Finite eleme… alerkin method)

①

Consider two … boundary value problem: (BVP)

$$-y''(x) \;…\; a(x)\, y = f(x), \quad x \in [0,1] \quad —(1)$$

$-y''(x) + a(x) y = f(x)$

$$y(0) = y(1) = 0. \quad —(2)$$

Let $a(x) \geq 0$ for all $x \in [0,1]$, a and f are continuous functions. The BVP has a unique solution $y \in C^2[0,1]$.

We look for an approximation y_h of y from the piecewise polynomial space V_h defined by the following:

Consider an integer $N > 0$. $h = \frac{b-a}{N+1}$ and

$$x_j = a + jh, \quad j = 0, 1, \ldots (N+1).$$

Let $I_j = [x_{j-1}, x_j]$, $j = 1, 2, \ldots N+1$.

There are $(N+2)$ mesh points and $(N+1)$ intervals.

For given $r \geq 0$ (integer) define

$$V_h = \{ v \in C^0[0,1] : v|_{I_j} \in P_r(I_j), \; v_h(0) = v_h(1) = 0 \}$$

dimensions of $P_r(I_j) = r+1$.

continuity conditions to be in V_h: N.

boundary conditions: 2

Total conditions: $N+2$.

dimension of V_h $= (N+1)(r+1) - (N+2) = r(N+1) - 1$.

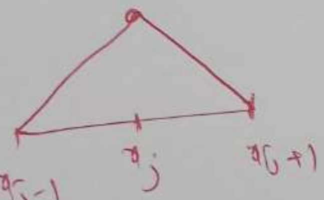
when $r=1$:

$$\dim(V_h) = N$$

<u>Basis for V_h when $r=1$:</u>

We associate a basis function ϕ_j to each interior mesh point x_j, $j=1,\dots,N$, defined by

$$\phi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{x_j - x_{j-1}}, & x \in [x_{j-1}, x_j] \\[2ex] \dfrac{x_{j+1} - x}{x_{j+1} - x_j}, & x \in [x_j, x_{j+1}] \\[2ex] 0 & \text{otherwise} \end{cases}$$



$\phi_j(x)$ satisfies

$$\phi_j(x_k) = \delta_{jk} = \begin{cases} 1, & \text{if } j=k \\ 0 & \text{otherwise} \end{cases}$$

for $1 \le j, k \le N$.

$\{\phi_j\}_{j=1}^{N}$ from a basis for V_h when $r=1$.

Consider one such $\phi_j(x)$, multiply ① by $\phi_j(x)$

Then integrate over $[0,1]$.

$$\int_0^1 \left(-y''(x) + a(x)\,y(x)\right)\phi_j(x) = \int_0^1 f(x)\,\phi_j(x) \quad ③$$

We get N equations like this. $(1 \le j \le N)$

Using the support of ϕ_j (i.e. the set where $\phi_j(x)$ is not zero): $\text{supp}\,\phi_j = \overline{\{x \in [0,1] : \phi_j(x) \neq 0\}}$.

~~we~~ Eq. ③ takes the form

$$\int_{x_{j-1}}^{x_{j+1}} \left(-y''(x) + a(x)\,y(x)\right)\phi_j(x) = \int_{x_{j-1}}^{x_{j+1}} f(x)\,\phi_j(x)$$

we look for an approximation y_n of y from V_n with $r=1$. Then (2)

$$y_n(x) = \sum_{i=1}^{N} \alpha_i \phi_i(x). \quad \text{— (4)}$$

First note that

$$\int_{x_{j-1}}^{x_{j+1}} -y''(x)\phi_j(x)\,dx = \int_{x_{j-1}}^{x_j} -y''(x)\phi_j(x) + \int_{x_j}^{x_{j+1}} -y''(x)\phi_j(x)$$

Integration by parts imply that

$$\int_{x_{j-1}}^{x_j} -y''(x)\phi_j(x) = \int_{x_{j-1}}^{x_j} y'(x)\phi_j'(x)\,dx + \Big[y'(x)\phi_j(x)\Big]_{x_{j-1}}^{x_j}$$

$$= \int_{x_{j-1}}^{x_j} y'(x)\phi_j'(x)\,dx + y'(x_j)\phi_j(x_j) \quad \text{— (5)}$$

since $(\phi_j(x_{j-1})=0)$.

Similarly

$$\int_{x_j}^{x_{j+1}} -y''(x)\phi_j(x)\,dx = \int_{x_j}^{x_{j+1}} y'(x)\phi_j'(x) + \Big[y'(x)\phi_j(x)\Big]_{x_j}^{x_{j+1}}$$

$$= \int_{x_j}^{x_{j+1}} y'(x)\phi_j'(x)\,dx - y'(x_j)\phi_j(x_j). \quad \text{— (6)}$$

[since $\phi_j(x_{j+1})=0$]

Therefore

$$\int_{x_{j-1}}^{x_{j+1}} -y''(x)\phi_j(x) = \int_{x_{j-1}}^{x_{j+1}} y'(x)\phi_j'(x)\,dx,$$

here $\phi_j'(x)$ is the derivative of $\phi_j(x)$ which is defined piece wise on $[x_{j-1}, x_{j+1}]$.

i.e. $\phi_j(x)$ has derivative on $[x_{j-1}, x_j]$ and on $[x_j, x_{j+1}]$, with left and right hand derivatives are being taken at x_j, respectively

$$\phi_j'(x) = \begin{cases} \dfrac{1}{x_j - x_{j-1}}, & x \in [x_{j-1}, x_j] \\ \dfrac{-1}{x_{j+1} - x_j}, & x \in [x_j, x_{j+1}] \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\int_0^1 -y''(x)\,\phi_j(x)\,dx = \int_0^1 y'(x)\,\phi_j'(x)\,dx$$

Using this we have

$$\int_0^1 \left(-y''(x) + a(x)y(x)\right)\phi_j(x) = \int_0^1 y'(x)\,\phi_j'(x)\,dx + \int_0^1 a(x)y(x)\phi_j(x)$$

&

$$\int_0^1 y'(x)\,\phi_j'(x) + a(x)y(x)\,\phi_j(x) = \int_0^1 f(x)\,\phi_j(x).$$

The finite element method is to find $y_h \in V_h$ $(r=1)$: such that

$$\int_0^1 y_h'(x)\,\phi_j'(x) + a(x)\,y_h(x)\,\phi_j(x) = \int_0^1 f(x)\,\phi_j(x) \quad (6)$$

for each $j = 1, 2, \ldots N$.

Since any element $v_h \in V_h$ can be written

$$\Rightarrow v_h = \sum_{j=1}^{N} \beta_j \phi_j(x).$$

multiplying (6) with β_j and taking sum over $j = 1, \ldots N$, we get

$$\int_0^1 y_h'(x)\,v_h'(x) + a(x)\,y(x)\,v_h(x) = \int_0^1 f(x)\,v_h(x)$$

for all $v_h \in V_h$.

Define
$$a(v,w) = \int_0^1 v'(x)w'(x)\,dx + a(x)y(x)w(x)$$
$$F(w) = \int_0^1 f(x)w(x).$$

Then y_h satisfies
$$a(y_h, v_h) = F(v_h) \quad \text{for all } v_h \in V_h.$$

Note that y satisfies
$$a(y, v_h) = F(v_h) \quad \text{for all } v_h \in V_h.$$
from above two equality (for y & y_h), we get
$$a(y-y_h, v_h) = 0 \quad \text{for all } v_h \in V_h.$$

[That is the error $y-y_h$ is orthogonal to the space V_h [note that $a(\cdot,\cdot)$ defines an innerproduct on V_h]. we will come back to this later].

Since $y_h \in V_h$, $y_h = \sum_{i=1}^{N} \alpha_i \phi_i$,

from ⑥, we have
$$\int_0^1 \left(\sum_{i=1}^{N} \alpha_i \phi_i'(x)\right)\phi_j'(x) + a(x)\left(\sum_{i=1}^{N} \alpha_i \phi_i(x)\right)\phi_j(x) = \int_0^1 f(x)\phi_j(x)$$
$$\Rightarrow \sum_{i=1}^{N} \alpha_i \int_0^1 \phi_i'(x)\phi_j'(x) + a(x)\phi_i(x)\phi_j(x) = \int_0^1 f(x)\phi_j(x). \quad \text{—⑦}$$

Define
$$a_{ji} = \int_0^1 \phi_i'(x)\phi_j'(x) + a(x)\phi_i(x)\phi_j(x)$$
$$b_j = \int_0^1 f(x)\phi_j(x)$$

$A = [a_{ji}]_{1\le i,j\le N}$; $b = [b_j]_{1\le j\le N}$, $\alpha = [\alpha_j]_{j=1}^{N}$

⑦ Can be written as
$$A\alpha = b.$$

To show that the system $A\alpha = b$ has a unique solution it is enough to show that the system $A\alpha = 0$ has only the trivial solution $\alpha = 0$.

Suppose that $\alpha \neq 0$, then $y_h \not\equiv 0$, $y_h \in V_h$ is a solution of the system

$$a(y_h, v_h) = 0 \quad \forall v_h \in V_h.$$

Then taking $v_h = y_h$, we have

$$a(y_h, y_h) = 0.$$

This implies

$$\int_0^1 (y_h'(x))^2 + a(x)(y_h(x))^2 = 0 \quad ;$$

$\Rightarrow$ $\int_0^1 (y_h'(x))^2\,dx = 0$ and $\int_0^1 a(x)(y_h(x))^2 = 0$ each is non negative

$\Rightarrow$ the piecewise derivative $y_h'(x)$ of $y_h(x)$ is zero

$\Rightarrow$ $y_h'(x) = 0 \quad \forall x \in [x_{j-1}, x_j]$, $j = 1, \dots N+1$.

$\Rightarrow$ $y_h(x) = c_j$, $\forall x \in [x_{j-1}, x_j]$, $j = 1, \dots N+1$

for some constants c_j.

But $y_h(x) \in V_h$, $y_h(x)$ must be continuous on $[0,1]$

we must have

$$c_j = c \quad \forall j = 1, \dots N+1$$

for some constant c.

$\therefore$ $y_h(x)$ is constant on $[0,1]$

But $y_h(0) = y_h(1) = 0$, we must have $c = 0$.

$$\therefore \; y_h(x) = 0 \quad \forall x \in [0,1].$$

Contradiction that $y_h \not\equiv 0$. $\therefore$ $\alpha = 0$ is the only solution.

Define

$$|||v|||^2 = a(v,v) = \int_0^1 \left((v'(x))^2 + a(x)(v(x))^2 \right) dx$$

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Then

$$|||v|||^2 = a(v,v) \quad \text{and} \quad |a(v,w)| \le |||v|||\,|||w|||.$$

we make use the following Cauchy-Schwarz inequality

$$\left| \int_0^1 f(x)g(x)\,dx \right| \le \left(\int_0^1 (f(x))^2 dx \right)^{1/2} \left(\int_0^1 (g(x))^2 dx \right)^{1/2}.$$

Since we have

$$a(y-y_h, v_h) = 0 \quad \forall v_h \in V_h$$

Note that

$$|||y-y_h|||^2 = a(y-y_h, y-y_h) = a(y-y_h, y-v_h) + \underbrace{a(y-y_h, v_h-y_h)}_{0}$$

$$= a(y-y_h, y-v_h)$$

$$\le |||y-y_h|||\;|||y-v_h|||.$$

$$\therefore \quad |||y-y_h||| \le |||y-v_h|||$$

The ~~right~~ left hand side is independent of v_h, we find that

$$|||y-y_h||| \le \inf_{v_h \in V_h} |||y-v_h||| \quad \text{— (BE)}$$

we can quantify the right hand side in terms of powers of mesh size (h) and ~~so~~ the derivatives of y.

we define $\pi_h y \in V_h$ as follows:

$$\pi_h y(x_j) = y(x_j), \quad j = 0, \dots N+1;$$

i.e. $\pi_h y$ is the piecewise linear interpolant ($r=1$) of y from V_h. [This is Lagrange interpolation on each $[x_{j-1}, x_j]$ with $r=1$.]

We focus on each $[x_{j-1}, x_j] = I_j$ (8)

$$\pi_h y\big|_{I_j} \in P_1(I_j); \quad \pi_h y(x_k) = y(x_k), \quad k = j-1, j.$$

Let $e = y - \pi_h y$. Consider $e\big|_{I_j} = (y - \pi_h y)\big|_{I_j}$

$$e(x_{j-1}) = y(x_{j-1}) - \pi_h y(x_{j-1}) = 0$$

$$e(x_j) = y(x_j) - \pi_h y(x_j) = 0.$$

Since $y \in C^2[0,1]$, $y\big|_{I_j} \in C^2[x_{j-1}, x_j]$; $\pi_h y\big|_{I_j} \in C^2[x_{j-1}, x_j]$.

By Rolle's theorem, there exists a $\xi \in (x_{j-1}, x_j)$ such that

$$e'(\xi) = 0.$$

$$|e'(x)| = \left| \int_\xi^x e''(s)\,ds \right| = \left| \int_\xi^x y''(s)\,ds \right|; \quad (\pi_h y)''(x) = 0 \;\; \forall x \in (x_{j-1}, x_j).$$

$$\le (x-\xi)^{1/2} \left(\int_\xi^x |y''(s)|^2 ds \right)^{1/2}$$

$$\Rightarrow \quad |e'(x)|^2 \le h \int_{x_{j-1}}^{x_j} |y''(s)|^2 ds$$

Integrating over $[x_{j-1}, x_j]$, we find

$$\int_{x_{j-1}}^{x_j} |e'(x)|^2 dx \le h^2 \int_{x_{j-1}}^{x_j} |y''(s)|^2 ds$$

Sum over all $j = 1, \dots, N+1$, and then taking square root,

$$\left(\int_0^1 |e'(x)|^2 dx \right)^{1/2} \le h \left(\int_0^1 |y''(s)|^2 ds \right)^{1/2}. \quad \text{— IE}$$

Since $e(x_{j-1}) = 0$;

$$|e(x)| = \left| \int_{x_{j-1}}^x e'(s)\,ds \right| \le \int_{x_{j-1}}^{x_j} |e'(s)|\,ds$$

$$\le h^{1/2} \left(\int_{x_{j-1}}^{x_j} |e'(s)|^2 ds \right)^{1/2}$$

$\therefore \quad |e(x)| \le h^{1/2} \left(\int_{x_{j-1}}^{x_j} |e'(s)|^2 ds \right)^{1/2}$

Squaring and integrating over $[x_{j-1}, x_j]$,

$$\left(\int_{x_{j-1}}^{x_j} |e(x)|^2 dx \right) \le h^2 \left(\int_{x_{j-1}}^{x_j} |e'(s)|^2 ds \right)$$

sum over $j = 1, 2, \ldots N+1$, and taking square root

$$\left(\int_0^1 |e(x)|^2 dx \right)^{1/2} \le h \left(\int_0^1 |e'(s)|^2 ds \right)^{1/2}$$

Using (IE) on page 8.

$$\left(\int_0^1 |e(x)|^2 dx \right)^{1/2} \le h^2 \left(\int_0^1 |y''(s)|^2 ds \right)^{1/2} \quad \text{— (IL)}$$

Using (IE) and (IL), we find that $\left(M = \max_{x \in [0,1]} |a(x)|\right)$

$$|||y - \pi_h y||| = \left(\int_0^1 |(y - \pi_h y)'(s)|^2 ds + \int_0^1 a(s) |(y - \pi_h y)(s)|^2 ds \right)^{1/2}$$

$$\le \left(\int_0^1 |(y - \pi_h y)'(s)|^2 ds + M \int_0^1 |(y - \pi_h y)(s)|^2 ds \right)^{1/2}$$

$$\le \left(h^2 \int_0^1 |y''(s)|^2 ds + M h^4 \int_0^1 |y''(s)|^2 ds \right)^{1/2}$$

$$\le \tilde{m} h \left(\int_0^1 |y''(s)|^2 ds \right)^{1/2}.$$

Since $\inf_{v_h \in V_h} |||y - v_h||| \le |||y - \pi_h y|||$.

we find from (RE) on 7 that

$$|||y - y_h||| \le \tilde{m} h \left(\int_0^1 |y''(s)|^2 ds \right)^{1/2}.$$

Weak derivative (Remarks):

Suppose $y \in C^1[0,1]$ and $\phi \in C_0^\infty[0,1] := \{ v \in C^\infty[0,1];\ v = 0$ outside $[a,b] \subset [0,1],\ 0<a,\ b<1 \}$

Then by integration by parts

$$\int_0^1 y'(x)\phi(x)\,dx = -\int_0^1 y(x)\phi'(x)\,dx + \Big[y(x)\phi(x)\Big]_{x=0}^{x=1}$$

$$= -\int_0^1 y(x)\phi'(x)\,dx, \quad \text{since } \phi(0)=\phi(1)=0.$$

we have

$$\int_0^1 y'(x)\phi(x)\,dx = -\int_0^1 y(x)\phi'(x)\,dx. \quad \text{for all } \phi \in C_0^\infty(0,1) \qquad \text{— (E)}$$

Note that the RHS of (E) does not involve the derivative of y, the RHS makes sense (finite value, and the integrand is integrable) if y is merely continuous on $[0,1]$. While LHS requires a derivative of y.

The RHS $-\int_0^1 y(x)\phi'(x)\,dx$ can be used to define a weak derivative of y as a functional, through action of y on ϕ'.

i.e. $$(y')(\phi) = -\int_0^1 y(x)\phi'(x)\,dx. \quad \forall\ \phi \in C_0^\infty(0,1)$$

Here y' is not defined as a function but through action of y on ϕ' under integration.

Usual function $x \longrightarrow y'(x)$

Here function $\phi \longrightarrow (y')(\phi)$.

For a Riemann integrable function y, the integral $-\int_0^1 y(x)\,\phi'(x)$ is defined.

If there exists a Riemann integrable function $g(x)$ on $[0,1]$ such that

$$-\int_0^1 y(x)\,\phi'(x) = \int_0^1 g(x)\,\phi(x)\,dx. \quad \text{for all } \phi \in C_0^\infty(0,1)$$

then the weak derivative of y is defined to be $g(x)$ and we denote

weak derivative of $y = g$. [we can define this for any interval $[a,b]$]

Example: Consider $y(x) = |x|$, $x \in [-1,1]$.

y is continuous on $[-1,1]$ so it is integrable

Consider

$$-\int_{-1}^1 |x|\,\phi'(x)\,dx = -\int_{-1}^0 (-x)\,\phi'(x)\,dx - \int_0^1 x\,\phi'(x)\,dx$$

Integration by parts: simplify the RHS:

$$-\int_{-1}^0 (-x)\,\phi'(x)\,dx = \int_{-1}^0 x\,\phi'(x)\,dx = -\int_{-1}^0 \phi(x)\,dx + \Big[x\,\phi(x)\Big]_{x=-1}^{x=0}$$

$$= \int_{-1}^0 (-1)\,\phi(x)\,dx$$

$$-\int_0^1 x\,\phi'(x)\,dx = \int_0^1 \phi(x)\,dx + \Big[x\,\phi(x)\Big]_{x=0}^{x=1} = \int_0^1 (1)\,\phi(x)\,dx$$

$$\therefore -\int_{-1}^1 |x|\,\phi'(x)\,dx = \int_{-1}^1 g(x)\,\phi(x)\,dx, \quad \text{where } g(x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$$

g is integrable (Riemann integrable) and

$$-\int_{-1}^1 y(x)\,\phi'(x)\,dx = \int_{-1}^1 g(x)\,\phi(x)\,dx;$$ weak derivative of y is $g(x)$.