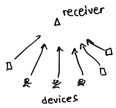
Distributed Power Control and Optimal Denoising for Over the Air Computation Networks

Devices: $\mathcal{N} = \{1, \dots, n, \dots, N\}$

Sensing Reading: X_n



Desired Average at the receiver:

$$\boxed{\tilde{f} = \frac{1}{N} \sum_{n \in \mathcal{N}} X_n \quad (1)}$$

X_n is normalised as $s_n \triangleq \psi(X_n)$, $\forall n \in \mathcal{N}$.
Linear function $\psi(\cdot)$ is the normalization operator to ensure that s_n has zero mean and unit variance.

Received signal at the receiver:

$$\boxed{f = \frac{1}{N} \sum_{n \in \mathcal{N}} s_n \quad (2)}$$

Transmit coefficient of node n: b_n

Received signal at the receiver:

$$\boxed{y = \sum_{n \in \mathcal{N}} h_n b_n s_n + w \quad (3)}$$

h_n: channel gain of the communication link among the node n and the receiver

w: Additive White Gaussian Noise (AWGN) with $w \sim \mathcal{CN}(0, \sigma^2)$

Transmit coefficient to achieve signal alignment:

$$b_n = \frac{\sqrt{P_n}\, h_n^{+}}{|h_n|}$$

$P_n[W]$: transmission power of node n

$$(3) \Rightarrow \boxed{y = \sum_{n \in \mathcal{N}} |h_n| \sqrt{P_n}\, s_n + w \quad (4)}$$

The receiver applies a denoising factor $\eta > 0$ to the received signal, thus, the scaled signal is:

$$\boxed{\hat{f} = \frac{\mathrm{Re}(y)}{N \cdot \sqrt{\eta}} \quad (5)}$$

Our goal is to minimize the distortion of the recovered signal

$$MSE(\vec{P}, \eta) = E\left[(\hat{f} - f)^2\right] \overset{(2)}{\underset{(5)}{=}} \frac{1}{N^2} E\left[\frac{\mathrm{Re}(y)}{N\sqrt{\eta}} - \sum_{n \in \mathcal{N}} s_n\right]$$

$$\overset{(4)}{=} \frac{1}{N^2}\left[\sum_{n \in \mathcal{N}} \left(\frac{|h_n|\sqrt{P_n}}{\sqrt{\eta}} - 1\right)^2 + \frac{\sigma^2}{\eta}\right]$$

$$\Rightarrow \boxed{MSE(\vec{P}, \eta) = \frac{1}{N^2}\left[\sum_{n \in \mathcal{N}} \left(\frac{|h_n|\sqrt{P_n}}{\sqrt{\eta}} - 1\right)^2 + \frac{\sigma^2}{\eta}\right]} \quad (6)$$

↳ Observations:

1) η: "centralized" variable (controlled by the receiver)
2) $\vec{P}$: "distributed" variable (controlled by $\forall$ node)

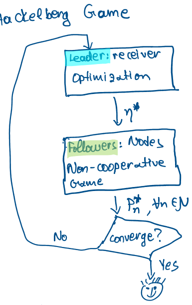
Goal:

$$\boxed{\begin{aligned} &\min_{\vec{P}, \eta} \; MSE(\vec{P}, \eta) \\ &\text{s.t. } \; \eta > 0 \\ &\qquad P_n \le P_n^{Max}, \; \forall n \in \mathcal{N} \end{aligned}} \quad (7)$$

→ centralized power control drawbacks:
1) single point of failure
2) security and privacy
3) nodes can be owned by ≠ providers

Address it as:

Stackelberg Game



Leader: Given $\vec{P}^*$ determine η^*

$$\min_{\eta > 0} \frac{1}{N^2}\left[\sum_{n \in \mathcal{N}} \left(\frac{|h_n|\sqrt{P_n}}{\sqrt{\eta}} - 1\right)^2 + \frac{\sigma^2}{\eta}\right]$$

Proposition 1: The optimal denoising factor of the receiver for any given $\vec{P} = [P_1, \dots, P_n, \dots, P_N]$, $n \in \mathcal{N}$ is given by:

$$\eta^* = \left[\sum_{n \in \mathcal{N}} \left(|h_n|^2 P_n + \frac{2\sigma^2}{N}\right) \dots\right]^2 \quad (8)$$

Proposition 1: The optimal denoising factor ... $\vec{P} = [P_1, \dots, P_n, \dots, P_N]$, $n \in N$ is given by:

$$\boxed{\eta^* = \left[\frac{\sum_{n\in N}\left(|h_n|^2 P_n + \frac{\sigma^2}{N}\right)}{\sum_{n\in N}\left(|h_n|\sqrt{P_n}\right)}\right]^2} \quad (8)$$

Proof:

Set $x = \frac{1}{\sqrt{\eta}} \Rightarrow$

$$g(x) = \frac{1}{N^2}\left[\sum_{n\in N}\left(|h_n|\sqrt{P_n}\,x - 1\right)^2 + \sigma^2 x^2\right]$$

$$\frac{\partial g(x)}{\partial x} = \frac{1}{N^2}\left[\sum_{n\in N}(|h_n|\sqrt{P_n}\,x - 1)|h_n|\sqrt{P_n} + 2\sigma^2 x\right]$$

$$\frac{\partial^2 g(x)}{\partial x^2} = \frac{1}{N^2}\left[\sum_{n\in N}|h_n|^2 P_n + 2\sigma^2\right] > 0$$

$$\frac{\partial g(x)}{\partial x} = 0 \Leftrightarrow \sum_{n\in N}(|h_n|\sqrt{P_n}\,x - 1)|h_n|\sqrt{P_n} + 2\sigma^2 x = 0 \Leftrightarrow$$

$$\sum_{n\in N}(|h_n|\sqrt{P_n}\,x - 1)|h_n|\sqrt{P_n} + \frac{2\sigma^2 x N}{N} = 0 \Leftrightarrow$$

$$\sum_{n\in N}\left[(|h_n|\sqrt{P_n}\,x - 1)|h_n|\sqrt{P_n} + \frac{2\sigma^2 x}{N}\right] = 0 \Rightarrow$$

$$\sum_{n\in N}\left[|h_n|^2 P_n x - |h_n|\sqrt{P_n} + \frac{2\sigma^2 x}{N}\right] = 0 \Rightarrow$$

$$\sum_{n\in N}\left(|h_n|^2 P_n + \frac{2\sigma^2}{N}\right)x = \sum_{n\in N}|h_n|\sqrt{P_n} \Rightarrow$$

$$x = \frac{\sum_{n\in N}|h_n|\sqrt{P_n}}{\sum_{n\in N}\left(|h_n|^2 P_n + \frac{2\sigma^2}{N}\right)} \Rightarrow$$

$$\frac{1}{\sqrt{\eta}} = \frac{\sum_{n\in N}|h_n|\sqrt{P_n}}{\sum_{n\in N}\left[|h_n|^2 P_n + \frac{2\sigma^2}{N}\right]} \Rightarrow$$

$$\boxed{\eta^* = \left[\frac{\sum_{n\in N}\left(|h_n|^2 P_n + \frac{2\sigma^2}{N}\right)}{\sum_{n\in N}\left(|h_n|\sqrt{P_n}\right)}\right]^2}$$

Q.E.D.

Given the optimal denoising factor η^*, the goal of the nodes is to distributedly optimize their transmission power P_n^*, $\forall n \in N$ in order to minimize the MSE

$$\Rightarrow \boxed{\begin{aligned}&\min_{\vec{P}} \frac{1}{N^2}\left[\sum_{n\in N}\left(\frac{|h_n|\sqrt{P_n}}{\sqrt{\eta}} - 1\right)^2 + \frac{\sigma^2}{\eta}\right] \\ &\text{s.t.} \quad P_n \le P_n^{Max}, \forall n \in N\end{aligned}} \quad (9)$$

Followers:

Analyze the function:

$$h(\vec{P}) = \sum_{n\in N}\left(\frac{|h_n|\sqrt{P_n}}{\sqrt{\eta}} - 1\right)^2 = \sum_{n\in N}\left(\frac{|h_n|^2 P_n}{\eta} - \frac{2|h_n|\sqrt{P_n}}{\sqrt{\eta}} + 1\right) = \frac{|h_n|^2 P_n}{\sqrt{\eta}} - \frac{2|h_n|\sqrt{P_n}}{\sqrt{\eta}} + 1$$

$$+ \sum_{\substack{n'\in N \\ n'\ne n}}\left(\frac{|h_{n'}|^2 P_{n'}}{\eta} - \frac{2|h_{n'}|\sqrt{P_{n'}}}{\sqrt{\eta}} + 1\right)$$

I want $P_n \downarrow$ $\quad P_{n'} \downarrow (\to h_{n'}^2)$

$\sqrt{P_n} \uparrow$ $\quad \sqrt{P_{n'}} \uparrow (\to h_{n'})$

Define each node's utility: (for simplicity $|h_n| \to h_n$ → change it in the paper)

$$\boxed{U_n(P_n, \vec{P}_{-n}) = \frac{h_n\sqrt{P_n}}{\sqrt{\eta}} \frac{\sum_{\substack{n'\in N\\ n'\ne n}}(\sqrt{P_{n'}}\, h_{n'})}{\sqrt{\eta}} - c\,\frac{h_n^2 P_n}{\eta} \frac{\sum_{\substack{n'\in N\\ n'\ne n}}(P_{n'} h_{n'}^2)}{\eta}} \quad (10)$$

$$\boxed{\begin{aligned}&\max_{P_n} U_n(P_n, \vec{P}_{-n}) \\ &\text{s.t.} \quad P_n \le P_n^{Max}\end{aligned}}$$

Non-cooperative game: $G = [N, \{P_n\}_{\forall n\in N}, \{U_n\}_{\forall n\in N}]$

N ↓ players

$\{P_n\}$ ↓ strategy set

$P_n = [0, P_n^{Max}]$

Definition 1: The non-cooperative game $G = [\mathcal{N}, \{P_n\}_{\forall n \in \mathcal{N}}, \{U_n\}_{\forall n \in \mathcal{N}}]$ is smooth supermodular if, for all n,

1) P_n is a compact cube in a Euclidean space
2) $U_n(P_n, \vec{P}_{-n})$ is twice continuously differentiable
3) supermodular in P_n for fixed $\vec{P}_{-n}$, i.e., $\frac{\partial^2 U_n}{\partial P_n' \partial P_n} \geq 0$, $P_n \neq P_n'$
4) with increasing differences in $(P_n, \vec{P}_{-n})$, i.e., $\frac{\partial^2 U_n}{\partial P_n \partial P_{n'}} \geq 0$, $n \neq n'$, $\forall n, n' \in \mathcal{N}$, $\forall P_n \in P_n$, $\forall P_{n'} \in P_{n'}$. ∎

In a supermodular game, there always exist extremal equilibria: a largest element $\overline{\vec{P}} = \sup\{\vec{P} \in P : BR(\vec{P}) \geq \vec{P}\}$ and a smallest element $\underline{\vec{P}} = \inf\{\vec{P} \in P : BR(\vec{P}) \leq \vec{P}\}$, with $P = P_1 \times P_2 \times \dots \times P_N$

Theorem 1: The non-cooperative game's $G = [\mathcal{N}, \{P_n\}_{\forall n \in \mathcal{N}}, \{U_n\}_{\forall n \in \mathcal{N}}]$ utility, as defined in (10) is supermodular in P_n for fixed $\vec{P}_{-n}$

Proof:
$$\frac{\partial U_n}{\partial P_n} = \frac{1}{2\sqrt{\eta}} \frac{h_n}{\sqrt{P_n}} \frac{\sum_{\substack{n' \in \mathcal{N} \\ n' \neq n}} \left(\sqrt{P_{n'}}\, h_{n'}\right)}{\sqrt{\eta}} - \frac{c h_n^2}{\eta} \frac{\sum_{n' \neq n} \left(P_{n'} h_{n'}^2\right)}{\eta}$$

$$\frac{\partial^2 U_n}{\partial P_n' \partial P_n} = 0$$

∎ Q.E.Δ.

Theorem 2: The non-cooperative game's $G = [\mathcal{N}, \{P_n\}_{\forall n \in \mathcal{N}}, \{U_n\}_{\forall n \in \mathcal{N}}]$ utility, as defined in (10) has increasing differences in $(P_n, \vec{P}_{-n})$ for $c \leq \frac{\eta}{4 h_n h_{n'} \sqrt{P_n P_{n'}}}$ $\forall P_n \in P_n$, $P_{n'} \in P_{n'}$

Theorem 1 and Theorem 4, derivatives are contradicting or?

Proof:
$$\frac{\partial U_n}{\partial P_n} = \frac{1}{2\sqrt{\eta}} \frac{h_n}{\sqrt{P_n}} \frac{\sum_{\substack{n' \in \mathcal{N} \\ n' \neq n}} \left(\sqrt{P_{n'}}\, h_{n'}\right)}{\sqrt{\eta}} - \frac{c h_n^2}{\eta} \frac{\sum_{n' \neq n} \left(P_{n'} h_{n'}^2\right)}{\eta}$$

$$\frac{\partial^2 U_n}{\partial P_{n'} \partial P_n} = \frac{\cancel{h_n}}{2\cancel{\eta}\sqrt{P_n}} \frac{1}{2} \frac{\cancel{h_{n'}}}{\cancel{\eta}\sqrt{P_{n'}}} - c \frac{h_n^{\cancel{2}} h_{n'}^{\cancel{2}}}{\eta^{\cancel{2}}} \geq 0 \iff \frac{1}{4\sqrt{P_n}\sqrt{P_{n'}}} \geq \frac{c h_n h_{n'}}{\eta}$$

$$c \leq \frac{\eta}{4 h_n h_{n'} \sqrt{P_n P_{n'}}}$$

(practically holds because of the orders of magnitude)

∎ Q.E.Δ.

Theorem 3: The non-cooperative game $G = [\mathcal{N}, \{P_n\}_{\forall n \in \mathcal{N}}, \{U_n\}_{\forall n \in \mathcal{N}}]$ has at least one Pure Nash Equilibrium [Topkis 1979 - Kakutani's fixed point theorem].

Q.E.Δ.

Theorem 4: The non-cooperative game $G = [\mathcal{N}, \{P_n\}_{\forall n \in \mathcal{N}}, \{U_n\}_{\forall n \in \mathcal{N}}]$ has a unique Pure Nash Equilibrium.

Q.E.Δ.

Proof:
$$\frac{\partial U_n}{\partial P_n} = \frac{1}{2} \frac{h_n}{\sqrt{P_n}\sqrt{\eta}} \frac{\sum_{\substack{n' \in \mathcal{N} \\ n' \neq n}} \left(\sqrt{P_{n'}}\, h_{n'}\right)}{\sqrt{\eta}} - \frac{c h_n^2}{\eta} \frac{\sum_{n' \neq n} \left(P_{n'} h_{n'}^2\right)}{\eta} = 0 \implies$$

$$\frac{1}{2} \frac{\cancel{h_n}}{\sqrt{P_n}\sqrt{\eta}} \frac{\sum_{\substack{n' \in \mathcal{N} \\ n' \neq n}} \left(\sqrt{P_{n'}}\, h_{n'}\right)}{\cancel{\sqrt{\eta}}} = \frac{c h_n^{\cancel{2}}}{\cancel{\eta}} \frac{\sum_{n' \neq n} \left(P_{n'} h_{n'}^2\right)}{\eta}$$

$$\frac{1}{2} \frac{1}{\sqrt{P_n}} \sum_{n' \neq n} \left(\sqrt{P_{n'}}\, h_{n'}\right) = c \frac{h_n}{\eta} \sum_{n' \neq n} \left(P_{n'} h_{n'}^2\right)$$

$$\left[\frac{\eta}{2 c h_n} \cdot \frac{\sum_{n' \neq n} \left(\sqrt{P_{n'}}\, h_{n'}\right)}{\sum_{n' \neq n} \left(P_{n'} h_{n'}^2\right)}\right]^2 = P_n$$

$$\frac{\partial^2 U_n}{\partial P_n^2} = -\frac{1}{2} \frac{h_n P_n^{-3/2}}{\sqrt{\eta}} \frac{\sum_{\substack{n' \in \mathcal{N} \\ n' \neq n}} \sqrt{P_{n'}}}{\sqrt{\eta}} < 0$$

$\Downarrow$

concave

$$\boxed{P_n^* = \left[\frac{\eta}{2 c h_n} \cdot \frac{\sum_{n' \neq n} \left(\sqrt{P_{n'}}\, h_{n'}\right)}{\sum_{n' \neq n} \left(P_{n'} h_{n'}^2\right)}\right]^2} \quad (11)$$

∎ Q.E.Δ.

Theorem 5: The non-cooperative game $G = [\mathcal{N}, \{P_n\}_{\forall n \in \mathcal{N}}, \{U_n\}_{\forall n \in \mathcal{N}}]$ converges monotonically to the unique Pure Nash Equilibrium (PNE)

Proof: The PNE is unique, thus, it is globally stable $P_n^* \in [0, P_n^{Max}]$. Given that the PNE is unique, the ... $\{U_n\}_{\forall n \in \mathcal{N}}]$ is dominance solvable [Milgrom

Proof: The PNE is unique, thus, it is globally stable $P_n^* \in [0, P_n^{Max}]$. Given that the PNE is unique, the non-cooperative game $G = [\mathcal{N}, \{P_n\}_{\forall n \in \mathcal{N}}, \{U_n\}_{\forall n \in \mathcal{N}}]$ is dominance solvable [Milgrom and Roberts 1990]. The best response will converge monotonically downward (upward) to the unique PNE starting at any point in the intersection of the upper (lower) contour sets of the largest (smallest) best responses of players $\overline{P}$ and $\underline{P}$. ∎

Best Response Algorithm:

$P_n = [P_{n1}, P_{n2}, \ldots, P_{nS}]$

Input: $\mathcal{N}, \{P_n\}_{\forall n \in \mathcal{N}}, c, \{h_n\}_{\forall n \in \mathcal{N}}, \eta$

Output: $\vec{P}^*$

Initialization: $ite = 0$, Convergence $= 0$, Randomly select $P_n^{ite=0} \in P_n, \forall n \in \mathcal{N}$ $\quad \vec{P}^{ite=0} = [P_1^{(0)}, P_2^{(0)}, \ldots, P_N^{(0)}]$

1. while Convergence $== 0$ do
2. $\quad ite = ite + 1$
3. $\quad P_n^{*ite} = BR(\vec{P}_{-n}^{ite-1}) = \arg\max_{P_n} U_n(P_n^{ite}, \vec{P}_{-n}^{ite-1}), \forall n \in \mathcal{N}$ $\rightarrow \vec{P}^{ite=1}$
4. $\quad$ Compute $U_n(P_n^{*ite}, \vec{P}_{-n}^{ite-1}), \forall n \in \mathcal{N}$
5. $\quad$ If $|U_n(P_n^{ite}, \vec{P}_{-n}^{ite}) - U_n(P_n^{ite-1}, \vec{P}_{-n}^{ite-1})| \leq \varepsilon$, ε small positive real number then
6. $\qquad$ Convergence $= 1$
7. $\quad$ end If
8. End while

$\rightarrow \vec{P}^* = [P_1^*, \ldots, P_n^*, \ldots, P_N^*]$

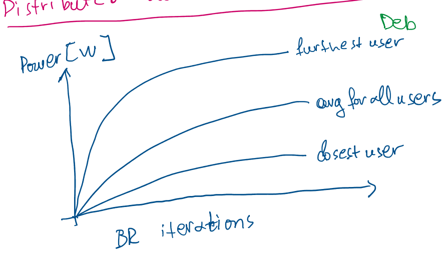
Theorem 6: The stackelberg game among the receiver and the nodes has a Stackelberg Equilibrium.

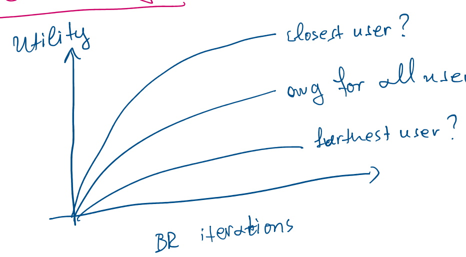
Proof: The optimization problem of the server has a unique solution η^* and the non-cooperative game G has a unique PNE. Thus, the Stackelberg game has at least one Stackelberg Equilibrium. ∎

Q.E.D.

(I) Distributed Power Control and Optimal Denoising

Deb



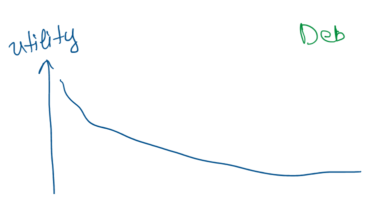


Deb

Non-cooperative game

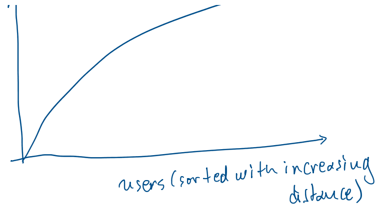
Deb

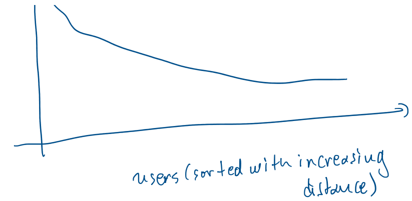
Deb

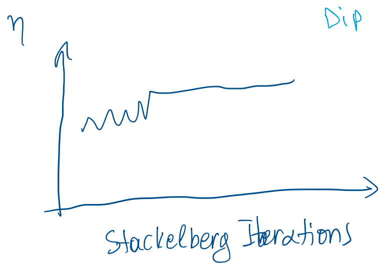


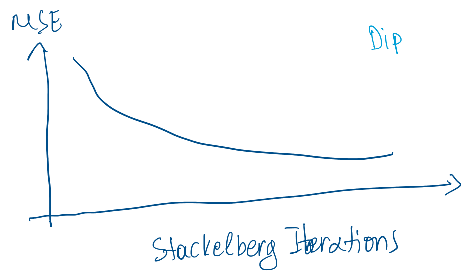
Deb = Debaleena

Dip = Dipanjan

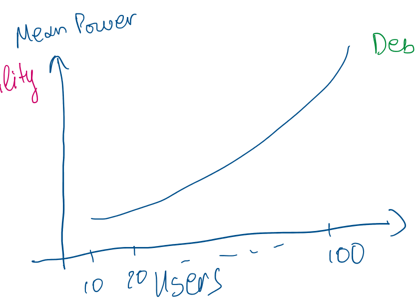


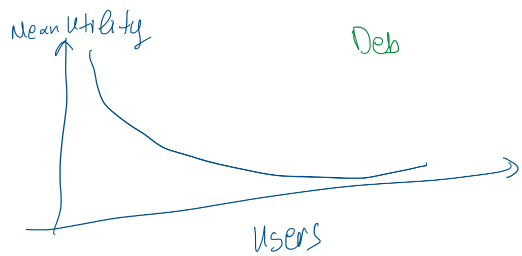


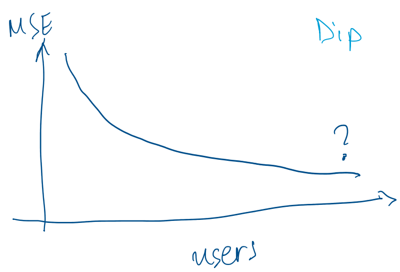


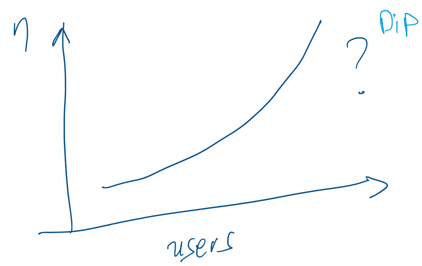


(II) Scalability

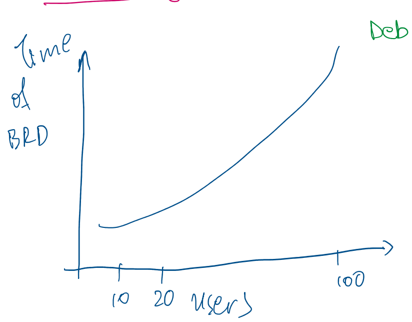


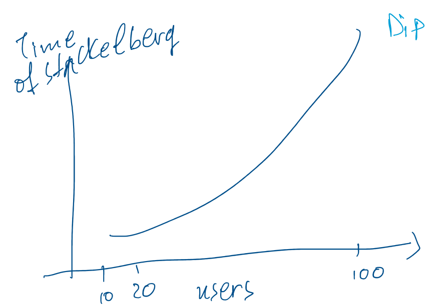






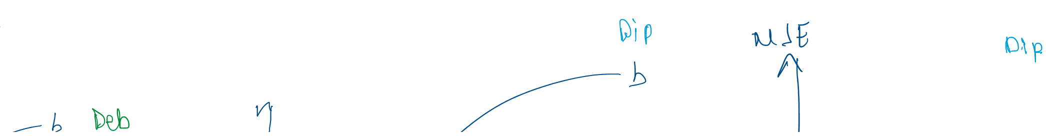
(III) Complexity :



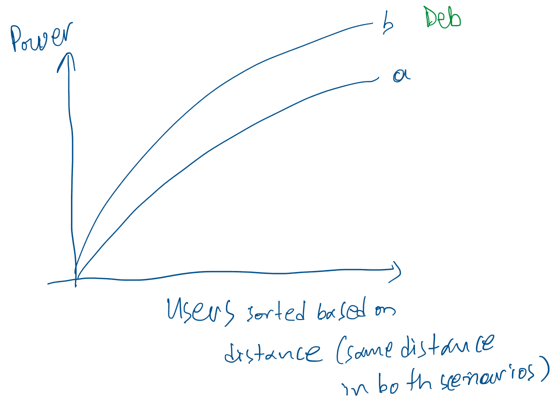


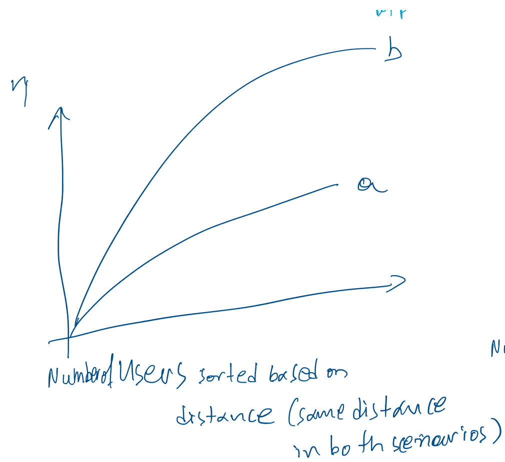
(IV) Sensitivity Analysis :

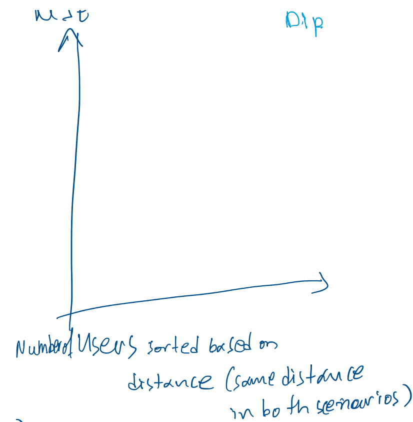
a) Indoor Channel gain } Check 3GPP
b) Outdoor Channel gain



b) Outdoor Channel gain







(v) Comparative :

A : Centralized Solution

B. Proposed