Q) the equation $y' = -2y$, $y(0) = 1 = \alpha$. [Stability]

$y(t) = e^{-2t}$

Using $w_0 = \alpha$, $w_1 = \alpha_1$

$w_{i+1} = w_{i-1} + 2h f(t_i, w_i)$

$i = 1, 2, \ldots N-1$.

we see that $w_{i+1} = w_{i-1} + 2h(-2w_i)$

$\Rightarrow w_{i+1} + 4h w_i - w_{i-1} = 0$.

If $w_i = \beta^i$, then

$\beta^{i+1} + 4h\beta^i - \beta^{i-1} = 0 \Rightarrow \beta^{i-1}\left[\beta^2 + 4h\beta - 1\right] = 0$

$$\beta = \frac{-4h \pm \sqrt{16h^2 + 4}}{2} = -2h \pm \sqrt{1+4h^2}$$

ω $\beta_1 = -2h + \sqrt{1+4h^2} = 1 - 2h + O(h^2)$

$\beta_2 = -(2h+1) + O(h^2) = -2h - \sqrt{1+4h^2}$

Hence $w_n = c_1\beta_1^n + c_2(-1)^n\left[(1+2h) + O(h^2)\right]^n$.

$w_n = c_1\beta_1^n + c_2(-1)^n\beta_2^n$.

$\lim_{h\to 0} \beta_1^n = e^{-2t_n}$; $\lim_{h\to 0} \beta_2^n = e^{2t_n}$.

as $h \to 0$ $w_n \longrightarrow c_1 e^{-2t_n} + c_2(-1)^n e^{2t_n}$.

Considering $w_n = c_1\beta_1^n + c_2(-1)^n\beta_2^n$

$w_0 = c_1 + c_2$; $w_1 = c_1\beta_1 - c_2\beta_2$

$c_1 + c_2 = \alpha$; $c_1\beta_1 - c_2\beta_2 = \alpha_1$

$c_2 = \alpha - c_1$; $c_1\beta_1 - (\alpha - c_1)\beta_2 = \alpha_1$

$c_1(\beta_1 + \beta_2) = \alpha_1 + \alpha\beta_2$

$$c_1 = \frac{\alpha_1 + \alpha\beta_2}{\beta_1 + \beta_2}$$

$$c_2 = \alpha - c_1 = \alpha - \frac{\alpha_1 + \alpha\beta_2}{\beta_1 + \beta_2}$$

$$= \frac{\alpha\beta_1 + \cancel{\alpha\beta_2} - \alpha_1 - \cancel{\alpha\beta_2}}{\beta_1 + \beta_2} = \frac{\beta_1 - \alpha_1}{\beta_1 + \beta_2}.$$

$$\left(\frac{\quad}{\beta_1+\beta_2}\right)\beta_1^n + \left(\frac{\quad}{\beta_1+\beta_2}\right)\beta_2^n(-1)^n.$$

Our initial approximations are $\omega_0 = \alpha_0$ and ω_1

$$c_1\beta_1 - c_2\beta_2 = \alpha_1 \qquad [\alpha = 1]$$

$$\left(\frac{\alpha_1+\beta_2}{\beta_1+\beta_2}\right)\beta_1 - \left(\frac{\beta_1-\alpha_1}{\beta_1+\beta_2}\right)\beta_2 = \alpha_1$$

$$(\alpha_1+\beta_2)\beta_1 - (\beta_1-\alpha_1)\beta_2 = \alpha_1(\beta_1+\beta_2)$$

~~$c_1 \approx 1$; $c_2 \approx 0$~~

if $\alpha_1 = \beta_1$ then $c_1 = 1$;

and $c_2 = 0$.

If the approximation $\alpha_1 \neq \beta_1$, then the error introduced at $t = t_1$ has the influence on the solution for $t > t_1$.

Since $w_n \to c_1 e^{-2t_n} + c_2(-1)^n e^{2t_n}$

but $y_n = e^{-2t_n}$

So we must have $c_1 = 1$

and $c_2 = 0$.

the multi step method

$$w_0 = \alpha,\ w_1 = \alpha_1,\ w_2 = \alpha_2,\ \cdots\ w_{m-1} = \alpha_{m-1}$$

$$w_{i+1} = a_{m-1} w_i + a_{m-2} w_{i-1} + \cdots + a_0 w_{i+1-m} + h f(t_i, h, w_{i+1}, \ldots w_{i+1-m}) \quad -(*)$$

the characteristic polynomial is defined by

$$P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - \cdots - a_1\lambda - a_0.$$

The stability of multi step method with respect to the round-off errors is dictated by the magnitude of the zeros of the characteristic polynomial.

To see this consider simple model problem

$$y' = 0,\ y(a) = \alpha, \quad \text{where } \alpha \neq 0. \quad -(1)$$

This problem has exact solution $y(t) = \alpha$.

from $y(t_{i+1}) = y(t_i) + \int_{t_i}^{t_{i+1}} f(t, y(t))\,dt$ (or)

$$y(t_{i+1}) = y(t_i) + \int_{t_i}^{t_{i+1}} P(t)\,dt$$

In theory any multi step method will yield exact approximate solution $w_n = \alpha$ for all n. The only deviation from exact solution is due to the round off error of the method.

for (1). the multi-step method is of the form

$$w_{i+1} = a_{m-1} w_i + a_{m-2} w_i + \cdots + a_0 w_{i+1-m} \quad -(2)$$

and $P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - \cdots - a_1\lambda - a_0.$

Suppose λ is one of zeros of the characteristic polynomial $P(\lambda)$.

Then $w_n = \lambda^n$, for each n, is a solution of (2) since

$$\lambda^{i+1} - a_{m-1}\lambda^i - a_{m-2}\lambda^{i-1} - \cdots - a_0\lambda^{i+1-m}$$

$$= \lambda^{i+1-m}\left[\lambda^m - a_{m-1}\lambda^{m-1} - \cdots - a_0\right] = 0.$$

In fact if $\lambda_1, \lambda_2, \ldots \lambda_m$ are distinct zeros of the characteristic polynomial for (*), it can be seen that every solution of (2) can be expressed in the form

$$w_n = \sum_{i=1}^{m} c_i \lambda_i^n \quad -(3)$$

for some unique ...

Since the exact solution of ① is $y(t)=\alpha$, the choice $w_n=\alpha$ for all n, is a solution to ②

Using this fact in ② gives

$$0=\alpha-\alpha a_{m-1}-\alpha a_{m-2}-\cdots-\alpha a_0=\alpha\left[1-a_{m-1}-a_{m-2}-\cdots-a_0\right].$$

This implies that $\lambda=1$ is one of the zeros of the characteristic polynomial. We will assume that in the representation ② this solution is described by $\lambda_1=1$ and $c_1=\alpha$, so all solutions to ① are expressed as

$$w_n=\alpha+\sum_{i=2}^{m} c_i \lambda_i^n.$$

If all the calculations were exact, all the constants $c_2, c_3, \ldots c_m$ would be zero. In practice the constants $c_2, c_3 \ldots c_m$ are not zero due to roundoff error. In fact the round off ~~errors~~ error grows exponentially unless $|\lambda_i|\le 1$ for each of the roots $\lambda_2, \lambda_3, \ldots \lambda_m$.