Consider IVP:

$$y' = f(t, y), \quad y(t_0) = y_0. \quad \text{—— ①}$$

where f is any continuous real-valued function defined on some rectangle

$$R: \ |t - t_0| \le a, \quad |y - y_0| \le b, \quad (a, b > 0).$$

Our object is to ~~find~~ show that on some interval I containing t_0, there is a solution ϕ of ① satisfying $\phi(t_0) = y_0$, $(t, \phi(t)) \in R$ for all $t \in I$ and $\phi'(t) = f(t, \phi(t))$ for all $t \in I$.

Note that ① is equivalent to [we prove this fact] an integral equation

$$y(t) = y_0 + \int_{t_0}^{t} f(x, y(x))\, dx \quad \text{— ②}$$

on I. By a solution of ② on I means a real valued continuous function ϕ on I such that $(t, \phi(t)) \in R$ for all $t \in I$, and

$$\phi(t) = y_0 + \int_{t_0}^{t} f(x, \phi(x))\, dx$$

for all $t \in I$.

<u>Theorem:</u> A function ϕ is a solution of ① on an interval I if and only if it is a solution of ② on I.

<u>proof:</u> Suppose ϕ is a solution of ① on I. Then

$$\phi'(t) = f(t, \phi(t)) \quad \text{for all } t \in I.$$

Since ϕ is continuous on I, and f is continuous on R the function $F(t) = f(t, \phi(t))$ is continuous on I.

Integrating, we find

$$\phi(t) = \phi(t_0) + \int_{t_0}^{t} f(x, \phi(x))\, dx$$

since $\phi(t_0) = y_0$, we get that ϕ is a solution of (2).

Conversely suppose that ϕ is a solution of (2) on I. By fundamental theorem of calculus, by differentiating, we get

$$\phi'(t) = f(t, \phi(t)) \quad \text{for all } t \in I.$$

Also note that $\phi(t_0) = y_0$. Hence ϕ is a solution of (1).

---

we will find solutions of the equation (2).

since ϕ appears on both sides of (2), we may perform some iteration, by ~~starting~~ ~~[illegible]~~ starting with $\phi_0(t) = y_0$.

Define

$$\phi_1(t) = y_0 + \int_{t_0}^{t} f(x, \phi_0(x))\, dx.$$

and in general

$$\phi_{k+1}(t) = y_0 + \int_{t_0}^{t} f(x, \phi_k(x))\, dx, \quad k \geq 0. \quad \text{—(3)}$$

we may generate a sequence $\{\phi_k(t)\}$ and hope that ϕ_k converges to some function ϕ, which is a solution of (2).

ϕ_k are called successive approximations.

Theorem: The successive approximations ϕ_k, defined by (3), exist as continuous functions on

$$I: |t - t_0| \le \alpha; \quad \alpha = \text{minimum}\{a, b/M\},$$

where $M>0$ is such that $|f(t,y)| \le M$ for all $(t,y) \in R$, and $(t, \phi_k(t)) \in R$ for all $t \in I$. Indeed ϕ_k satisfy

$$(*) \text{ --- } |\phi_k(t) - y_0| \le M|t - t_0| \quad \text{for all } t \in I.$$

Proof: Note that the inequality

$$|\phi_k(t) - y_0| \le M\,|t - t_0|, \quad t \in I$$

implies that

$$|\phi_k(t) - y_0| \le M \cdot \alpha \le M \cdot b/M = b.$$

and $(t, \phi_k(t)) \in R$ for all $t \in I$.

we need to show (*). for $k=0$, $\phi_0(t) = y_0$.

(*) holds true. For $k=1$,

$$\phi_1(t) - y_0 = \int_{t_0}^{t} f(x, \phi_0(x))\,dx$$

$$\Rightarrow |\phi_1(t) - y_0| = \left| \int_{t_0}^{t} f(x, \phi_0(x))\,dx \right| \le \int_{t_0}^{t} |f(x, \phi_0(x))|\,dx$$

$$\le M|t - t_0|.$$

(*) holds for $k=1$.

Now for Induction, suppose that (*) holds for $k=m$.

Then consider

$$\phi_{m+1}(t) - y_0 = \int_{t_0}^{t} f(x, \phi_m(x))\,dx$$

$$\rightarrow |\phi_{m+1}(t) - y_0| \le M\,|t - t_0| \quad \text{for all } t \in I.$$

Hence (*) holds for $k = m+1$. Hence holds for all k.

Theorem: (Existence theorem): Let f be a continuous real-valued function on the rectangle

$$R: \ |t-t_0| \le a, \quad |y-y_0| \le b, \qquad (a, b > 0).$$

and let

$$|f(t,y)| \le M \quad \text{for all} \quad (t,y) \in R.$$

Further suppose f satisfies a Lipschitz condition with constant K in R. Then the successive approximations

$$\phi_0(t) = y_0, \quad \phi_{k+1}(t) = y_0 + \int_{t_0}^{t} f(x, \phi_k(x))\, dx, \quad k \ge 0$$

converge on the interval

$$I: \ |t-t_0| \le \alpha = \min\{a, b/M\}$$

to a solution ϕ of the IVP $y' = f(t,y)$, $y(t_0) = y_0$ on I.

Proof: By previous theorem, we know that each $\phi_k(t)$ is defined on I, and $(t, \phi(t)) \in R$ for all $t \in I$. write ϕ_k as

$$\phi_k = \phi_0 + (\phi_1 - \phi_0) + (\phi_2 - \phi_1) + \cdots + (\phi_k - \phi_{k-1})$$

~~and hence~~ (or)

$$\phi_k(t) = \phi_0(t) + \sum_{p=1}^{k} (\phi_p(t) - \phi_{p-1}(t)).$$

which is a partial sum of the series

$$\phi_0(t) + \sum_{p=1}^{\infty} [\phi_p(t) - \phi_{p-1}(t)] \quad \text{--- (4)}$$

The convergence of $\{\phi_k\}$ is equivalent to the convergence of the series.

Note that

$$|\phi_1(t) - \phi_0(t)| = |\phi_1(t) - y_0| \leq M\,|t - t_0|$$

for all $t \in I$, by previous theorem.

From the definition of ϕ_2 and ϕ_1, we find

$$\phi_2(t) - \phi_1(t) = \int_{t_0}^{t} [f(x, \phi_1(x)) - f(x, \phi_0(x))]\,dx.$$

Therefore

$$|\phi_2(t) - \phi_1(t)| \leq \int_{t_0}^{t} |f(x, \phi_1(x)) - f(x, \phi_0(x))|\,dx$$

$$\leq K \int_{t_0}^{t} |\phi_1(x) - \phi_0(x)|\,dx$$

$$\leq KM \int_{t_0}^{t} |x - t_0|\,dx = KM \frac{|t-t_0|^2}{2}.$$

We can prove by induction that

$$|\phi_p(t) - \phi_{p-1}(t)| \leq \frac{MK^{p-1}|t-t_0|^p}{p!} \quad \text{--- } (*)$$

From $(*)$, we see that the series

$$\phi_0(t) + \sum_{p=1}^{\infty} [\phi_p(t) - \phi_{p-1}(t)] \quad \text{--- } (4)$$

is absolutely convergent, since from $(*)$.

$$|\phi_p(t) - \phi_{p-1}(t)| \leq \left(\frac{M}{K}\right) \frac{K^p|t-t_0|^p}{p!} \quad \text{--- } (+)$$

the RHS of $(+)$ is the p^{th} term in the power series for expansion of $e^{K|t-t_0|}$. Since $e^{K|t-t_0|}$ is convergent, the series (4) is convergent on I. Therefore the k-th partial sum of (4), which is $\phi_k(t)$, converges to a limit $\phi(t)$ as $k \to \infty$ for each $t \in I$.

properties of the limit ϕ:

(1) ϕ is continuous on I.

proof. for t_1, t_2 in I.

$$|\varphi_{k+1}(t_1) - \varphi_{k+1}(t_2)| = \left| \int_{t_1}^{t_2} f(x, \phi_k(x))\, dx \right|$$

$$\leq M|t_1 - t_2|$$

which implies, by letting $k \to \infty$,

$$|\phi(t_1) - \phi(t_2)| \leq M|t_1 - t_2| \quad \text{—} (5)$$

This shows that ϕ is continuous on I.

Also letting $t_1 = t$, $t_2 = t_0$ in (5), we get

$$|\phi(t) - \phi(t_0)| \leq M|t - t_0|$$

$$\Rightarrow |\phi(t) - y_0| \leq M|t - t_0| \leq b.$$

which implies $(t, \varphi(t)) \in R$ for all $t \in I$.

(2) estimates for $|\phi(t) - \varphi_k(t)|$. since

$$\phi(t) = \varphi_0(t) + \sum_{p=1}^{\infty} [\varphi_p(t) - \varphi_{p-1}(t)]$$

and

$$\varphi_k(t) = \varphi_0(t) + \sum_{p=1}^{k} [\varphi_p(t) - \varphi_{p-1}(t)],$$

we see that

$$\phi(t) - \varphi_k(t) = \sum_{p=k+1}^{\infty} [\varphi_p(t) - \varphi_{p-1}(t)]$$

and $$|\phi(t) - \varphi_k(t)| \leq \sum_{p=k+1}^{\infty} |\varphi_p(t) - \varphi_{p-1}(t)|.$$

using (4), $$|\varphi(t) - \varphi_k(t)| \leq \frac{M}{K} \sum_{p=k+1}^{\infty} \frac{(K\alpha)^p}{p!}$$

$$= \frac{M}{K} \frac{(K\alpha)^{k+1}}{(k+1)!} \sum_{p=0}^{\infty} \frac{(K\alpha)^p}{p!}$$

$$= \frac{M}{K} \frac{(K\alpha)^{k+1}}{(k+1)!} e^{K\alpha}.$$

we see that $\dfrac{(K\alpha)^{k+1}}{(k+1)!} \longrightarrow 0$ as $k \longrightarrow \infty$.

Let $\epsilon_k = \dfrac{(K\alpha)^{k+1}}{(k+1)!}$. Then $\epsilon_k \longrightarrow 0$ as $k \longrightarrow \infty$.

(3) The limit ϕ is a solution: we must show that ϕ satisfies

$$\phi(t) = y_0 + \int_{t_0}^{t} f(s, \phi(s))\, ds.$$

for all $t \in I$. Since ϕ is continuous on I, f is continuous on R, we see that $F(t) = f(t, \phi(t))$ is continuous on I. Now

$$\phi_{k+1}(t) = y_0 + \int_{t_0}^{t} f(s, \phi_k(s))\, ds$$

and $\phi_{k+1}(t) \longrightarrow \phi(t)$, as $k \longrightarrow \infty$. We must show that $\int_{t_0}^{t} f(s, \phi_k(s))\, ds \longrightarrow \int_{t_0}^{t} f(s, \phi(s))\, ds$

for each t in I.

we have

$$\left| \int_{t_0}^{t} f(s, \phi(s))\, ds - \int_{t_0}^{t} f(s, \phi_k(s))\, ds \right|$$

$$\leq \int_{t_0}^{t} | f(s, \phi(s)) - f(s, \phi_k(s)) |\, ds$$

$$\leq K \int_{t_0}^{t} |\phi(s) - \phi_k(s)|\, ds$$

$$\leq K e^{K\alpha} \epsilon_k |t - t_0| \longrightarrow 0$$

as $k \longrightarrow \infty$.

Hence $\phi(t)$ is a solution of 2 on I.

Uniqueness: suppose ϕ and ψ satisfy

$$\phi(t) = y_0 + \int_{t_0}^{t} f(x, \phi(x))\,dx, \qquad \psi(t) = y_0 + \int_{t_0}^{t} f(x, \psi(x))\,dx$$

Then
$$\phi(t) - \psi(t) = \int_{t_0}^{t} [f(x, \phi(x)) - f(x, \psi(x))]\,dx$$

$$|\phi(t) - \psi(t)| \le K \int_{t_0}^{t} |\phi(x) - \psi(x)|\,dx.$$

Let
$$E(t) = \int_{t_0}^{t} |\phi(x) - \psi(x)|\,dx.$$

Then
$$E'(t) \le K E(t) \quad \text{and} \quad E(t_0) = 0.$$

$$\Rightarrow \quad E'(t) - K E(t) \le 0.$$

$$\left[e^{-K(t-t_0)} E(t)\right]'(t) = e^{-K(t-t_0)} E'(t) - K e^{-K(t-t_0)} E(t)$$

$$= e^{-K(t-t_0)} [E'(t) - K E(t)] \le 0.$$

Integrating from t_0 to t.

$$e^{-K(t-t_0)} E(t) - e^{-K(t_0-t_0)} E(t_0) \le 0.$$

$$\Rightarrow \quad E(t) \le 0.$$

But $E(t) \ge 0 \Rightarrow E(t) \equiv 0, \ t \in I.$

$$\therefore \quad \phi(t) \equiv \psi(t).$$