Mathematical Physics 2 - Spring 2024

Assignment - 3

Instructor: Amin A. Nizami GA: Debadyuiti Ghosh

Due: 19th March, 2024 (in class) Marks: 40

Question 1: Inter-species competition

Here we look at a model of inter-species competition in populations (N_1 and N_2) with finite carrying capacity. It is given by the non-linear dynamical system

$$\dot{N}_1 = r_1 N_1(1 - N_1/K_1) - b_1 N_1 N_2,$$
$$\dot{N}_2 = r_2 N_2(1 - N_2/K_2) - b_2 N_1 N_2.$$

This model has both *inter*- and *intra*- species competition. Here all the parameters as well N_1, N_2 are positive. Working with rescaled parameters ($x_i = N_i/K_i$), as in the lectures

(a) Write the equations in terms of x_1 and x_2 **[2]**

(b) Plot the vector field on the x_1 and x_2 axes. **[3]**

(c) Find all the fixed points. **[2]**

(d) Find the conditions on the parameters under which the two species can stably co-exist. Plot a rough sketch of the phase portrait in this case. Interpret your result. **[3]**

(e) What other kinds of qualitatively different long-term behaviours are possible for different parameter values ? Enumerate and explain. **[3]**

Question 2: Conservative Systems

In one of *Class Lectures*, we considered an example of a mechanical system, the double-well oscillator

$$m\ddot{x} = F = \frac{-dV(x)}{dx}$$

where the particle has a mass, $m = 1$ and the potential is given by

$$V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 \tag{1}$$

This system was rewritten as the vector fields in 2D and after finding the fixed points we constructed the corresponding phase portrait and interpreted the result physically.

(a) Consider a modified potential, $V(x) = -\frac{1}{2}x^2 + \frac{1}{3}x^3$. Identify the conserved quantity here. **[0.5]**

(b) Find and classify the fixed points for this case [*Part(a)*]. **[2]**

(c) Plot the nullclines & phase portrait for this system [*Part(a)*]. **[3]**

(d) Find the equation of the *homoclinic orbit* from the energy expression. Also point out the *homoclinic orbit* in the above plot [*Part(b)*]. What does this represent physically? **[2]**

(e) Now consider another kind of variation of the original system (1). Let's add a small amount of damping. The new system is then $\dot{x} = y,\ \dot{y} = -by + x - x^3$, where $0 < b \ll 1$.

Plot the phase portrait following similar steps as before and sketch the *basin of attraction* for the stable fixed point, of this system. **[2.5]**

Question 3: Index Theory (Properties)

(a) Find the index for the following type of fixed points **[3]**

- (i) Star
- (ii) Center
- (iii) Stable & unstable spiral

(b) Using the results from one of the *Class Lectures* and of [*Part(a)*], show that when a closed orbit encloses C centers, P spirals, N nodes and S saddles then : **[2]**

$$N + P + C = 1 + S$$

(c) Consider the following system :

$$\dot{x} = x(1 - y)$$
$$\dot{y} = y(4 - x - y^2),$$

Find the fixed points and classify them. Then explain why there can't be any closed orbits for this system using index theory. **[3]**

Question 4: Index Theory, again

(a) A two-dimensional version of a saddle node bifurcation is given by the system

$$\dot{x} = a + x^2$$
$$\dot{y} = -y$$

- (i) Find and classify all the fixed points as a varies from $-\infty$ to $+\infty$. **[3]**
- (ii) Show that the sum of the indices of all the fixed points stays constant as a varies. **[2]**

(b) Consider the family of linear systems

$$\dot{x} = x\cos\alpha - y\sin\alpha$$
$$\dot{y} = x\sin\alpha + y\cos\alpha,$$

where $\alpha \in [0, \pi]$. Also let C be a closed curve that does not pass through the origin.

- (i) Classify the fixed point at the origin as a function of α. **[2]**
- (ii) Use the integral formula for the index to show that I_C is independent of α. **[2]**