The trapezoidal rule:

Let $y(t)$ be the unique solution of IVP

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha.$$

Integrating the differential equation over $[t_i, t_{i+1}]$ we find

$$\int_{t_i}^{t_{i+1}} y'(s)\,ds = \int_{t_i}^{t_{i+1}} f(s, y(s))\,ds$$

$$y(t_{i+1}) - y(t_i) = \int_{t_i}^{t_{i+1}} f(s, y(s))\,ds.$$

We use the trapezoidal rule to approximate the integral on RHS to find

$$y(t_{i+1}) - y(t_i) \approx \frac{h}{2}\left[f(t_i, y(t_i)) + f(t_{i+1}, y(t_{i+1}))\right]$$

The trapezoidal method:

$$w_0 = \alpha$$

$$w_{i+1} = w_i + \frac{h}{2}\left[f(t_i, w_i) + f(t_{i+1}, w_{i+1})\right], \quad \text{—} (1)$$

for $i = 0, 1, \ldots N-1$.

Note that w_{i+1} appears on both sides of the difference equation, and thus the method is called an implicit method.

Local truncation error:

local truncation error formula is

$$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - \frac{1}{2}\left[f(t_i, y_i) + f(t_{i+1}, y_{i+1})\right]$$

where $y_i = y(t_i)$ and $y_{i+1} = y(t_{i+1})$.

Consider

$$y(t_{i+1}) - y(t_i) - \frac{h}{2}\left[f(t_i, y(t_i)) + f(t_{i+1}, y(t_{i+1}))\right]$$

to examine the local truncation error

$$y(t_{i+1}) - y(t_i) - \frac{h}{2}\left[f(t_i, y(t_i)) + f(t_{i+1}, y(t_{i+1}))\right]$$

$$= y(t_i) + h\,y'(t_i) + \frac{h^2}{2} y''(t_i) + O(h^3)$$ [expanding $y(t_{i+1})$ about $y(t_i)$]

$$- y(t_i) - \frac{h}{2} y'(t_i) - \frac{h}{2} y'(t_{i+1})$$ [$y'(t_i) = f(t_i, y(t_i))$, $y'(t_{i+1}) = f(t_{i+1}, y(t_{i+1}))$]

$$= \frac{h}{2} y'(t_i) - \frac{h}{2} y'(t_{i+1}) + \frac{h^2}{2} y''(t_i) + O(h^3)$$

$$= \frac{h}{2} y'(t_i) - \frac{h}{2}\left[y'(t_i) + h\,y''(t_i) + O(h^2)\right] + \frac{h^2}{2} y''(t_i) + O(h^3)$$

[expanding $y'(t_{i+1})$ in $y'(t_i)$]

$$= O(h^3).$$

Therefore

$$z_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - \frac{1}{2}\left[f(t_i, y_i) + f(t_{i+1}, y_{i+1})\right]$$

$$= O(h^2).$$ [provided y has 3 derivatives]

Local truncation of the trapezoidal rule is of order 2.

Indeed we have that

$$y(t_{i+1}) - y(t_i) - \frac{h}{2}\left[f(t_i, y(t_i)) + f(t_{i+1}, y(t_{i+1}))\right]$$

$$= \frac{h^3}{6} y'''(\xi_i) - \frac{h^3}{4} y'''(\zeta_i)$$

for some ξ_i and ζ_i lie between t_i and t_{i+1}.

If $|y'''(t)| \le \tilde{M}$ for all $t \in [a, b]$, for some $\tilde{M} > 0$, then

$$|z_{i+1}(h)| \le h^2 \frac{\tilde{M} \cdot 5}{12} = M h^2;$$ where $M = \frac{\tilde{M} \cdot 5}{12}$.

Theorem: Suppose $y(t)$ be the unique solution of IVP

$$y'(t) = f(t, y(t)), \quad a \le t \le b, \quad y(a) = \alpha.$$

Assume that $y \in C^3[a,b]$ and f satisfies a Lipschitz condition on $D = \{(t,y): a \le t \le b, -\infty < y < \infty\}$ with Lipschitz constant L. Let $y_i = y(t_i)$, $i = 0, 1, \ldots N$.

If $w_0, w_1, \ldots w_N$ denote approximations obtained by the trapezoidal rule, then, if $hL < 2$, we have

$$|y_i - w_i| \le \frac{ch^2}{L}\left[\exp\left(\frac{(t_{i+1}-a)L}{1-\frac{hL}{2}}\right) - 1\right]; \quad \text{(some constant } c \text{ depending on } y'''(t))$$

Proof: From the local truncation error analysis

$$y(t_{i+1}) = y(t_i) + \tfrac{1}{2}h\left[f(t_i, y(t_i)) + f(t_{i+1}, y(t_{i+1}))\right] + O(h^3).$$

we have $\quad w_0 = \alpha, \quad w_{i+1} = w_i + \frac{h}{2}\left[f(t_i, w_i) + f(t_{i+1}, w_{i+1})\right].$

Denote $e_i = y_i - w_i$, $i = 0, 1, \ldots N$. Then

$$e_{i+1} = e_i + \tfrac{1}{2}h\left[f(t_i, y_i) - f(t_i, w_i)\right] + \tfrac{1}{2}h\left[f(t_{i+1}, y_{i+1}) - f(t_{i+1}, w_{i+1})\right] + O(h^3)$$

Therefore

$$|e_{i+1}| \le |e_i| + \frac{hL}{2}|e_i| + \frac{hL}{2}|e_{i+1}| + ch^3.$$

This implies

$$|e_{i+1}| \le \left(\frac{1+\frac{hL}{2}}{1-\frac{hL}{2}}\right)|e_i| + \left(\frac{c}{1-\frac{hL}{2}}\right)h^3.$$

Note that since $hL < 2 \Rightarrow \frac{hL}{2} < 1$

$$\Rightarrow 1 - \frac{hL}{2} > 0.$$

Remarks: Here c is $\frac{5}{12}M$, $\quad M = \max_{a \le t \le b}|y'''(t)|.$

since

$$\frac{1+\frac{hL}{2}}{1-\frac{hL}{2}} = 1+\frac{hL}{\left(1-\frac{hL}{2}\right)} = 1+s, \qquad s=\frac{hL}{\left(1-\frac{hL}{2}\right)}$$

$$|e_{i+1}| \le (1+s)\,|e_i| + \left(\frac{c}{1-\frac{hL}{2}}\right)h^3 .$$

Using the Lemma, that if

$$a_{i+1} \le (1+s)\,a_i + t$$

$$\text{then} \quad a_{i+1} \le e^{(i+1)s}\left(a_0+\frac{t}{s}\right)-\frac{t}{s},$$

we find

$$|e_{i+1}| \le e^{(i+1)s}\left(|e_0| + \frac{ch^3}{\left(1-\frac{hL}{2}\right)}\cdot\frac{1}{s}\right) - \frac{ch^3}{\left(1-\frac{hL}{2}\right)s} .$$

since $y_0 = w_0 = \alpha$, $|e_0| = 0$.

and
$$\frac{ch^3}{\left(1-\frac{hL}{2}\right)s} = \frac{ch^3}{\left(1-\frac{hL}{2}\right)\left(\frac{hL}{\left(1-\frac{hL}{2}\right)}\right)} = \frac{ch^3}{hL} = \frac{ch^2}{L} .$$

$$(i+1)s = \frac{(i+1)hL}{\left(1-\frac{hL}{2}\right)} = \frac{(t_{i+1}-a)L}{\left(1-\frac{hL}{2}\right)} .$$

Therefore

$$|e_{i+1}| \le \left[\exp\left(\frac{(t_{i+1}-a)L}{\left(1-\frac{hL}{2}\right)}\right) - 1\right]\frac{ch^2}{L} .$$

Thus the global error in the trapezoidal rule is of order 2.

Fixed point iteration for the trapezoidal rule:

Recall the trapezoidal rule

$$w_0 = \alpha$$

$$w_{i+1} = w_i + \frac{h}{2}\left[f(t_i, w_i) + f(t_{i+1}, w_{i+1})\right]$$

$i = 0, 1, \ldots N-1.$

Since f can be nonlinear in w, solving for w_{i+1} requires a fixed point iteration.

If we denote $w_{i+1}^{(k)}$ $(k \geq 1)$ be the solution of

$$w_{i+1}^{(k)} = w_i + \frac{h}{2}\left[f(t_i, w_i) + f(t_{i+1}, w_{i+1}^{(k-1)})\right]$$

where w_i is known to be computed, and $w_{i+1}^{(0)}$ is initial approximation of w_{i+1}. $[w_{i+1}^{(0)}$ may be taken as $w_i]$.

Note that if $\frac{hL}{2} < 1$, we have

$$w_{i+1}^{(k+1)} - w_{i+1}^{(k)} = \frac{h}{2}\left[f(t_{i+1}, w_{i+1}^{(k)}) - f(t_{i+1}, w_{i+1}^{(k-1)})\right]$$

$$\left|w_{i+1}^{(k+1)} - w_{i+1}^{(k)}\right| \leq \frac{hL}{2}\left|w_{i+1}^{(k)} - w_{i+1}^{(k-1)}\right|$$

The fixed point iteration converges.

Consider IVP:

$$y'(t) = f(t, y(t)), \quad a \le t \le b, \quad y(a) = \alpha.$$

Using Taylor's theorem

$$y(t_i) = y(t_{i+1}) + (t_i - t_{i+1})\, y'(t_{i+1}) + \frac{(t_i - t_{i+1})^2}{2} y''(\xi_i)$$

where ξ_i lies between t_i and t_{i+1}.

Dropping the remainder term, we get backward Euler's method:

$$w_0 = \alpha$$

$$w_{i+1} = w_i + h f(t_{i+1}, w_{i+1}), \quad i = 0, 1, \ldots N-1.$$

It can be easily seen that local truncation error for the backward Euler's method is $O(h)$.

Recall $h = \frac{t_{i+1} - t_i}{h} = \frac{b-a}{N}$. Assume that $0 < hL < 1$.

We have, if $e_i = y_i - w_i$, then

$$e_{i+1} = e_i + h\big(f(t_{i+1}, y_{i+1}) - f(t_{i+1}, w_{i+1})\big) + O(h^2)$$

$$|e_{i+1}| \le |e_i| + hL\,|e_{i+1}| + ch^2. \qquad c = \frac{M}{2},\ M = \max_{t \in [a,b]} |y''(t)|.$$

$$\Rightarrow \quad |e_{i+1}| \le \left(\frac{1}{1-hL}\right)|e_i| + \left(\frac{c}{1-hL}\right)h^2$$

But $\frac{1}{1-hL} = 1 + \frac{hL}{1-hL} = 1 + s, \quad s = \frac{hL}{1-hL}.$

Applying the lemma on (4/10).

$$|e_{i+1}| \le e^{(i+1)s}\left(|e_0| + \left(\frac{ch^2}{1-hL}\right)\cdot\frac{1}{s}\right) - \frac{ch^2}{(1-hL)s}$$

Since $y_0 = w_0 = \alpha$; $e_0 = 0$.

$$(i+1)s = \frac{(i+1)hL}{1-hL} = \frac{(t_{i+1} - a)L}{1-hL}; \qquad \frac{ch^2}{(1-hL)s} = \frac{ch}{L} = \frac{Mh}{2L}.$$

$$\therefore \quad |e_{i+1}| \le \frac{Mh}{2L}\left(\exp\left(\frac{(t_{i+1} - a)L}{1-hL}\right) - 1\right); \quad \boxed{hL < 1}$$