Newton's (Newton-Raphson) Method:

Suppose that $f \in C^2[a,b]$ (twice differentiable functions whose second order derivative is continuous on $[a,b]$). Let $p_0 \in [a,b]$ be an approximation to p such that $f'(p_0) \neq 0$ and $|p - p_0|$ is small. Consider Taylor polynomial for $f(x)$ expanded about p_0 and evaluated at $x = p$,

$$f(p) = f(p_0) + (p - p_0) f'(p_0) + \frac{(p-p_0)^2}{2} f''(\xi(p))$$

where $\xi(p)$ lies between p and p_0. Since $f(p) = 0$, this equation gives

$$0 = f(p_0) + (p - p_0) f'(p_0) + \frac{(p-p_0)^2}{2} f''(\xi(p))$$

Newton method is derived by assuming that since $|p - p_0|$ is small, the term involving $(p-p_0)^2$ is much smaller, so

$$0 \approx f(p_0) + (p - p_0) f'(p_0).$$

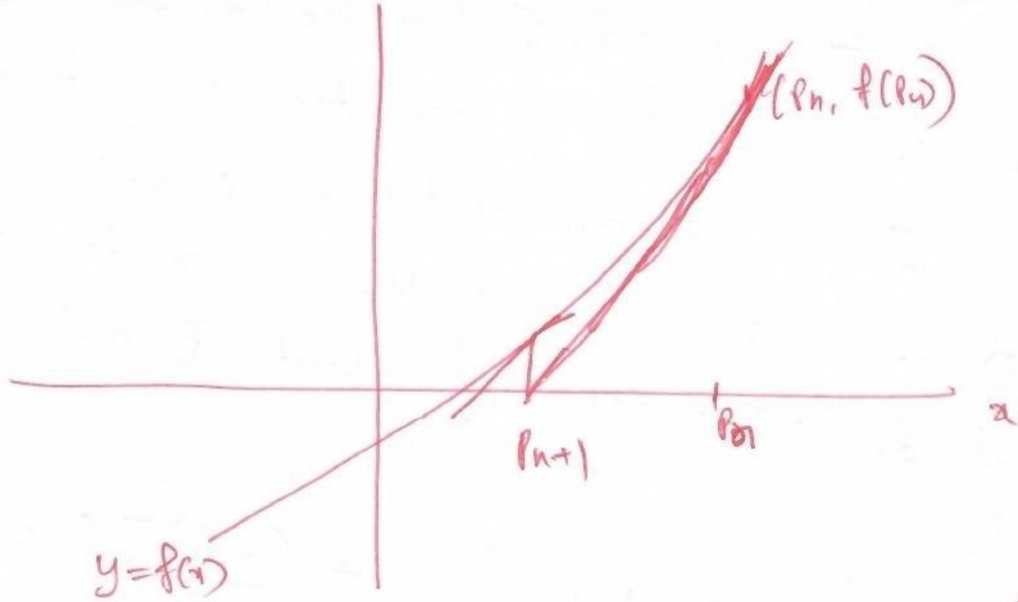
Solving for p, gives

$$p \approx p_0 - \frac{f(p_0)}{f'(p_0)} \equiv p_1$$

This sets the stage for Newton's method, which starts with an initial approximation p_0, and generates a sequence $\{p_n\}_{n=0}^{\infty}$ by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \quad \text{for } n \geq 1.$$

Graphical interpretation:



Consider a point p_n, locate point $(p_n, f(p_n))$ on the graph of $y=f(x)$. Draw tangent to $y=f(x)$ at $(p_n, f(p_n))$ and extend tangent to meet the x-axis. The x-intercept of tangent line is taken as p_{n+1}. Then repeat the procedure.

equation of tangent line through $(p_n, f(p_n))$ is

$$y = f'(p_n)(x-p_n) + f(p_n)$$

x-intercept: set $y=0$, then

$$x = p_n - \frac{f(p_n)}{f'(p_{n+1})} = p_{n+1}.$$

Convergence of Newton's Method:

Theorem: Let $f \in C^2[a,b]$. If $p \in (a,b)$ is such that $f(p)=0$ and $f'(p) \neq 0$, then there is a $\delta > 0$ such that Newton's method generate a sequence $\{p_n\}_{n=1}^{\infty}$ converges to p for any initial approximation $p_0 \in [p-\delta, p+\delta]$.

Proof: The proof is based on analyzing Newton's method as fixed-point iteration $p_n = g(p_{n-1})$ for $n \geq 1$, with

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

Let $K \in (0,1)$. We need to find an interval $[p-\delta, p+\delta]$ that g maps into itself and for which $|g'(x)| \leq K$ for all $x \in (p-\delta, p+\delta)$.

Since $f'(p) \neq 0$ and f' is continuous on $[a,b]$, there is a $\delta_1 > 0$ such that

$$f'(x) \neq 0 \quad \text{for all } x \in [p-\delta_1, p+\delta_1] \subseteq [a,b]$$

$\boxed{\subseteq [a,b]}$

This implies g is defined and continuous on $[p-\delta_1, p+\delta_1]$. Also

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2} = 1 - \frac{[f'(x)]^2}{[f'(x)]^2} + \frac{f(x)f''(x)}{[f'(x)]^2}$$

$$\therefore g'(x) = \frac{f(x)f''(x)}{[f'(x)]^2} \quad \text{for all } x \text{ in } [p-\delta_1, p+\delta_1].$$

Since $f \in C^2[a,b]$, we have $g \in C^1[p-\delta_1, p+\delta_1]$. (5)

By assumption $f(p)=0$; so

$$g'(p) = \frac{f(p)\, f''(p)}{[f'(p)]^2} = 0.$$

Since g' is continuous, and $0<k<1$, there exists a δ; with $0<\delta<\delta_1$ such that

$$|g'(x)| \le k \quad \text{for all } x \in [p-\delta, p+\delta].$$

It remains to show that g maps $[p-\delta, p+\delta]$ into itself. First note that $g(p)=p$.

[this is because $g(p) = p - \frac{f(p)}{f'(p)} = p$, since $f(p)=0$].

For $x \in [p-\delta, p+\delta]$, consider

$$|g(x)-p| = |g(x)-g(p)| \le |g'(\xi)|\,|x-p| \le k\,|x-p|$$

where ξ lies between x and p.

$$\therefore \; |g(x)-p| \le |x-p| < \delta \quad [\because k<1]$$

$$\therefore \; g(x) \in [p-\delta, p+\delta].$$

$\therefore$ g maps $[p-\delta, p+\delta]$ into itself.

By applying fixed-point theorem, we conclude that the sequence $\{p_n\}$, generated by starting with $p_0 \in [p-\delta, p+\delta]$ and $p_{n+1} = g(p_n)$, converges to p, (zero of f).

$\blacksquare$

Newton's Method Algorithm:

To find a solution to $f(x)=0$ given an initial approximation p_0.

INPUT: p_0, TOL, N_0 = maximum number of iterations.

OUTPUT: Approximate solution p (or) message failure.

Step 1: set $i=1$.

Step 2: while $i \le N_0$, do steps 3-6.

Step 3: set $p = p_0 - \frac{f(p_0)}{f'(p_0)}$ (compute p_i)

Step 4: If $|p-p_0| < TOL$, then

OUTPUT (p) (The procedure successful)

STOP.

Step 5: set $i=i+1$;

Step 6: set $p_0 = p$ (update p_0).

Step 7: OUTPUT (The method failed after N_0 iterations).

stop.