5.3 AXIOMS AND LAWS OF BOOLEAN ALGEBRA

Axioms or *postulates* of Boolean algebra are a set of logical expressions that we accept without proof and upon which we can build a set of useful theorems. Actually, axioms are nothing more than the definitions of the three basic logic operations that we have already discussed: AND, OR, and INVERT. Each axiom can be interpreted as the outcome of an operation performed by a logic gate.

AND operation	OR operation	NOT operation
Axiom 1: $0 \cdot 0 = 0$	Axiom 5: $0 + 0 = 0$	Axiom 9: $\overline{1} = 0$
Axiom 2: $0 \cdot 1 = 0$	Axiom 6: $0 + 1 = 1$	Axiom 10: $\overline{0} = 1$
Axiom 3: $1 \cdot 0 = 0$	Axiom 7: $1 + 0 = 1$	
Axiom 4: $1 \cdot 1 = 1$	Axiom 8: $1 + 1 = 1$	

5.3.1 Complementation Laws

The term *complement* simply means to invert, i.e. to change 0s to 1s and 1s to 0s. The five laws of complementation are as follows:

Law 1: $\overline{0} = 1$

Law 2: $\overline{1} = 0$

Law 3: If $A = 0$, then $\overline{A} = 1$

Law 4: If $A = 1$, then $\overline{A} = 0$

Law 5: $\overline{\overline{A}} = A$ (double complementation law)

Notice that the double complementation does not change the function.

5.3.2 AND Laws

The four AND laws are as follows:

Law 1: $A \cdot 0 = 0$ (Null law)

Law 2: $A \cdot 1 = A$ (Identity law)

Law 3: $A \cdot A = A$

Law 4: $A \cdot \overline{A} = 0$

5.3.3 OR Laws

The four OR laws are as follows:

Law 1: $A + 0 = A$ (Null law)

Law 2: $A + 1 = 1$ (Identity law)

Law 3: $A + A = A$

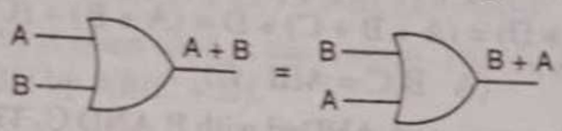
Law 4: $A + \overline{A} = 1$

5.3.4 Commutative Laws

Commutative laws allow change in position of AND or OR variables. There are two commutative laws.

Law 1: $A + B = B + A$

This law states that, A OR B is the same as B OR A, i.e. the order in which the variables are ORed is immaterial. This means that it makes no difference which input of an OR gate is connected to A and which to B. We give below the truth tables illustrating this law.



A	B	A + B
0	0	0
0	1	1
1	0	1
1	1	1

=

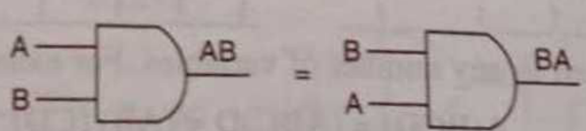
B	A	B + A
0	0	0
0	1	1
1	0	1
1	1	1

This law can be extended to any number of variables. For example,

$$A + B + C = B + C + A = C + A + B = B + A + C$$

Law 2: $A \cdot B = B \cdot A$

This law states that A AND B is the same as B AND A, i.e. the order in which the variables are ANDed is immaterial. This means that it makes no difference which input of an AND gate is connected to A and which to B. The truth tables given below illustrate this law.



A	B	A · B
0	0	0
0	1	0
1	0	0
1	1	1

=

B	A	B · A
0	0	0
0	1	0
1	0	0
1	1	1

This law can be extended to any number of variables. For example,

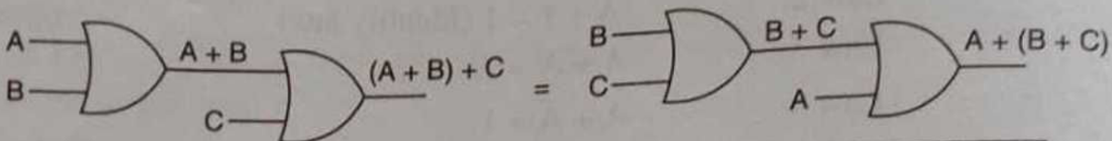
$$A \cdot B \cdot C = B \cdot C \cdot A = C \cdot A \cdot B = B \cdot A \cdot C$$

5.3.5 Associative Laws

The associative laws allow grouping of variables. There are two associative laws.

Law 1: $(A + B) + C = A + (B + C)$

A OR B ORed with C is the same as A ORed with B OR C. This law states that the way the variables are grouped and ORed is immaterial. The truth tables given next illustrate this law.



A	B	C	A + B	(A + B) + C
0	0	0	0	0
0	0	1	0	1
0	1	0	1	1
0	1	1	1	1
1	0	0	1	1
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

=

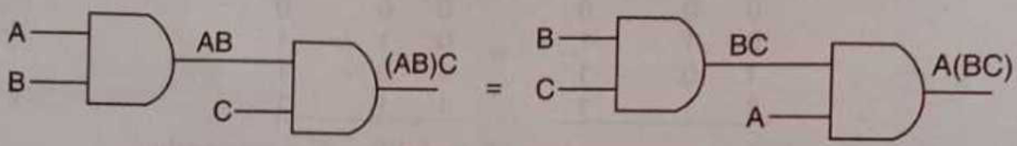
A	B	C	B + C	A + (B + C)
0	0	0	0	0
0	0	1	1	1
0	1	0	1	1
0	1	1	1	1
1	0	0	0	1
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

This law can be extended to any number of variables. For example,

$$A + (B + C + D) = (A + B + C) + D = (A + B) + (C + D).$$

Law 2: $(A \cdot B)C = A(B \cdot C)$

A AND B ANDed with C is the same as A ANDed with B AND C. This law states that the way the variables are grouped and ANDed is immaterial. See the truth tables below:



A	B	C	AB	(AB)C
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	0	0
1	0	0	0	0
1	0	1	0	0
1	1	0	1	0
1	1	1	1	1

=

A	B	C	BC	A(BC)
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	1	0
1	0	0	0	0
1	0	1	0	0
1	1	0	0	0
1	1	1	1	1

This law can be extended to any number of variables. For example,

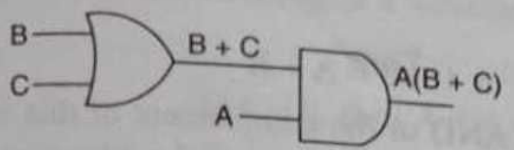
$$A(BCD) = (ABC)D = (AB)(CD)$$

5.3.6 Distributive Laws

The distributive laws allow factoring or multiplying out of expressions. There are two distributive laws.

Law 1: $A(B + C) = AB + AC$

This law states that ORing of several variables and ANDing the result with a single variable s equivalent to ANDing that single variable with each of the several variables and then ORing the roducts. The truth table given below illustrates this law.



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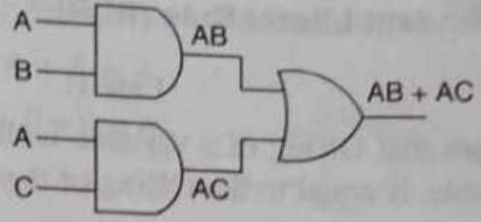


A	B	C	B + C	A(B + C)
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	1	0
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

=

A	B	C	AB	AC	AB + AC
0	0	0	0	0	0
0	0	1	0	0	0
0	1	0	0	0	0
0	1	1	0	0	0
1	0	0	0	0	0
1	0	1	0	1	1
1	1	0	1	0	1
1	1	1	1	1	1

This law applies to single variables as well as combinations of variables. For example,

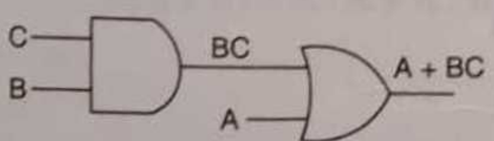
$$ABC(D + E) = ABCD + ABCE$$
$$AB(CD + EF) = ABCD + ABEF$$

The distributive property is often used in the reverse. That is, given AB + AC, we replace it by A(B + C); and ABC + ABD by AB(C + D).

Law 2: $$A + BC = (A + B)(A + C)$$

This law states that ANDing of several variables and ORing the result with a single variable is equivalent to ORing that single variable with each of the several variables and then ANDing the sums. This can be proved algebraically as shown below. Also, the truth tables given next illustrate this law.

$$\begin{aligned} RHS &= (A + B)(A + C) \\ &= AA + AC + BA + BC \\ &= A + AC + AB + BC \\ &= A(1 + C + B) + BC \\ &= A \cdot 1 + BC \qquad\qquad (\because \quad 1 + C + B = 1 + B = 1) \\ &= A + BC \\ &= LHS \end{aligned}$$



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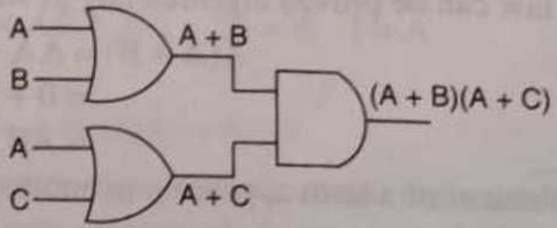


A	B	C	BC	A + BC
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	1	1
1	0	0	0	1
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

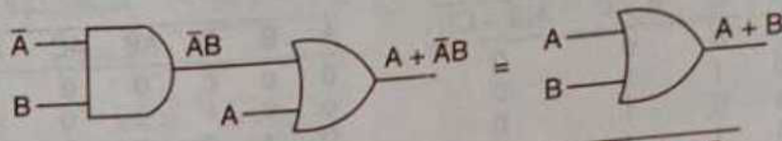
=

A	B	C	A + B	A + C	(A + B)(A + C)
0	0	0	0	0	0
0	0	1	0	1	0
0	1	0	1	0	0
0	1	1	1	1	1
1	0	0	1	1	1
1	0	1	1	1	1
1	1	0	1	1	1
1	1	1	1	1	1

5.3.7 Redundant Literal Rule (RLR)

Law 1: $A + \bar{A}B = A + B$

This law states that ORing of a variable with the AND of the complement of that variable with another variable, is equal to the ORing of the two variables. See the truth tables given below.



A	B	$\bar{A}B$	$A + \bar{A}B$
0	0	0	0
0	1	1	1
1	0	0	1
1	1	0	1

=

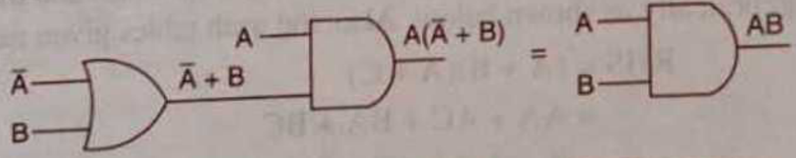
A	B	A + B
0	0	0
0	1	1
1	0	1
1	1	1

This law can be proved algebraically as shown below.

$$\begin{aligned} A + \bar{A}B &= (A + \bar{A})(A + B) \\ &= 1 \cdot (A + B) \\ &= A + B \end{aligned}$$

Law 2: $A(\bar{A} + B) = AB$

This law states that ANDing of a variable with the OR of the complement of that variable with another variable, is equal to the ANDing of the two variables. See the truth tables given below.



A	B	$\bar{A} + B$	$A(\bar{A} + B)$
0	0	1	0
0	1	1	0
1	0	0	0
1	1	1	1

=

A	B	AB
0	0	0
0	1	0
1	0	0
1	1	1

This law can be proved algebraically as shown below.

$$\begin{aligned} A(\bar{A} + B) &= A\bar{A} + AB \\ &= 0 + AB \\ &= AB \end{aligned}$$

Complement of a term appearing in another term is redundant.

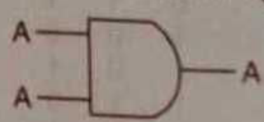
5.3.8 Idempotence Laws

Law 1: $A \cdot A = A$

Idempotence means the same value. We are already familiar with the following laws:

If $A = 0$, then $A \cdot A = 0 \cdot 0 = 0 = A$

If $A = 1$, then $A \cdot A = 1 \cdot 1 = 1 = A$

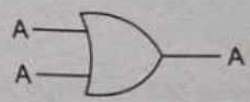


This law states that ANDing of a variable with itself is equal to that variable only.

Law 2: $A + A = A$

If $A = 0$, then $A + A = 0 + 0 = 0 = A$

If $A = 1$, then $A + A = 1 + 1 = 1 = A$



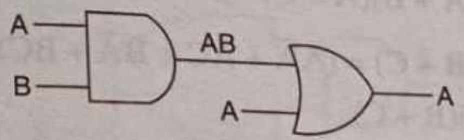
This law states that ORing of a variable with itself is equal to that variable only.

5.3.9 Absorption Laws

There are two laws:

Law 1: $A + A \cdot B = A$

This law states that ORing of a variable (A) with the AND of that variable (A) and another variable (B) is equal to that variable itself (A).



A	B	AB	A + AB
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	1

Algebraically, we have

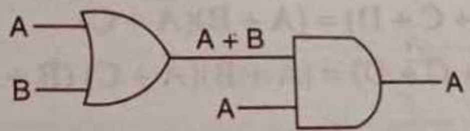
$$A + A \cdot B = A(1 + B) = A \cdot 1 = A$$

Therefore,

$$A + A \cdot \text{Any term} = A$$

Law 2: $A(A + B) = A$

This law states that ANDing of a variable (A) with the OR of that variable (A) and another variable (B) is equal to that variable itself (A).



A	B	A + B	A(A + B)
0	0	0	0
0	1	1	0
1	0	1	1
1	1	1	1

Algebraically, we have

$$A(A + B) = A \cdot A + A \cdot B = A + AB = A(1 + B) = A \cdot 1 = A$$

Therefore,

$$A(A + \text{Any term}) = A$$

If a term appears in toto in another term, then the latter term becomes redundant and may be removed from the expression without changing its value. Removal of a term is equivalent to replacing that term by 0 if it is in a sum or by 1 if it is in a product.

5.3.10 Consensus Theorem (Included Factor Theorem)

Theorem 1: $AB + \bar{A}C + BC = AB + \bar{A}C$

Proof:

$$\begin{aligned}
\text{LHS} &= AB + \bar{A}C + BC \\
&= AB + \bar{A}C + BC(A + \bar{A}) \\
&= AB + \bar{A}C + BCA + BC\bar{A} \\
&= AB(1 + C) + \bar{A}C(1 + B) \\
&= AB(1) + \bar{A}C(1) \\
&= AB + \bar{A}C \\
&= \text{RHS}
\end{aligned}$$

This theorem can be extended to any number of variables. For example,

$$AB + \bar{A}C + BCD = AB + \bar{A}C$$

$$\text{LHS} = AB + \bar{A}C + BCD = AB + \bar{A}C + BC + BCD = AB + \bar{A}C + BC = AB + \bar{A}C = \text{RHS}$$

Theorem 2: $(A + B)(\bar{A} + C)(B + C) = (A + B)(\bar{A} + C)$

Proof:

$$\begin{aligned}
\text{LHS} &= (A + B)(\bar{A} + C)(B + C) = (A\bar{A} + AC + B\bar{A} + BC)(B + C) \\
&= (AC + BC + \bar{A}B)(B + C) \\
&= ABC + BC + \bar{A}B + AC + BC + \bar{A}BC = AC + BC + \bar{A}B \\
\text{RHS} &= (A + B)\,(\bar{A} + C) \\
&= A\bar{A} + AC + BC + \bar{A}B \\
&= AC + BC + \bar{A}B = \text{LHS}
\end{aligned}$$

If a sum of products comprises a term containing A and a term containing $\bar{A}$, and a third term containing the left-out literals of the first two terms, then the third term is redundant, that is, the function remains the same with and without the third term removed or retained.

This theorem can be extended to any number of variables. For example,

$$(A + B)(\bar{A} + C)(B + C + D) = (A + B)(\bar{A} + C)$$

$$\begin{aligned}
\text{LHS} &= (A + B)(\bar{A} + C)(B + C)(B + C + D) = (A + B)(\bar{A} + C)\,(B + C) \\
&= (A + B)(\bar{A} + C)
\end{aligned}$$

5.3.11 Transposition Theorem

Theorem: $AB + \bar{A}C = (A + C)(\bar{A} + B)$

Proof:

$$\begin{aligned}
\text{RHS} &= (A + C)(\bar{A} + B) \\
&= A\bar{A} + C\bar{A} + AB + CB \\
&= 0 + \bar{A}C + AB + BC \\
&= \bar{A}C + AB + BC(A + \bar{A}) \\
&= AB + ABC + \bar{A}C + \bar{A}BC \\
&= AB + \bar{A}C \\
&= \text{LHS}
\end{aligned}$$