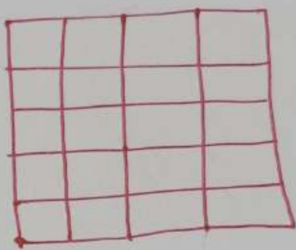
Consider
$$\left.\begin{aligned} -\Delta u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}\right] \text{(BVP)}$$

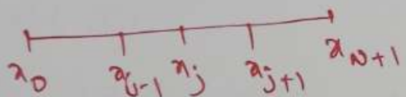
$\Omega \subset \mathbb{R}^2$ is a bounded domain with boundary $\partial\Omega$.

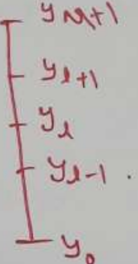
$$+\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad (\Delta \text{ - Laplace operator})$$

(BVP) is known as Dirichlet boundary value problem for Poisson equation.



Assume that $\Omega = [0,1] \times [0,1]$.

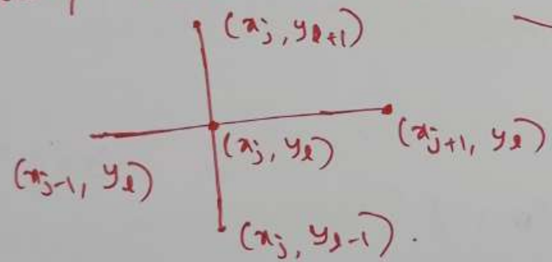




N, M integers. $N, M > 0$.

For simplicity take $N = M$. $\quad h = x_j - x_{j-1} = y_l - y_{l-1}$.

$\longrightarrow$ $x_j = jh$, $j = 0, \ldots, N+1$
$y_l = lh$, $l = 0, \ldots N+1$.



For any function v, denote $v_{j,l} = v(x_j, y_l)$.

Note that
$$\frac{\partial^2 u}{\partial x^2}(x_j, y_l) = \frac{u_{j+1,l} - 2u_{j,l} + u_{j-1,l}}{h^2} + O(h^2)$$

$$\frac{\partial^2 u}{\partial y^2}(x_j, y_l) = \frac{u_{j,l+1} - 2u_{j,l} + u_{j,l-1}}{h^2} + O(h^2).$$

An FDM for BVP is derived by using finite difference approximations of the second order derivatives (by dropping remainder terms of $O(h^2)$), and equating the PDE in BVP at each (x_j, y_l): $1 \le j \le N$, $1 \le l \le N$.

$$-\Delta u(x_j, y_l) = f(x_j, y_l) \qquad 1 \le j \le N, \quad 1 \le l \le N.$$ (2)

$$-\frac{u_{j+1,l} - 2u_{j,l} + u_{j-1,l}}{h^2} - \frac{u_{j,l+1} - 2u_{j,l} + u_{j,l-1}}{h^2} = f_{j,l}.$$

$1 \le j \le N$
$1 \le l \le N$.

where $u_{j,l} \simeq u(x_j, y_l)$.

Simplifying, we get

$$u_{j+1,l} + u_{j-1,l} - 4u_{j,l} + u_{j,l+1} + u_{j,l-1} + h^2 f_{j,l} = 0. \quad \text{—(1)}$$

$1 \le j \le N$
$1 \le l \le N$.

Note that from boundary condition $u = 0$ on $\partial\Omega$.

$u(x,y) = 0, \quad 0 \le x \le 1, \ y = 0, \ y = 1$
$u(x,y) = 0, \quad 0 \le y \le 1, \ x = 0, \ x = 1.$

This implies:

$$\left.\begin{aligned} u_{0,l} &= u_{N+1,l} = 0 \quad \text{for} \quad 0 \le l \le N+1. \\ \text{and} \quad u_{j,0} &= u_{j,N+1} = 0 \quad \text{for} \quad 0 \le j \le N+1. \end{aligned}\right\} \text{—(2)}$$

The FDM consists of finding $u_{j,l}$, $1 \le j, l \le N$.
from (1) & (2).

1.

Finite element method:

Consider (BVP):

$$-\Delta u = f \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega.$$

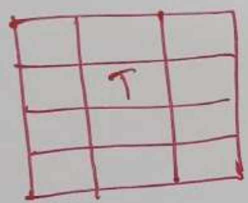
Ω is assumed to be a polygonal domain with boundary $\partial\Omega$.

Examples of Ω:

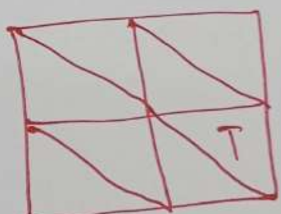
Step 1: Partition of Ω into rectangles (or) triangles:

$\mathcal{T}_h$ = partition of Ω.

$\mathcal{T}_h = \{T : T \subset \Omega\}$



(or)



~~Lowest~~

Step 2: Lowest order finite element space

[we can have higher order as well].

$$V_h = \{ v \in C^0(\bar{\Omega}) : v|_T \in R_1(T),\ \forall T \in \mathcal{T}_h,\ v_h = 0 \text{ on } \partial\Omega \}.$$

where $R_1(T)$ can be either $P_1(T)$ (if T is triangle) or $Q_1(T)$ (if T is a rectangle).

$$P_1(T) = \text{span}\{1, x, y\} = \{a_0 + a_1 x + a_2 y : a_j \in \mathbb{R},\ j = 0,1,2;\ x, y \in T\}.$$

$$Q_1(T) = \text{span}\{1, x, y, xy\} = \{a_0 + a_1 x + a_2 y + a_3 xy : a_j \in \mathbb{R},\ 0 \le j \le 3;\ x, y \in T\}.$$

$\dim(P_1(T)) = 3$; $\dim(Q_1(T)) = 4$.

Q_1 space can be treated of

$$Q_1 = \text{span}\{\{1,x\}\times\{1,y\}\} = \text{span}\{1,x,y,xy\}$$

The space $V_h = \{v \in C^0(\bar{\Omega}) : v|_T \in Q_1(T),\ \forall T \in \mathcal{T}_h,\ v_h = 0 \text{ on } \partial\Omega\}$

can be constructed by considering the product basis functions of of the dimension finite element

$$V_h^x = \{v_h \in C^0(\bar{I}) : v_h|_{I_j} \in P_1(x),\ 1 \le j \le N+1,\ v_h(0) = v_h(1) = 0\}$$

$I = [0,1]$.

$$I_j = [x_{j-1}, x_j], \quad 1 \le j \le N+1.$$

and

$$V_h^y = \{v_h \in C^0(\bar{I}) : v_h|_{I_j^*} \in P_1(y),\ 1 \le j \le N+1,\ v_h(0) = v_h(1) = 0\}$$

$\bar{I} = [0,1]$

$$I_j^* = [y_{j-1}, y_j], \quad 1 \le j \le N+1.$$

The dimension of V_h: [when Q_1 is used with (x_j, y_l), $0 \le j \le N+1$, $0 \le l \le N+1$, points are used].

If $\{\phi_j\}_{j=1}^N$ is the basis of V_h^x, with $\phi_j(x_k) = \delta_{jk}$

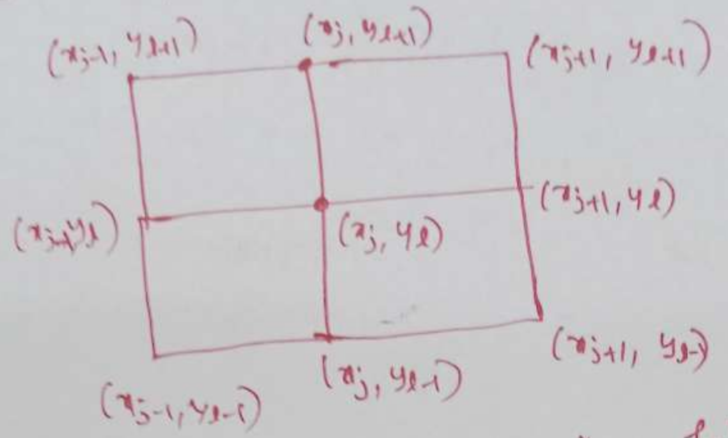
and $\{\psi_l\}_{l=1}^N$ is the basis of V_h^y, with $\psi_l(y_k) = \delta_{lk}$.

Then the basis for V_h is given by

$$\{\phi_j \psi_l\}_{1 \le j, l \le N} = \{\phi_j(x)\psi_l(y)\}_{1 \le j,l \le N}$$

The dimension of $V_h = N^2$ (when $Q_1(T)$ space is used on each T).

Support of a typical basis function $\phi_j \psi_l$



$(\phi_j \psi_l)(x,y) = \phi_j(x)\,\psi_l(y).$

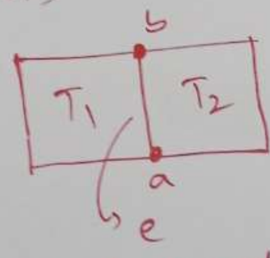
The function $v_h = \phi_j \psi_l$ has the following property:

$v_h(x_j, y_l) = 1;$

$v_h(x_k, y_m) = 0,\quad k \in \{j-1, j, j+1\},\ m \in \{l-1, l, l+1\}$
but when $k = j,\ m \neq l$.

i.e. v_h is zero at 8 nodes except at the center (x_j, y_l):

Suppose $v_h|_{T_1} \in Q_1(T_1)$ and $v_h|_{T_2} \in Q_1(T_2)$ and $v_h|_{T_1}(a) = v_h|_{T_2}(a)$
$v_h|_{T_1}(b) = v_h|_{T_2}(b)$.



Then $v_h|_{T_1}(x) = v_h|_{T_2}(x)$ for all $x \in e$

[e = line segment connecting a & b.]

Let $w = v_h|_{T_1} - v_h|_{T_2}$ on e. Then w is a linear polynomial of degree 1 in one variable.

and $w(a) = 0,\ w(b) = 0$.

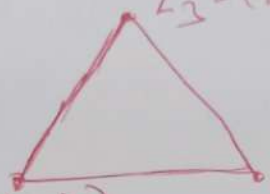
A linear polynomial of one variable has two zeros on e. Then w must be identically zero on e. $\therefore\ w = v_h|_{T_1}\big|_e - v_h|_{T_2}\big|_e = 0.$

$\therefore\ v_h|_{T_1}(x) = v_h|_{T_2}(x)$ for all $x \in e$.

$P_1(T) = \{ a_0 + a_1 x + a_2 y : a_j \in \mathbb{R},\ j = 0,1,2.\ x, y \text{ in } T \}$

Let T has three vertices named z_1, z_2, z_3.

$z_3 = (x_3, y_3)$



$z_1 = (x_1, y_1)$ $\quad$ $z_2 = (x_2, y_2)$

(x_k, y_k), $1 \le k \le 3$ are non collinear

We construct λ_j, $j = 1,2,3$, $\lambda_j = \lambda_j(x,y)$ such that

$$\lambda_j(z_k) = \delta_{jk} \quad 1 \le j, k \le 3. \qquad z_k = (x_k, y_k) \quad 1 \le k \le 3.$$

Let $\lambda_j(x,y) = a_0^j + a_1^j x + a_2^j y$.

$$\lambda_j(z_k) = \lambda_j(x_k, y_k) = a_0^j + a_1^j x_k + a_2^j y_k = \delta_{jk}.$$

writing into a matrix system

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a_0^j \\ a_1^j \\ a_2^j \end{bmatrix} = \delta_{jk}.$$

The matrix $\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$ is invertible since the determinant of this matrix is $2 \cdot \text{area}(T)$.

Since T is a triangle, it has non zero area.

We have three linear polynomials λ_j^T satisfying:

$$\lambda_1^T(z_1) = 1, \quad \lambda_1^T(z_2) = \lambda_1^T(z_3) = 0$$
$$\lambda_2^T(z_2) = 1, \quad \lambda_2^T(z_1) = \lambda_2^T(z_3) = 0$$
$$\lambda_3^T(z_3) = 1, \quad \lambda_3^T(z_1) = \lambda_2^T(z_2) = 0.$$

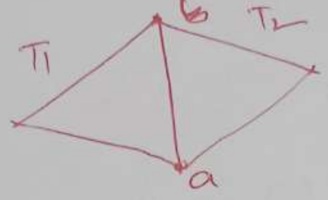
It can be easily seen that they are linearly independent & $\dim(P_1(T)) = 3$.

$$P_1(T) = \text{span}\{\lambda_1^T, \lambda_2^T, \lambda_3^T\}.$$

②

[illegible]

Suppose that ~~a~~ function V, is such that $V|_{T_1} \in P_1(T_1)$ and $V|_{T_2} \in P_1(T_2)$ where T_1 and T_2 share a common edge.



Also $V|_{T_1}(a) = V|_{T_2}(a)$ and $V|_{T_2}(b) = V|_{T_2}(b)$.

Then $V|_{T_1}(x) = V|_{T_2}(x)$ for all x on the edge connecting a and b.

Let $w(x) = V|_{T_1}(x) - V|_{T_2}(x)$ where $x \in$ edge $\{a, b\}$ $\rightarrow x = (x_1, x_2)$

Then the edge is a line, it may have the form $x_1 = mx_2 + c$ (or) $x_1 = m$. Hence w is a one variable function on e.

and $w(a) = w(b) = 0$. A one variable function w has two distinct zeros.

hence $w(x) = 0$ for all $x \in$ edge $\{a, b\}$.

It means that if V (a piecewise linear polynomial) is continuous at the two vertices of an edge, then V is continuous across the edge.

The rules for triangulation:

Ω is partitioned into triangles T.

T is a close set with three vertices.

Denotes the set of all triangles T as $\mathcal{T}_h$.

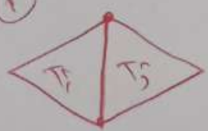
(1) $T_i^\circ \cap T_j^\circ$ = empty set for any T_i, T_j in $\mathcal{T}_h$.

[The interiors of T are disjoint].

(2) If $T_i \cap T_j \neq$ empty set,

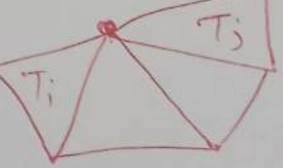
Then $T_i \cap T_j$ is either a common vertex of T_i and T_j or a common edge.

example (1)



common edge

example 2:



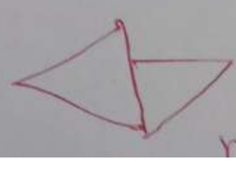
common node (vertex)

(3) $\bigcup_{T \in \mathcal{T}_h} T = \overline{\Omega}$.

[The triangles cover the whole domain including its boundary].

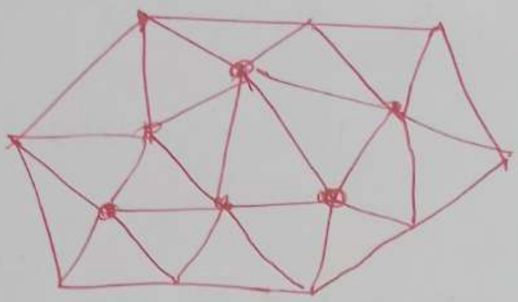
Not allowed:

not allowed (overlapping)



not allowed.

For a domain, consider the triangulation $\mathcal{T}_h$: Let N be the number of vertices in side Ω [that is the vertices interior to Ω].



Enumerate these vertices $z_1, z_2 \dots z_N$.

Then we can construct canonical basis functions for $V_h = \{v \in C^0(\bar{\Omega}) : v|_T \in P_1(T), \forall T \in \mathcal{T}_h, v = 0 \text{ on } \partial\Omega\}$

as follows:

$$\phi_j(z_k) = \delta_{jk}, \quad 1 \le j, k \le N$$

and $\phi_j \in V_h$ i.e. $\phi_j|_T \in P_1(T)$.

$\phi_j(x) = 0$ if $x \in \partial\Omega$.

For each interior vertex, we associate a basis function.

The dimension of V_h = the number of interior vertices.

Green's theorem:

For v and w C^1 functions on a domain D.

$$\int_D v_{x_i} w = - \int_D v w_{x_i} + \int_{\partial D} v w \nu_i$$

ν_i = ith component of unit outward normal vector ν to $\partial\Omega$

Then for $v \in C^1(D)$, $w \in (C^1(D))^2$

$$\int_D \nabla v \cdot w = - \int_D v (\nabla \cdot w) + \int_{\partial D} v w \cdot \nu$$

Here ∇ is gradient $\nabla\cdot$ is divergence

Note that $\Delta v = \nabla \cdot (\nabla v)$.

Recall the PDE: $-\Delta u = f$ in Ω

Since $-\Delta u = -\nabla \cdot (\nabla u)$

$$\int_\Omega - \nabla \cdot (\nabla u) \phi_j = - \sum_{T \in \mathcal{T}_h} \int_T \nabla \cdot (\nabla u) \phi_j$$

$$= \sum_{T \in \mathcal{T}_h} \int_T \nabla u \cdot \nabla \phi_j - \sum_{T \in \mathcal{T}_h} \int_{\partial T} \phi_j \nabla u \cdot \nu$$

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} \phi_j \nabla u \cdot \nu = \sum_{e \in \mathcal{E}_h^i} \int_e (\phi_j \nabla u \cdot \nu)\Big|_{T_1^e} + (\phi_j \nabla u \cdot \nu)\Big|_{T_2^e}$$

$$+ \sum_{e \in \mathcal{E}_h^\partial} \int_e (\phi_j \nabla u \cdot \nu)\Big|_{T^e}$$

where $\mathcal{E}_h^i$ is the set of all interior edges

$\mathcal{E}_h^\partial$ is the set of all boundary edges.

for each interior edge e, there are exactly two triangles T_1^e and T_2^e such that

$$\partial T_1^e \cap \partial T_2^e = e$$

for any boundary edge e, there is exactly one triangle T^e such such that $e \subset \partial T^e$.

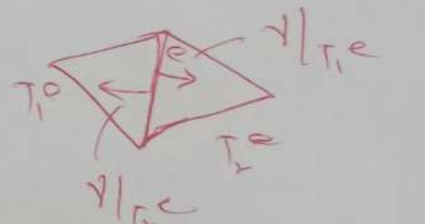
Since $\phi_j = 0$ on $\partial\Omega$, we must have

$$\sum_{e\in\mathcal{E}_h^{\partial}} \int_e \phi_j \nabla u\cdot\nu = 0.$$

for an interior edge, e, with $e = \partial T_1^e \cap \partial T_2^e$: consider

$$I = \int_e (\phi_j \nabla u\cdot\nu)|_{T_1^e} + (\phi_j(\nabla u\cdot\nu))|_{T_2^e}$$

Since ϕ_j is continuous on $\bar\Omega$, if we assume $u \in C^2(\bar\Omega)$, then



$$I = \int_e \phi_j \nabla u\cdot\nu|_{T_1^e} + \phi_j \nabla u\cdot\nu|_{T_2^e}.$$

Since $\nu|_{T_1^e} = -\nu|_{T_2^e}$, $I = 0$.

Therefore

$$-\int_\Omega \Delta u\,\phi_j = \int_\Omega \nabla u\cdot\nabla\phi_j \quad \text{for all } 1\le j\le N.$$

$N = \dim(V_h)$.

and

$$\int_\Omega \nabla u\cdot\nabla\phi_j = \int_\Omega f\phi_j.$$

The P_1 finite element method for

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

is to find

$$u_h \in V_h = \{ v_h \in C^0(\bar{\Omega}) : v_h|_T \in P_1(\mathcal{T}_h),\ T \in \mathcal{T}_h,\ v_h = 0 \text{ on } \partial\Omega \}$$

such that

$$\int_\Omega \nabla u_h \cdot \nabla \phi_j = \int_\Omega f \phi_j, \quad 1 \le j \le N$$

$$\dim(V_h) = N$$
$$V_h = \operatorname{span}\{\phi_j\}_{j=1}^{N}$$

This is equivalent to
find $u_h \in V_h$ such that

$$\int_\Omega \nabla u_h \cdot \nabla v_h = \int_\Omega f v_h \quad \text{for all } v_h \in V_h.$$

Note that if $u \in C^2(\bar{\Omega})$, we have that

$$\int_\Omega \nabla u \cdot \nabla v_h = \int_\Omega f v_h \quad \text{for all } v_h \in V_h.$$

from above two equations, we have that

$$\int_\Omega \nabla (u - u_h) \cdot \nabla v_h = 0 \quad \text{for all } v_h \in V_h.$$

This is known as Galerkin orthogonality.