UMC-202 Problems Sheet Fall 2024

1. Let T be the triangle with vertices $z_1 = (0,0), z_2 = (1,0), z_3 = (0,1)$. Construct the canonical basis $\{\lambda_j\}_{J=1}^{3}$, $\lambda_j \in P_1(T)$ satisfying $\lambda_j(z_k) = \delta_{jk}$. Show that $\{\lambda_j\}_{J=1}^{3}$ is a linearly independent set. Consider the interpolation $\Pi_T : C(T) \to P_1(T)$ by $\Pi_T u = \sum_{j=1}^{3} u(z_j)\lambda_j$. Show that $\Pi_T v = v$ for all $v \in P_1(T)$.

2. Let T be the rectangle with vertices $z_1 = (0,0), z_2 = (1,0), z_3 = (1,1), z_4 = (0,1)$. Construct the canonical basis $\{\psi_j\}_{J=1}^{4}$, $\psi_j \in Q_1(T)$ satisfying $\lambda_j(z_k) = \delta_{jk}$. Show that $\{\lambda_j\}_{J=1}^{4}$ is a linearly independent set. Consider the interpolation $\Pi_T : C(T) \to Q_1(T)$ by $\Pi_T u = \sum_{j=1}^{4} u(z_j)\lambda_j$. Show that $\Pi_T v = v$ for all $v \in Q_1(T)$.

3. Let $\hat{T}$ be the triangle with vertices $\hat{a}_1 = (0,0), \hat{a}_2 = (1,0), \hat{a}_3 = (0,1))$. Let T be the triangle with vertices $a_1 = (x_1, y_1), a_2 = (x_2, y_2), a_3 = (x_3, y_3)$. Construct an affine map $F_T(x,y) = (b_{11}x + b_{12}y + d_1, b_{21}x + b_{22}y + d_2)$ mapping $\hat{T}$ onto T by mapping $F_T(\hat{a}_j) = a_j, j = 1,2,3$. Find the inverse of the map F_T.

4. Let $\mathcal{T}_h$ be the triangulation of unit square $\Omega = [0,1] \times [0,1]$ into triangles. Let $V_h = \{v_h \in C(\Omega) : v|_T \in P_1(T), T \in \mathcal{T}_h\}$ with dimension N and $\{\phi_j\}_{j=1}^{N}$ be the canonical basis of V_h such that $\phi_j(z_k) = \delta_{jk}$, $1 \le j,k \le N$, where $\{z_k\}_{k=1}^{N}$ is the set of all vertices in $\mathcal{T}_h$. Recall $P_1(T) = \{a_0 + a_1 x + a_2 y : x, y \in T, \ \ a_0, a_1, a_2 \in \mathbb{R}\}$. Let n denote the unit outward normal vector to $\partial\Omega$, the boundary of Ω. Show that the solution $u \in C^2(\Omega)$ of $-\Delta u + u = f$ in Ω, $\nabla u \cdot n = g$ on $\partial\Omega$, satisfies

$$\int_\Omega (\nabla u \cdot \nabla v_h + u v_h)\, d\Omega = \int_\Omega f v_h \, d\Omega + \int_{\partial\Omega} g v_h \, dS, \text{ for all } v_h \in V_h,$$

where the integral on $\partial\Omega$ is the piece-wise line integration and ∇v_h is the piece-wise (triangle wise) gradient of v_h.