# CS Practice Questions -

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## Vectors and Matrices

$$X = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \quad y = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad z = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

① $y^T z = \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 + 3 \times 3 = 2 + 9 = 11.$

② $Xy = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \times 1 + 4 \times 3 \\ 1 \times 1 + 3 \times 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 10 \end{bmatrix}$

③ $\det(X) = 2 \times 3 - 4 \times 1 = 6 - 4 = 2.$

As $\det(X) \neq 0$, $X$ is invertible.

$$\operatorname{adj}(X) = \begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix}$$

$$X^{-1} = \frac{1}{2} \begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 3/2 & -2 \\ -1/2 & 1 \end{pmatrix}$$

④ $X$ is $2 \times 2$ matrix and $\det(X) \neq 0$.
So, rank of $X$ is 2.

## CALCULUS:

① $y = x^3 + x - 5.$

$$\frac{dy}{dx} = 3x^2 + 1$$

② $f(x_1, x_2) = x_1 \sin(x_2) e^{-x_1}$.

$$\frac{df}{\partial x_1} = \sin(x_2) \frac{\partial}{\partial x_1}(x_1 e^{-x_1})$$

$$= \sin(x_2)\left[e^{-x_1} - x_1 e^{-x_1}\right]$$

$$\frac{\partial f}{\partial x_2} = x_1 \cos(x_2) e^{-x_1}.$$

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \sin(x_2)\left[e^{-x_1} - x_1 e^{-x_2}\right] \\ x_1 \cos(x_2) e^{-x_1} \end{pmatrix}$$

PROBABILITY and STATISTICS:

Sample data: $S = \{1, 1, 0, 1, 0\}$

0 - Heads, 1 - Tails.

① Sample Mean: $\bar{X} = \frac{\sum_i x_i}{n} = \frac{1+1+0+1+0}{5} = \frac{3}{5}$.

② Variance: $\sigma^2 = \frac{\sum_i (\bar{X} - X_i)^2}{n-1}$

$$= \frac{\left(\frac{3}{5}-1\right)^2 + \left(\frac{3}{5}-1\right)^2 + \left(\frac{3}{5}-0\right)^2 + \left(\frac{3}{5}-1\right)^2 + \left(\frac{3}{5}-0\right)^2}{5-1}$$

$$= \frac{\frac{4}{25} + \frac{4}{25} + \frac{9}{25} + \frac{4}{25} + \frac{9}{25}}{4} = \frac{\cancel{30}^{\,3}}{\cancel{25}_{\,5}} \times \frac{1}{\cancel{4}_{\,2}} = \frac{3}{10}$$

③ $P(x=1) = 0.5$, $P(x=0) = 0.5$.

Probability for $S = \frac{1}{2^5}$.

(4) Probality: $p^3(1-p)^2$ where $p = p(x-1)$.

$$\frac{dP}{dp} = \frac{d(p^3(1-p)^2)}{dp}$$

$$= (1-p)^2\frac{dp^3}{dp} + p^3\frac{d(1-p)^2}{dp}$$

$$= 3p^2(1-p)^2 + p^3\, 2(1-p)\times -1$$

$$= 3p^2(1-p)^2 - 2p^3(1-p).$$

For max:

$$\frac{dP}{dp} = 0$$

$$\Rightarrow 3p^2(1-p)^{\cancel{2}} = 2p^{\cancel{3}}\cancel{(1-p)}$$

$$\Rightarrow 3 - 3p = 2p$$

$$\Rightarrow 3 = 5p \Rightarrow p = \frac{3}{5}.$$

$$\frac{d^2P}{dp^2} = \frac{d(3p^2(1-p)^2)}{dp} - \frac{d(2p^3(1-p))}{dp}$$

$$= -3p^2\times 2(1-p) + (1-p)^2\, 3\times 2p - [-2p^3 + 6p^2(1-p)]$$

$$= -6p^2(1-p) + 6p(1-p)^2 + 2p^3 - 6p^2(1-p)$$

$$= 2p^3 + 6p(1-p)^2 - 12p^2(1-p)$$

$$= 2\left(\frac{3}{5}\right)^3 + 6\times\frac{3}{5}\left(1-\frac{3}{5}\right)^2 - 12\left(\frac{3}{5}\right)^2\left(1-\frac{3}{5}\right)$$

$$= -0.72.$$

As $\frac{d^2 P}{dp^2} < 0$,

$p = \frac{3}{5}$ maximizes the probability.

⑤

$p(z = T \text{ and } y = b) = 0.1$.

$$p(z = T \mid y = b) = \frac{P(z = T \wedge y = b)}{P(y = b)}$$

$$= \frac{0.1}{0.1 + 0.15} = \frac{0.1}{0.25} = \frac{\cancel{10}\,2}{\cancel{25}\,5} = \frac{2}{5}$$

$$= 0.4.$$

Big - O Notation :

① $f(n) = \ln(n)$

$g(n) = \lg(n)$

We know:

$$\lg(n) = \frac{\ln(n)}{\ln(2)} \quad \Bigg| \quad \ln(n) = \frac{\lg(n)}{\lg(e)}.$$

So, $g(n) \in O(f(n))$

and also, $f(n) \in O(f(n))$, as both can be expressed with a positive multiplicative.

② $f(n) = 3^n$ , $g(n) = n^{100}$.

Let $n_0 = 1000$, $c_1 = 10$.

We can see,

$$0 \le g(n) \le c_1 f(n) \quad \forall\, n \ge n_0.$$

So, $g(n) \in O\, f(n)$.

③ $f(n) = 3^n$ , $g(n) = 2^n$.

As $2^n < 3^n$ for $\forall n \ge 1$

So, for $n_0 = 1$, $c = 1$,

$$g(n) \in O(f(n)).$$

④ $f(n) = 1000n^2 + 2000n + 4000$.

$g(n) = 3n^3 + 1$.

Let $n_0 = 1000$ and $c = 1$,

$$0 \le f(n) \le c\,g(n).$$

So, $f(n) \in O(g(n))$.

# ALGORITHMS :

Array contains n elements.

$\{0\ 0 \ldots 0\ 1\ 1 \ldots 1\}$

$\uparrow$ k.

We have to find k index.

We will use the divide and conquer algorithm.

For this, we will divide the array into two parts and check the transition condition. If the condition does not satisfy, we will repeat this process on one half that we get from checking the condition.

Algo -

1. function fmid(i,j) finds $mid = \lfloor \frac{i+j}{2} \rfloor$

   Store the two different arrays in new variables (x, y).

   If (x[mid] = = 1 && y[mid+1] = = 1)

   fmid(i, mid).

   else if (x[mid] = = 0 && y[mid+1] = = 0)

   fmid(mid+1 , j).

   return

   mid.

2. Now, the function fmid (i,j) will return the k index.

Running Time:

As the algorithm divides into two halves, we can write it as-

$$T(n) = T(n/2) + O(1).$$

Using Masters Theorem-

$$T(n) \in O(\log n).$$

## PROBABILITY and RANDOM VARIABLES:

Probability:

a) $P(A \cup B) = P(A \cap (B \cap A^c))$ — False.

RHS: $P(A \cap (B \cap A^c)) = P(\phi)$

b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ — True.

c) $P(A) = P(A \cap B) + P(A^c \cap B).$ — False.

d) $P(A|B) = P(B|A)$ — False.

e) $P(A_1 \cap A_2 \cap A_3) = P(A_3 | (A_2 \cap A_1))\, P(A_2|A_1)\, P(A_1)$

RHS: $\dfrac{P(A_3 \cap A_2 \cap A_1)}{\cancel{P(A_2 \cap A_1)}} \times \dfrac{\cancel{P(A_2 \cap A_1)}}{\cancel{P(A_1)}} \times \cancel{P(A_1)}.$

$= LHS.$

So, its True.

## Discrete and Continous Distributions -

Multivariate Gaussian : $\frac{1}{\sqrt{(2\pi)^d |\Sigma|}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right)$

Bernoulli : $p^x (1-p)^{1-x}$

Uniform : $\frac{1}{b-a}$ when $a \leq x \leq b$ ; 0 otherwise

Binomial : $\binom{n}{x} p^x (1-p)^{n-x}$.

## Mean, Variance and Entropy -

a) $\mathbb{E}X < \infty$.

$Var(X) = \mathbb{E}[(X - \mathbb{E}X)^2]$.

To show: $Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$.

$$
\begin{aligned}
Var(X) &= \mathbb{E}[(X - \mathbb{E}X)^2] \\
&= \mathbb{E}[X^2 + (\mathbb{E}X)^2 - 2X\,\mathbb{E}X] \\
&= \mathbb{E}(X^2) + \mathbb{E}((\mathbb{E}X)^2) - 2\,\mathbb{E}X\,\mathbb{E}(\mathbb{E}X). \\
&= \mathbb{E}(X^2) + (\mathbb{E}X)^2 - 2(\mathbb{E}X)^2. \\
&= \mathbb{E}(X^2) - (\mathbb{E}X)^2.
\end{aligned}
$$

b) Bernoulli $(p)$ random variable : Let $p = P(X=1)$

→ Mean : $P(X=1)\cdot 1 + P(X=0)\cdot 0$

$= p$.

$= \mathbb{E}X$.

ii, $Var(X) = E(X^2) - (EX)^2.$

$$E(X^2) = p(x=1)\cdot 1^2 + p(x=0)\cdot 0$$
$$= p.$$

$$Var(X) = p - p^2 = p(1-p).$$

iii, Entropy:

$$H(X) = -\sum_{x_i} Pr(x_i) \log_2 Pr(x_i)$$

$$= - Pr(X=0)\log_2 Pr(X=0) - Pr(X=1)\log_2 Pr(X=1)$$
$$= -(1-p)\log_2(1-p) - p\log_2(p).$$

Law of Large Numbers and Central Limit Theorem-

a) Die is rolled for 6000 times which is very large. So, by law of large numbers, the probability will approach to the ideal uniform distribution.

So, No. of times 3 shows $\approx \frac{N}{6} \approx \frac{6000}{6} \approx 1000$

b) Head $(X=1) = p(x=1) = \frac{1}{2}$.
Tail $(X=0) = p(x=0) = \frac{1}{2}$.
$EX = p(x=1)\cdot 1 + p(x=0)\cdot 1 = \frac{1}{2}$.
$E(X^2) = p(x=1)\cdot 1^2 + p(x=0)\cdot 0^2 = \frac{1}{2}$.
$\sigma^2 = E(X^2) - (EX)^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{2}.$

So, by Central limit Theorem,

$\mu = 1/2, \quad \sigma^2 = 1/4.$

$\bar{X}$ satisfies :

$$\sqrt{n}(\bar{X} - \mu) \xrightarrow{n \to \infty} \mathcal{N}(0, \sigma^2)$$

$$\Rightarrow \sqrt{n}\left(\bar{X} - \frac{1}{2}\right) \xrightarrow{n \to \infty} \mathcal{N}\left(0, \frac{1}{4}\right).$$

Linear Algebra:

Geometry -

a) $\vec{w}$ is a vector.

$\vec{w}^T \vec{x} + b = 0.$ ← line.

To prove: $\vec{w}$ is orthogonal to line.

Proof: Let $\vec{x}_1$ and $\vec{x}_2$ be point position vectors on line.

$\therefore \vec{w}^T \vec{x}_1 + b = 0, \quad \vec{w}^T \vec{X}_2 + b = 0.$

$\Rightarrow \vec{w}^T(\vec{x}_1 - \vec{x}_2) = 0.$ (Subtracting).

$\Rightarrow \langle \vec{w} \mid (\vec{x}_1 - \vec{x}_2) \rangle = 0.$

So, $\vec{w}$ is orthogonal to the line. ($\because \vec{x}_1 - \vec{x}_2 \parallel$ line).

b) Find distance from origin to line. Prove it is $\frac{b}{\|\vec{w}\|}$.

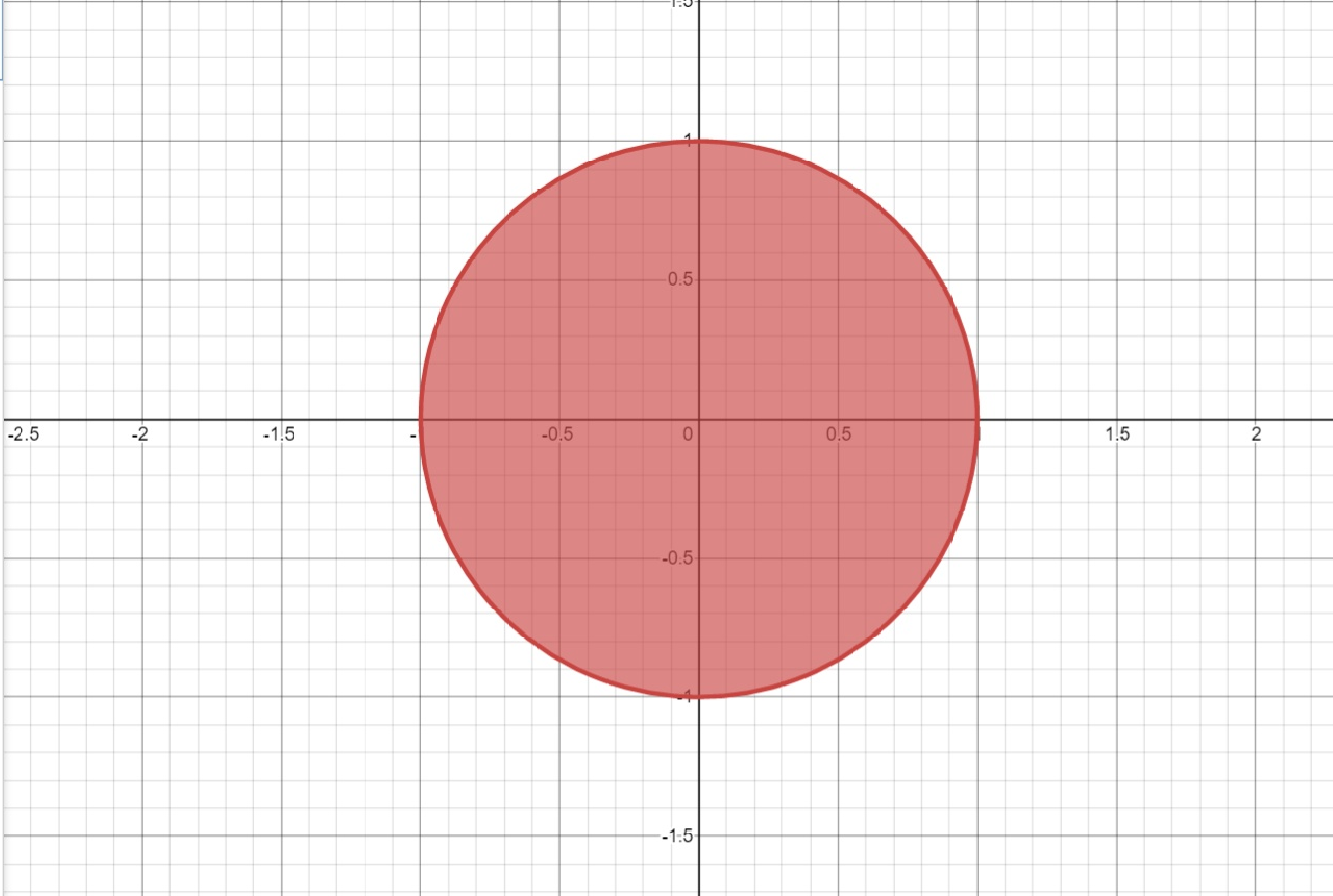
Let $\vec{x}$ be $n$ dimensional.

$\Rightarrow \vec{w}^T \vec{x} + b = 0 \Rightarrow w_1 x_1 + \cdots + w_n x_n + b = 0.$

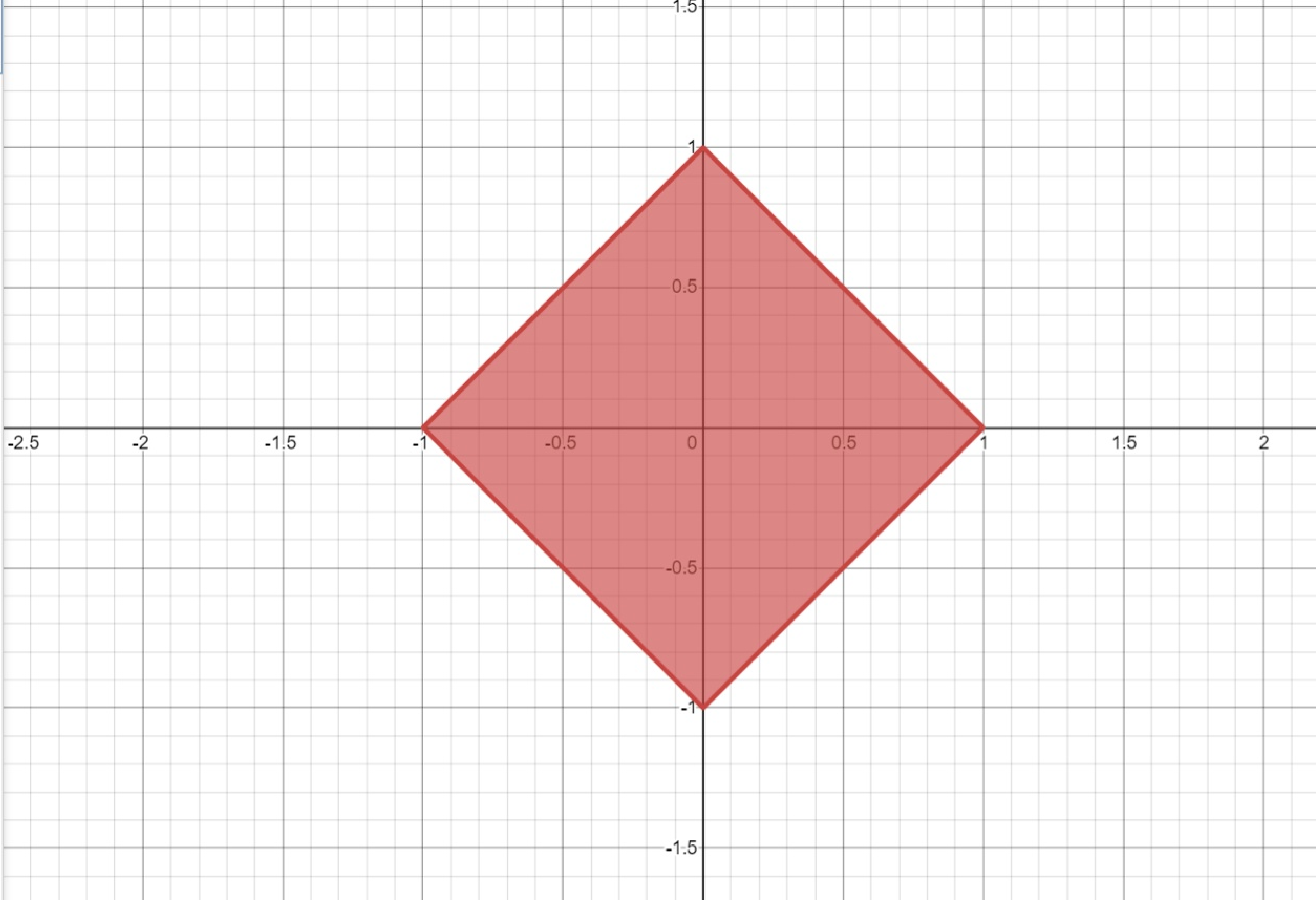
So, distance from origin:

$$d = \frac{|w_1 \cdot 0 + w_2 \cdot 0 + \cdots + w_n \cdot 0 + b|}{\sqrt{w_1^2 + w_2^2 + \cdots w_n^2}} = \frac{|b|}{\|\vec{w}\|} = \frac{b}{\|\vec{w}\|} \text{ for } b \geq 0.$$

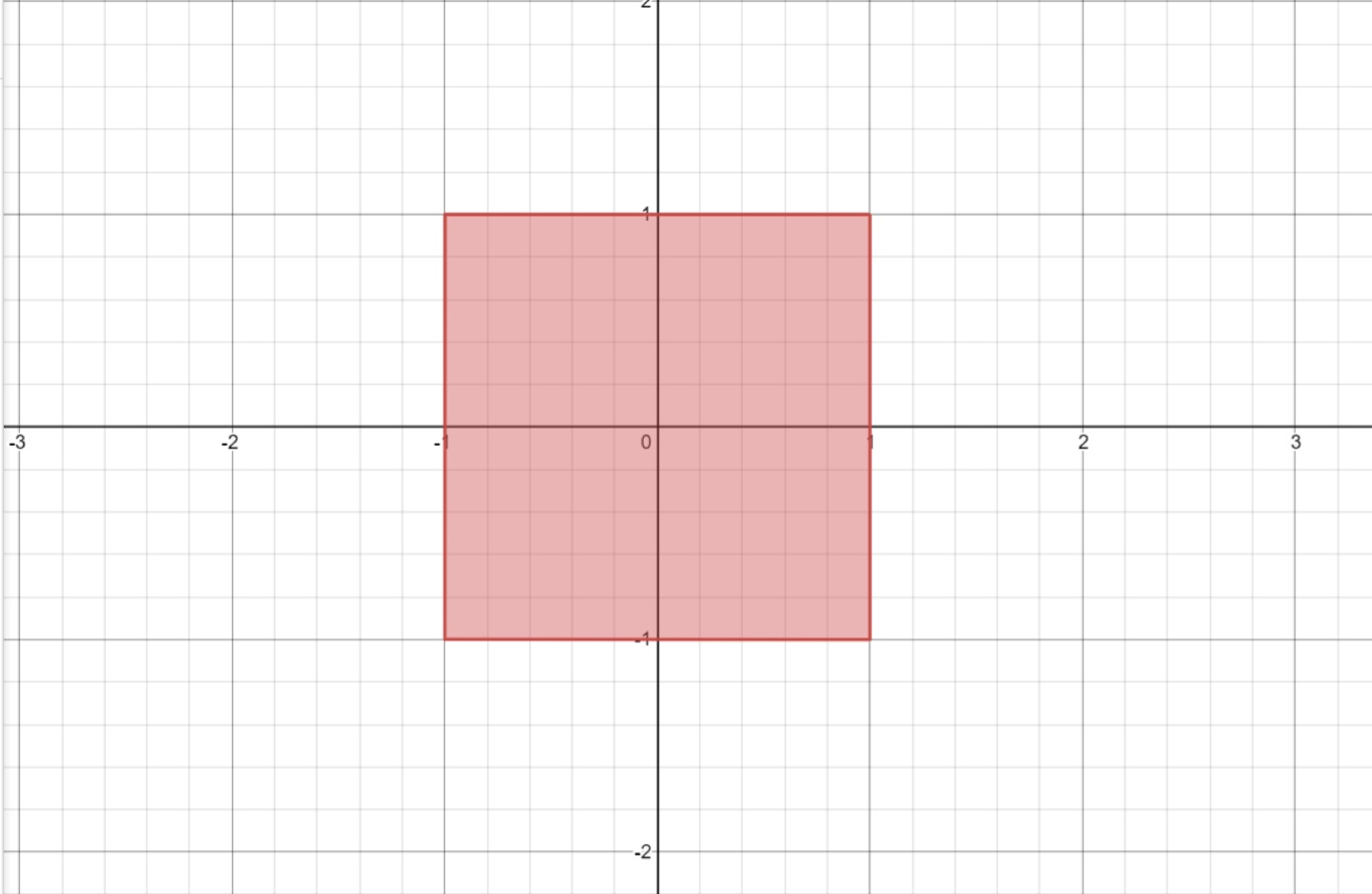
$\sqrt{(x^2 + y^2)} \leq 1$



$|x| + |y| \leq 1$



$\max(|x|, |y|) \leq 1$



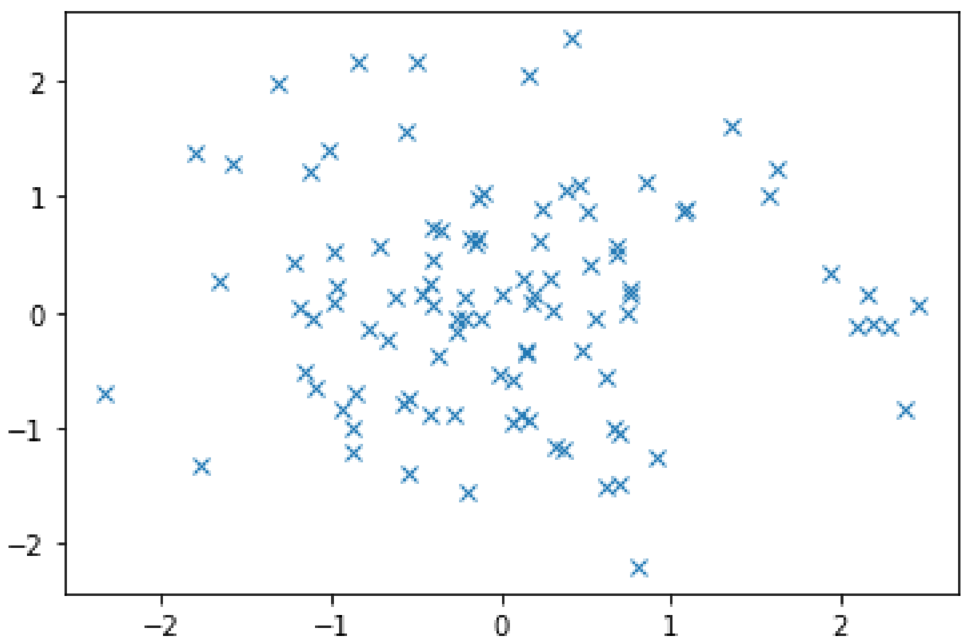
# assignment1

September 10, 2021

Programming Skills

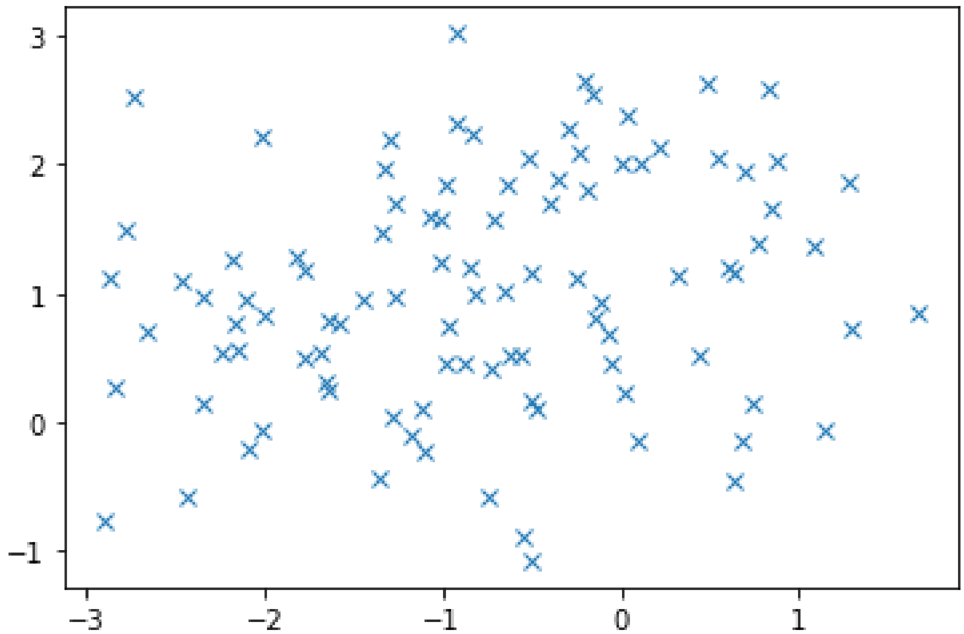
a) Draw 100 samples x = [x1 x2] from a 2-dimensional Gaussian distribution with mean [0, 0] and identity covariance matrix

[1]:
```
import numpy as np
import matplotlib.pyplot as plt

mean = [0, 0]
cov = [[1, 0], [0, 1]]
x, y = np.random.multivariate_normal(mean, cov, 100).T
plt.plot(x, y, 'x')
plt.show()
```



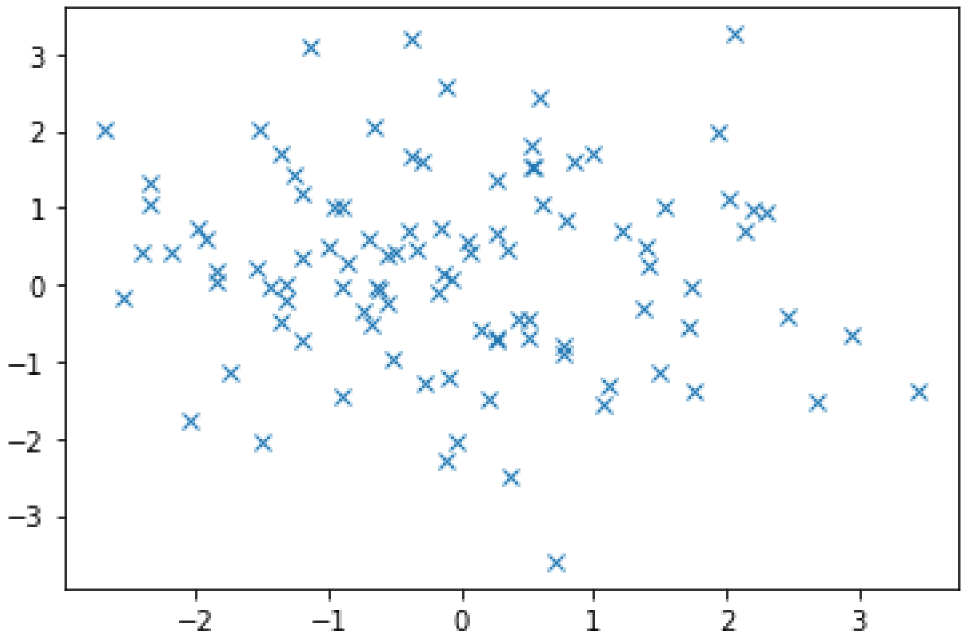

b)How does the scatter plot change if the mean is [−1, 1]? # The plot is shifted according to the mean as when the mean is calculated using the data points, it should be the same as the mean provided.

[2]:
```
mean = [-1,1]
x, y = np.random.multivariate_normal(mean, cov, 100).T
plt.plot(x, y, 'x')
plt.show()
```



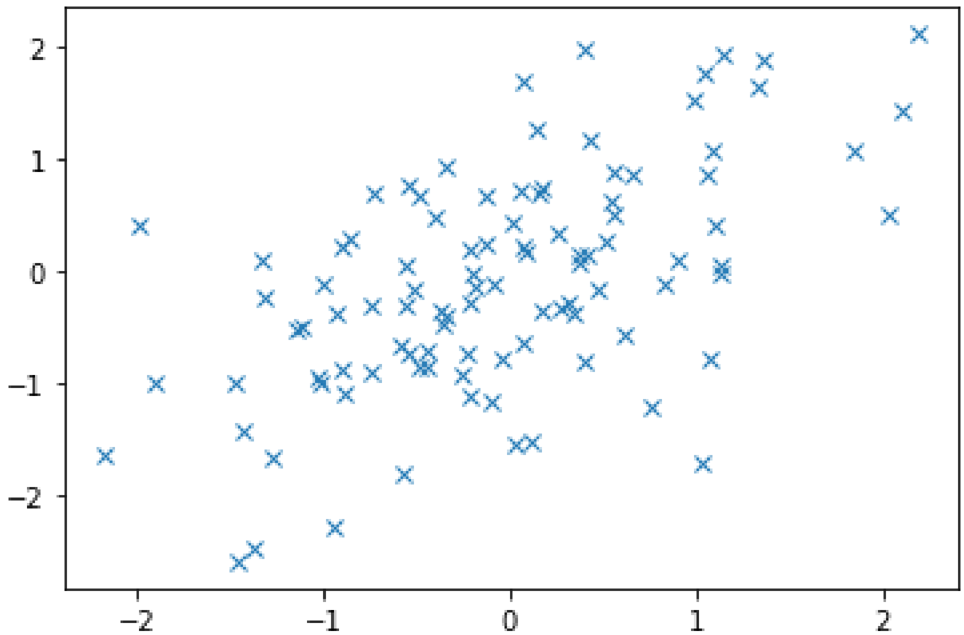

c) How does the scatter plot change if you double the variance of each component (x1 & x2)?
# The plot shows that the points are more scattered than the previous cases.

[3]:
```
mean = [0, 0]
cov = [[2, 0], [0, 2]]
x, y = np.random.multivariate_normal(mean, cov, 100).T
plt.plot(x, y, 'x')
plt.show()
```



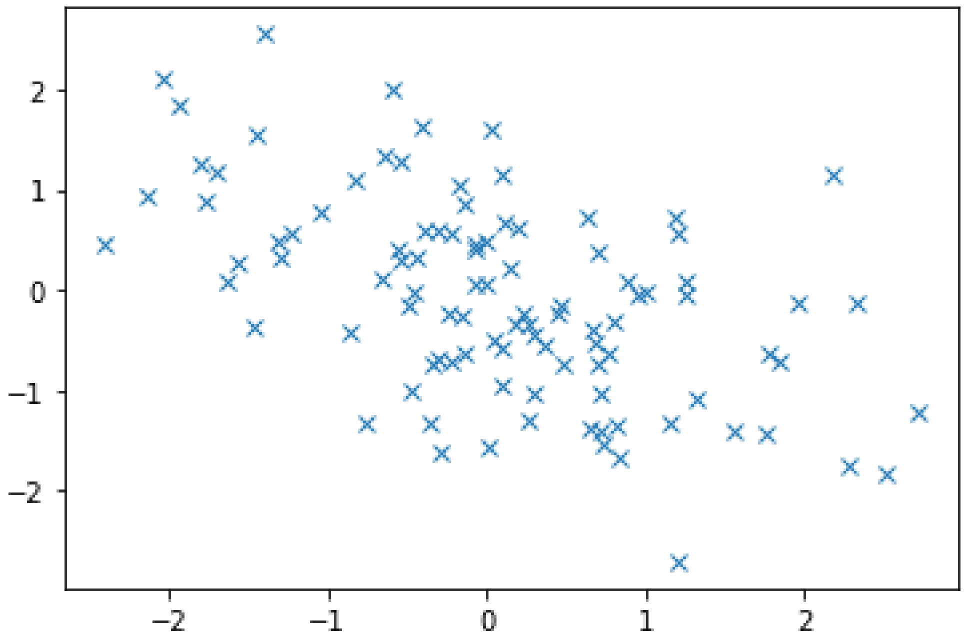

d) How does the scatter plot change if the covariance matrix is changed to the following? cov = [[1,0.5], [0.5,1]] # Seems like most points are located in the quadrant 1 and 3.

[4]:
```
cov = [[1, 0.5], [0.5, 1]]

x, y = np.random.multivariate_normal(mean, cov, 100).T
plt.plot(x, y, 'x')
plt.show()
```




e) How does the scatter plot change if the covariance matrix is changed to the following? cov = [[1, -0.5], [-0.5, 1]] # Seems like most points are located in the quadrant 2 and 4.

[7]:
```
cov = [[1, -0.5], [-0.5, 1]]

x, y = np.random.multivariate_normal(mean, cov, 100).T
plt.plot(x, y, 'x')
plt.show()
```


2
1
0
−1
−2
−2
−1
0
1
2

[ ]: