(1)

$\Rightarrow$ Consider
$$-\Delta u = f \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega.$$

$$V_h = \{ v_h \in C^0(\bar{\Omega}) : v_h|_T \in R_1(T), T \in \mathcal{T}_h, v_h = 0 \ \partial\Omega \}$$

$R_1(T) = P_1(T)$ if T is triangle
$R_1(T) = Q_1(T)$ if T is rectangle.

Let $\{\phi_j\}_{j=1}^N$ be the basis of V_h, $\{z_j\}_{j=1}^N$ be the set of interior vertices in $\mathcal{T}_h$, $N = \dim(V_h)$, $\phi_j(z_k) = \delta_{jk}$, $1 \le j, k \le N$.

We know that (proved it)
$$\int_\Omega \nabla u \cdot \nabla \phi_j = \int_\Omega f \phi_j \qquad 1 \le j \le N. \quad — (1)$$

Since any $v_h \in V_h$ can be written as $v_h = \sum_{j=1}^N \beta_j \phi_j$,
multiplying (1) with β_j and taking sum over $j=1$ to N,
$$\int_\Omega \nabla u \cdot \sum_{j=1}^N \beta_j \nabla \phi_j = \int_\Omega f \sum_{j=1}^N \beta_j \phi_j$$
$$\Rightarrow \quad \int_\Omega \nabla u \cdot \nabla v_h = \int_\Omega f v_h \quad \text{for any } v_h \in V_h \quad -(2)$$

FEM: Find $u_h \in V_h$ such that
$$\int_\Omega \nabla u_h \cdot \nabla v_h = \int_\Omega f v_h \quad \text{for all } v_h \in V_h. \quad -(3)$$

from (1) and (2), $\int_\Omega \nabla(u - u_h) \cdot \nabla v_h = 0$ for all $v_h \in V_h$ —(4)

Define
$$a(v, w) = \int_\Omega \nabla v \cdot \nabla w, \qquad L(w) = \int_\Omega f w$$

Then (2) is written as
$$a(u, v_h) = L(v_h) \quad \forall v_h \in V_h,$$
and (3) is written as
$$a(u_h, v_h) = L(v_h) \quad \forall v_h \in V_h.$$
Then (4) takes the form
$$a(u - u_h, v_h) = 0 \quad \forall v_h \in V_h$$

Define $|||v|||^2 = a(v,v)$.

Cauchy Schwarz inequality

$$\int_\Omega fg \le \left(\int_\Omega f^2\right)^{1/2}\left(\int_\Omega g^2\right)^{1/2}$$

implies that

$$|a(v,w)| \le \left(\int_\Omega |\nabla v|^2\right)^{1/2}\left(\int_\Omega |\nabla w|^2\right)^{1/2} = |||v|||\,|||w|||.$$

from (4), If $e = u - u_h$. then for any $v_h \in V_h$

$$|||e|||^2 = a(e,e) = a(u-u_h, u-u_h)$$

$$= a(u-u_h, u-v_h) + \underbrace{a(u-u_h, v_h-u_h)}_{0}$$

$$= a(u-u_h, u-v_h)$$

$$\le |||u-u_h|||\;|||u-v_h|||$$

$$\therefore \quad |||u-u_h||| \le |||u-v_h||| \quad \text{for all } v_h \in V_h$$

$$\therefore \quad |||u-u_h||| = \inf_{v_h \in V_h} |||u-v_h||| \le |||u-\pi_h u||| \quad (5)$$

where $\pi_h u \in V_h$ be an interpolation of u which we construct and quantify $|||u-\pi_h u|||$ interms of the mesh size h.

Let $\pi_h u \in V_h$ be an interpolation of u which we define soon. Then

$$|||u-\pi_h u|||^2 = \int_\Omega |D(u-\pi_h u)|^2 = \sum_T \int_T |D(u-\pi_h u)|^2$$

we prove that $\int_T |D(u-\pi_h u)|^2 \le c\, h_T^2 \left(\int_T |D^2 u|^2\right)$

Where D^2 is the Hessian of u

$h_T = \text{diam}(T)$, c is independent of T.

Then

$$|||u-\pi_h u|||^2 \le \sum_T C h_T^2 \int_T |D^2 u|^2$$

$$\le C h^2 \sum_T \int_T |D^2 u|^2 \quad \text{where} \quad h = \max_{T\in \mathcal{T}_h} h_T.$$

and

$$|||u-\pi_h u||| \le Ch\, |u|_{2,\Omega}, \quad |u|_{2,\Omega} = \left(\int_\Omega |D^2 u|^2\right)^{1/2}.$$

Thus combining this with (*), we get

$$|||u-u_h||| \le Ch\, |u|_{2,\Omega}.$$

We define $\pi_h u \in V_h$ by

$$\pi_h u(x) = \pi_T u(x), \quad \text{if } x \in T. \quad x=(x,y)\in \mathbb{R}^2$$

that is we define $\pi_h|_T = \pi_T$ on each triangle.

since $\pi_h u \in V_h$, $\quad \pi_h u(x) = \sum_{j=1}^{N} \alpha_j \phi_j(x)$

take $\alpha_j = u(z_j)$.

$$\pi_h u(x) = \sum_{j=1}^{N} u(z_j)\phi_j(x) \Rightarrow \pi_h u(z_j) = u(z_j)$$

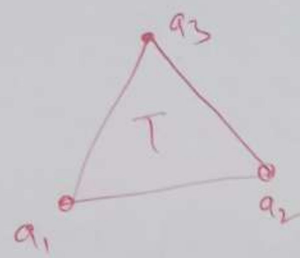
where $\{z_j\}_{j=1}^{N}$ is the set of interior vertices of $\mathcal{T}_h$

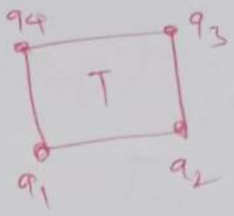
Fix a triangle, T, then for any $z\in T$,
there are only three ϕ_j functions are non zero on T if T is a triangle, and,
there are only four ϕ_j functions are non zero on T if T is a rectangle

$$z\in T;\ \pi_h u(z) = \pi_T u(z) = \sum_{j=1}^{N_T} u(z_j)\phi_j^T(z)$$

$N_T = 3$ if T is triangle

[illegible] if T is rectangle





$\Pi_T u \in P_1(T)$ if T is triangle

$\Pi_T u \in Q_1(T)$ if T is rectangle

Theorem: Let D be a connected open set in $\mathbb{R}^2$. Then for any $u \in C^2(\bar{D})$,

$$\inf_{p \in P_1(D)} \left(\int_D |D(u-p)|^2 \right)^{1/2} \le C_D \left(\int_D |D^2 u|^2 \right)^{1/2}$$

where C_D is a constant depending on D.

$$|v|_{1,D} = \left(\int_D |Dv|^2 \right)^{1/2}; \qquad |v|_{2,D} = \left(\int_D |D^2 v|^2 \right)^{1/2}.$$

Note that for any $p \in P_1(T)$,

$\Pi_T p(z) = p(z)$ for all $z \in T$. $[\Pi_T p = p, \ \forall p \in P_1]$

$\Pi_T p$ preserves all $p \in P_1(T)$.

Then for any v

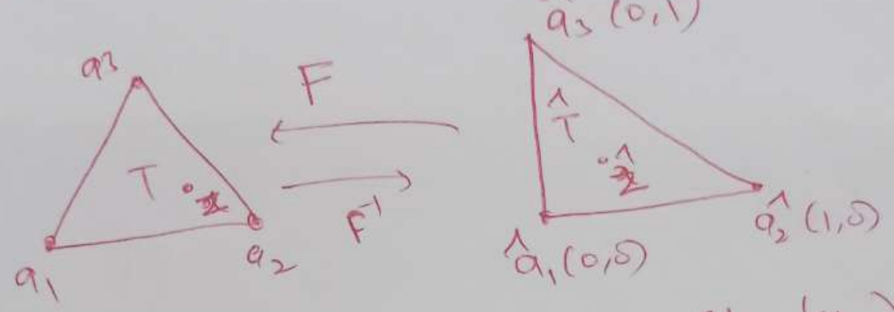
$$\begin{aligned}
|v - \Pi_h v|_{1,T} &= |v - p + p - \Pi_h v|_{1,T} \\
&= |v - p + \Pi_h p - \Pi_h v|_{1,T} \\
&= |(v-p) - \Pi_h (v-p)|_{1,T} \\
&= |(I - \Pi_h)(v-p)|_{1,T} \\
&\le \|I - \Pi_h\|_* \, |v-p|_{1,T} ;
\end{aligned}$$

$C^* = \|I - \Pi_h\|_*$

$$|v - \Pi_h v|_{1,T} \le C^* \inf_{p \in P_1(T)} |v-p|_{1,T} \le C_N C^* |v|_{2,T}.$$

The constants C^*, C_N are to be esti[...]
of the mesh size.

We estimate them by using scaling arguments



$F(\hat{x}) = B\hat{x} + d = x$ $\quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$; $\quad d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$

$x = B\hat{x} + d$. $\quad$ B is invertible

$\hat{x} = B^{-1}(x - d) := F^{-1}(x)$.

F is computed by $\quad F(\hat{a}_j) = a_j$. $\quad 1 \le j \le 3$.

F is an affine map.

If $v \in C(T)$, then $\hat{v} \in C(\hat{T})$ is defined by

$\hat{v}(\hat{x}) = v(x) = (v \circ F)(\hat{x}) = v(F(\hat{x}))$

ll'y $\quad v(x) = \hat{v}(\hat{x}) = \hat{v}(F^{-1}(x))$.

Let $\lambda_j(z) \in P_1(T)$ be such that $\lambda_j(a_k) = \delta_{jk}$

and $\hat{\lambda}_j(\hat{z}) \in P_1(\hat{T})$ be such that $\hat{\lambda}_j(\hat{a}_k) = \delta_{jk}$

But note that

$$\lambda_j(z) = \hat{\lambda}_j(\hat{z}).$$

If for $v \in C(T)$, $\quad \Pi_T v(z) = \sum_{j=1}^{3} v(a_j)\,\lambda_j(z)$

$$= \sum_{j=1}^{3} \hat{v}(\hat{a}_j)\,\hat{\lambda}_j(\hat{z})$$

$$= \hat{\Pi}_{\hat{T}}\hat{v}(\hat{z}).$$

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The interpolation of v in $P_1(T)$ is the same as interpolation of $\hat{v}$ in $P_1(\hat{T})$, where $v(z) = \hat{v}(\hat{z})$.

~~Therefore~~

For any $v \in C^2(T)$, $\hat{v}(\hat{z}) = v(z)$, then $\hat{v} \in C^2(\hat{T})$.

$$\frac{\partial v}{\partial z_j}(z) = \frac{\partial}{\partial z_j}\left(\hat{v}(F^{-1}(z))\right) = \sum_{k=1}^{2} \frac{\partial \hat{v}}{\partial \hat{z}_k}(\hat{z}) \cdot \frac{\partial \hat{z}_k}{\partial z_j}$$

$$|Dv(z)| \leq |\hat{D}\hat{v}(\hat{z})| \, \|B^{-1}\|$$

$A = \mathbb{R}^2 \to \mathbb{R}^2$

$\|A\| = \sup_{|z|=1} |Az|$

Similarly

$$|D^2 v(z)| \leq |\hat{D}^2 \hat{v}(\hat{z})| \, \|B^{-1}\|^2.$$

$D = \left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right)$

$\hat{D} = \left(\frac{\partial}{\partial \hat{z}_1}, \frac{\partial}{\partial \hat{z}_2}\right)$

On the other hand

$$|\hat{D}\hat{v}(\hat{z})| \leq |Dv(z)| \, \|B\|$$

$$|\hat{D}^2 \hat{v}(\hat{z})| \leq |D^2 v(z)| \, \|B\|^2.$$

D^2 Hessian w.r.t. z

$\hat{D}^2$ " " $\hat{z}$.

Note that $\int_T |v(z)| dz = \int_{\hat{T}} |\hat{v}(\hat{z})| \, |\det B| \, d\hat{z}$.

$z = F(\hat{z})$; $dz = |JF| \, d\hat{z}$.

↳ Jacobian of F.

By change of variables

$$\int_{\hat{T}} |\hat{v}(\hat{z})| \, d\hat{z} = \int_T |v(z)| \, |\det B^{-1}| \, dz$$

$$\int_T |D(v-\Pi_T v)|^2 dz \le \|B^{-1}\|^2 \int_{\hat T} |\hat D(\hat v-\hat\Pi_T \hat v)|^2 |\det B|\, d\hat z$$

$$= \|B^{-1}\|^2 |\det B| \int_{\hat T} |\hat D(\hat v-\hat\Pi_T \hat v)|^2 d\hat z$$

But $$\int_{\hat T} |\hat D(\hat v-\hat\Pi_T \hat v)|^2 \le C \int_{\hat T} |\hat D^2 \hat v|^2$$

$$\le C \|B\|^4 |\det B^{-1}| \int_T |D^2 v|^2 dz$$

$$\therefore\ |v-\Pi_T v|_{1,T}^2 \le C \|B\|^4 \|B^{-1}\|^2 |\det B|\,|\det B^{-1}|\ |v|_{2,T}.$$

since $|\det(B)|\cdot|\det B^{-1}| = 1$.

$$|v-\Pi_T v|_{1,T} \le C\|B\|^2 \|B^{-1}\|\ |v|_{2,T}.$$

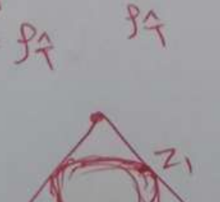
For a matrix A the norm $\|A\|$ can be estimated as follows:

$$\|A\| = \sup_{|z|=r} \frac{|Az|}{r}, \quad r>0.$$

Let ρ_T and h_T be the diameters of incircle and circumcircle of T, respectively. Similar $\rho_{\hat T}$ and $h_{\hat T}$ be the diameters of incircle and circumcircle of $\hat T$, respectively.

$$\|B\| = \sup_{|z|=\rho_{\hat T}} \frac{|Bz|}{\rho_{\hat T}}.$$

For any $z \in \mathbb{R}^2$ with $|z| = \rho_{\hat T}$, there exist $\hat z_1$ and $\hat z_2$ in $\hat T$ such that $\hat z_1 - \hat z_2 = \hat z$ and $|\hat z_1 - \hat z_2| = \rho_T$



For $|\hat{z}| = \rho_{\hat{T}}$, $\quad \hat{z}_1 - \hat{z}_2 = \hat{z}$,

and $\quad |B\hat{z}| = |B(\hat{z}_1 - \hat{z}_2)| = |B\hat{z}_1 - B\hat{z}_2| = |F(\hat{z}_1) - F(\hat{z}_2)|$

and $\quad |F(\hat{z}_1) - F(\hat{z}_2)| \le h_T$.

$$\therefore \quad \frac{|B\hat{z}|}{\rho_{\hat{T}}} \le \frac{h_T}{\rho_{\hat{T}}} \quad \Rightarrow \quad \|B\| \le \frac{h_T}{\rho_{\hat{T}}}$$

Similarly we can prove $\quad \|B^{-1}\| \le \dfrac{h_{\hat{T}}}{\rho_T}$.

$$\therefore \quad |v - \pi_T v|_{1,T} \le C \, \frac{h_T^2}{\rho_{\hat{T}}^2} \cdot \frac{h_{\hat{T}}}{\rho_T} \, |v|_{2,T}.$$

$$= C \cdot h_T \cdot \left(\frac{h_T}{\rho_T}\right) \cdot \left(\frac{h_{\hat{T}}}{\rho_{\hat{T}}^2}\right) |v|_{2,T}.$$

Since $\hat{T}$ is fixed, $\dfrac{h_{\hat{T}}}{\rho_{\hat{T}}^2}$ is constant

If the shape of triangles T, is such that $\dfrac{h_T}{\rho_T} \le C$

for all triangles $T \in \mathcal{T}_h$, then

$$|v - \pi_T v|_{1,T} \le C \, h_T \, |v|_{2,T}.$$

Then $\quad \left(\sum_T |v - \pi_T v|_{1,T}^2 \right)^{1/2} \le C h \left(\sum_T |v|_{2,T}^2 \right)^{1/2}$

$$|v - \pi_h v|_{1,\Omega} \le C h \, |v|_{2,\Omega},$$

where $\quad h = \max\limits_{T \in \mathcal{T}_h} h_T$.

finally

$$|u - u_h|_{1,\Omega} \le |u - \pi_h u|_{1,\Omega} \le C h \, |u|_{2,\Omega}.$$

$\boxed{C}$