

# General Theory of Relativity – The Power of Speculative Thought

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As readers would be aware, Einstein presented his Special Theory of Relativity in 1905. It soon became clear that while this theory could encompass mechanics and electromagnetism, gravity lay beyond its reach. The effort to reconcile special relativity with Newtonian gravitation theory turned out to be exceptionally demanding, and it led to the General Theory which transcends both these starting points.

Prof S Chandrasekhar remarks in an article about the General Theory of Relativity – “Only by a mixture of physical reasonableness, mathematical simplicity and aesthetic sensibility can one arrive at Einstein’s field equations. The general theory of relativity is, in fact, an example of the power of speculative thought” [1].

In the later period, Einstein in his Kyoto lecture narrated the story of his first idea of General Theory of Relativity (see pp.178, 179 of [2]). “I was sitting in the chair in the patent office at Bern when all of a sudden a thought occurred to me: “If a person falls freely he will not feel his own weight”. I was startled. This simple thought made a deep impression on me. It impelled me towards the theory of gravitation”. In the present context I quote from an interesting paper written by Einstein in 1920. Although it was not formally published in a journal due to some reasons, it is an interesting document revealing the thoughts and feelings of the great scientist about the beginning of GTR. Einstein writes – “Then there occurred to me .... the happiest thought of my life, in the following form. The gravitational field has only a relative existence in a very similar



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## Keywords

Special relativity, general relativity, equivalence principle.

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way to the electric field generated by magnetoelectric induction. Because, for an observer falling freely from the roof of a house there exists at least in his immediate surroundings no gravitational field. Indeed if the observer drops some bodies then these remain relative to him in a state of rest or of uniform motion, independent of their particular chemical or physical nature...”

This is really the ‘equivalence principle’, which forms the basis of the General Theory of Relativity. Beginning with the insight provided by the ‘equivalence principle’, let us sketch the progression of ideas. In a static homogeneous gravitational field the equation of motion is given by

$$m_i \frac{d^2 \vec{r}}{dt^2} = m_g \vec{g}, \quad (1)$$

where  $\vec{g}$  is independent of space and time coordinates,  $m_i$  is the inertial mass of the body, and  $m_g$  its gravitational mass.  $m_i$  and  $m_g$  are, in fact, the measures of the body’s ‘resistance’ to the action of force and its capability to respond to the gravitational field respectively. It is found observationally that  $m_g = m_i$ . This equality of the gravitational and inertial masses is referred to as the Eötvös Law. Eötvös and his collaborators concluded experimentally that the equality is true to one part in  $10^9$ . Subsequent repetitions of the experiment with refined instruments show that the equality holds to the accuracy of one part in  $10^{11}$  (See [3]). A further reduction in the uncertainty upto the extent of less than one part in  $10^{12}$  was due to Braginskii and Panov [4]. Now the equation of motion (1) reduces to  $d^2 \vec{r}/dt^2 = \vec{g}$ , which under the transformation  $\vec{r}' = \vec{r} - \frac{1}{2} \vec{g} t^2$  yields in a straight forward manner the interesting result

$$\frac{d^2 \vec{r}'}{dt^2} = 0. \quad (2)$$

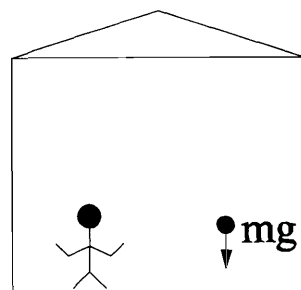
The gravitational field seems to vanish in the primed coordinate system. The primed frame is actually a frame



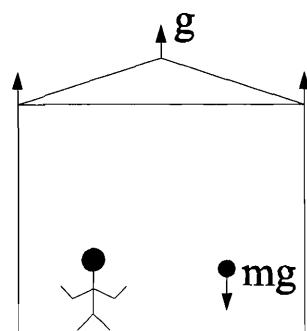
moving with the same acceleration  $\vec{g}$  with respect to the unprimed frame.

1. Suppose a man is at rest in an elevator and the elevator is at rest on the earth. The man can then determine the gravitational field by noting the acceleration of a freely falling body (See *Figure 1*).
2. Next, if there is no chance of getting information from outside, he might as well argue that the elevator is accelerating upwards with respect to the body at rest till the body meets the floor (See *Figure 2*).
3. Now imagine that the elevator is broken from the cable and is having a free fall under gravity. The bodies inside the elevator have a similar free fall with the same acceleration, so that they appear to be unaccelerated with respect to the elevator. The observer inside interprets that there is no gravitational field and that the bodies are in an inertial frame (See *Figure 3*).

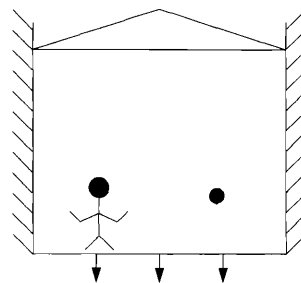
How can we then distinguish between an accelerated frame and a true gravitational field? Gravitational field is real and can not be eliminated at all places by simply choosing a non inertial frame due to the non-uniformity of the actual gravitational field. Gravitational field vanishes at a large distance from the source, whereas an accelerated frame can eliminate the gravitational field only locally. This frame when falling freely must have an acceleration due to gravity valid only in that small region. Again if we consider two freely falling particles wide apart on the surface of the earth, they will fall towards the center of the earth and hence their accelerations will not be parallel (See *Figure 4*). How will they appear in a freely falling elevator? If one of these two bodies appears to be at rest, the other seems to approach the former in course of time. So by choosing a



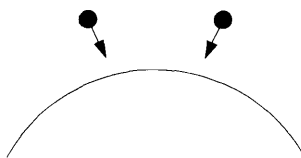
**Figure 1.**



**Figure 2.**



**Figure 3.**



**Figure 4.**

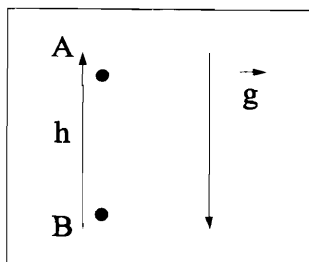


Figure 5.

single accelerated frame it is not possible to eliminate the gravitational field in both regions.

### Gravitational Redshift

When light is propagated against a gravitational field it loses energy progressively. This manifests as an increase in its wavelength, a phenomenon known as gravitational redshift.

Einstein was already aware of the gravitational redshift in 1907 but the derivation given by him was, however, in line with equivalence principle. Consider a source at the bottom B and the receiver at the top A of a system at rest in a uniform gravitational field. Now for a freely falling observer both the source and the receiver would appear to accelerate upwards with an acceleration  $g$  (see Figure 5). Let the separation between A and B be  $h$ . Light signal is sent at  $t = 0$  from B towards A. The signal reaches A at the time  $t$ , so that

$$ct = h + \frac{1}{2}gt^2 \quad (3)$$

Another signal emitted at a later time  $\Delta t_B$  would reach A at the time  $(t + \Delta t_A)$ .

So

$$c(t + \Delta t_A - \Delta t_B) = h + \frac{1}{2}g(t + \Delta t_A)^2 - \frac{1}{2}g(\Delta t_B)^2 \quad (4)$$

Neglecting small terms like  $(\Delta t_A)^2$  and  $(\Delta t_B)^2$  and using (3) in (4) we obtain

$$c(\Delta t_A - \Delta t_B) = gt \cdot \Delta t_A,$$

which in turn leads us to the relation

$$\Delta t_A = \frac{\Delta t_B}{1 - gt/c} = \frac{\Delta t_B}{1 - gh/c^2}. \quad (5)$$

We must note that here  $gt \ll c$ , because practically the velocity attained in time  $t$  is negligible compared to the



velocity of light  $c$ . Since the total number of light pulses emitted by the source in time  $\Delta t_B$  is the same as that received by A in time  $\Delta t_A$ , we must have

$$\frac{\nu_A}{\nu_B} = \frac{\Delta t_B}{\Delta t_A},$$

which using (5) leads to the relation

$$\frac{\lambda_B}{\lambda_A} = (1 - gh/c^2). \quad (6)$$

$\lambda_A > \lambda_B$  indicating redshift of the light traveling from the source to the receiver. From the equivalence principle it follows that there must be such redshift in the presence of the gravitational field.

We must remember in this context the experiment devised by Pound and Rebka [5] (1960) after the discovery of Mossbauer effect to measure the small gravitational redshift.  $\text{Fe}^{57}$  samples were used both as an emitter and an absorber of the  $\gamma$  radiation of energy 14.4 keV. The emitter and the absorber were placed at the bottom and at the top of a tower 22.6 m high. Normally there was no resonant absorption of  $\gamma$ -ray at the top. To attain the resonant absorption it was necessary to impart a small suitable velocity of the emitter towards the absorber. This motion would generate the appropriate blue-shift by Doppler effect and cancel the original gravitational red-shift creating a situation for resonant absorption of  $\gamma$ -rays.

Now we discuss in mathematical terms the basic steps towards the general theory of relativity – which an advanced reader may find useful. First we describe a non-inertial frame (with the example of a rotating frame), then we introduce ‘Christoffel symbols’ which then help us to define the ‘curvature’ of space-time. Even though complete derivations are omitted and would be out of place here, we would like the reader to be aware of the principal concepts and relations involved in this theory.

An inertial frame is that in which Newtonian laws are valid.



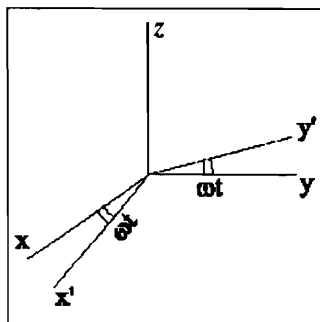


Figure 6.

## The Case of a Non-Inertial Frame

It is well known that examples of non-inertial frames are (i) a frame rotating with respect to an inertial frame (ii) a frame accelerated with respect to an inertial frame. An inertial frame is that in which Newtonian laws are valid. Ideally an inertial frame is specified as one in which a particle with no force on it appears to move with a uniform velocity in a straight line. The only way to visualize an inertial frame is to imagine it far away from any gravitating matter. But in presence of gravitation we can make it locally inertial. That is in a very small region. We now introduce a rotating frame to show how the metric tensor changes character in a non-inertial frame around  $z$  axis (see *Figure 6*).

$$\begin{aligned}x &= x' \cos \omega t - y' \sin \omega t \\y &= x' \sin \omega t + y' \cos \omega t \\z &= z'\end{aligned}$$

so that

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

goes over to

$$\begin{aligned}ds^2 &= [c^2 - \omega^2(x'^2 + y'^2)]dt^2 + 2\omega dt(y'dx' - x'dy') \\&\quad - (dx'^2 + dy'^2 + dz'^2).\end{aligned}\quad (7)$$

So in general non-inertial coordinates present the metric form as

$$ds^2 = g_{\mu\nu}(x')dx'^{\mu}dx'^{\nu}$$

Here  $g_{\mu\nu}$  are functions of space time co-ordinates in the most general case. This is true for accelerating frame also, that is, when the new coordinates describe a frame accelerated with respect to an inertial frame. Since non-inertial frames are to be introduced even locally to replace gravitational field, one is compelled to give up the laws of Euclidean geometry.

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## Curvature of Space-Time

In special relativity the equation of motion in Minkowski coordinates is given by

$$\frac{d^2\zeta^\alpha}{d\tau^2} = 0.$$

Here  $d\tau$  stands for the proper time interval. We now go over to  $x$  coordinate describing a non-inertial frame of reference, so that

$$\frac{d\zeta^\alpha}{d\tau} = \left( \frac{\partial\zeta^\alpha}{\partial x^\mu} \right) \left( \frac{dx^\mu}{d\tau} \right)$$

and hence

$$\frac{d^2\zeta^\alpha}{d\tau^2} = \left( \frac{\partial\zeta^\alpha}{\partial x^\mu} \right) \left( \frac{d^2x^\mu}{d\tau^2} \right) + \frac{\partial^2\zeta^\alpha}{\partial x^\nu \partial x^\mu} \left( \frac{dx^\mu}{d\tau} \right) \left( \frac{dx^\nu}{d\tau} \right) = 0. \quad (8)$$

Equation (8) can also be written as

$$\frac{d^2x^\lambda}{d\tau^2} = -\Gamma_{\mu\nu}^\lambda \left( \frac{dx^\mu}{d\tau} \right) \left( \frac{dx^\nu}{d\tau} \right) \quad (9)$$

where

$$\Gamma_{\mu\nu}^\lambda = \frac{\partial^2\zeta^\alpha}{\partial x^\nu \partial x^\mu} \left( \frac{\partial x^\lambda}{\partial \zeta^\alpha} \right)$$

Now since  $ds^2$  is invariant we can write

$$g_{\mu\nu} = \eta_{\alpha\beta} \left( \frac{\partial\zeta^\alpha}{\partial x^\mu} \right) \left( \frac{\partial\zeta^\beta}{\partial x^\nu} \right)$$

So that,

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \Gamma_{\lambda\mu}^\delta g_{\delta\nu} + \Gamma_{\lambda\nu}^\delta g_{\delta\mu}$$

which after simple manipulations lead to the following expression

$$\Gamma_{\lambda\mu}^\delta g_{\delta\nu} = \frac{1}{2} \left( \frac{\partial g_{\lambda\nu}}{\partial x^\mu} + \frac{\partial g_{\nu\mu}}{\partial x^\lambda} - \frac{\partial g_{\lambda\mu}}{\partial x^\nu} \right) \quad (10)$$

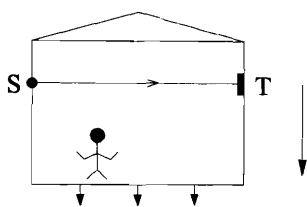


Figure 7.

$\Gamma_{\lambda\mu}^{\delta}$  is called the Christoffel symbol. Equation (9) is the geodesic equation of motion and  $\Gamma_{\mu\nu}^{\lambda}$  indicates the force term. The metric tensor components  $g_{\mu\nu}$  represent potential terms. Equation (9) stands for the equation of motion in a constant gravitational field that is in a locally Lorentz form and reduces to uniform (no acceleration) motion when transformed to inertial frame. So in a locally inertial frame the Christoffel symbols can be transformed to zero by a suitable coordinate transformation. In the true gravitational field, however, it is inhomogeneous and the gradients of all the Christoffel symbols do not vanish even locally, although the symbols themselves will vanish at that point by a suitable coordinate transformation. So the Christoffel symbols are used to construct the Riemann–Christoffel curvature tensor in the following form.

$$R_{\mu\nu\lambda}^{\alpha} = -\frac{\partial\Gamma_{\mu\nu}^{\alpha}}{\partial x^{\lambda}} + \frac{\partial\Gamma_{\mu\lambda}^{\alpha}}{\partial x^{\nu}} + \Gamma_{\mu\lambda}^{\delta}\Gamma_{\nu\delta}^{\alpha} - \Gamma_{\mu\nu}^{\delta}\Gamma_{\lambda\delta}^{\alpha}. \quad (11)$$

A space time is said to be flat when all the components of the curvature tensor vanish.

### Bending of Light

In 1911 when Einstein was a professor at the Karl-Ferdinand University, he predicted bending of light beam in a gravitational field. He was, however, already aware of this phenomenon in 1907. The difficulty then was that such prediction was not verifiable at that time by terrestrial experiment as a means of observation. The solar eclipse experiment performed much later in 1919 demonstrated that the light is bent by an amount close to Einstein's estimate made in 1915 after GTR was established. We see here that this result too follows from the principle of equivalence.

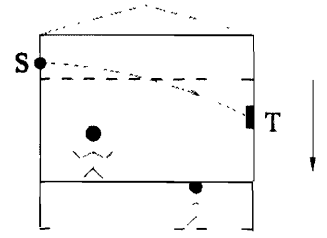
In *Figure 7* an observer is shown to be falling freely with the elevator. He notices that a laser beam being emitted from the sources on the left hand side wall hits the target

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(T) on the opposite wall along a straight horizontal line. The observer seems situated in an inertial frame.

In *Figure 8* the observer, who is stationary outside and watching the elevator falling, concludes that the light path is curved in between the source and the target within the elevator.



**Figure 8.**

In 1912 Einstein began a new phase of his gravitational research with the help of his mathematician friend Marcel Grossman by using the technique of tensor calculus. After a number of false starts Einstein published in late 1915 the definitive version of General Theory of Relativity. After the confirmation of Einstein's prediction on bending of light in the British eclipse expedition of 1919 *London Times* wrote on Nov. 7, 1919:

“Revolution in Science – New Theory of the Universe – Newtonian ideas overthrown”

### Einstein's Field Equations

Actually on November 25, 1915 Einstein presented his final version of the field equation during a lecture at the Prussian Academy.

It is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\kappa T_{\mu\nu}, \quad (12)$$

where

$$R_{\mu\nu} = R_{\mu\nu\lambda}^{\lambda} = -\frac{\partial\Gamma_{\mu\nu}^{\lambda}}{\partial x^{\lambda}} + \frac{\partial\Gamma_{\mu\lambda}^{\lambda}}{\partial x^{\nu}} + \Gamma_{\mu\lambda}^{\delta}\Gamma_{\nu\delta}^{\lambda} - \Gamma_{\mu\nu}^{\delta}\Gamma_{\lambda\delta}^{\lambda}$$

and

$$R = g^{\mu\nu}R_{\mu\nu}.$$

$R_{\mu\nu}$  and  $R$  are the Ricci tensor and the Ricci scalar respectively.  $T_{\mu\nu}$  is the energy momentum tensor.

The field equation (12) was arrived at originally by Einstein following certain logical arguments.

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Nov. 7, 1919

1. Minkowski metric in flat space should be replaced by Riemannian metric, which represents generally a curved space but may be said to be locally flat. This is analogous to the tangent space in the curved geometry. The principle of equivalence can be applied in such a tangent space.
2. The equations must be covariant tensor equations valid in all coordinate systems.
3. Both sides of the field equation must be consistent. In equation (12) the left hand side satisfies

$$(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)_{;\nu} = 0$$

which is called the Bianchi identity. On the right hand side we have

$$T^{\mu\nu}_{;\nu} = 0$$

which follows from the conservation principle.

4. Newtonian equation must follow from Einstein's equation for a slowly moving particle and in the weak field approximation. It is expected that the Poisson equation in Newtonian gravitational field:  $\Delta\phi = 4\pi G\rho$  should follow from Einstein's field equation.

### The Coupling Constant $\kappa$ from the Newtonian Approximation

Consider a static weak gravitational field, where  $g_{\mu\nu}$ 's depart only slightly from the Minkowski metric  $\eta_{\mu\nu}$ , that is

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \text{ with } h_{\mu\nu} \ll 1$$

and  $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$  in units such that  $c = 1$ .

We omit details of calculations. From Einstein's field equation (12) it is not difficult to arrive at the equation

$$R_{00} = -\frac{1}{2}\Delta h_{00} = -(\kappa/2)\rho, \quad (13)$$

where  $\Delta$  is the ordinary Laplacian operator, so that we get

$$\Delta h_{00} = \kappa \rho. \quad (14)$$

Now for a test particle moving slowly along a geodesic in a weak gravitational field the geodesic equation

$$\frac{d^2 x^\mu}{ds^2} = -\Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds}$$

(see equation (9)) yields

$$\frac{d^2 x^i}{dt^2} \approx -\frac{1}{2} \frac{\partial h_{00}}{\partial x^i}. \quad (15)$$

The corresponding Newtonian equation gives

$$\frac{d^2 x^i}{dt^2} = -\frac{\partial \phi}{\partial x^i},$$

where  $\phi$  is the Newtonian potential. After comparing,  $\frac{1}{2}h_{00}$  is identified with the potential  $\phi$ .

Hence from (14)  $\Delta\phi = (\kappa/2)\rho$ . Remembering that the Poisson equation in Newtonian physics is  $\Delta\phi = 4\pi G\rho$ , we can immediately identify  $\kappa$  with  $8\pi G$  (here  $G$  is the gravitational constant). So finally Einstein's field equation (12) can be expressed in the usual form

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu}. \quad (16)$$

## Summary

This brief discussion was intended to provide the reader with an introduction to Einstein's approach to address the problem of gravity under the general theory of relativity. In the words of Prof S Chandrasekhar, "One of the ways in which one may explore the physical content of the general theory of relativity is to allow one's sensibility to its aesthetic base guide in the formulation of problems with conviction in the harmonic coherence of its mathematical structure"

## Suggested Reading

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