Q. Find the mgf of a random variable whose moments are

$$\mu_r' = (r+1)!\,.\,2^r$$

Soln: The mgf is given by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r' = \sum_{r=0}^{\infty} \frac{t^r}{r!} (r+1)!\, 2^r$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} (r+1) r!\, 2^r$$

$$= \sum_{r=0}^{\infty} t^r (r+1) 2^r$$

$$\therefore M_X(t) = \left(t^0 (0+1) 2^0\right) + \left(t^1 (1+1) 2^1\right) + \left(t^2 (2+1) 2^2\right) + \dots$$

$$M_X(t) = 1 + 2.(2t) + 3(2t)^2 + 4(2t)^3 + \dots$$

$$= (1-2t)^{-2}$$

$$\left| \; q^{x-1} = \frac{q^x}{q} \right.$$

Q. Let the random variable X assume the value 'x' with the probability function is given by

$$P(x=x) = q^{x-1} p \qquad ; \quad x = 1, 2, 3, \dots$$

Find the mgf of X and hence mean and variance.

Soln.

$$M_X(t) = E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx} p(x) = \sum_{x=1}^{\infty} e^{tx} q^{x-1} p = p \sum_{x=1}^{\infty} e^{tx} \frac{q^x}{q}$$

$$M_X(t) = \frac{p}{q} \sum_{x=1}^{\infty} (qe^t)^x$$

$$= \frac{p}{q} \left[qe^t + (qe^t)^2 + (qe^t)^3 + (qe^t)^4 + \dots \right]$$

$$= \frac{p}{q} qe^t \left[1 + (qe^t) + (qe^t)^2 + (qe^t)^3 + \dots \right]$$

$$= \frac{pe^t}{1-qe^t}$$

$$1 + a + a^2 + a^3 + \dots$$

$$= \frac{1}{1-a}$$

$$\mu_1' = \frac{d}{dt} \left[M_X(t) \right] = \frac{d}{dt} \left[\frac{pe^t}{1-qe^t} \right]_{t=0}$$

$$= \left[\frac{(1-qe^t)(pe^t) - pe^t(-qe^t)}{(1-qe^t)^2} \right]_{t=0}$$

$$\mu_1' = \left.\frac{pe^t}{(1-qe^t)^2}\right|_{t=0} = \frac{p}{(1-q)^2} = \frac{p}{(p)^2} = \frac{1}{p}$$

Again,

$$\mu_2' = \frac{d^2}{dt^2}\left[M_x(t)\right]_{t=0}$$

$$= \frac{d}{dt}\left[\frac{pe^t}{(1-qe^t)^2}\right]_{t=0}$$

$$= \left.\frac{(1-qe^t)^2 (pe^t) - pe^t\left[2(1-qe^t)\right](-qe^t)}{(1-qe^t)^4}\right|_{t=0}$$

$$\mu_2' = \left.\frac{pe^t(1+qe^t)}{(1-qe^t)^3}\right|_{t=0}$$

$$= \frac{1+q}{p^2}$$

Hence,

$$\text{Mean} = \mu_1' \text{ (about origin)} = \frac{1}{p}$$

$$\text{Variance} = \mu_2 = \mu_2' - \mu_1'^2$$

$$= \frac{1+q}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{1+q-1}{p^2} = \frac{q}{p^2}$$

Q. A random variable X has probability function $p(x) = \frac{1}{2^x}$; $x = 1, 2, 3, \ldots$

find its mgf, mean and variance. | Ans - $\frac{e^t}{2-e^t}$

mean = 2

variance = 2

JOINT DISTRIBUTION & BIVARIATE DISTRIBUTION

Joint Probability

Two random variables X and Y are said to be jointly distributed if they are defined on same probability space. The joint probability function is denoted by $P_{XY}(x, y)$ or $f_{XY}(x, y)$.

Joint Probability Mass Function

Let X and Y be random variables on a sample space S with respective image sets $X(S) = \{x_1, x_2, \ldots, x_n\}$ and $Y(S) = \{y_1, y_2, \ldots, y_m\}$. The function p on $X(S) \times Y(S)$ defined by

$$p_{ij} = P(X = x_i \cap Y = y_j) = p(x_i, y_j)$$

is called joint probability function of X and Y

where $X(S) \times Y(S) = \{x_1, x_2, \ldots, x_n\} \times \{y_1, y_2, \ldots, y_m\}$

$x \rightarrow$ \ $y \downarrow$	y_1	y_2	y_3	$\ldots$	y_j	$\ldots$	y_m	Total
x_1	p_{11}	p_{12}	p_{13}	$\ldots$	p_{1j}	$\ldots$	p_{1m}	p_1
x_2	p_{21}	p_{22}	p_{23}	$\ldots$	p_{2j}	$\ldots$	p_{2m}	p_2
x_3	p_{31}	p_{32}	p_{33}	$\ldots$	p_{3j}	$\ldots$	p_{3m}	p_3
$\vdots$	$-$	$\ldots$	$-$					$\vdots$
x_i	p_{i1}	p_{i2}	p_{i3}	$\ldots$	p_{ij}	$\ldots$	p_{im}	p_i
$\vdots$	$-$	$\ldots$	$-$					$\vdots$
x_n	p_{n1}	p_{n2}	p_{n3}	$\ldots$	p_{nj}	$\ldots$	p_{nm}	p_n
Total	p_1	p_2	p_3	$\ldots$	p_j	$\ldots$	p_m	

$$\sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j) = 1$$

Marginal and Conditional Probability Function

Consider a joint distribution of two random variables X and Y then

$$f_X(x) = p_X(x_i) = P(X = x_i) = p_{i1} + p_{i2} + \dots + p_{ij} + \dots + p_{im}$$

$$= \sum_{j=1}^{m} p_{ij}$$

$$= p_i$$

Marginal Probability function of X.

$$f_Y(y) = P_Y(y_j) = P(Y = y_j) = \sum_{i=1}^{n} p_{ij} = p_j$$

$\hookrightarrow$ Marginal Probability function of Y.

<u>Conditional</u> Probability f^n of X when $Y = y_j$ is given

$$f_{X/Y}(x/y) = P(X = x_i / Y = y_j) = \frac{P(X = x_i \cap Y = y_j)}{P(Y = y_j)}$$

$$= \frac{P(x_i, y_j)}{P(y_j)}$$

$$= \frac{p_{ij}}{p_j}$$

Conditional probability f^n of y when $X = x_i$ is given

$$f_{Y/X}(y/x) = P(Y = y_j \mid X = x_i) = \frac{p(x_i, y_j)}{p(x_i)} = \frac{p_{ij}}{p_i}$$

Independent ; $P(X = x_i, Y = y_j) = P(X = x_i)\,P(Y = y_j)$

JOINT DISTRIBUTION FUNCTION

$X, Y \leftarrow$ two R.V.s

Then their joint distribution f^n $F_{X,Y}(x, y)$ is given by

$$F_{X,Y}(x,y) = P(X \le x, Y \le y) \quad \forall \; x, y \in \mathbb{R}$$

where $\sum_x \sum_y f_{X,Y}(x,y) = 1 \leftarrow$ Discrete R.V.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) = 1 \leftarrow \text{cont. R.V.}$$

Properties

① If $x_1 < x_2$ and $y_1 < y_2$ then (Rectangle Rule)

$$P[x_1 < X \le x_2,\; y_1 < Y \le y_2] = F(x_2, y_2) - F(x_2, y_1) - F(x_1, y_2) + F(x_1, y_1) \ge 0$$

2. (a) $F(-\infty, y) = \lim_{x \to -\infty} F(x,y) = 0 \quad \forall \quad y \in \mathbb{R}$

(b) $F(x, -\infty) = \lim_{y \to -\infty} F(x,y) = 0 \quad \forall \quad x \in \mathbb{R}$

(c) $\lim_{\substack{x \to \infty \\ y \to \infty}} F(x,y) = F(\infty, \infty) = 1$

3. $F(x,y)$ is right continuous in each argument i.e.

$$\lim_{h \to 0^+} F(x+h, y) = \lim_{h \to 0^+} F(x, y+h) = F(x,y)$$

4. If the density function $f(x,y)$ is continuous at (x,y) then

$$\frac{\partial^2 F}{\partial x \partial y} = f(x,y).$$

Expectation, Covariance, correlation coefficient

Let us consider (X, Y) as a two dimensional discrete random variable with joint discrete density function $f_{X,Y}(x,y)$.

The expectation of $g(X,Y)$ is denoted by $E[g(X,Y)]$ and defined as

$$E[g(X,Y)] = \sum_x \sum_y g(x,y) f_{X,Y}(x,y)$$

Particular Cases

(i) $E[X] = \sum x f_X(x)$

(ii) $E[Y] = \sum y f_Y(y)$

(iii) $E[XY] = \sum_x \sum_y xy f_{X,Y}(x,y)$

Covariance: $Cov(X,Y) = E[X - E(X)]\, E[Y - E(Y)] = E[XY] - E[X]E[Y]$

Correlation coefficient: $\rho(X,Y) = \dfrac{Cov(X,Y)}{\sqrt{Var\,X}\,\sqrt{Var\,Y}} = \dfrac{Cov(X,Y)}{\sigma_X \sigma_Y}$

where, $\sigma_X > 0$, $\sigma_Y > 0$

Note: $-1 \leq \rho(X,Y) \leq 1$

Conditional Expectation

If (X,Y) are joint discrete random variable then conditional expectation of $g(X,Y)$ given $X=x$ is defined as

$$E[g(X,Y)/X=x] = \sum_j g(x_i, y_j)\, f_{Y/X}(y_j \mid x)$$

for Particular,

$$E[Y/X=x] = \sum_j y_j\, f_{Y/X}(y_j \mid x) = \sum_j y_j\, P(Y=y_j \mid X=x)$$

Q. For the following bivariate probability distribution of X and Y find

(i) $P(X \le 2, Y = 3)$

(ii) $P(X \le 1)$

(iii) $P(Y = 4)$

(iv) $P(Y \le 5)$

(v) $P(X < 2, Y \le 3)$

X \ Y	1	2	3	4	5	6
0	0	0	1/32	2/32	2/32	3/32
1	1/16	1/16	1/8	1/8	1/8	1/8
2	1/32	1/32	1/64	1/64	0	2/64

Soln: The marginal distribution is given

$X \downarrow$ \ $Y \rightarrow$	1	2	3	4	5	6	$p_X(x)$
0	0	0	1/32	2/32	2/32	3/32	8/32
1	1/16	1/16	1/8	1/8	1/8	1/8	10/16
2	1/32	1/32	1/64	1/64	0	2/64	8/64
$p_Y(y)$	3/32	3/32	11/64	13/64	6/32	16/64	1

(1) $P(X \le 2, Y=3) = P(X=0, Y=3) + P(X=1, Y=3) + P(X=2, Y=3)$

$$= \frac{1}{32} + \frac{1}{8} + \frac{1}{64}$$

$$= \frac{11}{64}$$

(ii) $P(X \leq 1) = P(X=0) + P(X=1)$

$= \frac{8}{32} + \frac{10}{16}$

$= \frac{7}{8}$

(iii) $P(Y=4) = \frac{13}{64}$

(iv) $P(Y \leq 5) = P(Y=1) + P(Y=2) + P(Y=3) + P(Y=4) + P(Y=5)$

$= ?$

$P(Y \leq 5) = 1 - P(Y=5)$

$= 1 - P(Y=6)$

$= 1 - \frac{16}{64} = \frac{48}{64}$

(v) $P(X<2, Y \leq 3) = P(X=0, Y=1) + P(X=0, Y=2) + P(X=0, Y=3) + P(X=1, Y=1) + P(X=1, Y=2) + P(X=1, Y=3)$

$= ?$

Q. Let X and Y have joint p.d.f.

$Y \backslash X$	-1	0	1
0	b	$2b$	b
1	$3b$	$2b$	b
2	$2b$	b	$2b$

Find marginal distribution of X and Y. Also find conditional distribution of X given $Y = 1$.

Sol^n: Marginal Distribution table will be given by

$Y \backslash X$	-1	0	1	$p_Y(y)$
0	b	$2b$	b	$4b$
1	$3b$	$2b$	b	$6b$
2	$2b$	b	$2b$	$5b$
$p_X(x)$	$6b$	$5b$	$4b$	$15b$

Marginal Distribution of X is

$P(X=-1) = 6b$, $P(X=0) = 5b$, $P(X=1) = 4b$

Marginal Distribution of Y is

$P(Y=0) = 4b$, $P(Y=1) = 6b$, $P(Y=2) = 5b$

Conditional distribution of X when $Y=1$

$$P(X=x / Y=1) = \frac{P(X=x \cap Y=1)}{P(Y=1)} \left\{ \begin{array}{l} \dfrac{P(X=-1 \cap Y=1)}{P(Y=1)} \\[2ex] \dfrac{P(X=0 \cap Y=1)}{P(Y=1)} \\[2ex] \dfrac{P(X=1 \cap Y=1)}{P(Y=1)} \end{array} \right.$$

$$P(X=x \mid Y=1) = \begin{cases} \dfrac{3b}{6b} & \text{when } X=-1,\ Y=1 \\[2ex] \dfrac{2b}{6b} & \text{when } X=0,\ Y=1 \\[2ex] \dfrac{b}{6b} & \text{when } X=1,\ Y=1 \end{cases}$$

$$= \begin{cases} \dfrac{1}{2}, & \text{when } X=-1,\ Y=1 \\[1ex] \dfrac{1}{3}, & \text{when } X=0,\ Y=1 \\[1ex] \dfrac{1}{6}, & \text{when } X=1,\ Y=1 \end{cases}$$

Alternative

$P(X=-1 \mid Y=1) = \frac{1}{2}$, $P(X=0 \mid Y=1) = \frac{1}{3}$, $P(X=1, Y=1) = \frac{1}{6}$

Q. The joint probability distribution of X and Y is given in the following table

$X \backslash Y$	1	3	9
2	$1/8$	$1/24$	$1/12$
4	$1/4$	$1/4$	0
6	$1/8$	$1/24$	$1/12$

(a) find the marginal probability distribution of Y

(b) Find the conditional distribution of Y given $X=4$

(c) Find covariance of X and Y.

(d) Are X and Y independent?

Soln: Marginal distribution table is given by

X \ Y	1	3	9	$f_X(x)$
2	$1/8$	$1/24$	$1/12$	$6/24$
4	$1/4$	$1/4$	0	$2/4$
6	$1/8$	$1/24$	$1/12$	$6/24$
$f_Y(y)$	$4/8$	$8/24$	$2/12$	1

(a) Marginal Probability Distribution of Y is

$$P(Y=1) = \frac{4}{8} = \frac{1}{2}$$

$$P(Y=3) = \frac{8}{24} = \frac{1}{3}$$

$$P(Y=9) = \frac{2}{12} = \frac{1}{6}$$

i.e. $$P(Y=y) = \begin{cases} \frac{1}{2}, & y=1 \\ \frac{1}{3}, & y=3 \\ \frac{1}{6}, & y=9 \end{cases}$$

(b) The conditional distribution of Y given $X=4$ is

$$P(Y=y \mid X=4) = \frac{P(Y=y \cap X=4)}{P(X=4)}$$

Now,

$$P(Y=1 \mid X=4) = \frac{P(Y=1 \cap X=4)}{P(X=4)} = \frac{1/4}{2/4} = \frac{1}{2}$$

$$P(Y=3 \mid X=4) = \frac{P(Y=3 \cap X=4)}{P(X=4)} = \frac{1/4}{2/4} = \frac{1}{2}$$

$$P(Y=9 \mid X=4) = \frac{P(Y=9 \cap X=4)}{P(X=4)} = \frac{0}{2/4} = 0$$

i.e. $$P(Y=y \mid X=4) = \begin{cases} \frac{1}{2}, & y=1,\ x=4 \\ \frac{1}{2}, & y=3,\ x=4 \\ 0, & y=9,\ x=4 \end{cases}$$

c $$\operatorname{cov}(X,Y) = E[XY] - E[X]\,E[Y] \longrightarrow 1$$

Now, $$E[X] = \sum x f_X(x) = \left[\left(2 \times \frac{6}{24}\right) + \left(4 \times \frac{2}{4}\right) + \left(6 \times \frac{6}{24}\right)\right]$$

$$= 4$$

$$E[Y] = \sum y\, f_Y(y) = \left[\left(1 \times \frac{4}{8}\right) + \left(3 \times \frac{8}{24}\right) + \left(9 \times \frac{2}{12}\right)\right]$$

$$= 3$$

$$E[XY] = \sum xy\, f_{X,Y}(x, y)$$

$$= \left[\left(2 \times 1 \times \frac{1}{8}\right) + \left(2 \times 3 \times \frac{1}{24}\right) + \left(2 \times 9 \times \frac{1}{12}\right)\right] + \left[\left(4 \times 1 \times \frac{1}{4}\right) + \left(4 \times 3 \times \frac{1}{4}\right) + (4 \times 9 \times 0)\right]$$

$$+ \left[\left(6 \times 1 \times \frac{1}{8}\right) + \left(6 \times 3 \times \frac{1}{24}\right) + \left(6 \times 9 \times \frac{1}{12}\right)\right]$$

$$= 12$$

from ①,

$$\mathrm{cov}(X, Y) = E[XY] - E[X]E[Y] = 12 - (4 \times 3) = 12 - 12 = 0$$

$f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$ ← if true then indp.

<u>Check</u>

(d) $f_{X,Y}(2,1) = ?$

$f_X(2) = ?$

$f_Y(1) = ?$

Check if $f_{X,Y}(2,1) = f_X(2)\, f_Y(1)$ or not

and also check for the rest.

Q. Consider a sample of size 2 drawn without replacement from an urn containing three balls, numbered 1, 2 and 3. Let X be the smaller of the two numbers drawn and Y the larger

(a) Find the joint discrete density function X and Y.

(b) Find the conditional distribution of Y given $X=1$.

(c) Find $\rho(X,Y)$

Soln: Here,

X = smaller of the two numbers drawn

Y = larger of the two numbers drawn.

Possible outcomes are (1,2), (1,3), (2,1), (2,3), (3,1), (3,2)

(∵ Replacement is not allowed)

A/q X is smaller among the two no.s and Y is larger

∴ possible values of (X, Y) are (1,2), (1,3), (2,3)

→ Total no. of outcomes = 3

(a) The joint discrete density function of X and Y is given below:

① ② ③ urn

(1,2), (1,3), (2,1), (2,3)
(3,1), (3,2)
← Replacement not allowed

X smaller than Y

$x = 1$ — 2, 3

$x = 2$ — 3

$x = 3$ — no possible values

$(1,2),\ (1,3)$
$(2,3)$

$X \backslash Y$	2	3	$f_X(x)$
1	$1/3$	$1/3$	$2/3$
2	0	$1/3$	$1/3$
$f_Y(y)$	$1/3$	$2/3$	1

(b) Conditional distribution of Y given $X=1$

$$f_{Y/X}(Y=y \mid X=1) = \frac{f(1,y)}{f_X(1)} = \frac{f(1,y)}{2/3} = \frac{3}{2} f(1,y)$$

$$f_{Y/X}(Y=y \mid X=1) = \begin{cases} \frac{3}{2} f(1,2) & ; \text{ when } y=2 \\ \frac{3}{2} f(1,3) & ; \text{ when } y=3 \end{cases}$$

$$= \begin{cases} \frac{3}{2}\left(\frac{1}{3}\right) & ; \quad y=2 \\ \frac{3}{2}\left(\frac{1}{3}\right) & ; \quad y=3 \end{cases}$$

$$= \begin{cases} \frac{1}{2} & ; \quad y=2 \\ \frac{1}{2} & ; \quad y=3 \end{cases}$$

(c) $E[X] = \sum x f_X(x) = \left(1 \times \frac{2}{3}\right) + \left(2 \times \frac{1}{3}\right) = \frac{4}{3}$

$E[Y] = \sum y f_Y(y) = \left(2 \times \frac{1}{3}\right) + \left(3 \times \frac{2}{3}\right) = \frac{8}{3}$

$E[XY] = \sum xy f_{X,Y}(x,y)$

$= \left[1 \times 2 \times \frac{1}{3}\right] + \left[1 \times 3 \times \frac{1}{3}\right] + \left[2 \times 2 \times 0\right] + \left[2 \times 3 \times \frac{1}{3}\right]$

$= \frac{11}{3}$

$E[X^2] = \sum x^2 f_X(x) = \left(1^2 \times \frac{2}{3}\right) + \left(2^2 \times \frac{1}{3}\right) = \frac{6}{3} = 2$

$E[Y^2] = \sum y^2 f_Y(y) = \left(2^2 \times \frac{1}{3}\right) + \left(3^2 \times \frac{2}{3}\right) = \frac{22}{3}$

Now,

$$\text{var}(X) = E[X^2] - (E[X])^2 = 2 - \left(\frac{4}{3}\right)^2 = 2 - \frac{16}{9}$$

$$= \frac{2}{9}$$

$$\text{var}(Y) = E[Y^2] - (E[Y])^2 = \frac{22}{3} - \left(\frac{8}{3}\right)^2$$

$$= \frac{22}{3} - \frac{64}{9}$$

$$= \frac{66 - 64}{9}$$

$$= \frac{2}{9}$$

$$\text{Cov}(X, Y) = E(XY] - E[X]E[Y] = \frac{11}{3} - \left(\frac{4}{3} \times \frac{8}{3}\right) = \frac{33 - 32}{9} = \frac{1}{9}$$

$$\rho(X,Y) = \frac{\text{cov}(X,Y)}{\sqrt{\text{var}(X)}\sqrt{\text{var}(Y)}} = \frac{1/9}{\sqrt{2/9}\sqrt{2/9}}$$

$$= \frac{1/9}{2/9}$$

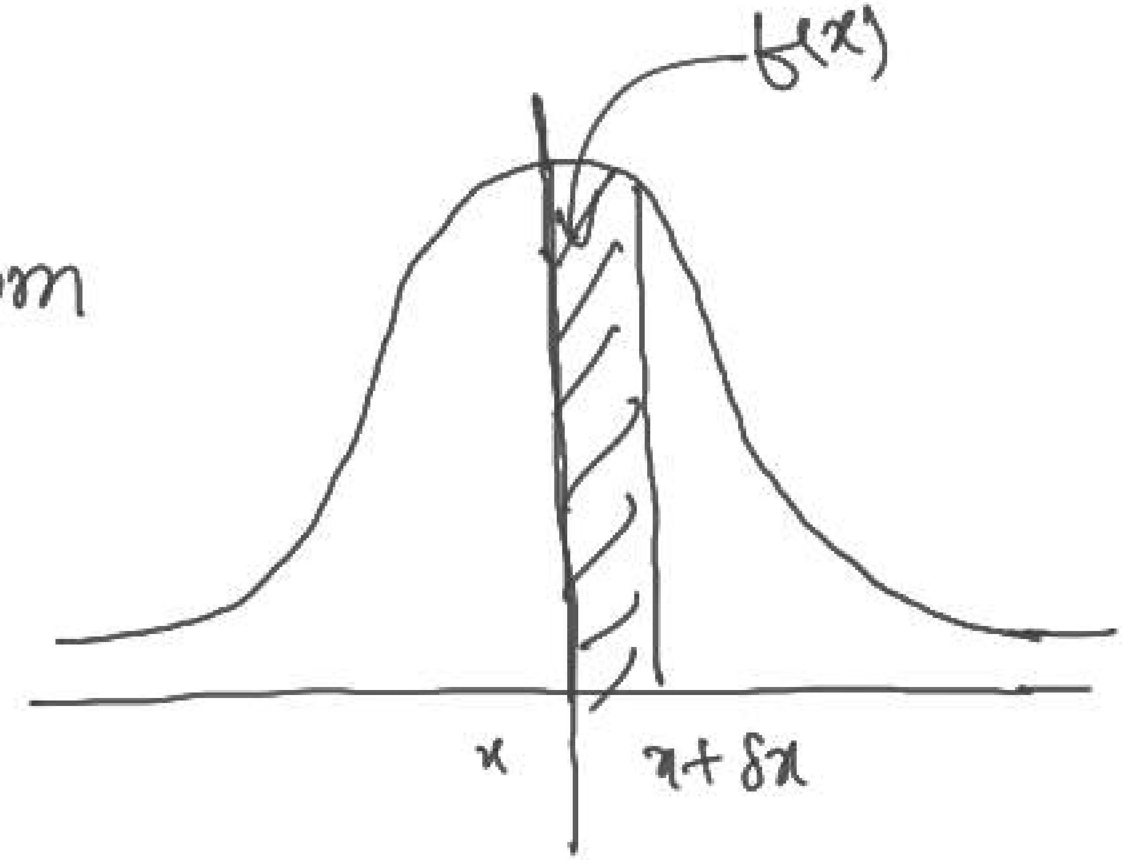
$$= \frac{1}{2}$$

Q. X and Y are two random variable having joint density function $= \frac{1}{27}(2x+y)$, where x and y can assume only integer values 0, 1 and 2. Find conditional distribution of Y

for $X = x$.

CONTINUOUS RANDOM VARIABLE

Probability Density Function



The Probability density function of random variable X is defined as

$$f_X(x) = P\left(x \le X \le x+\delta x\right)/\delta x$$

for small interval $[x, x+\delta x]$ of length dx around the point x.

$$\boxed{P(a \le X \le b) = \int_a^b f(x)\,dx}$$

Properties

1. $\int_{-\infty}^{\infty} f(x)\,dx = 1$ i.e. $P(-\infty < X < \infty) = 1$.

2. $f(x) \geq 0$, $-\infty < x < \infty$

$f_X(x)$ or $f(x)$

↳ p.d.f.

probability density f^n

Cumulative Distribution (Distribution Function)

$X \leftarrow$ R.V.

c.d.f. (cumulative distribution or Distribution F^n) is denoted by $F(x)$ and is given by

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$$

Expectation

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$E[X^n] = \int_{-\infty}^{\infty} x^n f(x)\,dx$$

Properties of $E[X]$ is same as discussed earlier.

$$\operatorname{var}(X) = E\{X - \bar{X}\}^2 = E\{X^2\} - [E(X)]^2$$

$$S.D.(x) = \sigma = \sqrt{\operatorname{var} X} = +\sqrt{E[X^2] - \{E(X)\}^2}$$

Q. A continuous random variable X has probability density f^n defined by

$$f(x) = \begin{cases} \frac{1}{16}(3+x)^2, & -3 \le x < -1 \\ \frac{1}{16}(6-2x^2), & -1 \le x < 1 \\ \frac{1}{16}(3-x)^2, & 1 \le x \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

Verify that $f(x)$ is density f^n and also find the mean of the random variable X.

Soln: $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{-3} f(x)\,dx + \int_{-3}^{-1} f(x)\,dx + \int_{-1}^{1} f(x)\,dx + \int_{1}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx$

$$= \int_{-\infty}^{-3} 0.\,dx + \int_{-3}^{-1} \frac{1}{16}(3+x)^2\,dx + \int_{-1}^{1} \frac{1}{16}(6-2x^2)\,dx + \int_{1}^{3} \frac{1}{6}(3-x)^2\,dx + \int_{3}^{\infty} 0.\,dx$$

$$= \frac{1}{16}\left\{ \int_{-3}^{-1} (9+x^2+6x)\,dx + \left[6x - \frac{2x^3}{3}\right]_{-1}^{1} + \int_{1}^{3} (9+x^2-6x)\,dx \right\}$$

$$= \frac{1}{16}\left\{ \left[9x + \frac{x^3}{3} + \frac{6x^2}{2}\right]_{-3}^{-1} + \left[6(1+1) - \frac{2}{3}(1^3-(-1)^3)\right] + \left[9x + \frac{x^3}{3} - \frac{6x^2}{2}\right]_{1}^{3} \right\}$$

$$\int_{-3}^{-1} (3+x)^2\,dx$$

$$\left[\frac{(3+x)^3}{3}\right]_{-3}^{-1}$$

Short cut

$$= \frac{1}{16}\left[9(-1+3) + \frac{1}{3}(-1+27) + 3(1-9) + 12 - \frac{4}{3} + 9(3-1) + \frac{1}{3}(27-1) - 3(9-1) \right]$$

$$= \frac{1}{16}(16)$$

$$= 1$$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$\therefore$ $f(x)$ is a density f^n.

Mean of the random variable X is

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{-\infty}^{-3} x f(x)\, dx + \int_{-3}^{-1} x f(x)\, dx + \int_{-1}^{1} x f(x)\, dx + \int_{1}^{3} x f(x)\, dx + \int_{3}^{\infty} x \cdot f(x)\, dx$$

$$= \int_{-\infty}^{-3} x \cdot 0 \cdot dx + \int_{-3}^{-1} x \cdot \frac{1}{16}(3+x)^2\, dx + \int_{-1}^{1} x \cdot \frac{1}{16}(6-2x^2)\, dx + \int_{1}^{3} x \cdot \frac{1}{16}(3-x)^2\, dx$$

$$+ \int_{3}^{\infty} x \cdot 0 \cdot dx$$

$$= ?$$

Q. Show that the continuous random variable X having $f(x) = \begin{cases} \frac{1}{2}(x+1) & , -1<x<1 \\ 0 & , \text{elsewhere} \end{cases}$

represents density, find the mean and s.d. of X.

Soln: Given,

$$f(x) = \begin{cases} \frac{1}{2}(x+1) , & -1 < x < 1 \\ 0 , & \text{elsewhere} \end{cases}$$

Now,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{-1} f(x)\,dx + \int_{-1}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{-1} 0.\,dx + \int_{-1}^{1} \frac{1}{2}(x+1)\,dx + \int_{1}^{\infty} 0.\,dx$$

$$= 0 + \frac{1}{2}\int_{-1}^{1}(x+1)\,dx + 0$$

$$= \frac{1}{2}\left[\frac{x^2}{2} + x\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\left(\frac{1}{2} - \frac{1}{2}\right) + (1+1)\right]$$

$$= 1$$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1$$

Thus $f(x)$ represents density f^n.

Mean of the random variable is

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{-\infty}^{-1} x f(x)\, dx + \int_{-1}^{1} x f(x)\, dx + \int_{1}^{\infty} x f(x)\, dx$$

$$= \int_{-\infty}^{-1} x \cdot 0 \cdot dx + \int_{-1}^{1} x \cdot \frac{1}{2}(x+1)\, dx + \int_{1}^{\infty} x \cdot 0 \cdot dx$$

$$= 0 + \frac{1}{2}\int_{-1}^{1} (x^2 + x)\, dx + 0$$

$$= \frac{1}{2}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\left(\frac{1}{3}+\frac{1}{3}\right) + \left(\frac{1}{2}-\frac{1}{2}\right)\right]$$

$$= \frac{1}{3}$$

$\therefore$ Mean, $E[X] = \frac{1}{3}$ ⟵ (1)

Again,

Variance, $\mathrm{Var}(X) = E[X^2] - (E[X])^2$ ⟵ (4)

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_{-\infty}^{-1} x^2 \cdot f(x)\, dx + \int_{-1}^{1} x^2 \cdot f(x)\, dx + \int_{1}^{\infty} x^2 \cdot f(x)\, dx$$

$$= \int_{-\infty}^{-1} x^2 \cdot 0 \cdot dx + \int_{-1}^{1} x^2 \cdot \frac{1}{2}(x+1)\, dx + \int_{1}^{\infty} x^2 \cdot 0 \cdot dx$$

$$= 0 + \frac{1}{2}\int_{-1}^{1} (x^3 + x^2)\, dx + 0$$

$$= \frac{1}{2}\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\left(\frac{1}{4}-\frac{1}{4}\right)+\left(\frac{1}{3}+\frac{1}{3}\right)\right]$$

$$= \frac{1}{3}$$

$$\therefore \quad var(X) = E[X^2] - (E[X])^2$$

$$= \frac{1}{3} - \left(\frac{1}{3}\right)^2$$

$$= \frac{1}{3} - \frac{1}{9}$$

$$= \frac{3-1}{9}$$

$$\text{var}(X) = \frac{2}{9}$$

$$\therefore \quad S.D.(X) = +\sqrt{\text{var}(X)} = +\sqrt{2/9} = \frac{\sqrt{2}}{3}$$

Q. If the probability density f^n is given by

$$f(x) = \begin{cases} kx^3 & , 0 \le x \le 3 \\ 0 & , \text{elsewhere} \end{cases}$$

Find the value of 'k' and also the probability between $x = \frac{1}{2}$ and $x = \frac{3}{2}$

Soln: Given,

$$f(x) = \begin{cases} kx^3 & , \quad 0 \le x \le 3 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

If $f(x)$ represents a density f^n

then $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\Rightarrow \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx = 1$

$\Rightarrow \int_{-\infty}^{0} 0.\,dx + \int_{0}^{3} kx^3\,dx + \int_{3}^{\infty} 0.\,dx = 1$

$$\Rightarrow 0 + k\int_0^3 x^3\,dx + 0 = 1$$

$$\Rightarrow k\left[\frac{x^4}{4}\right]_0^3 = 1$$

$$\Rightarrow k\left[\frac{3^4}{4} - 0\right] = 1$$

$$\Rightarrow k\left(\frac{81}{4}\right) = 1$$

$$\Rightarrow k = \frac{4}{81}$$

$$\therefore \quad f(x) = \begin{cases} \frac{4}{81}x^3, & 0 \leq x \leq 3 \\ 0, & \text{elsewhere} \end{cases}$$

Now,

$$P\left(\frac{1}{2} \leq x \leq \frac{3}{2}\right) = \int_{1/2}^{3/2} f(x)\,dx$$

$$= \int_{1/2}^{3/2} \frac{4}{81}x^3\,dx$$

$$= \frac{4}{81}\int_{1/2}^{3/2} x^3\,dx$$

$$= \frac{4}{81}\left[\frac{x^4}{4}\right]_{1/2}^{3/2}$$

$$= \frac{1}{81}\left[\left(\frac{3}{2}\right)^4 - \left(\frac{1}{2}\right)^4\right]$$

$$= \frac{1}{81}\left[\frac{81-1}{16}\right]$$

$$= \frac{1}{81}\left[\frac{80}{16}\right]$$

$$= \frac{5}{81}$$

Q. Is the f^n defined by

$$f(x) = \begin{cases} 0, & x < 2 \\ \frac{3+2x}{18}, & 2 \le x \le 4 \\ 0, & x > 4 \end{cases}$$

a probability density f^n? Find the probability that a variate having $f(x)$ as density f^n will fall in the interval $2 \le X \le 3$.

Q. A continuous random variable has the pdf

$$f(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Find the probabilities that it will take on a value

(i) betn 1 & 3

(ii) greater than 0.5

Soln: Given,

$$f(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

(i) $P(1 < X < 3) = \int_1^3 f(x)\,dx$

$$= \int_1^3 2e^{-2x}\,dx$$

$$= 2\int_1^3 e^{-2x}\,dx$$

$$= 2\left[\frac{e^{-2x}}{-2}\right]_1^3$$

$$= -\left[e^{-6} - e^{-2}\right]$$

$$= e^{-2} - e^{-6}$$

$$= 0.1338$$

(ii) $P(X > 0.5) = \int_{0.5}^{\infty} f(x)\,dx$

$$= \int_{0.5}^{\infty} 2e^{-2x}\,dx$$

$$= 2\left[\frac{e^{-2x}}{-2}\right]_{0.5}^{\infty}$$

$$= -\left[e^{-2x}\right]_{0.5}^{\infty}$$

$$= -\left(0 - e^{-1}\right)$$

$$= e^{-1}$$

$$= 0.3687$$

Q. Let $F(x)$ be the distribution function of a random variable X given by

$$F(x) = \begin{cases} cx^3 & , \quad 0 \leq x \leq 3 \\ 1 & , \quad x > 3 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

If $P(X=3) = 0$ then determine

(i) c

(ii) mean

(iii) $P(X>1)$

Soln Given,

$$F(x) = \begin{cases} cx^3 , & 0 \le x \le 3 \\ 1 , & x > 3 \\ 0 , & \text{elsewhere} \end{cases}$$

Now,

$$\because f(x) = \frac{d}{dx} F(x)$$

$$= \begin{cases} 3cx^2 , & 0 \le x \le 3 \\ 0 , & x > 3 \\ 0 , & \text{elsewhere} \end{cases}$$

$$= \begin{cases} 3cx^2 , & 0 \le x \le 3 \\ 0 , & \text{elsewhere} \end{cases}$$

$\because$ $f(x)$ is a density f^n

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} 0.dx + \int_{0}^{3} 3cx^2\,dx + \int_{3}^{\infty} 0.dx = 1$$

$$\Rightarrow 0 + 3c\left[\frac{x^3}{3}\right]_0^3 + 0 = 1$$

$$\Rightarrow c\,(27-0) = 1$$

$$\therefore \quad c = \frac{1}{27}$$

Now,

$$f(x) = \begin{cases} 3 \cdot \frac{1}{27} \cdot x^2 & , 0 \le x \le 3 \\ 0 & , \text{elsewhere} \end{cases}$$

$$= \begin{cases} x^2/9 & , 0 \le x \le 3 \\ 0 & , \text{elsewhere} \end{cases}$$

(ii) Mean, $E[x] = \int_{-\infty}^{\infty} x f(x)\,dx$

$$= \int_{-\infty}^{0} x f(x)\,dx + \int_{0}^{3} x f(x)\,dx + \int_{3}^{\infty} x f(x)\,dx$$

$$= \int_{-\infty}^{0} x \cdot 0 \cdot dx + \int_{0}^{3} x \cdot \frac{x^2}{9}\, dx + \int_{3}^{\infty} x \cdot 0 \cdot dx$$

$$= 0 + \frac{1}{9} \int_{0}^{3} x^3\, dx + 0$$

$$= \frac{1}{9} \left[\frac{x^4}{4} \right]_0^3$$

$$= \frac{1}{9} \left[\frac{81}{4} - 0 \right]$$

$$= \frac{9}{4}$$

Calculate $P(X > 1)$

Moment Generating Function (m.g.f.)

$E(e^{tX})$

$$M_X(t) = \int e^{tx} f(x)\, dx$$

Q. Find the m.g.f. of the random variable X having the probability density function

$$f(x) = \begin{cases} x & , \quad 0 \le x < 1 \\ 2-x & , \quad 1 \le x < 2 \\ 0 & , \quad \text{otherwise} \end{cases}$$

Find mean and variance of X using m.g.f.

Sol:

$$M_X(t) = E\left[e^{tX}\right]$$

$$= \int_0^1 e^{tx}\, x\, dx + \int_1^2 e^{tx}(2-x)\,dx$$

$$= \left\{ \left[x \int e^{tx}\,dx \right]_0^1 - \int_0^1 \left(\frac{d}{dx} x \int e^{tx}\,dx \right) dx \right\} +$$

$$\left\{ \left[(2-x) \int e^{tx}\,dx \right]_1^2 - \int_1^2 \left(\frac{d}{dx}(2-x) \int e^{tx}\,dx \right) dx \right\}$$

$$= \left[x\left(\frac{e^{tx}}{t}\right) \right]_0^1 - \int_0^1 \frac{e^{tx}}{t}\,dx + \left[(2-x)\cdot\frac{e^{tx}}{t} \right]_1^2 + \int_1^2 \frac{e^{tx}}{t}\,dx$$

$$M_x(t) = \left(\frac{1 \cdot e^t}{t} - 0\right) - \left[\frac{e^{tx}}{t^2}\right]_0^1 + \left[0 - \frac{e^t}{t}\right] + \left[\frac{e^{tx}}{t^2}\right]_1^2$$

$$= \frac{e^t}{t} - \left(\frac{e^t}{t^2} - \frac{1}{t^2}\right) - \frac{e^t}{t} + \left(\frac{e^{2t}}{t^2} - \frac{e^t}{t^2}\right)$$

$$= -\frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{e^t}{t^2}$$

$$= \frac{e^{2t}}{t^2} - \frac{2e^t}{t^2} + \frac{1}{t^2} = \frac{1}{t^2}\left[e^{2t} - 2e^t + 1\right] \text{ --- } (1)$$

$$= \frac{(e^t)^2 - 2e^t + 1}{t^2}$$

$$M_x(t) = \frac{(e^t - 1)^2}{t^2} \qquad (1)$$

Expanding $M_x(t)$ using (1)

$$M_x(t) = \frac{1}{t^2}\left[\left(1+2t+\frac{(2t)^2}{2!}+\frac{(2t)^3}{3!}+\frac{(2t)^4}{4!}+\dots\right) - 2\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\dots\right)+1\right]$$

$$= \frac{1}{t^2}\left[t^2+t^3+\frac{7}{12}t^4+\dots\right]$$

$$M_x(t) = 1+t+\frac{7}{12}t^2+\dots$$

Mean $= \mu_1' =$ coefficient of t in $M_X(t) = 1$

$\mu_2' =$ coefficient of $\frac{t^2}{2!}$ in $M_X(t) = 2!\,\frac{7}{12} = \frac{7}{6}$

variance $(\mu_2) = \mu_2' - \mu_1'^2$

$= \frac{7}{6} - (1)^2$

$= \frac{1}{6}$

NORMAL DISTRIBUTION

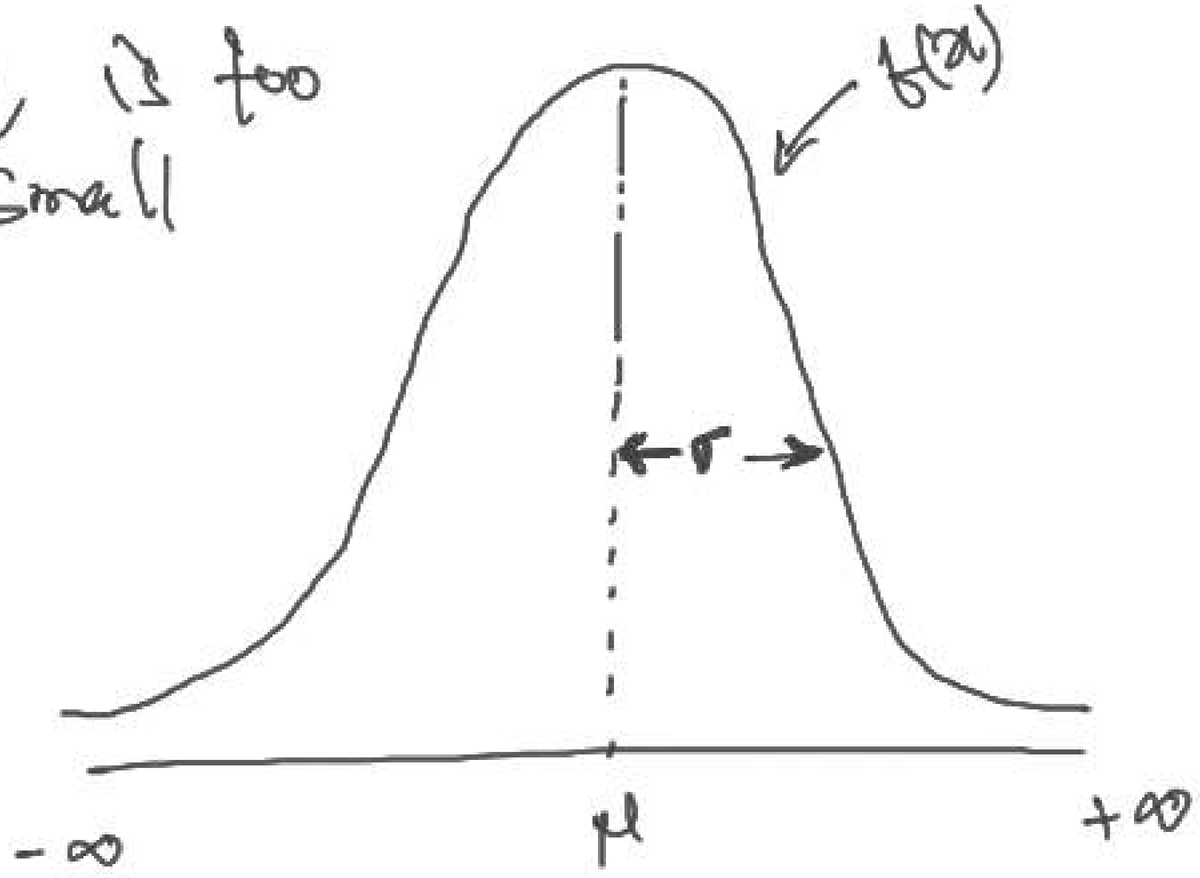
Limiting case of Binomial Distribution

$p \to q$ ← neither p nor q is too small

$n \to \infty$ ← infinitely large

Discovered by De Moivre in 1733

It is continuous R.V. distr.



$X \leftarrow$ continuous R.V.

Then X is said to have normal distribution if its p.d.f. is defined as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad , -\infty < x < \infty$$

$$\sigma > 0$$

Here $\mu, \sigma \leftarrow$ parameters of Normal Distribution

Binomial	Poison	Normal.
1. Discrete R.V.	Discrete R.V.	continuous R.V.
2. 2 parameters (p or q and n)	One parameter (λ)	Two parameters (μ, σ)
3. —	Limiting case of Binomial dist. $p \to 0, n \to \infty$	Limiting case binomial distn. $p \to q, n \to \infty$
4. $P(x) = {}^nC_x p^x q^{n-x}$	$P(x) = \frac{e^{-\lambda} \lambda^x}{x!}$	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ pdf

B.D	P.D.	N.D
5. mean = np	mean = λ	mean = μ
var = npq	var = λ	var = σ^2

values det. from formula given at (4)

Q. Prove that the mean and variance of the normal distribution

Mean = μ

Variance = σ^2

8. For normal distribution

Mean $= \mu$

Median $= \mu$

Mode $= \mu$

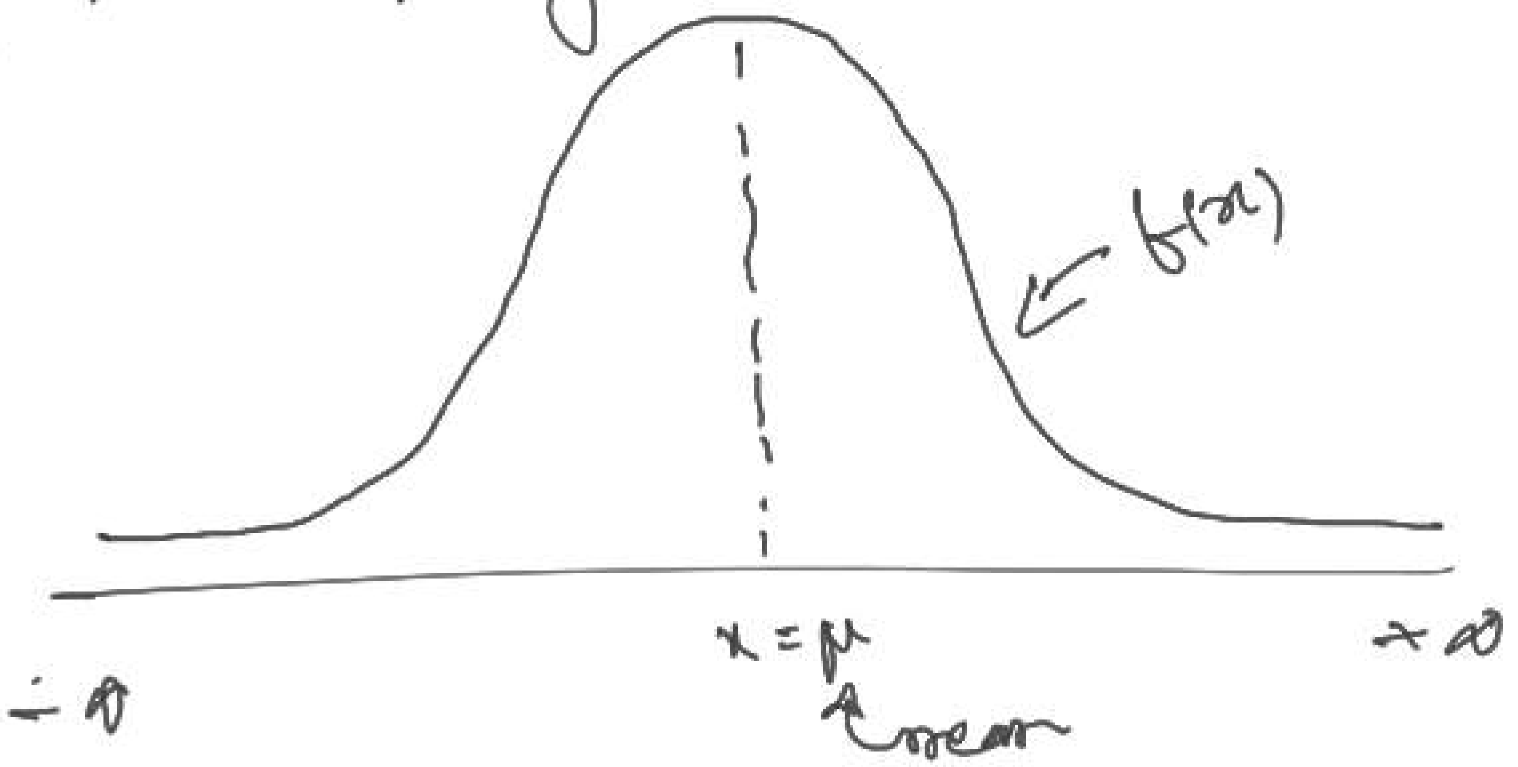
* In case of normal distribution, mean = median = mode.

Properties of Normal Distribution

① The normal probability curve with mean μ and standard deviation σ is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad , \quad -\infty < x < \infty$$

2. The curve is bell-shaped and symmetrical about the line line $x = \mu$



3. Mean, median and mode of the normal distribution coincides i.e. unimodal.

4. $f(x)$ decreases rapidly as x increases.

5. X-axis is an asymptote to the curve.

6. Maximum probability occurs at the point $x = \mu$

and $\max^m$ prob $= \dfrac{1}{\sigma\sqrt{2\pi}}$

7. $M.D.(\bar{x}) = \dfrac{4}{5}\sigma$ or $M.D.(\mu) = \dfrac{4}{5}\sigma$

8. The point of inflextion of curve are at $x = \mu \pm \sigma$

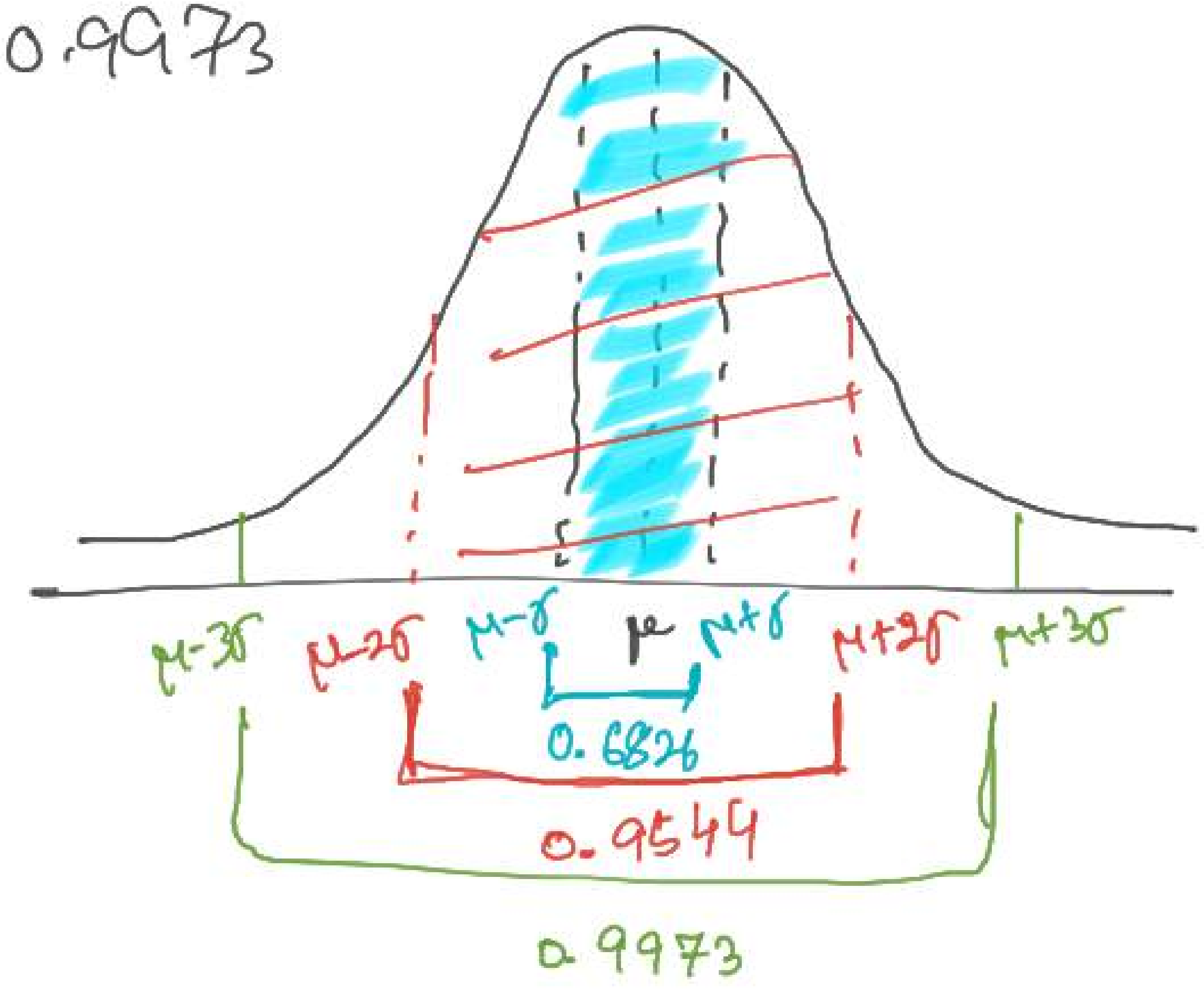
(i) Area of the normal curve between $(\mu - \sigma)$ and $(\mu + \sigma)$ is

0.6826 i.e. $P(\mu - \sigma < X < \mu + \sigma) = 0.6826$

(ii) $P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9544$

(iii) $P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973$

Area betn — & —



Q. If $\mu = 50$ and $\sigma = 10$ then find

(i) $P(50 \le x \le 80)$

(ii) $P(60 \le x \le 70)$

(iii) $P(30 \le x \le 40)$

(iv) $P(40 \le x \le 60)$

Normal variate

$$Z = \frac{x - \mu}{\sigma}$$

$$\boxed{z = \frac{x - \mu}{\sigma}}$$

Sol: Standard Normal variate, $Z = \dfrac{x - \mu}{\sigma}$

$$\Rightarrow Z = \frac{x - 50}{10} \quad \text{— (A)} \quad (\because \mu = 50, \ \sigma = 10)$$

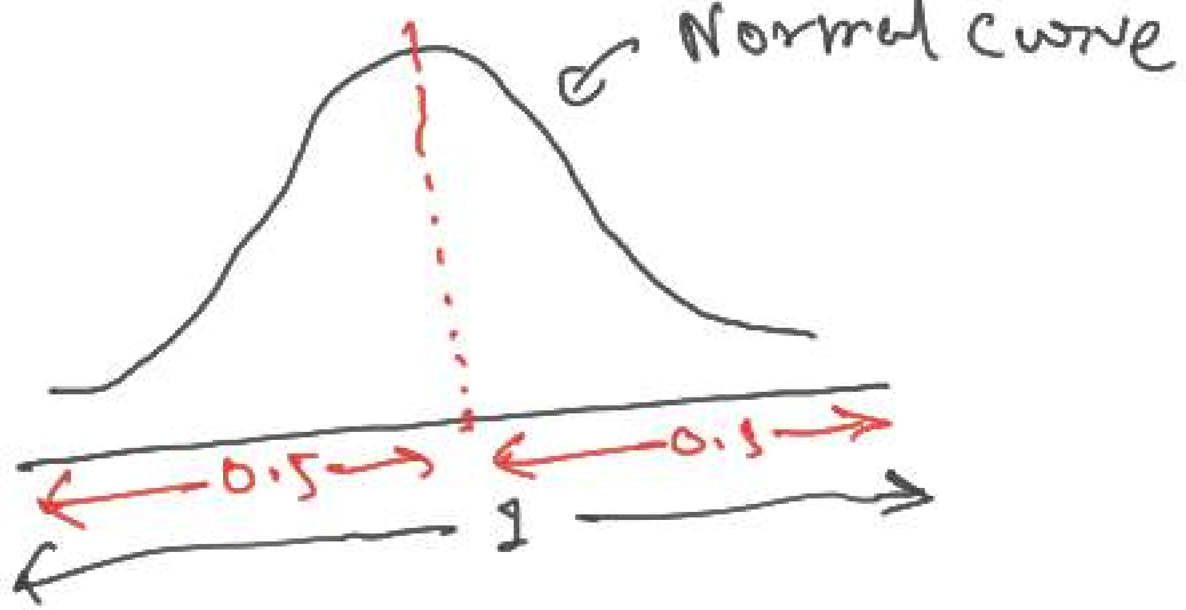
(1) $P(50 \le X \le 80)$

$$= P\left(\frac{50-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{80-\mu}{\sigma}\right)$$

$$= P\left(\frac{50-50}{10} \le Z \le \frac{80-50}{10}\right)$$

$$= P(0 \le Z \le 3)$$

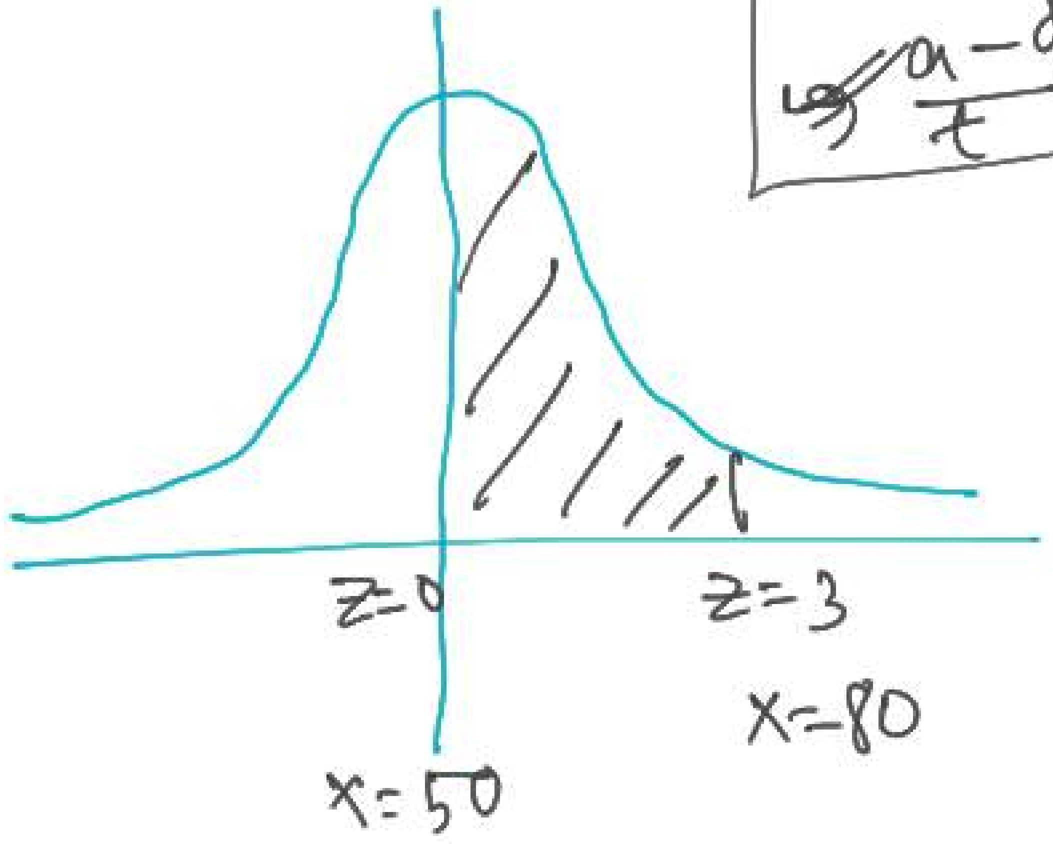
$$= 0.4987$$



$$a \le x \le b$$

$$\Rightarrow a-d \le x-d \le b-d$$

$$\Rightarrow \frac{a-d}{t} \le \frac{x-d}{t} \le \frac{b-d}{t}$$



(ii) $P(60 \le X \le 70) = P\left(\frac{60-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{70-\mu}{\sigma}\right)$

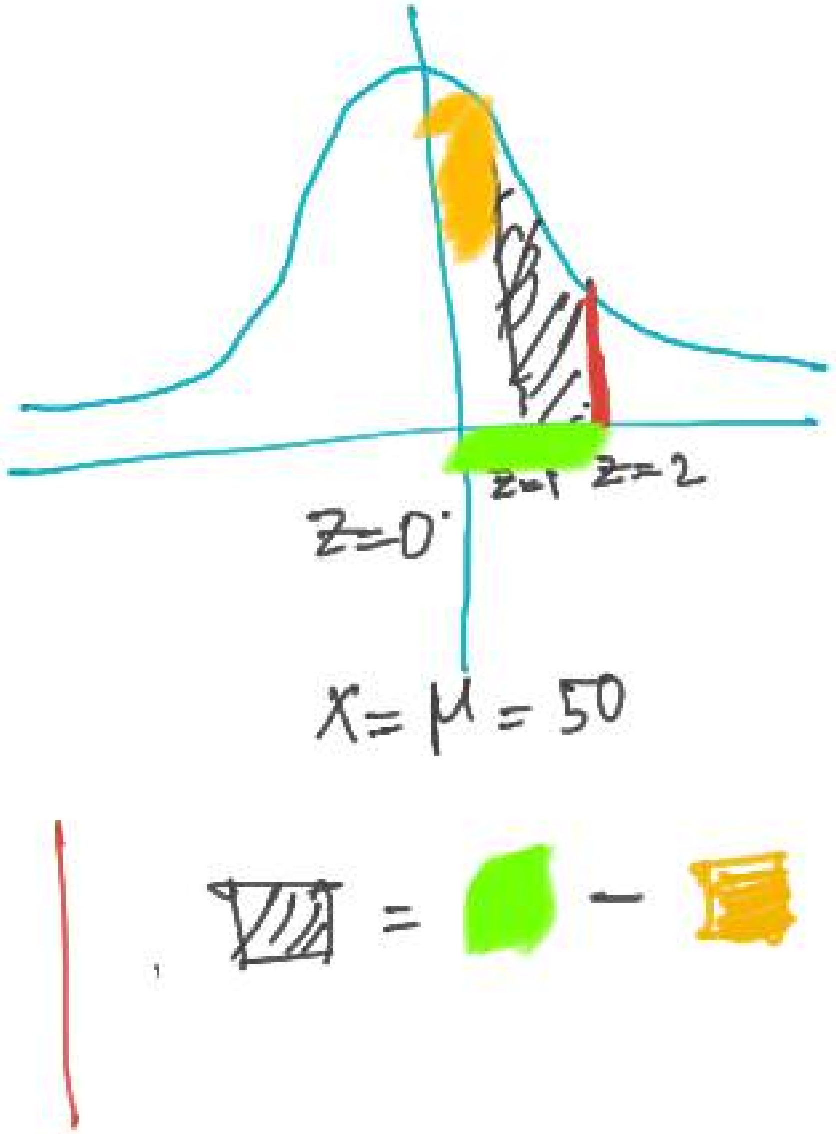
$= P\left(\frac{60-50}{10} \le Z \le \frac{70-50}{10}\right)$

$= P(1 \le Z \le 2)$

$= P(0 \le Z \le 2) - P(0 \le Z \le 1)$

$= 0.4772 - 0.3413$

$= 0.1359$

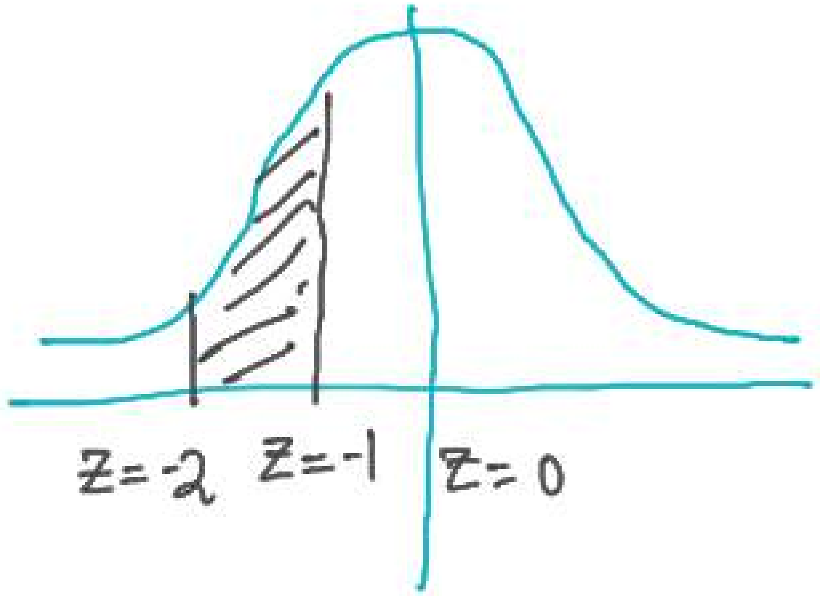


(iii) $P(30 \le X \le 40) = P\left(\frac{30-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{40-\mu}{\sigma}\right)$

$= P\left(\frac{30-50}{10} \le Z \le \frac{40-50}{10}\right)$

$= P(-2 \le Z \le -1)$

$= P(1 \le Z \le 2)$ ($\because$ The curve is symmetric)



$= 0.1359$

Due to Symmetricity

$P(-2 \le Z \le -1)$

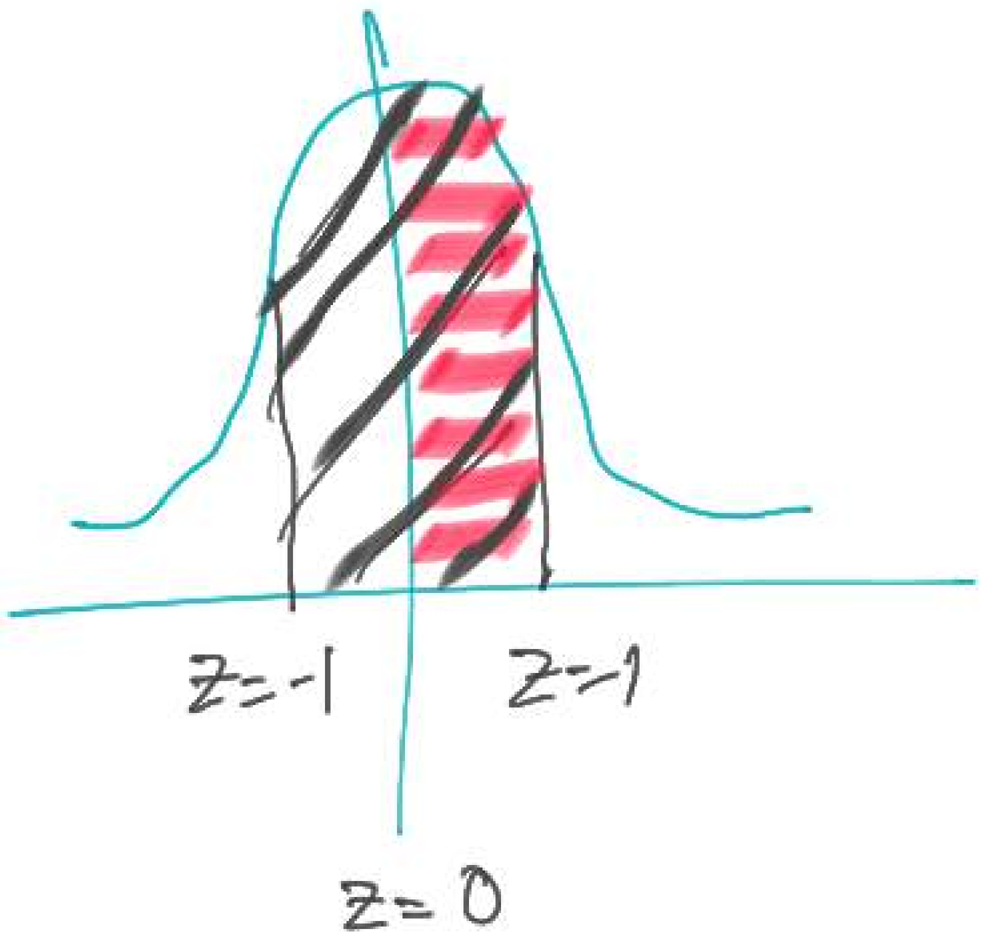
$= P(1 \le Z \le 2)$

(iv) $P(40 \leq X \leq 60) = P\left(\frac{40-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{60-\mu}{\sigma}\right)$

$$= P\left(\frac{40-50}{10} \leq Z \leq \frac{60-50}{10}\right)$$

$$= P(-1 \leq Z \leq 1)$$

$$= 2\,P(0 \leq Z \leq 1) \quad (\because \text{The curve is symmetric})$$



$$= 2(0.3413)$$

$$= 0.6826$$

Q. A sample of 100 dry battery cells tested to find the length of life produced the following results :

$$\mu = 12 \text{ hrs} \quad , \quad \sigma = 3 \text{ hrs}$$

Assuming the data to be normally distributed, what percentage of battery cells are expected to have life

(a) more than 15 hrs

(b) less than 6 hrs

(c) betn 10 and 14 hrs.

Soln :   let X = length of life of the battery cells.

Standard Normal Variate, $$Z = \frac{X - \mu}{\sigma} \quad \text{---} (1)$$

Here, $\mu = 12$

$\sigma = 3$

$$\therefore \quad Z = \frac{X - 12}{3} \quad \text{---} (11)$$

(a) Probability of battery cells having life more than 15 hrs

$= P(X > 15)$

$$= P\left(\frac{X-\mu}{\sigma} > \frac{15-\mu}{\sigma}\right)$$

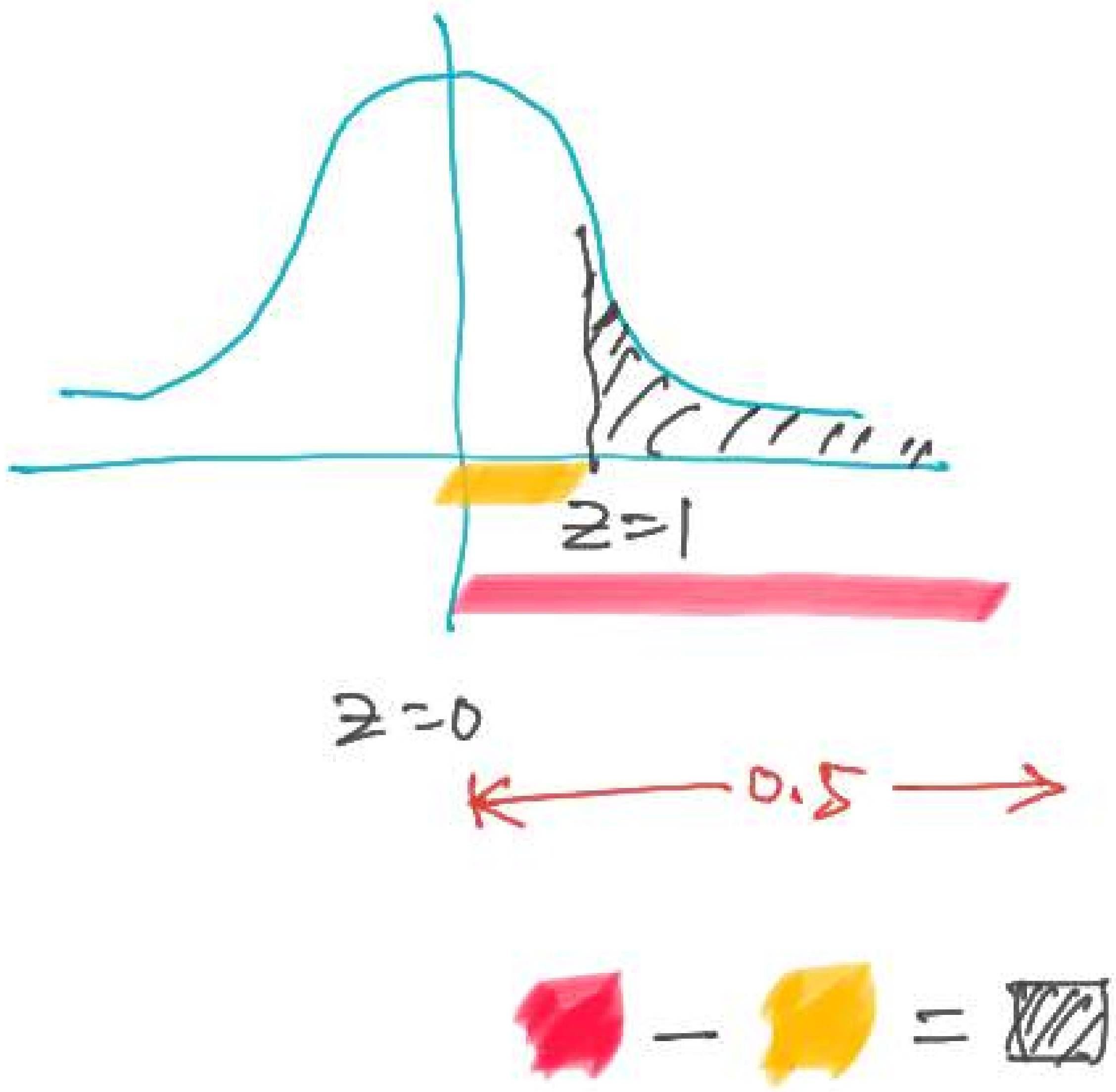
$$= P\left(Z > \frac{15-12}{3}\right)$$

$$= P(Z > 1)$$

$$= 0.5 - P(0 \leq Z \leq 1)$$

$$= 0.5 - 0.3413$$

$$= 0.1587$$



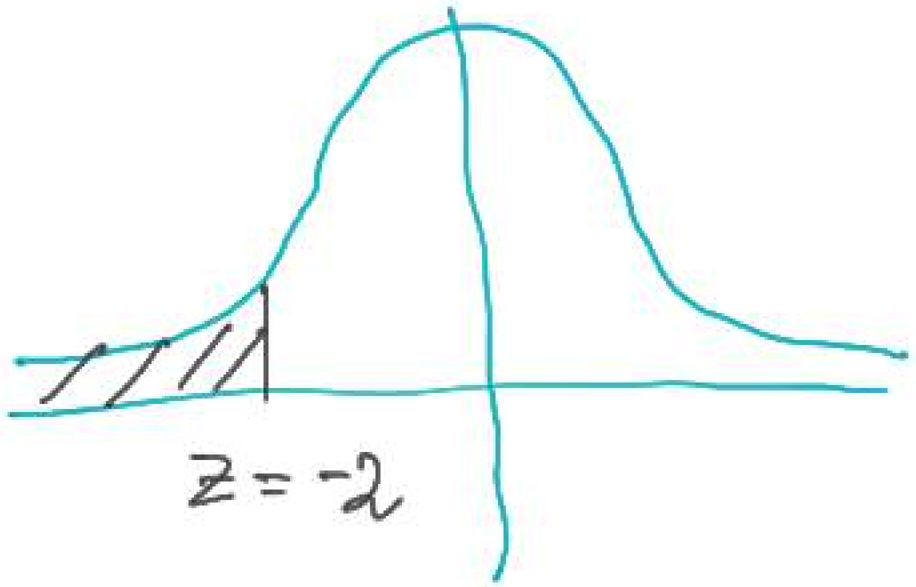
$\therefore$ Percentage of battery cells having life more than 15 hours $= 0.1587 \times 100 = 15.87\%$

(b) Probability of battery cells having life less than

6 hrs $= P(X < 6)$

$= P\left(\frac{X-\mu}{\sigma} < \frac{6-\mu}{\sigma}\right)$

$= P\left(Z < \frac{6-12}{3}\right)$



$= P(Z < -2)$

$= P(Z > 2)$ (∵ The curve is symmetric)

$= 0.5 - P(Z < 2)$ $= 0.5 - 0.4772$

$= 0.0228$

Percentage of battery cells having life less than 6 hrs

$= 0.0228 \times 100$

$= 2.28\ \%$

(c) Prob. of battery cells having life in betn 10 to 14 hrs

$= P(10 < X < 14)$

$= P\left(\frac{10-12}{3} < Z < \frac{14-12}{3}\right)$

$= P(-0.67 < Z < 0.67)$

$= 2P(0 < Z < 0.67)$

$= 2(0.2486)$

$= 0.4972$

$\therefore$ Percentage of battery cells having life in between 10 hrs and 14 hrs $= 0.4972 \times 100$

$= 49.72\%$

Q. The average height of soldiers of a country is given as 68.22 inches with variance 10.8 sq inch. How many soldiers out of 1000 would you expect to be over 72 inches tall? Given that the area under the normal curve between $z=0$ to $z=0.35$ is 0.1368 and between $z=0$ to $z=1.15$ is 0.3746.

Soln: Given,

$$\mu = 68.22$$

$$\sigma^2 = 10.8 \Rightarrow \sigma = \sqrt{10.8}$$

Let X = height of soldier

$$P(X > 72) = P\left(\frac{X-\mu}{\sigma} > \frac{72-\mu}{\sigma}\right)$$

$$= P\left(Z > \frac{72-68.22}{\sqrt{10.8}}\right)$$

$$= P(Z > 1.15)$$

$$= 0.5 - P(Z < 1.15)$$

$$= 0.5 - 0.3746$$

$$= 0.1254$$

$\therefore$ No. of soldiers out of 1000 whose height is over 72 inches

$= 1000 \times 0.1254$

$= 125$

Q. Students of a class were given a mathematics aptitude test. These marks were found to be normally distributed with mean 60 and standard deviation 5. What percent of students scored :

(i) more than 60 marks

(ii) less than 56 marks

(iii) between 45 and 65 marks.

Q. In a sample of 1000 cases, the mean of a certain test is 14 and standard deviation is 2.5. Assuming the distribution to be normal, find

(i) how many students score between 12 and 15?

(ii) how many score above 18?

(iii) How many below 8?

(iv) How many score 16?

Sol^n: Here, $\mu = 14$

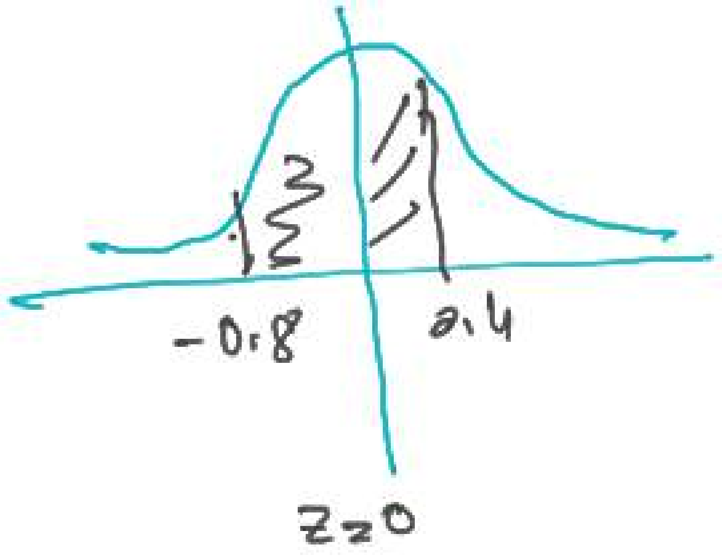
$\sigma = 2.5$

let X = no. of students getting a score

(i) $P(12 < X < 15) = P\left(\frac{12-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{15-\mu}{\sigma}\right)$

$= P\left(\frac{12-14}{2.5} < Z < \frac{15-14}{2.5}\right)$

$$= P(-0.8 < Z < 0.4)$$



$$= P(-0.8 < Z < 0) + P(0 < Z < 0.4)$$

$$= P(0 < Z < 0.8) + P(0 < Z < 0.4)$$

$$= 0.2881 + 0.1554$$

$$= 0.4435$$

$\therefore$ No. of students scoring in betn 12 and 15

$$= 1000 \times 0.4435 = 443.5 = 444$$

(a) $P(X>18) = P\left(\frac{X-\mu}{\sigma} > \frac{18-\mu}{\sigma}\right)$

$$= P\left(Z > \frac{18-14}{2.5}\right)$$

$$= P(Z > 1.6)$$

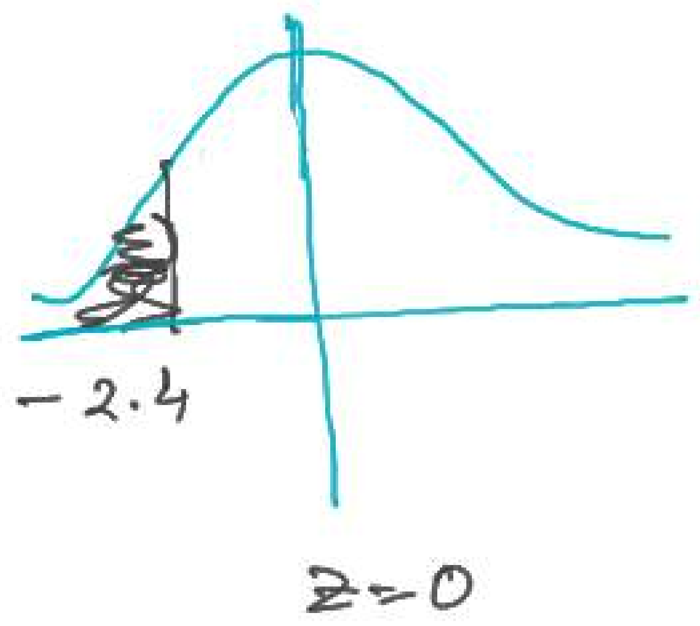
$$= 0.5 - P(Z < 1.6)$$

$$= 0.5 - 0.4452$$

$$= 0.0548$$

$\therefore$ No. of students getting a score above 18 $= 1000 \times 0.0548 = 54.8$
$= 55$

(iii) $P(X<8) = P\left(\frac{X-\mu}{\sigma} < \frac{8-\mu}{\sigma}\right)$



$= P\left(Z < \frac{8-14}{2.5}\right)$

$= P(Z < -2.4)$

$= P(Z > 2.4)$

$= 0.5 - P(Z < 2.4)$

$= 0.5 - 0.4918$

$= 0.0082$

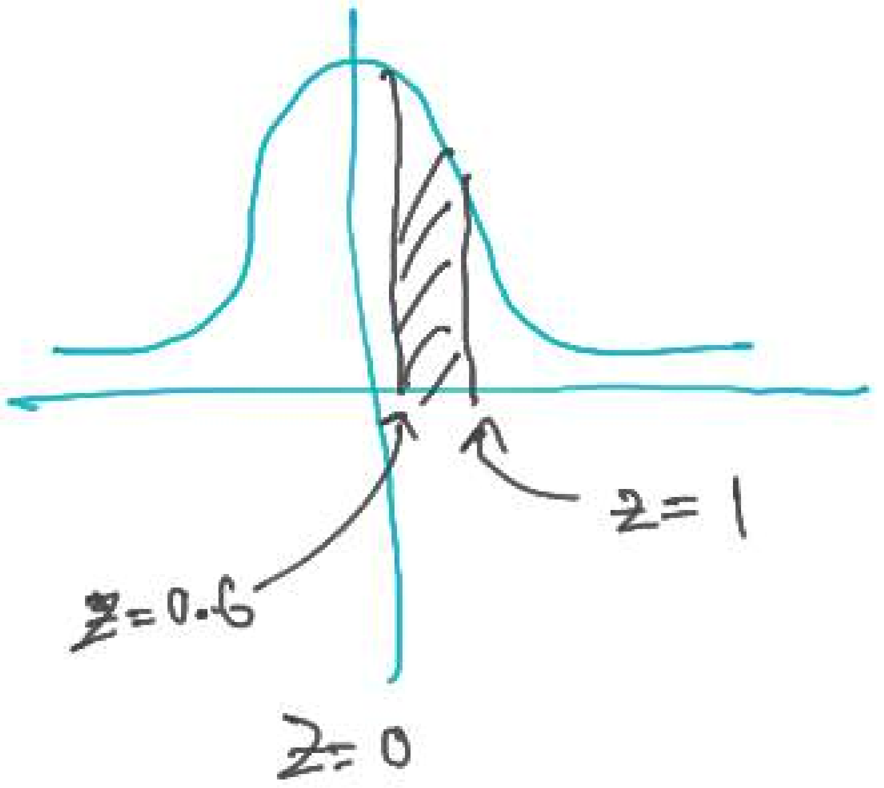
$\therefore$ No. of students scoring below $8 = 1000 \times 0.0082 = 8.2$
$= 8$

12

$$P(X=16) = P(15.5 < X < 16.5)$$

$$= P\left(\frac{15.5-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{16.5-\mu}{\sigma}\right)$$

$$= P\left(\frac{15.5-14}{2.5} < Z < \frac{16.5-14}{2.5}\right)$$



$$= P(0.6 < Z < 1)$$

$$= P(0 < Z < 1) - P(0 < Z < 0.6)$$

$$= 0.3413 - 0.2258$$

$$= 0.1155$$

$\therefore$ No. of students getting a score 16

$= 1000 \times 0.1155$

$= 115.5$

$= 116$

9. The distribution of a random variable is given by

$$f(x) = ce^{-\frac{1}{50}(9x^2 - 30x)} \quad ; \quad -\infty < x < \infty$$

Find the constant c, the mean and the variance of the random variable. Find also the upper 5% value of the R.V.

Soln: Given,

$$f(x) = ce^{-\frac{1}{50}(9x^2 - 30x)} \quad ; \quad -\infty < x < \infty$$

We know that

$$P(-\infty < X < \infty) = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow c \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = 1$$

$\therefore \mu = \frac{5}{3}$, $2\sigma^2 = \frac{50}{9} \Rightarrow \sigma^2 = \frac{25}{9}$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$-\frac{1}{50}(9x^2 - 30x)$$

$$= -\frac{9}{50}\left(x^2 - \frac{30}{9}x\right)$$

$$= -\frac{9}{50}\left(x^2 - \frac{10}{3}x\right)$$

$$= -\frac{9}{50}\left[x^2 - 2.x.\frac{5}{3} + \left(\frac{5}{3}\right)^2 - \left(\frac{5}{3}\right)^2\right]$$

$$-\frac{9}{50}\left(\left(x - \frac{5}{3}\right)^2 - \left(\frac{5}{3}\right)^2\right)$$

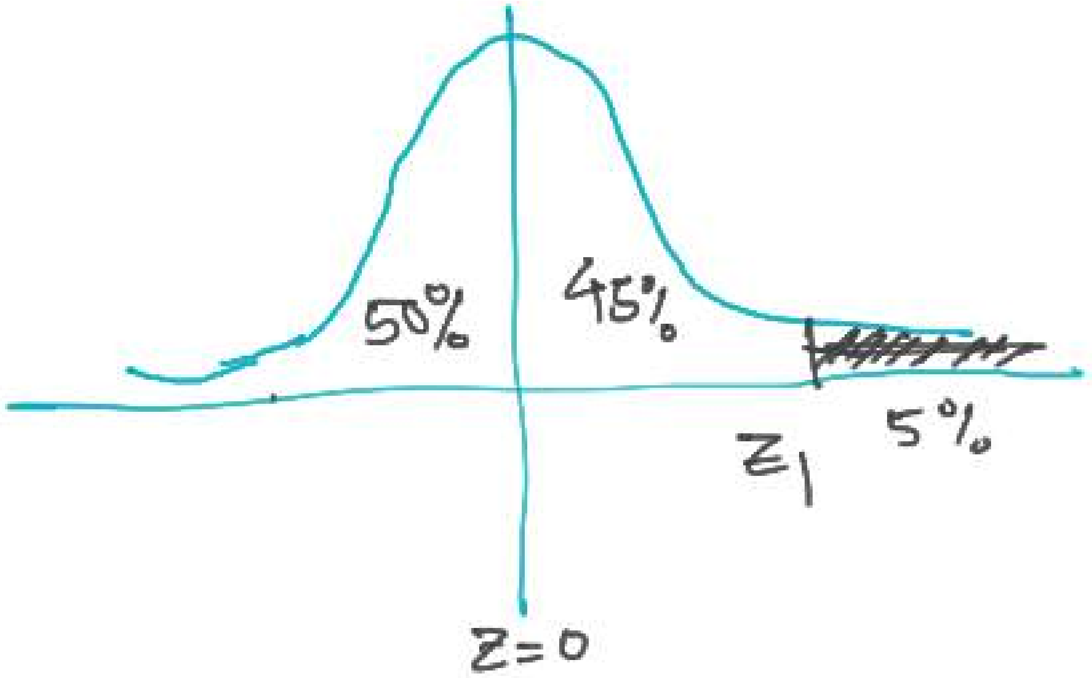
$\therefore \; \sigma = 5/3$

$\mu = 5/3$

$$C = \frac{1}{\sigma\sqrt{2\pi}}$$

$$\Rightarrow C = \frac{1}{(5/3)\sqrt{2\pi}} = \frac{3}{5\sqrt{2\pi}} = 0.239 = 0.24$$

let, $z = z_1$ be the co-ordinate of z at 45% mark



5% area

$$P(0 < z < z_1) = 0.5 - 0.05$$

$$= 0.45$$

Value of z corresponding to this area (From the area table)

$$z_2 = 1.66$$

Now,

standard normal variate, $z = \dfrac{x - M}{\sigma}$

$$\therefore \quad z_2 = \frac{x - M}{\sigma}$$

$$\Rightarrow 1.66 = \frac{x - 5/3}{5/3}$$

$$\Rightarrow x = 1.66 \times \frac{5}{3} + \frac{5}{3}$$

$$= 4.44$$