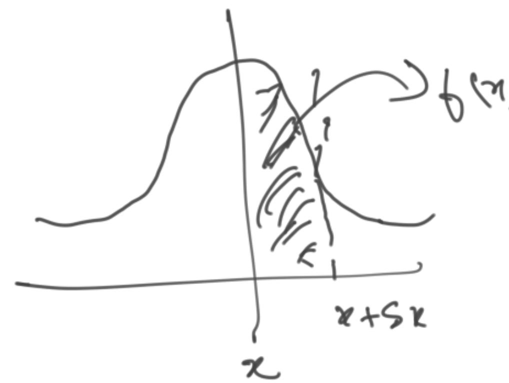
Continuous Probability Distribution

Probability Density Function (p.d.f.)



The probability density f^n of a R.V. X is defined as

$$f_X(x) = P(x \le X \le x + \delta x) / \delta x$$

for small interval $(x, x+\delta x)$

$$\boxed{P(a \le X \le b) = \int_a^b f(x)\,dx}$$

$\because$ Total prob. $= 1$

$\therefore \quad \displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$ ⟸ imp.

Properties

① $f(x) \geq 0$, $-\infty < x < \infty$

② $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Cumulative Probability Distribution (Dist f^n)

If X is a random variable, then $P(X \leq x)$

is called cummulative distribution f^n (c.d.f.)

and is denoted by $F(x)$

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$$

· Expectation of random variable

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

Properties

Same as that of Discrete RV

① $E[a] = a$

② $E[aX] = a\,E[X]$

③ $E[X - \bar{X}] = 0$

$$E[X+Y] = E[X] + E[Y]$$

$$E[XY] = E[X]\,E[Y]$$; X & Y are independent events

$$E[Y] = E[aX+b]$$

$$= a\,E[X] + b$$ where $y = ax+b$

Variance

$$\sigma^2 = Var(X) = E[X - \bar{X}]^2 = E[X^2] - (E[X])^2$$

Standard Deviation

$$S.D.(x) = \sigma = \sqrt{V(X)}$$

Q. A continuous R.V. X has a p.d.f. defined by

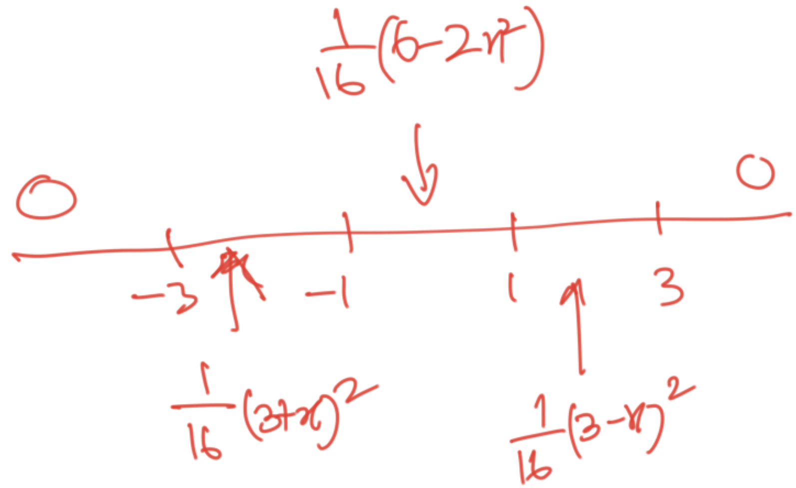
$$f(x) = \begin{cases} \frac{1}{16}(3+x)^2 & , \quad -3 \le x < -1 \\ \frac{1}{16}(6-2x^2) & , \quad -1 \le x < 1 \\ \frac{1}{16}(3-x)^2 & , \quad 1 \le x \le 3 \\ 0 & , \quad \text{otherwise} \end{cases}$$

Verify if $f(x)$ is a density fn and find mean if it is so.

If $f(x)$ is density f^n then $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Now,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-3}^{3} f(x)\,dx$$



$$= \int_{-3}^{-1} f(x)\,dx + \int_{-1}^{1} f(x)\,dx + \int_{1}^{3} f(x)\,dx$$

$$= \int_{-3}^{-1} \frac{1}{16}(3+x)^2\,dx + \int_{-1}^{1} \frac{1}{16}(6-2x^2)\,dx + \int_{1}^{3} \frac{1}{16}(3-x)^2\,dx$$

$$= \frac{1}{16}\left[\left[\frac{(3+x)^3}{3}\right]_{-3}^{-1} + \left[6x - \frac{2x^3}{3}\right]_{-1}^{1} - \left[\frac{(3-x)^3}{3}\right]_{1}^{3} \right]$$

$$= \frac{1}{16}\left\{ \left(\frac{8}{3} - 0\right) + \left[\left(6 - \frac{2}{3}\right) - \left(-6 + \frac{2}{3}\right)\right] - \left(0 - \frac{8}{3}\right) \right\}$$

$$= 1$$

$\therefore$ $f(x)$ is density f^n

Again,

$$\text{Mean} = E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{-3}^{3} x f(x)\, dx$$

$$= \int_{-3}^{-1} x f(x)\, dx + \int_{-1}^{1} x f(x)\, dx + \int_{1}^{3} x f(x)\, dx$$

A continuous R.V. X has the pdf

$$f(x) = \begin{cases} \frac{1}{2}(x+1) & , -1 < x < 1 \\ 0 & , \text{Otherwise} \end{cases}$$

Find mean and S.D.

$$\text{Mean} = E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_{-1}^{1} x \cdot \left\{ \frac{1}{2}(x+1) \right\} dx$$

$$= \frac{1}{2} \int_{-1}^{1} x(x+1) dx$$

$$= \frac{1}{2} \int_{-1}^{1} (x^2 + x) dx$$

$$= \frac{1}{2} \left[\frac{x^3}{3} + \frac{x^2}{2} \right]_{-1}^{1}$$

$$E[X] = \frac{1}{2}\left[\left(\frac{1}{3} - \left(-\frac{1}{3}\right)\right) + \left(\frac{1}{2} - \frac{1}{2}\right)\right]$$

$$= \frac{1}{3}$$

Now,

$$\text{Variance} = \text{Var}(X) = E[X^2] - (E[X])^2 \quad — (1)$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_{-1}^{1} x^2 \cdot \frac{1}{2}(x+1)\,dx$$

$$= \frac{1}{2}\int_{-1}^{1} x^2 (x+1)\,dx$$

$$E[x^2] = \frac{1}{2}\int_{-1}^{1}(x^3+x^2)\,dx$$

$$= \frac{1}{2}\left[\frac{x^4}{4}+\frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\left(\frac{1}{4}-\frac{1}{4}\right)+\left(\frac{1}{3}-\left(\frac{-1}{3}\right)\right)\right]$$

$$= \frac{1}{2}\left[\frac{2}{3}\right]$$

$$= \frac{1}{3}$$

From ①, $Var(X) = E[X^2] - (E[X])^2$

$$= \frac{1}{3} - \left(\frac{1}{3}\right)^2$$

$$= \frac{1}{3} - \frac{1}{9}$$

$$= \frac{3-1}{9}$$

$$= \frac{2}{9}$$

$\therefore$ S.D. $= \sqrt{var(X)} = \sqrt{2/9} = \frac{\sqrt{2}}{3}$

If the p.d.f.

$$f(x) = \begin{cases} kx^3, & 0 \leq x \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

Find the value k and find probability between $x = \frac{1}{2}$ and $x = \frac{3}{2}$

We know that, for pdf

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow 0 + \int_{0}^{3} kx^3\,dx + 0 = 1$$

$$\Rightarrow k\int_{0}^{3} x^3\,dx = 1$$

$$\Rightarrow k\left[\frac{x^4}{4}\right]_0^3 = 1 \Rightarrow k\left(\frac{81}{4} - 0\right) = 1$$

$$\Rightarrow k = 4/81$$

Now,

$$f(x) = \begin{cases} \frac{4}{81}x^3, & 0 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$$

$$P\left(\frac{1}{2} \le x \le \frac{3}{2}\right) = \int_{1/2}^{3/2} f(x)\,dx$$

$$= \int_{1/2}^{3/2} \frac{4}{81} x^3\,dx = \ldots$$

$$= 5/81$$

A continuous random variable X has pdf

$$f(x) = \begin{cases} kx & ; \quad 0 \le x < 2 \\ 2k & ; \quad 2 \le x < 4 \\ -kx + 6k & ; \quad 4 \le x < 6 \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

Find k and mean

A $\quad k = 1/8 \qquad E[X] = 3$

If X is a continuous R.V. and K is a constant then prove that

(i) $V(x+K) = V(X)$

(ii) $V(KX) = K^2 V(X)$

$$\boxed{\begin{array}{c} E[K] = K \\ Var(K) = 0 \end{array}}$$

we know that,

$$Var(X) = E[X^2] - (E[X])^2 \quad \text{——} \ (1)$$

(i) $Var(X+K) = E[(X+K)^2] - (E[X+K])^2$

$$\mathrm{Var}(X+K) = E[X^2 + 2KX + K^2] - (E[X] + K)^2$$

$$= E[X^2] + E[2KX] + E[K^2] - \left[(E[X])^2 + 2E[X]K + K^2\right]$$

$$= E[X^2] + 2K\,E[X] + K^2 - (E[X])^2 - 2K\,E[X] - K^2$$

$$= E[X^2] - (E[X])^2 = \mathrm{Var}(X) \quad (\text{Using } ①)$$

$$\therefore \ \mathrm{Var}(X+K) = \mathrm{Var}(X)$$

Again,

$$\begin{aligned}
Var(kX) &= E[(kX)^2] - (E[kX])^2 \\
&= E[k^2X^2] - (kE[X])^2 \\
&= k^2E[X^2] - k^2(E[X])^2 \\
&= k^2(E[X^2] - (E[X])^2) \\
&= k^2 Var(X) \quad (\text{Using } (1))
\end{aligned}$$

If the pdf of a R.V. is given by

$$f(x) = \begin{cases} k(1-x^2), & 0 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

find the value of K and the probabilities that it will take on a value

(a) betn 0.1 and 0.2

(b) greater than 0.5 $\longrightarrow \int_{0.5}^{\infty} f(x)\,dx$

(c) mean

(d) variance

If $f(x) = ke^{-|x|}$ is p.d.f. in $-\infty < x < \infty$, find the values of K and variance of the random variable and also find the probability betⁿ 0 & 4.

We know that,

$\because$ $f(x)$ is p.d.f

$\therefore$ $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

$$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} k e^{-(-x)}\,dx + \int_{0}^{\infty} k e^{-x}\,dx = 1$$

$$\Rightarrow k \int_{-\infty}^{0} e^{x}\,dx + k \int_{0}^{\infty} e^{-x}\,dx = 1$$

$$k\left[e^{x}\right]_{-\infty}^{0} - k\left[e^{-x}\right]_{0}^{\infty} = 1$$

$$\Rightarrow k[1-0] - k[0-1] = 1$$

$$\Rightarrow 2k = 1$$

$$\Rightarrow k = 1/2$$

$$\therefore \quad f(x) = \frac{1}{2}e^{-|x|}$$

Now,

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$\boxed{\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & f(x) \text{ even} \\ 0, & f(x) \text{ is odd} \end{cases}}$$

$$= \int_{-\infty}^{\infty} x \, \frac{1}{2} e^{-|x|}\, dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} x\, e^{-|x|}\, dx$$

$g(x) = x e^{-|x|}$, $g(-x) = -x e^{-|x|}$
$= -g(x)$

$\Rightarrow g(x)$ is odd f^n

$E[X] = 0$ $\quad$ ($\because$ integrand is odd fn)

Again,

$$E[X^2] = \int_{-\infty}^{\infty} x^2 \frac{1}{2} e^{-|x|}\, dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} x^2 e^{-|x|}\, dx = \frac{1}{2} \cdot 2 \int_{0}^{\infty} x^2 e^{-|x|}\, dx$$

$\phi(x) = x^2 e^{-|x|}$

$\phi(-x) = (-x)^2 e^{-|-x|}$

$= x^2 e^{-|x|} = \phi(x)$

even fn

$E[X] = 0$ ($\because$ integrand is odd fn)

Again,

$$E[X^2] = \int_{-\infty}^{\infty} x^2 \frac{1}{2} e^{-|x|} dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} x^2 e^{-|x|} dx = \frac{1}{2} \cdot 2 \int_{0}^{\infty} x^2 e^{-|x|} dx$$

$\phi(x) = x^2 e^{-|x|}$

$\phi(-x) = (-x)^2 e^{-|-x|}$

$= x^2 e^{-|x|} = \phi(x)$

even fn

A continuous R.V. X has distribution f^n

$$F(x) = \begin{cases} 0 & , x \le 1 \\ k(x-D)^4 & , 1 \le x \le 3 \\ 1 & , x > 3 \end{cases}$$

Determine (a) $f(x)$,

(b) mean at $D = 1$

$$f(x) = \frac{d}{dx} F(x) = \begin{cases} 4k(x-D)^3 & , 1 \le x \le 3 \\ 0 & , \text{elsewhere} \end{cases}$$

$$\text{Mean} = E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{1}^{3} x \cdot 4k(x-D)^3\, dx$$

$$= 4k \int_{1}^{3} x(x-D)^3\, dx$$

$$= 4k \int_{1}^{3} x\left(x^3 - D^3 - 3x^2 D + 3x D^2\right) dx$$

= calculate this