Electricity & Magnetism in Light of Relativity

DS : Vector Calculus-1

(MONSOON 2024)

ashoka UNIVERSITY

DEBADYUITI GHOSH

Problem 1 : Revising Dot and Cross products

(a) Can you explain what a dot and a cross product of two vectors represent, physically?

(b) Suppose that $\mathbf{A} \neq 0$ and if both the conditions are simultaneously satisfied: $\mathbf{A} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{C}$, and $\mathbf{A} \times \mathbf{B} = \mathbf{A} \times \mathbf{C}$, then prove that $\mathbf{B} = \mathbf{C}$

(c) Find a unit vector that is perpendicular to the plane of both vectors, $\mathbf{A} = 2\hat{\imath} - 6\hat{\jmath} - 3\hat{k}$ and $\mathbf{B} = 4\hat{\imath} + 3\hat{\jmath} - \hat{k}$. Do it in the first way you can think of. (*Don't say you have no clue!*)

Now, can you do it some other way (*Hint: without using the cross-product*)?

Problem 2 : Differentiation and integration of vectors

(a) For a constant vector $\mathbf{A}(\mathbf{t})$, show that $\mathbf{A}$ and $\frac{d\mathbf{A}}{dt}$ are perpendicular? [1]

(b) If the force $\mathbf{F}$ acts perpendicular to the velocity $\mathbf{v}$ then $|\mathbf{v}| =$ constant.

(c) **Line integral:**

Calculate the work done by a force field $\mathbf{F} = \sin y\, \hat{x} + x\,(1 + \cos y)\, \hat{y}$ in moving a particle around a circle C in the xy-plane centred at the origin with radius a.

(Note - conventionally, traversing any closed curve in an anti-clockwise sense is taken as the *positive* direction)

Problem 3 : Grad, Div, Curl and all that ⋆

(a) Calculate the following quantities for $\phi = \ln |\mathbf{r}|$:

(i) $\boldsymbol{\nabla}\phi$, (ii) $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\phi$ (iii) $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\phi$ and, (iv) $\nabla^2\phi$

(b) What is the unit normal for the ellipsoidal surfaces (with a constant factor A) defined by

$$\Phi(x,\, y,\, z) = A\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)$$

Does your answer make sense for $a = b = c$? (*the normal vector is obvious in this special case*)

[1] *Think:* how is it a constant vector if $\mathbf{A} = \mathbf{A}(t)$?

(c) **What's in a name: Curl?**

Suppose that $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, where $\boldsymbol{\omega}$ is a constant vector. Prove that $\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\nabla}\times\mathbf{v}$

Can you get a sense of how this connects to the name of this operation?

"If the original function is a vector then ∇ applied to it may give two parts [2]. The scalar part I would call the Convergence of the vector function, and the vector part I would call the Twist of the vector function. Here the word twist has nothing to do with a screw or helix. If the word turn or version would do they would be better than twist, for twist suggests a screw. Twirl is free from the screw notion and is sufficiently racy. Perhaps it is too dynamical for pure mathematicians, so for Cayley's sake I might say Curl." – Maxwell, 1870

(d) Prove that $\boldsymbol{\nabla} f(r) = f'(r)\,\hat{r}$ and $\nabla^2 f(r) = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$.

Problem 4 : Vector identities (worth remembering) ⋆

(a) **Curl of a gradient is zero.**
Using the definition of $\boldsymbol{\nabla}$, prove that (for any arbitrary ϕ), $\boxed{\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\phi) = 0}$

(b) **Divergence of a curl is zero.**
Again as previously, prove (for any arbitrary $\mathbf{A}$) that, $\boxed{\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) = 0}$

(c) **Curl of a curl :** For any arbitrary $\mathbf{A}$ prove the following identitiy,

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \nabla^2\mathbf{A}$$

Note : Any vector field $\mathbf{F}$, is determined *uniquely* by its divergence and curl, iff appropriate boundary conditions are given. (Helmholtz theorem)

Problem 5 : Miscellaneous practice problems ⋆

(a) A vector $\mathbf{v}$ is called *irrotational* if $\boldsymbol{\nabla}\times\mathbf{v} = 0$. Check that the vector

$$\mathbf{V} = (-4x - 3y + 4z)\mathbf{i} + (-3x + 3y + 5z)\mathbf{j} + (4x + 5y + 3z)\mathbf{k}$$

is irrotational. From Prob-4(a), we know we can write $\mathbf{V} = \boldsymbol{\nabla} f$, find the function f.

(b) For an *irrotational* vector $\mathbf{A}$, show that : $\boldsymbol{\nabla}\cdot(\mathbf{r}\times\mathbf{A}) = 0$

(c) The electric field in terms of the electrostatic potential ϕ is given as $\mathbf{E} = -\boldsymbol{\nabla}\phi$ and the corresponding charge density is $\rho = \epsilon_0\boldsymbol{\nabla}\cdot\mathbf{E}$.

If $\mathbf{a}$ and $\mathbf{b}$ are constant vectors, find the electric field and the charge density for the potential, $\phi = (\mathbf{a}\times\mathbf{r})\cdot(\mathbf{b}\times\mathbf{r})$

[2] Note: scalar part ('convergence') refers to dot product, while the vector part is the cross product