Bisection Method:

To find a solution to $f(x)=0$ given continuous function f on the interval $[a, b]$, where $f(a)$ and $f(b)$ have opposite signs.

Method: set $a_1 = a$ and $b_1 = b$.

Let $p_1 = a_1 + \frac{b_1 - a_1}{2} = \frac{a_1 + b_1}{2}$

If $f(p_1)=0$, then $p = p_1$ and we are done.

If $f(p_1) \neq 0$, then $f(p_1)$ has the ~~same~~ sign as either $f(a_1)$ or $f(b_1)$.

If $f(p_1) \cdot f(a_1) > 0$, then $p \in (p_1, b_1)$

set $a_2 = p_1$, $b_2 = b_1$

If $f(p_1) \cdot f(a_1) < 0$, then $p \in (a_1, p_1)$

set $a_2 = a_1$, $b_2 = p_1$

Continue like this to generate sequence $\{p_n\}$.

1 (2)

Bisection Algorithm: To find a solution to $f(x)=0$ given continuous function f on the interval $[a, b]$ where $f(a)$ and $f(b)$ have opposite signs.

INPUT: a, b, TOL, maximum number of iterations N_0.

OUTPUT: approximate solution p, (or) message failure.

Step 1: set $i=1$; $FA = f(a)$

Step 2: While $i \le N_0$, do steps 3 ~~to~~ 6.

Step 3: set $p = a + \frac{(b-a)}{2}$; (compute p_i)

$Fp = f(p)$

Step 4: If $Fp = 0$ (or) $\frac{(b-a)}{2} < TOL$, then

OUTPUT (p); (Procedure successful)

STOP;

Step 5: set $i = i+1$;

Step 6: If $FA \cdot Fp > 0$, then

set $a = p$; $FA = Fp$.

else set $b = p$;

Step 7: output (Method failure after N_0 iterations).

STOP.

Theorem: Suppose that $f \in C[a,b]$ and $f(a)\cdot f(b) < 0$. (3)

The Bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero p of f with

$$|p_n - p| \le \frac{b-a}{2^n}, \quad \text{when } n \ge 1.$$

Proof: For each $n \ge 1$, we have

$$b_n - a_n = \frac{1}{2^{n-1}}(b-a) \quad \text{and} \quad p \in (a_n, b_n).$$

[refer to page (1), where p_n is defined].

Since $p_n = \frac{1}{2}(a_n + b_n)$ for all $n \ge 1$, it follows that

$$|p_n - p| \le \frac{1}{2}(b_n - a_n) = \frac{b-a}{2^n}.$$

that is $$|p_n - p| \le \frac{1}{2^n}(b-a).$$

the sequence $\{p_n\}$ converges to p.

Definition: Suppose $\{\beta_n\}_{n=1}^{\infty}$ is a sequence known to converge to zero, and $\{\alpha_n\}_{n=1}^{\infty}$ converges to a number α. If a positive constant K exists with

$$|\alpha_n - \alpha| \le K|\beta_n| \quad \text{for large } n,$$

then we say that $\{\alpha_n\}_{n=1}^{\infty}$ converges to α with rate of convergence (or order of convergence) $O(\beta_n)$ [The symbol big oh of β_n] and written as $\alpha_n = \alpha + O(\beta_n)$.

We have for Bisection method that

$$|p_n - p| \le \frac{1}{2^n}(b-a).$$

the sequence $\{p_n\}$ converges to p with order of convergence $O(1/2^n)$, that is

$$p_n = p + O(1/2^n).$$

Fixed Point iteration:

<u>Definition:</u> A number p is a <u>fixed point</u> for a given function g if $g(p)=p$.

<u>observation:</u> A root finding problem $f(p)=0$ can be reformulated to a fixed point problem $p=g(p)$ in many ways.

For example, fixed point of following functions g_1 and g_2, defined by

$$g_1(x) = x - f(x) \quad \text{and} \quad g_2(x) = x + 4f(x),$$

is a zero of the function f.

Conversely, if g has a fixed point p, then the function $f(x) = x - g(x)$ has a zero at p.

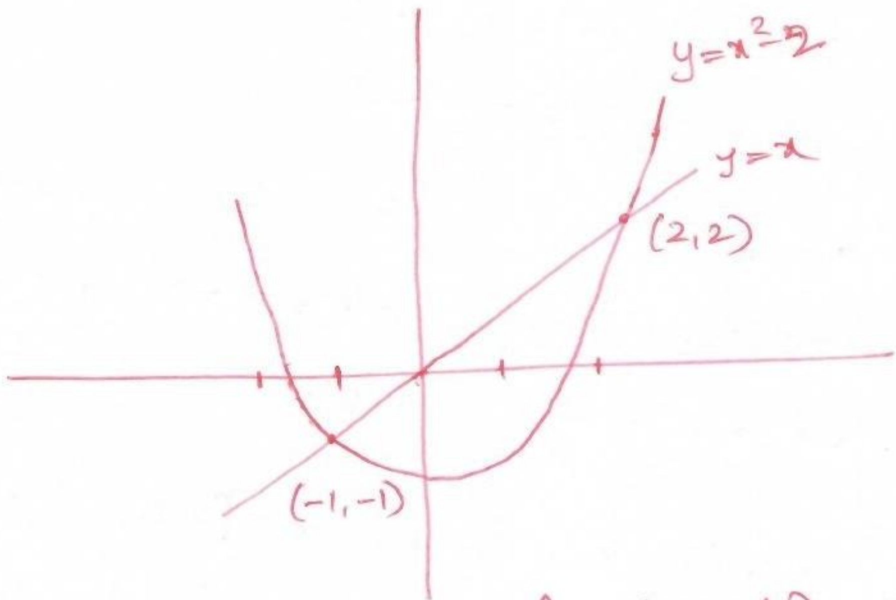
Determine fixed points of $g(x) = x^2 - 2$.

fixed point p of g has the following property $p = g(p)$. Then using $g(p) = p^2 - 2$.

$$p = p^2 - 2 \Rightarrow p^2 - p - 2 = 0$$

$$\Rightarrow (p+1)(p-2) = 0.$$

$$\therefore p = -1 \text{ (or) } p = 2.$$



Intersection points of $y = g(x)$ and $y = x$ are fixed points of function g.

<u>Theorem</u>: (i) If $g : [a,b] \longrightarrow [a,b]$ is continuous function, then g has a fixed point in $[a,b]$.

(ii) If in addition, $g'(x)$ exists on (a,b) and a positive constant $k < 1$ exists with

$$|g'(x)| \leq k \quad \text{for all } x \in (a,b)$$

then there exists exactly one fixed point in $[a,b]$.

proof: (i): If $g(a)=a$ (or) $g(b)=b$, we are done. 6

Suppose $g(a)\neq a$ and $g(b)\neq b$. Then $g(a)>a$ and $g(b)<b$: [Note that $g(x)\in[a,b]$.

$\Rightarrow a\leq g(x)\leq b$:

since $a\neq g(a)$ and $b\neq g(b)$

$a<g(x)<b$ for all $x\in[a,b]$]

Define $h(x)=g(x)-x$.

Then $h(a)=g(a)-a>0$

$h(b)=g(b)-b<0$.

$\therefore$ h has a zero p in (a,b) by intermediate value theorem. Then $h(p)=0 \Rightarrow g(p)=p$.

g has a fixed point.

(ii) Given. $|g'(x)|\leq k$ for all $x\in(a,b)$, $0<k<1$.

suppose that there are two fixed points for g in $[a,b]$, say, there are p_1 and p_2 such that $g(p_1)=p_1$ and $g(p_2)=p_2$, $(p_1\neq p_2)$ in $[a,b]$

Then by mean value theorem

$$|p_1-p_2|=|g(p_1)-g(p_2)| = |g'(\xi)|\,|p_1-p_2|$$

where ξ lies between p_1 and p_2.

$\therefore |p_1-p_2|\leq k\,|p_1-p_2|$; $[\because |g'(x)|\leq k]$

& $|p_1-p_2|<|p_1-p_2|$ $[\because k<1]$

A contradiction to $p_1\neq p_2$.

$\therefore$ g has a unique fixed point in $[a,b]$.

Fixed point iteration:

To approximate the fixed point of a function g, we choose an initial approximation P_0 and generate the sequence $\{P_n\}_{n=0}^{\infty}$ by letting $P_n = g(P_{n-1})$, for each $n \geq 1$. If the sequence converges to P and g is continuous, then

$$P = \lim_{n\to\infty} P_n = \lim_{n\to\infty} g(P_{n-1}) = g\left(\lim_{n\to\infty} P_{n-1}\right) = g(P).$$

and a solution is obtained for $x = g(x)$.

This technic is called fixed-point (or) functional iteration.

Algorithm: To find a solution to $P = g(P)$ given an initial approximation P_0;

INPUT: P_0, TOL, maximum number of iterations N_0.

OUTPUT: approximate solution P (or) message failure.

Step 1: set $i = 1$.

Step 2: while $i \leq N_0$, do steps 3–6.

Step 3: set $P = g(P_0)$. (compute P_i)

Step 4: if $|P - P_0| < TOL$, then

OUTPUT (P) [The procedure successful]

STOP.

Step 5: set $i = i + 1$.

Step 6: set $P_0 = P$ (update P_0).

Step 7: OUTPUT (the method failed after N_0 iterations),

STOP.

Fixed point theorem: (Theorem):

Let $g \in C[a,b]$ be such that $g(x) \in [a,b]$, for all $x \in [a,b]$. Suppose, in addition, that g' exists on (a,b) and that a constant $0<k<1$ exists with

$$|g'(x)| \le k, \text{ for all } x \in (a,b).$$

Then for any $p_0 \in [a,b]$, the sequence generated by $p_n = g(p_{n-1})$, $n \ge 1$ converges to the unique fixed point p in $[a,b]$.

Proof: By the previous theorem (on page 5).
g has a unique fixed point in $[a,b]$.
i.e. there exists a unique $p \in [a,b]$ such that

$$g(p) = p.$$

Since g maps $[a,b]$ into itself, the sequence $\{p_n\}_{n=0}^{\infty}$ is defined for all $n \ge 0$ and $p_n \in [a,b]$.
for all n: Using mean value theorem,

$$|p_n - p| = |g(p_{n-1}) - g(p)| = |g'(\xi_n)|\ |p_{n-1} - p|$$
$$\le k\ |p_{n-1} - p|,$$

where $\xi_n \in (a,b)$. Applying ~~mean value~~ this inequality inductively

$$|p_n - p| \le k^n |p - p_0|. \quad \text{—— } (*)$$

Since $0<k<1$, we have $\lim_{n\to\infty} k^n = 0$ and

hence
$$\lim_{n\to\infty} |p_n - p| = 0.$$

Hence $\{p_n\}$ converges to p. ▨

Corollary: If g satisfies the hypothesis of fixed point theorem, then

$$|p_n - p| \le k^n \max\{p_0 - a, b - p_0\}.$$

and $$|p_n - p| \le \frac{k^n}{1-k} |p_1 - p_0| \quad \text{for all } n \ge 1.$$

Proof: From * on page 8.

$$|p_n - p| \le k^n |p - p_0| \quad \text{— } (+)$$

But since $p \in [a, b]$, we have

$$|p - p_0| \le \max\{p_0 - a, b - p_0\}.$$

The first inequality is proved.

Note that $$|p - p_0| \le |p - p_1| + |p_1 - p_0| \quad \text{— } (1)$$

Using * for $n=1$; $$|p - p_1| \le k|p - p_0| \quad \text{— } (2)$$

Use (2) in (1):

$$|p - p_0| \le k|p - p_0| + |p_1 - p_0|$$

Simplifying $\Rightarrow$ $$|p - p_0| \le \frac{1}{1-k} |p_1 - p_0| \quad \text{— } (3)$$

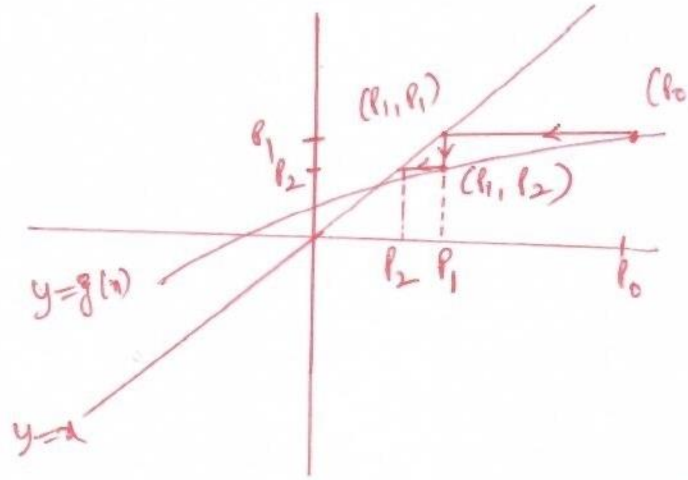
Use (3) in (+) $\therefore$ $$|p - p_n| \le \frac{k^n}{1-k} |p_1 - p_0|$$

Hence the proof.

Remark: The estimates in the corollary gives computable bounds on the error $|p - p_n|$.

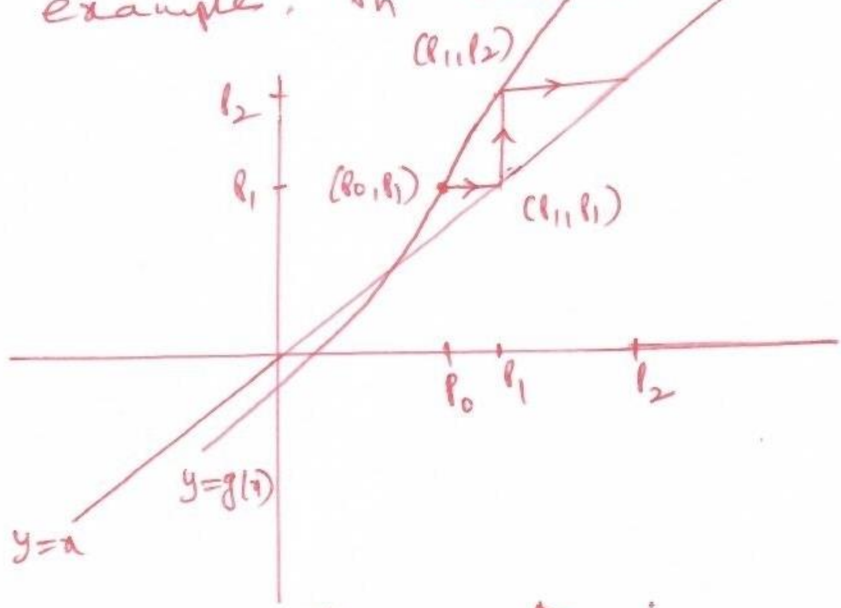
Graphical view of fixed point iteration:

(1)



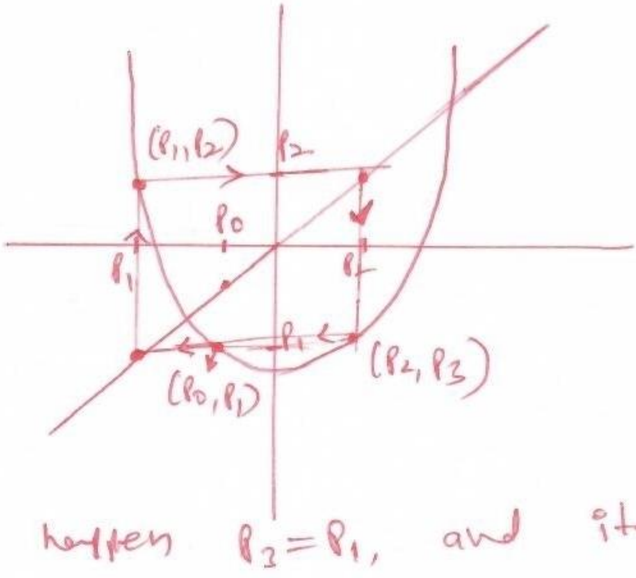
In this example, p_n converges to p.

(2)



In this case p_n diverges.

(3)



It may happen $p_3 = p_1$, and iterations enter into oscillations between p_1 and p_3.

Question: What are your observations on slopes & convergence

Examples:

Consider solving $f(x)=0$

where $f(x)=x^2-2$

We are looking for a root of $x^2-2=0$.

Rewrite it as $x^2=2 \iff x=\frac{2}{x}$

$\iff \frac{1}{2}x=\frac{1}{x}$.

$\iff \frac{1}{2}x+\frac{1}{2}x=\frac{1}{2}x+\frac{1}{x}$

$\iff x=\frac{1}{2}\left(x+\frac{2}{x}\right)$.

We look for a fixed point for function $g(x)=\frac{1}{2}\left(x+\frac{2}{x}\right)$.

Note that (by AM-GM inequality)

$g(x) \geq \sqrt{2}$ — (1)

$\frac{1}{x}$

and $g(x)-x=\frac{1}{2}\left(x+\frac{2}{x}\right)-x=\frac{2}{x}-\frac{x}{2}=\frac{2-x^2}{2x}$

If $x>\sqrt{2}$, then $x^2-2>0$ & $g(x)-x<0$

i.e. $g(x)<x$. — (2)

If $x\in[\sqrt{2},2]$, then $g(x)\in[\sqrt{2},2]$. Using (1) & (2).

$\therefore$ g maps $[\sqrt{2},2]$ into itself

$g'(x)=\frac{1}{2}\left(1-\frac{2}{x^2}\right) \Rightarrow |g'(x)|\leq\frac{1}{2}$; if $x\in[\sqrt{2},2]$

By fixed point theorem, the fixed point iteration $x_{n+1}=g(x_n)$, $n\geq 0$, converges if $x_0>\sqrt{2}$: Note that if p is fixed point of g, then $p=g(p) \iff p=\frac{1}{2}\left(p+\frac{2}{p}\right) \iff \frac{1}{2}p=\frac{1}{p}$

$\iff p^2=2$

$\iff p=\sqrt{2}$.