Stability:

One-step methods:

Definition: A one-step difference-equation method with local truncation error $\tau_i(h)$ at the ith step is said to be consistent with the differential equation it approximates if

$$\lim_{h \to 0} \max_{1 \le i \le N} |\tau_i(h)| = 0.$$

Definition: A one-step difference-equation method is said to be convergent with respect to the differential equation it approximates if

$$\lim_{h \to 0} \max_{1 \le i \le N} |w_i - y(t_i)| = 0,$$

where $y(t_i)$ denotes the exact value of the solution to the differential equation and w_i is the approximation obtained from the difference method at the ith step.

Example: Euler's method is convergent

we have proved earlier that

$$\max_{1 \le i \le N} |y(t_i) - w_i| \le \frac{Mh}{2L} \left| e^{L(b-a)} - 1 \right|$$

where M, L, a and b are constants, and

$$\lim_{h \to 0} \max_{1 \le i \le N} |y(t_i) - w_i| \le \lim_{h \to 0} \frac{Mh}{2L} \left| e^{L(b-a)} - 1 \right| = 0.$$

Definition: A difference-equation method is stable if small changes or perturbations in the initial conditions produce correspondingly small changes in the subsequent approximation.

The following theorem provides stability of one-step methods. The stability of a one-step method is somewhat analogous to the condition of a difference-equation being well-posed.

Theorem: Suppose the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

is approximated by a one-step difference method in the form

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + h\,\phi(t_i, w_i, h).$$

Suppose also that a number $h_0 > 0$ exists and that $\phi(t, w, h)$ is continuous and satisfies a Lipschitz condition in the variable w with Lipschitz constant L on the set

$$D = \{(t, w, h) \mid a \le t \le b, \; -\infty < w < \infty, \; 0 \le h \le h_0\}.$$

Then (i) The method is stable

(ii) The difference method is convergent if and only if it is consistent, which is equivalent to

$$\phi(t, y, 0) = f(t, y), \quad a \le t \le b.$$

(iii) If a function τ exists and for each $i = 1, 2, \ldots, N$, the local truncation error $\tau_i(h)$ satisfies $|\tau_i(h)| \le \tau(h)$, whenever $0 \le h \le h_0$, then

$$|y(t_i) - w_i| \le \frac{\tau(h)}{L} e^{L(t_i - a)}.$$

Example: The RK-method of order 2:

$w_0 = \alpha$,

$$w_{i+1} = w_i + \frac{h}{2}\left[f(t_i, w_i) + f(t_{i+1}, w_i + h f(t_i, w_i))\right],$$

for $i = 0, 1, \ldots N-1$.

We show that RK-method is stable by verifying the hypothesis of previous theorem

Solution: For this method

$$\phi(t, w, h) = \frac{1}{2} f(t, w) + \frac{1}{2} f(t+h, w + h f(t, w)).$$

If f satisfies a Lipschitz condition on

$D = \{(t, w) \mid a \le t \le b, -\infty < w < \infty\}$ in the variable w with constant L, then, since

$$\phi(t, w, h) - \phi(t, \bar{w}, h) = \frac{1}{2}\left(f(t, w) - f(t, \bar{w})\right) + \frac{1}{2}\left(f(t+h, w + h f(t, w)) - f(t+h, \bar{w} + h f(t, \bar{w}))\right),$$

implies

$$|\phi(t, w, h) - \phi(t, \bar{w}, h)| \le \frac{1}{2} L |w - \bar{w}| + \frac{1}{2} L |w + h f(t, w) - \bar{w} - h f(t, \bar{w})|$$

$$\le L |w - \bar{w}| + \frac{1}{2} L |h f(t, w) - h f(t, \bar{w})|$$

$$\le L |w - \bar{w}| + \frac{h}{2} L^2 |w - \bar{w}|$$

$$= \left(L + \frac{h}{2} L^2\right) |w - \bar{w}|.$$

Therefore, ϕ satisfies a Lipschitz condition in w on the set

$$\{(t, w, h) \mid a \le t \le b, -\infty < w < \infty, 0 \le h \le h_0\}$$

for any $h_0 > 0$, with constant

$$L' = L + \frac{h}{2} L^2.$$

If f is continuous on $\{(t,w) \mid a \le t \le b, -\infty < w < \infty\}$, (4)

then ϕ is continuous on

$$\{(t,w,h) \mid a \le t \le b, -\infty < w < \infty, \ 0 \le h \le h_0\}.$$

So the RK-2 method is stable.

Letting $h=0$, we have

$$\phi(t,w,0) = \frac{1}{2} f(t,w) + \frac{1}{2} f(t+0, w+0 \cdot f(t,w))$$

$$= \frac{1}{2} f(t,w) + \frac{1}{2} f(t,w) = f(t,w).$$

So the consistency condition in the theorem holds. Thus the method is convergent.

Moreover we have seen that for this method the local truncation error is $O(h^2)$.

i.e. $\tau(h) = O(h^2)$. The result in (iii) of the theorem implies that RK-2 method is also $O(h^2)$.

~~A~~ Multi-step methods:

For multi-step methods, the problems involved with consistency, convergence and stability are compounded because of the number of approximations involved at each step.

The general multi-step method for approximating the solution to the IVP: $y' = f(t, y)$, $a \le t \le b$, $y(a) = \alpha$. has the form

$$\left.\begin{aligned} &w_0 = \alpha, \quad w_1 = \alpha_1, \quad \cdots \quad w_{m-1} = \alpha_{m-1}, \\ &w_{i+1} = a_{m-1} w_i + a_{m-2} w_{i-1} + \cdots + a_0 w_{i+1-m} \\ &\qquad + h\, F(t_i, h, w_{i+1}, w_i, \cdots, w_{i+1-m}) \end{aligned}\right\} \text{—} (1)$$

for each $i = m-1, m, \cdots N-1$, where $a_0, a_1, \cdots a_{m+1}$ are constants, and $h = \frac{b-a}{N}$, and $t_i = a + ih$.

The local truncation error has the form

$$\left.\begin{aligned} \tau_{i+1}(h) = {}& \frac{y(t_{i+1}) - a_{m-1} y(t_i) - \cdots - a_0 y(t_{i+1-m})}{h} \\ & - F(t_i, h, y(t_{i+1}), y(t_i), \cdots, y(t_{i+1-m})) \end{aligned}\right\} \text{—} (2)$$

for each $i = m-1, m, \cdots, N-1$.

<u>Definition</u>: A multi-step method is convergent if the solution to the difference equation ~~to the~~ approaches the solution to the differential equation as the step size h approaches zero.

This means $\lim_{h \to 0} \max_{1 \le i \le N} |y(t_i) - w_i| = 0$.

Definition: A multi-step method of the form (1) is said to be consistent if

$$\lim_{h\to 0} |z_i(h)| = 0, \quad \text{for all } i = m, m+1, \dots N,$$

and

$$\lim_{h\to 0} |\alpha_i - y(t_i)| = 0, \quad \text{for } i = 1, 2, \dots m-1, \quad — (*)$$

where local truncation error z_i is defined by the equation in (2).

Note that the initial approximations $\alpha_1, \dots \alpha_{m-1}$ are obtained by some other methods. The condition in (*) implies that the method used for obtaining $\alpha_1, \dots \alpha_{m-1}$, must be consistent in order to have the multi-step method is consistent.

$\boxed{\text{An example of unstable method}}$

A method which is not stable is called unstable method.

Consider the IVP:

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha.$$

Using Taylor's theorem,

$$y(t_{n+1}) = y(t_n) + h\,y'(t_n) + \frac{h^2}{2} y''(t_n) + \frac{h^3}{6} y'''(\xi_n)$$

$$y(t_{n-1}) = y(t_n) - h\,y'(t_n) + \frac{h^2}{2} y''(t_n) - \frac{h^3}{6} y'''(\eta_n)$$

where $h = t_{n+1} - t_n = t_n - t_{n-1}$, $\xi_n \in (t_n, t_{n+1})$, $\eta_n \in (t_{n-1}, t_n)$.

From above two equations

$$y(t_{n+1}) = y(t_{n-1}) + 2h\,y'(t_n) + \frac{h^3}{6}\left(y'''(\xi_n) + y'''(\eta_n)\right)$$

Dropping the remainder term and using $y'(t_n) = f(t_n, y(t_n))$, we notice that (7)

$$y(t_{n+1}) \approx y(t_{n-1}) + 2h\, f(t_n, y(t_n))$$

The method is :-

$$w_0 = \alpha, \quad w_1 = \alpha_1,$$

$$w_{i+1} = w_{i-1} + 2h\, f(t_i, w_i)$$

for $i = 1, 2, \ldots N-1$. Here α_1 needs to be obtained from some other method.

The local truncation error for this method is

$$\tau_{i+h} = \frac{y(t_{i+1}) - y(t_{i-1})}{2h} - f(t_i, y(t_i)) = O(h^2).$$

We would expect more accurate solution with this method compared with Euler's method.

Let us examine the method for the equation:

~~$y' = 2y, \ y(a) = \alpha$~~ $y' = -2y, \ y(0) = 1.$

Then the method takes the form

$$w_{i+1} = w_{i-1} + 2h(-2w_i) \quad ; \quad w_0 = 1.$$

$$\Rightarrow \quad w_{i+1} + 4hw_i - w_{i-1} = 0$$

If $w_i = \beta^i$, then

$$\beta^{i+1} + 4h\beta^i - \beta^{i-1} = 0$$

$$\beta^{i-1}(\beta^2 + 4h\beta - 1) = 0$$

Therefore β is a root of $\beta^2 + 4h\beta - 1 = 0$

$$\beta = -2h \pm \sqrt{1+4h^2}$$

If we expand $\sqrt{1+4h^2}$ in a Taylor's series we get

$$\beta_1 = 1-2h+O(h^2)$$

$$\beta_2 = -(1+2h)+O(h^2)$$

The general solution of difference equation is given by

$$w_n = c_1\beta_1^n + c_2\beta_2^n \qquad \text{(used } i=n\text{)}$$

$$w_n = c_1\left(1-2h+O(h^2)\right)^n + c_2(-1)^n\left(1+2h+O(h^2)\right)^n$$

we know that

$$\lim_{\epsilon\to 0}(1+\epsilon)^{1/\epsilon} = e.$$

since $t_n = nh$, it follows that

$$\lim_{h\to 0}(1+2h)^n = \lim_{h\to 0}(1+2h)^{\frac{2t_n}{2h}} = e^{2t_n}$$

Similarly $\lim_{h\to 0}(1-2h)^n = e^{-2t_n}$.

as $h\to 0$, the solution of difference equation approaches

$$w_n = (c_1 e^{-2t_n}) + c_2(-1)^n e^{2t_n}.$$

The solution of $y' = -2y$, $y(0)=1$ is $y(t) = e^{-2t}$

$\therefore y(t_n) = e^{-2t_n}$, but the approximation w_n by extraneous term $c_2(-1)^n e^{2t_n}$ which may increase exponentially as n increases.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e = \lim_{n\to\infty}\left(1+\frac{1}{n}+\frac{1}{n^2}\right)^n.$$

$$\left(1+\frac{1}{n}+\frac{1}{n^2}\right)^n = \left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n^2(1+1/n)}\right)^n = a_n$$

since

$$1 \le \left(1+\frac{1}{n^2(1+1/n)}\right)^n \le \left(1+\frac{1}{n^2}\right)^n,$$

we have

$$\left(1+\frac{1}{n}\right)^n \le a_n \le \left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n^2}\right)^n$$

In calculus, it is already proved that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

Consider $\left(1+\frac{1}{n^2}\right)^n$:

$$\left(1+\frac{1}{n^2}\right)^n = 1+n\cdot\frac{1}{n^2}+\frac{n(n-1)}{2}\left(\frac{1}{n^2}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n^2}\right)^3+\cdots$$

$$\cdots+\frac{n(n-1)(n-2)\cdots(n-(n-1))}{n!}\left(\frac{1}{n^2}\right)^n$$

$$=1+\frac{1}{n}+\frac{1}{2!}(1-1/n)\cdot\frac{1}{n^2}+\frac{1}{3!}(1-1/n)(1-2/n)\cdot\frac{1}{n^3}+\cdots$$

$$\cdots+\frac{1}{n!}(1-1/n)(1-2/n)\cdots\left(1-\frac{n-1}{n}\right)\cdot\frac{1}{n^n}$$

$$=1+\frac{1}{n}\left[1+\frac{1}{2!}(1-1/n)\cdot\frac{1}{n}+\frac{1}{3!}(1-1/n)(1-2/n)\cdot\frac{1}{n^2}+\cdots\right.$$

$$\left.\cdots+\frac{1}{n!}(1-1/n)(1-2/n)\cdots\left(1-\frac{n-1}{n}\right)\cdot\frac{1}{n^{n-1}}\right]$$

$$\le 1+\frac{1}{n}\left[1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}\right]$$

$$\le 1+\frac{e-1}{n}$$

since $1 \le \left(1+\frac{1}{n^2}\right)^n \le 1+\frac{e-1}{n}$,

$$\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^n = 1.$$

Therefore $\lim_{n\to\infty}\left(1+\frac{1}{n}+\frac{1}{n^2}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$

~~Stability of numerical methods:~~ ~~Rough!~~

~~Definition: A one-step method~~

For the multistep method

$$w_0 = \alpha,\ w_1 = \alpha_1,\ \ldots\ w_{m-1} = \alpha_{m-1}$$

$$w_{i+1} = a_{m-1} w_i + a_{m-2} w_{i-1} + \cdots + a_0 w_{i+1-m} + h f(t_i, h, w_{i+1}, w_i, \ldots w_{i+1-m}),$$

define the characteristic polynomial, $P(\lambda)$, by

$$P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - \cdots - a_1\lambda - a_0.$$

The stability of the multistep method with respect to round off errors is dictated by the magnitudes of the zeros of the characteristic polynomial.

Definition: Let $\lambda_1, \lambda_2, \ldots \lambda_m$ denote the (not necessarily distinct) roots of the characteristic equation

$$P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - \cdots - a_1\lambda - a_0 = 0$$

associated with the $m$step multistep method

$$w_0 = \alpha,\ w_1 = \alpha_1,\ \ldots\ w_{m-1} = \alpha_{m-1}$$

$$w_{i+1} = a_{m-1} w_i + a_{m-2} w_{i-1} + \cdots + a_0 w_{i+1-m} + h f(t_i, h, w_{i+1}, w_i, \ldots w_{i+1-m}).$$

If $|\lambda_i| \le 1$, for each $i = 1, 2, \ldots m$ and all roots with absolute value 1 are simple roots, then the difference method is said to satisfy the root condition:

Definition: (i) Methods that satisfy the root condition and have $\lambda = 1$ as the only root of the characteristic equation with magnitude one are called strongly stable.

(ii) Methods that satisfy the root condition and have more than one distinct root with magnitude one are called weakly stable

(iii) Methods that do not satisfy the root condition are called unstable.

[strongly stable methods are stable methods without any restrictions (or) condition. weakly stable methods are stable subject to additional conditions on the sign of $\frac{\partial f}{\partial y}$ on the interval of integration].

Examples: Adams-Bashforth fourth-order method

for $y' = \lambda y$, $y(0) = \alpha$.

The difference equation is

$$w_{i+1} = w_i + \frac{h\lambda}{24}\left[55 w_i - 59 w_{i-1} + 37 w_{i-2} - 9 w_{i-3}\right]$$

Characteristic polynomial is [4 step method, $m=4$]

$$\beta^4 - \beta^3 = 0 \Rightarrow \beta^2(\beta - 1) = 0$$

$\Rightarrow \beta = 1$, (or) $\beta = 0$.

Therefore the method is ~~stab~~ strongly stable.

The method we discussed earlier ($m=2$).

$$w_{i+1} = w_{i-1} + 2h\lambda w_i$$

The characteristic polynomial is

$\beta^2 - 1 = 0 \Rightarrow \beta = \pm 1$;

The method is weakly stable

The solution of difference equation is given by

as $h \to 0$: $y_n = c_1 e^{\lambda t_n} + c_2 (-1)^n e^{-\lambda t_n}$.

From the analysis of equation

$$\beta^2 - 2\lambda h \beta + 1 = 0$$

$$y_n = c_1 \beta_1^n + c_2 \beta_2^n$$

$\beta_1 = 1 + h\lambda$, $\beta_2 = 1 - h\lambda$

If $\lambda > 0$, second term goes zero, If $\lambda < 0$, it