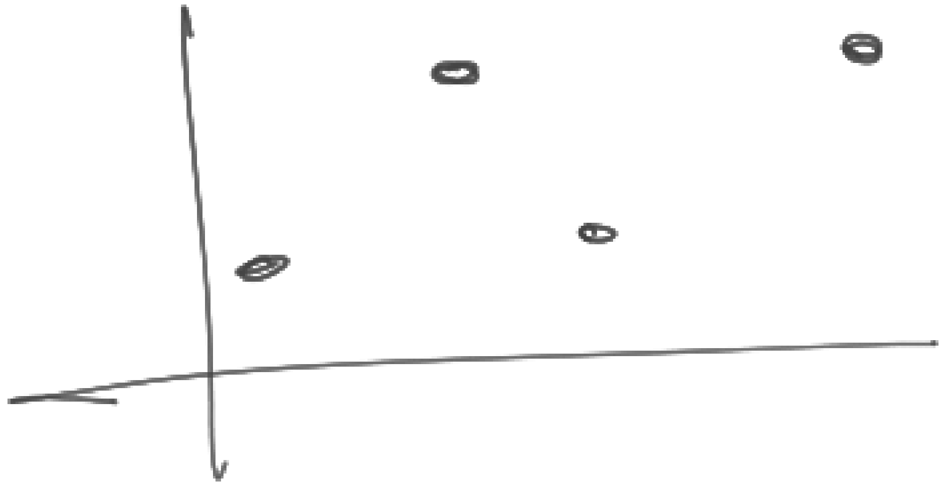
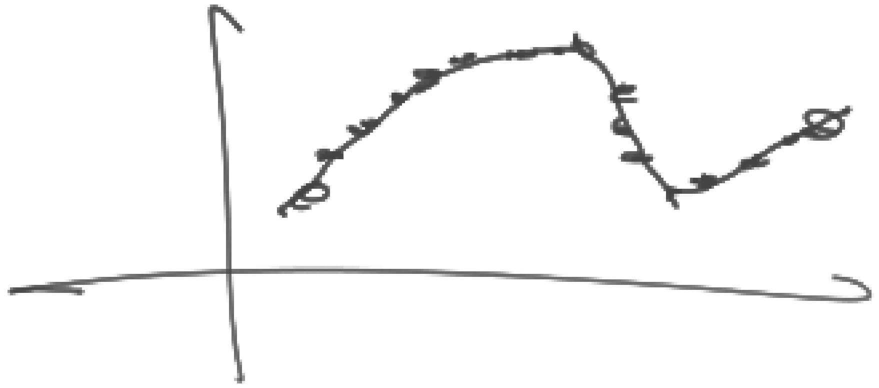




DISCRETE

CONTINUOUS

Baye's Th^m

Binomial

Integration

Normal

p

prob (success)

q

prob (failure)

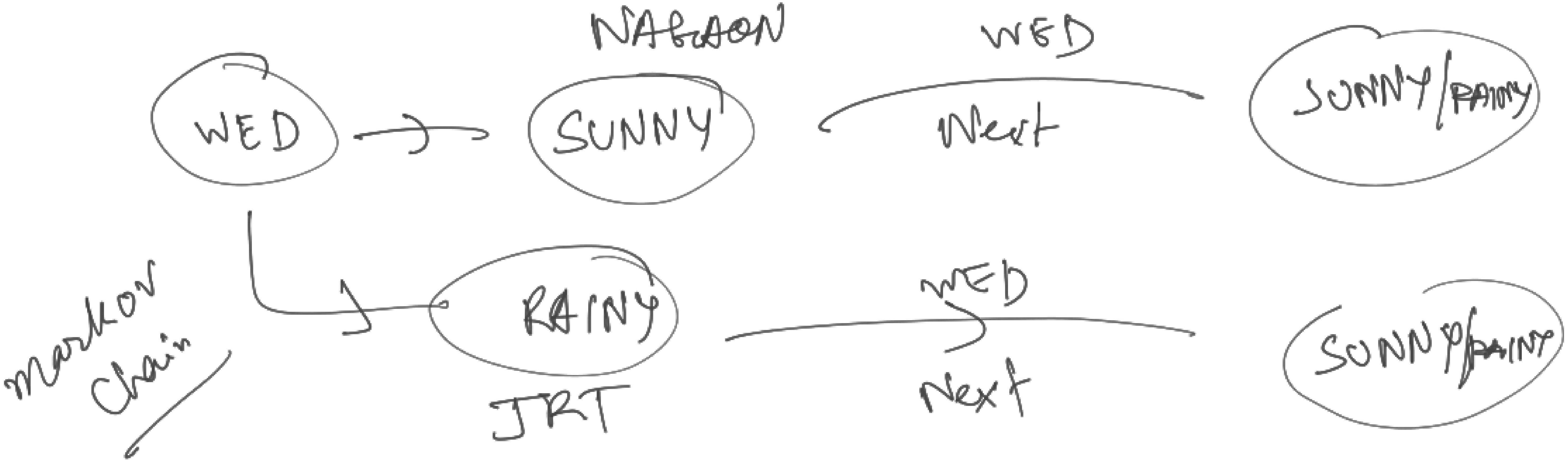
Poisson

Joint Distribution

Integration

H/T

Prob(Head) $= \frac{1}{2}$



Trial & Events

If an experiment is repeated under essentially the same conditions and it result in any one of the several possible outcomes, then the experiment is called a trial and the possible outcomes are known as events.

Eg: Tossing of a coin — Trial

H   T — Events

Sample Space

The set of all possible outcomes of an experiment is called sample space. It is denoted by S.

eg: Tossing of a coin

$$S = \{H, T\}$$

1st   2nd coin   2nd

$$\{(H, T)\} = \{HT\}$$

ordered pair   1st

Exhaustive Events

The outcomes of a random experiment is called exhaustive event if it covers all the possible outcomes of the event.

Throwing a die	Tossing a coin
$S = \{1, 2, 3, 4, 5, 6\}$	$S = \{H, T\}$
E_1 = getting an even no.	E_1 = getting a head
E_2 = getting an odd no.	E_2 = getting a tail.
$E_1 = \{2, 4, 6\}$, $E_2 = \{1, 3, 5\}$	$E_1 \cup E_2 = S$
$E_1 \cup E_2 = \{1, 2, 3, 4, 5, 6\} = S$ $\Rightarrow E_1$ & E_2 are exhaustive events	$\Rightarrow E_1$ & E_2 are exhaustive events

Favourable Events

The events which entail the required happening are called favourable events.

eg: In throwing of a pair of dice, the number of favourable events of getting a sum 7 is 6.

$(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)$

Mutually Exclusive Events

A T

H/T

Unbiased → 1. H/T

2. H/T

mutually exclusive M.E

Independent Events

A T

H/T H/T

1. H/T ⟷ independent 1. H/T

Independent events

Mutually Exclusive Events

Two or more events are said to be mutually exclusive if occurrence of one of them excludes the occurrence of the other. eg: while tossing a coin we either get a head or a tail but not both.

Independent Events

Two or more events are said to be independent if occurrence or non-occurrence of one doesn't depend on occurrence or non-occurrence of the other. eg: Two coins tossed at the same time, the outcome of one is independent of the outcome of the other.

Equally likely events

Two events are said to be equally likely if there is no reason to expect anyone with preference to other. eg: Head and tail are equally likely to come.

Classical Defination of Probability

$E \leftarrow$ event

$S \leftarrow$ sample space.

$$P(E) = \frac{\text{Number of favourable outcomes}}{\text{Total no. of outcomes}}$$

Probability of occurrence / happening of event E

$$P(E) = \frac{n(E)}{n(S)}$$

Axioms of Probability

(1) $0 \le P(E) \le 1$

(2) $P(S) = 1$

(3) $E_1, E_2, \ldots, E_n$ be n- multually exclusive events then

$$P(E_1) + P(E_2) + \ldots + P(E_n) = P(E_1) + P(E_2) + \ldots + P(E_n)$$

Q. Three coins are tossed. Find the probability of getting

(a) exactly one head

(b) atleast two heads

(c) atmost two heads

(d) no heads.

Two coins

HH, HT, TH, TT

Soln: Here, $S = \{HHH, HHT, HTH, THH, THT, HTT, TTH, TTT\}$

$n(S) = 8$

(a) Let E_1 = getting exactly one head. $= \{HTT, TTH, THT\}$

$n(E_1) = 3$

$\therefore$ Probability of getting exactly one head, $P(E_1) = \frac{n(E_1)}{n(S)} = \frac{3}{8}$

(b) E_2 = getting atleast two heads

$= \{HHH, HHT, HTH, THH\}$

$\therefore P(E_2) = \frac{4}{8} = \frac{1}{2}$

(c) E_3 = getting atmost 2 heads.

$= \{HHT, HTH, THH, THT, HTT, TTH, TTT\}$

$\therefore P(E_3) = \frac{7}{8}$

(d) E_4 = getting no head = $\{TTT\}$

$\therefore P(E_4) = \frac{1}{8}$

Q. A bag contains 4 white, 5 red and 7 black balls

(a) What is the probability that three ball's drawn at random are all red balls?

(b) What is the probability that one is white, one is red and one is black ball?

4W 5R 7B

3

16 Balls

3

Sol:

Total no. of balls = 16

Total no. of white balls = 4

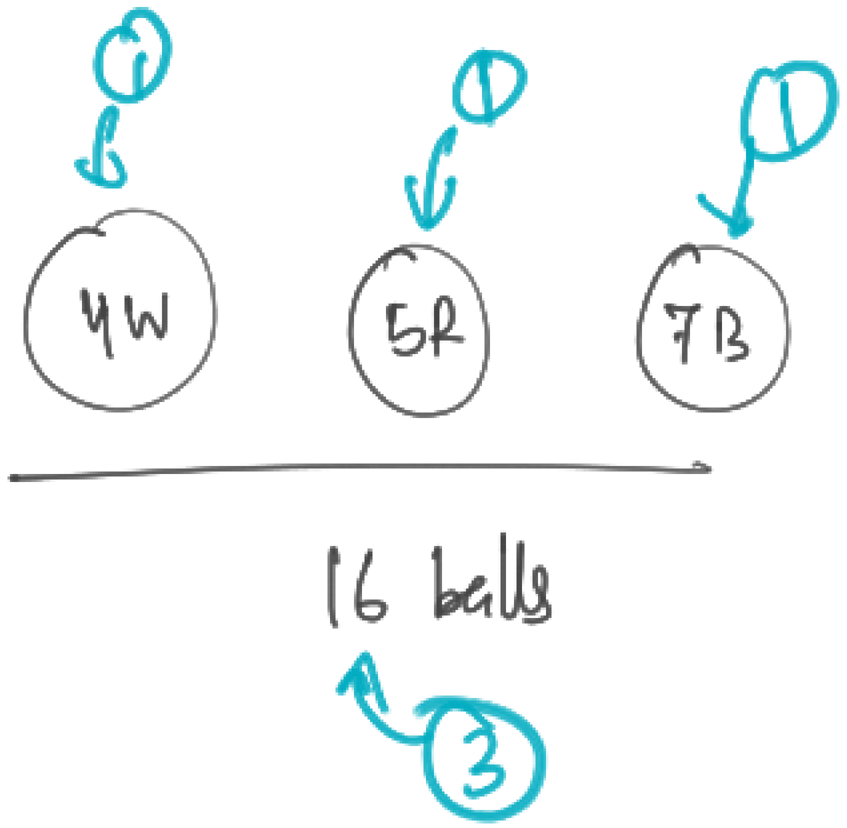
" " " red " = 5

" " " black " = 7

(a) Probability of getting 3 red balls $= \dfrac{^{5}C_{3}}{^{16}C_{3}} = ?$

(b) Probability of getting 1 white, 1 red, 1 black ball

$$= \frac{{}^{4}C_1 \times {}^{5}C_1 \times {}^{7}C_1}{{}^{16}C_3} = ?$$



Q. What is the probability that a non-leap year contains 53 sundays?

Note

① $P(\phi) = 0$

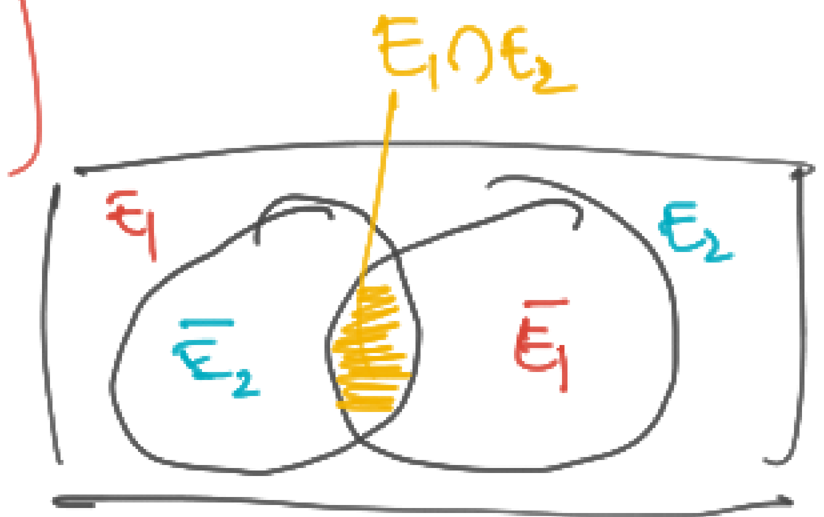
② If E be any event that probability of non-happening / non-occurrence is denoted by $P(\bar{E})$ or $P(E')$. It is given by

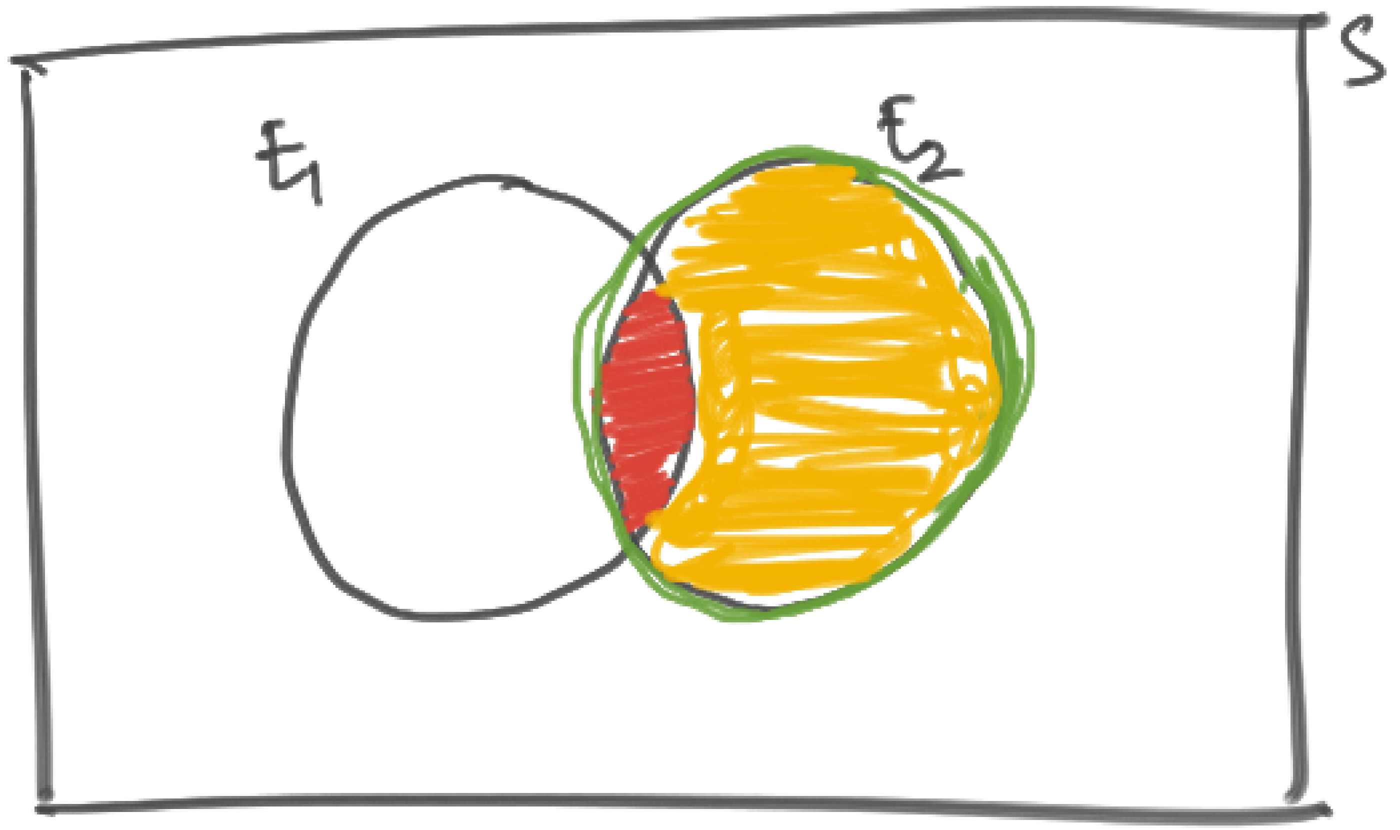
$$P(\bar{E}) = 1 - P(E)$$

$$\boxed{\sum_i P(E_i) = 1}$$

Prob. (occurrence) + Prob. (non-occurrence) = 1

③ $P(\bar{E}_1 \cap E_2) = P(E_2) - P(E_1 \cap E_2)$





$E_1 \cap E_2$

$E_2 - (E_1 \cap E_2) = \overline{E_1} \cap E_2$

M.E = mutually exclusive

④ $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

If E_1 and E_2 are M.E. then $E_1 \cap E_2 = \phi$ i.e. $P(E_1 \cap E_2) = 0$

$\therefore P(E_1 \cup E_2) = P(E_1) + P(E_2)$

⑤ $P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_3 \cap E_1)$
$+ P(E_1 \cap E_2 \cap E_3)$

If E_1, E_2, E_3 are M.E then ?

Conditional Probability

Let E_1 and E_2 be two events of a random experiment.

Then Probability of occurrence of E_1 given that E_2 has already occurred is denoted $P(E_1/E_2)$ and is defined as

$$P(E_1/E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}, \quad P(E_2) \neq 0.$$

Note

1. $P(S/E) = P(E/E) = 1$

$$P(S/E) = \frac{P(S\cap E)}{P(E)} = \frac{P(E)}{P(E)} = 1$$

2. $P((E_1 \cup E_2)/F) = P(E_1/F) + P(E_2/F) - P((E_1\cap E_2)/F)$

3. $P(\bar{E}/F) = 1 - P(E/F)$

Multiplicative law of Probability

The Probability of simultaneous occurrence of two events is equal to the probability of one multiplied by the conditional probability of the other.

For two events E_1 and E_2

$$P(E_1 \cap E_2) = P(E_1)\,P(E_2/E_1) \quad ; \quad P(E_1) \neq 0$$

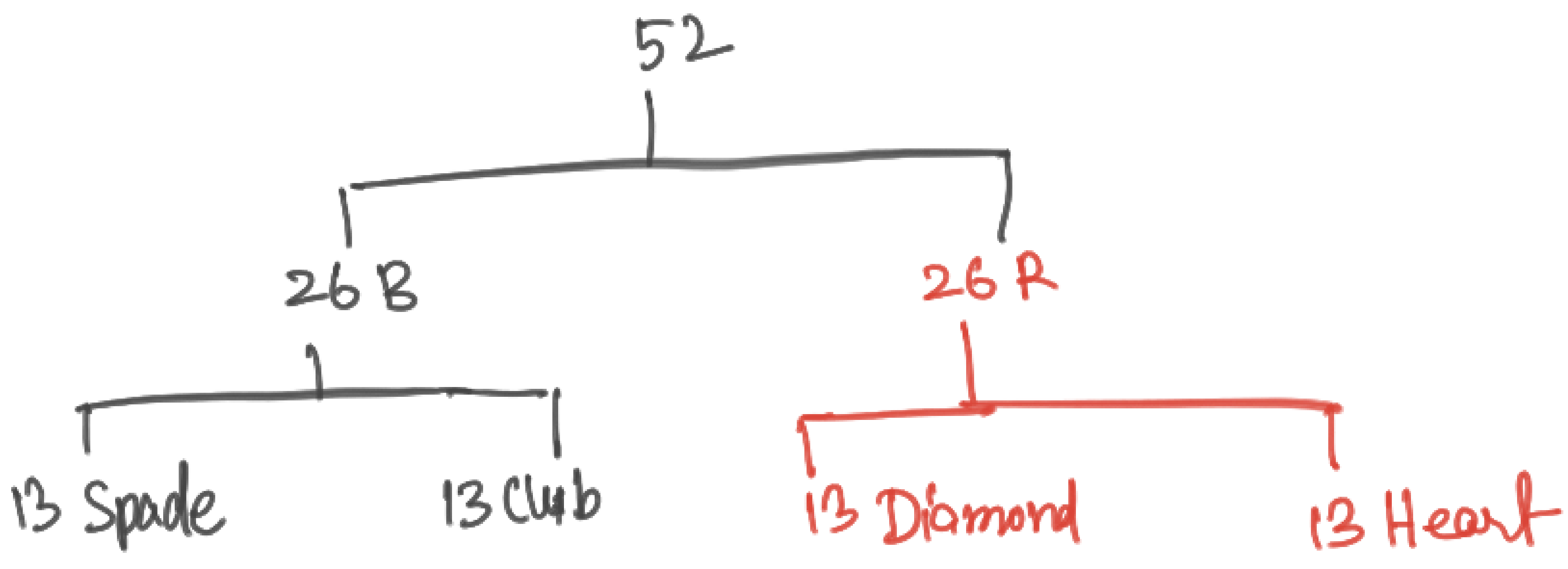
or $$P(E_1 \cap E_2) = P(E_2)\,P(E_1/E_2) \quad ; \quad P(E_2) \neq 0$$

Note

1. $P(E_1 \cap E_2)$ is also written as $P(E_1 E_2)$

2. If E_1 and E_2 are independent events then

$$P(E_1 \cap E_2) = P(E_1)\,P(E_2)$$



FACE CARDS

1 King

1 Queen

1 Jack

1 Ace (A)

2-10 numbers

Q. A card is drawn from a well shuffled deck of 52 cards and then a 2nd card is drawn. Find the probability that the first card is a spade and second card is a club if the first card is not replaced.

and $\rightarrow \cap \rightarrow \times$
or $\rightarrow \cup \rightarrow +$

Sol^n:

let, S = getting the first card a spade
C = getting the 2nd card a club

Now,

$$P(S \cap C) = P(S)\,P(C/S) \qquad (1)$$

Here,

P(S) = Probability that the 1st card is spade $= \frac{^{13}C_1}{^{52}C_1}$

$= \frac{13}{52}$

P(C/S) = Probability that 2nd card is club given that 1st card was spade

$= \frac{^{13}C_1}{^{51}C_1}$ (∵ Replacement doesn't occur)

$= \frac{13}{51}$

From (1)

$$P(S \cap C) = \left(\frac{13}{52}\right) \times \left(\frac{13}{51}\right) = ?$$

Q. A problem in physics is given to three students A, B and C whose chances of solving it are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ respectively. What is the probability that the problem will be solved?

Total Probability

$E_1, E_2, \ldots, E_n \leftarrow$ n - multually exclusive and exhaustive events

$A \leftarrow$ any arbitrary event associated with one / more of the above events

$P(E_i) \neq 0 \; (i = 1, 2, \ldots, n)$; $P(A) > 0$

$$P(A) = P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2) + \ldots + P(E_n)\, P(A/E_n)$$

Bayes' Theorem

$E_1, E_2, \ldots, E_n \leftarrow$ n - mutually exclusive and exhaustive events.

$A \leftarrow$ any arbitrary event associated with one / more of the above events

$P(E_i) \neq 0 \; (i = 1, 2, \ldots, n)$; $P(A) > 0$.

$$P(E_i/A) = \frac{P(E_i)\, P(A/E_i)}{P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2) + \ldots + P(E_n)\, P(A/E_n)}$$

Q. A bag X contains 2 white and 3 red balls and a bag Y contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and is found to be red. Find the probability that the drawn ball is from bag Y.

Soln Let,

A = the ball is red

E_1 = the ball is drawn from bag X

E_2 = the ball is drawn from bag Y.

Bag X (5) Bag Y (9)

By Bayes' Theorem,

$$P(E_2/A) = \frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)} \quad (1)$$

Here,

$P(E_1)$ = Probability that the ball is drawn from bag X

$= \frac{1}{2}$

$P(A/E_1)$ = Probability that the ball is red given that the ball is drawn from bag X $= \frac{3}{5}$

$P(E_2)$ = Probability that the ball is drawn from bag Y

$= \frac{1}{2}$

$P(A|E_2)$ = Probability that the ball drawn is red given that it is drawn from bag Y

$= \frac{5}{9}$

From (1),
$$P(E_2|A) = \frac{\frac{1}{2} \times \frac{5}{9}}{\left(\frac{1}{2} \times \frac{3}{5}\right) + \left(\frac{1}{2} \times \frac{5}{9}\right)} = ?$$

Q. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of accident is 0.01, 0.03 and 0.15 respectively. One of the insured person meets an accident. What is the probability that he/she is a scooter driver?

Q. A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.

Sol:

Let, A = the man reports it is a six.

E_1 = a six occurs

E_2 = a six doesn't occur.

By Bayes' Theorem,

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)P(A/E_1) + P(E_2)\,P(A/E_2)} \quad \text{—— (1)}$$

Here,

$P(E_1)$ = Probability that a six occurs $= \frac{1}{6}$

$P(A/E_1)$ = Probability that man reports it is a six given that six occurs

= Probability that the man speaks the truth.

$= \frac{3}{4}$

$P(E_2)$ = Probability that six doesn't occur $= \frac{5}{6}$

$P(A/E_2)$ = Probability that the man reports it is a six given that six doesn't occur.

$$= \frac{1}{4}$$

$$\boxed{\left(1 - \frac{3}{4} = \frac{1}{4}\right)}$$

From (1), we have

$$P(E_1/A) = \frac{\frac{1}{6} \times \frac{3}{4}}{\left(\frac{1}{6} \times \frac{3}{4}\right) + \left(\frac{5}{6} \times \frac{1}{4}\right)} = ?$$

Q. Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4, she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she is threw 1, 2, 3 or 4 with the die?

Q. If a machine is correctly set up, it produces 90% acceptable items. If it is incorrectly set up, it produces only 40% acceptable items. Past experience shows that 80% of the setups are done correctly. If after a certain setup, the machine produces 2 acceptable items, find the probability that the machine is set up correctly.

Soln. Let,

$A =$ machine produces 2 acceptable items.

$E_1 =$ machine set up is correct.

E_2 = machine set up is incorrect.

By Bayes' Theorem,

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)} \longleftarrow (1)$$

Here,

$P(E_1)$ = Prob - - - $= \frac{80}{100} = 0.8$

$P(E_2)$ = - - - - $= \frac{20}{100} = 0.2$

$P(A|E_1)$ = Probability that the machine produces two acceptable items given that the machine set up is correct

$$= \frac{90}{100} \times \frac{90}{100}$$

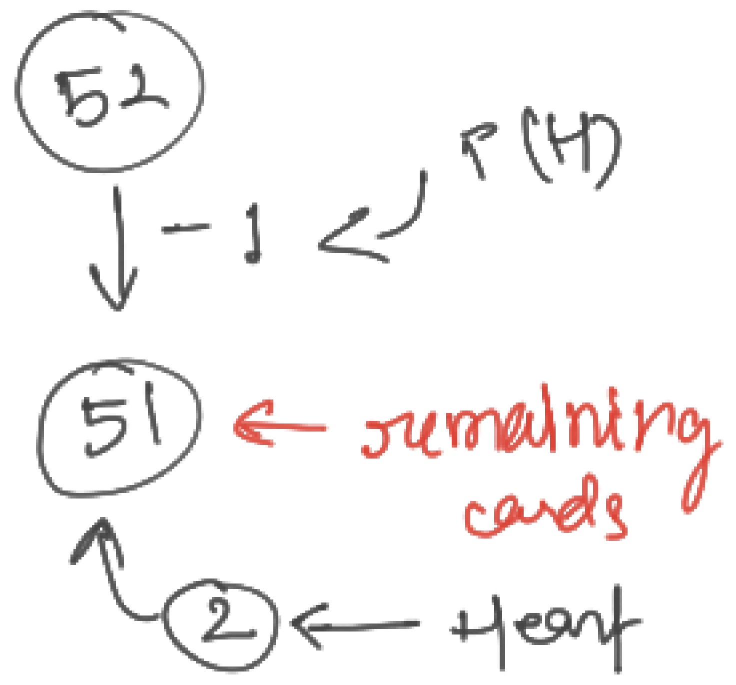
$$= 0.81$$

$P(A|E_2)$ = ... $= \frac{40}{100} \times \frac{40}{100} = 0.16$

From (1), we have

$$P(E_1/A) = \frac{0.8 \times 0.81}{(0.8 \times 0.81) + (0.2 \times 0.16)} \Rightarrow ?$$

Q. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be heart. Find the probability of the missing card to be a heart.



Solⁿ: Let A = two cards drawn from the remaining cards are heart.

E_1 = the missing card is heart

E_2 = " " " " diamond

E_3 = the missing card is club

E_4 = the missing card is spade.

By Bayes' Theorem,

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + P(E_3)P(A/E_3) + P(E_4)P(A/E_4)} \quad \text{--- (1)}$$

Here,

$P(E_1)$ = Probability that the missing card is heart $= \frac{{}^{13}C_1}{{}^{52}C_1}$

$= \frac{13}{52} = \frac{1}{4}$

$P(E_2) = \ldots = \ldots = \frac{1}{4}$

$P(E_3) = \ldots = \ldots = \frac{1}{4}$

$P(E_4) = \ldots = \ldots = \frac{1}{4}$

$P(A/E_1)$ = Probability that two cards drawn are heart given that the missing card is heart

$$= \frac{^{12}C_2}{^{51}C_2}$$

$P(A/E_2)$ = Prob. that two cards drawn are heart given that the missing card is diamond

$$= \frac{{}^{13}C_2}{{}^{51}C_2}$$

$$P(A/E_3) = \ldots \quad = \frac{{}^{13}C_2}{{}^{51}C_2}$$

$$P(A/E_4) = \ldots \quad = \frac{{}^{13}C_2}{{}^{51}C_2}$$

from ①

$$P(E_1/A) = \frac{\frac{1}{4} \cdot \frac{{}^{12}C_2}{{}^{51}C_2}}{\frac{1}{4} \cdot \frac{{}^{12}C_2}{{}^{51}C_2} + \frac{1}{4} \frac{{}^{13}C_2}{{}^{51}C_2} + \frac{1}{4} \frac{{}^{13}C_2}{{}^{51}C_2} + \frac{1}{4} \cdot \frac{{}^{13}C_2}{{}^{51}C_2}} = ?$$

Random Variable

A random variable is a rule that assigns a real number to each outcome of a random experiment. It is usually a f^n which is denoted by X.

eg: Tossing of a pair of coins

Define X = no. of tails

$$S = \{HH, HT, TH, TT\}$$

$\therefore$ X takes the value 0, 1, 2 i.e. $X = 0, 1, 2$

<u>Note</u> : Random variable is also called stochastic variable or variate.

Random Variable (R.V.)

- Discrete R.V.
- Continuous R.V.

Discrete Probability Distribution

If a random variable X have discrete set of values say $x_1, x_2, \ldots, x_n$ with respect to the probabilities $p_1, p_2, \ldots, p_n$ s.t. $\sum_i p_i = 1$ then occurrences of values x_i with respective probabilities p_i is called discrete probability distribution of X.

eg: If X denotes no. of tails in tossing of a pair of coins then probability distribution is given by

$S = \{HH, HT, TH, TT\}$

$X = 0, 1, 2$

$P(X=0)$ = Prob. that no. of tails is zero $= \frac{1}{4}$

$P(X=1)$ = " " " " " " one $= \frac{2}{4} = \frac{1}{2}$

$P(X=2)$ = " " " " " " two $= \frac{1}{4}$

X	0	1	2
$P(X)$	$1/4$	$1/2$	$1/4$

← Discrete Prob. Distribution

1.

X	1	2	3
P(X)	1/4	2/4	1/4

It is a Prob. Dist.

1. $0 \leq P(E) \leq 1$
2. $\sum_i P(E_i) = 1$

2.

X	1	2	3
P(X)	-1/4	3/4	2/4

Not a Prob. Dist.

3.

X	1	2	3
P(X)	1/4	1/4	1/4

Not a prob Dist.

Probability Function or Probability Mass Function (pmf)

Probability function or pmf of a random variable (R.V.) X is a mathematical function $p(x)$ which gives the probability corresponding to different possible discrete set of values say $x_1, x_2, \ldots, x_n$ of variable x.

i.e. $p(x_i) = p(x = x_i)$

The f^n $p(x)$ must satisfy the conditions

① $p(x_i) \geq 0$

② $\sum p(x_i) = 1$

Cumulative Distribution Function (Distribution Function)

If X is a R.V. the $P(X \leq x)$ is called the cumulative distribution function (cdf) and is denoted $F(x)$.

$$F(x) = P(X \leq x)$$

Expectation of a Discrete R.V.

If X is a R.V. which assumes the discrete set of values $x_1, x_2, \ldots, x_n$ with respective probabilities $p_1, p_2, \ldots, p_n$ then the expectation or expected value of X is denoted by $E[X]$ and is defined as

$$E[X] = \sum_{i=1}^{n} x_i p_i = x_1 p_1 + x_2 p_2 + \ldots + x_n p_n$$

$$E[X^2] = \sum_{i=1}^{n} x_i^2 p_i$$

$$E[X^m] = \sum_i x_i^m p_i$$

Properties

$X, Y \leftarrow$ R.V. $\quad a, b \leftarrow$ constants. $\quad \mu \leftarrow$ mean

$$\left.\begin{array}{l} (1)\ E[a] = a \\ (2)\ E[aX] = aE[X] \\ (3)\ E[X-\mu] = 0 \end{array}\right\}$$ Try to prove it.

④ $E[X \pm Y] = E[X] \pm E[Y]$

* ⑤ $E[XY] = E[X]E[Y]$ if X & Y are independent R.V.

⑥ if $Z = ax + b$ then $E[Z] = E[aX + b]$

or

$Z = aX + b$

$= a\,E[X] + b$

Variance and Standard Deviation

The variable of discrete R.V. X is expected value of

$(X-\mu)^2$ where μ is mean of the variable X.

$$\sigma^2 \leftarrow \text{Var}(X) = E[(X-\mu)^2]$$

Also denoted by σ^2

$$SD(X) = \sqrt{\text{Var}(X)} = \sqrt{E[X^2] - (E[X])^2}$$

$\hookrightarrow \sigma$

* Q. Prove that $\text{Var}(X) = E[X^2] - (E[X])^2$

Q. Find the expected value of getting head when a pair of coins is tossed.

Sol:

$$S = \{HH, HT, TH, TT\}$$

let, X = no. of heads

Possible values of X are — $X = 0, 1, 2$

Probability distribution table will be given by

X	0	1	2
$P(X)$	$1/4$	$2/4$	$1/4$

Expected value of getting head, $E[X] = \left(0 \times \frac{1}{4}\right) + \left(1 \times \frac{2}{4}\right) + \left(2 \times \frac{1}{4}\right)$

$$= 1$$

Alternate

$$E[X] = \sum_i x_i p_i = x_1 p_1 + x_2 p_2 + x_3 p_3$$

$$= \left(0 \cdot \frac{1}{4}\right) + \left(1 \cdot \frac{2}{4}\right) + \left(2 \cdot \frac{1}{4}\right) = 1.$$

Way ahead

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2$$

$$= \left(\sum_i x_i^2 p_i\right) - (1)^2$$

$$= (x_1^2 p_1 + x_2^2 p_2 + x_3^2 p_3) - 1$$

$$= \left[\left(0^2 \cdot \frac{1}{4}\right) + \left(1^2 \cdot \frac{2}{4}\right) + \left(2^2 \cdot \frac{1}{4}\right)\right] - 1$$

$$= \frac{1}{2} + 1 - 1$$

$$= \frac{1}{2}$$

(positive)

$$SD(X) = \sqrt{Var(X)} = \sqrt{1/2}$$

$$= 0.7071$$

Q. A random variable X has the following distribution

x	-2	-1	0	1	2	3
$P(X)$	0.1	k	0.2	$2k$	0.3	k

Determine

1. the value of k
2. $P(X<1)$
3. $P(X \geq 1)$
4. $P(X>1)$
5. μ
6. σ^2

Soln:

$\because$ It is a probability distribution

$\therefore \quad \Sigma P(X) = 1$

$\Rightarrow 0.1 + k + 0.2 + 2k + 0.3 + k = 1$

$\Rightarrow 0.6 + 4k = 1$

$\Rightarrow 4k = 0.4$

$\Rightarrow k = 0.1$

The probability Distribution is given by

X	-2	-1	0	1	2	3
$P(X)$	0.1	0.1	0.2	0.2	0.3	0.1

(2) $P(X<1) = P(X=-2) + P(X=-1) + P(X=0)$

$= 0.1 + 0.1 + 0.2$

$= 0.4$

③ $P(X \geq 1) = P(X=1) + P(X=2) + P(X=3)$

$= 0.2 + 0.3 + 0.1$

$= 0.6$

④ $P(X > 1) = P(X=2) + P(X=3)$

$= 0.3 + 0.1$

$= 0.4$

(5) Mean, $\mu = \sum x\, p(x)$

$= [(-2).(0.1)] + [(-1).(0.1)] + [(0).(0.2)] +$

$[(1).(0.2)] + [(2).(0.3)] + [(3).(0.1)]$

$= 0.8$

(6) Variance, $\sigma^2 = \sum x^2 p(x) - [\sum x\, p(x)]^2$

$= [(-2)^2.(0.1)] + [(-1)^2.(0.1)] + [(0)^2.(0.2)] + [(1)^2.(0.2)] + [(2).(0.3)] + [(3)^2.(0.1)]$

$- [0.8]^2$

Q. A player tossed two coins. If he gets two heads then he wins ₹4. If he gets one head then he wins ₹2 but if he gets two tails then he pays a penalty of ₹3. Calculate the expected value of the game to him.

HH HT TH
TT

Sol

Let X = no. of heads

X can take the values $0, 1, 2$

$P_0 = P(X=0)$ = Prob. of getting no heads $= \frac{1}{4}$

$p_1 = P(X=1) =$ Probability of getting one head $= \frac{2}{4}$

$p_2 = P(X=2) =$ Probability of getting two heads $= \frac{1}{4}$

Now,

$X=0 \Rightarrow x_0 = -3$ (negative sign denotes penalty)

$X=1 \rightarrow x_1 = 2$ | x_i denotes money.

$X=2 \Rightarrow x_2 = 4$

$\therefore E[X] = \sum_i x_i p_i = x_0 p_0 + x_1 p_1 + x_2 p_2 = \left[(-3)\left(\frac{1}{4}\right)\right] + \left[2.\left(\frac{2}{4}\right)\right] + \left[4.\left(\frac{1}{4}\right)\right] = 1.25$

$\therefore$ Expected value of the game = ₹ 1.25

Q. A fair coin is tossed until head or five tails occurs. Find expected number of tosses of the coins.

Soln:

p = prob. of success
q = prob. of failure

p = prob. of getting head = $\frac{1}{2}$

q = prob. of getting tail = $\frac{1}{2}$

X	1	2	3	4	5	6
outcome	H	TH	TTH	TTTH	TTTTH	TTTTTT
Probability	p	qp	q^2p	q^3p	q^4p	q^5

Expected no. of tosses is given by

$$E(x) = \sum x_i p_i$$

$$= 1.p + 2.qp + 3.q^2p + 4.q^3p + 5.q^4p + 6.q^5$$

Independent Random Variable

$$E[XY] = E[X]E[Y] \longrightarrow X \,\&\, Y \text{ are independent R.V.}$$

Covariance

| X or x

If X and Y are two random variables with respective means $\bar{X}$ and $\bar{Y}$, then the covariance between X and Y is denoted by cov (X, Y) and defined as

$$\mathrm{cov}(X,Y) = E[(X-\bar{X})(Y-\bar{Y})]$$

⇒ The expected value of the deviations of the two variables from their means is called their covariance.

* The covariance of two independent variables is equal to zero

Proof

if X and Y are two R.V. then

$\bar{X}, \bar{Y} \in$ mean

& constants

$$\operatorname{Cov}(X, Y) = E\left[(X-\bar{X})(Y-\bar{Y})\right]$$

$$= E\left[XY - X\bar{Y} - \bar{X}Y + \bar{X}\bar{Y}\right]$$

$$\mathrm{cov}(X,Y) = E[XY] - E[\bar{X}Y] - E[X\bar{Y}] + E[\bar{X}\bar{Y}]$$

$$= \underline{E[X]E[Y]} - \bar{X}E[Y] - \bar{Y}E[X] + \bar{X}\bar{Y}$$

($\because$ X & Y are independent)

$$= \bar{X}\bar{Y} - \bar{X}(\bar{Y}) - \bar{Y}(\bar{X}) + \bar{X}\bar{Y}$$

($\because$ $E[X] = \bar{X}$, $E[Y] = \bar{Y}$)

$$= \bar{X}\bar{Y} - \bar{X}\bar{Y} - \bar{Y}\bar{X} + \bar{X}\bar{Y}$$

$$= 0$$

* Correlation coefficient

$$\rho = \frac{\operatorname{cov}(X, Y)}{\sqrt{\operatorname{var}(X)}\,\sqrt{\operatorname{var}(Y)}}$$

Properties (covariance)

1. $\operatorname{cov}(X, X) = \operatorname{var}(X)$

2. If X and Y are independent then $\operatorname{cov}(X, Y) = 0$

3. $\text{cov}(X, Y) = \text{cov}(Y, X)$

4. $\text{cov}(aX, Y) = a\,\text{cov}(X, Y)$ $\qquad$ $a, c \leftarrow$ constants

5. $\text{cov}(X + c, Y) = \text{cov}(X, Y)$

6. $\text{cov}(X + Y, Z) = \text{cov}(X, Z) + \text{cov}(Y, Z)$.

Properties (correlation)

1. $-1 \le \rho(X, Y) \le 1$

2. If $\rho(X, Y) = 1$ then $Y = aX + b$ where $a > 0$

3. If $\rho(X,Y) = -1$ then $Y = aX + b$ where $a < 0$.

4. $\rho(aX+b, cY+d) = \rho(X,Y)$ for $a, c > 0$

Binomial Distribution

Bernoulli's Trial

$p \leftarrow$ prob. of success

$q \leftarrow$ prob. of failure

$n \leftarrow$ trials

$p \leftarrow$ probability of success

$q \leftarrow$ " " failure

$$\boxed{p + q = 1}$$

If there are 'r' success then $(n-r)$ failures in n trials then

$$P(X=r) = P(r) = {}^{n}C_{r}\, p^{r} q^{n-r} \quad ; \quad r = 0, 1, 2, \ldots, n$$

Binomial Distribution

$$f(r) = N\,P(r) = N\left[{}^{n}C_{r}\, p^{r} q^{n-r}\right]$$

N = no. of times the experiment is repeated (consisting of n trials)

Binomial Frequency Distribution

Properties

1. It is a discrete distribution.

2. It depends on two parameters p or q and n.

3. It is symmetrical if $p = q$.

4. Mean $= np$

Variance $= npq$

S.D. $= \sqrt{npq}$

5. Mode of Binomial Distribution = value of X that has the largest frequency.

$$\boxed{X \sim B(n, p)}$$

eg: $X \sim B(10, \frac{1}{2})$

$\Rightarrow n = 10$

$p = \frac{1}{2}$

$q = 1 - p = \frac{1}{2}$

Q. Prove that in case of Binomial distribution

Mean $(\mu) = np$

Variance $(\sigma^2) = npq$

Conditions for application of Binomial Distribution

1. The variable should be discrete.
2. A dichotomy must exists, i.e. there should be two alternatives either success or failure.
3. n must be finite and small.
4. Trials or events must be independent, i.e. happening of one event must not affect happening of other.

5. The Trials or Events must be repeated under identical conditions.

Recursion formula or Recurrence relation for Binomial Dist.

$$P(X=r) = {}^{n}C_{r}\, p^{r} q^{n-r}$$

$$\boxed{P(X=r+1) = \frac{n-r}{r+1} \cdot \frac{p}{q}\, P(X=r)} \quad ; \quad r = 1, 2, 3, \ldots$$

Q. Ten coins are thrown simultaneously. Find the probability of getting atleast seven heads.

$$P(X=r) = {}^nC_r \, p^r q^{n-r}$$

$n \leftarrow$ total outcome

$p \leftarrow$ prob. of success

$q \leftarrow$ prob. of failure

Soln let, X = no. of heads

p = prob. of getting a head $= \frac{1}{2}$

$q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

$n = 10$

$p = \frac{1}{2}$, $q = \frac{1}{2}$

Now,

Probability of getting atleast seven heads $= P(X \geq 7)$

$$= P(X=\overset{r}{7}) + P(X=\overset{r}{8}) + P(X=\overset{r}{9}) + P(X=\overset{r}{10})$$

$$= {}^{10}C_7\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^{10-7} + {}^{10}C_8\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^{10-8} + {}^{10}C_9\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right)^{10-9} + {}^{10}C_{10}\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^{10-10}$$

$$= ?$$

Q. $\underbrace{\text{Six dice}}_{n}$ are thrown together at a time, the process is repeated $\underbrace{\text{729 times}}_{N}$. How many times do you expect atleast three dice to have 4 or 6.

Soln: let X = no. of die giving (having) 4 or 6.

p = prob. of getting 4 or 6 $= \frac{2}{6} = \frac{1}{3}$

1, 2, 3, 4, 5, 6

die's outcome

$q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$

At a single time, six dice are thrown simultaneously

$$P(X \geq 3) = P(X=3) + P(X=4) + P(X=5) + P(X=6).$$

$$= {}^6C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^3 + {}^6C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^2 + {}^6C_5 \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^1 + {}^6C_6 \left(\frac{1}{3}\right)^6$$

$$= ?$$

The process is repeated 729 times so

Required no. of times atleast 3 dice will have 4 or 6

$= 729\ P(X \geq 3)$

$= 729\ (?)$

$= ?$

Q. If the sum of the mean and variance of binomial distribution of 5 trials is 4.8, find the distribution.

Soln: let, the binomial distribution be ${}^{n}C_{r}\, p^{r} q^{n-r}$

$n =$ no. of trials $= 5$

Mean of binomial distribution $= np$

variance of binomial distribution $= npq$

A/q

$np + npq = 4.8$

$$np(1+q) = 4.8$$

$$\Rightarrow 5p(1+q) = 4.8$$

$$\Rightarrow 5(1-q)(1+q) = 4.8 \qquad (\because p = 1-q)$$

$$\Rightarrow 5(1-q^2) = 4.8$$

$$\Rightarrow 1-q^2 = \frac{4.8}{5}$$

$$\Rightarrow 1-q^2 = \frac{48}{50}$$

$$\Rightarrow 50 - 50q^2 = 48$$

$$50q^2 = 2$$

$$\Rightarrow q^2 = \frac{2}{50}$$

$$\Rightarrow q^2 = \frac{1}{25}$$

$$\Rightarrow q = \frac{1}{5}$$ ($\because$ prob. cannot be negative).

$\therefore$ $p = 1 - q = \frac{4}{5}$

Hence, the required binomial distribution $= {}^5C_x \left(\frac{4}{5}\right)^x \left(\frac{1}{5}\right)^{5-x}$

Q. A student obtained the following answer to a certain problem given to him. Mean = 2.4, Variance = 3.2 for a binomial distribution. Comment on the result.

Soln: We know that,

Mean of Binomial distribution $= np$

Variance " " " $= npq$

Given,

$np = 2.4$

$npq = 3.2$

$\Rightarrow (2.4)q = 3.2$

$\Rightarrow q = \frac{3.2}{2.4} = 1.33 > 1$ which is not possible

$\therefore$ The given results are inconsistent.

q

Q. If on an average one ship in every ten is wrecked, find the probability that out of 5 ships expected to arrive, 4 atleast will arrive safely.

n

Soln Let, X = no. of ships arrived safely.

q = Prob. of ship getting wrecked $= \frac{1}{10}$

p = Prob. of ship arriving safely $= 1-q = 1-\frac{1}{10}$

$= \frac{9}{10}$

Here, $n=5$

Required Prob. $= P(X \geq 4) = P(X=4) + P(X=5)$

$=$?

Q. Probability of a man hitting a target is $\frac{1}{3}$.

(a) If he fires 6 times, what is the probability of hitting (i) atmost 5 times.

(ii) atleast 5 times.

(iii) exactly once.

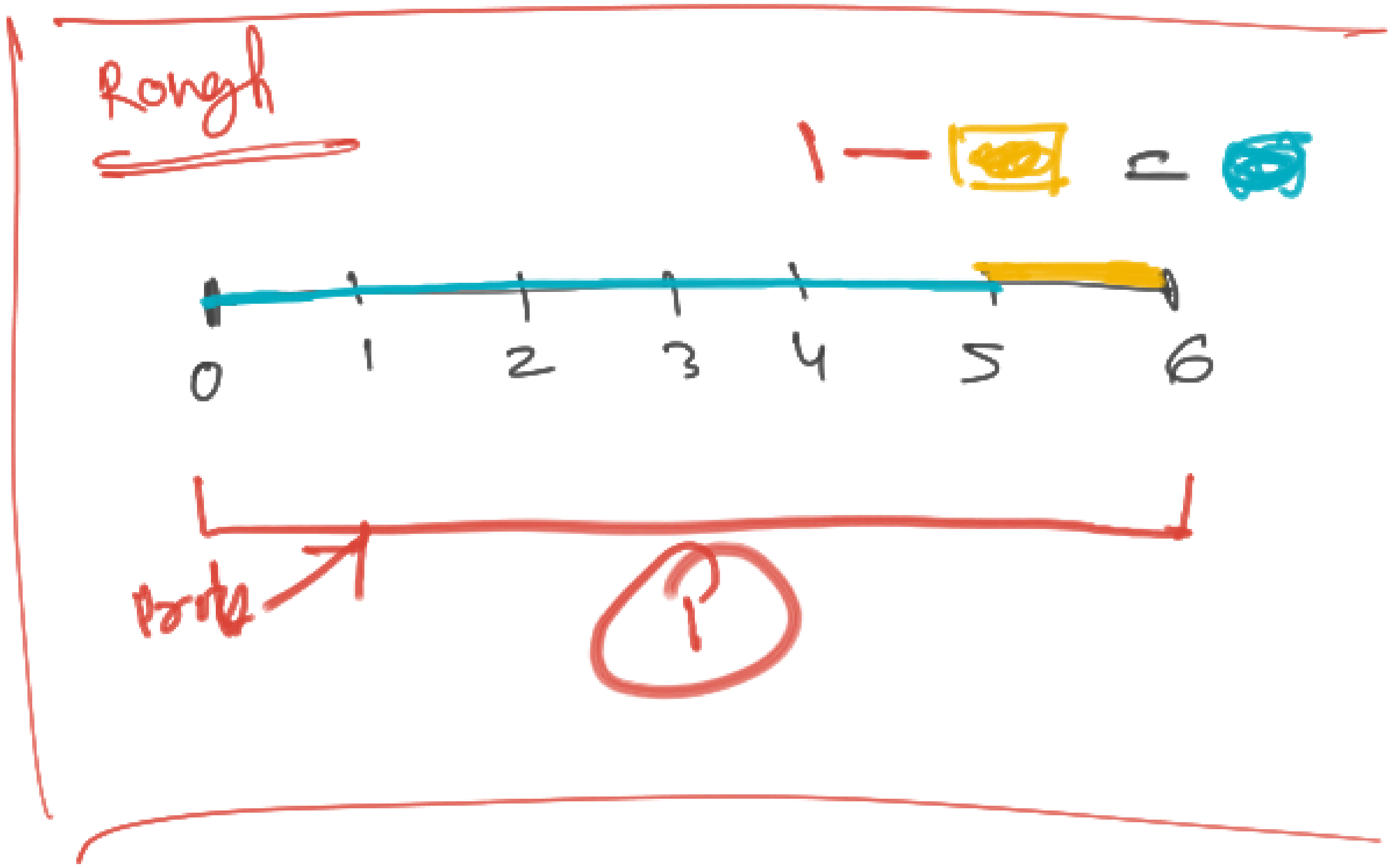
(b) If he fires so that the probability of his hitting the target atleast once is greater than $\frac{3}{4}$, find n.

Sol :

$\because$ Probability of the man hitting a target is $\frac{1}{3}$

$\therefore \quad p = \frac{1}{3}$

$q = 1 - p = \frac{2}{3}$



(a) Given, $n = 6$

Let X = no. of times he hit the target

(i) $P(X \le 5) = 1 - P(X > 5) = 1 - P(X = 6) = 1 - \left[{}^{6}C_{6} (p)^{6} (q)^{6-6} \right]$

$= 1 - \left[1 \cdot \left(\frac{1}{3}\right)^{6} \left(\frac{2}{3}\right)^{0} \right] = 1 - \left(\frac{1}{3}\right)^{6} = ?$

(ii) $P(x \geq 5) = P(x=5) + P(x=6)$

$$= \left[{}^6C_5 \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^{6-5} \right] + \left[{}^6C_6 \left(\frac{1}{3}\right)^6 \left(\frac{2}{3}\right)^6 \right]$$

$= ?$

(iii) $P(x=1) = {}^6C_1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^{6-1} = ?$

(b)

A|q

$$P(X \geq 1) > \frac{3}{4}$$

$$\Rightarrow 1 - P(X < 1) > \frac{3}{4}$$

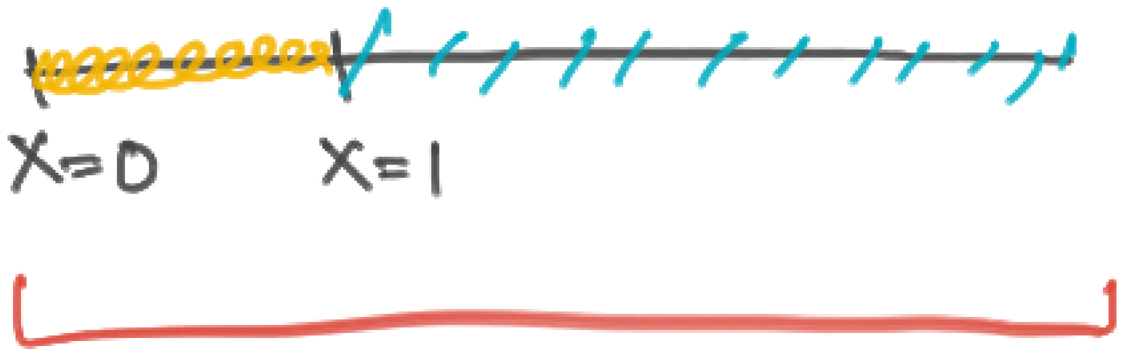
$$\Rightarrow 1 - P(X = 0) > \frac{3}{4}$$

$$\Rightarrow 1 - {}^{n}C_{0} \left(\frac{1}{3}\right)^{0} \left(\frac{2}{3}\right)^{n-0} > \frac{3}{4}$$

$$\Rightarrow 1 - \left(\frac{2}{3}\right)^{n} > \frac{3}{4}$$

$$\Rightarrow 1 - \frac{3}{4} > \left(\frac{2}{3}\right)^{n} \Rightarrow \left(\frac{2}{3}\right)^{n} < \frac{1}{4} \Rightarrow 2^{n} \cdot 4 < 3^{n}$$

$$\Rightarrow 2^{n} \cdot 2^{2} < 3^{n}$$



$$\Sigma P(X) = 1$$

P[illegible] $= 1 - P$[illegible]

$$P(X \geq 1) = 1 - P(X < 1)$$

$$\boxed{P(X = r) = {}^{n}C_{r}\, p^{r} q^{n-r}}$$

$\Rightarrow 2^{n+2} < 3^n$

The above inequality holds for $n = 4$

Hence, the man must fire 4 times so that the probability of hitting the target atleast once is greater than $\frac{3}{4}$

$n=1, \quad 2^{1+2} = 2^3 = 8$

$8 \nless 3 \quad 3^1 = 3$

not possible

$n=2, \quad 2^{2+2} = 2^4 = 16$

$16 \nless 9 \quad 3^2 = 9$

not possible

$n=3,$

$2^{3+2} = 2^5 = 32$

$3^3 = 27$

$32 \nless 27$

not possible

$n=4, \quad 2^{4+2} = 64$

$3^4 = 81$

$2^{4+2} < 3^4$ possible

POISSON DISTRIBUTION

French Mathematician — S.D. Poisson

Poisson Distribution — Discrete Probability Distribution

Characteristics

1. $n \to \infty$ i.e. sufficiently large (infinitely large
   $p \to 0$ i.e. sufficiently small (infinitely small

parameter, $\lambda = np \leftarrow$ finite no.

2. It consists of a single parameter i.e. λ.

Q. Probability of a man hitting a target is $\frac{1}{3}$.

(a) If he fires 6 times, what is the probability of hitting (i) atmost 5 times.

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Sol^n :

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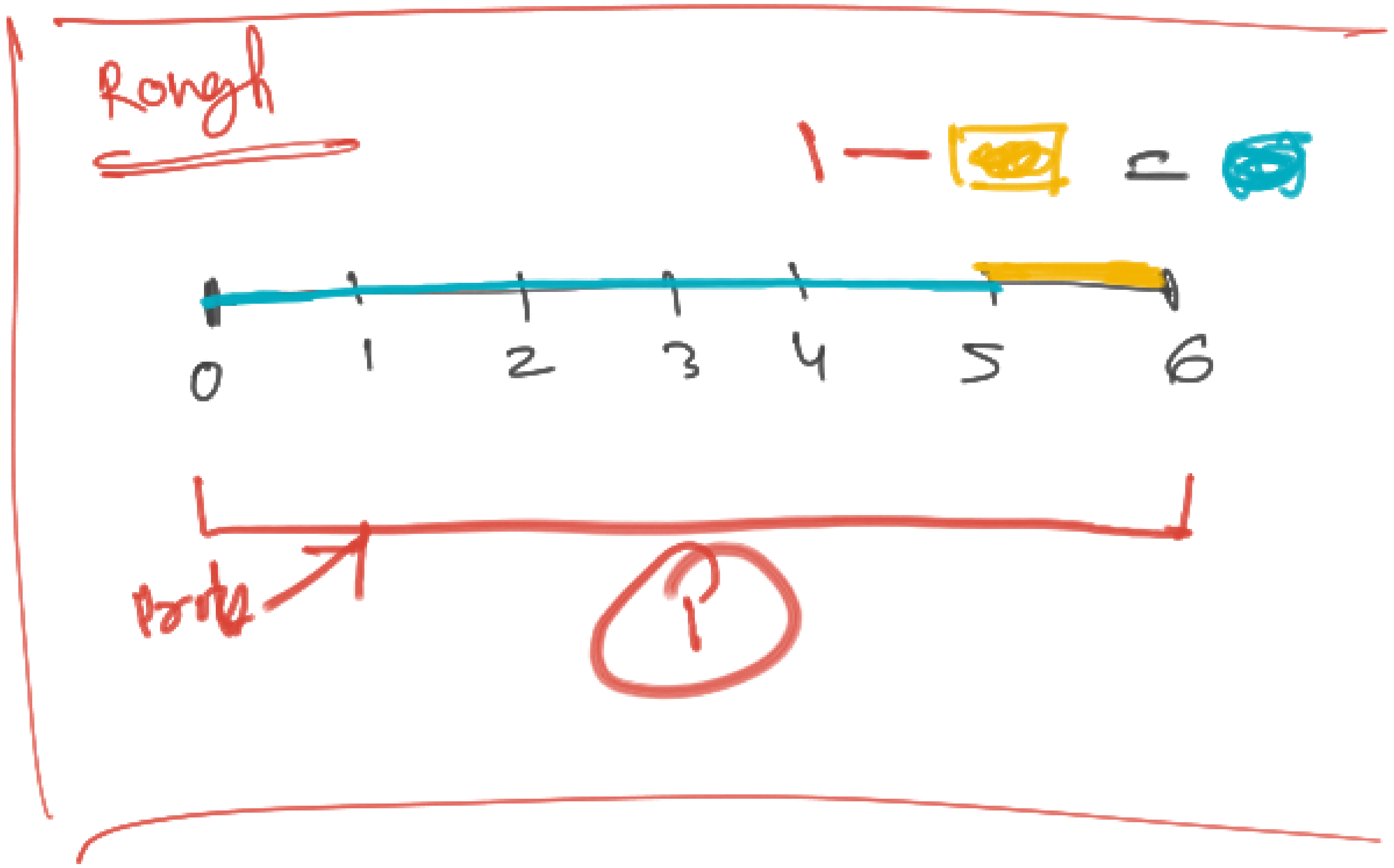
$\therefore \quad p = \frac{1}{3}$

$q = 1 - p = \frac{2}{3}$



(a) Given, $n = 6$

Let X = no. of times he hit the target

(i) $P(X \leq 5) = 1 - P(X > 5) = 1 - P(X = 6) = 1 - \left[{}^6C_6 \, (p)^6 (q)^{6-6}\right]$

$= 1 - \left[1 \cdot \left(\frac{1}{3}\right)^6 \left(\frac{2}{3}\right)^0\right] = 1 - \left(\frac{1}{3}\right)^6 = ?$

(ii) $P(X \geq 5) = P(X=5) + P(X=6)$

$$= \left[{}^6C_5 \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^{6-5}\right] + \left[{}^6C_6 \left(\frac{1}{3}\right)^6 \left(\frac{2}{3}\right)^6\right]$$

$$= ?$$

(iii) $P(X=1) = {}^6C_1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^{6-1} = ?$

(b)

A|q

$$P(X \geq 1) > \frac{3}{4}$$

$$\Rightarrow 1 - P(X < 1) > \frac{3}{4}$$

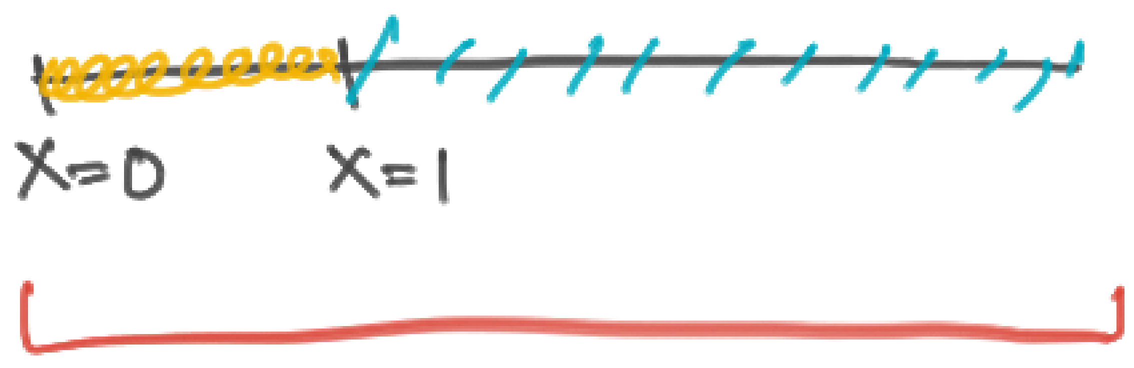
$$\Rightarrow 1 - P(X = 0) > \frac{3}{4}$$

$$\Rightarrow 1 - {}^{n}C_{0} \left(\frac{1}{3}\right)^{0} \left(\frac{2}{3}\right)^{n-0} > \frac{3}{4}$$

$$\Rightarrow 1 - \left(\frac{2}{3}\right)^{n} > \frac{3}{4}$$

$$\Rightarrow 1 - \frac{3}{4} > \left(\frac{2}{3}\right)^{n} \Rightarrow \left(\frac{2}{3}\right)^{n} < \frac{1}{4} \Rightarrow 2^{n} \cdot 4 < 3^{n}$$

$$\Rightarrow 2^{n} \cdot 2^{2} < 3^{n}$$



$$\Sigma P(X) = 1$$

P[illegible] = 1 − P[illegible]

$$P(X \geq 1) = 1 - P(X < 1)$$

$$\boxed{P(X = r) = {}^{n}C_{r}\, p^{r} q^{n-r}}$$

$\Rightarrow 2^{n+2} < 3^n$

The above inequality holds for $n = 4$

Hence, the man must fire 4 times so that the probability of hitting the target atleast once is greater than $\frac{3}{4}$

$n = 1$, $2^{1+2} = 2^3 = 8$

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not possible

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$16 \nless 9$ $\quad 3^2 = 9$

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$n = 3$,

$2^{3+2} = 2^5 = 32$

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$n = 4$, $2^{4+2} = 64$

$3^4 = 81$

$2^{4+2} < 3^4$ possible

POISSON DISTRIBUTION

French Mathematician — S.D. Poisson

Poisson Distribution — Discrete Probability Distribution

Characteristics

1. $n \to \infty$ i.e. sufficiently large (infinitely large
   $p \to 0$ i.e. sufficiently small (infinitely small

parameter, $\lambda = np$ ← finite no.

2. It consists of a single parameter i.e. λ.

In Poisson Distribution, Probability of 'r' success in a random experiment with total 'n' trials is given by

$$P(r) = \frac{e^{-\lambda} \lambda^{r}}{r!} \qquad \text{where} \quad \lambda = np \ \text{(parameter)}$$

p = probability of success

Conditions

① The R.V. must be discrete.

② A dichotomy exists i.e. the happening of the event must be of two alternatives — success and failure.

3. It is applicable in those cases where the number of trials (n) is very large and the probability of success (p) is very small, but mean (λ) must be finite.

Parameter
$\lambda = np$
↳ mean

Q. Prove that in case of Poisson Distribution

Mean $= \lambda$
Variance $= \lambda$.

Recurrence Formula

$$P(r) = \frac{e^{-\lambda} \lambda^r}{r!}$$

$$P(r+1) = \frac{\lambda}{r+1} P(r) \quad ; \quad r = 0, 1, 2, \ldots$$

Q. Suppose a book of 585 pages contains 43 typographical errors. If these errors are randomly distributed throughout the book, what is the probability that 10 pages, selected at random, will be free from errors? (Use $e^{-0.735} = 0.4795$)

↳ zero error

Soln:

Let X = no. of errors in 10 pages

$n = 10$

$p = \frac{43}{585} = 0.0735$

$\therefore \lambda = np = 0.0735 \times 10 = 0.735$

Here, n is very large in comparison to p

So we can use Poisson Distribution

We know that,

$$P(x=r) = \frac{e^{-\lambda} \lambda^r}{r!}$$

Probability of getting no error i.e. free from errors.

$$P(x=0) = \frac{e^{-\lambda} \lambda^0}{0!}$$

$$= \frac{e^{-0.735} \; 1}{1} = 0.4795$$

Q. If a random variable X follows a Poisson Distribution such $P(X=2) = 9P(X=4) + 90P(X=6)$, find the mean and variance of X.

Soln: For Poisson Distribution,

$$P(X=r) = \frac{e^{-\lambda}\lambda^r}{r!}$$

A/q

$$P(X=2) = 9P(X=4) + 90P(X=6)$$

$$\Rightarrow \frac{e^{-\lambda}\lambda^2}{2!} = 9\left[\frac{e^{-\lambda}\lambda^4}{4!}\right] + 90\left[\frac{e^{-\lambda}\lambda^6}{6!}\right]$$

$$\Rightarrow \frac{1}{2} = \frac{9\lambda^2}{4.3.2} + \frac{90\lambda^4}{6.5.4.3.2}$$

$$\Rightarrow 1 = \frac{3\lambda^2}{4} + \frac{\lambda^4}{4}$$

$$\Rightarrow \lambda^4 + 3\lambda^2 - 4 = 0$$

$$\Rightarrow \lambda^4 + 4\lambda^2 - \lambda^2 - 4 = 0$$

$$\Rightarrow \lambda^2(\lambda^2+4) - 1(\lambda^2+4) = 0$$

$$\Rightarrow (\lambda^2-1)(\lambda^2+4) = 0 \qquad \Rightarrow \lambda^2 - 1 = 0 \qquad (\because \lambda > 0$$
$$\Rightarrow \lambda = 1 \qquad \lambda^2+4 \neq 0)$$

For Poisson Distribution,

$$\text{Mean} = \lambda = 1$$

$$\text{Variance} = \lambda = 1.$$

Q. If a random variable has a Poisson distribution s.t. $P(1) = P(2)$, find

(a) mean of the distribution

(b) $P(4)$

Soln. For Poisson distribution

$$P(X=r) = \frac{e^{-\lambda} \lambda^{r}}{r!}$$

A/q

$$P(1) = P(2)$$

$$\Rightarrow P(X=1) = P(X=2)$$

$$\Rightarrow \frac{e^{-\lambda} \lambda^{1}}{1!} = \frac{e^{-\lambda} \lambda^{2}}{2!}$$

$$\Rightarrow 1 = \frac{\lambda}{2} \quad \Rightarrow \lambda = 2$$

(a) Mean $= \lambda = 2$

(b) $P(4) = P(X=4) = \dfrac{e^{-\lambda} \lambda^4}{4!}$

$$= \frac{e^{-2} (2)^4}{4!}$$

$= ?$

Q. A certain screw making machine on average produces 2 defective screws out of 100, and packs them in boxes of 500. Find the probability of getting 15 defective screws.

Soln

let X = no. of defective screws

$n = 500$, $p = \frac{2}{100} = 0.02$

$\therefore \lambda = np = ?$

Probability of getting 15 defective screws $= P(X=15) = \frac{e^{-\lambda}\lambda^{15}}{15!} = ?$

μ_X = mean wrt X

MOMENTS OF RANDOM VARIABLE

The moments of a random variable or its distribution are expected values of powers or related functions of the random variable.

General –

The rth moment of X is $\mu_r' = \sum x^r P(X=x) = E[X^r]$

The rth central moment of X is $\mu_r = E(X-\mu_X)^r$

In particular –

1st moment – $\mu_1' = \sum x P(X=x) = E[X]$ i.e. 1st moment = mean

2nd central moment – $\mu_2 = E(X-\mu_X)^2 = \sigma^2$ i.e. 2nd central moment = variance

Relation betn Moment & central Moment

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = \mu_2' - \mu_1'^2$$

$$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3 = \mu_3' - 3\mu_1'\mu_2' + 2\mu_1'^3$$

$$\mu_4 = \mu_4' + 6\mu_1'^2\mu_2' - 4\mu_1'\mu_2' - 3\mu_1'^4$$

Q. Let X be a R.V. have pmf

$$P_X(x) = \begin{cases} 1/2 , & x=1 \\ 1/3 , & x=2 \\ 1/6 , & x=3 \\ 0 ; & \text{otherwise} \end{cases}$$

find 3rd moment of X.

Soln: Third moment of $X = E[X^3]$

$$= \sum x^3 p(x)$$

$$= \left(1^3 \cdot \frac{1}{2}\right) + \left(2^3 \cdot \frac{1}{3}\right) + \left(3^3 \cdot \frac{1}{6}\right)$$

$$= 7.67$$

Q. Let X be a discrete random variable with probability mass function (pmf)

$$P_X(x) = \begin{cases} 3/4 & , \ x = 1 \\ 1/4 & , \ x = 2 \\ 0 & , \ \text{otherwise} \end{cases}$$

Find the third central moment of X.

Soln: Third central moment, $\mu_3 = E(X - \mu_X)^3$ — (1)

Now, $\mu_X = E[X] = \sum x p(x) = \left(1 \cdot \frac{3}{4}\right) + \left(2 \cdot \frac{1}{4}\right) = \frac{5}{4}$

$$\therefore \quad \mu_3 = E(X-\mu_X)^3 = \sum (x-\mu_X)^3\, p(x)$$

$$= \sum \left(x - \frac{5}{4}\right)^3 p(x)$$

$$= \left[\left(1-\frac{5}{4}\right)^3 \cdot \left(\frac{3}{4}\right)\right] + \left[\left(2-\frac{5}{4}\right)^3 \cdot \left(\frac{1}{4}\right)\right]$$

$$= \frac{3}{32}$$

MOMENT GENERATING FUNCTION (m.g.f.)

The moment generating function (m.g.f.) of a random variable X having the probability function $f(x)$ is given by

$$M_x(t) = E[e^{tx}] = \sum_x e^{tx} f(x) \longrightarrow \text{Discrete R.V.}$$

where t = real constant

$$\boxed{M_x(t) = \int e^{tx} f(x)\,dx \hookrightarrow \text{continuous R.V.}}$$

$$M_x(t) = E(e^{tx}) = E\left[1 + tX + \frac{t^2 X^2}{2!} + \dots + \frac{t^r X^r}{r!} + \dots\right]$$

$$= 1 + tE[X] + \frac{t^2}{2!}E[X^2] + \dots + \frac{t^r}{r!}E[X^r] + \dots$$

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \dots}$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \dots + \frac{t^r}{r!}\mu_r' + \dots \quad (*)$$

coefficient of $\frac{t^r}{r!}$ in the expansion $(*)$ is μ_r' i.e. rth moment of X about origin

$\therefore$ It generates moments

$\therefore$ It is called m.g.f.

$$\mu_r' = \frac{d^r}{dt^r}\left[M_x(t)\right]$$

m.g.f. of x about the point $x = a$ is

$$M_x(t) \text{ (about } x = a) = E\left[e^{t(x-a)}\right]$$

m.g.f of x about mean

$$M_x(t) \text{ (about mean)} = E\left[e^{t(x-\bar{x})}\right] \text{ or } E\left[e^{t(x-\mu)}\right]$$

Properties

(1) $M_{cx}(t) = M_x(ct)$ where $c =$ constant

(2) The moment generating function of the sum of a number of independent random variables is equal to the product of their respective moment generating function i.e.

if $x_1, x_2, \ldots, x_n$ are independent random variables then the moment generating function of their sum $x_1 + x_2 + \ldots + x_n$ is given by

$$M_{X_1+X_2+\dots+X_n}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \dots M_{X_n}(t).$$

EFFECT OF CHANGE OF ORIGIN AND SCALE ON M.G.F.

X be a R.V. with $M_X(t) = E[e^{tX}]$

let $U = \dfrac{X-a}{h}$ where a and h are constants

$\Rightarrow X = a + hU$

$$M_U(t) = e^{-at/h} M_X\left(\frac{t}{h}\right) \quad \text{or} \quad M_X(t) = e^{at} M_U(th)$$