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1 Hochschule Bonn-Rhein-Sieg

2 Neural Networks, WS17/18

3 Assignment 05 (06-november-2017)

3.1 Debaraj Barua, Naresh Kumar Gurulingan

```
In [1]: import numpy as np
        import matplotlib.pyplot as plt
        import sympy as sp
        import IPython
        sp.init_printing()
```

3.2 Question 1:

Summary of Haykin NN edition 2 : chapter 3.1-3.5.

Answer 1:

Single Layer Perceptrons

- Perceptron is the simplest form of neural network with varying weights that enables classification of linearly separable patterns.
- It consists of a single neuron with adjustable synaptic weight and bias, and is able to classify two linearly separable classes.
- These free parameters are adjsuted during training to learn the weights that enables linear separation of the classes
- These parameters represent a decision hyperplane between these classes.

Adaptive Filtering Problem - Considering a dynamical system, whose mathematical characterization is unknown.

- The adaptive filtering problem addresses how to design a multiple input and single output model of the aforemetntioned dynamic system by building it around a single linear neuron.

- This model operates under the influence of an algorithm that controls how the synaptic weights of the neuron are updated.
  - The algorithm starts from arbitary setting of the synaptic weight.
  - Adjustments to these weights are made continuously in respinse to statistical variations in the system's behaviour.
  - Computation of these adjustments are completed in a time interval that is one sampling period long.
- The operation conssists of two continuous processes:
  - *Filtering process*, which involves computation of the two signals:
    * Output signal, on the basis of the input vector on the model represented by the neuron.
    * Error Signal, which is the difference of the output signal from the desired output (produced by the unknown system)
  - Adaptive process, which invovles automatic adjustment of the weights of the model, in accordance to the error produced.
- Thus the two processes work together to constitute a feedback loop around the neuron.

Unconstrained Optimization Techniques Considering a cost function $\varepsilon(w)$ that is continuously differentiable function of some unknown weight vector w. This function is a measure of how to choose the weight vector w of an adaptive filtering algorithm so it behaves in an optimum manner. We want to find an optimal solutuion w^*, which satisfies,

$$\varepsilon(w^*) \leq \varepsilon(w)$$

So, we need to minimize the cost function with respect to weight vector. That is, for optimality,

$$\nabla\varepsilon(w^*) = 0$$

Methods:

- *Method of Steepest descent* is applied to the weight vector w are in the direction of the steepest descent. This will be acting in the direction opposite to the gradient vector.

$$w(n+1) = w(n) - \nabla\varepsilon(w)$$

- *Newton's Method* esentially minimizes the quadratic approximation of cost function around the current weight.

$$w(n+1) = w(n) - H^{-1}(n)g(n)$$

  Where, $H(n) = \nabla\nabla\varepsilon(w(n))$, and $g(n) = \nabla\varepsilon(w(n))$

- *Gauss-Newton Method* is a special case of the Newton's method and is applicable only to a cost function that is expressed as the sum of error squares.

$$\varepsilon(w) = \frac{1}{2}\sum_{i=1}^{n} e^2(i)$$

  The weight vector is trained by all the data from the entire observation interval till n. Thus it is a batch learning procedure

Linear Least-Squares Filter

- Characteristics:
  - The single neuron around which it is built is linear.
  - The cost function $\varepsilon(w)$ used to design the filter consists of sum if error squares, similar to that of Gauss-Newton method.

$$w(n+1) = X^{+}(n)d(n)$$

    where,

$$\begin{aligned} d(n) &= \text{Desired output vector} \\ &= \begin{bmatrix} d(1) \\ d(2) \\ . \\ . \\ d(n) \end{bmatrix} \\ X^{+}(n) &= \text{pseudoinverse of the data matrix } X(n) \\ X(n) &= \text{Input data matrix} \\ &= \begin{bmatrix} x(1) \\ x(2) \\ . \\ . \\ x(n) \end{bmatrix} \end{aligned}$$

- Wiener Filter:
  - It is the limiting form of the linear least-squares filter for an Ergodic environment.
  - In such a case, we can substitute the long term sample aversages or time-averages for expectations.
  - Then above equation can be rewritten using second order statistics as:

$$w_o = R_x^{-1} r_{xd}$$

    where,

$$\begin{aligned} R_x &= \text{Correlation matrix of input vector } x(i) \\ r_{xd} &= \text{Cross-correlation vector between the input vector } x(i) \text{ and the desired output } d(i) \\ w_o &= \text{Weiner Solution, to the linear optimum filtering problem} \end{aligned}$$

Least Mean Square Algorithm

- Characteristics:
  - It is based on the use of instantaneous values of the cost function, i.e.,

$$\varepsilon(w) = \frac{1}{2}e^2(n)$$

  - It can be formulated as,

$$\hat{w}(n+1) = \hat{w}(n) + \eta x(n)e(n)$$

- Convergence consideration:
  - LMS algorithm is convergent in mean square in the below mentioned region:

$$0 < \eta < \frac{2}{\lambda_{max}}$$

    Where, λ_{max} is the largest eigen value of the correlation matrix of the input vector.

- Virtues and Limitations:
  - Virtues:
    * Simple
    * Independent and Robust; i.e., small model uncertainity and small disturbance.
    * LMS is optimal in accordance to H^{∞} or minimax criterion.
  - Limitations:
    * Slow rate of convergence.
    * Sensitivity to variation in Eigen structure of input.

3.3 Question 2 (Haykin Edition:2, Ex: 3.1):

Explore the method of steepest descent involving a single weight w by considering the following cost function:

$$\varepsilon(w) = \frac{1}{2}\sigma^2 - r_{xd}w + \frac{1}{2}r_x w^2$$

where, σ^2,r_{xd} and r_x are constants.

Answer 2:

```
In [2]: s,r_xd,r_x,w=sp.symbols('s,r_xd,r_x,w')

        e=1./2.*s**2-r_xd*w+1./2.*r_x*w**2

        nabla=sp.diff(e,w)
        nabla
```

Out [2]:

$$1.0r_xw - r_{xd}$$

So, Gradient,

$$\begin{aligned}\nabla\varepsilon(w) &= \frac{\partial\varepsilon(w)}{\partial w}\\ &= -r_{xd} + r_xw\end{aligned}$$

According to steepest descent method, weight vector is updated in the direction of the steepest descent, i.e., in the direction opposite to the gradient vector.

So, $\Delta w = -\eta\nabla\varepsilon(w)$, where η is the learning rate

Now, updated weight is:

$$\begin{aligned}w(n+1) &= w(n) + \Delta w\\ &= w(n) - \eta(-r_{xd} + r_xw)\end{aligned}$$

3.4 Question 3 (Haykin Edition:2, Ex: 3.2):

Consider the cost function

$$\varepsilon(w) = \frac{1}{2}\sigma^2 - r_{xd}^Tw + \frac{1}{2}w^TR_xw$$

where, σ^2,r_{xd} and r_x are constants.

$$r_{xd} = \begin{bmatrix}0.8182\\0.354\end{bmatrix}$$

$$R_x\begin{bmatrix}1 & 0.8182\\0.8182 & 1\end{bmatrix}$$

3.4.1 3(a)

Find the optimum value of w^* for which $\varepsilon(w)$ reaches it's minimum value.

Answer 3(a): $\varepsilon(w)$ will reach maximum value when it's Jacobian will be zero.

In [3]:
```
s=sp.symbols('s')
w=sp.MatrixSymbol('w',2,1)
r_xd=sp.Matrix([[0.8182],[0.354]])
R_x=sp.Matrix([[1,0.8182],[0.8182,1]])

e=1./2.*s**2-r_xd.T.dot(sp.Matrix(w))+1./2.*sp.Matrix(w).T.dot(R_x.dot(sp.Matrix(w)))
e=sp.Matrix([e])
symbolic_jacobian=e.jacobian(w)
print "Jacobian Matrix is:"
symbolic_jacobian
```

Jacobian Matrix is:

Out[3]:

$$\begin{bmatrix} -0.8182 + 0.5\left(2w_{0,0} + 1.6364w_{1,0}\right) & -0.354 + 0.5\left(1.6364w_{0,0} + 2w_{1,0}\right) \end{bmatrix}$$

```
In [4]: print "optimum values of weight, w*:"
        sp.Matrix([sp.solve(symbolic_jacobian,sp.Matrix(w))])
```

optimum values of weight, w*:

Out[4]:

$$\begin{bmatrix} \{w_{0,0} : 1.59902944424901, \quad w_{1,0} : -0.954325891284541\} \end{bmatrix}$$

3.4.2 3(b)

Use the method of steepest descent to compute w^* for the following two values of learning rate parameter:

(i) $\eta=0.3$
(ii) $\eta=1.0$

For each case, plot the trajectory traced the evolution of the weight vector $w(n)$ in the W-Plane.
Note: The trajectories obtained for cases (i) and (ii) of part (b) should correspond to the picture in Fig. 3.2

```
In [5]: IPython.display.Image("images/HaykinFig3.2.png")
```

Out[5]:

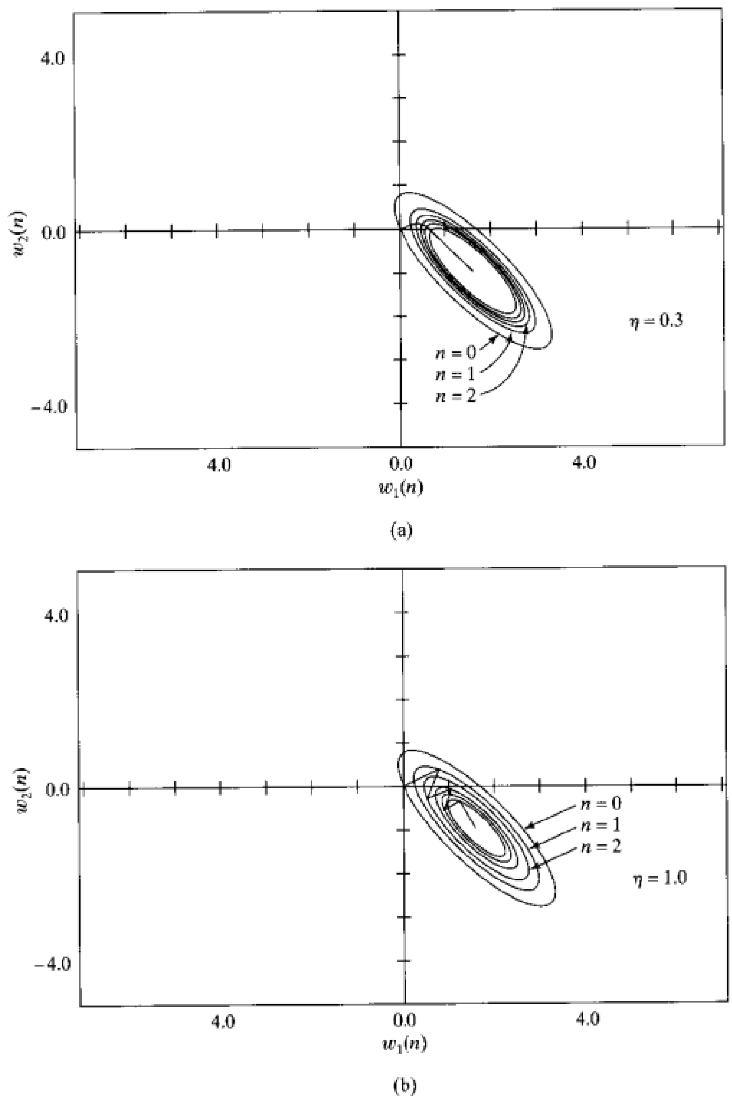


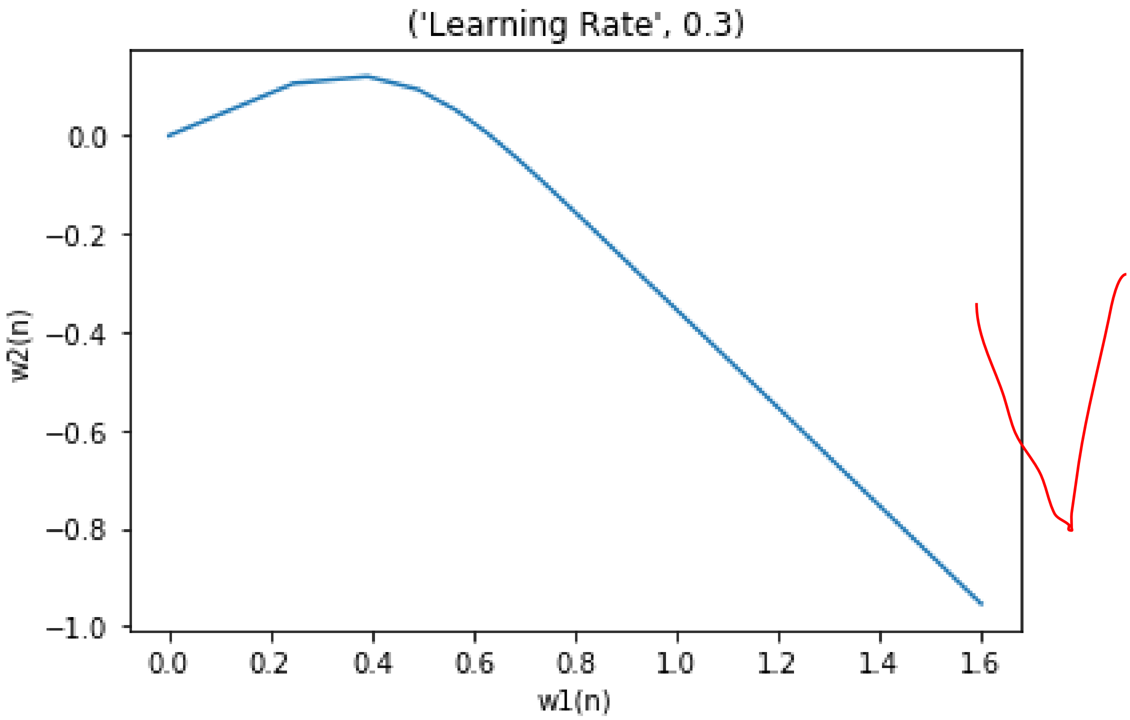
FIGURE 3.2 Trajectory of the method of steepest descent in a two-dimensional space for two different values of learning-rate parameter: (a) $\eta = 0.3$, (b) $\eta = 1.0$. The coordinates w_1 and w_2 are elements of the weight vector **w**.

Answer 3(b):

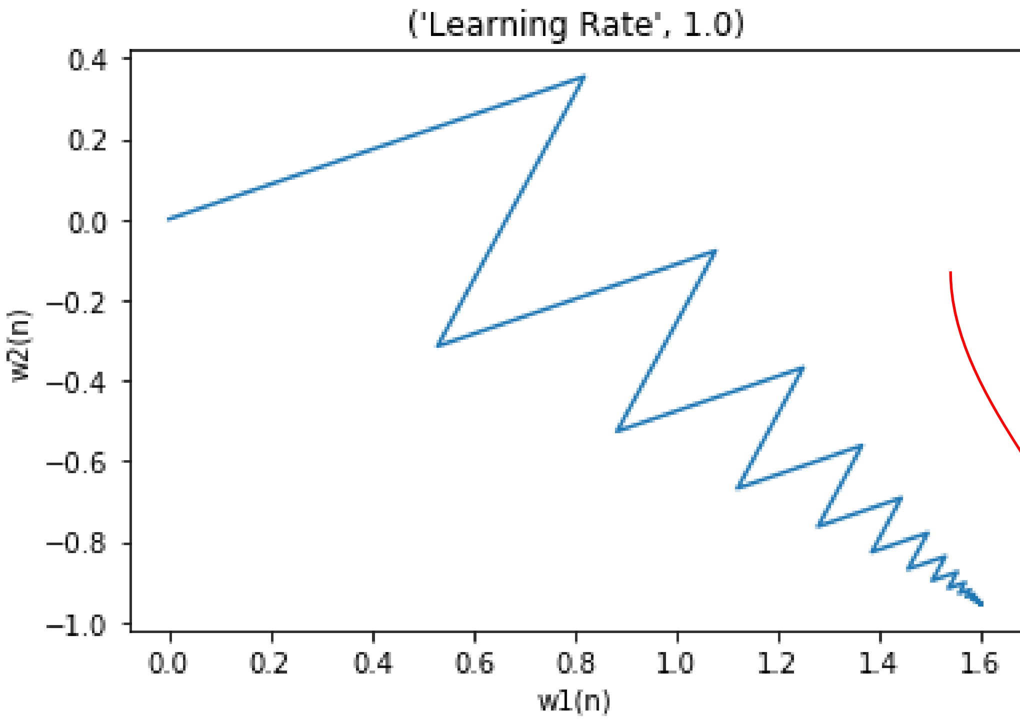
In [6]:
```
#Lambda function for substitution of weight into symbolic jacobian matrix.
delta_wt_func=sp.lambdify((w),symbolic_jacobian)

def steepest_desc(weight, eta, max_iteration):
    found_optimum=False
    iteration=0
    wt_matrix=np.array(weight).T
```

```
        print "Learning Rate: ", eta
        while (found_optimum==False and iteration<max_iteration):
            iteration+=1
            delta_wt=delta_wt_func(weight)

            if(abs(delta_wt.all())<1e-5):
                """
                If absolute change in weight is less than 10^(-5),
                we consider we have reached the optimal solution
                """
                print "Reached optimum value"
                print "Absolute change in weight is less than 10^(-5), exiting loop."
                found_optimum=True
                break
            else:
                weight+=-eta*delta_wt.T
                wt_matrix=np.vstack((wt_matrix,weight.T))

        if iteration==max_iteration:
            print "Max iteration reached!"
        print "optimum values of weight, w*:",np.array(weight).T
        print "Total iterations required:",iteration
        print wt_matrix.shape
        plt.plot(wt_matrix[:,0],wt_matrix[:,1])
        plt.xlabel("w1(n)")
        plt.ylabel("w2(n)")
        plt.title(("Learning Rate",eta))
        plt.show()

    initial_weight = sp.Matrix([[0],[0]])
    print initial_weight.shape
    eta1 = 0.3
    eta2 = 1.0
    max_iteration =1000
    steepest_desc(initial_weight,eta1,max_iteration)
    steepest_desc(initial_weight,eta2,max_iteration)
```

```
(2, 1)
Learning Rate:  0.3
Reached optimum value
Absolute change in weight is less than 10^(-5), exiting loop.
optimum values of weight, w*: [[1.59897673946915 -0.954273186504682]]
Total iterations required: 181
(181, 2)
```




```
Learning Rate:  1.0
Reached optimum value
Absolute change in weight is less than 10^(-5), exiting loop.
optimum values of weight, w*: [[1.59899794732667 -0.954307093426509]]
Total iterations required: 55
(55, 2)
```



3.5 Question 4 (Haykin Edition:2, Ex: 3.4):

The correlation matrix R_x of the input vector $x(n)$ in the LMS algorithm is defined by

$$R_x = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

Define the range of values for the learning-rate parameter η of the LMS algorithm for it to be convergent in the mean square.

Answer 4: LMS algorithm is convergent in mean square in the below mentioned region:

$$0 < \eta < \frac{2}{\lambda_{max}}$$

Where, λ_{max} is the largest eigen value of the correlation matrix of the input vector.

```
In [7]: R_x=np.array([[1,0.5],[0.5,1]])

        eigval,eigvec=np.linalg.eig(R_x)
        max_eigvalue=eigval[np.argmax(eigval)]

        print "Ramge of values for learning rate: (0,",2./max_eigvalue,")"
```

```
Ramge of values for learning rate: (0, 1.33333333333 )
```

3.6 Question 5 (Haykin Edition:2, Ex: 3.8):

The ensemble-averaged counterpart of the sum of error squares viewed as a cost function is the mean-square value of the error signal:

$$\begin{aligned} J(w) &= \frac{1}{2}E[e^2(n)] \\ &= \frac{1}{2}E[(d(n) - x^T(n)w)^2] \end{aligned}$$

3.6.1 5(a)

Assuming that the input vector $x(n)$ and the desired response $d(n)$ are drawn from a stationary environment, show that

$$J(w) = \frac{1}{2}\sigma_d^2 - r_{xd}^T w + \frac{1}{2}w^T R_x w$$

where,

$$\begin{aligned} \sigma_d^2 &= E[d^2(n)] \\ r_{xd} &== E[x(n)d(n)] \\ R_x &== E[x(n)x^T(n)] \end{aligned}$$

3.6.2 Answer:

Given:

$$\begin{aligned} J(w) &= \frac{1}{2}E[(d(n) - x^T(n)w)^2] \\ &= \frac{1}{2}E[d^2(n) - 2 \cdot d(n)x^T(n)w + (x^T(n)w)^2] \end{aligned}$$

Since expectation is a linear operator:

$$\begin{aligned} &= \frac{1}{2}E[d^2(n)] - \frac{1}{2} \cdot 2 \cdot E[d(n)x^T(n)w] + \frac{1}{2}E[(x^T(n)w)^2] \\ &= \frac{1}{2}E[d^2(n)] - E[d(n)x^T(n)w] + \frac{1}{2}E[(x^T(n)w)^T(x^T(n)w)] \\ &= \frac{1}{2}E[d^2(n)] - E[d(n)x^T(n)w] + \frac{1}{2}E[w^T x(n)x^T(n)w] \\ &= \frac{1}{2}E[d^2(n)] - E[d(n)x^T(n)]w + \frac{1}{2}w^T E[x(n)x^T(n)]w \end{aligned}$$

Substituting $\sigma_d^2, r_{xd}^T = E[d(n)x^T(n)]$ and R_x, we have:

$$J(w) = \frac{1}{2}\sigma_d^2 - r_{xd}^T w + \frac{1}{2}w^T R_x w$$

which is the required proof.

3.6.3 5(b)

For this cost function, show that the gradient vector and the Hessian matrix of $J(w)$ are as follows, respectively:

$$g = -r_{xd} + R_x w$$
$$H = R_x$$

3.6.4 Answer:

The gradient vector is obtained by partial differentiation of J(w) with respect to w:

$$\nabla J(w) = g = -r_{xd} + R_x w$$

The Hessian matrix is the partial derivative of gradient vector with respect to w:

$$H = \nabla g = R_x$$

3.6.5 5(c)

In the *LMS/Newton algorithm*, the gradient vector g is replaced by its instantaneous value (*Widrow and Stearns, 1985*). Show that this algorithm, incorporating a learning rate parameter η, is described by

$$\hat{w}(n+1) = \hat{w}(n) + \eta R_x^{-1} x(n)(d(n) - x^T(n)w(n))$$

The inverse of the correlation matrix R_x, assumed to be positive definite, is calculated ahead of time.

3.6.6 Answer:

From Newton's Method:

$$\Delta w = -H^{-1} g$$

For the given cost function, from 5(b), we know that Hessian matrix is:

$$H = R_x$$

We also know the instantaneous value of the gradient vector is given by:

$$\hat{g} = -x(n)(e(n))$$
$$\hat{g} = -x(n)(d(n) - x^T(n)w(n))$$

Therefore,

$$\Delta w = -R_x^{-1}(-x(n)(d(n) - x^T(n)w(n)))$$
$$\Delta w = R_x^{-1}x(n)(d(n) - x^T(n)w(n))$$

Hence, after incorporating learning rate η,

$$\hat{w}(n+1) = \hat{w}(n) + \eta R_x^{-1}x(n)(d(n) - x^T(n)w(n))$$

which is the required proof.
