Q. If X and Y are two random variables having joint density function

$$f(x,y) = \begin{cases} \frac{1}{8}(6-x-y) , & 0<x<2 , \ 2<y<4 \\ 0 , & \text{otherwise} \end{cases}$$

Find (i) $P(X<1 \cap Y<3)$

(ii) $P(X+Y<3)$

(iii) $P(X<1 \mid Y<3)$ or $P(X<1 / Y<3)$

(iv) marginal & conditional distributions.

Soln: Given,

$$f(x,y) = \begin{cases} \frac{1}{8}(6-x-y) & , \quad 0<x<2 \ , \ 2<y<4 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(i) $$P(X<1 \cap Y<3) = \int_{-\infty}^{1}\int_{-\infty}^{3} f(x,y)\,dy\,dx$$

$$= \int_{0}^{1}\int_{2}^{3} \frac{1}{8}(6-x-y)\,dy\,dx$$

$$= \frac{1}{8}\int_{0}^{1}\left\{\int_{2}^{3}(6-x-y)\,dy\right\}dx$$

$$= \frac{1}{8}\int_0^1 \left\{ \left[6y - xy - \frac{y^2}{2} \right]_2^3 \right\} dx$$

$$= \frac{1}{8}\int_0^1 \left[6(3-2) - x(3-2) - \frac{1}{2}(9-4) \right] dx$$

$$= \frac{1}{8}\int_0^1 \left(6 - x - \frac{5}{2} \right) dx$$

$$= \frac{1}{8}\left[6x - \frac{x^2}{2} - \frac{5}{2}x \right]_0^1$$

$$= \frac{1}{8}\left[6(1-0) - \frac{1}{2}(1^2 - 0^2) - \frac{5}{2}(1-0) \right]$$

$$= \frac{1}{8}\left[6 - \frac{1}{2} - \frac{5}{2}\right]$$

$$= \frac{1}{8}\left[\frac{12-6}{2}\right]$$

$$= \frac{6}{16}$$

$$= \frac{3}{8}$$

(ii) $P(X+Y<3) = \int_0^1 \int_2^{3-x} \frac{1}{8}(6-x-y)\,dy\,dx$

$$= \frac{1}{8}\int_0^1 \left\{\int_2^{3-x} (6-x-y)\,dy\right\} dx$$

$$= \frac{1}{8}\int_0^1 \left[6y - xy - \frac{y^2}{2}\right]_2^{3-x} dx$$

$$= \frac{1}{8}\int_0^1 \left[6(3-x-2) - x(3-x-2) - \frac{1}{2}\left\{(3-x)^2-4\right\}\right] dx$$

$$= \frac{1}{8}\int_0^1 \left[6-6x - x + x^2 + \frac{1}{2}\left\{(1-x)(5-x)\right\}\right] dx$$

$$= \frac{1}{8}\int_0^1 \left[6 - 7x + x^2 + \frac{1}{2}\left\{5 - 6x + x^2\right\}\right] dx$$

$$= \frac{1}{8}\left[6x - \frac{7x^2}{2} + \frac{x^3}{3} + \frac{5}{2}x - 3.\frac{x^2}{2} + \frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{8}\left[6(1-0) - \frac{7}{2}(1-0) + \frac{1}{3}(1-0) + \frac{5}{2}(1-0) - \frac{3}{2}(1-0) + \frac{1}{3}(1-0)\right]$$

$$= \frac{1}{8}\left[6 - \frac{7}{2} + \frac{1}{3} + \frac{5}{2} - \frac{3}{2} + \frac{1}{3}\right]$$

$$= \frac{5}{24}$$

(iv) $P[X < 1 \mid Y < 3]$

$$P(X<1 \mid Y<3] = \frac{P(X<1 \cap Y<3)}{P(Y<3)} \rightarrow (1)$$

Now,

$$P(Y<3) = \int_0^2 \int_2^3 \frac{1}{8}(6-x-y)\,dy\,dx$$

$$= ??$$

$\left(P(Y<3) = \frac{5}{8}\right)$

(iv) Marginal distribution of X is given by

$$f_X(x) = \int_2^4 f(x,y)\, dy$$

$$= \int_2^4 \frac{1}{8}(6-x-y)\, dy$$

$$= \frac{1}{8}\left[6y - xy - \frac{y^2}{2}\right]_2^4$$

$$= ??$$

Marginal distribution of Y is

$$f_Y(y) = \int_0^2 f(x,y)\,dx$$

$$= \int_0^2 \frac{1}{8}(6-x-y)\,dx$$

$$= \frac{1}{8}\left[6x - \frac{x^2}{2} - xy\right]_0^2$$

$$= \frac{1}{8}\left[6(2-0) - \frac{1}{2}(4-0) - y(2-0)\right]$$

$$= \frac{1}{8}\left[12-2-2y\right] = \frac{1}{8}(10-2y) = \frac{1}{4}(5-y)$$

$\therefore$ Marginal distribution of $Y = \begin{cases} \frac{1}{4}(5-y) & , \quad 2<y<4 \\ 0 & , \text{ otherwise} \end{cases}$

The conditional distributions of X and Y

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{1}{8}(6-x-y)}{\frac{1}{4}(5-y)} = ? \; ; \quad \begin{matrix} 0<x<2 \\ 2<y<4 \end{matrix}$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{1}{8}(6-x-y)}{?} \; ; \quad \begin{matrix} 0<x<2 \\ 2<y<4 \end{matrix}$$