Interpolation and Lagrange polynomial:

Consider two points in $\mathbb{R}^2$
(x_0, y_0) and (x_1, y_1).

Define $L_0(x) = \dfrac{x - x_1}{x_0 - x_1}$ and $L_1(x) = \dfrac{x - x_0}{x_1 - x_0}$.

Both L_0 and L_1 are polynomials of degree 1.

If $P_1(x) = L_0(x) y_0 + L_1(x) y_1$

then $P_1(x_0) = y_0$ and $P_1(x_1) = y_1$, since
$L_0(x_0) = 1$, $L_0(x_1) = 0$, $L_1(x_0) = 0$ and $L_1(x_1) = 1$.

$P_1(x)$ is the polynomial of degree 1 interpolating data points (x_0, y_0) and (x_1, y_1).

We may think $P_1(x)$ is an approximation of a function $f(x)$, with $f(x_0) = y_0$ and $f(x_1) = y_1$.

i.e. $P_1(x)$ <u>interpolates</u> $(x_0, f(x_0))$, $(x_1, f(x_1))$.

To generalize the concept of linear interpolation, consider the construction of a polynomial of degree at most n that passes through $(n+1)$ points
$(x_0, f(x_0))$, $(x_1, f(x_1))$ ..... $(x_n, f(x_n))$.

<u>Remark</u>: A polynomial (in one variable) of degree at most n is given by

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

where $a_0, a_1, \ldots a_n$ are real numbers.

we first construct $(n+1)$ polynomials

$L_{n,k}(x)$, $k = 0, 1, 2 \ldots n$

that have the property:

$$L_{n,k}(x_j) = 0 \text{ when } k \neq j$$

$$L_{n,k}(x_k) = 1.$$

Recall that $x_0, x_1, \ldots x_n$ are given $(n+1)$ distinct points.

Since $L_{n,k}(x_j) = 0$ when $k \neq j$, $L_{n,k}$ has the term (in its numerator)

$$(x-x_0)(x-x_1) \ldots (x-x_{k-1})(x-x_{k+1}) \ldots (x-x_n)$$

and since $L_{n,k}(x_k) = 0$, we can write

$$L_{n,k}(x) = \frac{(x-x_0)(x-x_1) \ldots (x-x_{k-1})(x-x_{k+1}) \ldots (x-x_n)}{(x_k-x_0)(x_k-x_1) \ldots (x_k-x_{k-1})(x_k-x_{k+1}) \ldots (x_k-x_n)}$$

for $k = 0, 1, 2 \ldots n$.

Theory: If $x_0, x_1, \ldots x_n$ are $n+1$ distinct numbers and f is a function whose values are given at these numbers, then a unique polynomial $P(x)$ of degree at most n exists with

$$f(x_k) = P(x_k) \text{ for each } k = 0, 1, \ldots n.$$

and

$$P(x) = \sum_{k=0}^{n} f(x_k) L_{n,k}(x), \text{ where}$$

$$L_{n,k}(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x-x_i)}{(x_k-x_i)}.$$

Remarks. often $L_{n,k}(x)$ is written as $L_k(x)$ when n is fixed.

Proof: By construction

$$L_{n,k}(x_j) = \delta_{kj} \;; \quad \delta_{kj} = \begin{cases} 1, & \text{if } k=j \\ 0, & \text{if } k \neq j. \end{cases}$$

$0 \leq k, j \leq n.$

the polynomial $P(x)$ defined by

$$P(x) = \sum_{k=0}^{n} f(x_k)\, L_{n,k}(x)$$

has the property that $f(x_k) = P(x_k)$, $k = 0, 1, 2 \ldots n$.

This proves the existence of a polynomial of degree at most n. Satisfying the given property i.e. interpolating $(x_k, f(x_k))$, $k = 0, \ldots n$.

Suppose that there is another polynomial $Q(x)$ of degree at most n and satisfies $Q(x_k) = f(x_k)$, for $k = 0, 1, 2, \ldots n$.

Then we have

$$P(x_k) - Q(x_k) = 0 \quad \text{for } k = 0, 1, 2, \ldots n.$$

The polynomial $E(x) = P(x) - Q(x)$ is of degree at most n, and has $(n+1)$ distinct zeros ~~roots~~ $x_0, x_1, \ldots x_n$.

[A polynomial of degree at most n, can have n ~~roots~~ zeros].

$\therefore E(x) = 0$ for all x.

$\Rightarrow P(x) = Q(x)$ for all x.

$\therefore P$ is unique.

Theorem: Suppose $x_0, x_1, x_2 \ldots x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then for each x in $[a, b]$, a number $\xi(x)$ between $x_0, x_1, \ldots x_n$, and hence in (a, b), exists with

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n). \quad \text{—} (*)$$

where $P(x)$ is the interpolating polynomial of degree n such that $P(x_i) = f(x_i)$, $i = 0, 1, 2 \ldots n$.

Proof: Note first that if $x = x_k$, for any $k = 0, 1, \ldots n$, then $f(x_k) = P(x_k)$, and choose any $\xi(x_k)$ in (a, b) to obtain $(*)$.

If $x \neq x_k$, for all $k = 0, 1, 2 \ldots n$, define the function g for t in $[a, b]$ by

$$g(t) = f(t) - P(t) - [f(x) - P(x)]\frac{(t-x_0)(t-x_1)\cdots(t-x_n)}{(x-x_0)(x-x_1)\cdots(x-x_n)}$$

$$= f(t) - P(t) - [f(x) - P(x)]\prod_{i=0}^{n}\frac{(t-x_i)}{(x-x_i)}.$$

Since $f \in C^{n+1}[a, b]$, and $P \in C^{\infty}[a, b]$, it follows that $g \in C^{n+1}[a, b]$. For $t = x_k$, we have

$$g(x_k) = f(x_k) - P(x_k) - [f(x) - P(x)]\prod_{i=0}^{n}\frac{(x_k - x_i)}{(x - x_i)}$$

$$= 0 - [f(x) - P(x)] \cdot 0 = 0.$$

Moreover

$$g(x) = [f(x) - P(x)] - [f(x) - P(x)] \cdot \prod_{i=0}^{n}\frac{(x-x_i)}{(x-x_i)}$$

$$= f(x) - P(x) - [f(x) - P(x)] = 0.$$

Thus $g \in C^{n+1}[a, b]$, and g is zero at $(n+2)$ distinct points $x, x_0, x_1, \ldots x_n$.

By generalized Rolle's theorem, there exists a number ξ in (a, b) for which $g^{(n+1)}(\xi) = 0$.

So

$$0 = g^{(n+1)}(\xi) = f^{(n+1)}(\xi) - p^{(n+1)}(\xi) - [f(x) - P(x)] \frac{d^{n+1}}{dt^{n+1}} \left[\prod_{i=0}^{n} \frac{(t-x_i)}{(x-x_i)} \right] \quad (+)$$

However $P(x)$ is polynomial of degree at most n,

$p^{(n+1)}(\xi) = 0$ [$p^{(n+1)}$ is identically zero]

Also $\prod_{i=0}^{n} \frac{(t-x_i)}{(x-x_i)}$ is a polynomial of degree $(n+1)$,

so

$$\prod_{i=0}^{n} \frac{(t-x_i)}{(x-x_i)} = \frac{t^{n+1}}{\prod_{i=0}^{n}(x-x_i)} + \text{(lower order terms in } t)$$

[order $\leq n$]

and

$$\frac{d^{n+1}}{dt^{n+1}} \left[\prod_{i=0}^{n} \frac{(t-x_i)}{(x-x_i)} \right] = \frac{(n+1)!}{\prod_{i=0}^{n}(x-x_i)}$$

equation (+) becomes

$$0 = f^{(n+1)}(\xi) - [f(x) - P(x)] \frac{(n+1)!}{\prod_{i=0}^{n}(x-x_i)}$$

$$\therefore f(x) = P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n}(x-x_i)$$

Note that ξ depends on x, ξ lies between $x, x_0, \ldots x_n$.

Interpolation Newton form:

A polynomial $p(x)$ of degree $\leq n$ is of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n. \quad -(1)$$

with certain coefficients.

Another form (shifted power form) is useful in computation is

$$p(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n \quad -(2)$$

Note that $a_i = p^{(i)}(c)/i!$, $i = 0, 1, \ldots n$.

Form in (2) provides Taylor expansion for $p(x)$ around the center c.

A more general shifted power form is the Newton form.

$$p(x) = a_0 + a_1(x-c_1) + a_2(x-c_1)(x-c_2) + \cdots$$
$$\cdots + a_n(x-c_1)(x-c_2)\cdots(x-c_n). \quad -(3)$$

Here $c_1, c_2, \ldots c_n$ are called centers. The form (3) reduces to the form in (2) if we choose $c_1, c_2, \ldots c_n$ all equal c.

<u>Remark:</u> from (3), we notice that

$$p(x) = a_0 + (x-c_1)\Big[a_1 + (x-c_2)\Big\{a_2 + (x-c_3)\Big\langle a_3 + \cdots + (x-c_{n-1})\big(a_{n-1} + (x-c_n)\,a_n\big)\Big\rangle\Big\}\Big].$$

Suppose $P_n(x)$ is the polynomial interpolating (2)
a function $f(x)$ at the interpolation points [degree $P_n(x) \le n$]
$x_0, x_1, \ldots x_n$, i.e.

$$P_n(x_k) = f(x_k), \quad k = 0, 1, \ldots n.$$

Write P_n in the Newton form using $x_0, x_1, \ldots x_{n-1}$
as centers:

$$P_n(x) = A_0 + A_1(x-x_0) + A_2(x-x_0)(x-x_1) + \cdots \cdots + A_n(x-x_0)(x-x_1)\cdots(x-x_{n-1}) \quad (4)$$

For any k between [integer] 0 and n, let $q_k(x)$
be the sum of the first $k+1$ terms in this
form

$$q_k(x) = A_0 + A_1(x-x_0) + A_2(x-x_0)(x-x_1) + \cdots + A_k(x-x_0)\cdots(x-x_{k-1}).$$

Then every remaining term in (4) has the
factor $(x-x_0)(x-x_1)\cdots(x-x_k)$, and we write (4)
in the form

$$P_n(x) = q_k(x) + (x-x_0)(x-x_1)\cdots(x-x_k)\,r(x) \quad (5)$$

for some polynomial $r(x)$ of no further interest.

Note that the second term on RHS of (5)
is zero at $x = x_0, x_1, \ldots x_k$, hence

$$f(x_j) = P_n(x_j) = q_k(x_j), \quad j = 0, 1, \ldots k.$$

Hence $q_k(x)$ interpolates $f(x)$ at $x_0, x_1, \ldots x_k$.

But $P_k(x)$ also interpolates $f(x)$ at $x_0, x_1, \ldots x_k$

we have $q_k(x) = P_k(x)$ [since degree $q_k(x) \le k$, degree $P_k(x) \le k$].

$\therefore P_n(x) = P_k(x) + (x-x_0) \cdots (x-x_n)\, r(x).$

This suggests that the Newton form (4) for the interpolating polynomial $P_n(x)$ can be built up step by step as one constructs the sequence $P_0(x), P_1(x), P_2(x), \cdots$ with $P_k(x)$ obtained from $P_{k-1}(x)$ by adding the next term in the Newton form (4) i.e

$$P_k(x) = P_{k-1}(x) + A_k (x-x_0) \cdots (x-x_{k-1}).$$

It also shows that the coefficient A_k in the Newton form (4) for the interpolating polynomial is the leading coefficient, i.e the coefficient of x^k, in the polynomial $P_k(x)$ of degree $\le k$ which agrees with $f(x)$ at $x_0, x_1, \cdots x_k$. This coefficient depends only on the values of $f(x)$ at the points $x_0, x_1, \cdots x_k$, it is called the k^{th} divided difference of $f(x)$ at the points $x_0, x_1, \cdots x_k$. and is denoted by

$$f[x_0, x_1, \cdots x_k].$$

With this definition, we arrive at the Newton's formula for the interpolating polynomial

$$P_n(x) = f[x_0] + f[x_0, x_1](x-x_0) + f[x_0, x_1, x_2](x-x_0)(x-x_1)$$
$$+ \cdots + f[x_0, x_1, \cdots x_n](x-x_0)(x-x_1) \cdots (x-x_{n-1})$$

This can be written more compactly as

$$P_n(x) = \sum_{i=0}^{n} f[x_0, x_1, \cdots x_i] \prod_{j=0}^{i-1} (x - x_j) \quad \text{--- } (6)$$

if we make the convention that

$$\prod_{m=r}^{s} a_m = \begin{cases} a_r a_{r+1} \cdots a_s, & \text{for } r \le s \\ 1, & \text{for } r > s. \end{cases}$$

For $n = 1$, (6) reads

$$P_1(x) = f[x_0] + f[x_0, x_1](x - x_0)$$

and $f[x_0] = f(x_0)$ & since $f(x_1) = P_1(x_1)$

we obtain

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_0 - x_1}.$$

Divided Differences:

Higher order divided differences may be found by the formula:

$$f[x_0, x_1, \cdots x_k] = \frac{f[x_1, x_2, \cdots x_k] - f[x_0, x_1, \cdots x_{k-1}]}{x_k - x_0}.$$

whose validity is proved below.

Let $P_i(x)$ be the polynomial of degree $\leq i$ which agrees with $f(x)$ at $x_0, x_1, \ldots x_i$ as before, and let $q_{k-1}(x)$ be the polynomial of degree $\leq k-1$ which agrees with $f(x)$ at $x_1, x_2, \ldots x_k$. Then

$$P(x) = \frac{x-x_0}{x_k-x_0} q_{k-1}(x) + \frac{x_k-x}{x_k-x_0} P_{k-1}(x)$$

is a polynomial of degree $\leq k$, and it is easy to check that

$$P(x_j) = f(x_j), \quad j = 0, 1, \ldots k.$$

By the uniqueness of interpolating polynomial we must have $P(x) = P_k(x)$. Therefore

$$f[x_0, x_1, \ldots x_k] = \text{leading coefficient of } P_k(x)$$

$$= \frac{\text{leading coefficient of } q_{k-1}(x)}{x_k - x_0} - \frac{\text{leading coefficient of } P_{k-1}(x)}{x_k - x_0}$$

$$= \frac{f[x_1, x_2, \ldots x_k] - f[x_0, \ldots x_{k-1}]}{x_k - x_0}$$

[by definition].

Algorithm: Divided differences

Given the first two column of the table containing $x_0, x_1, \ldots x_n$ and correspondingly $f[x_0], f[x_1], \ldots f[x_n]$.

x_i	$f[\cdot]=f()$	$f[\cdot]$	$f[\cdot\cdot]$	$f[\cdots]$	
x_0	$f[x_0]$				
		$f[x_0, x_1]$			
x_1	$f[x_1]$		$f[x_0, x_1, x_2]$	$f[x_0, x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3, x_4]$
		$f[x_1, x_2]$			
x_2	$f[x_2]$		$f[x_1, x_2, x_3]$	$f[x_1, x_2, x_3, x_4]$	
		$f[x_2, x_3]$			
x_3	$f[x_3]$		$f[x_2, x_3, x_4]$		
		$f[x_3, x_4]$			
x_4	$f[x_4]$				

For $k = 1, 2, \ldots n$ do
  for $i = 0, 1 \ldots n-k$, do

$$f[x_i, \ldots x_{i+k}] = \frac{f[x_{i+1}, \ldots, x_{i+k}] - f[x_i, \ldots x_{i+k-1}]}{x_{i+k} - x_i}$$

Theorem Let $f:[a,b]\to\mathbb{R}$ be in $C^k[a,b]$. If $x_0, x_1, \dots x_k$ are $k+1$ distinct points in $[a,b]$, then there exists a $\xi\in(a,b)$ such that

$$f[x_0, x_1, \dots x_k] = \frac{f^{(k)}(\xi)}{k!}$$

Proof: Let $P_{k-1}(x)$ be polynomial of degree $\le k-1$ such that $P_{k-1}(x_j) = f(x_j)$, $j=0,1,2 \dots k-1$.

Then for any $x\in[a,b]$, we have [from Error formula]

$$f(x) = P_{k-1}(x) + \frac{f^{(k)}(\xi)}{k!}(x-x_0)(x-x_1)\dots(x-x_{k-1})$$

Let P_k be polynomial of degree $\le k-1$ such that $P_k(x_j) = f(x_j)$, $j=0,1,\dots k-1$ and $P_k(x_k) = f(x_k)$.

Then $$P_k(x_k) = P_{k-1}(x_k) + \frac{f^{(k)}(\xi)}{k!}(x_k-x_0)(x_k-x_1)\dots(x_k-x_{k-1}) \quad \text{--- (7')}$$

On the other hand from Newton's form, the form of $P_k(x)$ polynomial of degree $\le k$, such that $P_k(x_j) = f(x_j)$, $j=0,1,2 \dots k$. is

$$P_k(t) = P_{k-1}(t) + A_k(t-x_0)(t-x_1)\dots(t-x_{k-1})$$

and when $t = x_k$;

$$P_k(x_k) = P_{k-1}(x_k) + A_k(x_k-x_0)(x_k-x_1)\dots(x_k-x_{k-1}) \quad \text{--- (8)}$$

And. $A_k = f[x_0, x_1, \dots x_k]$.

from (7) - (8), we have

$$f[x_0, x_1, \dots x_k] = \frac{f^{(k)}(\xi)}{k!}$$