Let $P_n(x)$ be the polynomial of degree $\le n$ interpolating $f(x)$, at points $x_0, x_1, \ldots x_n$ is given in Newton form by

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + \cdots + f[x_0, x_1, \ldots x_n] \prod_{i=0}^{n-1} (x - x_i)$$

$$= f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots x_k](x - x_0) \cdots (x - x_{k-1}). \quad —(i)$$

This formula can be expressed in a simplified form when the nodes are arranged consequitively with equal spacing. Let $h = x_{i+1} - x_i$, for each $i = 0, 1, \ldots n-1$ and let $x = x_0 + sh$. Then $x - x_i = (s-i)h$. So (i) becomes

$$P_n(x) = P_n(x_0 + sh) = f[x_0] + sh\, f[x_0, x_1] + s(s-1)h^2 f[x_0, x_1, x_2]$$

$$+ \cdots + s(s-1) \cdots (s-n+1) h^n f[x_0, x_1, x_2, \ldots x_n].$$

$$= f[x_0] + \sum_{k=1}^{n} s(s-1) \cdots (s-k+1) h^k f[x_0, x_1, \ldots x_k].$$

Using binomial coefficient notation,

$$\binom{s}{k} = \frac{s(s-1) \cdots (s-k+1)}{k!}$$

we can express $P_n(x)$ compactly as

$$P_n(x) = P_n(x_0 + sh) = f[x_0] + \sum_{k=1}^{n} \binom{s}{k} k!\, h^{k} f[x_0, x_1, \ldots x_k].$$

Forward differences:

Newton forward-difference formula:

The Newton forward difference notation Δ, by introducing forward difference

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{1}{h}\left(f(x_1) - f(x_0)\right) = \frac{1}{h} \Delta f(x_0).$$

$$f[x_0, x_1, x_2] = \frac{1}{2h}\left[\frac{\Delta f(x_1) - \Delta f(x_0)}{h}\right] = \frac{1}{2h^2} \Delta^2 f(x_0)$$

and, in general,

$$f[x_0, x_1, \ldots x_k] = \frac{1}{k!\, h^k} \Delta^k f(x_0)$$

Using this notation we obtain Newton Forward difference formula:

$$P_n(x) = f(x_0) + \sum_{k=1}^{n} \binom{s}{k} \Delta^k f(x_0).$$

Backward Differences:

If the interpolation nodes are reordered from last to first as $x_n, x_{n-1}, \ldots x_0$, we can write interpolating polynomial as

$$P_n(x) = f[x_n] + f[x_n, x_{n-1}](x - x_n) + f[x_n, x_{n-1}, x_{n-2}](x - x_n)(x - x_{n-1})$$
$$+ \ldots + f[x_n, \ldots x_0](x - x_n)(x - x_{n-1}) \ldots (x - x_1)$$

If the nodes are equally spaced with $x = x_n + sh$

$[h = x_i - x_{i-1},\ i = n, \ldots 1]$

and $x = x_i + (s+n-i)h$, then

$x_i = x_n - (n-i)h$

$$P_n(x) = P_n(x_n + sh) = f[x_n] + sh\, f[x_n, x_{n-1}] + s(s+1)h^2 f[x_n, x_{n-1}, x_{n-2}]$$
$$+ \ldots + s(s+1) \ldots (s+n-1) h^n f[x_n, \ldots x_0].$$

This is referred as Newton Backward difference formula.

Definition: Given the sequence $\{P_n\}_{n=0}^{\infty}$ define backward difference ∇P_n by

$$\nabla P_n = P_n - P_{n-1}, \quad \text{for } n \geq 1$$

Higher powers are defined recursively by

$$\nabla^i P_n = \nabla(\nabla^{i-1} P_n), \quad \text{for } k \geq 2$$

The above definition implies that

$$f[x_n, x_{n-1}] = \frac{1}{h} \nabla f(x_n),$$

$$f[x_n, x_{n-1}, x_{n-2}] = \frac{1}{2h^2} \nabla^2 f(x_n)$$

and in general

$$f[x_n, x_{n-1}, \ldots x_{n-k}] = \frac{1}{k!\,h^k} \nabla^k f(x_n).$$

Consequently

$$P_n(x) = f[x_n] + s\nabla f(x_n) + \frac{s(s+1)}{2} \nabla^2 f(x_n) + \cdots$$
$$+ \frac{s(s+1)\cdots(s+n-1)}{n!} \nabla^n f(x_n).$$

If we define

$$\binom{-s}{k} = \frac{-s(-s-1)\cdots(-s-k+1)}{k!} = (-1)^k \frac{s(s+1)\cdots(s+k-1)}{k!}$$

then

$$P_n(x) = f[x_n] + (-1)^1\binom{-s}{1}\nabla f(x_n) + (-1)^2\binom{-s}{2}\nabla^2 f(x_n) + \cdots$$
$$\cdots + (-1)^n\binom{-s}{n}\nabla^n f(x_n).$$

Compactly:

$$P_n(x) = f[x_n] + \sum_{k=1}^{n} (-1)^k \binom{-s}{k} \nabla^k f(x_n)$$

Centered difference formula:

We choose x_0 near the point being approximated and label notation as follows: label the nodes directly below x_0 as $x_1, x_2, \ldots$ and those directly above as $x_{-1}, x_{-2}, \ldots$. With this convention, Stirling formula is given as follows: (PTO)

Suppose there are $2m+1$ nodes $x_{-m}, x_{-m+1}, \ldots x_{-1}, x_0, x_1, \ldots x_m$.
Then Stirling formula is given by, for polynomial of degree $\le 2m$, $P_{2m}(x)$.

$$P_{2m}(x) = f[x_0] + \frac{sh}{2}\left(f[x_{-1}, x_0] + f[x_0, x_1]\right) + s^2h^2 f[x_{-1}, x_0, x_1]$$

$$+ \frac{s(s^2-1)h^3}{2}\left(f[x_{-2}, x_{-1}, x_0, x_1] + f[x_{-1}, x_0, x_1, x_2]\right)$$

$$+ \cdots + s^2(s^2-1)(s^2-4)\cdots(s^2-(m-1)^2)\, h^{2m} f[x_{-m}, x_{-m+1}, \ldots x_m]$$

where $x = x_0 + sh$. $x_{-i} = x_0 - ih$, $i = 1, 2, \ldots m$

$x_i = x_0 + ih$, $i = 1, 2, \ldots m$.

To obtain the above formula consider the following table for illustration:

x_i	$f(x_i)$	1st divided difference	2nd divided difference	3rd divided difference	4th divided difference
x_{-2}	$f(x_{-2})$				
		$f[x_{-2}, x_{-1}]$			
x_{-1}	$f(x_{-1})$		$f[x_{-2}, x_{-1}, x_0]$		
		$f[x_{-1}, x_0]$		$f[x_{-2}, x_{-1}, x_0, x_1]$	
x_0	$f(x_0)$		$f[x_{-1}, x_0, x_1]$		$f[x_{-2}, x_{-1}, x_0, x_1, x_2]$
		$f[x_0, x_1]$		$f[x_{-1}, x_0, x_1, x_2]$	
x_1	$f(x_1)$		$f[x_0, x_1, x_2]$		
		$f[x_1, x_2]$			
x_2	$f(x_2)$				

Here $m=2$.
we construct $P^f_{2m}(x)$ using $f[x_0]$, $f[x_0, x_1]$, $f[x_{-1}, x_0, x_1]$
$f[x_{-1}, x_0, x_1, x_2]$ and $f[x_{-2}, x_{-1}, x_0, x_1, x_2]$.

as

$$P^f_{2m}(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_{-1}, x_0, x_1](x-x_0)(x-x_1)$$

$$+ f[x_{-1}, x_0, x_1, x_2](x-x_0)(x-x_1)(x-x_{-1}) +$$

$$+ f[x_{-2}, x_{-1}, x_0, x_1, x_2](x-x_0)(x-x_1)(x-x_{-1})(x-x_2)$$

Similarly we construct $p^b_{2m}(x)$ using $f[x_0]$, (4b)

$f[x_{-1}, x_0]$, $f[x_{-1}, x_0, x_1]$, $f[x_{-2}, x_{-1}, x_0, x_1]$, $f[x_{-2}, x_{-1}, x_0, x_1, x_2]$

&

$$
\begin{aligned}
p^b_{2m}(x) = {} & f[x_0] + f[x_{-1}, x_0](x-x_0) + f[x_{-1}, x_0, x_1](x-x_0)(x-x_{-1}) \\
& + f[x_{-2}, x_{-1}, x_0, x_1](x-x_0)(x-x_{-1})(x-x_1) + \\
& + f[x_{-2}, x_{-1}, x_0, x_1, x_2](x-x_0)(x-x_{-1})(x-x_1)(x-x_{-2}).
\end{aligned}
$$

by using $x - x_i = (s-i)h$, $i = 0, 1, \ldots m$ and
$x - x_{-i} = (s+i)h$, $i = 1, 2, \ldots m$, we get

$$
\begin{aligned}
p^f_{2m}(x) = {} & f[x_0] + sh\, f[x_0, x_1] + s(s-1)h^2 f[x_{-1}, x_0, x_1] + s(s^2-1)h^3 \\
& f[x_{-1}, x_0, x_1, x_2] + s(s-1)(s+1)(s-2)h^4 f[x_{-2}, x_{-1}, x_0, x_1, x_2]
\end{aligned}
$$

$$
\begin{aligned}
p^b_{2m}(x) = {} & f[x_0] + sh\, f[x_{-1}, x_0] + s(s+1)h^2 f[x_{-1}, x_0, x_1] + s(s^2-1)h^3 \\
& f[x_{-2}, x_{-1}, x_0, x_1] + s(s+1)(s-1)(s+2)h^4 f[x_{-2}, x_{-1}, x_0, x_1, x_2]
\end{aligned}
$$

Taking mean of $p^f_{2m}(x)$ and $p^b_{2m}(x)$, we get

$$
\begin{aligned}
p_{2m}(x) = {} & f[x_0] + \frac{sh}{2}\{f[x_0, x_1] + f[x_{-1}, x_0]\} + s^2h^2 f[x_{-1}, x_0, x_1] \\
& + \frac{s(s^2-1)h^3}{2}\{f[x_{-1}, x_0, x_1, x_2] + f[x_{-2}, x_{-1}, x_0, x_1]\} \\
& + s^2(s^2-1)h^4 f[x_{-2}, x_{-1}, x_0, x_1, x_2].
\end{aligned}
$$

Suppose the number of nodes is even, say, $2m$, and (4c)

$x_{-m}, x_{-m+1}, \ldots x_{-1}, x_1, x_2, \ldots x_m$. are the nodes.

The degree of polynomial to interpolate is $\leq 2m-1$.

Let $m=2$ for illustration

x_i	$f(x_i)$	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot]$
x_{-2}	$f[x_{-2}]$			
		$f[x_{-2}, x_{-1}]$		
x_{-1}	$f[x_{-1}]$		$f[x_{-2}, x_{-1}, x_1]$	
		$f[x_{-1}, x_1]$		$f[x_{-2}, x_{-1}, x_1, x_2]$
x_1	$f[x_1]$		$f[x_{-1}, x_1, x_2]$	
		$f[x_1, x_2]$		
x_2	$f[x_2]$			

Let the point x be lying between x_{-1} and x_1.
We can use forward differences for x_{-1} and backward differences for x_1 and then take mean of the resulting polynomial to get a polynomial of degree $2m-1$ ($m=2, \Rightarrow 2m-1=3$) which is of

Stirling type.

e.g.

$$p^f(x) = f[x_{-1}] + f[x_{-1}, x_1](x-x_{-1}) + f[x_{-2}, x_{-1}, x_1](x-x_{-1})(x-x_1) + f[x_{-2}, x_{-1}, x_1, x_2](x-x_{-1})(x-x_1)(x-x_{-2}).$$

$$p^b(x) = f[x_1] + f[x_{-1}, x_1](x-x_1) + f[x_{-1}, x_1, x_2](x-x_1)(x-x_{-1}) + f[x_{-2}, x_{-1}, x_1, x_2](x-x_1)(x-x_{-1})(x-x_2).$$

$$P(x) = \frac{1}{2}\left[p^f(x) + p^b(x)\right]$$

Theorem

Let $x_0, x_1, x_2, \ldots x_n$ be the points. 5

Let $(i_0, i_1, \ldots i_n)$ be some permutation of $(0, 1, \ldots n)$. Then

$$f[x_0, x_1, \ldots x_n] = f[x_{i_0}, x_{i_1}, \ldots x_{i_n}]. \quad \text{—} (2)$$

proof: Recall that

$$P_n(x) = P_{n-1}(x) + (x-x_0)(x-x_1)\cdots(x-x_{n-1}) f[x_0, x_1, \ldots x_n].$$

The coefficient of leading term is $f[x_0, x_1, \ldots x_n]$.

To prove (2), we go back to Lagrange form of $P_n(x)$. Note that if

$$\psi_n(x) = (x-x_0)(x-x_1)\cdots(x-x_n), \text{ then}$$

$$\psi_n'(x_j) = (x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_n),$$

and if x is not a node point,

$$P_n(x) = \sum_{j=0}^{n} \frac{\psi_n(x)}{(x-x_j)\,\psi_n'(x_j)} f(x_j)$$

If we look at the coefficient of leading term in $P_n(x)$ we get

$$f[x_0, x_1, \ldots x_n] = \sum_{j=0}^{n} \frac{f(x_j)}{\psi_n'(x_j)}.$$

From this formula, we see that

$$\sum_{j=0}^{n} \frac{f(x_j)}{\psi_n'(x_j)} = \sum_{j=0}^{n} \frac{f(x_{i_j})}{\psi_n'(x_{i_j})}$$

for any permutation $(i_0, i_1, \ldots i_n)$ of $(0, 1, 2, \ldots n)$.

Therefore ⑥

$$f[x_0, x_1, \ldots x_n] = f[x_{i_0}, x_{i_1}, \ldots x_{i_n}]$$

for any permutation $(i_0, i_1, \ldots i_n)$ of $(0, 1, \ldots n)$.

Convergence of secant method:

Recall that secant method is given by

$$x_{n+1} = x_n - f(x_n) \cdot \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}, \quad n \geq 1 \quad \text{③}$$

for given initial two approximations x_0 and x_1.

If α is the root of the equation $f(x)=0$, i.e. $f(\alpha)=0$, then from ③.

$$\alpha - x_{n+1} = (\alpha - x_n) + f(x_n) \cdot \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

$$= \frac{\big(f(x_n) - f(x_{n-1})\big)(\alpha - x_n) + f(x_n)\,(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$= \frac{f[x_{n-1}, x_n]\,(\alpha - x_n)(x_n - x_{n-1}) + \{f(x_n) - f(\alpha)\}(x_n - x_{n-1})}{f[x_{n-1}, x_n]\,(x_n - x_{n-1})}$$

$$= \frac{f[x_{n-1}, x_n](\alpha - x_n) - f[x_n, \alpha]\,(\alpha - x_n)}{f[x_{n-1}, x_n]}$$

$$= (\alpha - x_n) \frac{- f[x_{n-1}, x_n, \alpha]\,(\alpha - x_{n-1})}{f[x_{n-1}, x_n]}$$

$$= -(\alpha - x_{n-1})(\alpha - x_n) \frac{f[x_{n-1}, x_n, \alpha]}{f[x_{n-1}, x_n]}.$$

The quantities $f[x_{n-1}, x_n]$ and $f[x_{n-1}, x_n, \alpha]$ are first and second order Newton divided differences. There exists ζ_n and ξ_n such that

$$f[x_{n-1}, x_n] = f'(\xi_n) \quad \text{and} \quad f[x_{n-1}, x_n, \alpha] = \frac{f''(\zeta_n)}{2}.$$

with ξ_n between x_{n-1} and x_n

ζ_n between x_{n-1}, x_n and α.

<u>Theorem</u>: Assume $f(x)$, $f'(x)$ and $f''(x)$ are continuous for all values of x in some interval containing α, and assume $f'(\alpha) \neq 0$. Then if the initial guesses x_0 and x_1 are choosen sufficiently close to α, the iterates x_n will converge to α. The order of convergence will be $p = \frac{1+\sqrt{5}}{2}$. $[f(\alpha)=0$, α is the zero of $f]$

<u>Proof</u>: There is some $\epsilon > 0$ such that $f'(x) \neq 0$ on $I = [\alpha - \epsilon, \alpha + \epsilon]$. Then define

$$M = \frac{\max_{x \in I} |f''(x)|}{2 \min_{x \in I} |f'(x)|}.$$

Then for all $x_0, x_1 \in [\alpha - \epsilon, \alpha + \epsilon]$, using

$$e_{n+1} = -e_n \cdot e_{n-1} \frac{f''(\zeta_n)}{f'(\xi_n)} \; ; \quad (5)$$

ζ_n lies between x_{n-1}, x_n & α.
ξ_n lies between x_{n-1} and x_n.
$e_n = \alpha - x_n$.

we have

$$|e_2| \leq |e_1| \cdot |e_0| M.$$

$$M|e_2| \leq M|e_1| \cdot M|e_0|.$$

Further assume that x_1 and x_0 are so chosen that

$$\delta = \max\{M|e_1|, M|e_0|\} < 1.$$

Then $M|e_2| < 1$, since $M|e_2| \le \delta^2$

Also $M|e_2| \le \delta^2 < \delta$ implies that

$$|e_2| < \frac{\delta}{M} = \max\{|e_1|, |e_0|\} \le \epsilon.$$

and thus $x_2 \in [\alpha-\epsilon, \alpha+\epsilon]$. We apply this argument inductively to show that $x_n \in [\alpha-\epsilon, \alpha+\epsilon]$ and $M|e_n| \le \delta$ for $n \ge 2$.

To prove convergence and obtain order of convergence, continue applying (S) to get

$$M|e_3| \le M|e_2| \cdot M|e_1| \le \delta^2 \cdot \delta = \delta^3.$$

$$M|e_4| \le M|e_3| \cdot M|e_2| \le \delta^5.$$

For

$$M|e_n| \le \delta^{q_n},$$

$$M|e_{n+1}| \le M|e_n| \cdot M|e_{n-1}| \le \delta^{q_n + q_{n-1}} = \delta^{q_{n+1}}$$

Thus,

$$q_{n+1} = q_n + q_{n-1}, \quad n \ge 1.$$

with $q_0 = q_1 = 1$. This is a Fibonacci sequence of numbers whose explicit formula can be given:

$$q_n = \frac{1}{\sqrt{5}}\left[\gamma_0^{n+1} - \gamma_1^{n+1}\right] \quad n \ge 0$$

$$\gamma_0 = \frac{1+\sqrt{5}}{2} \simeq 1.618, \quad \gamma_1 = \frac{1-\sqrt{5}}{2} \simeq -0.618$$

Thus
$$q_n = \frac{1}{\sqrt{5}}(1.618)^{n+1} \quad \text{for large } n,$$
$[\because \gamma_1^{n+1} \to 0 \text{ as } n \to \infty]$

Returning to
$$M|e_n| \le \delta^{q_n}$$
we get
$$|e_n| \le \frac{\delta^{q_n}}{M}, \quad n \ge 0. \quad \text{—} (B)$$

Since $q_n \to \infty$ as $n \to \infty$, and $0 < \delta < 1$, we get $x_n \to \alpha$ as $n \to \infty$.

For the order of convergence, Let B_n denote the RHS of (B). Then
$$\frac{B_{n+1}}{B_n^{\gamma_0}} = \frac{\frac{1}{M}\delta^{q_{n+1}}}{\left[\frac{1}{M}\right]^{\gamma_0}\delta^{\gamma_0 q_n}} = M^{\gamma_0 - 1} \cdot \delta^{q_{n+1} - \gamma q_n}$$
$$\le \delta^{-1} M^{\gamma_0 - 1} = C,$$

because $q_{n+1} - \gamma_0 q_n = \gamma_1^{n+1} > -1$.

Thus
$$B_{n+1} \le C B_n^{\gamma_0}$$
which implies an order of convergence $p = \gamma_0 = \frac{1+\sqrt{5}}{2}$.

Similar argument for Newton's method (10)
can be derived to prove order of convergence
is 2, with assumption that $f \in C^2$.

Recall Newton's method.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0.$$

If α is zero of f, i.e. $f(\alpha)=0$.

Then using.

$$f(\alpha) = f(x_n) + (\alpha - x_n) f'(x_n) + \frac{(\alpha - x_n)^2}{2} f''(\xi_n)$$

where ξ_n lies between x_n and α.

Using $f(\alpha)=0$, we solve for α,

$$\alpha = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{(\alpha - x_n)^2}{2} \cdot \frac{f''(\xi_n)}{f'(x_n)}$$

$$\therefore \quad \alpha - x_{n+1} = -(\alpha - x_n)^2 \cdot \frac{f''(\xi_n)}{f'(x_n)}.$$

i.e.

$$e_{n+1} = -e_n^2 \cdot \frac{f''(\xi_n)}{f'(x_n)}, \quad e_n = \alpha - x_n.$$