Mathematical Physics 2 - Spring 2024

Assignment - 1

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Due: Feb.16, 2024 Total Marks: 50

Problem 1. Warming-up : Solving various ODEs

Use the mathematical methods used in the lectures to find the analytical solution of the following ODEs. If necessary, the solution could be left in the implicit form. **[6*2.5=15]**

(a) $y'\,(4-3x-3y) = x+y$

(b) $y = (y^3+x)\,y'$, (take y to be the independent variable)

(c) $x\,\dfrac{dy}{dx} + y = \dfrac{y^2}{x^{3/2}}$

(d) $(y-x)\,\dfrac{dy}{dx} + 2x + 3y = 0$

(e) $\dfrac{dy}{dx}\left[1 + \dfrac{3y^2+2y}{\cos(x+y)}\right] = (-1)$

(f) $\dfrac{dy}{dx} = \dfrac{\sqrt{y}-y}{\tan x}$

Note that some of these may be solvable by more than one methods.

Problem 2. The Riccati Equation

A natural extension of the first-order linear ODE is the *Riccati equation*:

$$y' = p(x) + q(x)y + r(x)y^2 \tag{1}$$

(a) Suppose you know (or can guess) a particular solution $y_1(x)$ of this equation, prove that the general solution has the form **[2]**

$$y(x) = y_1(x) + z(x)$$

where $z(x)$ is the general solution to the Bernoulli Equation

$$z' - (q + 2ry_1)z = rz^2$$

(b) Use this to find the general solution to

$$y' = \frac{y}{x} + x^3y^2 - x^5,$$

which has $y_1(x) = x$ as an obvious particular solution. **[3]**

(c) If Equation (1) has a known solution $y_1(x)$, show that the general solution has the form of the one-parameter family of curves **[3]**

$$y = \frac{cf(x) + g(x)}{cF(x) + G(x)}$$

(d) Show conversely that the differential equation that the above one-parameter family of curves satisfies is a Riccati Equation. **[2]**

Problem 3. Relativistic Particle in a Gravitational Field

The Special Theory of Relativity asserts that the relativistic 'momentum' of a particle moving with velocity v is given by:

$$p = m(v)\, v = \frac{m_0 v}{\sqrt{1 - v^2/c^2}}$$

where c is the speed of light and m_0 its rest mass.

(a) Suppose the particle starts from rest in empty space and moves for a long time under the influence of a constant gravitational field, find v as a function of time. What happens to v as $t \to \infty$? **[2.5]**

(b) Let $M = m - m_0$ be the increase in the mass of the particle. If the corresponding increase E in its energy is taken to be the work done on it by the force F, so that

$$E = \int_0^v F\, dx = \int_0^v \frac{d}{dt}(mv)\; dx = \int_0^v v\, d(mv)$$

verify that $E = Mc^2$. **[3.5]**

(c) From this, deduce that **[3]**

$$m(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

Problem 4. Vector field flow plots

In one of the *Class Lectures*, we graphically solved 1D non-linear differential equations. Instead of analytically solving for the unknown function, we plotted the vector field flow, and thus the nature of some typical solutions was found. For the following non-linear ODEs, find the fixed point(s), and use the vector field plot to sketch some corresponding solutions. **[3.5*2=7]**

(a) $\dfrac{dx}{dt} = \cos(x^2)$. In this case, find the late time ($t \to \infty$) value of the solution for the 3 different initial conditions: $x(0) = -1.5,\ x(0) = 2,\ x(0) = 3.5$.

(b) $\dfrac{dx}{dt} = 2x^4 - 7x^3 + 4x^2 + 7x - 6$. In this case, find the late time ($t \to \infty$) value of the solution for the 3 different initial conditions: $x(0) = 2.7,\ x(0) = -7,\ x(0) = 0$.

Problem 5. Multiple Decay Modes

An atom A decays into a metastable state B further decaying into two ground states (C_1 and C_2) with decay constants as shown in the figure 1. Assume that initially (for $t = 0$) only N_0 number of A atoms are present.

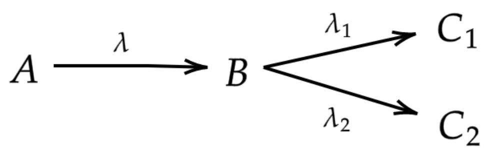

Figure 1: A decays into B which in turn decays into C_1 and C_2. The decay constants for each step - λ, λ_1 and λ_2 - are indicated next to the arrows

(a) Write down the differential equations which describe the rate of change of the number of each of the atom types. Make sure you get the signs right and that your equations make physical sense. **[3]**

(b) Determine the ratio of the number of A atoms to B atoms at the instant when the number of B atoms becomes maximum. Give your answer in terms of the given parameters. **[2]**

(c) Solve to get the number as a function of time for each of the atom types. **[4]**

Problem 6. Investigating Periodic Harvesting

We have already encountered the logistic equation for population growth in class. This can be used to model the population of a fish species in a lake over time. We have also encountered the modification for constant harvesting rate (i.e. depletion, in this case due to fishing) in class. Here, we shall study the case of periodic harvesting, described by the equation

$$\frac{dx}{dt} = rx(1-x) - h_0(1 + \eta \cos \omega t)$$

Here, η and ω represent the amplitude and frequency of fishing, i.e. how many fish are caught every time and how often fishing happens.

(a) Take $r = 5, \eta = 0.5, h_0 = 1, \omega = \pi$. For a range of initial conditions $x_0 \in [0, 1]$, is the fish population stable or does it go extinct? Is this the same for any starting population, or is there a cutoff initial population for survival? **[3]**

(b) Repeat the analysis, but now change h_0 to 1.2. What change do you see in the behaviour for the same range of initial populations? If a bifurcation occurs for a certain value of h_0, around what value does it occur? **[3]**

(c) Now let's play with the frequency of fishing. Fix the values of $r = 5, \eta = 0.5, h_0 = 1, x_0 = 0.5$, and take a range of $\omega \in [0, 2\pi]$ - do you see any change in behaviour at any point? Do fish populations always collapse or always remain stable? What happens if you instead start with $x_0 = 0.4$? **[3]**

[*Note:* This equation cannot be solved analytically, so you have to analyse it on the computer. Plotting graphs / slope fields could be helpful and/or solving the ODE numerically and tracking trajectories in $t \in [0, 50]$. Include the code in your solution.]