The Gauss-Seidel Method: (successive displacements):

For finding solution x of the linear system

$$Ax = b,$$

where $A = [a_{ij}]_{1\le i,j\le n}$, $x = [x_j]_{1\le j\le n}$, $b = [b_j]_{1\le j\le n}$,

rewrite the ith row of the system by

$$x_i = \frac{1}{a_{ii}}\left\{ b_i - \sum_{j=1}^{i-1} a_{ij}x_j - \sum_{j=i+1}^{n} a_{ij}x_j \right\}, \quad i = 1, 2 \ldots n. \quad (1)$$

with $\sum_{j=1}^{0} = \sum_{j=n+1}^{n} = 0$. Assume that $a_{ii} \neq 0$, $i = 1, 2, \ldots n$

For given initial approximation $x^{(0)}$, define

$$x_i^{(m+1)} = \frac{1}{a_{ii}}\left\{ b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(m+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(m)} \right\}, \quad i = 1, 2 \ldots n \quad (2)$$

from (1) & (2), if $e^{(m)} = x - x^{m}$, then

$$e_i^{(m+1)} = -\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} e_j^{(m+1)} - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} e_j^{(m)}, \quad i = 1, 2 \ldots n. \quad (3)$$

Define

$$\alpha_i := \sum_{j=1}^{i-1} \left|\frac{a_{ij}}{a_{ii}}\right|; \quad \beta_i := \sum_{j=i+1}^{n} \left|\frac{a_{ij}}{a_{ii}}\right|, \quad i = 1, 2, \ldots n.$$

with $\alpha_1 = \beta_n = 0$. Define

$$\mu = \max_{1\le i\le n} (\alpha_i + \beta_i).$$

we assume $\mu < 1$. Then define

$$\eta = \max_{1\le i\le n} \frac{\beta_i}{1-\alpha_i}$$

from (3),

$$|e_i^{(m+1)}| \le \alpha_i \|e^{(m+1)}\|_\infty + \beta_i \|e^{(m)}\|_\infty \quad i = 1, 2 \ldots n$$

Let k be the subscript for which

$$\|e^{(m+1)}\|_\infty = |e_k^{(m+1)}|.$$

$$|e_k^{(m+1)}| \le \alpha_k \|e^{(m+1)}\|_\infty + \beta_k \|e^{(m)}\|_\infty.$$

and hence

$$\|e^{(m+1)}\|_\infty \le \alpha_k \|e^{(m+1)}\|_\infty + \beta_k \|e^{(m)}\|_\infty.$$

$$\Rightarrow \quad \|e^{(m+1)}\|_\infty \le \frac{\beta_k}{1-\alpha_k}$$

and thus

$$\|e^{(m+1)}\|_\infty \le \eta \|e^{(m)}\|_\infty.$$

Note that for each i,

$$(\alpha_i+\beta_i) - \frac{\beta_i}{1-\alpha_i} = \frac{(1-\alpha_i)(\alpha_i+\beta_i)-\beta_i}{1-\alpha_i} = \frac{\alpha_i - (\alpha_i+\beta_i)\alpha_i}{1-\alpha_i}$$

$$= \frac{\alpha_i\left(1-(\alpha_i+\beta_i)\right)}{1-\alpha_i}.$$

Since

$$\alpha_i+\beta_i \le \mu, \quad -(\alpha_i+\beta_i) \ge -\mu \to 1-(\alpha_i+\beta_i) \ge 1-\mu$$

$$\therefore \quad (\alpha_i+\beta_i) - \frac{\beta_i}{1-\alpha_i} = \frac{\alpha_i\left(1-(\alpha_i+\beta_i)\right)}{1-\alpha_i} \ge \frac{\alpha_i(1-\mu)}{1-\alpha_i} \ge 0$$

we have

$$(\alpha_i+\beta_i) \ge \frac{\beta_i}{1-\alpha_i}$$

and

$$\eta \le \mu < 1.$$

Therefore $e^{m} \to 0$ as $m \to \infty$, since $\mu < 1$.
We expect the ~~the rate of convergence~~ the Gauss-Seidel method converges faster than Gauss-Jacobi method.

General frame work for iterative methods:

To solve $Ax=b$, form a split of A

$$A = N - P$$

and write $Ax=b$ as $\qquad Nx = b + Px. \quad \text{—} (1)$

The matrix N is chosen in such a way that the linear system $Nz=f$ is easily solvable for any f. For example, N might be diagonal, ~~triangular, or~~ tridiagonal.

Define the iterative method by

$$Nx^{(m+1)} = b + Px^{(m)}, \quad m \geq 0 \quad \text{—} (2)$$

with $x^{(0)}$ given.

Example: (1) Gauss Jacobi method

$N = \text{diag}\,[a_{11}, a_{22}, \dots a_{nn}]$, $P = N - A$.

(2) Gauss Seidel method:

$$N = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ a_{31} & a_{32} & a_{33} \cdots & 0 \\ & & \cdots & a_{nn} \\ a_{n1} & \cdots & \cdots & \end{bmatrix}; \qquad P = N - A.$$

For error analysis, from (1) & (2), for $e = x - x^{n}$

$$Ne^{(m+1)} = Pe^{(m)} \Rightarrow e^{(m+1)} = (N^{-1}P)\,e^{(m)}$$

$e^{(m+1)} = Me^{(m)}$, where $M = N^{-1}P$.

By induction,

$$e^{(n)} = M^n e^{(0)}, \quad n \geq 0.$$

In order that $e^n \to 0$ as $n \to \infty$, it is necessary that $M^n \to 0$ as $n \to \infty$.

<u>Theo:</u> Let A be a square matrix of order n. Then A^m converges to zero matrix as $m \to \infty$, if and only if $r_\sigma(A) < 1$, where

$$r_\sigma(A) = \max_{\lambda \in \sigma(A)} |\lambda|, \quad \sigma(A) = \text{set of all eigenvalues of } A.$$

Therefore the iteration method converges if $r_\sigma(M) < 1$.

Recall the Gauss-Seidel method.

$$x_i^{(m+1)} = \frac{1}{a_{ii}} \left\{ b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(m+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(m)} \right\} \quad — (*)$$

Successive over Relaxation Method : (SOR)

Introduce an acceleration parameter ω, and consider the following modification of $(*)$

$$z_i^{(m+1)} = \frac{1}{a_{ii}} \left\{ b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(m+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(m)} \right\}$$

$$x_i^{(m+1)} = \omega z_i^{(m+1)} + (1-\omega) x_i^{(m)}, \quad i = 1, 2, \ldots n,$$

for $m \geq 0$.

Let us examine this SOR for Gauss-Seidel method.

Decompose the matrix A as

$$A = D + L + U$$

with $D = \text{diag}[a_{11}, a_{22}, \dots a_{nn}]$, L lower triangular, U upper triangular, with both L and U having zeros on the diagonal. This implies

$$z^{(m+1)} = D^{-1}\left[b - Lx^{(m+1)} - Ux^{(m)}\right]$$

$$x^{(m+1)} = \omega z^{(m+1)} + (1-\omega)x^{(m)}, \quad m \geq 0.$$

Eliminating $z^{(m+1)}$ and solving for $x^{(m+1)}$.

$$x^{(m+1)} = \omega\left[D^{-1}(b - Lx^{(m+1)} - Ux^{(m)})\right] + (1-\omega)x^{(m)}$$

$$\left[I + \omega D^{-1}L\right]x^{(m+1)} = \omega D^{-1}b + \left[(1-\omega)I - \omega D^{-1}U\right]x^{(m)}$$

for the error

$$e^{(m+1)} = M(\omega)e^{(m)}, \quad m \geq 0.$$

$$M(\omega) = \left[I + \omega D^{-1}L\right]^{-1}\left[(1-\omega)I - \omega D^{-1}U\right].$$

The parameter ω to be choosen to minimize $r_\sigma(M(\omega))$. In order to obtain convergence It is required that $r_\sigma(M(\omega)) < 1$.