Q. Consider a sample of size 2 drawn without replacement from an urn containing three balls, numbered 1, 2 and 3. Let X be the smaller of the two numbers drawn and Y the larger

(a) Find the joint discrete density function X and Y.

(b) Find the conditional distribution of Y given $X=1$.

(c) Find $\rho(X,Y)$

Soln: Here,

X = smaller of the two numbers drawn

Y = larger of the two numbers drawn.

Possible outcomes are (1,2), (1,3), (2,1), (2,3), (3,1), (3,2)

($\because$ Replacement is not allowed)

A/q X is smaller among the two no.s and Y is larger

$\therefore$ possible values of (X, Y) are (1,2), (1,3), (2,3)

→ Total no. of outcomes = 3

(a) The joint discrete density function of X and Y is given below:

① ② ③ urn

(1,2), (1,3), (2,1), (2,3)
(3,1), (3,2)

↑ Replacement not allowed

X smaller than Y

$x = 1$ — 2
 — 3

$x = 2$ — 3

$x = 3$ — no possible values

$X \backslash Y$	2	3	$f_X(x)$
1	$1/3$	$1/3$	$2/3$
2	0	$1/3$	$1/3$
$f_Y(y)$	$1/3$	$2/3$	1

$(1,2)$, $(1,3)$
$(2,3)$

(3) Conditional distrebution of Y given $X=1$

$$f_{Y/X}(Y=y \mid X=1) = \frac{f(1,y)}{f_X(1)} = \frac{f(1,y)}{2/3} = \frac{3}{2} f(1,y)$$

$$f_{Y/X}(Y=y \mid X=1) = \begin{cases} \frac{3}{2} f(1,2) & ; \text{ when } y=2 \\ \frac{3}{2} f(1,3) & ; \text{ when } y=3 \end{cases}$$

$$= \begin{cases} \frac{3}{2}\left(\frac{1}{3}\right) & ; \quad y=2 \\ \frac{3}{2}\left(\frac{1}{3}\right) & ; \quad y=3 \end{cases}$$

$$= \begin{cases} \frac{1}{2} & ; \quad y=2 \\ \frac{1}{2} & ; \quad y=3 \end{cases}$$

(c) $E[X] = \sum x f_X(x) = \left(1 \times \frac{2}{3}\right) + \left(2 \times \frac{1}{3}\right) = \frac{4}{3}$

$E[Y] = \sum y f_Y(y) = \left(2 \times \frac{1}{3}\right) + \left(3 \times \frac{2}{3}\right) = \frac{8}{3}$

$E[XY] = \sum xy f_{X,Y}(x,y)$

$= \left[1 \times 2 \times \frac{1}{3}\right] + \left[1 \times 3 \times \frac{1}{3}\right] + \left[2 \times 2 \times 0\right] + \left[2 \times 3 \times \frac{1}{3}\right]$

$= \frac{11}{3}$

$E[X^2] = \sum x^2 f_X(x) = \left(1^2 \times \frac{2}{3}\right) + \left(2^2 \times \frac{1}{3}\right) = \frac{6}{3} = 2$

$E[Y^2] = \sum y^2 f_Y(y) = \left(2^2 \times \frac{1}{3}\right) + \left(3^2 \times \frac{2}{3}\right) = \frac{22}{3}$

Now,

$$\mathrm{var}(X) = E[X^2] - (E[X])^2 = 2 - \left(\frac{4}{3}\right)^2 = 2 - \frac{16}{9}$$

$$= \frac{2}{9}$$

$$\mathrm{var}(Y) = E[Y^2] - (E[Y])^2 = \frac{22}{3} - \left(\frac{8}{3}\right)^2$$

$$= \frac{22}{3} - \frac{64}{9}$$

$$= \frac{66-64}{9}$$

$$= \frac{2}{9}$$

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y] = \frac{11}{3} - \left(\frac{4}{3} \times \frac{8}{3}\right) = \frac{33-32}{9} = \frac{1}{9}$$

$$\rho(X,Y) = \frac{\text{cov}(X,Y)}{\sqrt{\text{var}(X)}\sqrt{\text{var}(Y)}} = \frac{1/9}{\sqrt{2/9}\sqrt{2/9}}$$

$$= \frac{1/9}{2/9}$$

$$= \frac{1}{2}$$

Q. X and Y are two random variable having joint density function $= \frac{1}{27}(2x+y)$, where x and y can assume only integer values 0, 1 and 2. Find conditional distribution of Y

for $X = x$.