Composite Numerical Quadratures :

Let $f:[a,b] \longrightarrow \mathbb{R}$ be a continuous function. Then f is integrable on $[a,b]$. Often, it is difficult to find antiderivative of f so that the integral

$$I(f) = \int_a^b f(x)\,dx \text{ is computed.}$$

The numerical quadratures on $[a,b]$ may not give satisfactory result as the interval $[a,b]$ is big or the function f may not be smooth on $[a,b]$. However we can use composite numerical quadrature rules, which are defined on subintervals of $[a,b]$. Let $a = x_0 < x_1 < \cdots < x_N = b$ be a partition of $[a,b]$ with break points $\{x_j\}_{j=0}^{N}$.

Then

$$I(f) = \int_a^b f(x)\,dx = \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} f(x)\,dx.$$

We call $I_j = [x_{j-1}, x_j]$, $1 \le j \le N$.

Note that finding $I(f)$ is the same as finding $I_j(f) = \int_{x_{j-1}}^{x_j} f(x)\,dx$ and then summing all these integrals on the subintervals.

We may use the numerical integration formulas that are developed previously to numerically integrate $I_j(f)$ and then sum all these to obtain composite numerical integration.

Composite rectangle rule::

Note that the rectangle rule gives

$$\int_{x_{j-1}}^{x_j} f(x)\,dx \approx (x_j - x_{j-1})\, f(x_{j-1})$$

with error $E^R = \frac{(x_j - x_{j-1})^2}{2} f'(\eta_j); \quad \eta_j \in (x_{j-1}, x_j)$.

Then composite rectangle rule is given by

$$\int_a^b f(x)\,dx \approx \sum_{j=1}^{N} (x_j - x_{j-1})\, f(x_{j-1}).$$

with Error. $E^{CR} = \sum_{j=1}^{N} \frac{(x_j - x_{j-1})^2}{2} f'(\eta_j), \quad \eta_j \in I_j$.

Assume that the break points $\{x_j\}$ are equally spaced, i.e. $h = h_j = x_j - x_{j-1}$, $1 \le j \le N$. Then the rule is written as

$$I^{CR}(f) = h \sum_{j=1}^{N} f(x_{j-1}).$$

&

$$E^{CR}(f) = \sum_{j=1}^{N} \frac{h^2}{2} f'(\eta_j), \quad \eta_j \in I_j.$$

If $|f'(x)| \le M \quad \forall x \in I_j$ and $1 \le j \le N$.

Then, since $h = \frac{b-a}{N}$, we find

$$|E^{CR}(f)| \le \frac{M}{2} \sum_{j=1}^{N} h^2 = \frac{M}{2} N h^2 = \frac{M(b-a)}{2} . h.$$

$$\therefore \quad |E^{CR}(f)| \le \frac{M|b-a|}{2} . h.$$

Note: The composite rectangle rule is exact, i.e. The rule gives exact integration value, for all polynomials of degree 0; [piecewise] i.e. when f is piecewise constant.

Composite Midpoint rule:

Note that the midpoint rule gives

$$\int_{x_{j-1}}^{x_j} f(x)dx \approx (x_j - x_{j-1})\, f\left(\frac{x_{j-1}+x_j}{2}\right).$$

with error $E^{M}(f) = \frac{(x_j - x_{j-1})^3}{24} f''(\eta_j)$, $\eta_j \in (x_{j-1}, x_j)$

Then the composite midpoint rule is given by

$$\int_a^b f(x)\,dx \approx \sum_{j=1}^{N} (x_j - x_{j-1})\, f\left(\frac{x_{j-1}+x_j}{2}\right).$$

with error $E^{CM}(f) = \sum_{j=1}^{N} \frac{(x_j - x_{j-1})^3}{24} f''(\eta_j)$, $\eta_j \in I_j$

Assume that the break points $\{x_j\}$ are equally spaced i.e. $h = h_j = x_j - x_{j-1}$, $1 \le j \le N$. Then the rule becomes

$$I^{CM}(f) = h \sum_{j=1}^{N} f\left(\frac{x_{j-1}+x_j}{2}\right).$$

and

$$E^{CM}(f) = \sum_{J=1}^{N} \frac{h^3}{24} f''(\eta_j). \quad \eta_j \in I_j$$

If $|f''(x)| \le M$, $\forall x \in I_j$, $1 \le j \le M$. Then

$$|E^{CM}(f)| \le \frac{M}{24} \sum_{j=1}^{N} h^3 = \frac{M}{24} N h^3 = \frac{M(b-a)}{24} \cdot h^2.$$

$$\therefore |E^{CM}(f)| \le \frac{M(b-a)}{24} h^2.$$

<u>Note</u>: The composite midpoint rule is exact for all piecewise polynomials of degree 1. i.e. when f is linear polynomial on each $I_j = [x_{j-1}, x_j]$.

Composite trapezoidal rule:

The trapezoidal rule gives

$$\int_{x_{j-1}}^{x_j} f(x)\,dx \approx \frac{(x_j - x_{j-1})}{2}\left[f(x_j) + f(x_{j-1})\right]$$

with error, $E^T(f) = \frac{(x_j - x_{j-1})^3}{12} f''(\eta_j), \quad \eta_j \in I_j.$

The composite trapezoidal rule is given by

$$I^{CT}(f) = \sum_{j=1}^{N} \frac{(x_j - x_{j-1})}{2}\left[f(x_{j-1}) + f(x_j)\right]$$

with Error $\quad E^{CR} = \sum_{j=1}^{N} \frac{(x_j - x_{j-1})^3}{12} f''(\eta_j), \quad \eta_j \in I_j.$

If the break points $\{x_j\}$ are equally spaced with $h = h_j = x_j - x_{j-1}, \ 1 \le j \le N$. Then

$$I^{CT}(f) = \frac{h}{2}\left[f(a) + f(b) + 2\sum_{j=1}^{N-1} f(x_j)\right]$$

and $\quad E^{CT}(f) = \sum_{j=1}^{N} \frac{h^3}{12} f''(\eta_j), \quad \eta_j \in I_j$

Let $|f''(\eta)| = \max_{1 \le j \le N} |f''(\eta_j)|$, for some $\eta \in [a, b]$.

Then

$$|E^{CT}(f)| \le |f''(\eta)| \sum_{j=1}^{N} \frac{h^3}{12} = \frac{|f''(\eta)|}{12} N h^3$$

$$\therefore \ |E^{CT}| \le \frac{|f''(\eta)|}{12}(b-a) \cdot h^2, \quad \eta \in [a, b].$$

<u>Note:</u> The composite trapezoidal rule is exact when f is polynomial of degree 1 on each I_j.

Composite Simpson rule:

The Simpson's rule gives

$$\int_{x_{j-1}}^{x_j} f(x)\,dx \approx \frac{(x_j - x_{j-1})}{6}\left[f(x_{j-1}) + 4f\left(\frac{x_{j-1}+x_j}{2}\right) + f(x_j)\right]$$

with error $E^S(f) = -\dfrac{f^{(4)}(\eta_j)}{90}\left(\dfrac{x_j - x_{j-1}}{2}\right)^5$, $\eta_j \in I_j$.

Let $x_{j-1/2} = \dfrac{x_{j-1}+x_j}{2}$, $1 \le j \le N$. Then composite Simpson rule is given by.

$$I^{CS}(f) = \frac{(x_j - x_{j-1})}{6}\left[f(a) + f(b) + 2\sum_{j=1}^{N-1} f(x_j) + 4\sum_{j=1}^{N} f(x_{j-1/2})\right].$$

with error.

$$E^{CS}(f) = -\sum_{j=1}^{N}\left(\frac{x_j - x_{j-1}}{2}\right)^5 \frac{f^{(4)}(\eta_j)}{90}, \quad \eta_j \in I_j.$$

Assuming $\{x_j\}$ are equally spaced, i.e. $h = h_j = x_j - x_{j-1}$, $1 \le j \le N$, we write

$$I^{CS}(f) = \frac{h}{6}\left[f(a) + f(b) + 2\sum_{j=1}^{N-1} f(x_j) + 4\sum_{j=1}^{N} f(x_{j-1/2})\right].$$

and $$E^{CS}(f) = -\sum_{j=1}^{N}\left(\frac{h}{2}\right)^5 \frac{f^{(4)}(\eta_j)}{90}, \quad \eta_j \in I_j$$

Let for some $\eta \in [a,b]$, $|f^{(4)}(\eta)| = \max_{1\le j\le N} |f^{(4)}(\eta_j)|$, then

$$|E^{CS}(f)| \le |f^{(4)}(\eta)| \frac{(b-a)}{2}\left(\frac{h}{2}\right)^4.$$

Note: The composite Simpson rule is exact when f is a cubic polynomial on each I_j.

Note: The break points may be chosen in such a way that they are close in the regions where $f^{(4)}$ is large, and get a better approximation. This can be done for any of composite rules.

<u>Definition:</u> The degree of accuracy (or) precision of a quadrature formulas is the largest positive integer n such that the formula is exact for x^k, for each $k = 0, 1, 2, \ldots n$.

<u>Note:</u> The degree of precision of a quadrature formula is n if and only if the error is zero for all polynomials of degree $k = 0, 1, 2 \ldots n$, but is not zero for some polynomial of degree $n+1$.

Gaussian Quadrature

Earlier numerical quadrature formulas have certain degree of accuracy, for example, Midpoint rule rectangle rule has degree 0, trapezoidal rule has degree 1 has degree 1, and simpson rule has degree 3, of accuracy.

The quadrature formula take the form

$$\int_a^b f(x)\,dx \approx \sum_{j=1}^{n} c_j f(x_j)$$

where x_j are nodes in $[a, b]$ and the weights c_j are constants obtained by integrating ~~the~~ certain Lagrange polynomials associated with the nodes $\{x_j\}$.

Guassian quadrature formula is desired based on the idea that we choose $c_1 \ldots c_n$ and the nodes $x_1, x_2 - - x_n$ in such a way that the quadrature rule is exact for higher degree polynomials (say, $2n-1$ degree polynomials).

We begin by illustrating this when $n=2$.

For convenience we take the interval to be $[-1, 1]$. so.

$$\int_{-1}^{1} f(x)\,dx \approx c_1 f(x_1) + c_2 f(x_2).$$

gives the exact result when $f(x)$ is a polynomial of degree $2(2)-1 = 3$, or less. Since integration is a linear operation, we can check for $f(x)$ is $1, x, x^2, x^3$. We need.

$$\int_{-1}^{1} 1\,dx = c_1 \cdot 1 + c_2 \cdot 1 \Rightarrow c_1 + c_2 = 2$$

$$\int_{-1}^{1} x\,dx = c_1 x_1 + c_2 x_2 \Rightarrow c_1 x_1 + c_2 x_2 = 0$$

$$\int_{-1}^{1} x^2\,dx = c_1 x_1^2 + c_2 x_2^2 \Rightarrow c_1 x_1^2 + c_2 x_2^2 = \frac{2}{3}$$

$$\int_{-1}^{1} x^3\,dx = c_1 x_1^3 + c_2 x_2^3 \Rightarrow c_1 x_1^3 + c_2 x_2^3 = 0.$$

we need to find unknowns c_1, c_2, x_1 and x_2 from above four equations.

Since the interval $[-1, 1]$ is symmetric around 0, we can assume $x_2 = -x_1$. Then find that

$$c_1 = 1, \quad c_2 = 1, \quad x_1 = \frac{1}{\sqrt{3}}, \quad x_2 = -\frac{1}{\sqrt{3}}.$$

Then $\int_{-1}^{1} f(x)\,dx \approx f(-1/\sqrt{3}) + f(1/\sqrt{3})$ is exact when f is cubic polynomial.

For general formula:

Let $P(x)$ be a polynomial of degree $\leq 2n-1$ on $[-1,1]$. Let P_n be a polynomial of degree $\leq n$. Dividing $P(x)$ by P_n, we obtain two polynomials of degree $< n$, say, $Q(x)$ and $R(x)$ such that

$$P(x) = Q(x)\, P_n(x) + R(x),$$

$\deg(P(x)) \leq 2n-1$
$\deg(P_n(x)) = n$
$\deg(Q(x)) < n$
$\deg(R(x)) < n$.

Then

$$\int_{-1}^{1} P(x)\,dx = \int_{-1}^{1} Q(x)\, P_n(x) \cancel{dx} + \int_{-1}^{1} R(x)\,dx.$$

If $P_n(x)$ is orthogonal to all polynomials of degree $< n$, that is, $\int_{-1}^{1} x^k P_n(x)\,dx = 0$, for $k = 0, 1, \ldots n-1$.

then $\int_{-1}^{1} Q(x)\, P_n(x)\,dx = 0$. and

$$\int_{-1}^{1} P(x)\,dx = \int_{-1}^{1} R(x)\,dx. \quad \text{—(1)}$$

Integrating $P(x)$ is nothing but integrating $R(x)$. Note that $R(x)$ is polynomial of degree $< n$. If we have a quadrature that is exact for polynomials of degree $< n$, then we can use that formula to find (1) exactly. The quadrature uses $R(x_i)$ for some nodes x_i in question. If we pick x_i such that $P(x_i) = R(x_i)$, then the quadrature for RHS of (1) is the quadrature for LHS of (1) using $P(x_i)$ and the rule is exact for polynomials of degree $< 2n-1$.

If we want that

$P(x_i) = R(x_i)$, then since

$P(x) = Q(x)\, P_n(x_i) + R(x_i)$, we must have $P_n(x_i) = 0$.

for all x_i. [i.e. x_i's are zeros of $P_n(x)$ in $[-1,1]$]

There fore we need to look for zeros of $P_n(x)$,

where $P_n(x)$ is orthogonal to all polynomials of degree $\le n-1$. i.e $\int_{-1}^{1} P_n(x)\, x^k\, dx = 0$, $k = 0, 1, \dots, n-1$.

Legendre polynomials

Properties of Legendre polynomials: $\{P_n(x)\}$.

(1) For each n, $P_n(x)$ is a monic polynomial of degree n.

(2) $\int_{-1}^{1} P(x)\, P_n(x)\, dx = 0$, for all polynomials of degree $< n$.

The first few Legendre polynomials:

$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = x^2 - \frac{1}{3}$; $P_3(x) = x^3 - \frac{3}{5}x$.

and $P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}$.

The roots of these polynomials are distinct, lie in the interval $[-1, 1]$, have a symmetry with respect to the origin.

Let $x_1, x_2, \dots x_n$ denote the zeros of the nth degree Legendre polynomial $P_n(x)$.

We use these points as nodes for finding quadrature formula. The question remaing to find weights $c_1, c_2 \dots c_n$

Case $n=1$:

$x_1 = 0$ is the zero of $P_1(x) = x$.

We need $\int_{-1}^{1} P(x)\,dx = c_1 P(x_1) = c_1 P(0)$

exact for all polynomials of degree 1. ($\deg(P(x)) = 1$)

We can take $P(x)$ is 1 and x to find c_1.

Let $P(x) = 1$. Then

$$2 = \int_{-1}^{1} P(x)\,dx = c_1 P(0) = c_1 \Rightarrow c_1 = 2.$$

$\therefore$ $\int_{-1}^{1} P(x) \simeq 2P(0)$ is exact for all polynomials $P(x)$ of degree 1.

Case $n \geq 2$:

Theorem: Suppose that $x_1, x_2, \dots x_n$ are zeros of the nth degree Legendre polynomial $P_n(x)$ and that the numbers c_i, are defined for each $i = 1, 2, \dots n$, by

$$c_i = \int_{-1}^{1} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}\,dx$$

If $P(x)$ is any polynomial of degree less than $2n$, then

$$\int_{-1}^{1} P(x)\,dx = \sum_{i=1}^{n} c_i P(x_i).$$

Proof. Let us first consider the case that $P(x)$ is of degree less than n.

Rewrite $P(x)$ in terms of Lagrange polynomials of degree $(n-1)$ with nodes $x_1, x_2, \dots x_n$, the zeros of $P_n(x)$. The error term of this form involves the nth derivative of $P(x)$.

Since $P(x)$ is polynomial of degree $(n-1)$, the error is zero in the Lagrange form [Lagrange interpolation].

So,
$$P(x) = \sum_{i=1}^{n} P(x_i) L_{n-1,i}(x) = \sum_{i=1}^{n} \left(\prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)} \right) P(x_i)$$

and
$$\int_{-1}^{1} P(x)\, dx = \int_{-1}^{1} \left(\sum_{i=1}^{n} \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)} \right) P(x_i)\, dx$$

$$= \sum_{i=1}^{n} \left(\int_{-1}^{1} \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)}\, dx \right) P(x_i)$$

$$= \sum_{i=1}^{n} c_i P(x_i).$$

Hence the result is true for polynomials of degree less than n.

Now consider $P(x)$ to be of degree at least n but less than $2n$. Divide $P(x)$ by the nth Legendre polynomial $P_n(x)$. This gives two polynomials $Q(x)$ and $R(x)$, each of degree less than n, such that
$$P(x) = Q(x) P_n(x) + R(x)$$

Since x_i, $i=1,2,\dots n$ is zero of $P_n(x)$, we have.
$$P(x_i) = Q(x) P_n(x_i) + R(x_i) = R(x_i).$$

Since $P_n(x)$ has the property that it is orthogonal to the polynomials of degree $< n$, we have
$$\int_{-1}^{1} Q(x) P_n(x)\, dx = 0.$$

Also since $R(x)$ is a polynomial of degree less than n, the first case implies that

$$\int_{-1}^{1} R(x)\,dx = \sum_{i=1}^{n} c_i R(x_i)$$

Combining the above two observations, we find

$$\int_{-1}^{1} P(x)\,dx = \int_{-1}^{1} \left(Q(x) P_n(x) + R(x)\right) dx = \int_{-1}^{1} R(x)\,dx = \sum_{i=1}^{n} c_i R(x_i)$$

But $R(x_i) = P(x_i)$, $i = 1, 2 \ldots n$.

we get $$\int_{-1}^{1} P(x)\,dx = \sum_{i=1}^{n} c_i P(x_i)$$

Therefore the quadrature formula is exact for all polynomials of degree less than $2n$.

Table:

n	Roots x_i	Coefficients c_i
2	0.5773502692	1.0000000
	-0.5773502692	1.000000
3	0.7745966692	0.555555556
	0.000000	0.888888889
	-0.7745966692	0.555555556
4	0.8611363116	0.3478548451
	0.3399810436	0.6521451549
	-0.3399810436	0.6521451549
	-0.8611363116	0.3478548451
5	0.9061798459	0.2369268850
	0.5384693101	0.4786286705
	0.000000	0.568888889
	-0.5384693101	0.4786286705
	-0.9061798459	0.2369268850

Gaussian Quadrature on general interval:

we derive Gaussian Quadrature formula on general intervals, say, $[a, b]$.

Consider the map $\phi : [-1, 1] \longrightarrow [a, b]$ given by $\phi(\hat{x}) = \left(\frac{b-a}{2}\right)\hat{x} + \left(\frac{a+b}{2}\right)$, $\hat{x} \in [-1, 1]$

Then
$$\phi(-1) = -\frac{(b-a)}{2} + \frac{a+b}{2} = a$$
$$\phi(1) = \left(\frac{b-a}{2}\right) + \frac{a+b}{2} = b.$$

The points in $[-1, 1]$ are denoted by $\hat{x}$ and the points in $[a, b]$ are denoted by x.

we have the map
$$x = \frac{(b-a)}{2}\hat{x} + \left(\frac{a+b}{2}\right).$$

Then if $f : [a, b] \longrightarrow \mathbb{R}$ be an integrable function on $[a, b]$, then
$$\int_a^b f(x)\,dx = \int_{-1}^{1} f\left(\left(\frac{b-a}{2}\right)\hat{x} + \left(\frac{a+b}{2}\right)\right)\left(\frac{b-a}{2}\right)d\hat{x}.$$

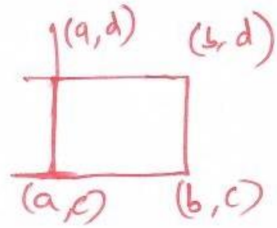
If $g(\hat{x}) = (f \circ \phi)(\hat{x})$, then
$$\int_a^b f(x)\,dx = \int_{-1}^{1} g(\hat{x})\left(\frac{b-a}{2}\right)d\hat{x} = \frac{(b-a)}{2}\int_{-1}^{1} g(\hat{x})\,d\hat{x}.$$

we can apply Gaussian Quadrature formula for $g(x)$ on $[-1, 1]$ to approximate
$$\int_{-1}^{1} f(x)\,dx.$$

Multiple Integrals:

Consider the double integral:

$$\iint_R f(x,y)\,dx\,dy \quad \text{———} \ (*)$$



where $R = \{(x,y) : a \le x \le b,\ c \le y \le d\}$

We write the double integral as an iterated integral.

$$\iint_R f(x,y)\,dx\,dy = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

~~For ex~~ We can extend the numerical quadrature formula derived before to these double integrals.

For example: Trapezoidal can be derived for $(*)$ as follows.

First $\displaystyle\int_c^d f(x,y)\,dy \approx \left(\frac{d-c}{2}\right)\left[f(x,c) + f(x,d)\right]$

Then

$$\int_a^b \left(\int_c^d f(x,y)\,dy \right) dx \approx \left(\frac{d-c}{2}\right) \int_a^b \left[f(x,c) + f(x,d)\right] dx$$

$$\approx \left(\frac{d-c}{2}\right)\left[\frac{(b-a)}{2}\left[f(a,c) + f(b,c)\right] + \frac{(b-a)}{2}\left[f(a,d) + f(b,d)\right] \right]$$

$$= \frac{(b-a)}{2}\frac{(d-c)}{2}\left[f(a,c) + f(b,c) + f(a,d) + f(b,d) \right].$$

Rodrigues formula for Legendre polynomials:

$$P_n(x) = K \frac{d^n}{dx^n}(x^2-1)^n \; ; \quad \left[\begin{array}{l} K \text{ is taken such that} \\ \text{coefficient of } x^n \text{ is} \\ \quad 1. \end{array} \right]$$

Note that

$$\int_{-1}^{1} P_n(x)\, x^m \, dx = K \int_{-1}^{1} \frac{d^n}{dx^n}(x^2-1)^n x^m \, dx = \; ; \quad \text{let } m < n.$$

$$= -K \int_{-1}^{1} \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n (m x^{m-1}) \, dx \quad \left[\text{integration by parts}\right]$$

$$= +(-1)^{n-1} K \int_{-1}^{1} \frac{d}{dx}(x^2-1)^n \frac{d^{n-1}}{dx^{n-1}}(x^m)\, dx$$

$$\int_{-1}^{1} P_n(x) x^m \, dx = (-1)^n K \int_{-1}^{1} (x^2-1)^n \frac{d^n}{dx^n}(x^m)\, dx$$

$$= 0 \quad \text{if } m < n.$$

we have used the fact that the functions

$(x^2-1)^n, \; \frac{d}{dx}(x^2-1)^n, \; \ldots, \; \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n$ are

zero when $x = 1$ (or) $x = -1$.

Leibnitz rule: $$(fg)^{(k)} = \sum_{j=0}^{k} {}^kC_j \, f^{(j)} g^{(k-j)}.$$

For $k \le n-1$; $$\frac{d^k}{dx^k}(x^2-1)^n = \frac{d^k}{dx^k}\left[(x+1)^n (x-1)^n\right]$$

$$= \sum_{j=0}^{k} {}^kC_j \left[(x+1)^n\right]^{(j)} \left[(x-1)^n\right]^{(k-j)}$$

For $k \le n-1$, both $j \le n-1$ & $k-j \le n-1$.

Hence $\left[(x+1)^n\right]^{(j)}$ and $\left[(x-1)^n\right]^{(k-j)}$ have a factor

$(x+1)$ and $(x-1)$ respectively.

There fore

$$\frac{d^k}{dx^k}(x^2-1)^n \quad \text{is zero for } x=1,\ x=-1.$$

when $k \le n-1$.

Leibnitz rule

$$(fg)^n = \sum_{j=0}^{n} nc_j\, f^{(j)}\, g^{(n-j)}$$

$$\frac{d^n}{dx^n}(x^2-1)^n = \frac{d^n}{dx^n}\left[(x+1)^n(x-1)^n\right]$$

$$= \sum_{j=0}^{n} nc_j \left[(x+1)^n\right]^{(j)} \left[(x-1)^n\right]^{(n-j)}$$

$$= \sum_{j=0}^{n} nc_j \frac{n!}{(n-j)!}(x+1)^{n-j} \cdot \frac{n!}{j!}(x-1)^{(j)}$$

Coefficient of x^n is

$$\sum_{j=0}^{n} (nc_j)^2 \cdot n!$$

There foe $k = \dfrac{1}{n! \sum_{j=0}^{n} (nc_j)^2}$.

But $\displaystyle\sum_{j=0}^{n} (nc_j)^2 = \frac{2n!}{(n!)^2}$

$$\therefore\ K = \frac{1}{n!\,\frac{2n!}{(n!)^2}} = \frac{n!}{2n!}$$

$$P_n(x) = \frac{n!}{2n!}\frac{d^n}{dx^n}(x^2-1)^n.$$