UMC-202 Problems Sheet Fall 2024

1. Determine the number of iterations necessary to solve $f(x) = x^3 + x - 2 = 0$ with accuracy 10^{-3} using bisection method starting with $a_1 = 0$ and $b_1 = 3$.

2. Consider the function $g(x) = e^{-x} + \frac{x}{2}$, $x \in \mathbb{R}$. Show that g has a fixed point $\alpha > 0$ and for x_0 sufficiently close to α, the sequence $\{x_n\}_{n \geq 1}$ of fixed point iteration $x_n = g(x_{n-1})$ converges to α.

3. A sequence $\{x_n\}_{n=0}^{\infty}$ is said to be a Cauchy sequence, if for every $\epsilon > 0$, there is a positive integer N such that
$$|x_m - x_n| < \epsilon, \quad \forall\, m, n \geq N.$$
A function $g : [a, b] \to \mathbb{R}$ is said to be a contraction if there is some $0 < k < 1$ such that
$$|g(x) - g(y)| \leq k|x - y|, \quad \forall\, x, y \in [a, b].$$
Problem: Suppose that $g : [a, b] \to [a, b]$ is a contraction and for any $x_0 \in [a, b]$, let the sequence $\{x_n\}_{n \geq 1}$ be generated by $x_n = g(x_{n-1}), (n \geq 1)$. Show that $\{x_n\}$ is a Cauchy sequence. Further, show that $\{x_n\}$ converges to the fixed point of g in $[a, b]$.

4. Let $x_0, x_1, \ldots, x_n$ be $(n+1)$ distinct points in $\mathbb{R}$ and $L_{n,k}(x)$, $0 \leq k \leq n$, be the Lagrange polynomials of degree less than or equal to n satisfying $L_{n,k}(x_j) = \delta_{jk}$ for $0 \leq j, k \leq n$, where $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ if $j \neq k$. Show that $\{L_{n,k}(x)\}_{k=0}^{n}$ forms a basis for the real linear space $\mathcal{P}_n$ consisting of all polynomials of degree less than or equal to n (show that they are linearly independent and span $\mathcal{P}_n$). Furthermore show that $\sum_{k=0}^{n} L_{n,k}(x) = 1$ for all x.

5. Consider the data points (x_i, y_i), $0 \leq i \leq 3$ given by $(0, 0), (1, 1), (2, \alpha), (3, 1)$. Let $p_3(x)$ be the interpolating polynomial of degree less than or equal to 3 interpolating this data. If $p_3(1.5) = 2$, then find α and justify.

6. Let $p_3(x)$ be the interpolating polynomial of degree less than or equal to 3 interpolating the data $(x_i, f(x_i))$, $1 \leq i \leq 4$, where

x_i	-1	0	1	2
$f(x_i)$	0	1	0	9

If $f^{(4)}(x) = 1$ (fourth order derivative of f) for all x, then find $f(x)$ and the error $|P_3(1.5) - f(1.5)|$.

7. Assume that f, f', f'' are continuous in the interval $[\alpha - 1, \alpha + 1]$, $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. Then for x_0 sufficiently close to α, the sequence $\{x_n\}$ $(n \geq 1)$ generated by
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0,$$
converges to α. Moreover show that
$$\lim_{n\to\infty} \frac{|\alpha - x_{n+1}|}{|\alpha - x_n|^2} = \frac{|f''(\alpha)|}{2|f'(\alpha)|}.$$