The theorem on error bounds for Euler's method ignores the effect of round-off errors in the choice of step size h. As h becomes smaller, more calculations are necessary and more round-off error is expected.

In actuality, the difference-equation form

$$w_0 = \alpha, \quad w_{i+1} = w_i + h f(t_i, w_i), \quad i = 0, 1, \ldots N-1$$

is not used to calculate the approximation to the solution $y_i = y(t_i)$ at a mesh point t_i. We use instead an equation of the form

$$(*) \left[\begin{array}{l} u_0 = \alpha + \delta_0 \\ u_{i+1} = u_i + h f(t_i, u_i) + \delta_{i+1}, \quad i = 0, 1, \ldots N-1. \end{array} \right.$$

where δ_i denotes the round-off error associated with u_i. We can prove the following theorem like earlier one.

<u>Theorem</u>: Let $y(t)$ denote the unique solution to the IVP: $y' = f(t, y)$, $a \le t \le b$, $y(a) = \alpha$. and $u_0, u_1, \ldots u_N$ be the approximations obtained using $(*)$. If $|\delta_i| < \delta$ for each $i = 0, 1, \ldots N$ and the hypothesis of earlier theorem (Error bounds for Euler's method) hold for IVP, then

$$|y(t_i) - u_i| \le \frac{1}{L}\left(\frac{hM}{2} + \frac{\delta}{h}\right)\left[e^{L(t_i - a)} - 1\right] + |\delta_0| e^{L(t_i - a)}.$$

for each $i = 0, 1, \ldots N$.

<u>Proof</u>: We can see that, using notation $y_i = y(t_i)$

$$y_{i+1} - u_{i+1} = y_i - u_i + h\left[f(t_i, y_i) - f(t_i, u_i)\right] + \delta_{i+1} + \frac{h^2}{2} y''(\xi_i)$$

where ξ_i lies between t_i and t_{i+1}. Then

$$|y_{i+1} - u_{i+1}| \le (1 + hL)|y_i - u_i| + |\delta_{i+1}| + \frac{Mh^2}{2}.$$

$$\le (1 + hL)|y_i - u_i| + \frac{2\delta + Mh^2}{2}$$

Applying the Lemma 2, (on 14-10-2023) ②

$$|y_{i+1}-u_{i+1}| \leq e^{(i+1)hL}\left(|y_0-u_0| + \frac{2\delta+Mh^2}{2hL}\right) - \frac{2\delta+Mh^2}{2hL}$$

$$\leq e^{L(t_{i+1}-a)}\left(|\delta_0| + \frac{2\delta+Mh^2}{2hL}\right) - \frac{2\delta+Mh^2}{2hL}$$

$$= \frac{1}{L}\left(\frac{Mh}{2} + \frac{\delta}{h}\right)\left(e^{L(t_{i+1}-a)} - 1\right) + |\delta_0| e^{L(t_{i+1}-a)}.$$

hence the proof ▨

The error bound in the theorem is not linear in L.

we see that

$$\lim_{h\to 0}\left(\frac{Mh}{2} + \frac{\delta}{h}\right) = \infty.$$

the error bound is expected to become large for sufficiently small values of h.

Let $E(h) = \frac{Mh}{2} + \frac{\delta}{h}$; then $E'(h) = \frac{M}{2} - \frac{\delta}{h^2}$

if $h < \sqrt{\frac{2\delta}{M}}$, then $E'(h) < 0$. and $E(h)$ is decreasing

if $h > \sqrt{\frac{2\delta}{M}}$, then $E'(h) > 0$ and $E(h)$ is increasing

The minimum value of $E(h)$ occurs when

$$h = \sqrt{\frac{2\delta}{M}},$$

The total error in the approximation increase if h is decreased beyond this value $h = \sqrt{\frac{2\delta}{M}}$.

normally the value of δ is sufficiently small that this lower bound for h does not affect the operation of Euler's method.

Consider (IVP):

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha.$$

Higher order Taylor Methods:

Def: The difference method

$$w_0 = \alpha,$$
$$w_{i+1} = w_i + h\,\phi(t_i, w_i), \text{ for each } i = 0, 1, \ldots N-1.$$

has local truncation error

$$z_{i+1}(h) = \frac{y_{i+1} - (y_i + h\,\phi(t_i, y_i))}{h} = \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i),$$

for each $i = 0, 1, \ldots N-1$, where y_i and y_{i+1} denote the solution at t_i and t_{i+1} respectively.

Example: Euler's method has local truncation error at the ith step.

$$z_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - f(t_i, y_i), \quad i = 0, 1, \ldots N-1.$$

The error is a local error because it measures the accuracy of the method at a specific step, assuming that the method was exact at the previous step. It depends on the differential equation, the step size and the particular step in the approximation.

From the previous analysis we see that Euler's method has

$$z_{i+1}(h) = \frac{h}{2} y''(\xi_i), \quad i = 0, 1, \ldots N-1,$$

where ξ_i lies between t_i and t_{i+1}.

If $|y''(t)| \le M$, for all $t \in [a, b]$, then

$$|z_{i+1}(h)| \le \frac{Mh}{2}.$$

So the local truncation error in Euler's method is $O(h)$.

We may derive higher-order methods by using more terms in Taylor's theorem.

Suppose the solution $y(t)$ to the IVP

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

has $(n+1)$ continuous derivatives. We expand $y(t)$ in terms of n^{th} Taylor polynomial about t_i, and evaluate at t_{i+1},

$$y(t_{i+1}) = y(t_i) + h y'(t_i) + \frac{h^2}{2} y''(t_i) + \cdots + \frac{h^n}{n!} y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!} y^{(n+1)}(\xi_i) \quad \text{— (+)}$$

for some $\xi_i \in (t_i, t_{i+1})$.

Successive differentiation of $y(t)$ gives

$$y'(t) = f(t, y), \quad y''(t) = f'(t, y(t)), \; \cdots \; y^{(k)}(t) = f^{(k-1)}(t, y(t))$$

Substituting these in (+) gives

$$y(t_{i+1}) = y(t_i) + h f(t_i, y(t_i)) + \frac{h^2}{2} f'(t_i, y(t_i)) + \cdots + \frac{h^n}{n!} f^{(n-1)}(t_i, y(t_i)) + \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i)) \quad \text{— (*)}$$

The difference equation (or the method) corresponding to (*) is obtained by deleting the remainder term.

Taylor method of order n

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + h\,T^{(n)}(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots N-1.$$

where

$$T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \cdots + \frac{h^{(n-1)}}{(n)!} f^{(n-1)}(t_i, w_i)$$

$\hookrightarrow n!$

Note: Euler's method is Taylor's method of order one.

Theorem: If Taylor's method of order n is used to approximate the solution to

$$y'(t) = f(t, y(t)), \quad a \le t \le b, \quad y(a) = \alpha,$$

with step size h and if $y \in C^{n+1}[a,b]$, then the local truncation error is $O(h^n)$.

Proof: Note that the Taylor's ~~method~~ theorem gives

$$y_{i+1} - y_i - h f(t_i, y_i) - \frac{h^2}{2} f'(t_i, y_i) - \cdots - \frac{h^n}{n!} f^{(n-1)}(t_i, y_i) =$$

$$= \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i))$$

for some $\xi_i \in (t_i, t_{i+1})$. So the local truncation error is

$$z_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - T^{(n)}(t_i, y_i) = \frac{h^n}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i)),$$

for each $i = 0, 1, \ldots N-1$. Since ~~$y \in C^{n+1}[a,b]$, we have~~ $y \in C^{n+1}[a,b]$, we have $y^{(n+1)}(t) = f^{(n)}(t, y(t))$ bounded on $[a,b]$ and $z_i(h) = O(h^n)$ for each $i = 1, 2, \ldots N$.

In the above Taylor's method, we used derivatives of f. They can be computed as follows: we have

$$y' = f(t, y)$$

$$f' = f_t + f_y y' = f_t + f f_y$$

$$f'' = (f')' = (f_t + f f_y)_t + (f_t + f f_y)_y y'$$

$$= f_{tt} + f_t f_y + f f_{yt} + (f_{ty} + f_y f_y + f f_{yy}) f$$

$$= f_{tt} + f_t f_y + f f_{yt} + f f_{ty} + f f_y^2 + f^2 f_{yy}$$

$$= f_{tt} + 2 f f_{ty} + f_t f_y + f f_y^2 + f^2 f_{yy}.$$

and similarly we can compute higher derivatives of f. But as we see that the higher derivatives of f lead higher partial derivatives of f. For practical reasons, we must limit the number of terms in the expression in $T^{(n)}$. to a reasonable number i.e. reasonable value of n.

Euler's method is not very useful in practical problems (1)
because it requires a very small step size for reasonable
accuracy. Taylor's algorithm of higher order is unacceptable
as a general - purpose procedure because of the need
to obtain higher derivatives of $y(t)$.

The Runge-kutta methods attempt to obtain
greater accuracy, and at the same time avoid the
need for higher derivatives, by evaluating $f(t,y)$ at
selected points on each subinterval.

We shall derive the simplest of the Runge-kutta
methods: A formula of the following form is sought:

$$y_{n+1} = y_n + a k_1 + b k_2 \quad \text{—— (1)}$$

where $k_1 = h f(t_n, y_n)$

$k_2 = h f(t_n + \alpha h, \; y_n + \beta k_1)$

and a, b, α, β are constants to be determined so that
(1) will agree with the Taylor algorithm of as high
an order as possible. On expanding $y(t_{n+1})$ in a
Taylor series through the terms of order h^3, we find

$$y(t_{n+1}) = y(t_n) + h y'(t_n) + \frac{h^2}{2} y''(t_n) + \frac{h^3}{6} y'''(t_n) + \cdots$$

$$= y(t_n) + h f(t_n, y_n) + \frac{h^2}{2}(f_t + f f_y)_n$$

$$+ \frac{h^3}{6}\left(f_{tt} + 2 f f_{ty} + f_{yy} f^2 + f_t f_y + f_y^2 f\right)_n$$

$$+ O(h^4) \quad \text{—— (2)}$$

where subscript n means that all functions
involved are evaluated at (t_n, y_n).

On the other hand, using Taylor's expansion for ② functions of two variables, we find

$$\frac{k_2}{h} = f(t_n+\alpha h,\ y_n+\beta k_1) = f(t_n, y_n) + \alpha h f_t + \beta k_1 f_y$$
$$+ \frac{\alpha^2 h^2}{2} f_{tt} + \alpha h \beta k_1 f_{ty} + \frac{\beta^2 k_1^2}{2} f_{yy} + O(h^3) \quad — ③$$

where all the derivatives are evaluated at (t_n, y_n).

We substitute this expression for k_2 in ①

and since $k_1 = hf(t_n, y_n)$, we find upon rearranging terms in powers of h, that

~~$y_{n+1} = y_n + ahf + bh$~~

$$y_{n+1} = y_n + ah\, f(t_n, y_n) + bh\, f(t_n+\alpha h,\ y_n+\beta k_1)$$
$$= y_n + (a+b)hf + bh^2(\alpha f_t + \beta f f_y)$$
$$+ bh^3\left(\frac{\alpha^2}{2} f_{tt} + \alpha\beta f f_{ty} + \frac{\beta^2}{2} f^2 f_{yy}\right) + O(h^4) \quad — ④.$$

On comparing ④ with ②'', we see that to make the corresponding powers of h and h^2 agree, we must have

$$a+b = 1$$
$$b\alpha = b\beta = \frac{1}{2}.$$

There are many solutions for a, b, α and β. Most popular one is $a=\frac{1}{2},\ b=\frac{1}{2},\quad \alpha=1,\ \beta=1$

(or) $a = b = \frac{1}{2}$ and $\alpha = \beta = 1$

Algorithm: Runge-kutta method of order 2

$$w_0 = \alpha,$$
$$w_{i+1} = w_i + \frac{h}{2} f(t_i, w_i) + \frac{h}{2} f(t_i + h, y_i + h f(t_i, w_i))$$

for $i = 0, 1, 2, \dots N-1$.

Local truncation error in R-K Method of order 2

Recall that the local truncation error is defined by

$$z_{i+h}(h) = \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i),$$ where for R-K method

$$\phi(t_i, y_i) = \frac{1}{2}\left[f(t_i, y_i) + f(t_i + h, y_i + h f(t_i, y_i))\right].$$

from (2),

$$\frac{y_{i+1} - y_i}{h} - \frac{1}{2} f(t_i, y_i) - \frac{1}{2} f(t_i + h, y_i + h f(t_i, y_i))$$

$$= \frac{1}{2} f(t_i, y_i) + \frac{h}{2}(f_t + f f_y)_i - \frac{1}{2} f(t_i + h, y_i + h f(t_i, y_i))$$

$$+ O(h^2) \text{ terms} \quad \text{—— (4)}$$

from (3),

$$\text{RHS of (4)} = -\left[\frac{h^2}{4} f_{tt} + \frac{h^2}{2} f f_{ty} + \frac{h^2}{4} f^2 f_{yy}\right] + O(h^3) + O(h^4)$$

If all the derivatives of f (of order 2) are bounded then, the local ~~local~~ truncation error is of the form

$$|z_{i+1}(h)| \leq C h^2,$$

The method is of order 2.

Midpoint Method [Another form of R-K method of order 2

By choosing $a=0,\ b=1,\ \alpha=\beta=\frac{1}{2}$,

we get

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + h f\left(t_i + \frac{h}{2},\ w_i + \frac{h}{2} f(t_i, w_i)\right)$$

for $i=0,1,\ldots N-1$.

The local truncation error will be of the order $O(h^2)$ just same like the previous case.

Remark: The method obtained by choosing $a=b=\frac{1}{2}$ and $\alpha=\beta=1$ is also called Modified Euler Method

Runge - Kutta method of order four :

$$w_0 = \alpha$$

$$k_1 = h f(t_i, w_i)$$

$$k_2 = h f\left(t_i + \frac{h}{2},\ w_i + \frac{1}{2} k_1\right)$$

$$k_3 = h f\left(t_i + \frac{h}{2},\ w_i + \frac{1}{2} k_2\right)$$

$$k_4 = h f(t_{i+1},\ w_i + k_3)$$

$$w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

for each $i = 0, 1, \ldots N-1$.

This method has local truncation error $O(h^4)$, provided $y(t)$ has five continuous derivatives.