Numerical Methods for ODEs:

Consider initial value problem (IVP)

$$y' = f(t, y), \quad y(t_0) = y_0;$$

the function $f(t, y)$ is continuous for all (t, y) in some domain D of the ty-plane, and (t_0, y_0) is a point in D.

Definition: We say that a function $y(t)$ is a solution of (IVP) on $[a, b]$, if for all $a \le t \le b$,

(1). $(t, y(t)) \in D$

(2). $y(t_0) = y_0$

(3). $y'(t)$ exists and $y'(t) = f(t, y(t))$.

Example ① The general first-order linear differential equation is

$$y' = a_0(t)\, y + g(t), \qquad a \le t \le b$$

in which $a_0(t)$ and $g(t)$ are assumed to continuous on $[a, b]$. The domain D for this problem is

$$D = \{(t, y) : a \le t \le b, \ -\infty < y < \infty\}$$

②. Consider $y' = -y^2$, $y(0) = 1$. Solving it

we get $y(t) = \dfrac{1}{1+t}$.

we can see that $y(t) \to \infty$, as $t \to -1$ from right.

Thus the global smoothness of $f(t, y) = -y^2$ does not guarantee a similar behaviour in solution.

Example 3: Consider

$$y' = y^{1/2}; \quad y(0)=0.$$

on solving we get $y(t) = \frac{t^2}{4}$.

But $y(t) \equiv 0$ is also a solution.

In this case we have two solutions.

Definition: (Lipschitz condition):

A function $f(t, y)$ is said to satisfy a Lipschitz condition in the variable y on a set $D \subset \mathbb{R}^2$ if a constant $L > 0$ ~~such that~~ exists with

$$|f(t, y_1) - f(t, y_2)| \leq L\,|y_1 - y_2|,$$

whenever (t, y_1) and (t, y_2) are in D. The constant L is called a Lipschitz constant for f.

Examples: $f(t, y) = t\,|y|$ satisfies a Lipschitz condition on the interval $D = \{(t, y) \mid 1 \leq t \leq 2,\ -3 \leq y \leq 4\}$

Solution: For (t, y_1) and (t, y_2) in D,

$$|f(t, y_1) - f(t, y_2)| = \big|\, t\,|y_1| - t\,|y_2| \,\big| = |t|\, \big|\, |y_1| - |y_2| \,\big|$$

$$\leq |t|\, |y_1 - y_2| \leq 2\, |y_1 - y_2|.$$

$\therefore$ f satisfies a Lipschitz condition on D in the ~~interval~~ variable y with Lipschitz constant 2.

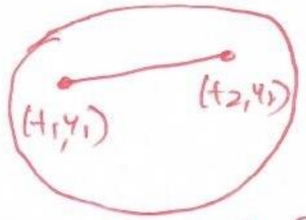
The smallest possible Lipschitz constant for this function ~~on~~ D is 2, since

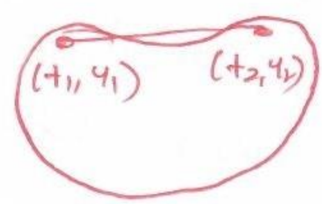
$$|f(2,1) - f(2,0)| = |2 - 0| = 2\,|1 - 0|.$$

Definition: A set $D \subset \mathbb{R}^2$ is said to be convex if whenever (t_1, y_1) and (t_2, y_2) belong to D, then $\left((1-\lambda)t_1 + \lambda t_2,\ (1-\lambda)y_1 + \lambda y_2\right)$ also belong to D for every λ in $[0,1]$.

Geometrically the above definition states that a set is convex provided that whenever two points belong to the set, the entire line segment between the points also belongs to the set.

non convex set.





Convex set

Theorem: Suppose $f(t,y)$ is defined on a convex set $D \subset \mathbb{R}^2$. If a constant $L > 0$ exists with

$$\left|\frac{\partial f}{\partial y}(t, y)\right| \le L, \text{ for all } (t,y) \in D,$$

then f satisfies a Lipschitz condition on D with Lipschitz constant L in the variable y.

Remark: The proof follows from mean value theorem. For (t, y_1) and (t, y_2),

$$f(t, y_1) - f(t, y_2) = \frac{\partial f}{\partial y}(t, \xi)(y_1 - y_2)$$

where (t, ξ) is a point on the line segment connecting (t, y_1) and (t, y_2)

~~Non Lipschitz~~

Function fail to satisfy Lipschitz condition:

Consider $f(t,y) = y^{1/2}$ on

$$D = \{(t,y) : t \in \mathbb{R},\ 0 \le y \le 1\}$$

Note that f is continuous on D, but f does not satisfy a Lipschitz condition on D.

We can prove this by contradiction.

Suppose there exists a constant $L > 0$ such that

$$|f(t,y_1) - f(t,y_2)| \le L\,|y_1 - y_2|,$$

for all (t,y_1) and (t,y_2) in D.

Then we must have

$$|y_1^{1/2} - y_2^{1/2}| \le L\,|y_1 - y_2|$$

$$\Rightarrow |y_1^{1/2} - y_2^{1/2}| \le L\,|y_1^{1/2} - y_2^{1/2}|\,|y_1^{1/2} + y_2^{1/2}|$$

Let $y_1 \ne y_2$, then by cancelling $|y_1^{1/2} - y_2^{1/2}|$ on both sides

$$1 \le L\,|y_1^{1/2} + y_2^{1/2}|$$

for all (t,y_1) and (t,y_2) in D.

taking $y_1 = 0$, and, $y_2 = \frac{1}{n^2}$, $n \in \mathbb{N}$, we get

$$1 \le L \cdot \frac{1}{n} \Rightarrow n \le L,\ n \in \mathbb{N}$$

there is ~~no~~ no such L satisfying this.

Hence a contradiction and thus f does not satisfy a Lipschitz condition on D.

Existence and uniqueness theorem for (IVP)

Theorem: Let f be continuous real valued function on the rectangle

$$R : |t-t_0| \le a, \quad |y-y_0| \le b, \quad (a, b > 0)$$

and let

$$|f(t,y)| \le M \qquad (\text{for some } M > 0)$$

for all (t, y) in R. Further suppose that f satisfies a Lipschitz condition with constant L in R. The there is an interval

$$I : |t-t_0| \le \alpha = \min\left\{a, \frac{b}{M}\right\}$$

such that the (IVP) has a unique solution on I.

Remark: This theorem is known as Picard's theorem.

Definition: (Well-posed problem):

The initial value problem

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

is said to be a well-posed problem, if

(1) A unique solution, $y(t)$, to the problem exists,

(2) There exists constants $\epsilon_0 > 0$ and $k > 0$ such that for any ϵ, with $0 < \epsilon < \epsilon_0$, whenever $\delta(t)$ is continuous with $|\delta(t)| < \epsilon$ for all t in $[a, b]$, and when $|\delta_0| < \epsilon$, the (IVP)

$$\frac{dz}{dt} = f(t, z) + \delta(t), \quad a \le t \le b, \quad z(a) = \alpha + \delta_0$$

has a unique solution $z(t)$ that satisfies

$$|z(t) - y(t)| < k\epsilon \quad \text{for all } t \text{ in } [a, b]$$

Remark: The first condition (1) requires the (IVP) to have a unique solution, and the solution exists.

The second condition tells us the continuous dependence of the solution on the given data. A small change in the data implies a small change in the solution.

Theorem: suppose $D = \{(t, y) \mid a \le t \le b, \text{ and } -\infty < y < \infty\}$. (7)
If f is continuous and satisfies a Lipschitz condition in the variable y on the set D, then the (IVP)

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha$$

is well-posed.

Euler's Method:

Consider the interval $[a, b]$ and the mesh points

$$t_i = a + ih \quad \text{for each } i = 0, 1, 2, \ldots N.$$

where h is the common distance between the points $h = (b-a)/N = t_{i+1} - t_i$.

h is called the step size.

We use Taylor's theorem to derive the method. suppose $y(t)$ is the solution of (IVP), has two second derivatives on $[a, b]$, so that ~~a~~for each $i = 0, 1, 2 \ldots N-1$.

$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)\, y'(t_i) + \frac{(t_{i+1} - t_i)^2}{2} y''(\xi_i)$$

for some number ξ_i in (t_i, t_{i+1}). Since ~~h=t_{i+1}-t_i~~

$h = t_{i+1} - t_i$, we have

$$y(t_{i+1}) = y(t_i) + h y'(t_i) + \frac{h^2}{2} y''(\xi_i)$$

and since $y(t)$ satisfies the differential equation 8

$$y(t_{i+1}) = y(t_i) + h f(t_i, y(t_i)) + \frac{h^2}{2} y''(\xi_i).$$

Euler's method constructs $w_i \approx y(t_i)$, for each $i = 1, 2 \dots N$, by ~~delet~~ deleting the remainder term.

Thus Euler's method is

$$w_0 = \alpha$$

$$w_{i+1} = w_i + h f(t_i, w_i), \text{ for each } i = 0, 1, \dots, N-1.$$

Error bounds for Euler's method:

Lemma: for all $x \geq -1$, and any positive m, we have $0 \leq (1+x)^m \leq e^{mx}$.

proof: Using Taylor's theorem for $f(x) = e^x$,

$$e^x = 1 + x + \frac{1}{2} x^2 e^{\xi}$$

[i.e. $f(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(\xi)$].

where ξ is between x and zero. Thus

$$0 \leq 1 + x \leq 1 + x + \frac{1}{2} x^2 e^{\xi} = e^x$$

and because $1 + x \geq 0$, we have

$$0 \leq (1+x)^m \leq (e^x)^m = e^{mx}.$$

$\boxed{\text{MC}}$

Lemma: If s and t are positive real numbers, $\{a_i\}_{i=0}^{k}$ is a sequence satisfying $a_0 \geq -t/s$, and

$$(*) \text{ — } \quad a_{i+1} \leq (1+s)a_i + t, \text{ for each } i = 0,1,2 \ldots k-1.$$

then
$$a_{i+1} \leq e^{(i+1)s}\left(a_0 + \frac{t}{s}\right) - \frac{t}{s}.$$

Proof: For each i, $(*)$ implies that

$$\begin{aligned}
a_{i+1} &\leq (1+s)a_i + t \\
&\leq (1+s)\left[(1+s)a_{i-1} + t\right] + t \\
&= (1+s)^2 a_{i-1} + \left[1 + (1+s)\right]t \\
&\leq (1+s)^2\left[(1+s)a_{i-2} + t\right] + \left[1 + (1+s)\right]t \\
&= (1+s)^3 a_{i-2} + \left[1 + (1+s) + (1+s)^2\right]t \\
&\ \vdots \\
&\leq (1+s)^{i+1} a_0 + \left[1 + (1+s) + \cdots + (1+s)^i\right]t,
\end{aligned}$$

But
$$1 + (1+s) + (1+s)^2 + \cdots + (1+s)^i = \sum_{j=0}^{i}(1+s)^j$$

is equal to
$$\frac{1-(1+s)^{i+1}}{1-(1+s)} = \frac{1}{s}\left[(1+s)^{i+1} - 1\right].$$

Thus
$$a_{i+1} \leq (1+s)^{i+1} a_0 + \frac{(1+s)^{i+1} - 1}{s} \cdot t = (1+s)^{i+1}\left(a_0 + \frac{t}{s}\right) - \frac{t}{s}.$$

Using previous Lemma (Page 8).
$$a_{i+1} \leq e^{(i+1)s}\left(a_0 + \frac{t}{s}\right) - \frac{t}{s}. \quad \left[\begin{array}{l}\text{since} \\ a_0 + \frac{t}{s} \geq 0\end{array}\right]$$

Theorem: Suppose f is continuous and satisfies a Lipschitz condition with constant L on

$$D = \{(t, y) : a \le t \le b \text{ and } -\infty < y < \infty\}$$

and that a constant M exists with

$$|y''(t)| \le M \quad \text{for all } t \in [a, b],$$

where $y(t)$ denotes the unique solution to the (IVP) $y' = f(t, y)$, $a \le t \le b$, $y(a) = \alpha$.

Let $w_0, w_1, \dots w_N$ be the approximation generated by the Euler's method for some positive integer N. Then for each $i = 0, 1, \dots N$,

$$|y(t_i) - w_i| \le \frac{hM}{2L}\left[e^{L(t_i - a)} - 1\right].$$

Proof: when $i = 0$, the result is true, since $y(t_0) = w_0 = \alpha$.

For $i \neq 0$: we have

$$y(t_{i+1}) = y(t_i) + h f(t_i, y(t_i)) + \frac{h^2}{2} y''(\xi_i) \quad \text{— (1)}$$

For $i = 0, 1, \dots N-1$, we have

$$w_{i+1} = w_i + h f(t_i, w_i). \quad \text{— (2)}$$

Using the notation $y_i = y(t_i)$ and $y_{i+1} = y(t_{i+1})$, we get from (1) & (2).

$$y_{i+1} - w_{i+1} = y_i - w_i + h\left[f(t_i, y_i) - f(t_i, w_i)\right] + \frac{h^2}{2} y''(\xi_i)$$

Hence

$$|y_{i+1} - w_{i+1}| \le |y_i - w_i| + h\,|f(t_i, y_i) - f(t_i, w_i)| + \frac{h^2}{2}|y''(\xi_i)|$$

Since f satisfies Lipschitz condition in the second variable with constant L, and $|y''(t)| \le M$, we have

$$|y_{i+1} - w_{i+1}| \le (1 + hL)\, |y_i - w_i| + \frac{h^2}{2} M.$$

Referring to previous Lemma (page 9) and letting $s = hL$, $t = h^2M/2$ and $a_j = |y_j - w_j|$ for each $j = 0, 1, \dots N$, we find that

$$|y_{i+1} - w_{i+1}| \le e^{(i+1)hL}\left(|y_0 - w_0| + \frac{h^2 M}{2hL}\right) - \frac{h^2 M}{2hL}$$

Because $|y_0 - w_0| = 0$ and $\cancel{(i+1)h = t_{i+1}}$

$t_{i+1} = (i+1)h + a \Rightarrow (i+1)h = t_{i+1} - a$, this implies that

$$|y_{i+1} - w_{i+1}| \le \frac{hM}{2L}\left(e^{(t_{i+1} - a)L} - 1\right)$$