UMC-202 Problems Sheet Fall 2024

1. Consider the initial value problem(IVP):

$$y''(t) = f(t, y') + y, \;\; t \in [a, b], \quad y(a) = \alpha, \; y'(a) = \beta,$$

where f is a smooth functions of its variables. Convert the IVP into an initial value problem for a system of first order differential equations and derive the Adams-Bashforth two step method for the resulting system. Express the local truncation error in terms of the mesh size and the derivatives of y.

2. For the two point boundary value problem

$$y''(x) + a(x)y(x) = f(x), \; x \in [0, 1], \quad y(0) = \alpha, \; y(1) = \beta,$$

use the centered difference scheme of order two to get a linear system $A\alpha = b$. Determine A and b, and show that the Guass-Jacobi and Gauss-Seidel iterative methods converge for solving the system $A\alpha = b$ when $-a(x) \geq k > 0$ for all $x \in [0, 1]$. (Theorems proved for Gauss-Jacobi, Gauu-Seidel methods in class can be stated and used).

3. Consider the two point boundary value problem (BVP):

$$y''(x) + a(x)y'(x) - y(x) = f(x), x \in [0, 1], y(0) = \alpha, \; y(1) = \beta.$$

For BVP, define the linear shooting method using initial value problems(IVPs) and Euler's method (for IVPs) to deduce a numerical algorithm to determine numerical solution of BVP.

4. Let $I = [0, 1]$, N be a positive integer and $h = 1/(N+1)$. Let $x_j = jh$, $j = 0, 1, \ldots, N+1$ be the mesh points and $I_k = [x_{k-1}, x_k]$, $k = 1, \ldots, N+1$. Consider the real vector space

$$V_h = \{v \in C[0, 1] : v|_{I_k} \in P_2(I_k), k = 1, 2 \ldots, N+1, v_h(0) = v_h(1) = 0\},$$

where $P_2(I_k)$ is the set of all quadratic polynomials of degree less than or equal to two and restricted to I_k. Find the dimension of V_h and construct a canonical Lagrange basis for V_h.

5. Let $\{x_j\}_{j=1}^{N+1}$, $x_j = hj$, $j = 1, 2, \ldots, N+1$, $h = 1/(N+1)$ be a partition of the interval $I = [0, 1]$. Let $I_j = [x_{j-1}, x_j]$ and

$$V_h = \{v \in C[0, 1] : v|_{I_j} \in P_1(I_j), j = 1, 2 \ldots, N+1, v(0) = 0\}.$$

Suppose that $\{\phi_J\}_{j=1}^{N+1}$ be a basis of V_h such that $\phi_j(x_k) = \delta_{jk}$. Show that the solution $u \in C^2(I)$ of
$-u" + u' + u = f$ in I, $u(0) = 0$ and $u'(1) = 2$, satisfies

$$\int_I \left(u'\phi_j' + u'\phi_j + u\phi_j\right) \, dx = \int_I f\phi_j \, dx + 2\phi_j(1) \;\text{ for all } j = 1, 2, \ldots, N+1.$$

and thereby show that

$$\int_I (u'v_h' + u'v_h + uv_h) \, dx = \int_I fv_h \, dx + 2v_h(1) \;\text{ for all } v_h \in V_h.$$