# Moment of Random Variable

The moments of a random variable (or its distribution) are expected values of powers or related $f^n$ of the random variable.

The $r$th moment of $X$ is $\mu_r' = E(X^r)$

$= \sum x^r P(X = x)$

$= \sum x^r p(x)$

1st moment $\longrightarrow$ $\mu_1' = E(X^1) = E(X)$

$= \sum x\, p(x)$

## Moment Generating Function

The moment generating f$^n$ (m.g.f.) of a R.V. X having the probability f$^n$ $f(x)$ is given by

$$M_x(t) = E(e^{tX})$$

$$= \sum_x e^{tx} f(x) \quad \longleftarrow \quad \text{Discrete R.V.}$$

$$= \int e^{tx} f(x)\,dx \quad \longleftarrow \quad \text{cont. R.V.}$$

Here $t$ is real const.

$$M_X(t) = E(e^{tX}) = E\left[1 + tX + \frac{(tX)^2}{2!} + \dots + \frac{(tX)^r}{r!} + \dots\right)$$

$$= E\left[1 + tX + \frac{t^2X^2}{2!} + \dots + \frac{t^rX^r}{r!} + \dots\right)$$

$$= 1 + tE[X] + \frac{t^2}{2!}E[X^2] + \dots + \frac{t^r}{r!}E[X^r) + \dots$$

$$1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \dots + \frac{t^r}{r!}\mu_r' + \dots$$

Coefficient of $\frac{t^r}{r!}$ will give $r^{th}$ moment about the origin.

Note ① $\frac{d^r}{dt^r}[M_X(t)] = \mu_r'$

3rd moment, $r = 3$

$\frac{d^3}{dt^3}(M_X(t)) = \mu_3'$ ← 3rd moment

Moment generating $f^n$ of X about the point $x = a$

$$M_X(t) \text{ (about } x = a) = E\left[e^{t(X-a)}\right]$$

$$M_X(t) \text{ (about mean)} = E\left[e^{t(X-\bar{X})}\right]$$

Here, $\bar{X}$ = mean

## Properties

(i) $M_{cX}(t) = M_X(ct)$

(2) $M_{X_1+X_2+\cdots+X_n}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_n}(t)$

Q. Find m.g.f. of a random variable whose moments are

$$\mu_r' = (r+1)!\, 2^r$$

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r' = \sum_{r=0}^{\infty} \frac{t^r}{r!} (r+1)!\, 2^r$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} (r+1) \cdot r!\, 2^r$$

$$= \sum_{r=0}^{\infty} t^r (r+1)\, 2^r = \sum_{r=0}^{\infty} (r+1)(2t)^r$$

$$M_X(t) = \left[(0+1)(2t)^0\right] + \left[(1+1)(2t)^1\right] + \left[(2+1)(2t)^2\right] + \cdots$$

$$= 1 + 2\cdot(2t) + 3(2t)^2 + \cdots$$

$$= (1-2t)^{-2}$$

2nd moment $= \mu_2' = E[X^2] = \sum x^2 p(x)$

3rd moment $= \mu_3' = E[X^3] = \sum x^3 p(x)$

$\rightarrow$ $r^{th}$ central moment $= \mu_r = E[X - \mu_x]^r$

2nd central moment $= \mu_2 = E(X - \mu_x)^2$

3rd " " $= \mu_3 = E[X - \mu_x]^3$

Q. The 2nd central moment is : (a) mean (b) variance (c) S.D. (d) none

The $r^{th}$ central moment of X is $\mu_r = E(X - \mu_x)^r$

Q. Let X be a discrete random variable having probability mass $f^n$

$$P_X(x) = \begin{cases} 1/2 & , x = 1 \\ 1/3 & , x = 2 \\ 1/6 & , x = 3 \\ 0 & , \text{otherwise} \end{cases}$$

Find 3rd moment of X

3rd moment is given by $\mu_3' = E[X^3]$

$$= \sum x^3 p(x)$$

$$= \left\{(1)^3 \times \left(\frac{1}{2}\right)\right\} + \left\{(2)^3 \times \frac{1}{3}\right\} + \left\{3^3 \times \frac{1}{6}\right\}$$

$$= 23/3$$

Let X be a discrete R.V. with p.m.f.

$$P_X(x) = \begin{cases} 3/4 & , \quad x=1 \\ 1/4 & , \quad x=2 \\ 0 & , \quad \text{otherwise} \end{cases}$$

Find the 3rd central moment of X.

3rd central moment is given by $\mu_3 = E[X-\mu_x]^3$

Now,

$$\mu_x = E[X] = \Sigma x\, p(x) = \left(1\times\frac{3}{4}\right) + \left(2\times\frac{1}{4}\right) + 0$$

$$= \frac{3}{4} + \frac{2}{4} = \frac{5}{4}$$

$$\mu_3 = E\left[X - \frac{5}{4}\right]^3 = \sum \left(x - \frac{5}{4}\right)^3 p(x)$$

$$= \left(1 - \frac{5}{4}\right)^3 \left(\frac{3}{4}\right) + \left(2 - \frac{5}{4}\right)^3 \left(\frac{1}{4}\right)$$

$$= \frac{3}{32}$$

Q. Show that mgf of a R.V. X having the probability density $f^n$

$$f(x) = \begin{cases} 1/3 & , -1 < x < 2 \\ 0 & , \text{elsewhere} \end{cases}$$

is

$$M_X(t) = \begin{cases} \dfrac{e^{2t} - e^{-t}}{3t} & , t \neq 0 \\ 1 & , t = 0 \end{cases}$$

$$M_X(t) = E(e^{tX})$$

$$= \int_{-1}^{2} e^{tx} f(x)\,dx$$

$$= \int_{-1}^{2} e^{tx} \left(\frac{1}{3}\right) dx \qquad 1$$

$$= \frac{1}{3} \int_{-1}^{2} e^{tx}\,dx$$

$$= \frac{1}{3} \left[ \frac{e^{tx}}{t} \right]_{-1}^{2} = \frac{1}{3t} \left( e^{2t} - e^{-t} \right) \quad ; t \neq 0$$

Sub. $t=0$ in (1) we get

$$M_X(t) = \int_{-1}^{2} e^{0 \cdot x} \left(\tfrac{1}{3}\right) dx$$

$$= \frac{1}{3}\int_{-1}^{2} dx$$

$$= \frac{1}{3}\left[x\right]_{-1}^{2}$$

$$= \frac{1}{3}\left[2-(-1)\right]$$

$$= \frac{3}{3} = 1$$

$$\therefore M_X(t) = \begin{cases} \dfrac{e^{2t}-e^{-t}}{3t}, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

Q. Find the mgf of the R.V. X having the probability density $f^n$

$$f(x) = \begin{cases} x & , 0 \le x < 1 \\ 2-x & , 1 \le x < 2 \\ 0 & , \text{otherwise} \end{cases}$$

Also find the mean and variance of X using mgf.

$$M_X(t) = E[e^{tX}]$$

$$= \int_0^1 e^{tx} f(x)\,dx + \int_1^2 e^{tx} f(x)\,dx$$

$$M_X(t) = \left[ \frac{e^{tx} \cdot x}{t} \right]_0^1 - \int_0^1 \frac{e^{tx}}{t}(1)\,dx + \left[ \frac{e^{tx}}{t}(2-x) \right]_1^2 - \int_1^2 \frac{e^{tx}}{t}(-1)\,dx$$

$$= \frac{e^t}{t} - \left[ \frac{e^{tx}}{t^2} \right]_0^1 - \frac{e^t}{t} + \left[ \frac{e^{tx}}{t^2} \right]_1^2$$

$$= \frac{e^t}{t} - \left[ \frac{e^t}{t^2} - \frac{1}{t^2} \right] - \frac{e^t}{t} + \left[ \frac{e^{2t}}{t^2} - \frac{e^t}{t^2} \right]$$

$$= \frac{e^{2t}}{t^2} - \frac{2e^t}{t^2} + \frac{1}{t^2}$$

1

$$= \frac{1}{t^2}\left(e^{2t} - 2e^t + 1\right) = \frac{(e^t-1)^2}{t^2}$$

11

Expanding $M_X(t)$ in (1) we get

$$M_X(t) = \frac{1}{t^2}\left[\left(1+2t+\frac{(2t)^2}{2!}+\frac{(2t)^3}{3!}+\cdots\right)-2\left(1+t+\frac{t^2}{2!}+\cdots\right)+1\right]$$

$$\frac{1}{t^2}\left(t^2+t^3+\frac{7}{12}t^4+\cdots\right)$$

$$= 1+t+\frac{7}{12}t^2+\cdots \quad \text{(iii)}$$

Mean, $\mu_1' =$ coefficient of $t$ in $M_X(t)$

$= 1$ (From (iii))

$\mu_2' =$ coefficient of $\frac{t^2}{2!}$ in $M_X(t)$ $= \frac{7}{12} \times 2! = \frac{7}{6}$

$$\text{Variance}\,(\mu_2) = \mu_2' - (\mu_1')^2 = \frac{7}{6} - (1)^2 = \frac{1}{6}$$