## Piecewise polynomial Approximation:



Let $I = [a, b]$. Consider breakpoints $\{x_i\}$ in $[a, b]$

$$a = x_0 < x_1 < x_2 < \cdots < x_N = b$$

where $N$ is a positive integer.

Let $I_j = [x_{j-1}, x_j]$, $1 \le j \le N$.

We can define piecewise polynomial vector space

$$V_D^k = \{ p : I \to \mathbb{R} : p|_{I_j} \in P_k(I_j),\ k \ge 0,\ 1 \le j \le N \}$$

where $P_k(I_j)$ is the space of all polynomials of degree $\le k$, restricted to $I_j$.

for any $j$, $\dim(P_k(I_j)) = k+1$.

$$\dim(V_D^k) = N(k+1) = Nk + N.$$

Also we can define piecewise polynomial function space but continuous on $I$ as

$$V_c^k = \{ p \in C[a,b] : p|_{I_j} \in P_k(I_j),\ k \ge 1,\ 1 \le j \le N \}$$

on each $I_j$, we have $\dim(P_k(I_j)) = k+1$.

There are $N$ intervals and continuity conditions at each break point $x_j$, $1 \le j \le N-1$. [$N-1$ conditions].

$$\underline{\dim(V_c^k) = N(k+1) - (N-1)} = Nk + N - N + 1 = Nk+1.$$

Examples: $V_c^1$ [i.e. $k=1$].

$$\dim(V_c^1) = N+1.$$


we can associate a basis function $\phi_j(x)$ to each node point $x_j$, $0 \le j \le N$.

Define:

$$\phi_0(x) = \begin{cases} \dfrac{x_1 - x}{x_1 - x_0}, & x \in [x_0, x_1] \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_N(x) = \begin{cases} \dfrac{x - x_{N-1}}{x_N - x_{N-1}}, & x \in [x_{N-1}, x_N] \\ 0, & \text{otherwise} \end{cases}$$

$$\phi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{x_j - x_{j-1}}, & x \in [x_{j-1}, x_j] \\ \dfrac{x_{j+1} - x}{x_{j+1} - x_j}, & x \in [x_j, x_{j+1}] \\ 0 & \text{otherwise} \end{cases}$$

for $1 \le j \le N-1$.

There $N+1$ functions, $\{\phi_j\}_{j=0}^{N}$.

each $\phi_j$ has the property that

$$\phi_j(x_i) = \delta_{ij}, \quad \text{where } \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

Note that $\{\phi_j\}_{j=0}^{N}$ is a linearly independent set.

suppose there are $c_0, c_1, \ldots c_N$ (constants) not all zero such that

$$\sum_{j=0}^{N} c_j \phi_j(x) = 0.$$

taking $x = x_i$, we get $\sum_{j=0}^{N} c_j \phi_j(x_i) = 0$

$\Rightarrow c_i = 0$. [we can take any $i \in \{0, 1, \ldots N\}$.

Contradiction to the assumption that not all $c_i$ are zero.

Therefore $\{\phi_j\}_{j=0}^{N}$ is linearly independent set.

suppose $g \in V_c^1$. Then $g$ is continuous on $[a, b]$ and $g|_{I_j}$ is a linear polynomial on $I_j$ for $1 \le j \le N$. Consider the function $g_h$ defined by $g_h(x) = \sum_{j=0}^{N} g(x_j)\phi_j(x)$. Clearly $g_h \in V_h^1$. Since $g_h(x)$ is a linear combination of $\{\phi_j\}_{j=0}^{N}$.



Note that $g_h(x_i) = g(x_i)$ for $i = 0, 1, \dots N$.

On any interval $I_j = [x_{j-1}, x_j]$, we have

$$g_h(x_{j-1}) = g(x_{j-1}) \text{ and } g_h(x_j) = g(x_j).$$

Then $e_h = g - g_h$ satisfies $e_h(x_{j-1}) = e_h(x_j) = 0$.

Since $g_h$ is linear polynomial on $I_j$ and $g$ is also linear polynomial on $I_j$, $e_h$ is linear polynomial on $I_j$. But $e_j$ has two zero $x_{j-1}$ and $x_j$.

This implies that $g_h(x) = g(x)$ on $[x_{j-1}, x_j]$ for $j = 1, 2, \dots N$. and hence $g_h(x) = g(x)$, for $x \in I$.

$$\therefore g(x) = \sum_{j=0}^{N} g(x_j)\phi_j(x).$$

Thus any function in $V_c^1$ is spanned by $\{\phi_j\}_{j=0}^{N}$.

$\therefore \{\phi_j\}$ is a basis for $V_c^1$.

<u>Remark:</u> Also note that since $\dim(V_c^1) = N+1$ and the set of $N+1$ functions $\{\phi_j\}_{j=0}^{N}$ is linearly independent, $\{\phi_j\}_{j=0}^{N}$ must be a basis for $V_c^1$.

Similarly one may construct a basis for $V_c^2$, i.e. when $k=2$. Recall

$$V_c^2 = \{ p \in C[a,b] : p|_{I_j} \in P_2(I_j),\ 1 \le j \le N \}$$

and $\dim(V_c^2) = 2N+1$ and $\dim(P_2(I_j)) = 3$, $1 \le j \le N$.

For this we introduce midpoints on each $I_j = [x_{j-1}, x_j]$

$$x_{j-1/2} = \frac{x_{j-1} + x_j}{2}, \quad \text{for } 1 \le j \le N.$$

Let $y_1 = x_{j-1}$, $y_2 = x_j$, $y_3 = x_{j-1/2}$.


We construct Lagrange polynomials of degree 2 with node points $y_1, y_2, y_3$ as follows.

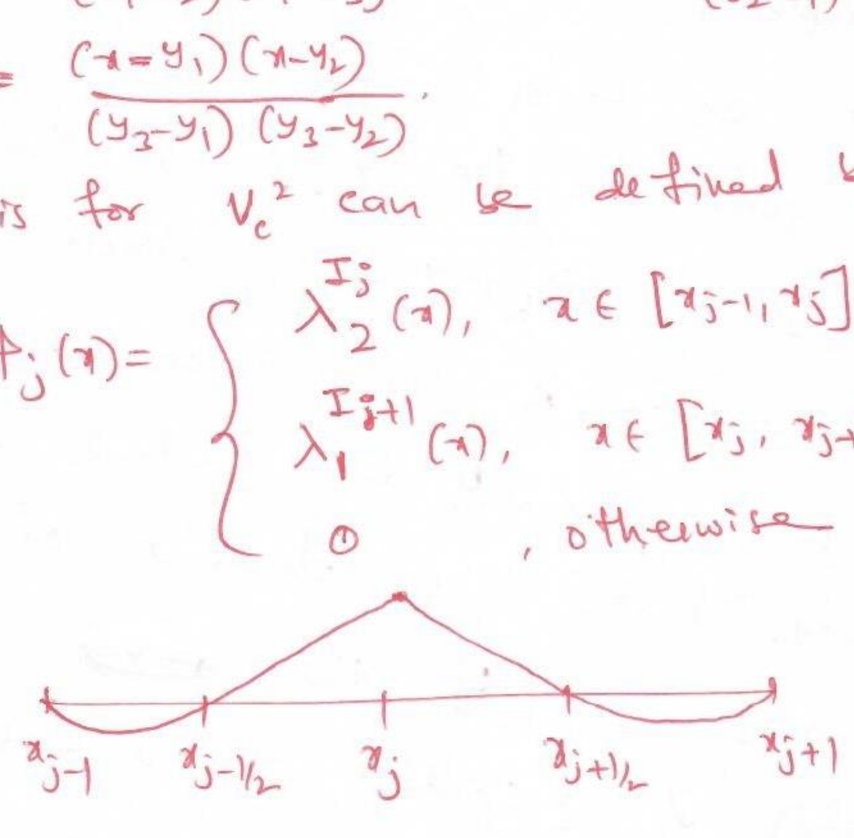
$$\lambda_1^{I_j}(x) = \frac{(x-y_2)(x-y_3)}{(y_1-y_2)(y_1-y_3)}; \quad \lambda_2^{I_j}(x) = \frac{(x-y_1)(x-y_3)}{(y_2-y_1)(y_2-y_3)} \quad \text{and}$$

$$\lambda_3^{I_j}(x) = \frac{(x-y_1)(x-y_2)}{(y_3-y_1)(y_3-y_2)}.$$

A basis for $V_c^2$ can be defined by $\{\phi_j\}_{j=0}^{N} \cup \{\phi_{j-1/2}\}_{j=1}^{N}$

$$\phi_j(x) = \begin{cases} \lambda_2^{I_j}(x), & x \in [x_{j-1}, x_j] \\ \lambda_1^{I_{j+1}}(x), & x \in [x_j, x_{j+1}] \\ 0, & \text{otherwise} \end{cases}$$

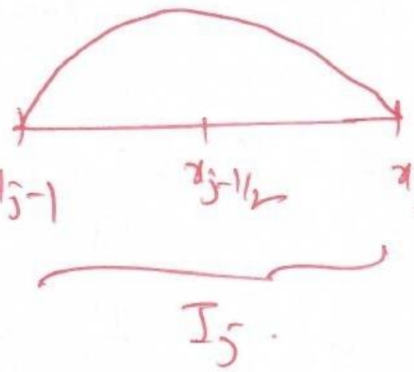
for $1 \le j \le N$




Graph of $\phi_j(x)$. $\phi_j(x) \neq 0$ for $x \in (x_{j-1}, x_j]$ & $x \in [x_j, x_{j+1})$

$$\phi_{j-1/2}(x) = \begin{cases} \lambda_2^{I_j}(x), & x \in [x_{j-1}, x_j] \\ 0 & \text{otherwise} \end{cases}$$

for $1 \le j \le N$.




Graph of $\phi_{j-1/2}$.

General principle to construct basis for $V_c^k$ is that, on each $I_j$, consider $(k-1)$ interior points in $[x_{j-1}, x_j]$, call them $x_{j-1} = y_0 < y_1 < y_2 < \dots < y_k = x_j$. Construct Lagrange polynomials of degree $k$ using these nodes. These Lagrange polynomials are called local basis functions. But since elements of $V_c^k$ are continuous on $[a, b]$, the ~~global~~ basis functions for $V_c^k$ [called global basis functions] need to be defined carefully by looking at the nodes shared by the local basis functions. For example, at each nodes $x_1, x_2, \dots x_{N-1}$, we need to define

basis function to be continuous. Such basis function will have to come from the intervals ~~shared by~~ sharing the node $x_j$.
[for basis function correspondly to the node $x_j$]

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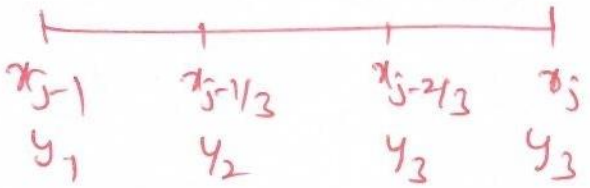
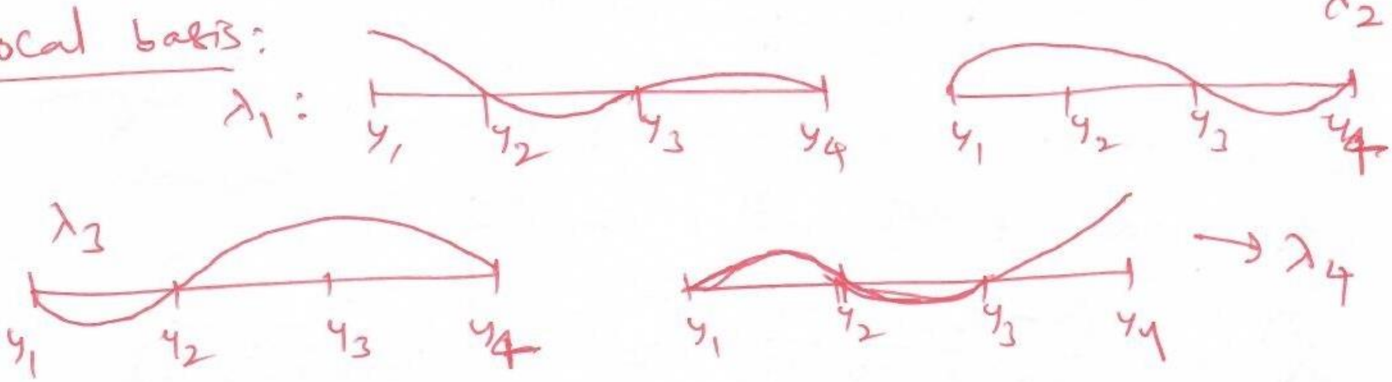
<u>illustration for $k=3$:</u> Consider $I_j = [x_{j-1}, x_j]$




<u>Local basis:</u>




$\lambda_2$ and $\lambda_3$ are zero at $y_1$ and $y_4$.
They can be used as global basis function by defining to be zero outside $I_j$.

But $\lambda_1$ is 1 at $y_1$, to define the global basis function using $\lambda_1$, we need to consider $\lambda_4$ coming from the interval left to $I_j$.
Similarly $\lambda_4$ is 1 at $y_4$, to define the global basis function using $\lambda_4$, we need to consider $\lambda_1$ coming from the interval right to $I_j$.

Approximation using Piecewise polynomials:

Suppose $f: I \to \mathbb{R}$ be a function such that $f \in C^{(k+1)}[I_j]$ for each $1 \le j \le N$.

Recall $V_D^k = \{ p : I \to \mathbb{R} : p|_{I_j} \in P_k(I_j),\ 1 \le j \le N \}$

Let $g_k(x)$ be an element of $V_D^k$. Then $g_k|_{I_j} \in P_k(I_j)$. Suppose that $g_k(x)$ interpolates $f(x)$ at $(k+1)$ distinct points $x_0^j, x_1^j, \ldots x_k^j$ in $[x_{j-1}, x_j]$. Then for $x \in [x_{j-1}, x_j]$

$$f(x) = g_k(x) + f[x_0^j, x_1^j, \ldots x_k^j, x]\, \psi_k(x),$$

$$\text{where} \quad \psi_k(x) = \prod_{i=0}^{k} (x - x_i^j).$$

(or)

$$f(x) = g_k(x) + \frac{f^{(k+1)}(\xi)}{(k+1)!}\, \psi_k(x).$$

where $\xi$ lies between $x_0^j, x_1^j, \ldots x_k^j, x$.

This can be done for any interval $I_j$.

If $x_0, x_1, \ldots x_k$ are equally spaced, i.e. $h = h_j = x_j - x_{j-1}$, is same for all $j = 1, 2, \ldots N$.

Then $|\psi_k(x)| \le h^{k+1}$ and for any $x \in I = [a, b]$.

$$|f(x) - g_k(x)| \le \left[ \sup_{1 \le j \le N} \sup_{x \in I_j} \left| \frac{f^{(k+1)}(x)}{(k+1)!} \right| \right] h^{k+1}$$

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**Remark:** Note here that $x_i^j$ need not include the end points $x_{j-1}$ and $x_j$.

Suppose that $f : I \longrightarrow \mathbb{R}$ be continuous and $f|_{I_j} \in C^{(k+1)}(I_j)$. for each $1 \leq j \leq N$.

In this case we want to use the space

$$V_c^k = \{ p \in C(I) : p|_{I_j} \in P_k(I_j), \ 1 \leq j \leq N \}$$

Let $g_k(x)$ be an element of $V_c^k$. Then $g_k$ is continuous on $I$ and $g_k|_{I_j} \in P_k(I_j)$.

Let $g_k$ be interpolating $f$ at the node points $x_{j-1} = x_0^j < x_1^j < \cdots < x_k^j = x_j$. $1 \leq j \leq N$.

Then

$$f(x) = g_k(x) + f[x_0^j, x_1^j, \ldots x_k^j, x] \, \psi_k(x)$$

for $x \in [x_{j-1}, x_j]$ and $\psi_k(x) = \prod_{i=0}^{k} (x - x_i^j)$

(or)

$$f(x) = g_k(x) + \frac{f^{(k+1)}(\xi)}{(k+1)!} \psi_k(x), \quad \xi \text{ lies between } x_0^j, \ldots x_k^j, x.$$

We do this for each interval $I_j$.

If $h = h_j = x_j - x_{j-1}$, $j = 1, 2 \ldots N$. [i.e. $x_j$ are equally spaced].

Then

$$|\psi_k(x)| \leq h^{k+1} \text{ and}$$

$$|f(x) - g_k(x)| \leq \left[ \sup_{1 \leq j \leq N} \sup_{x \in I_j} \left| \frac{f^{(k+1)}(x)}{(k+1)!} \right| \right] h^{k+1}.$$