ASSIGNMENT-1

(1) (a) $y'(4-3x-3y) = x+y$ (Separable)

Put $z = x+y$

$$\Rightarrow \frac{dz}{dx} = \frac{4-2z}{4-3z}$$

$$\boxed{x+3y+2\ln|x+y-2| = c}$$

(b) $y = (y^3+x)y'$ (Linear)

$$\frac{dx}{dy} - \frac{x}{y} = y^2 \qquad I.F = e^{-\int \frac{1}{y}dy} = \frac{1}{y}$$

$$\Rightarrow \frac{x}{y} = \int y\,dy = \frac{y^2}{2} + c$$

$$\boxed{x = \frac{1}{2}y^3 + cy}$$

(c) $x\frac{dy}{dx} + y = \frac{y^2}{x^{3/2}}$ (Bernoulli)

$$\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^{5/2}} \quad (n=2)$$

Take $z = y^{1-2} = 1/y$

$$\Rightarrow \frac{dz}{dx} - \frac{z}{x} = -x^{-5/2} \qquad I.F = e^{-\int \frac{1}{x}dx} = \frac{1}{x}$$

$$\Rightarrow \frac{z}{x} = -\int x^{-7/2}dx = +\frac{2}{5}x^{-5/2} + c$$

$$\boxed{y = \left(\frac{2}{5}x^{-3/2} + cx\right)^{-1}}$$

(d) $(y-x)\frac{dy}{dx} + 2x + 3y = 0$ (Homogeneous)

$$\frac{dy}{dx} = \frac{2x+3y}{x-y}$$

Take $z = \frac{y}{x} \longrightarrow \frac{dy}{dx} = x\frac{dz}{dx} + z$

$$\Rightarrow x\frac{dz}{dx} = \frac{z^2+2z+2}{1-z}$$

$$\Rightarrow \ln|Ax| = 2\tan^{-1}(1+z) - \frac{1}{2}\ln|z^2+2z+2|$$

$$\Rightarrow \ln|Cx^2(z^2+2z+2)| = 4\tan^{-1}(1+z)$$

$$\therefore \boxed{c(y^2+2xy+2x^2) = \exp\left[4\tan^{-1}\left(\frac{x+y}{x}\right)\right]}$$

① (e) $\frac{dy}{dx}\left[1+\frac{3y^2+2y}{\cos(x+y)}\right] = (-1)$ (Exact)

$$\underbrace{\left[\cos(x+y) + 3y^2 + 2y\right]}_{N} dy + \underbrace{\cos(x+y)}_{M} dx = 0 \Rightarrow \partial_y M = \partial_x N = -\sin(x+y)$$

$$\boxed{G(x,y) = \sin(x+y) + y^3 + y^2 = c}$$

(f) $\frac{dy}{dx} = \frac{\sqrt{y}-y}{\tan x}$ (Separable/ Bernoulli)

$$\int \frac{dy}{\sqrt{y}(1-\sqrt{y})} = \int \cot x \, dx$$

$\left\{ \begin{array}{l} 1-\sqrt{y} = t \\ \Rightarrow dt = \frac{-1}{2\sqrt{y}} dy \end{array} \right.$

$$\Rightarrow -2\ln|A(1-\sqrt{y})| = \ln|\sin x|$$

$$\boxed{\sin x \,(1-\sqrt{y})^2 = c}$$

$\Rightarrow \frac{dy}{dx} + (\cot x)\, y = (\cot x)\, y^{1/2}$

Taking $z = y^{1/2}$

$\Rightarrow \frac{dz}{dx} + \left(\frac{1}{2}\cot x\right) z = \frac{1}{2}\cot x$

which is Linear, with $I.F = e^{\frac{1}{2}\int \cot x\, dx} = \sqrt{\sin x}$

$\Rightarrow z\sqrt{\sin x} = \frac{1}{2}\int \frac{\cos x}{\sqrt{\sin x}} dx = \sqrt{\sin x} + C_0$

$$\boxed{\sin x(\sqrt{y}-1)^2 = c}$$

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② $y' = p(x) + q(x)y + r(x)y^2$

(a) $y(x) = y_1(x) + z(x)$ is a solution

$\Rightarrow (y_1' + z') = p + q(y_1+z) + r(y_1+z)^2$ since y_1 satisfies Riccati eqn.

$\Rightarrow \boxed{z' = (q+2ry_1)z + rz^2}$

This is the form of Bernoulli eqn.

(b) $y' = \underbrace{x^3}_{r(x)} y^2 + \underbrace{\frac{y}{x}}_{q(x)} \underbrace{- x^5}_{p(x)}$ with $y_1(x) = x$ (known soln.)

For the general soln. $\Rightarrow y = x + z(x)$

$$z' - \left(2x^4 + \frac{1}{x}\right)z = x^3 z^2$$

Taking $u = 1/z \Rightarrow u' + \left(2x^4 + \frac{1}{x}\right)u = -x^3$ ⟶ Linear $\quad I.F = xe^{2x^5/5}$

$$\Rightarrow \frac{1}{u} = z = \frac{-2x}{1+Ae^{-2x^5/5}}$$

Thus, $y = x + z \Rightarrow \boxed{\frac{y}{x} = \frac{Ae^{-2x^5/5} - 1}{Ae^{-2x^5/5}+1}}$ or, $\boxed{\frac{y-x}{y+x} = ce^{2x^5/5}}$

(c) Put $u = \frac{1}{z}$ in $z' - (q+2ry_1)z = rz^2$ (as, $n=2$)

$\Rightarrow u' + (q+2ry_1)u = (-r)$ Linear $\quad I.F = e^{\int (q+2ry_1)dx} \equiv \mu(x)$

$$\Rightarrow u = \frac{-1}{\mu}\Big\{\underbrace{\int (\mu r)dx}_{h(x)} + c\Big\}$$

$$\Rightarrow z = \frac{\mu(x)}{h(x)+c}$$

$$\boxed{y(x) = y_1(x) + z(x) = \frac{cy_1(x) + \{h(x)y_1(x) + \mu(x)\}}{c + h(x)}}$$

which is of the form, $y(x) = \frac{cf(x) + g(x)}{cF(x) + G(x)}$

② (d) $y = \dfrac{c\,f(x) + g(x)}{c\,F(x) + G(x)}$

$$\Rightarrow\ y' = \frac{1}{(cF+G)^2}\left[(cf'+g')(cF+G) - (cf+g)(cF'+G')\right] = \frac{cf'+g'}{cF+G} - \left(\frac{cF'+G'}{cF+G}\right)y$$

Multiplying numerator & denominator with $(cf+g)$,

$$\boxed{y' = \left(\frac{cf'+g'}{cf+g}\right)y - \left(\frac{cF'+G'}{cf+g}\right)y^2}$$ i.e., a Riccati eqn.

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③ $p = mv = \dfrac{m_0 v}{\sqrt{1-v^2/c^2}}$

(a) Constant gravitational field $\longrightarrow F = m\alpha$ (α is the const. acceleration)

$$F = \frac{dp}{dt} \quad\Rightarrow\quad m\alpha = \frac{d}{dt}(mv) = m_0\frac{d}{dt}\left(\frac{v}{\sqrt{1-v^2/c^2}}\right)$$

$$\Rightarrow\ \cancel{m}\alpha = \frac{m_0}{\cancel{\sqrt{1-v^2/c^2}}}\left[\left(\frac{1}{1-v^2/c^2}\right)\frac{dv}{dt}\right]$$

$$\Rightarrow\ \alpha\int_0^t dt = \int_0^v \frac{dv}{1-v^2/c^2} = c\tanh^{-1}\left(\frac{v}{c}\right)\Bigg|_0^v$$

$$\Rightarrow\ \frac{v}{c} = \tanh\left(\frac{\alpha t}{c}\right) = \frac{1-e^{-2\alpha t/c}}{1+e^{-2\alpha t/c}}$$

$$\left\{\begin{aligned} \tanh x &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \\ &= \frac{1-e^{-2x}}{1+e^{-2x}} \end{aligned}\right.$$

$$\boxed{v(t) = c\left[\frac{1-e^{-2\alpha t/c}}{1+e^{-2\alpha t/c}}\right]} \longrightarrow \text{For } t\to\infty,\quad v = c$$

(b) $E = \displaystyle\int_0^v F\,dx = (m-m_0)c^2$

$$E = m_0\int_0^v \frac{v\,dv}{(1-v^2/c^2)^{3/2}} = m_0c^3\int_0^v \frac{v}{(c^2-v^2)^{3/2}}dv = m_0c^3\left[\frac{1}{(c^2-v^2)^{1/2}} - \frac{1}{c}\right] \Rightarrow \boxed{E = Mc^2}$$

(c) Converse proof

Given that, $E = Mc^2 = (m-m_0)c^2$

$$dE = c^2dm = v\,d(mv) = v^2dm + mv\,dv$$

$$\Rightarrow \int_{m_0}^m \frac{dm}{m} = \int_0^v \frac{v\,dv}{c^2-v^2} \quad\Rightarrow\quad \ln\left|\frac{m}{m_0}\right| = \frac{-1}{2}\ln\left|\frac{c^2-v^2}{c^2}\right| \quad\Rightarrow\quad \boxed{\frac{m}{m_0} = \frac{1}{\sqrt{1-v^2/c^2}}}$$

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④ (a) $\dfrac{dx}{dt} = \cos(x^2)$

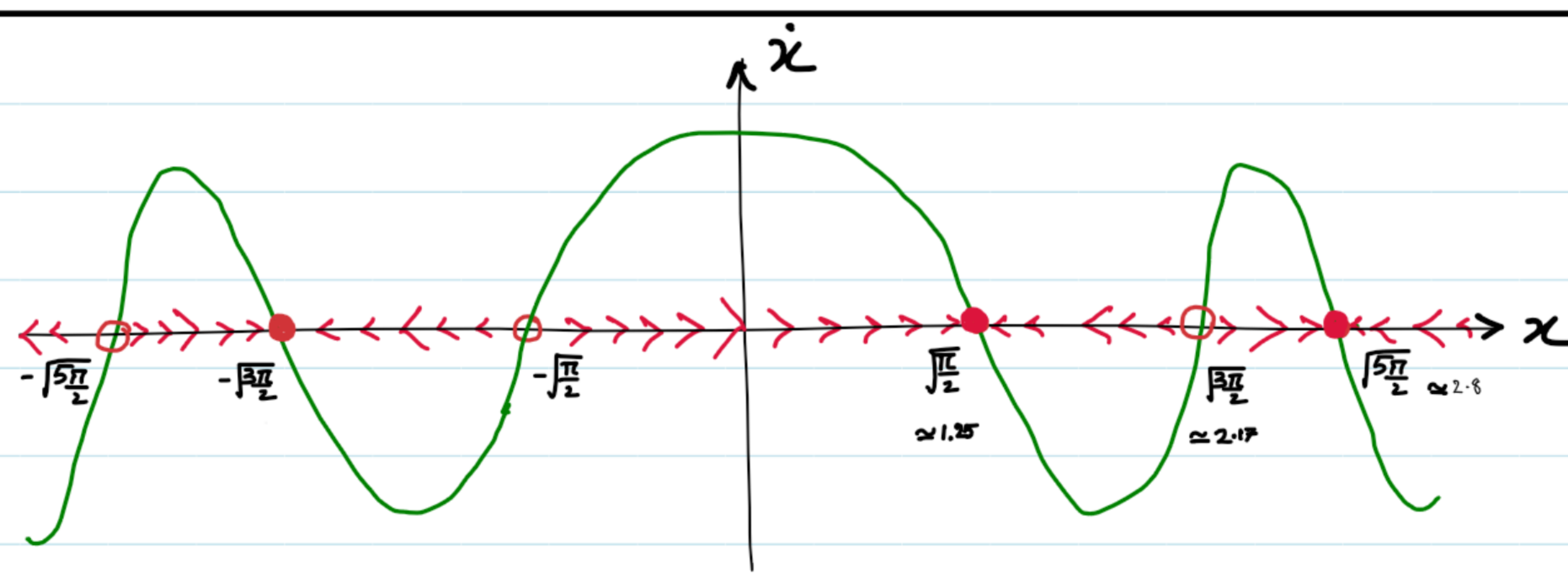
$\boxed{\text{Fixed points} = \pm\sqrt{2n+1}\times\sqrt{\frac{\pi}{2}}}$

$= \pm\sqrt{\pi/2},\ \pm\sqrt{3\pi/2},\ \pm\sqrt{5\pi/2}\ \ldots$

For the initial conditions: $x(0) = -1.5 \longrightarrow x(\infty) = -\sqrt{3\pi/2}$

$x(0) = 2 \longrightarrow x(\infty) = \sqrt{\pi/2}$

$x(0) = 3.5 \longrightarrow x(\infty) = 3\sqrt{\pi/2}$



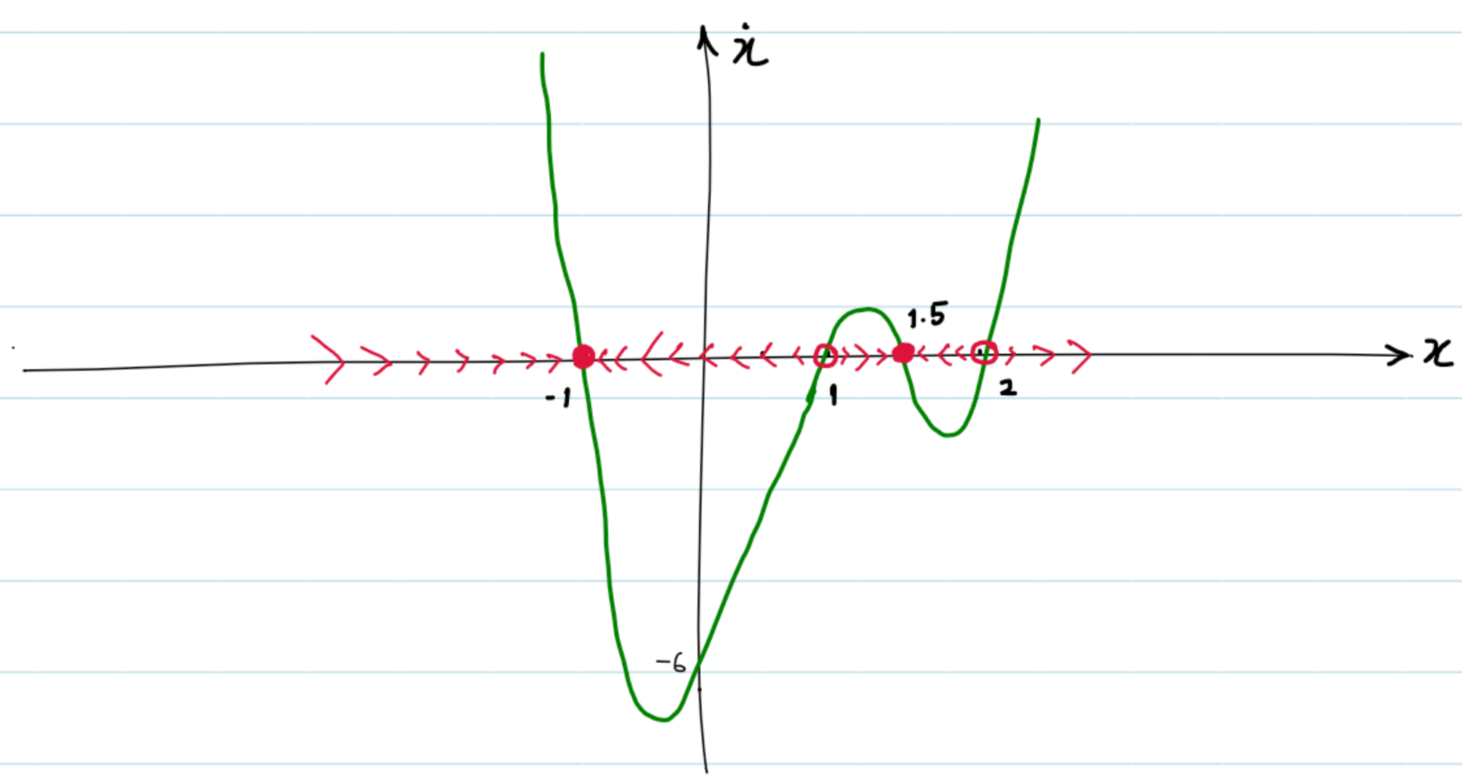
(b) $\dfrac{dx}{dt} = 2x^4 - 7x^3 + 4x^2 + 7x - 6 = (x-1)(x+1)(x-2)(2x-3)$

$\boxed{\text{Fixed points} = \pm1, 1.5, 2}$

For the initial conditions: $x(0) = 2.7 \longrightarrow x(\infty) = \infty$

$x(0) = -7 \longrightarrow x(\infty) = (-1)$

$x(0) = 0 \longrightarrow x(\infty) = (-1)$



5. Multiple Decay modes

$$A \xrightarrow{\lambda} B \begin{cases} \xrightarrow{\lambda_1} C_1 \\ \xrightarrow{\lambda_2} C_2 \end{cases}$$

(a) Differential eqn.s:

$$\left.\begin{aligned} \frac{dN_A}{dt} &= -\lambda N_A \\ \frac{dN_B}{dt} &= -(\lambda_1+\lambda_2) N_B + \lambda N_A \\ \frac{dN_{C_{1,2}}}{dt} &= \lambda_{1,2} N_B \end{aligned}\right\}$$

(b) When $N_B \to$ max. $\frac{dN_B}{dt} = 0$

$$\Rightarrow \frac{N_A}{N_B} = \frac{\lambda_1+\lambda_2}{\lambda}$$

(c) No. of atoms for each atom: (Integrating eqn.s from part (a))

$$N_A = N_0 e^{-\lambda t}$$

$$N_B = \frac{\lambda N_0}{\lambda - (\lambda_1+\lambda_2)} \left[e^{-(\lambda_1+\lambda_2)t} - e^{-\lambda t} \right]$$

$$N_{C_{1,2}}(t) = \frac{\lambda_{1,2}\, \lambda N_0}{\lambda - (\lambda_1+\lambda_2)} \left[\frac{e^{-\lambda t} - 1}{\lambda} - \frac{e^{-(\lambda_1+\lambda_2)t} - 1}{\lambda_1+\lambda_2} \right]$$