Theorem: A multistep method of the form

$$w_0=\alpha,\ w_1=\alpha_1,\ w_2=\alpha_2, \dots \quad w_{m-1}=\alpha_{m-1}$$

$$w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \dots + a_0 w_{i+1-m} + hf(t_i, h, w_{i+1}, \dots w_{i+1-m})$$

is stable if it satisfies the root condition.
If it satisfies condition (i), then it is strongly stable,
if is satisfy condition (ii), then it is weakly stable.
Moreover, if the difference method is consistent with the differential equation, then the method is stable if and only it is convergent.

Stiff differential equations:

The error bounds for all the numerical methods approximating solutions of initial value problems contist higher derivative of the solution of the differential equation. If the derivatives of the solutions ~~of~~ can be reasonably bounded, then the method will have a reasonable ~~and~~ error bound. Even if the derivative grows as the steps increase, the error may be kept under control, provided the solution ~~also~~ also grows in magnitude. Problems frequently arise, however, when the magnitude of the derivative increases but the solution does not. In that situation, the error can grow so large that it dominates the calculations.

Initial value problems for which this is likely to happen are called "stiff equations".

Simple test problem:

$$y' = \lambda y, \quad y(0) = \alpha, \quad \text{where } \lambda < 0.$$

The exact solution is $y(t) = \alpha e^{\lambda t}$.

Note that $\lim_{t\to\infty} y(t) = 0$. [0 is the steady solution].

Consider Euler's method applied to the ~~test~~ ~~method~~ equation. Let $h = \frac{b-a}{N}$, $t_j = jh$, $j = 0, 1, \dots N$.

The method is given by

$$w_0 = \alpha,$$

$$w_{j+1} = w_j + h(\lambda w_j), \quad j = 0, 1, \dots N-1$$

So,

$$w_{j+1} = (1 + h\lambda) w_j = (1+h\lambda)^{j+1} w_0 = (1+h\lambda)^{j+1} \alpha,$$

for $j = 0, 1, \dots N-1$.

Since the exact solution is $y(t) = \alpha e^{\lambda t}$, the absolute error is

$$|y(t_j) - w_j| = |e^{jh\lambda} - (1+h\lambda)^j|\,|\alpha|$$

$$= |(e^{h\lambda})^j - (1+h\lambda)^j|\,|\alpha|. \quad — (1)$$

The accuracy of the method is determined by how well the term $1+h\lambda$ approximates $e^{h\lambda}$. When $\lambda < 0$, the exact solution $(e^{h\lambda})^j$ decays to zero as j increases, but from (1), the approximation will have the property only if $|1+h\lambda| < 1$, which implies that $-2 < h\lambda < 0$. This effectively restricts the step size h for Euler's method to satisfy $h < \frac{2}{|\lambda|}$.

Boundary value problems:

The two point boundary value problem involve a second order differential equation of the form

$$y'' = f(x, y, y') \quad \text{for } a \le x \le b,$$

together with the boundary conditions

$$y(a) = \alpha \text{ and } y(b) = \beta.$$

Example: If the equation is of the form

$$y'' = p(x)\, y' + q(x)\, y + r(x), \quad a \le x \le b$$

with $y(a) = \alpha$ and $y(b) = \beta$, it is called a linear equation.

Theorem: The linear equation

$$\left.\begin{aligned} &y'' = p(x)\, y' + q(x)\, y + r(x), \quad a \le x \le b \\ &\text{with} \quad y(a) = \alpha \text{ and } y(b) = \beta, \end{aligned}\right\} \text{(1)}$$

satisfies (i) $p(x), q(x), r(x)$ are continuous on $[a, b]$

(ii) $q(x) > 0$ on $[a, b]$

then the boundary value problem (1) has a unique solution.

Linear shooting method:

To approximate the unique solution of (1), we consider two IVPs,

$$y'' = p(x)\, y' + q(x)\, y + r(x), \text{ with } a \le x \le b, \; y(a) = \alpha, \; y'(a) = 0 \qquad \text{(2)}$$

and

$$y'' = p(x)\, y' + q(x)\, y, \text{ with } a \le x \le b, \; y(a) = 0, \text{ and } y'(a) = 1. \qquad \text{(3)}$$

Let y_1 and y_2 be the solutions of (2) and (3) respectively. (4)
Assume that $y_2(b) \neq 0$. [If $y_2(b)=0$, then the theorem implies that $y_2 \equiv 0$, by uniqueness, since $y \equiv 0$ is a solution of the homogeneous equation in (3) with $y(a)=0=y(b)$]

Define $$y(x) = y_1(x) + \frac{\beta - y_1(b)}{y_2(b)} \cdot y_2(x).$$

Then $$y(a) = y_1(a) + \frac{\beta - y_1(b)}{y_2(b)} \cdot y_2(a) = y_1(a) = \alpha.$$

$$y(b) = y_1(b) + \frac{\beta - y_1(b)}{y_2(b)} \cdot y_2(b) = y_1(b) + \beta - y_1(b) = \beta.$$

Furthermore, $y_1(x)$ satisfies
$$y_1'' = p(x) y_1' + q(x) y_1 + r(x),$$
and $y_2(x)$ satisfies $y_2'' = p(x) y_2' + q(x) y_2$.

For any constant $c \neq 0$, the function $w = y_1 + c y_2$ satisfies $w'' = p(x) w' + q(x) w + r(x)$. To see this

$$w'' = y_1'' + c y_2'' = p(x) y_1' + q(x) y_1 + r(x) + c\left(p(x) y_2' + q(x) y_2\right)$$
$$= p(x)(y_1 + c y_2)' + q(x)(y_1 + c y_2) + r(x)$$
$$= p(x) w' + q(x) w + r(x).$$

In particular, when $c = \frac{\beta - y_1(b)}{y_2(b)}$, we have that

$y(x) = y_1(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2(x)$ satisfies the equation $y'' = p(x) y' + q(x) y + r(x)$ and boundary conditions $y(a) = \alpha$, $y(b) = \beta$.

Therefore solving y is equivalent to solving y_1 and y_2. We can apply the methods developed for the second order (through first-order system) IVP to approximate y_1 and y_2, then find approximation to y.

Theorem: Suppose the function f in the boundary value problem

$$y'' = f(x, y, y'), \quad a \le x \le b, \quad y(a) = \alpha \text{ and } y(b) = \beta.$$

is continuous on the set

$$D = \{(x, y, y') \mid \text{for } a \le x \le b \text{ with } -\infty < y < \infty, \text{ and } -\infty < y' < \infty\}$$

and the partial derivatives (of f), f_y and $f_{y'}$ are also continuous on D. If

(i) $f_y(x, y, y') > 0$ for all $(x, y, y') \in D$ and

(ii) a constant M exists, with

$$|f_{y'}(x, y, y')| \le M \quad \text{for all } (x, y, y') \in D,$$

then the boundary-value problem has a unique solution.

Example: $y'' + e^{-xy} + \sin y' = 0, \quad 1 \le x \le 2, \quad y(1) = y(2) = 0$

has a unique solution.

solution: $f(x, y, y') = -e^{-xy} - \sin y'$ and for all

$x \in [1, 2]$, $f_y(x, y, y') = x e^{-xy} > 0$ and

$$|f_{y'}(x, y, y')| = |-\cos y'| \le 1.$$

So the problem has a unique solution