11/01/23.

Time Series Analysis & Forecasting

References :-

(1) Time Series Analysis by J.D Hamilton (More theoritical)

(2) Time Series Analysis with applications in R by Shumway & stoffer (Applied)

(8) Time Series Analysis by Brockwell & Davis (Theoritical)

(4) Time Series Analysis by chatfield

Marks Distribution

Mid Sem 30%
End Sem 60%
Project 10%

Programming Language : R or Python.

Topics

(i) Basics of time Series
- (a) Mean, variance, auto covariance, auto-correlation
- (b) Some components of time series like trend, seasonality, cyclicality & Irregularity.

(ii) Stationary vs Non-stationary time series

(iii) Some standard stationarity Time series processes
- (1) AR "
- (2) MA "
- (3) ARMA "

(iv) Some standard non-stationary time series process
- (1) ARIMA process
- (2) Seasonal ARIMA process

(v) Estimation of ARMA parameters

— (a) Yule - Walker estimation method

(b) Maximum - Likelihood estimation

(iii) Conditional least Squares Method

(vi) Few more properties of a time Series

— (a) Causality

(b) Invertibility

(c) Ergodicity

(vii) Forecasting

— (a) Exponential smoothing

(b) Double exponential smoothing

(c) Triple exponential smoothing

(d) Forecasting for AR, MA & ARMA process

(e) Forecasting for an ARIMA & SARIMA process

(viii) Spectral Analysis / Frequency - Domain Analysis of a time Series

(ix) ARCH & GARCH models

(X) Test of Stationarity
— (a) Dickey Fuller
(b) Augmented Dickey Fuller

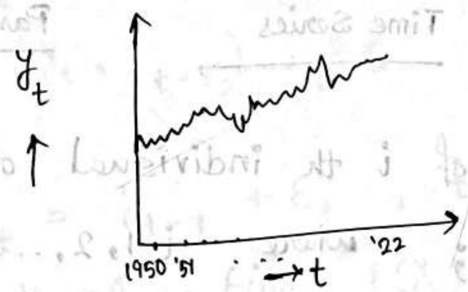
Basics of Time Series :

Definition :- A time series is a sequence of observations observed over a certain time points. Usually a time series process is denoted by $\{y_t\}_{t \in T}$ or $\{y_t\}_{t=1}^{N}$ or $\{y_t\}$ where $T = \{1, 2, \dots, N\}$

Example :

~~yearly GDP~~ (i) y_t : yearly GDP of a country at year t where $T = 1950, 1951, \dots, 2022$

& $\{y_t\}_{t \in T} = \{y_{1950}, y_{1951}, \dots, y_{2022}\}$: is the set of time series observations



Types of Data

There are mainly 3 types of data.

1) Cross Sectional Data — Data are observed over different subjects / indivisuals at a point of time.

Ex :- Income of N indivisuals at year 2020

let, y_i be the income of i th indivisual at 2020
where, $i = 1(1)N$

(2) Time Series Data :- The data are observed on a single subject over a period of time.

Ex :- let, y_t be the income of an indivisual observed over from 1990 to 2010.

(3) Panel / Longitudinal Data :-
The data are on N indivisuals over T time points.

i	y_i
1	y_1
2	y_2
$\vdots$	$\vdots$
N	y_N

cross-section.

t	y_t
1	y_1
2	y_2
$\vdots$	$\vdots$
T	y_T

Time Series

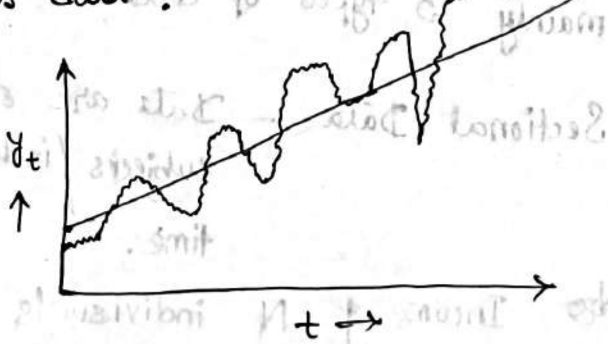
$i \backslash t$	1	2	$\cdots t$	$\cdots T$
1	y_{11}	y_{12}	$\cdot y_{1t}$	$\cdot y_{1T}$
2	y_{21}	y_{22}	$\cdot y_{2t}$	$\cdots y_{2T}$
$\vdots$ i	y_{i1}	y_{i2}	$\cdots y_{it}$	$\cdots y_{iT}$
$\vdots$ N	y_{N1}	y_{N2}	$\cdots y_{Nt}$	$\cdots y_{NT}$

Panel Data

Ex :- Income (y_{it}) of i th indivisual observed at t th time point; where $i = 1, 2, \ldots, N$
$t = 1, 2, \ldots, T$

<Q> What are the main features that we observe in a time series data?

For example :-



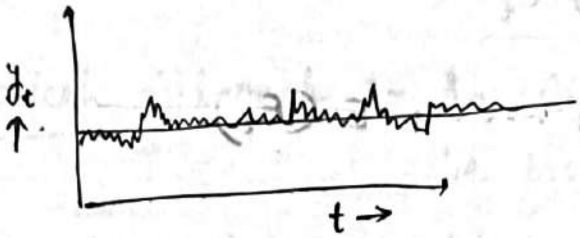
(a) Mean of y_t
(b) Variance of y_t
(c) auto - corr btw y_t & $y_{t'}$ when $t \neq t'$

$E(y_t) = \alpha + \beta t \quad , t = 1(1)N$

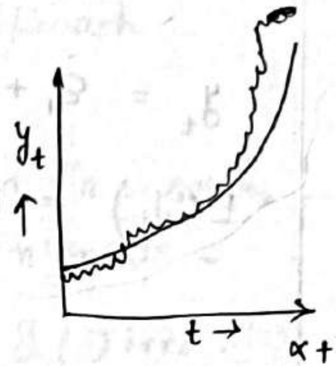
$E(y_1) = \alpha + \beta$

$E(y_2) = \alpha + 2\beta$

$\left\{ \begin{array}{l} \text{increasing mean} \\ \text{increasing variance} \end{array} \right.$



mean constant.



$\alpha + \beta t$.

$E(y_t) = e$

$E(y_t) = \alpha$.

Exercise

⟨1⟩ Let $y_t = \alpha + \varepsilon_t$ where $\varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$, $t = 1(1)N$

Find $E(y_t) = ?\ \alpha$, $V(y_t) = ?\ \sigma^2$

Random walk. $cov(y_t, y_{t+5}) = ?\ 0$

⟨2⟩ $y_t = y_{t-1} + \varepsilon_t$, $t = 1(1)N$ where $\varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$

& $y_0 = 0$. Find $E(y_t) = ?\ 0$

$V(y_t) = ?\ t\sigma^2$

$cov(y_t, y_{t+2}) = ?\ 2\sigma^2$

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Modelling time Series

There are two approaches through which one can model a time series $\{y_t\}$:- (i) Classical Approach
(ii) Stochastic Approach

1) Classical Approach :- In Classical approach a time series can be broken into four components -
(a) Trend , (b) Seasonality , (c) Cyclicality & (d) irregular component

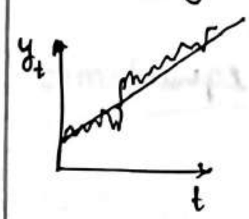
Eg :- let $\{y_t\}$ be a time series then y_t can be written as -

$$y_t = \underbrace{T_t}_{\text{Trend}} + \underbrace{S_t}_{\text{Seasonal}} + \underbrace{C_t}_{\text{cyclical}} + \underbrace{I_t}_{\text{irregular}} \text{ components.}$$

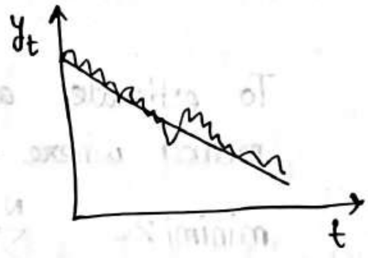
(a) Trend :-

Q1) what is trend of a time series ?
Q2) How to estimate trend ?

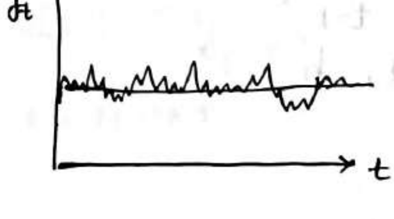
→ Trend a of time series $\{y_t\}$ is defined as long-term movement . e.g whether a time series is increasing or decreasing in the long-run .



increasing trend



Decreasing trend.



constant trend

How to estimate trend ?

There are two ways one can estimate trend —

(i) Parametric method or curve fitting

& (ii) Non-parametric method

(i) Parametric Methods.

we assume a functional form of T_t e.g, we can assume that $T_t = a + bt$ or $T_t = a + bt + ct^2$ or $T_t = e^{\alpha + \beta t}$ or, $T_t = a_0 + a_1 t + \dots + a_p t^p$: a pth degree polynomial function of t.

After assuming a functional form of T_t we need to estimate the parameters like $a_0, a_1, \dots, a_p$ or α, β

To estimate those parameters we use the following steps:

Step 1. Given the time series data $\{y_t\}_{t=1}^{N}$ or $\{y_1, y_2, \dots, y_N\}$.

Step 2. Plot y_t against t

Step 3 Assume a functional form of T_t.

i.e $T_t = a + bt$; $t = 1, 2, \dots, N$.

Step 4 To estimate a, b we use least square method where

minimize $\sum_{t=1}^{N} (y_t - a - bt)^2$

w.r.t a, b.

$$-2\sum_{t=1}^{N}(y_t - a - bt) = 0$$

$$\Rightarrow \sum_{t=1}^{N} y_t - Na - b\sum_{t=1}^{N} t = 0 \quad \text{——— ①}$$

$$\Rightarrow \boxed{a = \bar{y} - b\bar{t}}$$

$$-2\sum_{t=1}^{N}(y_t - a - bt)(-t) = 0$$

$$\Rightarrow \cancel{b\sum_{t=1}^{N} y_t - Nab + b^2\sum_{t=1}^{N} t = 0} \quad \text{——— ②}$$

~~multiplying ① by~~

$$\Rightarrow \sum_{t=1}^{N} t y_t + a\sum_{t=1}^{N} t + b\sum_{t=1}^{N} t^2 = 0$$

$$\hat{b} = \frac{\sum_{t=1}^{N} t y_t - N\bar{t}\bar{y}}{\sum_{t=1}^{N}(t-\bar{t})^2} = \boxed{\frac{\widehat{cov}(y_t, t)}{\widehat{var}(t)}}$$

$$\hat{a} = \bar{y} - \hat{b}\bar{t}.$$

Similarly suppose T_t has the following functional form : where p is known integer

$$T_t = a_0 + a_1 t + \cdots + a_p t^p$$

estimate $a_0, a_1, \ldots, a_p$ minimize $\sum_{t=1}^{N}(y_t - a_0 - a_1 t - \cdots - a_p t^p)^2$ w.r.t $a_0, a_1, \ldots, a_p$

Simulation exercise 1.

1) Generate a time series data of a size $N = 100$ from the following model : $y_t = 5 + 0.6t + \varepsilon_t$ where $\varepsilon_t \overset{iid}{\sim} N(0, 5^2)$; $t = 1(1)100$

Non-parametric Method :-

We estimate the trend using a k-point moving average method given by $\hat{T}_t = \frac{1}{k} \sum_{i=-d}^{d} y_{t+i}$ where $k = 2d+1$

For example :-

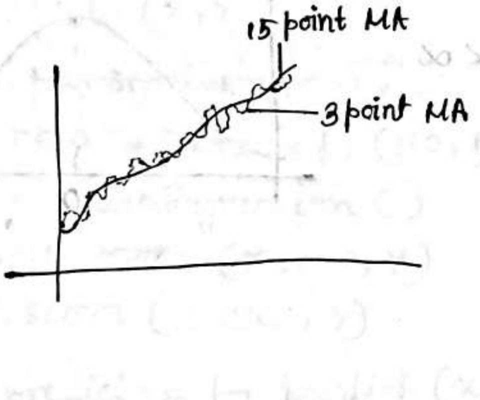
a 3-point moving average method is given by

$\hat{T}_t = \frac{1}{3} \sum_{i=-1}^{1} y_{t+i}$ and

5 point MA method $\hat{T}_t = \frac{1}{5} \sum_{i=-2}^{2} y_{t+i}$

$\boxed{\text{NOTE}}$

Large values of k ⇒ smoother trend



but larger error i.e large value of $\sum_{t=1}^{N} (y_t - \hat{T}_t)^2$

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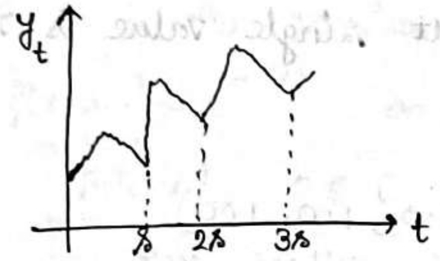
(b) Seasonality :

Seasonality of a time series is a regular pattern observed over s periods of time where s is called the period of seasonality.

Note that after s time points, the same pattern repeats. For example, for monthly data, usual value of $s = 12$ & for quarterly data $s = 4$



Estimation of Seasonality

Method 1 : let $\{y_1, y_2, \cdots, y_N\}$ be a time series
(Average Method) data of size N. After detrending the time series we get $y_t^* = y_t - T_t$; $t = 1, 2, \ldots, N$

Suppose, we have a monthly time series data then we can write the data (detrended) in the following table

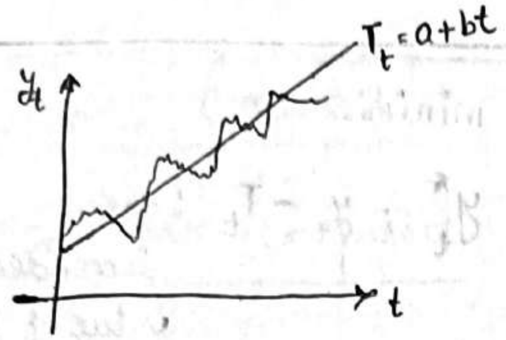
year \ Month	Jan	Feb	Mar	....	Dec
year 1	$y_1 - T_1$	$y_2 - T_2$	$y_3 - T_3$	....	$y_{12} - T_{12}$
year 2	$y_{13} - T_{13}$	$y_{14} - T_{14}$	$y_{15} - T_{15}$	....	$y_{24} - T_{24}$
⋮					
year T	$y_{12(T-1)+1} - T_{12(T-1)+1}$			....	$y_{12T} - T_{12T}$
	$\hat{S}_1$				$\hat{S}_{12} =$

12T-12

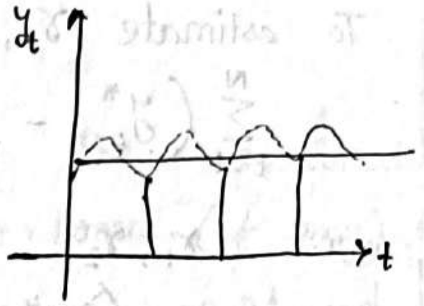
We make the assumptions to estimate the seasonality :

$$S_t = S_{t-s}$$

$$\hat{S}_1 = \frac{1}{T} \sum_{i=1}^{T} y^*_{12(i-1)+1} = \frac{1}{T} \sum_{i=1}^{T} y_{12(i-1)+1} - T_{12(i-1)+1}$$



after detrending



where $T \to$ no of years.

$$\hat{S}_2 = \frac{1}{T}\sum_{i=1}^{T} y^*_{12(i-1)+2}$$

$$\hat{S}_3 = \frac{1}{T}\sum_{i=1}^{T} y^*_{12(i-1)+3}$$

$$\vdots$$

$$\hat{S}_{12} = \frac{1}{T}\sum_{i=1}^{T} y^*_{12(i-1)+12}$$

$$\hat{S}_{13} = \hat{S}_1$$

$$\hat{S}_{14} = \hat{S}_2$$

$$\left(y^*_1 + y^*_{13} + y^*_{25} + \cdots + y^*_{12(T-1)+1}\right)$$

In general suppose you are having a time series data with period of seasonality as s and T numbers of years, then

$$\hat{S}_t = \begin{cases} \frac{1}{T}\sum_{i=1}^{T} y_{s(i-1)+t} & \text{if } 1 \le t \le s \\ \hat{S}_{t(\text{mod } s)} & \text{if } t > s \end{cases}$$

Method 2. (Dummy Variable Method)

Suppose $\{y^*_t\}_{t=1}^{N}$ is the detrended data and s is the period of seasonality then we fit the following regression

$$S_t = \gamma_1 D_{1t} + \gamma_2 D_{2t} + \cdots + \gamma_s D_{st} = \sum_{i=1}^{s} \gamma_i D_{it}$$

where,

$$D_{it} = \begin{cases} 1 & \text{if } t(\text{mod } s) = i \text{ or } t(\text{mod } s) = 0 \\ 0 & ; \text{ow} \end{cases}$$

$D_{it} \downarrow$ Dummy variable

$S_1 = \gamma_1$

$S_2 = \gamma_2$

$\vdots$

$S_s = \gamma_s$

$S_{1+s} = \gamma_1$

$S_{2+s} = \gamma_2$

To estimate $\gamma_1, \gamma_2, \dots, \gamma_s$ we minimize

$$\sum_{t=1}^{N}\left(y_t^* - \sum_{i=1}^{s}\gamma_i D_{it}\right)^2$$ where $y_t^* = y_t - T_t$: the detrended value of y_t

w.r.t $\gamma_1, \gamma_2, \dots, \gamma_s$.

$$\hat{\gamma}_1 = \frac{\widehat{cov}(y_t^*, D_{1t})}{\widehat{Var}(D_{1t})} = \frac{\sum_{t=1}^{N}(y_t^* - \bar{y}^*)(D_{1t} - \bar{D}_1)}{\sum_{t=1}^{N}(D_{1t} - \bar{D}_1)^2}$$

$$\hat{\gamma}_2 = \frac{\widehat{cov}(y_t^*, D_{2t})}{\widehat{Var}(D_{2t})}$$ where, $\bar{y}^* = \frac{1}{N}\sum_{t=1}^{N} y_t^*$ &

$$\bar{D}_1 = \frac{1}{N}\sum_{t=1}^{N} D_{1t}$$

$\vdots$

$$\hat{\gamma}_s = \frac{\widehat{cov}(y_t^*, D_{st})}{\widehat{Var}(D_{st})}$$

* Do the method 1 and method 2 give same estimates of Seasonality ?

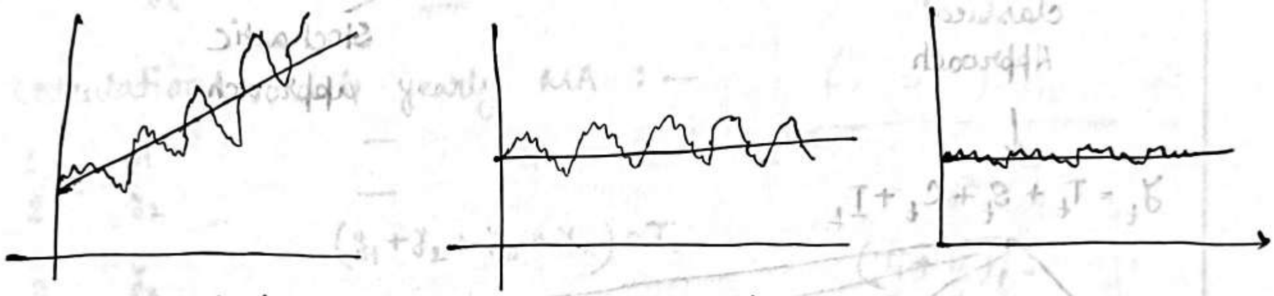
$$\hat{S}_1 = \frac{1}{T}\sum_{i=1}^{T} y^*_{s(i-1)+1}$$

* Show that $\hat{S}_i = \hat{\gamma}_i$ for $i = 1(1)s$.

(c) Cyclical Component :

(d) Irregular Component : After removing trend, seasonality and cyclical component (if any) from y_t, what we have is called irregular component and denoted by $I_t = y_t - \hat{T}_t - \hat{S}_t - \hat{C}_t$

** In case of seasonality the period of repition is usually less than a year whereas for cyclical component the period of repetition is more than one year and it is estimated by some sine ~~cosing~~ - cosine function.



↓

It looks like there is no pattern infact ~~it looks like~~ it seems completely random graph.

Summary :

Suppose you are having a time series $\{y_1, y_2, \dots, y_N\}$ and you would like to know the value of y_t at some future point $t = N+h$. That is given $\{y_1, y_2, \dots, y_N\}$ what shall be the value of y_{N+h} for some $h \geq 1$

Since we know that $y_t = T_t + S_t + C_t + I_t$ where $t = 1(1)N$

Then can we use the formula to forecast the value of y_{N+h} = ?

$$\underbrace{\hat{y}_{N+h}}_{\text{forecasted value of } y_t \text{ at } t = N+h} = \hat{T}_{N+h} + \hat{S}_{N+h} + \hat{C}_{N+h} + \hat{I}_{N+h}$$

$\hat{T}_{N+h} = a + b(N+h)$ if we have linear trend

$$\hat{S}_{N+h} = \hat{S}_{N+h \,(\mathrm{mod}\, s)}$$

$$= \begin{cases} \hat{\gamma}_{N+h\,(\mathrm{mod}\, s)} & \text{if } (N+h)\,(\mathrm{mod}\, s) \neq 0 \\ \hat{\gamma}_s & \text{if } (N+h)\,(\mathrm{mod}\, s) = 0 \end{cases}$$

Modelling Time Series

- classical Approach
  - $y_t = T_t + S_t + C_t + I_t$
    - (a) Estimate trend using parametric and non-parametric method
    - (b) Estimate seasonality by average method or Dummy variable method
    - (c) Estimate cyclicality using sine cosine function.
    - (d) After discussing stochastic approach we discuss estimation of irregular component.
- Stochastic approach.
- Forecasting

STOCHASTIC APPROACH

Let, $\{y_t\}_{t=1}^{T}$ be a time series process ~~with~~ where each y_t is assumed to be random. Hence, we can define the probability distribution for each y_t, $t=1(1)T$ and hence can define $E(y_t)$, $V(y_t)$, $cov(y_{t_1}, y_{t_2})$ $corr(y_{t_1}, y_{t_2})$ and other higher order moments.

For example

Let, $\{y_t\}$ be a time series process defined as

$$y_t = \alpha + \beta t + \varepsilon_t \quad , \; t=1(1)T \text{ where } \varepsilon_t \overset{iid}{\sim} N(0,\sigma^2)$$

Since, ε_t's are random, therefore y_t's are also random / stochastic.

Hence, we can compute the expectation, variance & covariances of y_t's.

Here,

$$\begin{aligned} E(y_t) &= E(\alpha + \beta t + \varepsilon_t) \\ &= \alpha + \beta t + E(\varepsilon_t) \\ &= \alpha + \beta t \qquad [\because E(\varepsilon_t) = 0] \end{aligned}$$

$$\begin{aligned} Var(y_t) &= V(\alpha + \beta t + \varepsilon_t) \\ &= Var(\varepsilon_t) = \sigma^2 \end{aligned}$$

$$\begin{aligned} cov(y_{t_1}, y_{t_2}) &= cov(\alpha + \beta t_1 + \varepsilon_{t_1}, \alpha + \beta t_2 + \varepsilon_{t_2}) \\ &= cov(\varepsilon_{t_1}, \varepsilon_{t_2}) \\ &= \begin{cases} 0 & \text{if } t_1 \neq t_2 \\ \sigma^2 & \text{if } t_1 = t_2 \end{cases} \end{aligned}$$

Exercise.

$\{y_t\}_{t=1}^{T}$ be a TS process

✓(i) $y_t = \varepsilon_t + \theta \varepsilon_{t-1}$ then $E(y_t) = ?$ $V(y_t) = ?$

$cov(y_{t_1}, y_{t_2}) = ?$

✓(ii) $y_t = \theta \varepsilon_t$ then $E(y_t) = ?$ $V(y_t) = ?$ $cov(y_{t_1}, y_{t_2}) = ?$

$\varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$

✓(iii) $y_t = \phi y_{t-1} + \varepsilon_t$ & $y_0 = 0$ find $E(y_t) = ?$

$V(y_t) = ?$

$cov(y_{t_1}, y_{t_2}) = ?$

Stationarity

There are two types of stationarity — (i) weak or covariance stationarity & (ii) strong stationarity.

i) Weak or covariance stationarity :-

A process $\{y_t\}$ is said to be weakly stationary or covariance stationary if it satisfies the following three properties —

(i) $E(y_t)$ or the unconditional mean is independent of t.

(ii) $V(y_t)$ the ~~unconditional~~ marginal variance is independent of t &

(iii) $cov(y_t, y_{t+h})$, the covariance between y_t & y_{t+h} is independent of t but may depend on h.

For example :-

the process $y_t = \varepsilon_t + \mu$ where $\varepsilon_t \overset{iid}{\sim} N(0,\sigma^2)$ is an example of a covariance stionary process.

whereas the process $\{y_t\}$, where $y_t = \alpha + \beta t + \varepsilon_t$, with $\varepsilon_t \overset{iid}{\sim} N(0,\sigma^2)$ is not a covariance stationary process

Exercises :-

Check whether the following process are covariance stationary or not.

(i) $y_t = y_{t-1} + \varepsilon_t$, $\varepsilon_t \overset{iid}{\sim} N(0,\sigma^2)$ & $y_0 = 0$

✓ (ii) $y_t = \varepsilon_t + \theta\varepsilon_{t-1}$, $\varepsilon_t \overset{iid}{\sim} N(0,\sigma^2)$ & θ is a constant

(iii) $y_t = \alpha + \beta t + \varepsilon_t + \theta\varepsilon_{t-1}$, $\varepsilon_t \overset{iid}{\sim} N(0,\sigma^2)$

✓ (iv) $y_t = 0.5\,y_{t-1} + \varepsilon_t$, $\varepsilon_t \overset{iid}{\sim} N(0,\sigma^2)$

(v) $y_t = 5y_{t-1} + \varepsilon_t$, $\varepsilon_t \overset{iid}{\sim} N(0,\sigma^2)$

✓ (vi) $y_t = \alpha + \rho y_{t-1} + \varepsilon_t + \theta\varepsilon_{t-1}$, $\varepsilon_t \overset{iid}{\sim} N(0,\sigma^2)$

NOTE :-

In time Series analysis, we always denote the autocovariance function between y_t & y_{t+h} ~~as~~ by $\gamma_t(h)$ i.e, $cov(y_t, y_{t+h}) = \gamma_t(h)$.

If $\{y_t\}$ is covariance stationary, then $\gamma_t(h)$ will be independent of t and hence for covariance process it will become simply $\gamma(h)$.

NOTE :-

If $h=0$ then $cov(y_t, y_{t+h}) = Var(y_t) = \gamma_t(0)$.

Therefore, Variance of y_t will be denoted by $\gamma_t(0)$.

However if $\{y_t\}$ is covariance stationary then it will be simply $\gamma(0)$.

NOTE :-

whenever we say that $\{y_t\}$ is stationary it means that $\{y_t\}$ is covariance stationary.

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Show that for a stationary process $\gamma(h) = \gamma(-h)$ for all integers h.

$\{y_t\}$ is covariance stationary so $\operatorname{cov}(y_t, y_{t+h}) = \gamma_t(h)$ is independent of t. Hence we can write

$$\operatorname{cov}(y_t, y_{t+h}) = \gamma(h) \quad \forall h$$

$$\gamma(-h) = \operatorname{cov}(y_t, y_{t-h})$$

$$\begin{aligned}
\gamma(h) &= \operatorname{cov}(y_t, y_{t+h}) \\
&= \operatorname{cov}(y_{\overline{t+h}-h}, y_{\overline{t+h}}) \\
&= \operatorname{cov}(y_{t'-h}, y_{t'}) \quad \text{where } t' = t+h \\
&= \operatorname{cov}(y_{t'}, y_{t'-h}) \\
&= \gamma(-h). \quad [\text{independent of } t].
\end{aligned}$$

For Stochastic process generally $\{y_t\}_{-\infty}^{\infty}$

Strict Stationarity or Strong stationarity

A process a said to be strictly stationary if the joint distribution of $(y_{t_1}, y_{t_2}, \dots, y_{t_K})$ is same as the joint distribution of $(y_{t_1+h}, y_{t_2+h}, \dots, y_{t_K+h})$ $\forall$ k & h. ie —

$$f(y_{t_1} = y_1, y_{t_2} = y_2, \dots, y_{t_K} = y_K) = f(y_{t_1+h} = y_1, y_{t_2+h} = y_2, \dots, y_{t_K+h} = y_K)$$

for all k & h.

or, $$f_{t_1, t_2, \dots, t_K}(y_1, y_2, \dots, y_K) = f_{t_1+h, t_2+h, \dots, t_K+h}(y_1, y_2, \dots, y_K)$$

$\forall$ k & h.

Example :-

let $\{y_t\}$ be a TS process given by $y_t = \varepsilon_t + \theta\varepsilon_{t-1}$, $\varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$. This is an example of a strict stationary process.

$\boxed{\text{NOTE}}$

(*) strictly stationarity implies weakly / covariance stationarity not the other way round.

Some Important covariance Stationary Process.

1) While Noise Process :-

A process $\{\varepsilon_t\}$ is said to be WN if $E(\varepsilon_t) = 0$
$V(\varepsilon_t) = \sigma^2$, $cov(\varepsilon_t, \varepsilon_{t+h}) = 0 \;\forall\; h \neq 0$ (uncorrelated across time).

$\varepsilon_t \sim WN(0, \sigma^2)$

2) Moving Average process of order 1 or $MA(1)$.

A process $\{y_t\}$ is said to be $MA(1)$ process if it can be written as - $y_t = \alpha + \varepsilon_t + \theta\varepsilon_{t-1}$ where $\varepsilon_t \sim WN(0,\sigma^2)$

The term MA comes from the fact that y_t is constructed from a weighted sum of the two most recent values of ε_t's.

α, θ are any constant.

For any value of α, θ $MA(1)$ is stationarily.

where, $E(y_t) = \alpha$

$V(y_t) = (1+\theta^2)\sigma^2$

$$cov(y_t, y_{t+h}) = \begin{cases} (1+\theta^2)\sigma^2 & \text{if } h=0 \\ \theta\sigma^2 & \text{if } h=1 \\ 0 & \text{if } h \geq 2 \end{cases}$$

3) Moving Average process of order q or $MA(q)$.

The process is written as $y_t = \alpha + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \dots + \theta_q\varepsilon_{t-q}$

where $\varepsilon_t \sim WN(0,\sigma^2)$ $= \alpha + \sum_{j=0}^{q} \theta_j \varepsilon_{t-j}$; $\theta_0 = 1$.

$\alpha, \theta_1, \dots, \theta_q$ are some constants.

Here also it can be shown that for any values of $\alpha, \theta_1, \theta_2, \dots, \theta_q$ an $MA(q)$ is always stationary.

The mean of y_t is $E(y_t) = \alpha$

$$V(y_t) = V(\varepsilon_t) + \theta_1^2 V(\varepsilon_{t-1}) + \theta_2^2 V(\varepsilon_{t-2}) + \dots + \theta_q^2 V(\varepsilon_{t-q})$$

$$= (1+\theta_1^2 + \theta_2^2 + \dots + \theta_q^2)\sigma^2 = \left(\sum_{j=0}^{q}\theta_j^2\right)\sigma^2$$

$$cov(y_t, y_{t+h}) = \begin{cases} \left(\sum_{j=0}^{q}\theta_j^2\right)\sigma^2 & \text{if } h=0 \\ (\theta_h + \theta_{h+1}\theta_1 + \theta_{h+2}\theta_2 + \dots + \theta_q\theta_{q-h})\sigma^2 & \text{if } 1 \leq h \leq q \\ 0 & \text{if } h > q \end{cases}$$

MA(q) can be written as -

$$y_t = \alpha + \varepsilon_t + \theta_1 (L\varepsilon_t) + \theta_2 (L^2\varepsilon_t) + \cdots + \theta_q (L^q \varepsilon_t)$$

where L is called lag operator.

$$\Rightarrow y_t = \alpha + (1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q)\varepsilon_t$$

$$= \alpha + \Theta(L)\varepsilon_t$$

where, $\Theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q$ is known as characteristic polynomial of MA(q) process.

MA (∞) process :-

A process $\{y_t\}$ is said to be MA(∞) process if it can be written as

$$y_t = \alpha + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots$$

$$= \alpha + \sum_{j=0}^{\infty} \theta_j \varepsilon_{t-j}, \text{ where } \theta_0 = 1.$$

$$= \alpha + \sum_{j=0}^{\infty} \theta_j L^j (\varepsilon_t)$$

$$= \alpha + \Theta(L)\varepsilon_t$$

where, $\Theta(L) = \sum_{j=0}^{\infty} \theta_j L^j$ is the characteristic polynomial of an MA(∞) process.

Ex:- Find $E(y_t)$, $Var(y_t)$ & $\gamma_t(h)$ for MA(∞) process.

$$E(y_t) = \alpha + \sum_{j=0}^{\infty} \theta_j E(\varepsilon_{t-j}) = \alpha.$$

$$Var(y_t) = Var(\varepsilon_t) + \theta_1^2 Var(\varepsilon_{t-1}) + \theta_2^2 Var(\varepsilon_{t-2}) + \cdots$$

$$= (1 + \theta_1^2 + \theta_2^2 + \cdots)\sigma^2 = \sum_{j=0}^{\infty} \theta_j^2 \sigma^2$$

In order to make the variance to be finite we need to have $\sum_{j=0}^{\infty} \theta_j^2 < \infty$.

$$\gamma_t(h) = \text{cov}(y_t, y_{t+h})$$
$$= \text{cov}\left[\left(\alpha + \sum_{j=0}^{\infty}\theta_j \varepsilon_{t-j}\right), \left(\alpha + \sum_{j=0}^{\infty}\theta_j \varepsilon_{t+h-j}\right)\right].$$
$$= (\theta_h + \theta_{h+1}\theta_1 + \theta_{h+2}\theta_2 + \cdots)\sigma^2 \quad \text{if } h>0$$
$$= \left(\sum_{j=0}^{\infty}\theta_{j+h}\theta_j\right)\sigma^2$$

$$\text{cov}\left[(\alpha + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \cdots), (\alpha + \varepsilon_{t+h} + \theta_1\varepsilon_{t+h-1} + \theta_2\varepsilon_{t+h-2} + \cdots)\right]$$

$\underline{h=1}$

$$= (\theta_1 + \theta_1\theta_2 + \theta_2\theta_3 + \cdots)\sigma^2$$

In general h:—

$$(\theta_h + \theta_h\theta_{h+1} + \theta_2\theta_{h+2} + \cdots)\sigma^2.$$

$\boxed{\text{NOTE :-}}$

An MA(∞) process will be stationary if $\sum_{j=0}^{\infty}\theta_j^2 < \infty$ (Square Summability)

When $Var(y_t)$ is stationary given $\sum_{j=0}^{\infty}\theta_j^2 < \infty$ then $\left(\sum_{j=0}^{\infty}\theta_{j+h}\theta_j\right)$ is also finite.

$$|\gamma(h)| < \infty$$

$$\Rightarrow \left|\sum_{j=0}^{\infty}\theta_{j+h}\theta_j\right| \leq \sqrt{\left(\sum\theta_j^2\right)\left(\sum\theta_{j+h}^2\right)}.$$

$\boxed{\text{Note :-}}$

Although the square summability of θ_j in an MA(∞) process is ~~stationary~~ good enough to ensure that an MA(∞) process is stationary, however, we will put a much stronger condition on θ_j's which is called absolute summability

i.e $\sum_{j=0}^{\infty}|\theta_j| < \infty \Rightarrow \sum_{j=0}^{\infty}\theta_j^2 < \infty$

$\sum\theta_j^2 < \infty$

$\sum|\theta_j| < \infty$

$$\downarrow \qquad\qquad \downarrow$$

Stronges ~~weak~~ Weaker.

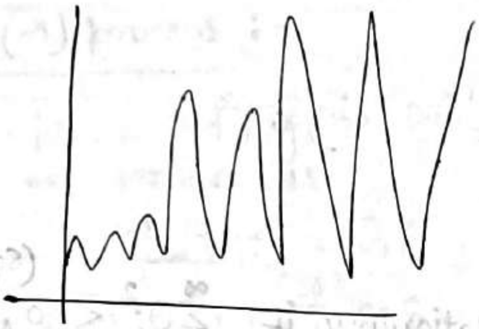
Autoregressive Process of order 1 or AR(1).

A time series $\{y_t\}$ is said to follow an AR(1) process if it can be written as —

$$y_t = \alpha + \phi y_{t-1} + \varepsilon_t \quad \text{where } \varepsilon_t \sim WN(0, \sigma^2)$$

and α, ϕ are constants. [AR(1) with drift].

→ Find the condition on α & ϕ for which an AR(1) process is stationary.



AR(1) process with drift & $|\phi| > 1$.

AR(1) without drift & $|\phi| > 1$

→ **Exercise :-**

Generate a time series data of $N = 100$ from the process and plot it against t.

(i) $y_t = 2 + 1{\cdot}5\, y_{t-1} + \varepsilon_t$, $\varepsilon_t \overset{iid}{\sim} N(0, 25)$

(ii) $y_t = 1{\cdot}5\, y_{t-1} + \varepsilon_t$, $\varepsilon_t \overset{iid}{\sim} N(0, 25)$

AR(1) with drift has mean changing & without drift has mean $= 0$.

If $|\phi| = 1$. then $y_t = \alpha + y_{t-1} + \varepsilon_t \longrightarrow$ Random Walk Process.

Python **Exercise**

Generate a TS of size $N = 100$ from the following process

$y_t = 2 + y_{t-1} + \varepsilon_t$, $\varepsilon_t \overset{iid}{\sim} N(0, 25)$, $y_0 = 0$

(ii) $y_t = y_{t-1} + \varepsilon_t$, $\varepsilon_t \overset{iid}{\sim} N(0,25)$, $y_0 = 0$

~~Plot~~ it against t.

If $|\phi| < 1$ then we guess that the process may be stationary

Proof. (stationary condition of AR(1) process).

$$y_t = \alpha + \phi y_{t-1} + \varepsilon_t$$

$$\Rightarrow y_t = \alpha + \phi L y_t + \varepsilon_t$$

$$\Rightarrow (1 - \phi L) y_t = \alpha + \varepsilon_t$$

$$\Rightarrow \Phi(L) y_t = \alpha + \varepsilon_t$$

where, $\Phi(L) = 1 - \phi L$ is the characteristic polynomial of an AR(1) process.

$$\Rightarrow y_t = \frac{1}{\Phi(L)}\alpha + \frac{1}{\Phi(L)}\varepsilon_t$$ if $\Phi(z) = 0$ has all the roots lie outside of an unit circle.

$$= \Phi^{-1}(L)\alpha + \Phi^{-1}(L)\varepsilon_t.$$

$$= \frac{1}{1-\phi L}\cdot\alpha + \frac{1}{1-\phi L}\varepsilon_t.$$

$\Phi(z) = 1 - \phi(z) = 0$

$\Rightarrow z = \frac{1}{\phi}$ only one root of $\Phi(z) = 0$

$|z| > 1$.

$\Rightarrow \left|\frac{1}{\phi}\right| > 1$.

$\Rightarrow |\phi| < 1$.

~~$(1 + \phi L + \phi^2 L^2)\alpha$.~~

~~$\alpha + \phi\alpha + \phi^2\alpha$.~~

~~$\frac{\alpha}{1-\phi}$.~~

$$\frac{1}{1-\phi L}\cdot\alpha = \frac{1}{(1-\phi L)(\frac{1}{\alpha})} = \frac{\alpha}{1-\phi}.$$

$$= (1-\phi L)^{-1}\alpha = (1 + \phi L + \phi^2 L^2 + \cdots)\alpha$$

$$= (1 + \phi + \phi^2 + \cdots)\alpha = \frac{\alpha}{1-\phi}$$ when $|\phi| < 1$.

$$\therefore y_t = \frac{\alpha}{1-\phi} + \sum_{j=0}^{\infty}\phi^j L^j \varepsilon_t = \mu + \sum_{j=0}^{\infty}\phi^j \varepsilon_{t-j}$$

$\longrightarrow$ MA(∞) process with $|\phi| < 1$.

NOTE :-

An AR(1) process can be represented as MA(∞) ~~from~~ form if $|\phi| < 1$ or $\Phi(z) = 0$ has all the roots lie outside an unit circle.

NOTE :-

Since any MA(∞) process is always stationary therefore therefore an AR(1) process is also stationary if $|\phi| < 1$.

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AR(1) Process :-

$$y_t = \alpha + \phi y_{t-1} + \varepsilon_t$$

If $|\phi| \geq 1$, there does not exist a covariance stationary process for $\{y_t\}$ with finite variance. That is if $|\phi| \geq 1$ then $Var(y_t) \to \infty$ as $t \to \infty$.

When $|\phi| < 1$, the process is stationary and can be written as : $y_t = \frac{\alpha}{1-\phi} + \sum_{j=0}^{\infty} \phi^j \varepsilon_{t-j}$

Note that ;

$$\sum_{j=0}^{\infty} |\phi|^j = \frac{1}{1-|\phi|} < \infty \text{ if } |\phi| < 1.$$

Therefore, the coefficients of MA(∞) representation is absolutely summable.

AR(p) process.

An AR(p) process can be written as

$$y_t = \alpha + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t$$ where

$\varepsilon_t \sim WN(0, \sigma^2)$ where , $\alpha, \phi_1, \phi_2, \cdots \phi_p$ are all constants.

Condition for stationarity of an AR(p) process :—

$$y_t = \alpha + \phi_1 L y_t + \phi_2 L^2 y_t + \cdots + \phi_p L^p y_t + \varepsilon_t .$$

$$\Rightarrow y_t = \alpha + \sum_{j=1}^{p} \phi_j L^j y_t + \varepsilon_t .$$

$$\Rightarrow \left[1 - \sum_{j=1}^{p} \phi_j L^j\right] y_t = \alpha + \varepsilon_t .$$

$$\Rightarrow \Phi(L) y_t = \alpha + \varepsilon_t$$

where $\Phi(L) = (1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p)$ is called the characteristic ~~equation~~ polynomial of an AR(p) process.

If all the roots of the characteristic equation $\Phi(z) = 0$ lie outside an unit circle, then the process becomes stationary.

Suppose, $z_1, z_2, \ldots, z_p$ are the roots of $\Phi(z) = 0$ then if. $|z_i| > 1 \;\forall\; i = 1(1)p$ then we say that AR(p) is stationary.

for p = 1, or AR(1) process.

$$\Phi(z) = 0$$

$$\Rightarrow 1 - \phi_1 z = 0$$

$$\Rightarrow z = \frac{1}{\phi_1}$$

$$|z| > 1$$

$\left|\frac{1}{\phi_1}\right| > 1 \Rightarrow |\phi_1| < 1$. then the process is stationary.

For $p=2$, AR(2) process.

$$1-\phi_1 L-\phi_2 L^2 \rightarrow \text{polynomial.}$$

$\Phi(z)=0$

$\Rightarrow 1-\phi_1 z-\phi_2 z^2=0.$

$\Rightarrow \phi_2 z^2+\phi_1 z-1=0$

$$z=\frac{-\phi_1 \pm \sqrt{\phi_1^2+4\phi_2}}{2\cdot\phi_2} \therefore \text{two roots are.}$$

$$z_1=\frac{-\phi_1+\sqrt{\phi_1^2+4\phi_2}}{2\phi_2} \text{ and } z_2=\frac{-\phi_1-\sqrt{\phi_1^2+4\phi_2}}{2\phi_2}.$$

The AR(2) process is stationary if $|z_1|>1$ & $|z_2|>1$.

$$\Rightarrow \left|\frac{-\phi_1+\sqrt{\phi_1^2+4\phi_2}}{2\phi_2}\right|>1. \text{ and } \left|\frac{-\phi_1-\sqrt{\phi_1^2+4\phi_2}}{2\phi_2}\right|>1.$$

When roots are complex $z=$ ~~a~~ $a\pm ib$

Then $|z|=\sqrt{a^2+b^2}$

1) Exercise: let $y_t=2+0.5y_{t-1}+0.7y_{t-2}+\varepsilon_t$ where $\varepsilon_t\sim WN(0,\sigma^2)$. Check whether the process is stationary or not.

2) Let $y_t=5+1.5y_{t-1}+\varepsilon_t$ where $\varepsilon_t\sim WN(0,\sigma^2)$ Check whether the process is stationary.

3) Let $y_t=5+\varepsilon_t+1.5\varepsilon_{t-1}$ where $\varepsilon_t\sim WN(0,\sigma^2)$ Check whether stationary or not.

4) $y_t=3+\varepsilon_t+1.5\varepsilon_{t-1}+(1.5)^2\varepsilon_{t-2}+\cdots$ where $\varepsilon_t\sim WN(0,\sigma^2)$ Check whether the process is stationary or not.

Problem: Suppose, $\{y_t\}$ is a stationary $AR(p)$ process. Then find $E(y_t)$, $V(y_t)$ & $\gamma(h) = cov(y_t, y_{t+h})$.

Ans: – $y_t \sim$ Stationary $AR(p)$

$$y_t = \alpha + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t \quad \text{where } \varepsilon_t \sim WN(0, \sigma^2)$$

$$\Rightarrow E(y_t) = \alpha + \phi_1 E(y_{t-1}) + \phi_2 E(y_{t-2}) + \cdots + \phi_p E(y_{t-p}) + E(\varepsilon_t)$$

$$\Rightarrow \mu = \alpha + \phi_1 \mu + \phi_2 \mu + \cdots + \phi_p \mu + 0$$

$$\Rightarrow (1 - \phi_1 - \phi_2 - \cdots - \phi_p)\mu = \alpha$$

$$\Rightarrow \mu = \frac{\alpha}{1 - \sum_{j=1}^{p} \phi_j} \Rightarrow \alpha = \left(1 - \sum_{j=1}^{p} \phi_j\right)\mu.$$

$$\cancel{V(y_t) = \phi_1^2 V(y_{t-1}) + \phi_2^2 V(y_{t-2}) + \cdots + \phi_p^2 V(y_{t-p}) + V(\varepsilon_t)}$$

$$\cancel{\Rightarrow (1 - \phi_1^2 - \phi_2^2 - \cdots - \phi_p^2) V(y_t) = \sigma^2} \quad \cancel{[V(y_t) \text{ does not depend on } t]}.$$

$\Rightarrow$

$$y_t = \left(1 - \sum_{j=1}^{p} \phi_j\right)\mu + \sum_{j=1}^{p} \phi_j y_{t-j} + \varepsilon_t$$

$$\Rightarrow (y_t - \mu) = \sum_{j=1}^{p} \phi_j (y_{t-j} - \mu) + \varepsilon_t$$

$$\Rightarrow (y_t - \mu)^2 = \left\{ \sum_{j=1}^{p} \phi_j (y_{t-j} - \mu) + \varepsilon_t \right\} (y_t - \mu)$$

[Multiplying both sides by $(y_t - \mu)$].

$$\Rightarrow (y_t - \mu)^2 = \sum_{j=1}^{p} \phi_j (y_{t-j} - \mu)(y_t - \mu) + \varepsilon_t (y_t - \mu)$$

$$\Rightarrow E\left[(y_t - \mu)^2\right] = E\left[\sum_{j=1}^{p} \phi_j (y_{t-j} - \mu)(y_t - \mu) + \varepsilon_t (y_t - \mu)\right].$$

$$= \sum_{j=1}^{p} \phi_j E\left[(y_{t-j} - \mu)(y_t - \mu)\right] + E\left[\varepsilon_t (y_t - \mu)\right]$$

$$= \sum_{j=1}^{p} \phi_j \, cov(y_t, y_{t-j}) + E\left[\varepsilon_t (y_t - \mu)\right]$$

$$= \sum_{j=1}^{p} \phi_j \gamma(j) + E\left[\varepsilon_t (y_t - \mu)\right]$$

$$y_t = \alpha + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \varepsilon_t.$$

$$E(y_t) = \alpha + \phi_1 E(y_{t-1}) + \cdots + \phi_p E(y_{t-p}) + E(\varepsilon_t).$$

$$\Rightarrow E(y_t) = (\alpha + \phi_1 \mu + \cdots + \phi_p \mu) + 0$$

$$\Rightarrow y_t - \alpha - \phi_1 y_{t-1} - \cdots - \phi_p y_{t-p} = \varepsilon_t.$$

$$\therefore (y_t - \mu) = \varepsilon_t.$$

$$E(\varepsilon_t^2) = \sigma^2.$$

$$E\left[\varepsilon_t (y_t - \mu)\right]$$

$$= E\left[\varepsilon_t \left(\sum_{j=1}^{p} \phi_j (y_{t-j} - \mu) + \varepsilon_t\right)\right]$$

$$= \sum_{j=1}^{p} \phi_j \, E\left\{\varepsilon_t (y_{t-j} - \mu)\right\} + E(\varepsilon_t^2)$$

Assuming ε_t & y_{t-j} are uncorrelated for all $j > 0$, $\quad cov(y_{t-j}, \varepsilon_t) = 0.$

we get –

$$E\left[\varepsilon_t (y_{t-j} - \mu)\right] = 0 \quad \forall j > 0 \qquad y_{t-j} = f(\varepsilon_t, \varepsilon_t$$

$$\therefore E\left[\varepsilon_t (y_t - \mu)\right] = E(\varepsilon_t^2) = \sigma^2$$

$$\therefore \gamma(0) = \sum_{j=1}^{p} \phi_j \gamma(j) + \sigma^2$$

$$= \phi_1 \gamma(1) + \phi_2 \gamma(2) + \cdots + \phi_p \gamma(p) + \sigma^2.$$

$$(y_t - \mu) = \sum_{j=1}^{p} \phi_j (y_{t-j} - \mu) + \varepsilon_t$$

Multiply both sides by $(y_{t-h} - \mu)$ & taking expectation.

$$E\left[(y_t - \mu)(y_{t-h} - \mu)\right] = \sum_{j=1}^{p} \phi_p \left[E(y_{t-j} - \mu)(y_{t-h} - \mu)\right] + E\left[\varepsilon_t (y_{t-h} - \mu)\right]$$

$$\Rightarrow \gamma(h) = \sum_{j=1}^{p} \phi_j \gamma(h-j) + \underline{\underline{E\left[\varepsilon_t (y_{t-h} - \mu)\right]}} \; ; h>0$$

$$\because E\left[\varepsilon_t (y_{t-j} - \mu)\right] = 0 .$$

$$\gamma(h) = \sum_{j=1}^{p} \phi_j \gamma(h-j) \quad \forall \quad h>0$$

$$= \phi_1 \gamma(h-1) + \phi_2 \gamma(h-2) + \cdots + \phi_p \gamma(h-p)$$

<u>Mean</u> : $E(y_t) = \mu = \dfrac{\alpha}{1 - \sum_{j=1}^{p} \phi_j}$

<u>Variance</u> : $V(y_t) = \gamma(0) = \sum_{j=1}^{p} \phi_j \gamma(j) + \sigma^2$

<u>Autocovariance</u> :

$$\gamma(h) = \sum_{j=1}^{p} \phi_j \gamma(h-j) \quad \forall \quad h>0 .$$

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Assignment 1 :-

(1) Find the autocorrelation function of ARMA(1,1) process.

(2) Let $y_t = \alpha + \beta t + \gamma t^2 + \varepsilon_t$, $t = 1(1)n$ & $\varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$. Find the LS estimators of α, β, γ. Find the variance formula of it (i.e find $Var(\hat{\alpha}_{OLS})$, $Var(\hat{\beta}_{OLS})$ & $Var(\hat{\gamma}_{OLS})$.

(3) Show that for any stationary process, $\gamma(h) = \gamma(-h) \; \forall \; h \neq 0$

(4) Let $y_t = 2.5 + \varepsilon_t + 0.5\,\varepsilon_{t-1}$, $\varepsilon_t \sim WN(0, \sigma^2)$. Show that
$$\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$$

(5) Let $y_t = \varepsilon_t + 2\varepsilon_{t-1} + 3\varepsilon_{t-2}$ Find the auto-correlation of ∇y_t where $\nabla y_t = y_t - y_{t-1}$ and $\varepsilon_t \sim WN(0, \sigma^2)$

(6) $y_t = 10 + 2.5\, y_{t-1} + 0.5\, y_{t-2} + \varepsilon_t + 0.7\,\varepsilon_{t-1} + 0.9\,\varepsilon_{t-2} + 1.5\,\varepsilon_{t-3}$
check whether the above process is stationary or not.

(7) Let $y_t = \alpha + \beta t + \gamma_1 D_{1t} + \gamma_2 D_{2t} + \gamma_3 D_{3t} + \gamma_4 D_{4t} + \varepsilon_t$, $\varepsilon_t \sim WN(0, \sigma^2)$. Find the LS estimators of $\gamma_1, \gamma_2, \gamma_3, \gamma_4$.

(8) Suppose you are given a time series data of size 10 given by $\{1, 3, 4, 2, 7, 9, 10, 6, 5, 7\}$. Suppose one fits a linear trend $y_t = \alpha + \beta t$ and another one uses a 3-point moving average method to estimate trend. Whose method is better and why?

ARMA process :-

A time series $\{y_t\}$ is said to follow an ARMA(p, q) process if it can be written as –

$$y_t = \alpha + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}$$

$$= \alpha + \underbrace{\sum_{i=1}^{p} \phi_i y_{t-i}}_{AR(p)} + \underbrace{\sum_{j=1}^{q} \theta_j \varepsilon_{t-j}}_{MA(q)} \; ; \theta_0 = 1.$$

where, $\varepsilon_t \sim WN(0, \sigma^2)$ & $\alpha, \phi_1, \ldots, \phi_p, \theta_1, \ldots, \theta_q$ are constants.

$$\Rightarrow y_t - \sum_{i=1}^{p} \phi_i y_{t-i} = \alpha + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j}$$

$$\Rightarrow y_t - \sum_{i=1}^{p} \phi_i L^i y_t = \alpha + \sum_{j=1}^{q} \theta_j L^j \varepsilon_t$$

$$\Rightarrow \left(1 - \sum_{i=1}^{p} \phi_i L^i\right) y_t = \alpha + \left(\sum_{j=0}^{q} \theta_j L^j\right) \varepsilon_t$$

$$\Rightarrow \Phi(L) y_t = \alpha + \Theta(L) \varepsilon_t$$

where, $\Phi(L) = 1 - \phi_1 L - \phi_2 L^2 + \cdots + \phi_p L^p$ is called p-order polynomial of AR(p) process.

$\Theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q$ is the q^{th} order polynomial of MA(q) process.

$$\Rightarrow y_t = \frac{\alpha}{\Phi(L)} + \frac{\Theta(L)\, \varepsilon_t}{\Phi(L)} \text{ where } \Psi(L) = \frac{\Theta(L)}{\Phi(L)}$$

$$= \alpha^* + \Psi(L)\, \varepsilon_t$$

Condition for Stationarity

If all the roots of $\Phi(z) = 0$ lie outside an unit circle then the process becomes stationary irrespective of the values of MA(q) parameters $\theta_1, \theta_2, \ldots, \theta_q$.
That is if $z_1, z_2, \ldots, z_p$ be the p-roots of $\Phi(z) = 0$ then $|z_i| > 1 \;\forall\; i = 1(1)p$

$$\mu = \frac{\alpha}{\Phi(L)} = \frac{\alpha}{1-\phi_1 L_1 - \cdots - \phi_p L^p} = \frac{1}{1-\phi_1 L_1 - \cdots - \phi_p L^p \left(\frac{1}{\alpha}\right)}$$

$$= \frac{1}{1-\phi_1 L_1 \left(\frac{1}{\alpha}\right) - \cdots - \phi_p L^p \left(\frac{1}{\alpha}\right)} = \frac{\alpha}{\left(1-\sum_{i=1}^{p}\phi_j\right)}$$

$$\Rightarrow \boxed{\alpha = \mu\left(1-\sum_{j=1}^{p}\phi_j\right)}$$

$$\Psi(L)\,\varepsilon_t = \frac{\Theta(L)}{\Phi(L)}\varepsilon_t = \left(1-\phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p\right)^{-1}\left(1+\theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q\right)\varepsilon_t$$

$$= \{(z_1 - L)(z_2 - L)\cdots(z_p - L)\}^{-1}\{1+\theta_1 L + \cdots + \theta_q L^q)\varepsilon_t$$

$$= (z_1-L)^{-1}(z_2-L)^{-1}\cdots(z_p-L)^{-1}(1+\theta_1 L + \cdots + \theta_q L^q)\varepsilon_t$$

$$= \left(\sum_{j=0}^{\infty}\psi_{1j}L^j\right)\left(\sum_{j=1}^{\infty}\psi_{2j}L^j\right)\cdots\left(\sum_{j=1}^{\infty}\psi_{pj}L^j\right)\left(1+\sum_{j=1}^{q}\theta_j L^j\right)\varepsilon_t$$

$$= \underbrace{\left(\sum_{j=0}^{\infty}\psi_j L^j\right)}_{\Psi(L)}\varepsilon_t$$

$$\therefore y_t = \frac{\alpha}{1-\sum_{j=1}^{p}\phi_j} + \left(\sum_{j=0}^{\infty}\psi_j L^j\right)\varepsilon_t \longrightarrow MA(\infty) \text{ process}$$

Mean, variances, covariances

$$y_t = \alpha + \sum_{i=1}^{p}\phi_i y_{t-i} + \sum_{j=0}^{q}\theta_j\varepsilon_{t-j}\,; \quad \theta_0 = 1,\ \varepsilon_t \sim WN(0,\sigma^2)$$

$$\Rightarrow y_t = \left(1-\sum_{i=1}^{p}\phi_i\right)\mu + \sum_{i=1}^{p}\phi_i y_{t-i} + \sum_{j=0}^{q}\theta_j\varepsilon_{t-j}$$

$$\Rightarrow (y_t - \mu) = \sum_{i=1}^{p}\phi_i(y_{t-i}-\mu) + \sum_{j=0}^{q}\theta_j\varepsilon_{t-j}$$

$$\Rightarrow (y_t-\mu)^2 = \sum_{i=1}^{p}\phi_i(y_{t-i}-\mu)(y_t-\mu) + \sum_{j=1}^{q}\theta_j\varepsilon_{t-j}(y_t-\mu)$$

$$Var(y_t-\mu) = \sum_{i=1}^{p}\phi_i\,\gamma(i) + \;??$$

Suppose, $x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + w_t$ AR(p)

$E(x_t x_{t-h}) = \phi_1 E(x_{t-1} x_{t-h}) + \phi_2 E(x_{t-2} x_{t-h}) + \cdots + \phi_p E(x_{t-p} x_{t-h}) + E(w_t x_{t-h})$

Exercise : $\gamma(h) = \phi_1 \gamma(h-1) + \phi_2 \gamma(h-2) + \cdots + \phi_p \gamma(h-p)$ $h > 0$.

Show that for an ARMA (p, q) process $\gamma(h) = \phi_1 \gamma(h-1) + \cdots + \phi_p \gamma(h-p)$ if $h > q$.

[For $0 \le h \le q$ the expression is complicated in nature].

$(y_t - \mu)(y_{t-h} - \mu)$

$= \left[\sum_{i=1}^{p} \phi_i (y_{t-i} - \mu) + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j}\right]\left[\sum_{i=1}^{p} \phi_i (y_{t-h-i} - \mu) + \sum_{j=1}^{q} \theta_j \varepsilon_{t-h-j}\right]$

$= [\phi_1 (y_{t-1} - \mu) + \phi_2 (y_{t-2} - \mu) + \cdots \phi_p (y_{t-p} - \mu) + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q}]$

$[\phi_1 (y_{t-h-1} - \mu) + \phi_2 (y_{t-h-2} - \mu) + \cdots + \phi_p (y_{t-h-p} - \mu) + \theta_1 \varepsilon_{t-h-1} + \theta_2 \varepsilon_{t-h-2} + \cdots + \theta_q \varepsilon_{t-h-q}$

$=$

$\gamma(h) = \text{cov}(y_{t+h}, y_t)$

$= E\left[\sum_{i=1}^{p} \phi_i (y_{t-i+h} - \mu) + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j+h}\right] y_t$ Assuming $E(y_t) = \mu = 0$

$= \sum_{i=1}^{p} \phi_i E(y_{t-i+h} y_t) + \sum_{j=1}^{q} \theta_j E(\varepsilon_{t+h-j} y_t)$

$= \sum_{i=1}^{p} \phi_i \gamma(h-i) + \sigma^2 \sum_{j=h}^{q} \theta_j \psi_{j-h}$ when $i > 0$ the y_t and ε_{t+i} are uncorrelated

for $i < 0$ $\psi_i = 0$

$\gamma(h) = \phi_1 \gamma(h-1) + \cdots + \phi_p \gamma(h-p)$ if $h > q$

Autoregressive Integrated Moving Average process or ARIMA process : -

First let's define the following operators :

i) Lag operator :- A lag operator is denoted by L & is defined as $L y_t = y_{t-1}$

It can be shown that from the above definition.

$L^2 y_t = y_{t-2}$

$L^3 y_t = y_{t-3}$

$L^d y_t = y_{t-d}$

(ii) Differencing operator : is denoted by ∇ and is defined as

$$\nabla y_t = y_t - y_{t-1}$$

Under the above definition

$$\nabla y_t = y_t - y_{t-1}$$
$$= y_t - Ly_t$$
$$= (1-L) y_t$$

$\therefore \boxed{\nabla = 1 - L}$

Exercise 1.

Find $\nabla^2 y_t$, $\nabla^3 y_t$ & hence $\nabla^d y_t$ for any integer d.

Definition : ARIMA (p,d,q)

A time series $\{y_t\}$ is said to follow an ARIMA process of order p, d, q if $\nabla^d y_t$ follows an ARMA (p,q) process.

That is $y_t^* = \nabla^d y_t \sim \text{ARMA}(p,q)$

$$\Rightarrow y_t^* = \alpha + \phi_1 y_{t-1}^* + \cdots + \phi_p y_{t-p}^* + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}$$

$$\Rightarrow \Phi(L) y_t^* = \alpha + \Theta(L) \varepsilon_t \text{ where,}$$

$$\Phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p$$

$$\Theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q$$

$$\Rightarrow \Phi(L) \nabla^d y_t = \alpha + \Theta(L) \varepsilon_t .$$

If $\nabla^d y_t \sim$ stationary ARMA (p,q)

then, $$\nabla^d y_t = \frac{\alpha}{\Phi(L)} + \frac{\Theta(L) \varepsilon_t}{\Phi(L)}$$

Seasonal ARIMA process : (SARIMA) $(p,d,q) \times (P,D,Q)_s$

Seasonal differencing operator (∇_s) is defined as

~~$\nabla_s y_t = y_t - y_{t-s} = y_t - L^s y_t = (1-L^s) y_t$~~

~~$\Rightarrow \nabla_s = 1 - L^s$~~

if $\nabla_s^D \nabla^d y_t$ follows a ~~ARMA~~ SARMA process of order $(p,q) \times (P,Q)_s$.

What is SARMA process of order $(p,q) \times (P,Q)_s$?

A process $\{y_t\}$ is said to be SARMA $(p,q)\times(P,Q)_s$ if it can be written as

$$\Phi_s(L) \Phi(L) y_t = \alpha + \Theta_s(L) \Theta(L) \varepsilon_t$$

where $\varepsilon_t \sim WN(0, \sigma^2)$

$\Phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p$

characteristic polynomial of AR part.

**

SARIMA

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$\Phi_s(L) = 1 - \Phi_1 L^s - \Phi_2 L^{2s} - \dots - \Phi_p L^{ps}$: Characteristic polynomial of Seasonal AR part.

$\theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \dots + \theta_q L^q$: ch polynomial of MA part.

$\Theta_s(L) = 1 + \Theta_1 L^s + \Theta_2 L^{2s} + \dots + \Theta_Q L^{Qs}$: ch polynomial of Seasonal MA part.

& s is the seasonal frequency & $\phi_1, \phi_2 \dots \phi_p, \Phi_1, \Phi_2, \dots, \Phi_P, \theta_1, \dots, \theta_q, \Theta_1, \Theta_2, \dots, \Theta_Q$ are all constants

$\Rightarrow \Phi_s(L)\ \phi(L) y_t = \alpha + \Theta_s(L)\theta(L)\varepsilon_t$

$\Rightarrow (1 - \Phi_1 L^s - \Phi_2 L^{2s} - \dots + \Phi_P L^{Ps})(1 - \phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p) y_t$

$= \alpha + (1 + \Theta_1 L^s + \Theta_2 L^{2s} + \dots + \Theta_Q L^{Qs})\ ~~where~~$

$(1 + \theta_1 L + \dots + \theta_q L^q)\varepsilon_t$

In particular suppose $y_t \sim \text{SARMA} \underset{p,q}{(1,0)} \times \underset{P,Q}{(1,0)}_{\underset{s}{12}}$

According to the above definition we can write

$\underbrace{\Phi_s(L)}_{\text{SAR poly}}\ \underbrace{\phi(L) y_t}_{\text{AR poly}} = \alpha + \underbrace{\Theta_s(L)}_{\text{SMA poly}}\underbrace{\theta(L)}_{\text{MA poly}}\varepsilon_t.$

$\Phi_s(L) = 1 - \Phi_1 L^{12}$

$\phi(L) = 1 - \phi_1 L$

$\Theta_s(L) = 1$

$\theta(L) = 1$

Then,

$(1 - \Phi_1 L^{12})(1 - \phi_1 L) y_t = \alpha + 1 \times 1\,\varepsilon_t$

$\Rightarrow (1 - \phi_1 L - \Phi_1 L^{12} + \Phi_1 \phi_1 L^{13}) y_t = \alpha + \varepsilon_t$

$\Rightarrow y_t - \phi_1 y_{t-1} - \Phi_1 y_{t-12} + \Phi_1 \phi_1 y_{t-13} = \alpha + \varepsilon_t$

$\Rightarrow y_t = \alpha + \phi_1 y_{t-1} + \Phi_1 y_{t-12} - \Phi_1 \phi_1 y_{t-13} + \varepsilon_t$

AR(13) process.

How does SAR(1) process look like?

- SAR (1)
$= \text{SARMA } (1,0) \times (1 \times 0)_s$

$\text{SARMA } (p, q) \times (P, Q)_s$

or, $\text{SARIMA } (p, d, q) \times (P, D, Q)_s$

When $d = 0$ then SARIMA = SARMA

<u>SARIMA $(p, d, q) \times (P, D, Q)_s$</u>

$$y_t \sim \text{SARIMA } (p, d, q) \times (P, D, Q)_s$$

if $y_t^* = \nabla_s^D \nabla^d y_t \sim \text{SARMA}(p, q) \times (P, Q)_s$

$\Rightarrow y_t^* = (1 - L^s)^D (1 - L)^d y_t \sim \text{SARMA } (p, q) \times (P, Q)_s$

Then we write, $\Phi_s(L)\, \phi(L)\, y_t^* = \alpha + \Theta_s(L)\, \Theta(L)\, \varepsilon_t$

$\Rightarrow \Phi_s(L)\, \phi(L)\, \nabla_s^D \nabla^d y_t = \alpha + \Theta_s(L)\, \Theta(L)\, \varepsilon_t$

<u>Example</u>

Suppose $y_t \sim \text{SARIMA}(\underset{p}{1}, \underset{d}{1}, \underset{q}{0}) \times (\underset{p}{1}, \underset{d}{1}, \underset{q}{0})_{\underset{s}{12}}$ $\quad y_t^* = \nabla_{12} \nabla y_t$

$\Phi_s(L) = 1 - \Phi_1 L^{12}$

$\phi(L) = 1 - \phi_1 L$

$\Theta_s(L) = 1$

$\Theta(L) = 1$

$y_t^* = (1 - L^{12})(1 - L)\, y_t$
$= (1 - L^{12} - L + L^{13})\, y_t$

$\therefore$ The process is

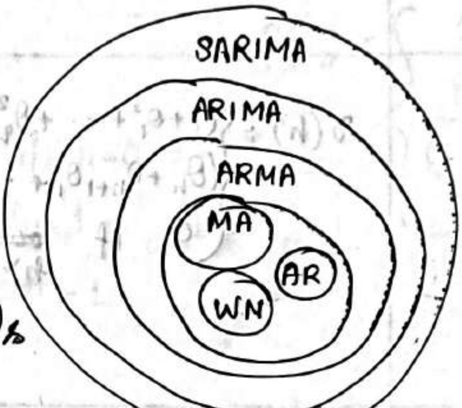
$$(1 - \Phi_1 L^{12})(1 - \phi_1 L)(y_t - y_{t-12} - y_{t-1} + y_{t-13})$$
$$= \alpha + \varepsilon_t$$

$$\Rightarrow (1 - \Phi_1 L^{12} - \phi_1 L + \Phi_1 \phi_1 L^{13})$$
$$(y_t - y_{t-12} - y_{t-1} + y_{t-13}) = \alpha + \varepsilon_t$$

Box - Jenkins Approach

Stochastic Approach

- **White Noise process**
  - $E(y_t) = \mu_t$
  - $var(y_t) = \gamma(0)$
  - $cov(y_t, y_{t+h}) = \gamma(h)$
  - $\rho(h) = \dfrac{\gamma(h)}{\gamma(0)}$
- **AR process**
  - μ_t
  - $\gamma(0)$
  - $\gamma(h)$
- **MA process**
  - μ_t
  - $\gamma(0)$
  - $\gamma(h)$
- **ARMA**
  - $\mu(t)$
  - $\gamma(0)$
  - $\gamma(h)$
- **ARIMA**
  - ?
- **SARIMA**
  - ?



(i) $y_t \sim SARIMA(1,0,0) \times (0,0,0)_s$

$\Rightarrow y_t \sim AR(1)$

(ii) $y_t \sim SARIMA(0,0,2) \times (0,0,0)_s$

$\Rightarrow y_t \sim MA(2)$

(iii) $y_t \sim SARIMA(1,0,1) \times (0,0,0)_s$

$\Rightarrow y_t \sim ARMA(1,1)$

(iv) $y_t \sim SARIMA(1,1,1) \times (0,0,0)_s$

$\Rightarrow y_t \sim ARIMA(1,1,1)$

when ∇y_t can capture Trend & Seasonality then take $\nabla_s y_t$ will take care of Seasonality

Take ∇y_t then analyse data then check $\nabla_s y_t$ this will give better result.

Recap of Autocovariance & Autocorrelation function :-

<u>ACVF or acvf</u> and <u>ACF or acf</u>

For any stationary time series, ACVF is defined as $\gamma(h) = Cov(y_t, y_{t+h})$ & ACF is denoted by

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)}$$

If $h=0$, $\gamma(h) = \gamma(0)$: variance of y_t & $\rho(h) = \rho(0) = 1$.

	Process	ACVF ($\gamma(h)$)	ACF ($\rho(h)$)
1)	White Noise Process	$\gamma(h) = \begin{cases} \sigma^2 & \text{if } h=0 \\ 0 & \text{o/w } h>0 \end{cases}$	$\rho(h) = \begin{cases} 1 & \text{if } h=0 \\ 0 & \text{o/w } h>0 \end{cases}$
2)	MA process (q)	$\gamma(h) = \begin{cases} (1+\theta_1^2+\cdots+\theta_q^2)\sigma^2 & \text{if } h=0 \\ (\theta_h+\theta_{h+1}\theta_1+\cdots+\theta_{h+q}\theta_q) & 1\le h\le q \\ 0 & \text{if } h>q \end{cases}$	$\begin{cases} \rho(h) = 1 & \text{if } h=0 \\ \dfrac{\theta_h+\theta_{h+1}\theta_1+\cdots+\theta_{h+q}\theta_q}{1+\theta_1^2+\cdots+\theta_q^2} & \text{if } 1\le h\le q \\ 0 & h>q \end{cases}$
3)	AR(p) process	$\gamma(h) = \begin{cases} \phi_1\gamma(h-1)+\phi_2\gamma(h-2)+\cdots+\phi_p\gamma(h-p)+\sigma^2 & \text{if } h=0 \\ = \phi_1\gamma(h-1)+\phi_2\gamma(h-2)+\cdots+\phi_p\gamma(h-p) & \text{if } h>0 \end{cases}$	$\rho(h) = \begin{cases} \phi_1\rho(h-1)+\cdots+\phi_p\rho(h-p) & \forall\, h>0 \\ 1 & \text{if } h=0 \end{cases}$
4)	ARMA (p, q) process.	$\gamma(h) = \phi_1\gamma(h-1)+\phi_2\gamma(h-2)+\cdots+\phi_p\gamma(h-p)$ for $h>q$	

Correlogram

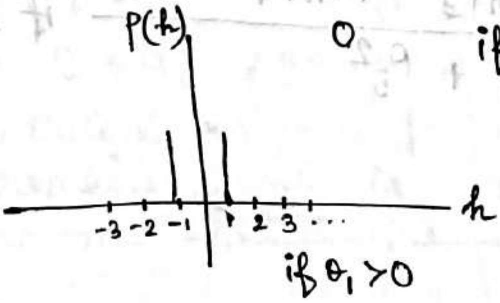
A plot of acf or $\rho(h)$ against h is called correlogram of a time series process.

Example

let, $y_t \sim MA(1)$ process. Then what will be the correlogram of it?

$$\gamma(h) = cov(y_t, y_{t+h}) = \begin{cases} (1+\theta_1^2)\sigma^2 & \text{if } h=0 \\ \theta_1\sigma^2 & \text{if } h=1 \\ 0 & \text{if } h>1 \end{cases}$$

$$\rho(h) = \begin{cases} 1 & \text{if } h=0 \\ \dfrac{\theta_1}{(1+\theta_1^2)} & \text{if } h=1 \\ 0 & h>1 \end{cases}$$



if $\theta_1 > 0$

Exercise

1) let, $y_t \sim AR(1)$ process. Then draw the correlogram for it.

2) $y_t \sim MA(5)$ then draw the correlogram for it.

22/02/23.

Parameters Estimation of MA(q), AR(p) & ARMA(p, q) process.

When we define a MA(q), AR(p) or ARMA(p, q) process we have some parameters which are unknown in practice for example for MA(1) process - $y_t = \alpha + \varepsilon_t + \theta\varepsilon_{t-1}$, $\varepsilon_t \sim WN(0, \sigma^2)$

when α, θ, σ^2 are unknown constants / parameters, we need to estimate these unknown constants or parameters from the data.

How to estimate these parameters given a time series data $\{y_1, \dots, y_N\}$ of size N ?

$\boxed{\text{NOTE :-}}$

There are mainly three estimation methods in time series to estimate those parameters of an MA(q) or AR(p) or ARMA(p, q) process. (for SARIMA & ARIMA method are same)

The three methods are ~~same~~ —

1) Maximum Likelihood estimation
2) Yule - Walker estimation (Methods of moment estimation)
3) Least Square estimation.

⟨1⟩ Maximum Likelihood Estimation (MLE)

we are given a time series data of size N $\{y_1, y_2, \dots, y_N\}$ from AR(1) process; $y_t = \alpha + \phi y_{t-1} + \varepsilon_t$ where $\varepsilon_t \sim WN(0, \sigma^2)$ with unknown parameters are α, ϕ, σ^2.

MLE for AR(1) process :-

let, $\{x_1, x_2, \dots, x_n\}$ are iid ~~sample~~ $N(\mu, \sigma^2)$ where μ, σ^2 are unknown. we want to find μ, σ^2 from the given data.

$$f(x_1)\,f(x_2|x_1)\,f(x_3|x_1,x_2)\cdots f(x_n|x_1,\ldots,x_{n-1}) =$$

Step 1:-

$$f(x_1,\cdots,x_n) = f(x_1)\cdots\cdots f(x_n) \quad [\because \text{iid}] = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2}$$

$$= (\sqrt{2\pi}\,\sigma)^{-n} e^{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2}$$

Step 2:-

$$\max_{\theta} L(\underset{\sim}{\theta})$$

$$= \max_{\mu,\sigma^2} L(\mu,\sigma^2)$$

$$= \min_{\mu,\sigma^2} -\log L(\mu,\sigma^2)$$

$$\begin{bmatrix} \hat{\mu}_{MLE} \\ \hat{\sigma}_{MLE} \end{bmatrix} = \arg\min_{\mu,\sigma^2} -\log L(\mu,\sigma^2)$$

Step 1:-

$$L(\underset{\sim}{\theta}) = f(y_1,\cdots,y_N), \quad \underset{\sim}{\theta} = \begin{bmatrix} \alpha \\ \phi \\ \sigma^2 \end{bmatrix}$$

where, $f(y_1,\cdots,y_N)$

$$= f(y_1)\,f(y_2|y_1)\,f(y_3|y_1,y_2)\cdots\cdots f(y_N|y_1,\cdots,y_{N-1}).$$

let us consider,

$$f(y_2|y_1) = ?$$

$$y_2 = \alpha + \phi y_1 + \varepsilon_2, \quad \varepsilon_2 \sim N(0,\sigma^2).$$

$$E(y_2|y_1) = E(\alpha + \phi y_1 + \varepsilon_2 | y_1)$$

$$= E(\alpha|y_1) + \phi E(y_1|y_1) + E(\varepsilon_2|y_1)$$

$$= \alpha + \phi y_1$$

$E(X/Y)$

$= \sum x f(x/y).$

$X_2 \backslash X_1$	0	1	
0	1/4	1/4	1/2
1	1/4	1/4	1/2
	$P(X_1=0)=\frac{1}{2}$	$P(X_1=1)=\frac{1}{2}$	

$E(X_1) = 0 \times 1/2 + 1 \times 1/2$

$= 1/2.$

$y_2 = \alpha + \varepsilon_2 + \theta\varepsilon_1$
$\Rightarrow \varepsilon_2 = y_2 - \alpha - \theta\varepsilon_1$

$$E(y_2|y_1) = \alpha + \phi E(y_1|y_1) + E(\varepsilon_2|y_1)$$
$$= \alpha + \phi y_1 + 0 = \alpha + \phi y_1$$

$$Var(y_2|y_1) = Var(\alpha + \phi y_1 + \varepsilon_2|y_1) = Var(\varepsilon_2|y_1)$$
$$= Var(\varepsilon_2) = \sigma^2$$

$$y_2|y_1 \sim N(\alpha + \phi y_1, \sigma^2)$$

$$\Rightarrow f(y_2|y_1) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(y_2 - \alpha - \phi y_1)^2}$$

Similarly find $f(y_3|y_2, y_1)$

$$y_3 = \alpha + \phi y_2 + \varepsilon_3$$

$$E(y_3|y_1, y_2) = E(\alpha + \phi y_2 + \varepsilon_3|y_2, y_1)$$
$$= \alpha + \phi E(y_2|y_2, y_1) + E(\varepsilon_3|y_1, y_2)$$
$$= \alpha + \phi y_2 + 0$$

$$V(y_3|y_1, y_2) = V(\alpha + \phi y_2 + \varepsilon_3|y_2, y_1) = Var(\varepsilon_3|y_2, y_1)$$
$$= Var(\varepsilon_3) = \sigma^2$$

$$\therefore (y_3|y_1, y_2) \sim N(\alpha + \phi y_2, \sigma^2)$$
$$\equiv (y_3|y_2) \sim N(\alpha + \phi y_2, \sigma^2)$$

$$L(\underset{\sim}{\theta}) = f(y_1) f(y_2|y_1) f(y_3|y_1, y_2) \cdots f(y_n|y_1, \ldots, y_{n-1}).$$
$$= f(y_1) f(y_2|y_1) f(y_3|y_2) \cdots f(y_n|y_{n-1})$$
$$= f(y_1) \prod_{t=2}^{n} f(y_t|y_{t-1}).$$
$$= f(y_1) \prod_{t=2}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(y_t - \alpha - \phi y_{t-1})^2}$$
$$= f(y_1) \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^{n-1} e^{-\frac{1}{2\sigma^2}\sum_{t=2}^{n}(y_t - \alpha - \phi y_{t-1})^2}.$$
$$\simeq \frac{1}{(\sqrt{2\pi}\,\sigma)^{n-1}} e^{-\frac{1}{2\sigma^2}\sum_{t=2}^{n}(y_t - \alpha - \phi y_{t-1})^2}$$

Step 2. $\max\limits_{\alpha, \phi, \sigma^2} L(\underset{\sim}{\theta})$ or $\min\limits_{\alpha, \phi, \sigma^2} -\ln L(\underset{\sim}{\theta})$

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Maximum Likelihood Estimation (MLE)

MLE method consists of two steps –

① we write the full likelihood function or approximate likelihood function and then at step ② –

② we maximize that full likelihood function or maximum likelihood function w.r.t model parameters $\underset{\sim}{\theta}$ to obtain the MLE of θ.

MLE for AR(1) process

$$y_t = \alpha + \phi y_{t-1} + \varepsilon_t \quad ; \quad t = 1(1)N$$

For finding MLE of $\underset{\sim}{\theta} = (\alpha, \phi, \sigma^2)$ we assume that $\varepsilon_t \sim$ iid $N(0, \sigma^2)$ instead of $WN(0, \sigma^2)$. So here we are making a distributional assumption of ε_t i.e normality of ε_t.

Step1 (likelihood function)

The data $\{y_1, y_2, \ldots, y_N\}$ of size N.

$$L(\underset{\sim}{\theta}) = f(y_1, \ldots, y_N)$$
$$= f(y_1) f(y_2|y_1) f(y_3|y_2, y_1) \ldots f(y_N|y_1, \ldots, y_{N-1})$$

$$(y_t | y_{t-1}, y_{t-2}, \ldots, y_1) \stackrel{d}{\equiv} (y_t | y_{t-1}), \quad t = 2, 3, \ldots, N \text{ for an AR(1) process.}$$

we have,
$$E(y_t | y_{t-1}) = \alpha + \phi y_{t-1}$$
$$V(y_t | y_{t-1}) = \sigma^2 \quad \forall \; t = 2(1)N$$

Hence,
$$f(y_t | y_{t-1}) = N(\alpha + \phi y_{t-1}, \sigma^2)$$
$$= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(y_t - \alpha - \phi y_{t-1})^2}, \quad t = 2(1)N$$

The marginal distribution of y_1 or $f(y_1)$ for AR(1) process:-

$$y_1 \sim (\quad)? \qquad f(y_1) = \frac{1}{\sqrt{2\pi}\sqrt{\frac{\sigma^2}{1-\phi^2}}} e^{-\frac{1-\phi^2}{2\sigma^2}\left(y_1 - \frac{\alpha}{1-\phi}\right)^2}$$

$$E(y_1) = \frac{\alpha}{1-\phi} \checkmark$$

$$V(y_1) = \frac{\alpha^2}{1-\phi^2} \checkmark$$

$$y_1 \stackrel{d}{\sim} N\left(\frac{\alpha}{1-\phi}, \frac{\sigma^2}{1-\phi^2}\right)$$

$N \to \infty$

full likelihood $\equiv$ approx likelihood.

1) $y_t = \beta t + \varepsilon_t$, $\varepsilon_t \sim WN(0,\sigma^2)$, $t = 1(1)N$. $\hat{\beta}$?

$\min \sum \varepsilon_t^2 = \sum (y_t - \beta t)^2$

$$\frac{\partial S(\underset{\sim}{\theta})}{\partial \beta} = 2 \sum (y_t - \beta t)(-t) = 0$$

$$\Rightarrow \sum t y_t - \beta \sum t^2 = 0$$

$$\Rightarrow \hat{\beta} = \frac{\sum t y_t}{\sum t^2}$$

2) $y_t = \alpha + \beta y_{t-1} + \gamma y_{t-2} + \varepsilon_t$, $\varepsilon_t \sim WN(0, \sigma^2)$. Find LS estimates of α, β, γ.

$$RSS = \sum_{t=3}^{N} \varepsilon_t^2 = \sum_{t=3}^{N} (y_t - \alpha - \beta y_{t-1} - \gamma y_{t-2})^2$$

$$\frac{\partial RSS}{\partial \alpha} = 2 \sum_{t=3}^{N} (y_t - \alpha - \beta y_{t-1} - \gamma y_{t-2})(-1) = 0$$

$$\Rightarrow \sum y_t$$

Least Square Method

~~$\alpha x + \beta y = b$~~ $\quad 2x + 3y = 5$

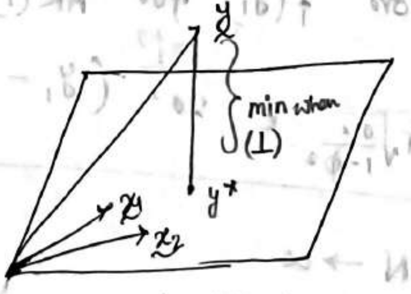
$\quad x + 5y = 3$

$$\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \qquad y = \frac{\begin{vmatrix} 5 & 3 \\ 3 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix}}$$

$$x = \frac{\begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix}} = \frac{6-5}{10-3} = \frac{1}{7} \qquad = \frac{6}{7}$$

$A \underset{\sim}{x} = \underset{\sim}{b}$

if $\underset{\sim}{b} \in \mathcal{C}(A)$ then system is consistent.



$\dim(\mathbb{R}^2) = 2$.

$\underset{\sim}{x}_1, \underset{\sim}{x}_2$ span $\mathbb{R}^2$.

$\alpha \underset{\sim}{x}_1 + \beta \underset{\sim}{x}_2 \neq \underset{\sim}{y}$ $\quad$ $\underset{\sim}{y}$ is in different plane

$\alpha \underset{\sim}{x}_1 + \beta \underset{\sim}{x}_2 \approx \underset{\sim}{y}$

or $\alpha \underset{\sim}{x}_1 + \beta \underset{\sim}{x}_2 = \underset{\sim}{y} + \underset{\sim}{\varepsilon}$

6 Questions $6 \times 10 = 60$

$$\Rightarrow [\underset{\sim}{x}_1 \ \ \underset{\sim}{x}_2] \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \underset{\sim}{y} + \underset{\sim}{\varepsilon}$$

we are doing some transformation to bring image of $\underset{\sim}{y}$ on the plane (orthogonal projection)

$$\Rightarrow X\underset{\sim}{\beta} = \underset{\sim}{y} + \underset{\sim}{\varepsilon}$$

$$\Rightarrow \underset{\sim}{y} = X\underset{\sim}{\beta} - \underset{\sim}{\varepsilon}$$

$$= X\underset{\sim}{\beta} + \underset{\sim}{\varepsilon}$$

or, $\underset{\sim}{y} \approx X\underset{\sim}{\beta}$

$$\Rightarrow \underbrace{X\underset{\sim}{\beta}}_{\text{linear combination}} \approx \underset{\sim}{y} \longrightarrow \underset{\sim}{y} \notin \mathcal{C}(X\underset{\sim}{\beta})$$

$$\Rightarrow X'(X\underset{\sim}{\beta}) = X'\underset{\sim}{y} \longrightarrow [X'\underset{\sim}{y} \in \mathcal{C}(X\underset{\sim}{\beta})] \text{ consistent}$$

$$\Rightarrow \hat{\underset{\sim}{\beta}}_{OLS} = (X'X)^{-1}(X'\underset{\sim}{y})$$

Minimize $\sum \varepsilon_t^2$

If $X\underset{\sim}{\beta} = \underset{\sim}{y}$ (consistent) then, $X'X\underset{\sim}{\beta} = X'y$ or, $X\underset{\sim}{\beta} \approx \underset{\sim}{y}$

$$= \underset{\sim}{\varepsilon}'\underset{\sim}{\varepsilon}$$

$$= (\underset{\sim}{y} - X\underset{\sim}{\beta})'(\underset{\sim}{y} - X\underset{\sim}{\beta})$$

$$\hat{\beta}_{OLS} = (X'X)^{-1}(X'\underset{\sim}{y}).$$

MLE for an AR(p) process:-

$$y_t = \alpha + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_p y_{t-p} + \varepsilon_t, \quad t = 1(1)N$$

Assume, $\varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$. Find MLE of $\alpha, \phi_1, \dots, \phi_p, \sigma^2$

Step 1. Write $L(\underset{\sim}{\theta})$ (either full/approx)

Step 2. Maximize $L(\underset{\sim}{\theta})$ w.r.t $\underset{\sim}{\theta}$.

$$L(\underset{\sim}{\theta}) = f(y_1, \dots, y_N)$$

$$= f(y_1) f(y_2|y_1) f(y_3|y_1, y_2) \dots f(y_N|y_1, \dots, y_{N-1})$$

$$= \underbrace{f(y_1, \dots, y_{p-1}, y_p)}_{\text{approx.}} f(y_{p+1}|y_p, y_{p-1}, \dots, y_1)$$

$$f(y_{p+2}|y_{p+1}, \dots, y_1) \dots f(y_N|y_{N-1}, \dots, y_1)$$

$$= f(y_1, \dots, y_p) \prod_{t=p+1}^{N} f(y_t | y_{t-1}, y_{t-2}, \dots, y_1)$$

$$= f(y_1, \dots, y_p) \prod_{t=p+1}^{N} f(y_t | y_{t-1}, y_{t-2}, \dots, y_{t-p}).$$

$$E(y_t \mid y_{t-1}, y_{t-2}, \ldots y_{t-p}) = \alpha + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p}.$$

$$V(y_t \mid y_{t-1}, y_{t-2}, \ldots, y_{t-p}) = \sigma^2$$

$$(y_t \mid y_{t-1}, y_{t-2}, \ldots, y_{t-p}) \sim N\left(\alpha + \sum_{j=1}^{p} \phi_j y_{t-j}, \sigma^2\right).$$

$$f(y_t \mid y_{t-1}, \ldots, y_{t-p}) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\frac{1}{2\sigma^2}\left(y_t - \alpha - \sum_{j=1}^{p} \phi_j y_{t-j}\right)}$$

Approx $\quad L(\underset{\sim}{\theta}) \approx \prod_{t=p+1}^{N} f(y_t \mid y_{t-1}, \ldots, y_{t-p}) \quad t = (p+1)1(N)$

(conditional likelihood function)

$$\mathcal{L}^*(\theta) = -\frac{T-p}{2}\log(2\pi) - \frac{T-p}{2}\log(\sigma^2) - \sum_{t=p+1}^{T} \frac{\varepsilon_t^2}{2\sigma^2}$$

10/3/23

Exercise :- (Computer).

Generate a time Series data of size $N = 500$ from the following process $y_t = 5 + 0.7 y_{t-1} + \varepsilon_t$, $\varepsilon_t \overset{iid}{\sim} N(0, 32)$

(a) After generating data $\{y_1, \dots, y_{500}\}$ fit $y_t = \alpha + \phi y_{t-1} + \varepsilon_t$ using MLE. Find the estimated values of α, ϕ & $\hat{\sigma}^2$.

(b) Fit $y_t = \alpha + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t$ using MLE. Find estimated values of α, ϕ_1 and ϕ_2.

$$RSS_{(a)} = \sum_{t=2}^{N} \left(y_t - \hat{\alpha}_{MLE} - \hat{\phi}_{MLE} y_{t-1} \right)^2$$

$$RSS_{(b)} = \sum_{t=3}^{N} \left(y_t - \hat{\alpha}_{MLE} - \hat{\phi}_1 y_{t-1} - \hat{\phi}_2 y_{t-2} \right)^2 \;??$$

$RSS_{(a)} < RSS_{(b)}$

MLE of Gaussian MA(1) process :-

The MA(1) process $y_t = \alpha + \theta \varepsilon_{t-1} + \varepsilon_t$, $\varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$

Step 1:-

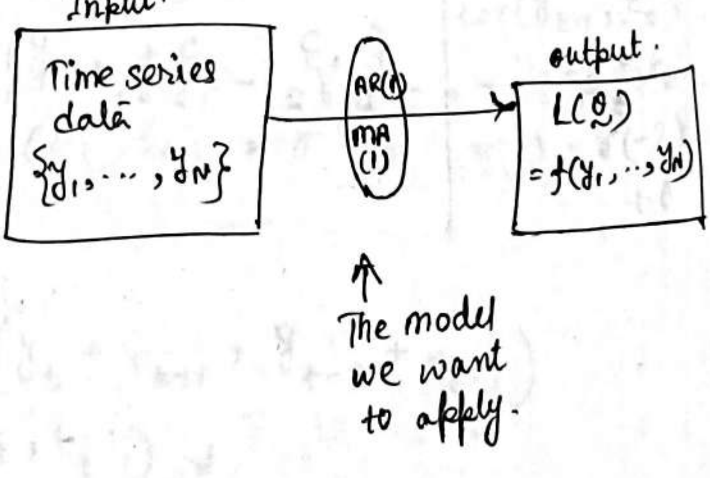
Write the Likelihood function $L(\underset{\sim}{\theta}) = f(y_1, \dots, y_N)$ — conditional / Exact.

Step 2 :-

Maximize $L(\underset{\sim}{\theta})$ w.r.t $\underset{\sim}{\theta}$.

$\cancel{y_1 = \alpha + \theta \varepsilon_0 + \varepsilon_1}$

$\cancel{E(y_1) = \alpha}$



$y_i = \alpha + \theta \varepsilon_{i-1} + \varepsilon_i$

$\alpha + \theta \varepsilon_0 + \varepsilon_1$

$\alpha + \theta \varepsilon_1 + \varepsilon_2$

Assume that $\varepsilon_0 = 0$

$\varepsilon_1 = y_1 - \alpha - \theta\varepsilon_0 = y_1 - \alpha$

$\varepsilon_2 = y_2 - \alpha - \theta\varepsilon_1$

$= y_2 - \alpha - \theta y_1 + \theta\alpha$

$= y_2 - \theta y_1 - \alpha + \theta\alpha$

$\varepsilon_3 = y_3 - \alpha - \theta\varepsilon_2 = y_3 - \alpha - \theta y_2 - \theta^2 y_1$

$\vdots$

$\varepsilon_N = y_N - \alpha - \theta\varepsilon_{N-1}$ ——— (i)

$f(\varepsilon_t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}\varepsilon_t^2}$

$f(\varepsilon_1) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y-\alpha)^2}$

The conditional likelihood function is —

$f(y_1, \dots, y_N|\varepsilon_0) = f(y_1|\varepsilon_0) \cdot f(y_2|y_1, \varepsilon_0) f(y_3|y_1, y_2, \varepsilon_0) \dots f(y_N|y_{N-1}, \dots, y_1, \varepsilon_0)$

$\equiv f(\varepsilon_1, \dots, \varepsilon_N|\varepsilon_0)$

$= f(\varepsilon_1|\varepsilon_0) \cdot f(\varepsilon_2|\varepsilon_0) \dots f(\varepsilon_N|\varepsilon_0) \equiv \prod_{t=0}^{N} f(\varepsilon_t|\varepsilon_0)$

$\cancel{= f(\varepsilon_1) f(\varepsilon_2) \dots f(\varepsilon_N)}$

$= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}\varepsilon_1^2} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}\varepsilon_2^2} \dots \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}\varepsilon_N^2}$

$= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^N e^{-\frac{1}{2\sigma^2}\sum_{t=1}^{N}\varepsilon_t^2}$ [Substitute ε_t from iteration (i)

$\cancel{y_3 - \alpha - \theta y_2 + \theta\delta}$

$\varepsilon_N = \dots - \theta^{t-1} y_1 - (1 + \theta + \dots + \theta^{t-1})\alpha$

$\varepsilon_t = y_t - \theta y_{t-1} - \theta^2 y_{t-2} - \dots - \theta^{t-1} y_1$

$$f(y_1, \dots, y_N \mid \varepsilon_0) = \frac{1}{(\sqrt{2\pi}\,\sigma)^N} e^{-\frac{1}{2\sigma^2}\sum_{t=1}^{N} \varepsilon_t^2}$$

$$= \frac{1}{(\sqrt{2\pi}\,\sigma)^N} e^{-\frac{1}{2\sigma^2}\sum_{t=1}^{N}\left\{\sum_{i=0}^{t-1}(-\theta)^i y_{t-i} - \alpha \sum_{i=0}^{t-1}(-\theta)^i\right\}}$$

Conditional likelihood function for a Gaussian MA(1) process.

Exercise :-

Write the conditional likelihood function of Gaussian MA(2) process.

MLE for a Gaussian MA(q) process.

$$y_t = \alpha + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}, \quad \varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$$

t=1. $\quad \varepsilon_0 = \varepsilon_{-1} = \cdots = \varepsilon_{-q+1} = 0$

$$y_1 = \alpha + \varepsilon_1 + \theta_1 \varepsilon_0 + \cdots + \theta_q \varepsilon_{-(q-1)}$$

$$\Rightarrow \varepsilon_1 = y_1 - \alpha - \theta_1 \varepsilon_0 - \theta_2 \varepsilon_{-1} - \theta_3 \varepsilon_{-2} - \cdots - \theta_q \varepsilon_{-(q-1)}$$

We need the assumption $\varepsilon_0 = \cdots = \varepsilon_{-(q-1)} = 0$.

$$\Rightarrow \varepsilon_1 = y_1 - \alpha$$

t=2.

$$\varepsilon_2 = y_2 - \alpha - \theta_1 \varepsilon_1 - \theta_2 \varepsilon_0 - \theta_3 \varepsilon_{-1} - \cdots - \theta_q \varepsilon_{-(q-2)}$$

$$= y_2 - \alpha(1-\theta_1) - \theta_1 y_1$$

t=3

$$\varepsilon_3 = y_3 - \alpha - \theta_1 \varepsilon_2 - \theta_2 \varepsilon_1 - \theta_3 \varepsilon_0 - \cdots - \theta_q \varepsilon_{-(q-3)}$$

$$= y_3 - \alpha - \theta y_2 - \alpha\theta_1(1-\theta_1) - \theta_1^2 y_1$$

$$- \theta_2 y_1 + \theta_2 \alpha$$

$\vdots$

$\varepsilon_N =$

conditional Likelihood function

$$f(y_1, \dots, y_N \mid \varepsilon_0, \varepsilon_{-1}, \dots \varepsilon_{-(q-1)})$$
$$= f(\varepsilon_1, \dots, \varepsilon_N \mid \varepsilon_0, \varepsilon_{-1}, \dots, \varepsilon_{-(q-1)})$$
$$= \prod_{t=1}^{N} f(\varepsilon_t \mid \varepsilon_0, \varepsilon_{-1}, \dots, \varepsilon_{-(q-1)})$$
$$= \prod_{t=1}^{N} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}\varepsilon_t^2}$$ where we need to substitute ε_t from iteration.

Hamilton ch: 5

MLE of ARMA (p, q)

$$y_t = \alpha + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q}$$

where $\varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$.

we have the data $\{y_1, \dots, y_N\}$

① conditional Likelihood function :-

$y_t \rightarrow \varepsilon_t$

$$\varepsilon_t = y_t - \alpha - \underbrace{\sum_{i=1}^{p} \phi_i y_{t-i}}_{\text{AR}(p)\text{ part}} - \sum_{j=1}^{q} \theta_j \varepsilon_{t-j}$$

t=1.

$$\varepsilon_1 = y_1 - \alpha - \phi_1 y_0 + \phi_2 y_{-1} + \dots + \phi_p y_{-(p-1)} - \theta_1 \varepsilon_0 - \theta_2 \varepsilon_{-1} - \dots - \theta_q \varepsilon_{-(q-1)}$$

we need the assumptions;

$$y_0 = y_{-1} = \dots = y_{-(p-1)} = \varepsilon_0 = \varepsilon_{-1} = \dots = \varepsilon_{-(q-1)} = 0$$

t=2

$$\varepsilon_2 = y_2 - \alpha - \phi_1 y_1 - \theta_1 (y_1 - \alpha)$$

t=3

$$\varepsilon_3 =$$

The Likelihood function $f(y_1, \dots, y_N | y_0, y_{-1}, \dots, y_{-(p-1)}, \varepsilon_0, \varepsilon_{-1}, \dots, \varepsilon_{-(q-1)})$

$$= \prod_{t=1}^{N} f(\varepsilon_t) \overset{(*)}{=} \prod_{t=1}^{N} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}\varepsilon_t^2}$$

BJ $f(y_{p+1}, y_{p+2}, \dots, y_N | y_p, y_{p-1}, \dots, y_1, \varepsilon, \dots \varepsilon_{-(q-1)})$

alternative L.F. $= \prod_{t=p+1}^{N} f(\varepsilon_t)$ $\quad (**)\ \prod_{t=1}^{p} f(\varepsilon_t) \prod_{t=p+1}^{N} f(\varepsilon_t)$

MID SEM QUESTION PAPER 2023 15/3/23.

Q1) $y_t = \alpha + \beta t + \varepsilon_t - \varepsilon_{t-1}$

$y_t^* = \nabla y_t = y_t - y_{t-1} = \alpha + \beta t + \varepsilon_t - \varepsilon_{t-1} - (\alpha + \beta(t-1) + \varepsilon_{t-1} - \varepsilon_{t-2})$

$= \beta + \varepsilon_t - 2\varepsilon_{t-1} + \varepsilon_{t-2}$

$\gamma(h) = cov(\nabla y_t, \nabla y_{t-h})$

$= cov(\beta + \varepsilon_t - 2\varepsilon_{t-1} + \varepsilon_{t-2}, \beta + \varepsilon_{t-h} - 2\varepsilon_{t-h-1} + \varepsilon_{t-h-2})$

If $h=0$

$\gamma(0) = cov(\beta + \varepsilon_t - 2\varepsilon_{t-1} + \varepsilon_{t-2}, \beta + \varepsilon_t - 2\varepsilon_{t-1} + \varepsilon_{t-2})$

$Var(\beta + \varepsilon_t - 2\varepsilon_{t-1} + \varepsilon_{t-2})$

$= \sigma^2 + 4\sigma^2 + \sigma^2 = 6\sigma^2$

If $h=1$

$cov(\beta + \varepsilon_t - 2\varepsilon_{t-1} + \varepsilon_{t-2}, \beta + \varepsilon_{t-1} - 2\varepsilon_{t-2} + \varepsilon_{t-3})$

$= -2\, Var(\varepsilon_{t-1}) - 2\, Var(\varepsilon_{t-2}) = -4\sigma^2$

If $h=2$

$cov(\beta + \varepsilon_t - 2\varepsilon_{t-1} + \varepsilon_{t-2}, \beta + \varepsilon_{t-2} - 2\varepsilon_{t-3} + \varepsilon_{t-4})$

$= \sigma^2$

If $h > 2$ $\quad cov(\cdot, \cdot) = 0$

$\rho(h) = 1$ if $h=0$

$= -\frac{2}{3}$ if $h=1, -1$

$= \frac{1}{6}$ if $h=2, -2$

$= 0$ if $|h|>2$

Q2) y_t be a covariance stationary

$$Var(\bar{y}_N) = \frac{1}{N}\sum_{h=-N}^{N}\left(1-\frac{|h|}{N}\right)\gamma(h)$$

$$\bar{y}_N = \frac{1}{N}\sum_{t=1}^{N} y_t$$

$$Var(\bar{y}_N) = Var\left(\frac{1}{N}\sum_{t=1}^{N} y_t\right) = \frac{1}{N^2}\left(Var\sum_{t=1}^{N} y_t\right)$$

$$Var(x_1+x_2+x_3) = \sum_{i=1}^{3} Var(x_i) + 2\sum_{\substack{i=1 \\ i<j}}^{3}\sum_{j=1}^{3} cov(x_i, x_j)$$

$$Var(a_1x_1, \dots a_nx_n) = \sum_{i=1}^{N} a_i^2 Var(x_i) + 2\sum_{\substack{i=1 \\ i<j}}^{N}\sum_{j=1}^{N} a_ia_j cov(x_i, x_j)$$

$$= \frac{1}{N^2}\left[\sum_{t=1}^{N} Var\, y_t + 2\sum_{\substack{t=1 \\ t<t'}}^{N}\sum_{t'=1}^{N} cov(y_t, y_{t'})\right]$$

$$= \frac{1}{N^2}\left[\sum_{t=1}^{N}\gamma(0) + 2\sum_{t=1}^{N}\sum_{t'=1}^{N}\gamma(t'-t)\right]$$

$$= \frac{1}{N^2}\Big[N\gamma(0) + 2\overset{\gamma(1)}{cov(y_1,y_2)} + 2\overset{\gamma(2)}{(y_1,y_3)} + \dots + 2\overset{\gamma(N-1)}{(y_1,y_N)}$$
$$+ 2\overset{\gamma(1)}{cov(y_2,y_3)} + 2\overset{\gamma(2)}{cov(y_2,y_4)} + \dots + 2\overset{\gamma(N-2)}{cov(y_2,y_N)}$$
$$+ 2\overset{\gamma(1)}{cov(y_3,y_4)} + 2\overset{\gamma(2)}{cov(y_3,y_5)} + \dots + 2\overset{\gamma(N-3)}{cov(y_3,y_N)}$$
$$+ \dots + 2\overset{\gamma(1)}{cov(y_{N-1},y_N)}$$

$$= \frac{1}{N^2}\Big[N\gamma(0) + 2\gamma(1) + 2\gamma(2) + \dots + 2\gamma(N-1) +$$
$$2\gamma(1) + 2\gamma(2) + \dots + 2\gamma(N-2) +$$
$$+ 2\gamma(1) + \dots + 2\gamma(N-3) +$$
$$\vdots$$
$$+ 2\gamma(1)\Big]$$

$$= \frac{1}{N^2}\left[N\gamma(0) + 2(N-1)\gamma(1) + 2(N-2)\gamma(2) + \cdots + 2\gamma(N-1)\right]$$

$$= \frac{1}{N^2}\left[N\gamma(0) + 2(N-1)\gamma(1) + 2(N-2)\gamma(2) + \cdots + 2(N-\overline{N-1})\gamma(N-1)\right]$$

$$Var(\bar{y}_N) = \frac{1}{N}\sum_{h=-N}^{N}\left(\frac{N-|h|}{N}\right)\gamma(h)$$

$$= \frac{1}{N^2}\sum_{h=-N}^{N}(N-|h|)\gamma(h)$$

$$\left(h = \{-N, -N+1, \ldots, -1, 0, 1, 2, \ldots, N\}\right)$$

$$= \frac{1}{N^2}\left[N\gamma(0) + \{(N-1)\gamma(1) + (N-1)\gamma(-1)\} + \{(N-2)\gamma(2) + (N-2)\gamma(-2)\} + \cdots \{\gamma(N-1) + \gamma(-\overline{N-1})\}\right]$$

$$= \frac{1}{N^2}\left[N\gamma(0) + \{(N-1)\gamma(1) + (N-2)\gamma(2) + \cdots + (N-\overline{N-1})\gamma(N-1)\} + \{(N-1)\gamma(-1) + (N-2)\gamma(-2) + \cdots + (N-\overline{N-1})\gamma(-\overline{N-1})\}\right]$$

$$= \frac{1}{N^2}\left[N\gamma(0) + \sum_{h=1}^{N-1}(N-h)\gamma(h) + \sum_{h=1}^{N-1}(N-h)\gamma(-h)\right]$$

$$= \frac{1}{N^2}\left[\sum_{h=0}^{N-1}(N-|h|)\gamma(h) + \sum_{h=-1}^{-(N-1)}(N-|h|)\gamma(h)\right]$$

$$= \frac{1}{N^2}\left[\sum_{h=-(N-1)}^{N-1}(N-|h|)\gamma(h)\right]$$

$$= \frac{1}{N^2}\left[\sum_{h=-N}^{N}(N-|h|)\gamma(h)\right]$$

Q3) $y_t = \phi y_{t-12} + \varepsilon_t$; $|\phi| < 1$.

$\Rightarrow y_t = \phi L^{12} y_t + \varepsilon_t$

$\Rightarrow \underbrace{(1 - \phi L^{12})}_{\text{AR poly.}} y_t = \varepsilon_t$

$1 - \phi z^{12} = 0$

$\Rightarrow z^{12} = \frac{1}{\phi} = \phi^* > 1$.

$\Rightarrow |z| = |\phi^{* 1/12}| > 1$.

The process is covariance stationary. AR(12)

Method :-

$y_t = \phi y_{t-12} + \varepsilon_t$

$\Rightarrow (1 - \phi L^{12}) y_t = \varepsilon_t$

$\Rightarrow y_t = (1 - \phi L^{12})^{-1} \varepsilon_t$

$= \sum_{j=0}^{\infty} (\phi L^{12})^j \varepsilon_t$

$\longrightarrow$ MA(∞) process

$= \varepsilon_t + \phi \varepsilon_{t-12} + \phi^2 \varepsilon_{t-24} + \cdots$

$(1 - x)^{-1} = 1 + x + x^2 + \cdots$

$(1 - ax)^{-1} = 1 + ax + a^2x^2 + \cdots$

$(1 - ax^{12})^{-1}$

$= 1 + ax^{12} + a^2 x^{24} + \cdots$

$= \sum_{j=0}^{\infty} (ax^{12})^j$

$\gamma(0) = \text{Var}(y_t)$

$= \text{Var}(\varepsilon_t + \phi \varepsilon_{t-12} + \phi^2 \varepsilon_{t-24} + \cdots)$

$= \sigma^2 (1 + \phi^2 + \phi^4 + \cdots) = \frac{\sigma^2}{1 - \phi^2}$

$\gamma(1) = \text{cov}(y_t, y_{t-1})$

$= \text{cov}(\varepsilon_t + \phi \varepsilon_{t-12} + \phi^2 \varepsilon_{t-24} + \cdots, \varepsilon_{t-1} + \phi \varepsilon_{t-13} + \phi^2 \varepsilon_{t-25} + \cdots)$

$= 0$

$\vdots$

$\gamma(11) = 0$

$\gamma(12) = \text{cov}(y_t, y_{t-12})$

$= \text{cov}(\varepsilon_t + \phi \varepsilon_{t-12} + \phi^2 \varepsilon_{t-24} + \cdots, \varepsilon_{t-12} + \phi \varepsilon_{t-12-12} + \phi^2 \varepsilon_{t-12-24} + \cdots)$

$= \phi \sigma^2 + \phi^3 \sigma^2 + \phi^5 \sigma^2 + \cdots$

$= \phi \sigma^2 (1 + \phi^2 + \phi^4 + \cdots) = \frac{\phi \sigma^2}{1 - \phi^2}$

$\gamma(13) = 0$ $\qquad \gamma(24) = cov(y_t, y_{t-24})$

$\vdots$

$\gamma(23) = 0$

$$= cov(\varepsilon_t + \phi\varepsilon_{t-12} + \phi^2\varepsilon_{t-24} + \cdots, \varepsilon_{t-24} + \phi\varepsilon_{t-24-12} + \phi^2\varepsilon_{t-24-24} + \cdots)$$

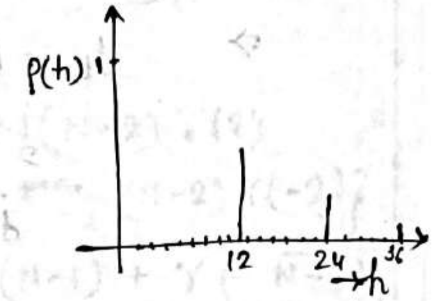
$$= \phi^2\sigma^2 + \phi^4\sigma^2 + \phi^6\sigma^2 + \cdots$$

$$= \frac{\phi^2\sigma^2}{1-\phi^2}$$

$$\gamma(h) = \begin{cases} \dfrac{\sigma^2}{1-\phi^2} & \text{if } h=0 \\ \dfrac{\phi^j\sigma^2}{1-\phi^2} & \text{if } h=\pm 12j \quad j=1,2,\ldots \\ 0 & \text{otherwise} \end{cases}$$

$$\sum\gamma(h) = \frac{\sigma^2}{1-\phi^2}(1+\phi+\phi^2+\cdots) = \frac{\sigma^2}{(1-\phi^2)(1-\phi)} < \infty$$

$$\rho(h) = \begin{cases} 1 & \text{if } h=0 \\ \phi^j & \text{if } h=\pm 12j \\ 0 & \text{otherwise} \end{cases}$$



Q5)

$y_t \sim ARMA(1,2)$

$y_t = \alpha + \phi y_{t-1} + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2}$, $\varepsilon_t \sim WN(0,\sigma^2)$

$\mu = \dfrac{\alpha}{1-\phi} \Rightarrow \alpha = \mu(1-\phi)$

$\Rightarrow (y_t - \mu) = \phi(y_{t-1} - \mu) + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2}$ — ⊛

Multiply by $(y_t - \mu)$ on both sides; Taking expectation.

$$E[(y_t-\mu)^2] = E[\phi(y_{t-1}-\mu)(y_t-\mu) + \varepsilon_t(y_t-\mu) + \theta_1\varepsilon_{t-1}(y_t-\mu) + \theta_2\varepsilon_{t-2}(y_t-\mu)]$$

$E(\varepsilon_t(y_t-\mu)) = \sigma^2 = \mathrm{cov}(\varepsilon_t, y_t) = \mathrm{cov}(\varepsilon_{t-1}, y_{t-1})$

$E[\varepsilon_{t-1}(y_t-\mu)] = \phi\,\mathrm{cov}(\varepsilon_{t-1}, y_{t-1}) + \theta_1\sigma^2$

$= \phi\sigma^2 + \theta_1\sigma^2 = \sigma^2(\phi+\theta_1)$

$E[\varepsilon_{t-2}(y_t-\mu)] = \phi\,\mathrm{cov}(\varepsilon_{t-2}, y_{t-1}) + \theta_2\sigma^2$

$= \phi(\phi\sigma^2 + \theta_1\sigma^2) + \theta_2\sigma^2$

$= (\phi^2 + \theta_2 + \theta_1\phi)\sigma^2$

$\gamma(0) = \phi\gamma(1) + \sigma^2 + \sigma^2(\theta_1\phi + \theta_1^2) + (\phi^2\theta_2 + \theta_2^2 + \theta_1\theta_2\phi)\sigma^2$ ——— ①

Multiply by $(y_{t-1}-\mu)$ on both sides of ⊛

$\gamma(1) = \phi\gamma(0) + E[(y_{t-1}-\mu)(\varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2})]$

$= \phi\gamma(0) + 0 + \theta_1 E[(y_{t-1}-\mu)\varepsilon_t] + \theta_2 E[(y_{t-1}-\mu)\varepsilon_{t-2}]$

$= \phi\gamma(0) + \theta_1\sigma^2 + \theta_2(\phi+\theta_1)\sigma^2$ ——— ②

Solve ① & ② for $\gamma(0)$ & $\gamma(1)$.

$(1-\phi^2)\gamma(0) = A + \phi B$

$\gamma(0) = \frac{A+\phi B}{1-\phi^2}$

$\gamma(1) = B + \phi\left(\frac{A+\phi B}{1-\phi^2}\right)$

$h = 2,$

$h > 3$

$\gamma(h) = \phi\gamma(h-1)$

From ①

$\gamma(0) - \phi\gamma(1) = A$

$\gamma(1) - \phi\gamma(0) = B$

where $A = \sigma^2 + \sigma^2(\theta_1\phi + \theta_1^2) + (\phi^2\theta_2 + \theta_2^2 + \theta_1\theta_2\phi)\sigma^2$

$B = \theta_1\sigma^2 + \theta_2(\phi+\theta_1)\sigma^2$

$\gamma(0) - \phi\cancel{\gamma(1)} = A$

$-\phi^2\gamma(0) + \phi\cancel{\gamma(1)} = \phi B$

$(1-\phi^2)\gamma(0) = A + \phi B$

$\gamma(0) = \frac{A+\phi B}{1-\phi^2}$

$\gamma(1) = B + \phi\left(\frac{A+\phi B}{1-\phi^2}\right)$

Multiplying (*) by $(y_{t-2} - \mu)$ and taking expectation;

$$\gamma(2) = \phi\gamma(1) + E[\varepsilon_t (y_{t-2} - \mu)] + \theta_1 E[\varepsilon_{t-1}(y_{t-2} - \mu)] + \theta_2 E[\varepsilon_{t-2}(y_{t-2} - \mu)]$$

(i) $E[\varepsilon_t (y_{t-2} - \mu)] = (\phi^2 + \theta_2 + \theta_1\phi)\sigma^2$

(ii) $E[\varepsilon_{t-1} (y_{t-2} - \mu)] = \sigma^2 (\phi + \theta_1)$

(iii) $E[\varepsilon_{t-2} (y_{t-2} - \mu)] = \sigma^2$

$$\begin{cases} \operatorname{cov}(y_{t-2}, \varepsilon_{t-2}) \\ = \operatorname{cov}(\alpha + \phi y_{t-3} + \varepsilon_{t-2} + \theta_1 \varepsilon_{t-3} + \theta_2 \varepsilon_{t-4}, \varepsilon_{t-2}) \\ = \sigma^2 \end{cases}$$

$$\begin{cases} \operatorname{cov}(y_{t-1}, \varepsilon_{t-1}) \\ = \operatorname{cov}(\alpha + \phi y_{t-2} + \varepsilon_{t-1} + \theta_1 \varepsilon_{t-2} + \theta_2 \varepsilon_{t-3}, \varepsilon_{t-1}) \\ = \sigma^2 \end{cases}$$

$$\therefore \gamma(2) = \phi B + \phi^2 \left(\frac{A + \phi B}{1 - \phi^2}\right) + (\phi^2 + \theta_2 + \theta_1\phi)\sigma^2 + \sigma^2(\theta_1^2 + \theta_1\phi) + \theta_2\sigma^2$$

$$= \phi B + \phi^2 \left(\frac{A + \phi B}{1 - \phi^2}\right) + C$$

where $C = \sigma^2 (\phi^2 + \theta_2 + \theta_1\phi + \theta_1^2 + \theta_1\phi + \theta_2)$

$= (\phi^2 + 2\theta_2 + \theta_1^2 + 2\theta_1\phi)\sigma^2$

$$\rho(h) = \begin{cases} 1 & \text{if } h = 0 \\ \dfrac{\gamma(1)}{\gamma(0)} & \text{if } h = 1 \\ \dfrac{\gamma(2)}{\gamma(0)} & \text{if } h = 2 \\ \phi\gamma(h-1) & \text{if } h > 3 \end{cases}$$

17/3/23

Method of Momments / Yule Walker Estimation :-

1) AR(p) (Covariance Stationary)

$$y_t = \alpha + \sum_{i=1}^{p} \phi_i y_{t-i} + \varepsilon_t \quad ; \quad \varepsilon_t \sim WN(0,\sigma^2)$$

Parameters :- $\alpha, \phi_1, \ldots, \phi_p, \sigma^2$. $\#(p+2)$

we use the population moments & sample moments to estimate the parameters.

Given :- $y_1, \ldots, y_N$:- N obs from AR(p) process.

For AR(p) process :- $E(y_t) = \dfrac{\alpha}{1 - \sum_{i=1}^{p} \phi_i} \longrightarrow \bar{y}$

$$\gamma(0) = \phi_1 \gamma(1) + \phi_2 \gamma(2) + \ldots + \phi_p \gamma(p) + \sigma^2 \longrightarrow \frac{1}{N-1} \sum_{t=1}^{N} (y_t - \bar{y})^2$$

$$\gamma(h) = \phi_1 \gamma(h-1) + \phi_2 \gamma(h-2) + \ldots + \phi_p \gamma(h-p) \text{ for } h \geq 1$$

$$\gamma(1) = \phi_1 \gamma(0) + \phi_2 \gamma(1) + \cdots + \phi_p \gamma(p-1)$$

$$\gamma(2) = \phi_1 \gamma(1) + \phi_2 \gamma(0) + \cdots + \phi_p \gamma(p-2)$$

$$\vdots$$

$$\gamma(p) = \phi_1 \gamma(p-1) + \phi_2 \gamma(p-2) + \cdots + \phi_p \gamma(0).$$

$$\underbrace{\begin{pmatrix} \gamma(1) \\ \gamma(2) \\ \vdots \\ \gamma(p) \end{pmatrix}}_{p\times 1} = \underbrace{\begin{pmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(p-1) \\ \gamma(1) & \gamma(0) & \cdots & \gamma(p-2) \\ \vdots & & & \\ \gamma(p-1) & \gamma(p-2) & \cdots & \gamma(0) \end{pmatrix}}_{p\times p} \underbrace{\begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{pmatrix}}_{p\times 1}$$

$$\boxed{\underset{\sim}{\gamma} = \Gamma \underset{\sim}{\Phi}}$$

$\underset{\sim}{\Phi} = \Gamma^{-1} \underset{\sim}{\gamma}$ if Γ is non-singular.

$$\hat{\gamma}(h) = \frac{1}{N-h} \sum_{t=1}^{N-h} (y_t - \bar{y})(y_{t+h} - \bar{y})$$

After replacing population moments by sample moments –

$$\hat{\underset{\sim}{\Phi}} = \hat{\Gamma}^{-1} \hat{\underset{\sim}{\gamma}}$$

$$1+\phi L+\phi^2L^2+\cdots = \frac{1}{1-\phi L}$$

$$E(y_t) = \bar{y}$$

$$\Rightarrow \frac{\hat{\alpha}}{1-\sum_{i=1}^{p}\hat{\phi}_i} = \bar{y} \Rightarrow \boxed{\hat{\alpha} = \bar{y}\left(1-\sum_{i=1}^{p}\hat{\phi}_i\right)}$$

$$\hat{\sigma}^2 = \hat{\gamma}(0) - \hat{\phi}_1\hat{\gamma}(1) - \hat{\phi}_2\hat{\gamma}(2) - \cdots - \hat{\phi}_p\hat{\gamma}(p)$$

$$= \hat{\gamma}(0) - \hat{\underset{\sim}{\gamma}}'\underset{\sim}{\phi}$$

Exercise

1) If $\{y_1, y_2, \ldots, y_N\}$ be a sample of size N from a covariance stationary process given by AR(1) process —

$$y_t = \alpha+\phi y_{t-1} + \varepsilon_t \;;\quad \varepsilon_t \sim WN(0,\sigma^2)$$

Find α, ϕ, σ^2.

Assuming covariance stationarity $E(y_t) = \alpha + \phi E(y_{t-1}) + E(\varepsilon_t)$

$$\Rightarrow E(y_t) = \frac{\alpha}{1-\phi L}$$

$$\frac{\alpha}{1-\phi L} = (1+\phi L+\phi^2L^2+\cdots)\alpha = \alpha+\phi\alpha+\phi^2\alpha+\cdots = \alpha(1+\phi+\phi^2+\cdots) = \frac{\alpha}{1-\phi}$$

$$y_t = \alpha + \phi L y_t + \varepsilon_t$$

$$\Rightarrow (1-\phi L)y_t = \alpha + \varepsilon_t$$

$$\Rightarrow y_t = \frac{\alpha}{1-\phi L} + \frac{\varepsilon_t}{1-\phi L}$$

$$\frac{\varepsilon_t}{1-\phi L} = (1+\phi L+\phi^2L^2+\cdots)\varepsilon_t = \varepsilon_t + \phi\varepsilon_{t-1} + \phi^2\varepsilon_{t-2}+\cdots$$

$$y_t = \frac{\alpha}{1-\phi} + \varepsilon_t + \phi\varepsilon_{t-1} + \phi^2\varepsilon_{t-2} + \cdots$$

$$V(y_t) = \sigma^2(1+\phi^2+\phi^4+\cdots) = \frac{\sigma^2}{1-\phi^2} \quad \text{——— (1)}$$

$$\mathrm{cov}(y_t, y_{t+h}) = \mathrm{cov}\left(\frac{\alpha}{1-\phi} + \varepsilon_t + \phi\varepsilon_{t-1} + \phi^2\varepsilon_{t-2}+\cdots, \frac{\alpha}{1-\phi} + \varepsilon_{t+h} + \phi\varepsilon_{t+h-1} + \phi^2\varepsilon_{t+h-2}+\cdots\right)$$

$$\gamma(1) = \mathrm{cov}\left(\frac{\alpha}{1-\phi} + \varepsilon_t + \phi\varepsilon_{t-1} + \phi^2\varepsilon_{t-2}+\cdots, \frac{\alpha}{1-\phi} + \varepsilon_{t+1} + \phi\varepsilon_t + \phi^2\varepsilon_{t-1}+\cdots\right)$$

$$= (\phi+\phi^3+\phi^5+\cdots)\sigma^2$$

$$= \frac{\sigma^2\phi}{1-\phi^2} \quad \text{——— (2)}$$

$$\frac{\sigma^2\phi}{1-\phi^2} = \frac{\sum_{t=1}^{N-1}(y_t-\bar{y})(y_{t+1}-\bar{y})}{N-1}, \quad \frac{\sum_{t=1}^{N}(y_t-\bar{y})^2}{N-1} = \frac{\sigma^2}{1-\phi^2}, \quad \frac{\alpha}{1-\phi} = \frac{1}{N}\sum_{t=1}^{N}y_t$$

(*) (*)

Dividing ② by ①

$$\hat{\phi} = \frac{\hat{\gamma}(1)}{\hat{\gamma}(0)} = \frac{\sum_{t=1}^{N-1} (y_t - \bar{y})(y_{t+1} - \bar{y})}{\sum_{t=1}^{N} (y_t - \bar{y})^2}$$

$$\hat{\alpha} = (1 - \hat{\phi}) \frac{1}{N} \sum_{t=1}^{N} y_t = \bar{y} - \hat{\phi}\,\bar{y}$$

$$\hat{\sigma}^2 = (1 - \hat{\phi}^2) \sum_{t=1}^{N} (y_t - \bar{y})^2 \Big/ (N-1)$$

(ii) MA(1)

$\{y_1, \dots, y_N\}$ a sample from an MA(1) process.

$y_t = \alpha + \varepsilon_t + \theta\varepsilon_{t-1}$; $\varepsilon_t \sim WN(0, \sigma^2)$

parameters :- α, θ, σ^2. # 3.

$E(y_t) = \alpha$

$\gamma(0) = (1+\theta^2)\sigma^2$

$\gamma(1) = \theta\sigma^2$

$\bar{y}$

$\frac{1}{N-1}\sum_{t=1}^{N}(y_t - \bar{y})^2 = \hat{\gamma}(0)$

$\frac{1}{N-\cancel{1}}\sum_{t=1}^{N-\cancel{1}}(y_t - \bar{y})(y_{t+\cancel{1}} - \bar{y}) = \hat{\gamma}(1)$

Then, $\bar{y} = \alpha$

$\hat{\gamma}(0) = (1+\theta^2)\sigma^2$

$\hat{\gamma}(1) = \theta\sigma^2$

$\frac{\hat{\gamma}(0)}{\hat{\gamma}(1)} = \frac{1+\theta^2}{\theta} \rightarrow$ quadratic equation giving 2 $\hat{\theta}$'s.

$\hat{\sigma}^2 = \frac{\hat{\gamma}(1)}{\hat{\theta}}$

* MLE is the best method when the likelihood fn is complicated then to solve the method for initial guess we use the YM estimation.

Exercise :-

2) $y_t = \alpha + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2}$; $\varepsilon_t \sim WN(0, \sigma^2)$

Then estimate α, θ_1, θ_2 & σ^2 using YW method.

YW Method is not very useful for MA(q) process. However one can use the method to get initial estimates of AR(p) or MA(q).

Conditional Least Square Estimates :- CLS

Least squares

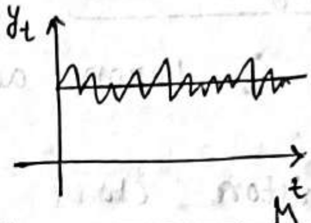
$$\min_{\underset{\sim}{\theta}} \sum_{t=1}^{N} \{y_t - E(y_t)\}^2$$

Conditional Least Squares

$$\min_{\underset{\sim}{\theta}} \sum_{t=1}^{N} \{y_t - E(y_t \mid y_{t-1}, y_{t-2}, \dots)\}^2$$

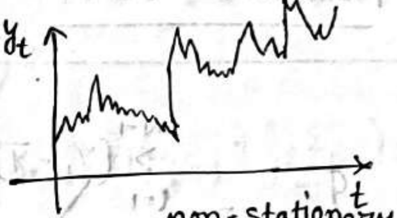
Recap

1) Modelling of stationary process –
   1) AR, MA, ARMA
   2) ACF (3) Estimation of parameters



Stationary

2) Modelling of non-stationary process –
   2) ARIMA , SARIMA



non-stationary.

Test for Stationarity

Given Time series data $\{y_1, \dots, y_N\}$ if the data comes from stationary process not.

$H_0: y_t \sim$ stationary Vs $H_1: y_t \not\sim$ stationary

(1) Dickey Fuller Test (DF Test)

Suppose, $y_t \sim AR(1)$

$y_t = \alpha + \phi y_{t-1} + \varepsilon_t$; $\varepsilon_t \sim WN(0, \sigma^2)$

$\{y_1, \dots, y_N\} \to$ Sample.

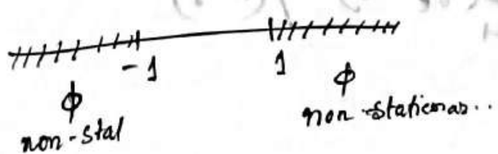
H_0: ~~[illegible]~~ $|\phi| \geq 1$ Vs. $H_1: |\phi| < 1$.
non-stationary. stationary.

↓ modified.

$H_0: \phi = 1$ Vs. $H_1: |\phi| < 1$.

whether AR(1) has unit root or not.

$1 - \phi z = 0$
$\Rightarrow z = \frac{1}{\phi} = 1$
$\phi = 1$



we ~~[illegible]~~ ~~[illegible]~~ don't check for $|\phi| > 1$ since the process is unbounded, non-stationary.

Step 1. Estimate $\min \sum_{t=1}^{N} (y_t - \alpha - \phi y_{t-1})^2$ w.r.t α, ϕ.

$$\hat{\phi} = \frac{\sum (y_t - \bar{y})(y_{t-1} - \bar{y})}{\sum (y_t - \bar{y})^2}$$

Rejection Rule :- $|\hat{\phi}| > c_\alpha$ ($\rightarrow$ from DF distribution).

This is known as unit root test. $\hat{\phi} \overset{a}{\sim}$ DF distribution

Hamilton Ch 17

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R Programming.

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Dickey Fuller Test.

Assume that $y_t = \alpha + \phi y_{t-1} + \varepsilon_t$; $\varepsilon_t \sim WN(0,\sigma^2)$
i.e the original sequence is $\{y_t\}$ an AR(1) process.
We test the following hypothesis :

$$H_0 : \phi = 1 \quad vs. \quad H_1 : \phi < 1.$$

We rewrite the above process in the following form –

$$\nabla y_t = y_t - y_{t-1} = \alpha + (\phi - 1) y_{t-1} + \varepsilon_t$$

Then our hypothesis becomes – $\underline{\underline{\delta = \phi - 1}}$.

$$H_0 : \delta = 0 \quad vs. \quad H_1 : \delta < 0$$

Note that : under $H_0 : \delta = 0$

$$\nabla y_t = \alpha + \varepsilon_t \quad \therefore \nabla y_t \sim \text{stationary}.$$

(Under H_0 y_t is non-stationary but ∇y_t is stationary)

Data	Hypothesis	Test Statistic	Rejection criteria.
y_1, y_2, $\vdots$, y_N	$H_0 : \delta = 0$ vs. $H_1 : \delta < 0$	$DF = \hat{\phi}_{OLS} = \dfrac{\sum_{t=1}^{N-1} y_t y_{t+1}}{\sum_{t=1}^{N} y_t^2}$ (Assuming $\alpha = 0$) under Ho the distrn. $\dfrac{(\hat{\phi}_{OLS} - 1)}{se(\hat{\phi}_{OLS})} \overset{H_0}{\sim} DF$	

→ Dickey Fuller test is not very useful in practice.

Augmented Dickey-Fuller Test.

Assumption is that $y_t \sim AR(p)$

$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t$ where $\varepsilon_t \sim WN(0,\sigma^2)$

$$(y_t - \mu) = \phi_1(y_{t-1} - \mu) + \phi_2(y_{t-2} - \mu) + \cdots + \phi_p(y_{t-p} - \mu) + \varepsilon_t$$

$$y_t^* = \phi_1 y_{t-1}^* + \phi_2 y_{t-2}^* + \cdots + \phi_p y_{t-p}^* + \varepsilon_t$$

when there is $\alpha \neq 0$ in the data. then

$$\begin{pmatrix} y_1 - \bar{y} = y_1^* \\ y_2 - \bar{y} = y_2^* \\ \vdots \\ y_N - \bar{y} = y_N^* \end{pmatrix}$$ does not have any intercept.

we want to test whether an AR(p) process has a unit root or not.

$$\Rightarrow \left(1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p\right) y_t = \varepsilon_t$$

Polynomial :- $1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p = 0$

$$\Downarrow$$

$$(1-\lambda_1 z)(1-\lambda_2 z) \cdots (1-\lambda_p z) = 0$$

$$z_1 = \frac{1}{\lambda_1}, \quad z_2 = \frac{1}{\lambda_2}, \quad \ldots, \quad z_p = \frac{1}{\lambda_p}.$$

Hamilton.

$$(1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p) y_t = \varepsilon_t \quad \text{where, } \varepsilon_t \overset{iid}{\sim} (0, \sigma^2)$$

$$E(\varepsilon_t^4) < \infty$$

let, $\rho = \phi_1 + \phi_2 + \cdots + \phi_p = \sum_{i=1}^{p} \phi_i$

$$\delta_j = -(\phi_{j+1} + \phi_{j+2} + \cdots + \phi_p); \; j = 1, 2, \ldots, p-1$$

$$= -\sum_{j=1}^{p-j} \phi_{j+1}$$

where,

$$\begin{cases} \delta_1 = -(\phi_2 + \phi_3 + \cdots + \phi_p) \\ \delta_2 = -(\phi_3 + \cdots + \phi_p) \\ \vdots \\ \delta_{p-1} = -\phi_p \end{cases}$$

$$(1 - \rho L) - (\delta_1 L + \delta_2 L^2 + \cdots + \delta_{p-1} L^{p-1})(1-L)$$

$$= 1 - (\rho + \delta_1) L - (\delta_2 - \delta_1) L^2 - (\delta_3 - \delta_2) L^3 - \cdots - (\delta_{p-1} - \delta_{p-2}) L^{p-1} + \delta_{p-1} L^p$$

$$= 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p$$

Now, $\left(1-\phi_1 L-\phi_2 L^2-\cdots-\phi_p L^p\right) y_t = \varepsilon_t$

$$\Rightarrow \left\{(1-\rho L)-\left(\delta_1 L+\delta_2 L^2+\cdots+\delta_{p-1} L^{p-1}\right)(1-L)\right\} y_t = \varepsilon_t$$

$$\Rightarrow (1-\rho L) y_t-\left(\delta_1 L+\delta_2 L^2+\cdots+\delta_{p-1} L^{p-1}\right)(1-L) y_t = \varepsilon_t$$

$$\Rightarrow y_t-\rho y_{t-1}-\left(\delta_1 L+\delta_2 L^2+\cdots+\delta_{p-1} L^{p-1}\right) \nabla y_t = \varepsilon_t$$

$$\Rightarrow y_t = \rho y_{t-1}+\delta_1 \nabla y_{t-1}+\delta_2 \nabla y_{t-2}+\cdots+\delta_{p-1} \nabla y_{t-p+1} + \varepsilon_t$$

$\hookrightarrow$ AR(p) process.

$\Phi(z) = 0$

if $z = 1$. $\Phi(1) = 0$

$$1-\phi_1 \cdot L-\phi_2 1 \cdot-\cdots-\phi_p \cdot 1 = 0$$

$$\Rightarrow \sum_{i=1}^{p} \phi_i = 1$$

$$\Rightarrow \rho = 1.$$

Note that:-

under $H_0: \rho = 1$ the above AR(p) process has an unit root.
Under $H_0: \rho = 1$ we can rewrite the above process

$H_1: |\rho| < 1$.

$$y_t = y_{t-1}+\sum_{i=1}^{p-1} \delta_i \nabla y_{t-i}+\varepsilon_t$$

$$\Rightarrow \nabla y_t = \sum_{i=1}^{p-1} \delta_i \nabla y_{t-i}+\varepsilon_t$$

under H_0; $\nabla y_t \sim AR(p-1)$ This could be represented as MA(∞) process.

$$(1-\rho L)-\left(\delta_1 L+\delta_2 L^2+\cdots+\delta_{p-1} L^{p-1}\right)(1-L)$$

Note that :-

$\boxed{\Phi(L) y_t = \varepsilon_t}$ can be rewritten as -

$$y_t = \rho y_{t-1} + \sum_{j=1}^{p-1} \delta_j \nabla y_{t-j} + \varepsilon_t$$

$$\Rightarrow y_t - y_{t-1} = \rho y_{t-1} - y_{t-1} + \sum_{j=1}^{p-1} \delta_j \nabla y_{t-j} + \varepsilon_t$$

$$\Rightarrow \boxed{\nabla y_t = (\rho - 1) y_{t-1} + \sum_{j=1}^{p-1} \delta_j \nabla y_{t-j} + \varepsilon_t}$$

Under H_0, $\nabla y_t = \sum_{j=1}^{p-1} \delta_j \nabla y_{t-j} + \varepsilon_t$

$$\Rightarrow \left(1 - \delta_1 L - \delta_2 L^2 - \dots - \delta_{p-1} L^{p-1}\right) \nabla y_t = \varepsilon_t$$

with all roots $1 - \delta_1 L - \delta_2 L^2 - \dots - \delta_{p-1} L^{p-1} = 0$ lie outside unit circle.

$$\Rightarrow \nabla y_t = \Psi(L) \varepsilon_t \longrightarrow MA(\infty).$$

Wold Decomposition Theorem

Any covariance stationary ARMA process $\{y_t\}$ ~~with mean zero~~ can be written as an $MA(\infty)$ form -

$$y_t = \mu + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} \quad \text{with} \sum_{j=0}^{\infty} |\psi_j| < \infty$$

$\&$ $\varepsilon_t \sim WN(0, \sigma^2)$.

$$\Phi(L) y_t = \alpha + \Theta(L) \varepsilon_t$$

$$\Rightarrow y_t = \frac{\alpha}{\Phi(L)} + \frac{\Theta(L) \varepsilon_t}{\Phi(L)}.$$

$$= \mu + \Psi(L) \varepsilon_t.$$

Any zero mean covariance stationary process $\{y_t\}$ can be written as $y_t = \underbrace{\kappa_t}_{\text{(Linearly deterministic component of } y_t)} + \underbrace{\sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}}_{\text{(linearly indeterministic/ stochastic component)}}$

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Simple Exponential Smoothing. (SES)

(1) $L_t = \alpha y_t + (1-\alpha) L_{t-1}$ → Level updating Equation

(2) $F_{t+h} = L_t, \quad h = 1, 2, \ldots$ → forecasting equation.

$= \hat{y}_{t+h|t}$

$$L_t = \alpha y_t + (1-\alpha)\{\alpha y_{t-1} + (1-\alpha) L_{t-2}\}$$

$$= \alpha y_t + \alpha(1-\alpha) y_{t-1} + (1-\alpha)^2 L_{t-2}$$

$$\vdots$$

$$= \sum_{i=0}^{t-1} \alpha (1-\alpha)^i y_{t-i} + \underbrace{(1-\alpha)^t L_0}_{\text{Assuming } L_0 = 0}$$

$$F_{t+h} = \hat{y}_{t+h|t}$$

If $h = 1$, $F_{t+1} = \hat{y}_{t+1|t} = L_t$

$h = 2$, $F_{t+2} = \hat{y}_{t+2|t} = L_t$

$\hat{y}_{t+1|t} = \hat{y}_{t+2|t} = \ldots\ldots = $ → SES

$\hat{y}_{t+1|t}$ ↓ one step ahead forecasting

$\hat{y}_{t+2|t}$ ↓ Two step ahead forecasting

$\hat{y}_{t+2|t+1} \neq \hat{y}_{t+1|t}$

$\hat{y}_{t+1|t}$ = Forecasting Value of y_{t+1} given $y_t, y_{t-1}, \ldots, y_1$

$\hat{y}_{t+h|t}$ = forecasting value of y_{t+h} given $y_t, y_{t-1}, \ldots, y_1$

1) Show that for SES, $F_{t+1} = F_{t+2} = \ldots$

2) Show that for SES, $F_{t+1} = \sum_{j=0}^{t-1} \alpha (1-\alpha)^j y_{t-j}$ assuming $L_0 = 0$

How to choose the value of α (smoothing constant)

t	y_t	F_t
1	y_1	$F_1 = 0$
2	y_2	$F_2 = \alpha y_1$
$\vdots$	$\vdots$	$F_3 = \alpha y_2 + (1-\alpha)\alpha y_1$
N	y_N	$F_N = \alpha y_N + \alpha(1-\alpha) y_{N-1} + \cdots + \alpha(1-\alpha)^{N-1} y_1$
$N+1$		
$N+2$		
$\vdots$		

not interested

If $\alpha = 1$

$$L_t = \alpha y_t + (1-\alpha) L_{t-1} = y_t$$

$$F_{t+h} = L_t = y_t$$

[By replacing $t = N$ in $F_{t+h} = L_t$]

$$F_{N+h} = L_N = \alpha y_N + (1-\alpha) L_{N-1} = y_N$$

$$\Rightarrow F_{N+1} = F_{N+2} = \cdots = y_N$$

If $\alpha = 0$

$$F_{N+h} = L_N = \alpha y_N + (1-\alpha) L_{N-1} = L_{N-1} = \cdots = L_1 = y_1$$

$$F_{N+1} = F_{N+2} = \cdots = y_1$$

$$\sum_{t=1}^{N} \left(y_t - \hat{y}_{t|t-1}\right)^2 = f(\alpha)$$

$$= \left(y_1 - \hat{y}_{1|0}\right)^2 + \left(y_2 + \hat{y}_{2|1}\right)^2 + \cdots + \left(y_N - \hat{y}_{N|N-1}\right)^2$$

$$= y_1^2 + \left(y_2 - \alpha y_1\right)^2 + \left(y_3 - \alpha y_2 - \alpha(1-\alpha) y_1\right)^2 + \cdots + \left(y_N - \sum_{j=0}^{N-1} \alpha (1-\alpha)^j y_{t-j}\right)^2$$

min $f(\alpha)$ w.r.t α.

Soln. $\hat{\alpha} = \arg\min f(\alpha)$

computer Assignment :

1> Download dataset

2> Find $\sum_{t=1}^{N}\left(y_t - \hat{y}_{t|t-1}\right)^2$ for $\alpha = 0.1$, $\alpha = 0.3$, $\alpha = 0.5$, $\alpha = 0.7$, $\alpha = 0.9$ using SES and choose the best value of α.

Double Exponential Smoothing (DES) / Holt's ES.

This forecasting method is suitable for time series with trend but no seasonality.

$$y_t = L_t + T_t + \varepsilon_t$$

This method consists of three equation —

1> Level updating equation :-

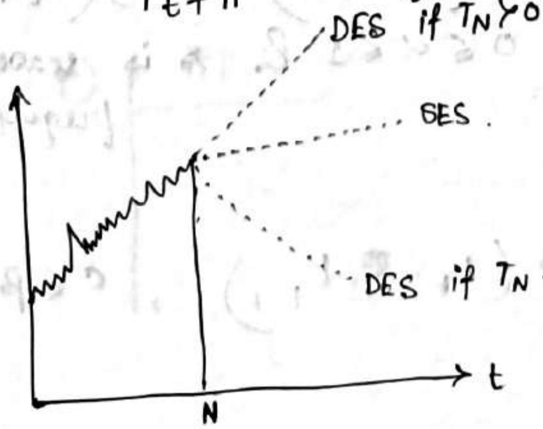
$$L_t = \alpha y_t + (1-\alpha)\left(L_{t-1} + T_{t-1}\right) \quad ; \quad 0 \leq \alpha \leq 1.$$

2> Trend updating equation :-

$$T_t = (1-\beta)T_{t-1} + \beta\left(L_t - L_{t-1}\right) \quad ; \quad 0 \leq \beta \leq 1.$$

3> Forecasting Equation :-

$$F_{t+h} = L_t + hT_t \quad \text{for } h = 1, 2, 3, \ldots$$



How to choose the smoothing constant α and β ?

$$MSE \quad \sum_{t=1}^{N} \left(y_t - \hat{y}_{t|t-1} \right)^2 = f(\alpha, \beta) \quad w.r.t \ \alpha, \beta$$

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} = \arg \min_{\alpha, \beta} f(\alpha, \beta)$$

$\downarrow$

Difficult to get $\hat{\alpha}$, $\hat{\beta}$

So we do some Grid search.

(α, β)	MSE
(0.1, 0.1)	
(0.1, 0.3)	
(0.1, 0.7)	
(0.1, 0.9)	
(0.3, 0.1)	
(0.3, 0.5)	
⋮	

Triple Exponential Smoothing / Holt-Winters' ES (TES

This forecasting method is suitable for a time series with trend and seasonality.

~~This method consists of three equations:~~

Under this method, there are two ways of making forecasting, namely

(1) Additive Method (Additive Seasonality)

(2) Multiplicative Method (Multiplicative Seasonality)

Additive Seasonality Method consists of 4 equations –

(i) Level updating : $L_t = \alpha(y_t - S_{t-s}) + (1-\alpha)(L_{t-1} + T_{t-1})$

$0 \leq \alpha \leq 1$ & s is seasonal frequency.

(ii) Trend updating :

$$T_t = (1-\beta) T_{t-1} + \beta (L_t - L_{t-1}), \quad 0 \leq \beta \leq 1.$$

(iii) Seasonal updating :

$$S_t = (1-\gamma) S_{t-s} + \gamma \left(y_t - L_t - T_{t-1}\right) \; ; \; 0 \le \gamma \le 1$$

(iv) Forecasting :

$$F_{t+h} = L_t + hT_t + S_{t+h-s(k+1)}$$

where $k = \left[\frac{h-1}{s}\right] \equiv$ integer part of $\frac{h-1}{s}$

$\left[\text{if } \frac{h-1}{s} = 3.5 \text{ then } \left[\frac{h-1}{s}\right] = 3\right]$

Multiplicative Seasonality Method

consists of 4 equations —

(1) Level updating : $L_t = \frac{\alpha y_t}{S_{t-s}} + (1-\alpha)\left(L_{t-1} + T_{t-1}\right)$

(2) Trend updating : $T_t = (1-\beta) T_{t-1} + \beta \left(L_t - L_{t-1}\right)$.

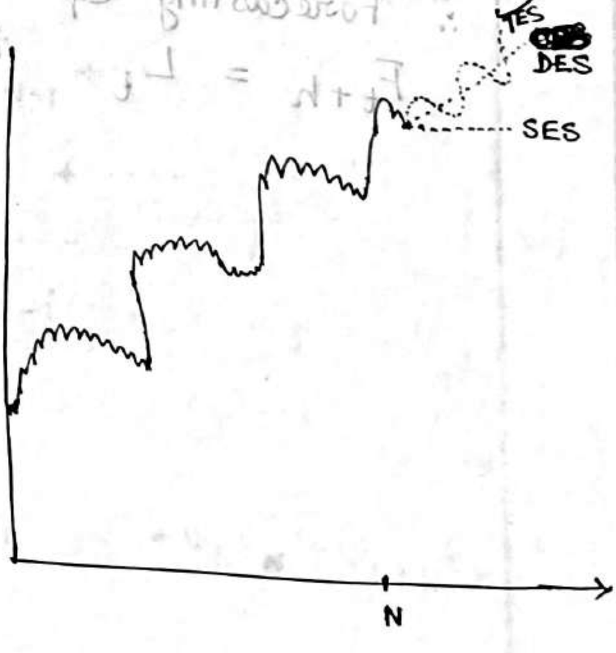
(3) Seasonal updating : $S_t = (1-\gamma) S_{t-s} + \gamma \left(\frac{y_t}{L_{t-1} + T_{t-1}}\right)$

(4) Forecasting updating :

$$F_{t+h} = \left(L_t + hT_t\right) S_{t+h-s(k+1)}$$

where $k = \left[\frac{h-1}{s}\right]$

(α, β, γ)	MSE



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Forecasting using an ARMA (p, q) process

(a) Forecasting using an MA(∞) process :- (9.2)(4.2

We have a time series data $\{y_1, y_2, \ldots, y_N\}$ from an MA(∞) process.

What will be the value of y_{N+h} ?

$y_t \sim MA(\infty)$

$$\Rightarrow y_t = \mu + \Psi(L)\varepsilon_t \quad \text{where} \quad \varepsilon_t \sim WN(0, \sigma^2)$$

$$\Psi(L) = \sum_{j=0}^{\infty} \psi_j L^j, \quad \sum_{j=0}^{\infty} |\psi_j| < \infty$$

$$= \mu + \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \cdots \qquad \psi_0 = 1$$

Then,

$$\Psi(L) = 1 + \psi_1 L + \psi_2 L^2 + \cdots$$

$$y_{N+h} = \mu + \varepsilon_{N+h} + \psi_1 \varepsilon_{N+h-1} + \psi_2 \varepsilon_{N+h-2} + \cdots + \psi_h \varepsilon_N + \psi_{h+1}\varepsilon_{N-1} + \cdots$$

calculation of $\hat{y}_{N+h|N}$

Let us assume that we know $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N$ & all the coefficient parameters $\mu, \psi_1, \psi_2, \ldots$, and

$$\varepsilon_{N+1} = \varepsilon_{N+2} = \cdots = \varepsilon_{N+h} = 0$$

Then,

$$\hat{y}_{N+h|\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{N-1}, \varepsilon_N} = \mu + 0 + \cdots + 0 + \psi_h \varepsilon_N + \psi_{h+1}\varepsilon_{N-1} + \cdots$$

$$\Rightarrow \hat{y}_{N+h|N} = \mu + \psi_h \varepsilon_N + \psi_{h+1}\varepsilon_{N-1} + \cdots$$

$$= \mu + \sum_{j=0}^{\infty} \psi_{h+j} \varepsilon_{N-j}$$

Forecast Errors :-

$$y_{N+h} = \mu + \varepsilon_{N+h} + \psi_1 \varepsilon_{N+h-1} + \cdots + \psi_h \varepsilon_N + \psi_{h+1}\varepsilon_{N-1} + \cdots$$

$$\hat{y}_{N+h|N} = \mu + \psi_h \varepsilon_N + \psi_{h+1}\varepsilon_{N-1} + \cdots$$

$$\hat{\varepsilon}_{N+h} = y_{N+h} - \hat{y}_{N+h|N} = \varepsilon_{N+h} + \psi_1 \varepsilon_{N+h-1} + \cdots + \psi_{h-1}\varepsilon_{N+1}$$

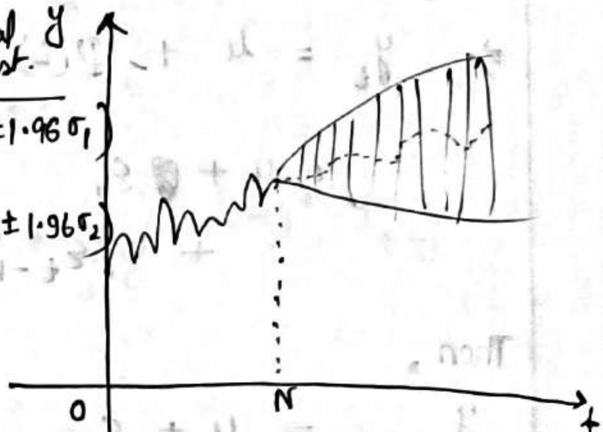
$$\sigma_h^2 = \text{MSE} = E\left(\hat{\varepsilon}^2_{N+h}\right) = E\left(\varepsilon_{N+h} + \psi_1 \varepsilon_{N+h-1} + \cdots + \psi_{h-1}\varepsilon_{N+1}\right)^2$$

$$= \sigma^2 + \psi_1^2\sigma^2 + \cdots + \psi_{h-1}^2\sigma^2$$

$$= \left(\sum_{j=0}^{h-1} \psi_j^2\right)\sigma^2$$

→ h-step ahead forecast error and we denote it by σ_h^2.

h	$\hat{y}_{N+h\|N}$ (point forecast)	σ_h^2	Interval forecast
1	$\hat{y}_{N+1\|N}$	σ^2	$(\hat{y}_{N+1\|N} \pm 1.96\sigma_1)$
2	$\hat{y}_{N+2\|N}$	$\sigma^2(1+\psi_1^2)$	$(\hat{y}_{N+2\|N} \pm 1.96\sigma_2)$
3	$\hat{y}_{N+3\|N}$	$\sigma^2(1+\psi_1^2+\psi_2^2)$	⋮
⋮			
h	$\hat{y}_{N+h\|N}$	$\sigma^2(1+\psi_1^2+\psi_2^2+\cdots+\psi_{h-1}^2)$	$(\hat{y}_{N+h\|N} \pm 1.96\sigma_h)$



Note that σ_h^2 increases as h increases and hence the width of forecast interval (which is $3.92\sigma_h$) also increases as h increases. (For any process).

(b) Forecasting using an MA(q) process:

Data $\{y_1, y_2, \ldots, y_N\} \equiv \{y_N, y_{N-1}, \ldots\}$

$y_t \sim MA(q)$

What will be the value of y_{N+h} given $y_N, y_{N-1}, \ldots$?

$y_t = \mu + \Theta(L)\varepsilon_t$ where, $\varepsilon_t \sim WN(0, \sigma^2)$

& $\Theta(L) = 1 + \theta_1 L + \cdots + \theta_q L^q$

$= \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}$

(*) y_{N+h} ↓ original value at $t = N+h$

$$y_{N+h} = \begin{cases} \mu + \varepsilon_{N+h} + \theta_1 \varepsilon_{N+h-1} + \cdots + \theta_h \varepsilon_N + \theta_{h+1}\varepsilon_{N-1} + \cdots + \theta_q \varepsilon_{N+h-q} & \text{if } 1 \le h \le q \\ \mu + \varepsilon_{N+h} + \theta_1 \varepsilon_{N+h-1} + \cdots + \theta_q \varepsilon_{N+h-q} & h > q \end{cases}$$

let; $\hat{y}_{N+h|N}$ be the forecast value of y_{N+h} given $\varepsilon_N, \varepsilon_{N-1}, \ldots$ (known).

$\varepsilon_{N+1} = \varepsilon_{N+2} = \cdots = \varepsilon_{N+h} = 0$ and $\mu, \theta_1, \theta_2, \ldots, \theta_q$ are known.

$$\hat{y}_{N+h|N} = \begin{cases} \mu + 0 + \cdots + 0 + \theta_h \varepsilon_N + \theta_{h+1}\varepsilon_{N-1} + \cdots + \theta_q \varepsilon_{N-(q-h)} & \text{for } h = 1, 2, \ldots, q \\ \mu & \text{if } h > q \end{cases}$$

↓

(*) **Show that.**

$y_{N+h} = \mu + \varepsilon_{N+h} + \theta_1 \varepsilon_{N+h-1} + \cdots + \theta_q \varepsilon_{N+h-q}$, $\forall\ h = 1, 2, \ldots$

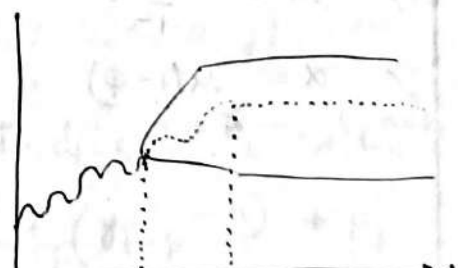
Forecast Error (σ_h^2)

$$\hat{\varepsilon}_{N+h} = y_{N+h} - \hat{y}_{N+h|N}$$

$$= \begin{cases} \varepsilon_{N+h} + \theta_1 \varepsilon_{N+h-1} + \cdots + \theta_{h-1}\varepsilon_{N+1} & \text{if } 1 \le h < q \\ \varepsilon_{N+h} + \theta_1 \varepsilon_{N+h-1} + \cdots + \theta_q \varepsilon_{N+h-q} & \text{if } h > q \end{cases}$$

$$\sigma_h^2 = E(\hat{\varepsilon}^2_{N+h}) = \begin{cases} \sigma^2 & \text{if } h = 1 \\ (1 + \theta_1^2 + \cdots + \theta_{h-1}^2)\sigma^2 & \text{if } 1 < h \le q \\ (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2)\sigma^2 & \text{if } h > q \end{cases}$$

h	$\hat{y}_{N+h\|N}$	σ_h^2	95% C.I
1	$\mu + \theta_1 \varepsilon_N + \theta_2 \varepsilon_{N-1} + \cdots + \theta_q \varepsilon_{N+1-q}$	σ^2	
2		$(1+\theta_1^2)\sigma^2$	
⋮			
q		$\sigma_q^2 (1+\theta_1^2+\theta_2^2+\cdots\theta_{q-1}^2)\sigma^2$	
$q+1$	μ	$(1+\theta_1^2+\cdots+\theta_q^2)\sigma^2 = \sigma_{q+1}^2$	
$q+2$	μ	⋮ σ_{q+1}^2	
⋮	⋮		



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3) Forecasting an AR(1) process:-

Data : $y_1, y_2, \ldots, y_N$ where $y_t \sim AR(1)$

$\varepsilon_t \sim WN(0,\sigma^2)$ $y_t = \alpha + \phi y_{t-1} + \varepsilon_t$,

$\Rightarrow (y_t - \mu) = \phi(y_{t-1} - \mu) + \varepsilon_t$

$(1-\phi L)(y_t - \mu) = \varepsilon_t$ where $\mu = E(y_t) = \frac{\alpha}{1-\phi}$

$\Rightarrow y_t - \mu = \left(\frac{1}{1-\phi L}\right)\varepsilon_t$

$\Rightarrow y_t = \mu + \psi(L)\varepsilon_t \longrightarrow MA(\infty)$

where $\psi(L) = \left(\frac{1}{1-\phi L}\right) = 1 + \phi L + \phi^2 L^2 + \cdots$

$$\hat{y}_{N+h|N} = \mu + \psi_h \varepsilon_N + \psi_{h+1}\varepsilon_{N-1} + \psi_{h+2}\varepsilon_{N-2} + \cdots$$

$$= \mu + (\psi_h + \psi_{h+1}L + \psi_{h+2}L^2 + \cdots)\varepsilon_N$$

$$= \mu + \left[\frac{\psi(L)}{L^h}\right]_+ \varepsilon_N$$

where,

$$\frac{\psi(L)}{L^h} = \frac{1}{L^h}\left[\psi_0 + \psi_1 L + \psi_2 L^2 + \cdots + \psi_h L^h + \psi_{h+1}L^{h+1} + \cdots\right]$$

$$= \psi_0 L^{-h} + \psi_1 L^{-h+1} + \psi_2 L^{-h+2} + \cdots + \psi_h + \psi_{h+1}L + \psi_{h+2}L^2 + \cdots$$

Assuming $L^{-h} = 0 \;\forall\; h = 1, 2, \ldots$

$$\left[\frac{\psi(L)}{L^h}\right]_+ = \left[\psi_h + \psi_{h+1}L + \psi_{h+2}L^2 + \cdots\right]$$

(Forecast error)

4) Forecasting for AR(p) process :-

Data : $y_1, y_2, \ldots, y_N$

$y_t \sim AR(p)$ process

$y_t = \alpha + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \varepsilon_t$, $\varepsilon_t \sim WN(0,\sigma^2)$

$\Rightarrow (y_t - \mu) = \phi_1(y_{t-1} - \mu) + \cdots + \phi_p(y_{t-p} - \mu) + \varepsilon_t$

$$\left[\mu = E(y_t) = \frac{\alpha}{1-\phi_1-\phi_2-\cdots-\phi_p}\right]$$

$$\Rightarrow (1-\phi_1 L-\phi_2 L^2-\cdots-\phi_p L^p)(y_t-\mu) = \varepsilon_t$$

$$\Rightarrow (y_t-\mu) = \left(\frac{1}{1-\phi_1 L-\cdots-\phi_p L^p}\right)\varepsilon_t = \Psi(L)\varepsilon_t$$

$\rightarrow$ MA(∞) process.

$$\Rightarrow y_t = \mu + \Psi(L)\varepsilon_t$$

$$\hat{y}_{N+h|N} = \mu + (\psi_h \varepsilon_N + \psi_{h+1}\varepsilon_{N-1} + \cdots)$$

MA
AR $\leftarrow$
ARMA

$$\boxed{\hat{y}_{N+h|N} = \mu + \left[\frac{\Psi(L)}{L^h}\right]_+ \varepsilon_N}, \text{ where } \left[\frac{\Psi(L)}{L^h}\right]_+ = [\psi_h + \psi_{h+1}L + \psi_{h+2}L^2 + \cdots]$$

$$(1-\phi_1 L-\phi_2 L^2-\cdots-\phi_p L^p)^{-1} = \psi_0 + \psi_1 L + \psi_2 L^2 + \cdots$$

$\Rightarrow \psi_j = f(\phi_1, \phi_2, \ldots, \phi_p) \rightarrow$ difficult to find for $p>1$

If $\underline{p=1}$

$$(1-\phi L)^{-1} = \psi_0 + \psi_1 L + \psi_2 L^2 + \cdots$$

$$\psi_j = \phi^j \quad \forall\, j = 0, 1, \ldots$$

5) Forecasting using ARMA (p, q)

consider, stationary and investible ARMA (p, q) process;

$$(1-\phi_1 L-\phi_2 L^2-\cdots-\phi_p L^p)(y_t-\mu) = (1+\theta_1 L+\theta_2 L^2+\cdots+\theta_q L^q)\varepsilon_t$$

The generalization of ARMA (1, 1) of 1 & 2 are;

$$(\hat{y}_{N+1|N}-\mu) = \phi_1(y_t-\mu) + \phi_2(y_{t-1}-\mu) + \cdots + \phi_p(y_{t-p+1}-\mu) + \theta_1\hat{\varepsilon}_N + \theta_2\hat{\varepsilon}_{N-1} + \cdots + \theta_q\hat{\varepsilon}_{N-q+1}$$

where, $\hat{\varepsilon}_N$ is generated from $\hat{\varepsilon}_N = y_N - \hat{y}_{N|N-1}$

$$(\hat{y}_{N+h|N}-\mu) = \phi_1(\hat{y}_{N+h-1|N}-\mu) + \phi_2(\hat{y}_{N+h-2|N}-\mu) + \cdots + \phi_p(\hat{y}_{N+h-p|N}-\mu) + \theta_h\hat{\varepsilon}_N + \theta_{h+1}\hat{\varepsilon}_{N-1} + \cdots + \theta_q\hat{\varepsilon}_{N+h-q}$$

for $h = 1, 2, \ldots, q$

$$= \phi_1(\hat{y}_{N+h-1|N} - \mu) + \phi_2(\hat{y}_{N+h-2|t} - \mu) + \dots + \phi_p(y_{N+h-p|N} - \mu)$$

for $s = q+1, q+2, \dots$

where $\hat{y}_{\tau|t} = y_\tau$ for $\tau \leq t$

for a horizon h greater than the MA order q the forecasts follow p th order difference equation solely governed by AR parameters

Forecasting with ARMA (1, 1) process.

$$(1-\phi L)(Y_t - \mu) = (1+\theta L)\varepsilon_t$$

The process is stationary when $|\phi| < 1$ and invertible when $|\theta| < 1$.

Now we know that, $\hat{Y}_{N+h|N} = \mu + \left[\frac{\psi(L)}{L^h}\right]_+ \varepsilon_N$

where $\varepsilon_t = \eta(L)(Y_t - \mu)$

Where $\psi(L) = \frac{1+\theta L}{1-\phi L}$ and $\eta(L) = \frac{1-\phi L}{1+\theta L}$:

$$\hat{Y}_{N+h|N} = \mu + \left[\frac{1+\theta L}{(1-\phi L)L^h}\right] \frac{1-\phi L}{1+\theta L}(Y_t - \mu)$$

$$\left[\frac{1+\theta L}{(1-\phi L)L^h}\right] = \frac{(1+\phi L + \phi^2 L^2 + \dots)}{L^h} + \frac{\theta L(1+\phi L + \phi^2 L^2 + \dots)}{L^h}$$

$$= (\phi^h + \phi^{h+1}L + \dots) + \theta(\phi^{h-1} + \phi^h L + \phi^{h+1}L^2 + \dots)$$

$$= \frac{\phi^h + \theta\phi^{h-1}}{1-\phi L}$$

$$\boxed{\therefore \hat{y}_{N+h|N} = \mu + \frac{\phi^h + \theta\phi^{h-1}}{1+\theta L}(y_N - \mu)}$$

$$\hat{y}_{N+1|N} = \mu + \left(\frac{\phi+\theta}{1+\theta L}\right)(y_N - \mu)$$

$$\Rightarrow \hat{y}_{N+1|N} - \mu = \frac{\phi(1+\theta L) + \theta(1-\phi L)}{1+\theta L}(y_N - \mu) = \phi(y_N - \mu) + \theta\hat{\varepsilon}_N$$

$$s_y(\omega) = \frac{1}{2\pi} g_y(z) = \frac{\sigma^2}{2\pi}\psi(z)\psi(z^{-1}) \qquad \gamma(h) = \int_{-\pi}^{\pi} s_y(\omega)\cos(\omega h)\,d\omega$$

$$= \frac{1}{2\pi}\sum \gamma(h) z^h$$

$$\frac{1}{2\pi}\left[\gamma(0) + 2\sum_{h=1}^{\infty}\gamma(h)\cos(\omega h)\right]$$

NOTE

So far we assume that ε_t's are known. But in practice ε_t's are not observed. We only observe y_t's i.e $\{y_1, y_2, \dots, y_N\}$. Then what will be the forecasting formula for all the 5 (cases)?

(1) MA(∞)

$$\hat{y}_{N+h/N} = \mu + \left[\frac{\Psi(L)}{L^h}\right]_+ \varepsilon_N \quad \text{where} \quad y_t = \mu + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$$

forecasting formula

$$y_t = \mu + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$$

$$= \mu + \sum_{j=0}^{\infty} \psi_j L^j \varepsilon_t$$

$$\Rightarrow (y_t - \mu) = \left(\sum_{j=0}^{\infty} \psi_j L^j\right) \varepsilon_t$$

Invertible process.

An MA(∞) is said to be invertible if it can be written as a AR(∞) form —

$$y_t = \alpha + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q}$$

$$\Rightarrow (1 + \theta_1 L + \dots + \theta_q L^q)\varepsilon_t = (y_t - \alpha)$$

$$\Rightarrow \varepsilon_t = \left(\frac{1}{1+\theta_1 L + \dots + \theta_q L^q}\right)(y_t - \alpha)$$

$$= \eta(L)(y_t - \alpha)$$

where $\eta(L) = \dfrac{1}{1+\theta_1 L + \dots + \theta_q L^q} = \sum_{j=0}^{\infty} \eta_j L^j$ & $\sum_{j=0}^{\infty} |\eta_j| < \infty$

MA

$$y_t = \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \dots$$

$$\varepsilon_t = (y_t - \alpha) + \eta_1 (y_{t-1} - \alpha) + \eta_2 (y_{t-2} - \alpha) + \dots$$

Suppose, MA(∞) process is investible then we know that $\boxed{\varepsilon_t = \eta(L)(y_t - \mu)}$

$$\hat{y}_{N+h|N} = \mu + \left[\frac{\Psi(L)}{L^h}\right]_+ \varepsilon_N = \mu + \left[\frac{\Psi(L)}{L^h}\right]_+ \eta(L)(y_N - \alpha)$$

$\alpha = \underline{E(y_t)}$ *

AR(p) : μ
MA(q) = α

For any MA or AR or ARMA process, the forecasting formula is —

$$\boxed{\hat{y}_{N+h|N} = \mu + \left[\frac{\Psi(L)}{L^h}\right]_+ \eta(L)(y_N - \mu)}$$ where $\mu = E(y_t)$

Case

Let, $y_t \sim AR(1)$ process

$$y_t = \alpha + \phi y_{t-1} + \varepsilon_t$$

$\Psi(L) = (1-\phi L)^{-1} = 1 + \phi L + \phi^2 L^2 + \cdots$

$\eta(L) = 1 - \phi L$

$$\Rightarrow (1-\phi L)(y_t - \mu) = \varepsilon_t$$

$$\Rightarrow (y_t - \mu) = \Psi(L)\varepsilon_t \longrightarrow MA(\infty)$$

$$\frac{\Psi(L)}{L^h} = \frac{1+\phi L + \phi^2 L^2 + \cdots}{L^h} = L^{-h} + \phi L^{-h+1} + \phi^2 L^{-h+2} + \cdots$$

$$= \phi^h + \phi^{h+1} L + \phi^{h+2} L^2 + \cdots$$

$$= \phi^h (1 + \phi L + \phi^2 L^2 + \cdots)$$

$$= \phi^h (1 - \phi L)^{-1}$$

$$\therefore \hat{y}_{N+h|N} = \mu + \phi^h (1-\phi L)^{-1}(1-\phi L)(y_N - \mu)$$

$$= \boxed{\mu + \phi^h (y_N - \mu)}$$

For an AR(1) process the forecasting formula is —

$$\hat{y}_{N+h|N} = \mu + \phi^h (y_N - \mu) \quad \forall \; h = 1, 2, \ldots$$

→ Find the forecast error σ_h^2 for AR(1) process.

→ Find $\hat{y}_{N+h|N}$ & σ_h^2 for ARMA(1,1) process

h	$\hat{y}_{N+h\mid N}$	
1	$\mu + \phi(y_N - \mu)$	
2	$\mu + \phi^2 (y_N - \mu)$	

Hamilton ch-6

19/4/23.

- Under the assumption that $\{y_t\}$ is covariance stationary we can describe y_t in the form of MA(∞) given by
$$y_t = \mu + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} \quad \text{(By wold decomposition theorem)}$$
- Autocorrelation between y_t & y_{t-h}.
This study is known as analysing the properties of $\{y_t\}$ in the time domain. (Time Domain Analysis).

Frequency Domain or Spectral Analysis

We describe the value of y_t as a weighted sum of periodic functions of the form of $\cos(\omega t)$ and $\sin(\omega t)$ where ω denotes the particular frequency:
$$y_t = \mu + \int_0^{\pi} \underbrace{\alpha(\omega)}_{\text{weight}} \cos(\omega t)\, d\omega + \int_0^{\pi} \underbrace{\delta(\omega)}_{\text{weight}} \sin(\omega t)\, d\omega$$

The goal is to determine how important cycles of different frequencies are in accounting for the behavior of y_t. (Frequency Domain / Spectral Analysis)

NOTE

- Any covariance stationary process has both time domain and frequency domain Analysis form.
And any feature of y_t that can be represented by one representation can equally well be described by other representation

- For some features of $\{y_t\}$ the time domain representation may be simpler while for other features the frequency domain representation may be simpler.
Try to find some examples of features that are simpler in time domain than ferequency domain —

For signalling analysis time domain analysis is easier. Beacause frequency domain can only anylse static signals.

$$y_t = \mu + \int^{\pi} \alpha(\omega)\cos(\omega t)\,d\omega + \int^{\pi} \delta(\omega)\sin(\omega t)\,d\omega$$

for different values of ω different waves are generated and y_t is written as a weighted sum of these waves.

$$y_t = \mu + \sum_{i=1}^{k} \alpha(\omega_i)\cos(\omega_i t) + \sum_{i=1}^{k} \delta(\omega_i)\sin(\omega_i t)$$

$$\simeq \mu + \int_0^{\pi} \alpha(\omega)\cos(\omega t)\,d\omega + \int_0^{\pi} \delta(\omega)\sin(\omega t)\,d\omega.$$

The population Spectrum :

$y_t \sim$ covariance stationary with $\mu = E(y_t)$ and

$\gamma(h) = \operatorname{cov}(y_t, y_{t-h})$ and $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$

Then auto covariance generating function is defined as

$g_y(z) = \sum_{h=-\infty}^{\infty} \gamma(h) z^h$ where z is a complex number $a+bi$

(Probability generating function p.g.f.
$E(t^x) = \sum_{x=-\infty}^{\infty} t^x p(x)$)

If we calculate $g_y(z)$ at $z = e^{-i\omega}$ and divide it by 2π then the result is called the population spectrum of y_t i.e

$$s_y(\omega) = \frac{1}{2\pi} g_y(e^{-i\omega})$$

$$= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma(h)(e^{-i\omega})^h$$

$$= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma(h) e^{-i\omega h}$$

$$= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma(h)\left[\cos(\omega h) - i\sin(\omega h)\right].$$

Since $\gamma(h) = \gamma(-h)$
for any covariance stationary y_t so;

$$s_y(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma(h)\left[\cos(\omega h) - i\sin(\omega h)\right]$$

$$= \frac{1}{2\pi} \sum_{h=-\infty}^{-1} \gamma(h)\left[\cos(\omega h) - i\sin(\omega h)\right]$$

$$+ \gamma_0(\cos 0 - i\sin 0) + \sum_{h=1}^{\infty} \gamma(h)\left[\cos(\omega h) - i\sin(\omega h)\right]$$

For covariance stationary process $\gamma(h) = \gamma(-h)$.

$$= \frac{1}{2\pi}\left\{\gamma_0 + 2\sum \gamma(h)\left[\cos(\omega h) + \cos(-\omega h) - i\sin(\omega h) - i\sin(-\omega h)\right]\right\}$$

$$\boxed{= \frac{1}{2\pi}\left\{\gamma_0 + 2\sum_{j=1}^{\infty}\gamma(h)\cos(\omega h)\right\}} \to \text{population spectrum for any cov-stationary process } \{y_t\}$$

as $\cos(-\theta) = \cos\theta$
$\sin(-\theta) = -\sin(\theta)$.

Properties of $s_y(\omega)$:-

1> $s_y(\omega)$ is continuous real valued function of ω and always exists.

$s_y(\omega) : [0,\pi] \to \mathbb{R}$

2> $s_y(\omega) \geq 0 \quad \forall\ \omega$

3> $s_y(\omega) = s_y(-\omega) \quad \forall\ \omega \to$ symmetric around 0

4> $s_y(\omega + 2\pi k) = s_y(\omega)$ for any integer k.

5> If we know the value of $s_y(\omega)$ for all ω between 0 & π, then we can know the value of $s_y(\omega)$ for any ω.

It is sufficient to know $s_y(\omega)$ for $0 \leq \omega \leq \pi$.

Example:-

$$y_t = \mu + \sum_{j=0}^{\infty}\psi_j \varepsilon_{t-j} \quad \& \sum_{j=0}^{\infty}|\psi_j| < \infty$$

$$\Rightarrow \sum_{h=-\infty}^{\infty}|\gamma(h)| < \infty$$

$$= \mu + \Psi(L)\varepsilon_t$$

$$\Psi(L) = 1 + \psi_1 L + \psi_2 L^2 + \dots$$

$s_y(\omega) = ?$

$$s_y(\omega) = \frac{1}{2\pi} g_y(e^{-i\omega})$$

where $g_y(z) = \sum_{h=-\infty}^{\infty}\gamma(h) z^h$ ← ACV g.fn.

$= \sigma^2 \Psi(z)\Psi(z^{-1})$ for any MA(∞) process

?? HW.

Example: $Y_t = \varepsilon_t$, $\varepsilon_t \sim WN(0, \sigma^2)$.

Show that $s_y(\omega) = \frac{\sigma^2}{2\pi}$.

Example: $Y_t = \alpha + \phi y_{t-1} + \varepsilon_t$ show that

ARMA (1,1) MA(1) $s_y(w) = \frac{\sigma^2}{2\pi} \cdot \frac{1}{1 + \phi^2 - 2\phi\cos(\omega)}$

Result

let, $s_y(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma(h) e^{-i\omega h}$ and $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$

Then $\gamma(h) = \int_{-\pi}^{\pi} s_y(w) e^{iwh} dw$

$= \int_{-\pi}^{\pi} s_y(\omega) \cos(\omega h) d\omega$

→ If we know the population spectrum $s_y(w)$ then we can compute $\gamma(h)$ for any h.

$\gamma(0) = \int_{-\pi}^{\pi} s_y(w) dw \longrightarrow Var(y_t)$.

Example

Let $s_y(\omega) = \frac{\sigma^2}{2\pi}$ Show that $\gamma(h) = \begin{cases} \sigma^2 & \text{if } h=0 \\ 0 & \text{if } h \neq 0 \end{cases}$

Example

let $s_y(\omega) = \frac{\sigma^2}{2\pi} \cdot \frac{1}{1+\phi^2-2\phi\cos(\omega)}$ then show that

$\gamma(h) = \frac{\sigma^2 \phi^h}{1-\phi^2}$ for all h.

Periodogram.

Periodogram of a time series process $\{y_t\}$ is sample version of the population spectrum.

$$\hat{s}_Y(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \hat{\gamma}(h) e^{-i\omega h}$$

$$= \frac{1}{N} \sum_{t=h+1}^{N} (y_t - \bar{y})(y_{t-h} - \bar{y}), \quad h = 0, 1, 2, \ldots$$

$$\hat{\gamma}(h) = \hat{\gamma}(-h), \quad \bar{Y} = \frac{1}{N} \sum_{t=1}^{N} Y_t$$

$$\hat{s}_Y(\omega) = \frac{1}{2\pi} \left[\hat{\gamma}(0) + 2 \sum_{h=1}^{\infty} \hat{\gamma}(h) \cos(\omega h) \right]$$

change point Analysis.

Box - Jenkin's Algorithm to fit a time series data most popular in R done by auto-arima function.

<u>Step 1</u> : Transform the data if necessary using log transformation or Box transformation or using ∇ (differencing) or ∇_s (seasonal differencing) so that y_t becomes stationary.

$y_t \to \log y_t$ $\qquad$ $y_t \to \nabla_s^D \nabla^d y_t$

$y_t \to \dfrac{y_t^\lambda - 1}{\lambda}$ $\qquad$ $\nabla_s^D \nabla^d \log y_t$

where $y_t^* \sim$ cov stationary

<u>Step 2</u> : Make an initial guess of (p, q) for ARMA(p, q) using ACF and PACF

<u>Step 3</u> : Estimate $\Phi(L)$ & $\Theta(L)$ of an ARMA(p, q)

<u>Step 4</u> : Perform model evaluation by comparing RSS