The shooting Method for Nonlinear Problems: ①

Consider the two point boundary value problem

$$y'' = f(x, y, y'), \quad a \le x \le b, \quad y(a) = \alpha \text{ and } y(b) = \beta. \quad \text{—(1)}$$

Unlike the linear two point boundary value problem, which was solved using two initial value problems, the nonlinear two point boundary value problem is solved using a sequence of initial value problems as follows involving a parameter t.

These sequence of problems have the form

$$y'' = f(x, y, y'), \text{ for } a \le x \le b, \quad y(a) = \alpha \text{ and } y'(a) = t. \quad \text{—(2)}$$

We do this by choosing the parameters $t = t_k$ in a manner to ensure that

$$\lim_{k \to \infty} y(b, t_k) = y(b) = \beta,$$

where $y(b, t_k)$ is the unique solution of (2) with $t = t_k$, and y is the solution of (1).

This technique is called a "shooting method", since the procedure involves solving a sequence of IVPs (firing the object with initial y and y', displacement & velocity, to hit the target $y(b) = \beta$).

If $y(x, t)$ denote the solution of (2), then the problem at hand is to find t such that

$$y(b, t) - \beta = 0. \quad \text{—(3)}$$

The solution $y(x, t)$ depends nonlinearly on t, finding t such that $y(b,t) - \beta = 0$ ②

is a problem of solving root of a nonlinear equation ③. ~~$y(b,t) - \beta$~~

we can apply any of the root finding methods to solve for a solution of ③.

Secant Method:

We choose two initial approximation t_0 and t_1, and then generate the remaining terms of the sequence by

$$t_k = t_{k-1} - \frac{(y(b,t_{k-1}) - \beta)}{y(b,t_{k-1}) - y(b,t_{k-2})} \cdot (t_{k-1} - t_{k-2}),$$

$k = 2, 3, \ldots$

Newton Method:

To use the Newton method to generate the sequence $\{t_k\}$, only one initial approximation t_0 is needed. The iteration has the form

$$t_k = t_{k-1} - \frac{y(b,t_{k-1}) - \beta}{\frac{dy}{dt}(b,t_{k-1})} \quad \text{——} \quad (N)$$

$k = 1, 2, \ldots$

Note that the nonlinear equation to solve is

$$F(t) = y(b,t) - \beta \Rightarrow \frac{dF}{dt}(t_{k-1}) = \frac{dy}{dt}(b,t_{k-1}).$$

Computation of $\frac{\partial y}{\partial t}(b, t_{k-1})$ requires a derivative of $y(b,t)$ with respect to t. But the explicit dependence of $y(b,t)$ on t is not known, and we know only $y(b, t_0)$, $y(b, t_1)$, ... $y(b, t_{k-1})$.

In order to find the derivative of $y(x,t)$ with respect to t, we rewrite IVP (2) emphasizing that the solution y depends on both x and the parameter t.

$$y''(x,t) = f(x, y(x,t), y'(x,t)), \text{ for } a \le x \le b, \; y(a,t) = \alpha,$$
$$\text{and } y'(a,t) = t. \quad \text{—— (4)}$$

The prime notation (y') is to denote the derivative of y with respect to x.

Take partial derivative of the differential equation in (4) with respect to t:

$$\frac{\partial}{\partial t} y''(x,t) = \frac{\partial f}{\partial t}(x, y(x,t), y'(x,t))$$
$$= \frac{\partial f}{\partial x}(x, y(x,t), y'(x,t)) \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}(x, y(x,t), y'(x,t)) \frac{\partial y}{\partial t}(x,t)$$
$$+ \frac{\partial f}{\partial y'}(x, y(x,t), y'(x,t)) \frac{\partial y'}{\partial t}(x,t).$$

since x and t are independent variables

$$\frac{\partial x}{\partial t} = 0.$$

The equation simplifies to

$$\frac{\partial y''}{\partial t}(x,t) = \frac{\partial f}{\partial y}(x, y(x,t), y'(x,t))\frac{\partial y}{\partial t}(x,t) + \frac{\partial f}{\partial y'}(x, y(x,t), y'(x,t))\frac{\partial y'}{\partial t}(x,t), \quad (5)$$

for $a \le x \le b$. The initial conditions give

$$\frac{\partial y}{\partial t}(a,t) = 0 \quad \text{and} \quad \frac{\partial y'}{\partial t}(a,t) = 1$$

Introduce the notation $z(x,t) = \frac{\partial y}{\partial t}(x,t)$ and assume that the order of differentiation can be reversed. We note that

$$\frac{\partial}{\partial t} y''(x,t) = \left[\frac{\partial y}{\partial t}(x,t)\right]'' = z''(x,t)$$ (prime is derivative w.r.t. x)

$$\frac{\partial y}{\partial t}(x,t) = z(x,t)$$

$$\frac{\partial y'}{\partial t}(x,t) = z'(x,t)$$, and

the equation (5) takes the form

$$\left.\begin{aligned} z''(x,t) &= \frac{\partial f}{\partial y}(x,y,y')\, z(x,t) + \frac{\partial f}{\partial y'}(x,y,y')\, z'(x,t) \\ &\text{with } z(a,t) = 0 \text{ and } z'(a,t) = 1. \end{aligned}\right\} (6)$$

The Newton method requires that two IVPs (4) and (6) be solved for each iteration.

The Newton method in (N) takes the form

$$t_k = t_{k-1} - \frac{(y(b, t_{k-1}) - \beta)}{z(b, t_{k-1})}, \quad k \ge 1.$$

* Note that first solve (4) for given t_{k-1}, and then solve (6) and use $y(b, t_{k-1})$ and $z(b, t_{k-1})$ in the Newton's method.