UMC-202 Problems Sheet Fall 2024

1. For each of the following Adams-Moulton implicit methods, derive the formula for the method and the form of their local truncation errors (see next page).

   - two step Adams-Moulton implicit method.
   - three step Adams-Moulton implicit method.
   - four step Adams-Moulton implicit method.

2. For given α amd α_1, consider the following explicit difference equation method

$$\begin{aligned} w_0 &= \alpha,\ w_1 = \alpha_1, \\ w_{i+2} &= -3w_i + 4w_{i+1} - 2hf(t_i, w_i), \end{aligned}$$

   for $i = 0, 1, \ldots, N-2$, where the uniform step size $h = (b-a)/N$, $t_i = a + ih$, $i = 0, 1, \ldots, N$, for the IVP

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha.$$

   Determine the local truncation error formula for the above difference equation.
   Hint: Use the numerical difference formula $y'(a) \approx \frac{-3y(a)+4y(a+h)-y(a+2h)}{2h}$.

3. Suppose $f = f(t, y)$ is a C^1 function (in each variable) on

$$D = \{(t, y) : a \le t \le b \text{ and } -\infty < y < \infty\}$$

   such that

$$\left|\frac{\partial f}{\partial y}(t, y)\right| \le L \text{ for all } t \in [a, b] \text{ and for all } y \in \mathbb{R}.$$

   Let $y(t)$ be the unique solution of the initial value problem(IVP):

$$y'(t) = f(t, y), \quad y(a) = \alpha,$$

   satisfying

$$|y''(t)| \le M \text{ for all } t \in [a, b].$$

   Let N be a positive integer and $t_i = a + ih$, for each $i = 0, 1, \ldots N$, with $h = (b-a)/N$. If $w_0, w_1, \ldots, w_N$ be the approximations of $y(t_i)$, generated by the backward Euler's method for each $i = 0, 1, \ldots N$, by,

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + h\, f(t_{i+1}, w_{i+1}). \end{aligned}$$

   Show, if $hL < 1$, that

$$|y(t_i) - w_i| \le \frac{hM}{2L}\left[\exp\left(\frac{L(t_{i+1} - a)}{1 - hL}\right) - 1\right].$$

Adams-Moulton Implicit Methods

Implicit methods are derived by using $(t_{i+1}, f(t_{i+1}, y(t_{i+1})))$ as an additional interpolation node in the approximation of the integral

$$\int_{t_i}^{t_{i+1}} f(t, y(t))\, dt.$$

Some of the more common implicit methods are as follows.

Adams-Moulton Two-Step Implicit Method

$$w_0 = \alpha, \quad w_1 = \alpha_1,$$
$$w_{i+1} = w_i + \frac{h}{12}[5f(t_{i+1}, w_{i+1}) + 8f(t_i, w_i) - f(t_{i-1}, w_{i-1})], \tag{5.37}$$

where $i = 1, 2, \ldots, N-1$. The local truncation error is $\tau_{i+1}(h) = -\frac{1}{24}y^{(4)}(\mu_i)h^3$, for some $\mu_i \in (t_{i-1}, t_{i+1})$.

Adams-Moulton Three-Step Implicit Method

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \tag{5.38}$$
$$w_{i+1} = w_i + \frac{h}{24}[9f(t_{i+1}, w_{i+1}) + 19f(t_i, w_i) - 5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})],$$

where $i = 2, 3, \ldots, N-1$. The local truncation error is $\tau_{i+1}(h) = -\frac{19}{720}y^{(5)}(\mu_i)h^4$, for some $\mu_i \in (t_{i-2}, t_{i+1})$.

Adams-Moulton Four-Step Implicit Method

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \quad w_3 = \alpha_3,$$
$$w_{i+1} = w_i + \frac{h}{720}[251f(t_{i+1}, w_{i+1}) + 646f(t_i, w_i) \tag{5.39}$$
$$- 264f(t_{i-1}, w_{i-1}) + 106f(t_{i-2}, w_{i-2}) - 19f(t_{i-3}, w_{i-3})],$$

where $i = 3, 4, \ldots, N-1$. The local truncation error is $\tau_{i+1}(h) = -\frac{3}{160}y^{(6)}(\mu_i)h^5$, for some $\mu_i \in (t_{i-3}, t_{i+1})$.