Q. Show that the continuous random variable X having $f(x) = \begin{cases} \frac{1}{2}(x+1) & , -1<x<1 \\ 0 & , \text{elsewhere} \end{cases}$

represents density, find the mean and s.d. of X.

<u>Soln</u>: Given,

$$f(x) = \begin{cases} \frac{1}{2}(x+1) , & -1<x<1 \\ 0 , & \text{elsewhere} \end{cases}$$

Now,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{-1} f(x)\,dx + \int_{-1}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{-1} 0.\,dx + \int_{-1}^{1} \frac{1}{2}(x+1)\,dx + \int_{1}^{\infty} 0.\,dx$$

$$= 0 + \frac{1}{2}\int_{-1}^{1}(x+1)\,dx + 0$$

$$= \frac{1}{2}\left[\frac{x^2}{2} + x\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\left(\frac{1}{2} - \frac{1}{2}\right) + (1+1)\right]$$

$$= 1$$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1$$

Thus $f(x)$ represents density f^n.

Mean of the random variable is

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_{-\infty}^{-1} x f(x)\,dx + \int_{-1}^{1} x f(x)\,dx + \int_{1}^{\infty} x f(x)\,dx$$

$$= \int_{-\infty}^{-1} x.0.dx + \int_{-1}^{1} x.\frac{1}{2}(x+1)\,dx + \int_{1}^{\infty} x.0.dx$$

$$= 0 + \frac{1}{2}\int_{-1}^{1} (x^2+x)\,dx + 0$$

$$= \frac{1}{2}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\left(\frac{1}{3}+\frac{1}{3}\right) + \left(\frac{1}{2}-\frac{1}{2}\right)\right]$$

$$= \frac{1}{3}$$

$\therefore$ Mean, $E[X] = \frac{1}{3}$ ——— (1)

Again,

Variance, $\mathrm{var}(X) = E[X^2] - (E[X])^2$ ——— (4)

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_{-\infty}^{-1} x^2 \cdot f(x)\,dx + \int_{-1}^{1} x^2 \cdot f(x)\,dx + \int_{1}^{\infty} x^2 \cdot f(x)\,dx$$

$$= \int_{-\infty}^{-1} x^2 \cdot 0 \cdot dx + \int_{-1}^{1} x^2 \cdot \frac{1}{2}(x+1)\,dx + \int_{1}^{\infty} x^2 \cdot 0 \cdot dx$$

$$= 0 + \frac{1}{2}\int_{-1}^{1} (x^3 + x^2)\,dx + 0$$

$$= \frac{1}{2}\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\left(\frac{1}{4}-\frac{1}{4}\right)+\left(\frac{1}{3}+\frac{1}{3}\right)\right]$$

$$= \frac{1}{3}$$

$$\therefore \quad \operatorname{var}(X) = E[X^2] - \left(E[X]\right)^2$$

$$= \frac{1}{3} - \left(\frac{1}{3}\right)^2$$

$$= \frac{1}{3} - \frac{1}{9}$$

$$= \frac{3-1}{9}$$

$$\text{var}(X) = \frac{2}{9}$$

$$\therefore \quad S.D.(X) = +\sqrt{\text{var}(X)} = +\sqrt{2/9} = \frac{\sqrt{2}}{3}$$

Q. If the probability density f^n is given by

$$f(x) = \begin{cases} kx^3, & 0 < x \leq 3 \\ 0, & \text{elsewhere} \end{cases}$$

Find the value of 'k' and also the probability between $x = \frac{1}{2}$ and $x = \frac{3}{2}$

Soln: Given,

$$f(x) = \begin{cases} kx^3 & , 0 \le x \le 3 \\ 0 & , \text{elsewhere} \end{cases}$$

If $f(x)$ represents a density f^n

then $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} 0.\,dx + \int_{0}^{3} kx^3\,dx + \int_{3}^{\infty} 0.\,dx = 1$$

$$\Rightarrow 0 + k\int_0^3 x^3\,dx + 0 = 1$$

$$\Rightarrow k\left[\frac{x^4}{4}\right]_0^3 = 1$$

$$\Rightarrow k\left[\frac{3^4}{4} - 0\right] = 1$$

$$\Rightarrow k\left(\frac{81}{4}\right) = 1$$

$$\Rightarrow k = \frac{4}{81}$$

$$\therefore \quad f(x) = \begin{cases} \frac{4}{81}x^3, & 0 \leq x \leq 3 \\ 0, & \text{elsewhere} \end{cases}$$

Now,

$$P\left(\frac{1}{2} \leq X \leq \frac{3}{2}\right) = \int_{1/2}^{3/2} f(x)\,dx$$

$$= \int_{1/2}^{3/2} \frac{4}{81}x^3\,dx$$

$$= \frac{4}{81} \int_{1/2}^{3/2} x^3\,dx$$

$$= \frac{4}{81}\left[\frac{x^4}{4}\right]_{1/2}^{3/2}$$

$$= \frac{1}{81}\left[\left(\frac{3}{2}\right)^4 - \left(\frac{1}{2}\right)^4\right]$$

$$= \frac{1}{81}\left[\frac{81-1}{16}\right]$$

$$= \frac{1}{81}\left[\frac{80}{16}\right]$$

$$= \frac{5}{81}$$

Q. Is the f^n defined by

$$f(x) = \begin{cases} 0 , & x < 2 \\ \frac{3+2x}{18} , & 2 \le x \le 4 \\ 0 , & x > 4 \end{cases}$$

a probability density f^n? Find the probability that a variate having $f(x)$ as density f^n will fall in the interval $2 \le X \le 3$.

Q. A continuous random variable has the pdf

$$f(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Find the probabilities that it will take on a value

(i) betn 1 & 3

(ii) greater than 0.5

Soln: Given,

$$f(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

(i) $P(1 < X < 3) = \int_1^3 f(x)\,dx$

$$= \int_1^3 2e^{-2x}\,dx$$

$$= 2\int_1^3 e^{-2x}\,dx$$

$$= 2\left[\frac{e^{-2x}}{-2}\right]_1^3$$

$$= -\left[e^{-6} - e^{-2}\right]$$

$$= e^{-2} - e^{-6}$$

$$= 0.1338$$

(ii) $P(X > 0.5) = \int_{0.5}^{\infty} f(x)\,dx$

$$= \int_{0.5}^{\infty} 2e^{-2x}\,dx$$

$$= 2\left[\frac{e^{-2x}}{-2}\right]_{0.5}^{\infty}$$

$$= -\left[e^{-2x}\right]_{0.5}^{\infty}$$

$$= -\left(0 - e^{-1}\right)$$

$$= e^{-1}$$

$$= 0.3687$$

Q. Let $F(x)$ be the distribution function of a random variable X given by

$$F(x) = \begin{cases} cx^3, & 0 \leq x \leq 3 \\ 1, & x > 3 \\ 0, & \text{elsewhere} \end{cases}$$

If $P(X=3) = 0$ then determine

(i) c

(ii) mean

(iii) $P(X>1)$

$\underline{Sol^n}$ Given,

$$f(x) = \begin{cases} cx^3 & , \quad 0 \le x \le 3 \\ 1 & , \quad x > 3 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

Now,

$$\because f(x) = \frac{d}{dx} F(x)$$

$$= \begin{cases} 3cx^2 & , \quad 0 \le x \le 3 \\ 0 & , \quad x > 3 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

$$= \begin{cases} 3cx^2 & , \quad 0 \le x \le 3 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

$\because$ $f(x)$ is a density f^n

$\therefore$ $$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} 0 . dx + \int_{0}^{3} 3cx^2\,dx + \int_{3}^{\infty} 0 . dx = 1$$

$$\Rightarrow 0 + 3c\left[\frac{x^3}{3}\right]_0^3 + 0 = 1$$

$$\Rightarrow c\,(27-0) = 1$$

$\therefore \quad c = \frac{1}{27}$

Now,

$$f(x) = \begin{cases} 3 \cdot \frac{1}{27} \cdot x^2 & , 0 \le x \le 3 \\ 0 & , \text{elsewhere} \end{cases}$$

$$= \begin{cases} x^2/9 & , 0 \le x \le 3 \\ 0 & , \text{elsewhere} \end{cases}$$

(ii) Mean, $E[X] = \int_{-\infty}^{\infty} x f(x)\,dx$

$$= \int_{-\infty}^{0} x f(x)\,dx + \int_{0}^{3} x f(x)\,dx + \int_{3}^{\infty} x f(x)\,dx$$

$$= \int_{-\infty}^{0} x \cdot 0 \cdot dx + \int_{0}^{3} x \cdot \frac{x^2}{9} dx + \int_{3}^{\infty} x \cdot 0 \cdot dx$$

$$= 0 + \frac{1}{9} \int_{0}^{3} x^3 dx + 0$$

$$= \frac{1}{9} \left[\frac{x^4}{4} \right]_0^3$$

$$= \frac{1}{9} \left[\frac{81}{4} - 0 \right]$$

$$= \frac{9}{4}$$

Calculate $P(X > 1)$