Q. Find the mgf of a random variable whose moments are

$$\mu'_r = (r+1)!\,.\,2^r$$

Soln.: The mgf is given by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu'_r = \sum_{r=0}^{\infty} \frac{t^r}{r!} (r+1)!\, 2^r$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} (r+1) r!\, 2^r$$

$$= \sum_{r=0}^{\infty} t^r (r+1) 2^r$$

$$\therefore M_X(t) = \left(t^0 (0+1) 2^0\right) + \left(t^1 (1+1) 2^1\right) + \left(t^2 (2+1) 2^2\right) + \ldots$$

$$M_x(t) = 1 + 2.(2t) + 3(2t)^2 + 4(2t)^3 + \dots$$

$$= (1-2t)^{-2}$$

$q^{x-1} = \frac{q^x}{q}$

Q. let the random variable X assume the value 'x' with the probability function is given by

$$P(x=x) = q^{x-1} p \quad ; \quad x = 1, 2, 3, \dots$$

Find the mgf of X and hence mean and variance.

Soln.

$$M_x(t) = E(e^{tx}) = \sum_{x=1}^{\infty} e^{tx} p(x) = \sum_{x=1}^{\infty} e^{tx} q^{x-1} p = p \sum_{x=1}^{\infty} e^{tx} \frac{q^x}{q}$$

$$M_X(t) = \frac{p}{q}\sum_{x=1}^{\infty}(qe^t)^x$$

$$= \frac{p}{q}\left[qe^t + (qe^t)^2 + (qe^t)^3 + (qe^t)^4 + \dots\right]$$

$$= \frac{p}{q}qe^t\left[1 + (qe^t) + (qe^t)^2 + (qe^t)^3 + \dots\right]$$

$$= \frac{pe^t}{1-qe^t}$$

$$1 + a + a^2 + a^3 + \dots$$

$$= \frac{1}{1-a}$$

$$\mu_1' = \frac{d}{dt}\left[M_X(t)\right] = \frac{d}{dt}\left[\frac{pe^t}{1-qe^t}\right]_{t=0}$$

$$= \left[\frac{(1-qe^t)(pe^t) - pe^t(-qe^t)}{(1-qe^t)^2}\right]_{t=0}$$

$$\mu_1' = \left.\frac{pe^t}{(1-qe^t)^2}\right|_{t=0} = \frac{p}{(1-q)^2} = \frac{p}{(p)^2} = \frac{1}{p}$$

Again,

$$\mu_2' = \frac{d^2}{dt^2}\left[M_x(t)\right]_{t=0}$$

$$= \frac{d}{dt}\left[\frac{pe^t}{(1-qe^t)^2}\right]_{t=0}$$

$$= \left.\frac{(1-qe^t)^2 (pe^t) - pe^t\left[2(1-qe^t)\right](-qe^t)}{(1-qe^t)^4}\right|_{t=0}$$

$$\mu_2' = \left.\frac{pe^t(1+qe^t)}{(1-qe^t)^3}\right|_{t=0}$$

$$= \frac{1+q}{p^2}$$

Hence,

Mean $= \mu_1'$ (about origin) $= \frac{1}{p}$

Variance $= \mu_2 = \mu_2' - \mu_1'^2$

$$= \frac{1+q}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{1+q-1}{p^2} = \frac{q}{p^2}$$

Q. A random variable X has probability function $p(x) = \frac{1}{2^x}$; $x = 1, 2, 3, \ldots$

find its mgf, mean and variance.

| Ans – $\frac{e^t}{2 - e^t}$

mean = 2

variance = 2

JOINT DISTRIBUTION & BIVARIATE DISTRIBUTION

Joint Probability

Two random variables X and Y are said to be jointly distributed if they are defined on same probability space. The joint probability function is denoted by $P_{XY}(x,y)$ or $f_{XY}(x,y)$.

Joint Probability Mass Function

Let X and Y be random variables on a sample space S with respective image sets $X(S) = \{x_1, x_2, \ldots, x_n\}$ and $Y(S) = \{y_1, y_2, \ldots, y_m\}$. The function p on $X(S) \times Y(S)$ defined by

$$p_{ij} = P(X = x_i \cap Y = y_j) = p(x_i, y_j)$$

is called joint probability function of X and Y

where $X(S) \times Y(S) = \{x_1, x_2, \ldots, x_n\} \times \{y_1, y_2, \ldots, y_m\}$

$x \to$ \ $y \downarrow$	y_1	y_2	y_3	$\cdots$	y_j	$\cdots$	y_m	Total
x_1	p_{11}	p_{12}	p_{13}	$\cdots$	p_{1j}	$\cdots$	p_{1m}	p_1
x_2	p_{21}	p_{22}	p_{23}	$\cdots$	p_{2j}	$\cdots$	p_{2m}	p_2
x_3	p_{31}	p_{32}	p_{33}	$\cdots$	p_{3j}	$\cdots$	p_{3m}	p_3
$\vdots$	$\cdots$							$\vdots$
x_i	p_{i1}	p_{i2}	p_{i3}	$\cdots$	p_{ij}	$\cdots$	p_{im}	p_i
$\vdots$	$\cdots$							$\vdots$
x_n	p_{n1}	p_{n2}	p_{n3}	$\cdots$	p_{nj}	$\cdots$	p_{nm}	p_n
Total	p_1	p_2	p_3	$\cdots$	p_j	$\cdots$	p_m	

$$\sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j) = 1$$