Error Analysis of iterative Methods:

Order of convergence:

Definition: suppose $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges to p, with $p_n \neq p$ for all n. If positive constants λ and α exist with

$$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^{\alpha}} = \lambda,$$

then $\{p_n\}_{n=0}^{\infty}$ converges to p of order α, with asymptotic error constant λ.

An iterative method of the form $p_n = g(p_{n-1})$ is said to be of order α if the sequence $\{p_n\}_{n=0}^{\infty}$ converges to the solution $p = g(p)$ of order α.

In general, a sequence with a high order of convergence converges more rapidly than a sequence with a lower order. The asymptotic constant affects the speed of convergence but not to the extent of the order. Two cases of order are given spetial attention.

(i) If $\alpha = 1$ (and $\lambda < 1$), the sequence is linearly convergent

(ii) If $\alpha = 2$, the sequence is quadratically convergent

Illustration of ~~lineat~~ comparision between linear and quadratic convergent sequences.

Illustration: suppose that $\{p_n\}_{n=0}^{\infty}$ is linearly convergent to 0 with

$$\lim_{n\to\infty} \frac{|p_{n+1}|}{|p_n|} = 0.5$$

and that $\{\tilde{p}_n\}_{n=0}^{\infty}$ is quadratically convergent to 0 with

$$\lim_{n\to\infty} \frac{|\tilde{p}_{n+1}|}{|\tilde{p}_n|^2} = 0.5.$$

For simplicity, assume that for each n, we have

$$\frac{|p_{n+1}|}{|p_n|} \approx 0.5 \quad \text{and} \quad \frac{|\tilde{p}_{n+1}|}{|\tilde{p}_n|^2} \approx 0.5.$$

For linearly convergent sequence, this means that

$$|p_n - 0| = |p_n| \approx 0.5\,|p_{n-1}| \approx (0.5)^2\,|p_{n-2}| \approx \cdots \approx (0.5)^n\,|p_0|$$

whereas the quadratically convergent sequence has

$$|\tilde{p}_n - 0| = |\tilde{p}_n| \approx (0.5)\,|\tilde{p}_{n-1}|^2 \approx (0.5)\left[(0.5)\,|\tilde{p}_{n-2}|^2\right]^2 = (0.5)^3\,|\tilde{p}_{n-2}|^4$$

$$\approx (0.5)^3\left[(0.5)\,|\tilde{p}_{n-3}|^2\right]^4 = (0.5)^7\,|\tilde{p}_{n-3}|^8$$

$$\approx \cdots \approx (0.5)^{2^n-1}\,|\tilde{p}_0|^{2^n}.$$

Quadratically convergent sequences are expected to converge much quicker than those that converge only linearly.

<u>Theorem</u>: Let $g \in C[a,b]$ be such that $g(x) \in [a,b]$ for all $x \in [a,b]$. Suppose, in addition, that g' is continuous on (a,b) and a positive constant $k<1$ exists with

$$|g'(x)| \leq k \quad \text{for all } x \in (a,b).$$

If $g'(p) \neq 0$, then for any number $p_0 \neq p$ in $[a,b]$, the sequence

$$p_n = g(p_{n-1}), \quad \text{for } n \geq 1,$$

converges only linearly to the unique fixed point p in $[a,b]$.

<u>proof:</u> we know by fixed point theorem from previous class that the sequence converges to p. By using Mean value theorem for g, for any n,

$$p_{n+1}-p = g(p_n)-g(p) = g'(\xi_n)(p_n-p),$$

where ξ_n is between p_n and p. Since $\{p_n\}_{n=0}^{\infty}$ converges to p, we also have $\{\xi_n\}_{n=0}^{\infty}$ converging to p. Since g' is continuous on (a,b), we have

$$\lim_{n\to\infty} g'(\xi_n) = g'(p).$$

Thus $$\lim_{n\to\infty} \frac{p_{n+1}-p}{p_n-p} = \lim_{n\to\infty} g'(\xi_n) = g'(p)$$

and $$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|} = |g'(p)|.$$

Hence if $g'(p) \neq 0$, fixed point iteration exhibits linear convergence with asymptotic error constant $|g'(p)|$.

<u>Remark:</u> Above theorem implies that higher-order convergence for fixed-point methods of the form $g(p)=p$ can occur only when $g'(p)=0$.

The next theorem describes additional conditions that ensure the quadratic convergence.

Theorem: Let p be a solution of the equation $x = g(x)$. suppose that $g'(p) = 0$ and g'' is continuous with $|g''(x)| < M$ on an open interval I containing p. Then there exists a $\delta > 0$ such that, for $p_0 \in [p-\delta, p+\delta]$, the sequence defined by $p_n = g(p_{n-1})$ when $n \geq 1$, converges at least quadratically to p. Moreover, for sufficiently large value of n,

$$|p_{n+1} - p| < \frac{M}{2} |p_n - p|^2.$$

proof: Choose $k \in (0,1)$ and $\delta > 0$ such that on the interval $[p-\delta, p+\delta]$, contained in I, we have $|g'(x)| \leq k$ and g'' continuous.

Since $|g'(x)| \leq k < 1$, we can easily see that the terms of the sequence $\{p_n\}_{n=0}^{\infty}$ are contained in $[p-\delta, p+\delta]$, whenever $p_0 \in [p-\delta, p+\delta]$.

Expanding $g(x)$ in Taylor polynomial for $x \in [p-\delta, p+\delta]$ gives

$$g(x) = g(p) + (x-p)\, g'(p) + \frac{g''(\xi)}{2} (x-p)^2$$

where ξ lies between x and p. The hypothesis $g(p) = p$ and $g'(p) = 0$ imply that

$$g(x) = p + \frac{g''(\xi)}{2} (x-p)^2.$$

In particular, when $x = p_n$,

$$p_{n+1} = g(p_n) = p + \frac{g''(\xi_n)}{2} (p_n - p)^2$$

with ξ_n between p_n and p. Thus

$$p_{n+1} - p = \frac{g''(\xi_n)}{2} (p_n - p)^2$$

Since $|g'(x)| \le k < 1$ on $[p-\delta, p+\delta]$ and g maps $[p-\delta, p+\delta]$ into itself, it follows that [fixed point theorem]. the sequence $\{p_n\}_{n=0}^{\infty}$ converges to p. and ξ_n is between p_n and p for each n, so $\{\xi_n\}_{n=0}^{\infty}$ also converges to p, and

$$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^2} = \frac{|g''(p)|}{2}$$

This implies that the sequence $\{p_n\}_{n=0}^{\infty}$ is quadratically convergent if $g''(p) \neq 0$ and of higher order if $g''(p)=0$.

Since g'' is continuous on $[p-\delta, p+\delta]$ and strictly bounded by M on this interval, for large enough n, we have,

$$|p_{n+1}-p| < \frac{M}{2}|p_n-p|^2.$$

For Newton's method, we have

$$g(x) = x - \frac{f(x)}{f'(x)}$$ and

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)f''(x)}{[f'(x)]^2},$$

which implies $g'(p)=0$, whenever $f(p)=0$, that is p is zero of f.

Therefore Newton's method exhibits quadratic convergence.

Newton's method requires $f(p_n)$ and $f'(p_n)$ at each iteration. To circumvent this difficulty, note that

$$f'(p_{n-1}) = \lim_{x \to p_{n-1}} \frac{f(x) - f(p_{n-1})}{x - p_{n-1}}.$$

If p_{n-2} is close to p_{n-1}, then

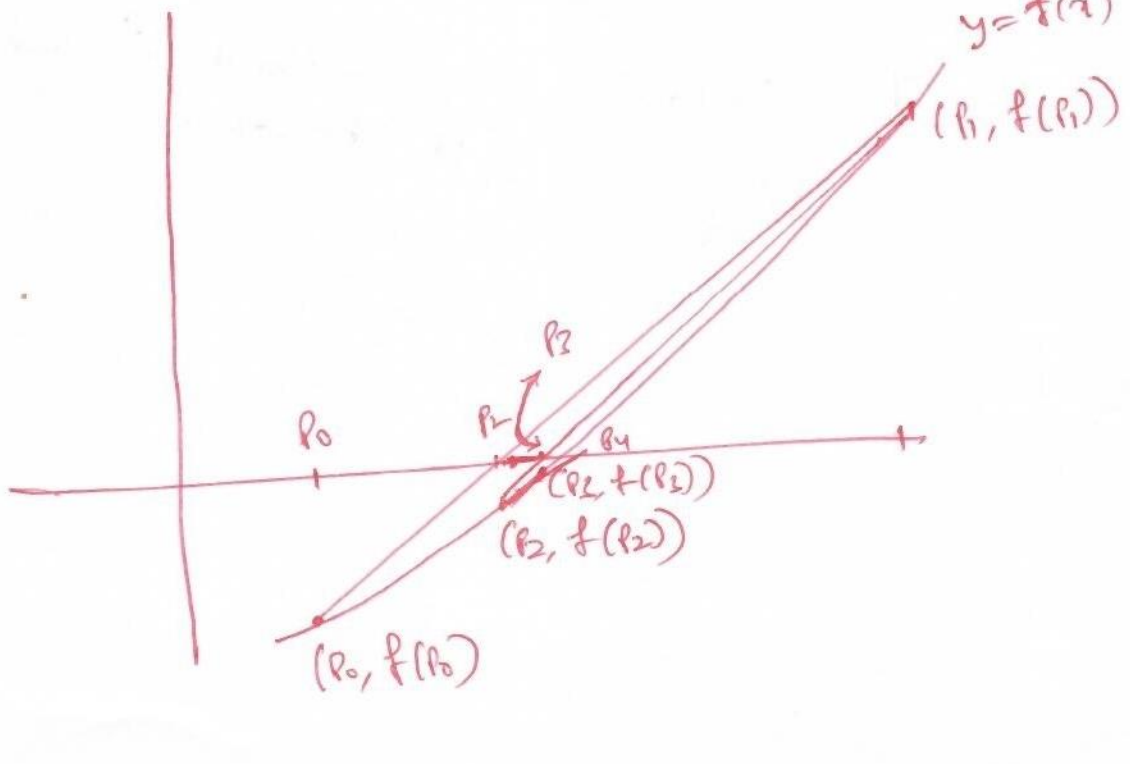
$$f'(p_{n-1}) \approx \frac{f(p_{n-2}) - f(p_{n-1})}{p_{n-2} - p_{n-1}}.$$

Using this approximation for $f'(p_{n-1})$ in Newton's method, we get

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}.$$

This technique is called the ~~simpson's method~~. secant method.

Starting with two initial approximations p_0 and p_1 the approximation p_2 is the x-intercept of the line joining $(p_0, f(p_0))$ and $(p_1, f(p_1))$. The approximation p_n is the x-intercept of the line joining $(p_{n-2}, f(p_{n-2}))$ and $(p_{n-1}, f(p_{n-1}))$, which we can see graphically:



Secant Method:

To find a solution to $f(x)=0$ given initial approximations p_0 and p_1

INPUT: p_0, p_1, TOL, N_0.

OUTPUT: approximate solution p (or) message failure.

step 1: set $i=2$;
$q_0=f(p_0)$;
$q_1=f(p_1)$.

Step 2: while $i \le N_0$ do steps 3-6.

step 3: set $p = p_1 - q_1(p_1-p_0)/(q_1-q_0)$ (Compute p_i)

step 4: if $|p-p_1| < TOL$, then
OUTPUT(p); (The procedure is successful)
STOP.

step 5: set $i=i+1$

step 6: set $p_0=p_1$; (update p_0, q_0, p_1, q_1)
$q_0=q_1$;
$p_1=p$;
$q_1=f(p)$.

Step 7 OUTPUT (The method failed after N_0 iterations).

STOP.

Remark: The Bisection method brackets root p of the equation $f(x)=0$, for each iteration a root lies between a_n and b_n, (nth step). and the error satisfies

$$|p_n - p| < \frac{1}{2}|b_n - a_n|.$$

Root bracketing is not guaranteed for either Newton's method or the secant method.

The method of False position: (Regula Falsi).

generates approximations in the same manner as the secant method, but it brackets a root, and it includes a test for this.

First choose two initial approximations p_0 and p_1 with $f(p_0)\cdot f(p_1)<0$. The approximation p_2 is the same as in the secant method, as the x-intercept of the line joining $(p_0, f(p_0))$ and $(p_1, f(p_1))$. To compute p_3, consider $f(p_2)\cdot f(p_1)$.

- If sign $f(p_2)\cdot f(p_1)<0$, then p_1 and p_2 bracket a root. Choose p_3 as the x-intercept of the line joining $(p_1, f(p_1))$ and $(p_2, f(p_2))$
- If not, chose p_3 as the x-intercept of the line joining $(p_0, f(p_0))$ and $(p_2, f(p_2))$ and then interchang the indices on p_0 and p_1.

In a similar manner, once p_3 is found, the sign of $f(p_3) \cdot f(p_2)$ determines whether we use p_2 and p_3, or, p_3 and p_1 to compute p_4. In the latter case a relabling of p_2 and p_1 is performed.

False Position: To find a solution to $f(x)=0$ given continuous function on $[p_0, p_1]$ where $f(p_0) \cdot f(p_1) < 0$.

INPUT: p_0, p_1, TOL, N_0.

OUTPUT: approximate solution p. (or) message failure.

Step 1: set $i=2$; $q_0 = f(p_0)$; $q_1 = f(p_1)$:

Step 2: While $i \le N_0$, do steps 3-7.

Step 3: set $p = p_1 - q_1 (p_1 - p_0) / (q_1 - q_0)$ (compute p_i)

Step 4: If $|p - p_1| < TOL$, then

OUTPUT (p) [successful procedure]

STOP.

Step 5: set $i = i+1$; $q = f(p)$.

Step 6: If $q \cdot q_1 < 0$ then set $p_0 = p_1$;
$q_0 = q_1$;

Step 7: set $p_1 = p$;
$q_1 = q$;

Step 8: OUTPUT (method failed after N_0 iterations).

STOP.