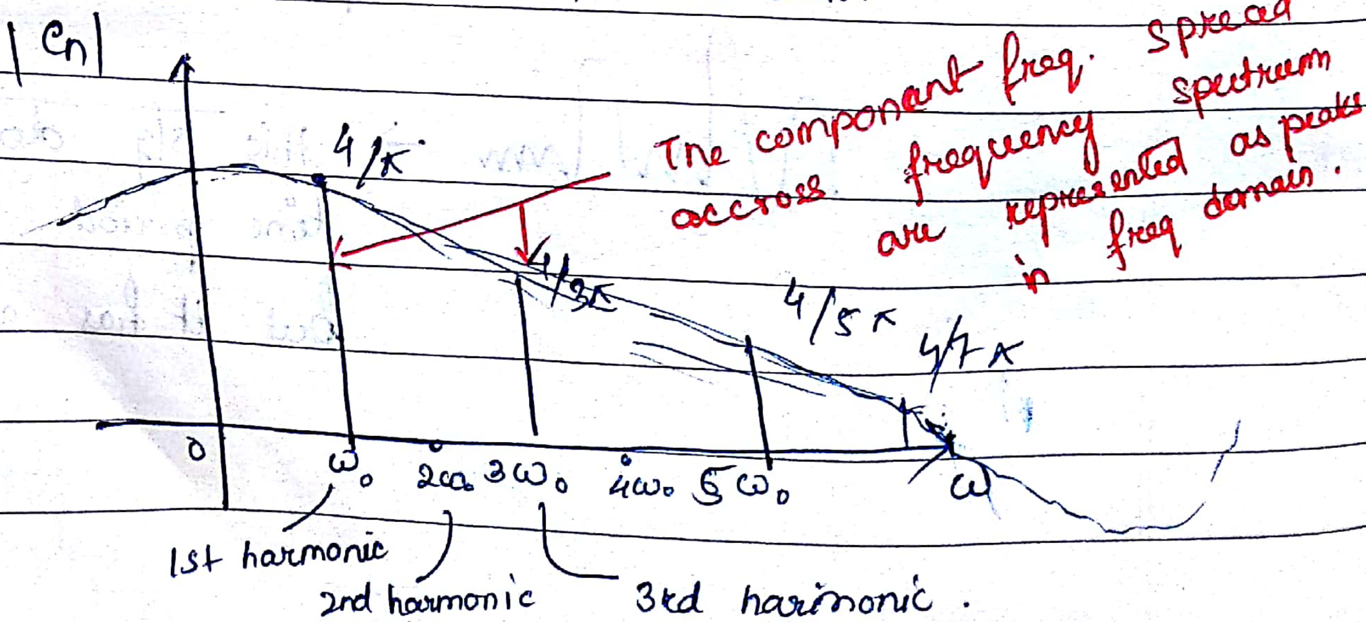


ultimately we will get a square wave [after seeing / observing the values of sin wave for diff. values of n & changing values of time period.]

We considered $x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j\omega_0 n t}$

$|C_n| = \frac{4}{\pi n}$, n = odd

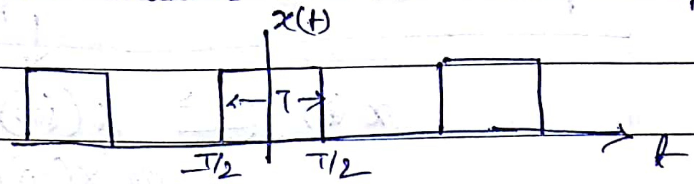
0 , n = even.



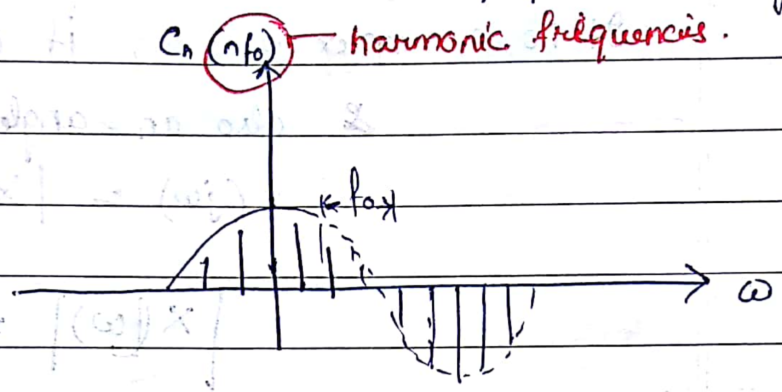
Frequency spectrum is the graphical display of the harmonic content of the s/g. $|C_n|$ is the mag. of the harmonics & plotted against ω.

As frequency increases, Harmonic magnitude decreases.

So what happens in Fourier Tx is that suppose a s/g $x(t)$



In frequency domain we can draw/plot the s/g as



$\frac{2\pi}{T/2}$ $\quad 2\,\frac{T}{2}$ $\quad \frac{2\pi}{T}$

If we further increase the time period the freq will decrease the space betn the spectral lines will become lesser & lesser and at one point infinity. ~~At the same time~~,

→ Fourier T_x is used for frequency analysis of s/g.

→ Representation:-

$$x(t) \rightleftharpoons \overset{\text{rad/s}}{X(j\omega)} \text{ or } \overset{\text{Hz}}{X(f)}$$

or $X(\omega)$

It is a complex no. So, it will have a magnitude & also an angle.

$$X(j\omega) = |X(j\omega)| \angle X(j\omega).$$

$$|X(\omega)| =$$

&

$$\angle X(\omega) = \tan^{-1}$$

<u>Conditions for Existence of Fourier Tx (Dirichlet Cond^n):-</u>

*1. S/g should have finite no. of maxima & minima over any finite interval.

* 2. S/g should have finite no. of discontinuities over any finite interval.

* 3. S/g should be absolutely integrable.

These are sufficient but not necessary cond^n.

eg:- $x(t) = u(t)$ → Power s/g which is not absolutely integrable.

Sol^n $X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

$= \int_{-\infty}^{\infty} u(t)\, e^{-j\omega t}\, dt$

$= \int_{-\infty}^{0} 0 . e^{-j\omega t}\, dt + \int_{0}^{\infty} 1 . e^{-j\omega t}\, dt$

$= 0 + \frac{\left(e^{-j\omega t}\right)_0^{\infty}}{-j\omega}$

$= \frac{e^{-j\omega\infty} - e^{-j\omega t 0}}{-j\omega}$

$= \frac{e^{-j\infty} - 1}{-j\omega}$

not defined

$= \frac{\cos\infty - j\sin\infty - 1}{-j\omega}$

∴ $X(j\omega)$ also not defined for using the formula we will use diff^n properties to find its F.T.

Formulae :-

F.T. pair

① $x(t) \longrightarrow X(j\omega)$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t).e^{-j\omega t}\,dt \quad —(I) \quad \leftarrow \text{F.T.}$$

② $X(j\omega) \longrightarrow x(t)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega).e^{j\omega t}\,d\omega \quad —(II) \quad \leftarrow \text{Inverse F.T.}$$

* Properties

① Linearity :-

$$x_1(t) \overset{F.T.}{\rightleftharpoons} X_1(j\omega)$$

$$x_2(t) \rightleftharpoons X_2(j\omega)$$

$$\alpha x_1(t) \rightleftharpoons \alpha X_1(j\omega)$$

$$\beta x_2(t) \rightleftharpoons \beta X_2(j\omega)$$

If we add then the transformed s/g also get added.

$$\alpha x_1(t) + \beta x_2(t) \rightleftharpoons \alpha X_1(j\omega) + \beta X_2(j\omega).$$

Pf :- $x(t) \rightleftharpoons X(j\omega)$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

If $\alpha x_1(t) + \beta x_2(t) = x(t)$

$$\therefore x(j\omega) = \int_{-\infty}^{\infty} \left[\alpha x_1(t) + \beta x_2(t)\right] e^{-j\omega t} \, dt$$

$$= \alpha \int_{-\infty}^{\infty} x_1(t) e^{-j\omega t} \, dt + \beta \int_{-\infty}^{\infty} x_2(t) e^{-j\omega t} \, dt$$

$$= \alpha X_1(j\omega) + \beta X_2(j\omega).$$

$$\rightleftharpoons x(t)$$

$$= \alpha x_1(t) + \beta x_2(t)$$

Fourier Transform of standard signals:

Impulse f^n $\delta(t)$

Signal	F.T.
$\delta(t)$	1
$e^{-at}u(t)$	$\frac{1}{a+j\omega}$
$e^{-a\|t\|}$	$\frac{2a}{a^2+\omega^2}$
$e^{j\omega_0 t}$	$2\pi\,\delta(\omega-\omega_0)$
Constant Amplitude(1)	$2\pi\,\delta(\omega)$
sgn (t)	$\frac{2}{j\omega}$
Unit step f^n $u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
Rectangular pulse (Gate pulse) $\Pi(t/\tau)$ or $rect(t/\tau)$	$\tau\ \mathrm{sinc}\ \frac{\omega\tau}{2}$
Triangular pulse $\Delta(t/\tau)$	$\tau/2\ \mathrm{sinc}^2\ \frac{\omega\tau}{2}$

~~cosec~~ $\cos \omega_0 t$	$\pi[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]$
$\sin \omega_0 t$	$-j\pi[\delta(\omega+\omega_0) - \delta(\omega+\omega_0)]$

Properties of Continuous Time Fourier Tx :

① Linearity Property	If $x_1(t) \xleftrightarrow{FT} X_1(\omega)$ $x_2(t) \longleftrightarrow X_2(\omega)$ then $ax_1(t) + bx_2(t) \xleftrightarrow{FT} aX_1(\omega) + bX_2(\omega)$
② Time Shifting Property	If $x(t) \xleftrightarrow{F.T.} X(\omega)$ then $x(t-t_0) \xleftrightarrow{F.T.} e^{-j\omega t_0} X(\omega)$
③ Frequency Shifting Property	If $x(t) \xleftrightarrow{F.T.} X(\omega)$ $e^{j\omega_0 t} x(t) \longleftrightarrow X(\omega-\omega_0)$
④ Time Reversal Property	If $x(t) \xleftrightarrow{F.T.} X(\omega)$ then $x(-t) \xleftrightarrow{F.T.} X(-\omega)$.
⑤ Time Scaling Property	If $x(t) \xleftrightarrow{FT} X(\omega)$ $x(at) \xleftrightarrow{F.T.} \frac{1}{\lvert a \rvert} X\left(\frac{\omega}{a}\right)$ $a > 1 \rightarrow x(at)$ is compressed $a < 1 \rightarrow x(at)$ is expanded version of $x(t)$.

(6) Differentiation in Time Domain Property	If $x(t) \xleftrightarrow{F.T.} X(\omega)$ $\frac{d}{dt}x(t) \xleftrightarrow{F.T.} j\omega X(\omega)$
(7) Differentiation in Frequency Domain Prop.	If $x(t) \xleftrightarrow{F.T.} X(\omega)$ then $t\,x(t) \xleftrightarrow{F.T.} j\frac{d}{d\omega}X(\omega)$
(8) Time Integration Property	If $x(t) \xleftrightarrow{F.T.} X(\omega)$ then $\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{F.T.} \frac{1}{j\omega}X(j\omega)$ if $X(0)=0$
(9) Convolution Property	If $x_1(t) \xleftrightarrow{F.T.} X_1(\omega)$ & $x_2(t) \xleftrightarrow{F.T.} X_2(\omega)$ then $x_1(t) * x_2(t) \xleftrightarrow{F.T.}$ $X_1(\omega)X_2(\omega)$
(10) Multiplication Property	If $x_1(t) \xleftrightarrow{F.T.} X_1(\omega)$ & $x_2(t) \xleftrightarrow{F.T.} X_2(\omega)$ then $x_1(t)\,x_2(t) \xleftrightarrow{F.T.}$ $\frac{1}{2\pi}X_1(\omega) * X_2(\omega)$
(11) Duality Property (Symmetry)	If $x(t) \xleftrightarrow{F.T.} X(\omega)$ $X(t) \xleftrightarrow{F.T.} 2\pi x(-\omega)$

(12)	Modulation Property	If $x(t) \xleftrightarrow{F.T.} X(\omega)$ then $x(t)\cos\omega_c(t) \xleftrightarrow{F.T.} \frac{1}{2}[X(\omega-\omega_c)+X(\omega+\omega_c)]$
(13)	Conjugation Property	If $x(t) \xleftrightarrow{F.T.} X(\omega)$ Then $x^*(t) \xleftrightarrow{F.T.} X^*(-\omega)$
(14)	Autocorrelation Property	If $x(t) \xleftrightarrow{F.T.} X(\omega)$ then $R(\tau) \xleftrightarrow{F.T.} \lvert X(\omega)\rvert^2$
(15)	Parsevals Relation/ Property	If $x_1(t) \xleftrightarrow{F.T.} X_1(\omega)$ & $x_2(t) \xleftrightarrow{F.T.} X_2(\omega)$ then $\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega) X_2^*(\omega)\, d\omega$ for complex $x_1(t)$ & $x_2(t)$.
(16)	Area under the curve	If $x(t) \xleftrightarrow{F.T.} X(\omega)$ $\int_{-\infty}^{\infty} x(t)\, dt = \frac{1}{2\pi} X(0)$, for $\omega=0$ $\int_{-\infty}^{\infty} X(\omega) d\omega = x(0)$, for $t=0$

Date
Page

(17) (i) If $x(t)$ is real, then $X_R(\omega) = \int_{-\infty}^{\infty} x(t) \cos \omega t \, dt$

then $X_I(t) = 0$ $\Rightarrow$ $X_I(\omega) = -\int_{-\infty}^{\infty} x(t) \sin \omega t \, dt$

& $X(-\omega) = X^*(\omega)$

(ii) If $x(t)$ is even & real,

$$X_R(\omega) = 2\int_{0}^{\infty} x_e(t) \cos \omega t \, dt$$

$$X_I(\omega) = 0$$

(iii) when $x(t)$ is odd & real,

$\cancel{x(t) = x_0 /}$

$$X_I(\omega) = jX(\omega) = -j2\int_{0}^{\infty} x_o(t) \sin \omega t \, dt$$

(iv) For non-symmetrical f^n –

$$X(\omega) = X_e(\omega) + X_o(\omega).$$

$\cos\omega_0 t$

$$X(\omega) = F\left[\tfrac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)\right]$$

$$= \tfrac{1}{2}\left[F\left(e^{j\omega_0 t}\right) + F\left(e^{-j\omega_0 t}\right)\right]$$

$$= \frac{1}{2}\left[2\pi\delta(\omega - \omega_0) + 2\pi\delta(\omega + \omega_0)\right]$$

$$= \pi\left[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)\right]$$

- Parsevals Theorem.
- F.T. of periodic s/g.
- Limitations & advantages of F.T.
- Numericals.

Q1. Find the F.T. of $x(t) = \delta(t+2) + \delta(t+1) + \delta(t-1) + \delta(t-2)$

$$\Rightarrow X(\omega) = \int_{-\infty}^{\infty} [\delta(t+2) + \delta(t+1) + \delta(t-1) + \delta(t-2)] e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \delta(t+2) e^{-j\omega t} dt + \int_{-\infty}^{\infty} \delta(t+1) e^{-j\omega t} dt$$

$$+ \int_{-\infty}^{\infty} \delta(t-1) e^{-j\omega t} dt$$

$$+ \int_{-\infty}^{\infty} \delta(t-2) e^{-j\omega t} dt$$

$$= e^{2j\omega} + e^{j\omega} + e^{-j\omega} + e^{-2j\omega}$$

$$= 2\left[\frac{e^{j\omega} + e^{-j\omega}}{2} + \frac{e^{j2\omega} + e^{-j2\omega}}{2}\right]$$

$$= 2[\cos\omega + \cos 2\omega]$$

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Q2. Find F.T. of $x(t) = e^{-|t|}$, $-2 \leq t \leq 2$
$= 0$, otherwise.

$\Rightarrow$ $$X(\omega) = \int_{-\infty}^{\infty} e^{-|t|} e^{-j\omega t} dt$$

$$= \int_{-2}^{0} e^{-(-t)} e^{-j\omega t} dt + \int_{0}^{2} e^{-(t)} e^{-j\omega t} dt$$

$$= \int_{-2}^{0} e^{t} e^{-j\omega t} dt + \int_{0}^{2} e^{-t} e^{-j\omega t} dt$$

$$= \int_{-2}^{0} e^{-(1-j\omega)t} dt + \int_{0}^{2} e^{-(1+j\omega)t} dt$$

$$= \left[\frac{e^{-(1-j\omega)}}{-(1-j\omega)}\right]_{0}^{2} + \left[\frac{e^{-(1+j\omega)}}{-(1+j\omega)}\right]_{0}^{2}$$

$$= \frac{e^{-2(1-j\omega)} - 1}{-(1-j\omega)} + \frac{e^{-2(1+j\omega)} - 1}{-(1+j\omega)}$$

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Q. Find F.T. of the sigs:—

(1) $e^{-2t} \cos 5t \, u(t)$

② $t e^{-at} u(t)$.

Q. Find F.T. of the s/gs:—

① $e^{-2t}\cos 5t\, u(t)$

$\Rightarrow$ $x(t) = e^{-2t}\cos 5t\, u(t)$

$$= e^{-2t}\left[\frac{e^{j5t} + e^{-j5t}}{2}\right] u(t)$$

$$\therefore X(\omega) = F\left[e^{-2t}\cos 5t\, u(t)\right].$$

$$= \int_{-\alpha}^{\alpha} e^{-2t}\left[\frac{e^{j5t} + e^{-j5t}}{2}\right] e^{-j\omega t}\, u(t)\, dt$$

$$= \frac{1}{2}\left[\int_{0}^{\alpha} e^{-2t}\, e^{j5t}\, e^{-j\omega t}\, dt + \int_{0}^{\alpha} e^{-2t} \cdot e^{-j5t} \cdot e^{-j\omega t}\, dt\right]$$

$$= \frac{1}{2}\left[\int_{0}^{\alpha} e^{-[2+j(\omega-5)]t} \cdot \int_{0}^{\alpha} e^{-[2+j(\omega+5)]} d\right.$$

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Page

$$= \frac{1}{2}\left[\frac{e^{-[2+j(\omega-5)]t}}{-[2+j(\omega-5)]} + \frac{e^{-[2+j(\omega+5)]t}}{-[2+j(\omega+5)]}\right]_0^{\infty}$$

$$= \frac{1}{2}\left[\frac{e^{-\infty} - e^{0}}{-[2+j(\omega-5)]} + \frac{e^{-\infty} - e^{0}}{-[2+j(\omega+5)]}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(2+j\omega)-j5} + \frac{1}{(2+j\omega)+j5}\right]$$

$$= \frac{1}{2}\left[\frac{2+j\omega}{(2+j\omega)^2+5^2}\right]$$

//

② $t e^{-at} u(t)$.

$$X(\omega) = F\left[t e^{-at} u(t)\right]$$

$$= \int_{-\alpha}^{\alpha} t e^{-at} u(t) e^{-j\omega t} \, dt$$

$$= \int_{0}^{\alpha} t e^{-(a+j\omega)t} \, dt$$

$$= \left[t \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right]_0^{\alpha}$$

$$- \int_0^{\alpha} \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \, dt$$

$$= \left[t \frac{e^{-a(+j\omega)t}}{-(a+j\omega)} \right]_0^{\alpha} - \left[\frac{e^{-(a+j\omega)}}{(a+j\omega)^2} \right]_0^{\infty}$$

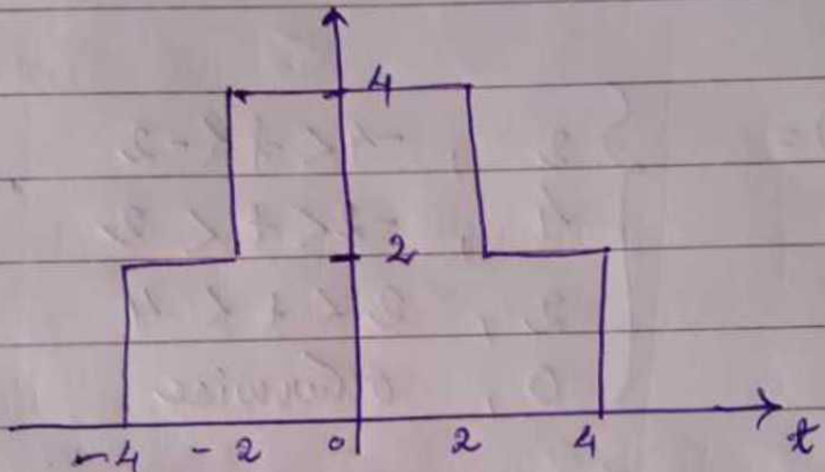
$$= 0 - 0 - 0 + \frac{1}{(a+j\omega)^2}$$

$$= \frac{1}{(a+j\omega)^2}$$

Fourier Transform

Numericals

Q.1. Find the F.T. of the s/g :

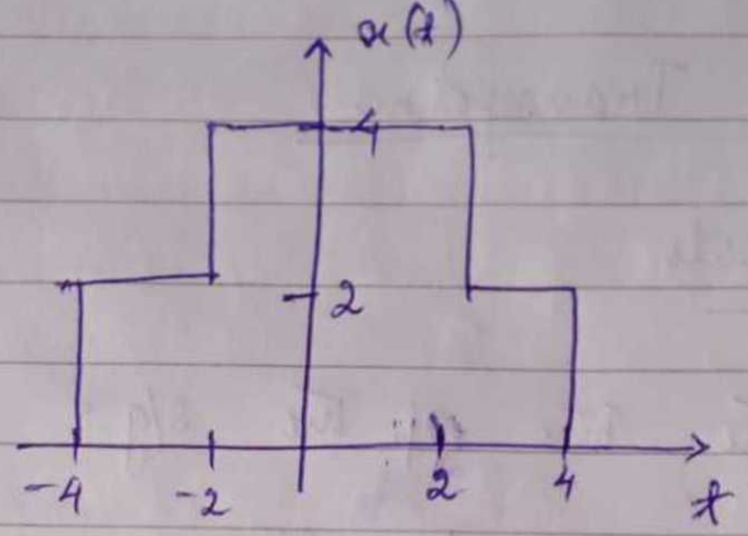


Q.2. Find inverse F.T. of $e^{-2\omega} u(\omega)$:

Q.3. Find the Fourier Transform of :

$$x(t) = \begin{cases} (1-t^2), & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$$

Q1. Sol^n.



$$x(t) = \begin{cases} 2, & -4 < t < -2 \\ 4, & -2 < t < 2 \\ 2, & 2 < t < 4 \\ 0, & \text{otherwise} \end{cases}$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-4}^{-2} 2e^{-j\omega t} dt + \int_{-2}^{2} 4e^{-j\omega t} dt$$

$$+ \int_{2}^{4} 2e^{-j\omega t} dt$$

$$= 2\left[\frac{e^{-j\omega}}{-j\omega}\right]_{-4}^{-2}$$

Q2 Sol^n:

$$x(t) = \frac{1}{2\pi} \int_{-\alpha}^{\alpha} e^{-2\omega} u(\omega) e^{j\omega t} d\omega .$$

$$= \frac{1}{2\pi} \int_{0}^{\alpha} e^{-2\omega} e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{0}^{\alpha} e^{-(2-jt)\omega} d\omega$$

$$= \frac{1}{2\pi} \left[\frac{e^{-(2-jt)}}{-(2-jt)} \right]_{0}^{\alpha}$$

$$= \frac{1}{2\pi(2-jt)}$$

Q.1 Find the fourier transform of $x(t) = \sin(8t + 0.1\pi)$

Soln

Let, $x_1(t) = \sin(t + 0.1\pi)$

$\therefore x_1(8t) = \sin(8t + 0.1\pi) = x(t)$

We know,

$$F(\sin t) = j\pi\left[\delta(\omega+1) - \delta(\omega-1)\right]$$

Using time shifting property [i.e.

$x(t-t_0) \xleftrightarrow{F.T} e^{-j\omega t_0} X(\omega)$] we have,

$$F\left[\sin (t + 0.1\pi)\right] = e^{j\omega(0.1\pi)} F(\sin t)$$

$$= e^{j\omega(0.1\pi)} j\pi\left[\delta(\omega+1) - \delta(\omega-1)\right]$$

Using time scaling property

[i.e. $x(at) \xleftrightarrow{F.T.} \frac{1}{|a|} X(\omega/a)$], we have

$$F\left[\sin(8t + 0.1\pi)\right] = \frac{1}{|8|} F\left[\sin(t + 0.1\pi)\right]\Big|_{\omega = \omega/8}$$

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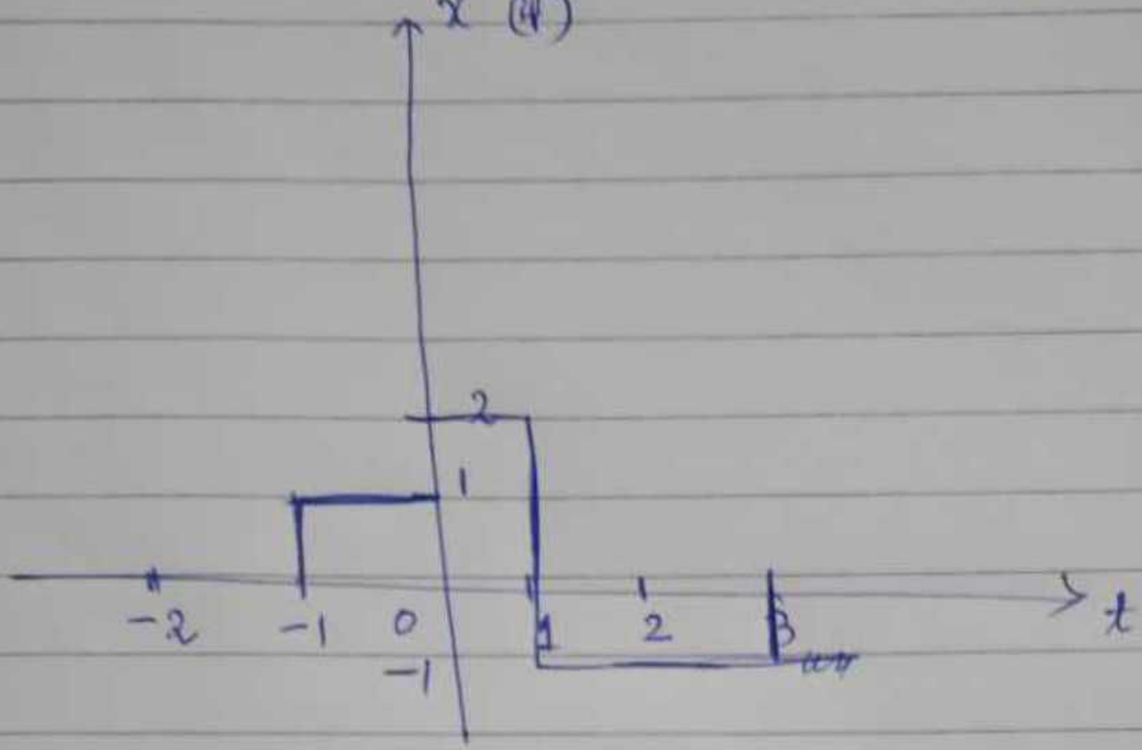
Q2. Compute the Fourier Tx for the s/g $x(t)$



Q3. Compute the Fourier Tx of the signal

$$x(t) = \begin{cases} 1+\cos \pi t & , |t| < 1 \\ 0 & |t| > 1 \end{cases}$$

Q.4. Find the Fourier Tx of s/g $x(t) = 5\cos^2 3t$.

Q5. Find inverse Fourier Tx of $X(\omega) = e^{-4\omega} u(\omega)$