# Mathematical Physics 2 - Spring 2024

## Assignment - 2

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**Due: March 5th, 2024** (in class) — Marks: 30

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## Question 1: Epidemic Modelling

In this problem, we shall analyse the Kermack-McKendrick model of epidemic evolution (1927). Here, we divide a population into three compartments: $x(t)$ = the number of healthy people (or susceptible people), $y(t)$ = the number of sick people, and $z(t)$ = the number of dead people, all at time $t$. The equations governing the dynamics of this model are given by

$$\dot{x} = -kxy$$
$$\dot{y} = kxy - ly$$
$$\dot{z} = ly$$

(a) Show that $x + y + z = N$, the total population, living and dead, which is a constant. **[0.5]**

(b) Show that $z$ satisfies the first order equation

$$\dot{z} = l(N - z - x_0 \exp(-kz/l)),$$

where $x_0 = x(t = 0)$. **[1]**

(c) Show that this equation can be turned into a dimensionless form **[1]**

$$\frac{du}{d\tau} = a - bu - \exp(-u)$$

(d) Show that $b = 1$ is a threshold case for an epidemic – if less than that, then $\dot{u}(t)$ is decreasing at $t = 0$, but if greater than that, then $\dot{u}(t)$ increases and then decreases after a peak. Does $b$ remind you of anything from the discussion around the coronavirus pandemic? **[2.5]**

(e) The $z$ dynamics do not affect the $x, y$ dynamics, so focussing only on the $XY$ phase plane, find all the fixed points and classify their stability using linearisation. **[3]**

(f) Draw the nullclines and the phase portrait. **[2]**

## Question 2: Biochemical Switch and Pattern Formation

Zebra stripes and butterfly wing patterns are two of the most spectacular examples of biological pattern formation. As one ingredient in a model of pattern formation, Lewis et al. (1977) considered a simple example of a biochemical switch, in which a gene $G$ is activated by a biochemical signal substance $S$. For example, the gene may normally be inactive but can be "switched on" to produce a pigment or other gene product when the concentration of $S$ exceeds a certain threshold. Let $g(t)$ denote the concentration of the gene product, and assume that the concentration $s_0$ of $S$ is fixed. The model is governed by

$$\dot{g} = k_1 s_0 - k_2 g + \frac{k_3 g^2}{k_4^2 + g^2},$$

where the $k$'s are positive constants. The production of $g$ is stimulated by $s_0$ at a rate $k_1$, and by an autocatalytic or positive feedback process (the nonlinear term). There is also a linear degradation of $g$ at a rate $k_2$.

(a) Show that the system can be put in the dimensionless form **[1]**

$$\frac{dx}{d\tau} = s - rx + \frac{x^2}{1+x^2},$$

where $r > 0$ and $s \geq 0$ are dimensionless quantities.

(b) Show that if $s = 0$, there are two positive fixed points $x*$ if $r < r_c$, where $r_c$ is to be determined. **[0.5]**

(c) Assume that initially there is no gene product, i.e., $g(0) = 0$, and suppose $s$ is slowly increased from zero (the activating signal is turned on); what happens to $g(t)$? What happens if $s$ then goes back to zero? Does the gene turn off again? **[2.5]**

## Question 3: Some 2D linear systems

For the following linear systems, write the corresponding matrix equation and find the eigenvalues and eigenvectors to get the general solutions. Find the fixed points of the system, classify the same (if applicable) and draw the complete phase portrait by starting in the vicinity of the fixed points. You need not use any computational help, although you may use it to check your plot.

(a)
$$\dot{x} = -2\,x - 3\,y$$
$$\dot{y} = 3\,x - 2\,y$$

[Hint: Consider $t \to \infty$ to determine the direction] **[2]**

(b)
$$\dot{x} = x + y - x^3$$
$$\dot{y} = -y$$
**[3]**

(c)
$$\dot{x} = \sin y$$
$$\dot{y} = \cos x$$
**[4]**

## Question 4: A simple 2D nonlinear system

Consider the dynamical system

$$\dot{x} = x^2 - y^2$$
$$\dot{y} = xy$$

(a) Find the fixed points for this system. What information does linearisation about the fixed point give here? **[1.5]**

(b) Plot the nullclines and the vector field. From this, identify the index of the fixed points. **[2.5]**

(c) Solve this system analytically and plot the integral curves. Compare with previous part. **[3]**