# NN_DebarajBarua_NareshKumarGurulingan_061117

November 11, 2017

# 1 Hochschule Bonn-Rhein-Sieg

# 2 Neural Networks, WS17/18

# 3 Assignment 05 (06-november-2017)

## 3.1 Debaraj Barua, Naresh Kumar Gurulingan

In [2]:
```
import numpy as np
import matplotlib.pyplot as plt
import sympy as sp
import IPython
sp.init_printing()
```

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## 3.2 Question 1:

Read Haykin NN edition 2 : chapter 3.1-3.5. Make a summary

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## 3.3 Question 2 (Haykin Edition:2, Ex: 3.1):

Explore the method of steepest descent involving a single weight $w$ by considering the following cost function:

$$\varepsilon(w) = \frac{1}{2}\sigma^2 - r_{xd}w + \frac{1}{2}r_x w^2$$

where, $\sigma^2$,$r_{xd}$ and $r_x$ are constants.

**Answer 2:**

In [3]:
```
s,r_xd,r_x,w=sp.symbols('s,r_xd,r_x,w')

e=1./2.*s**2-r_xd*w+1./2.*r_x*w**2

nabla=sp.diff(e,w)
nabla
```

Out[3]:

$$1.0r_x w - r_{xd}$$

So, Gradient,

$$\begin{aligned}\nabla\varepsilon(w) &= \frac{\partial\varepsilon(w)}{\partial w} \\ &= -r_{xd} + r_x w\end{aligned}$$

According to steepest descent method, weight vector is updated in the direction of the steepest descent, i.e., in the direction opposite to the gradient vector.

So, $\Delta w = -\eta\nabla\varepsilon(w)$, where $\eta$ is the learning rate

Now, updated weight is:

$$\begin{aligned}w(n+1) &= w(n) + \Delta w \\ &= w(n) - \eta(-r_{xd} + r_x w)\end{aligned}$$

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### 3.4 Question 3 (Haykin Edition:2, Ex: 3.2):

Consider the cost function

$$\varepsilon(w) = \frac{1}{2}\sigma^2 - r_{xd}^T w + \frac{1}{2}w^T R_x w$$

where, $\sigma^2$,$r_{xd}$ and $r_x$ are constants.

$$r_{xd} = \begin{bmatrix}0.8182\\0.354\end{bmatrix}$$

$$R_x \begin{bmatrix}1 & 0.8182\\0.8182 & 1\end{bmatrix}$$

#### 3.4.1 3(a)

Find the optimum value of $w^*$ for which $\varepsilon(w)$ reaches it's minimum value.

**Answer 3(a):** $\varepsilon(w)$ will reach maximum value when it's Jacobian will be zero.

In [11]:
```
s=sp.symbols('s')
w=sp.MatrixSymbol('w',2,1)
r_xd=sp.Matrix([[0.8182],[0.354]])
R_x=sp.Matrix([[1,0.8182],[0.8182,1]])

e=1./2.*s**2-r_xd.T.dot(sp.Matrix(w))+1./2.*sp.Matrix(w).T.dot(R_x.dot(sp.Matrix(w)))
e=sp.Matrix([e])
symbolic_jacobian=e.jacobian(w)
print "Jacobian Matrix is:"
symbolic_jacobian
```

```
Jacobian Matrix is:
```

Out[11]:

$$\left[\begin{matrix}-0.8182 + 0.5 \left(2 w_{0,0} + 1.6364 w_{1,0}\right) & -0.354 + 0.5 \left(1.6364 w_{0,0} + 2 w_{1,0}\right)\end{matrix}\right]$$

In [12]:
```python
print "optimum values of weight, w*:"
sp.Matrix([sp.solve(symbolic_jacobian,sp.Matrix(w))])
```

```
optimum values of weight, w*:
```

Out[12]:

$$\left[\begin{matrix}\left\{ w_{0,0} : 1.59902944424901, \quad w_{1,0} : -0.954325891284541\right\}\end{matrix}\right]$$

### 3.4.2 3(b)

Use the method of steepest descent to compute $w^*$ for the following two values of learning rate parameter:

<p>(i)  $\eta=0.3$</p>
<p>(ii) $\eta=1.0$</p>

For each case, plot the trajectory traced the evolution of the weight vector $w(n)$ in the W-Plane.

*Note*: The trajectories obtained for cases (i) and (ii) of part (b) should correspond to the picture in Fig. 3.2

In [187]:
```python
IPython.display.Image("images/HaykinFig3.2.png")
```

Out[187]:

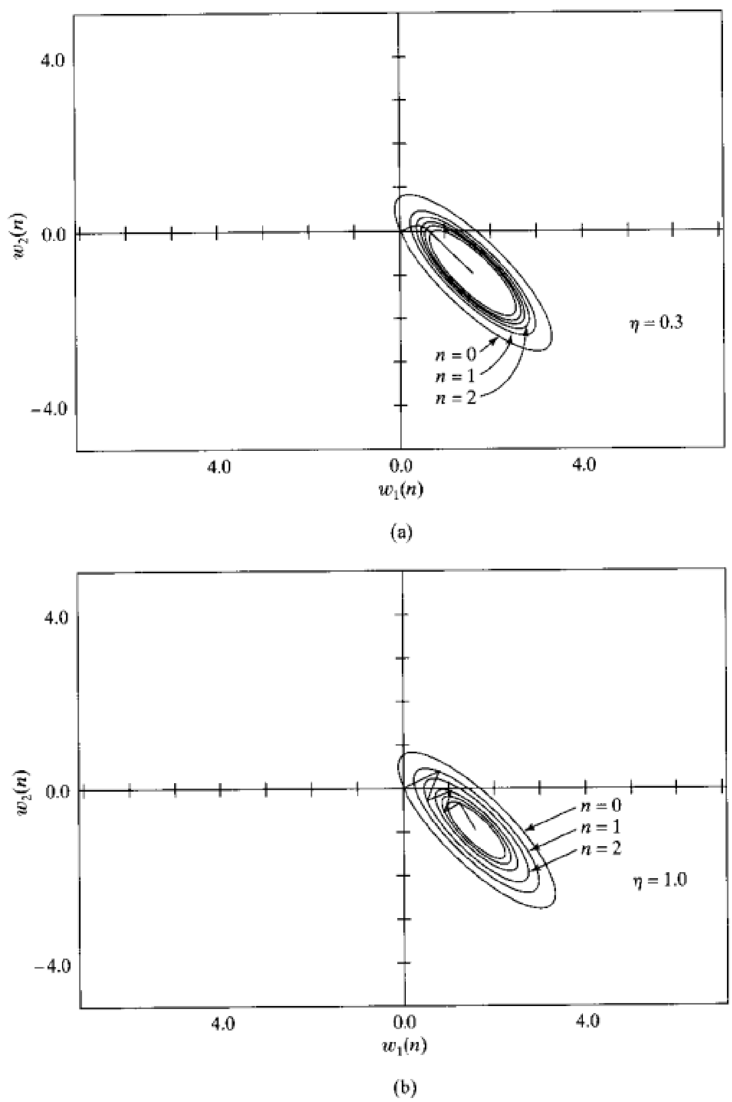


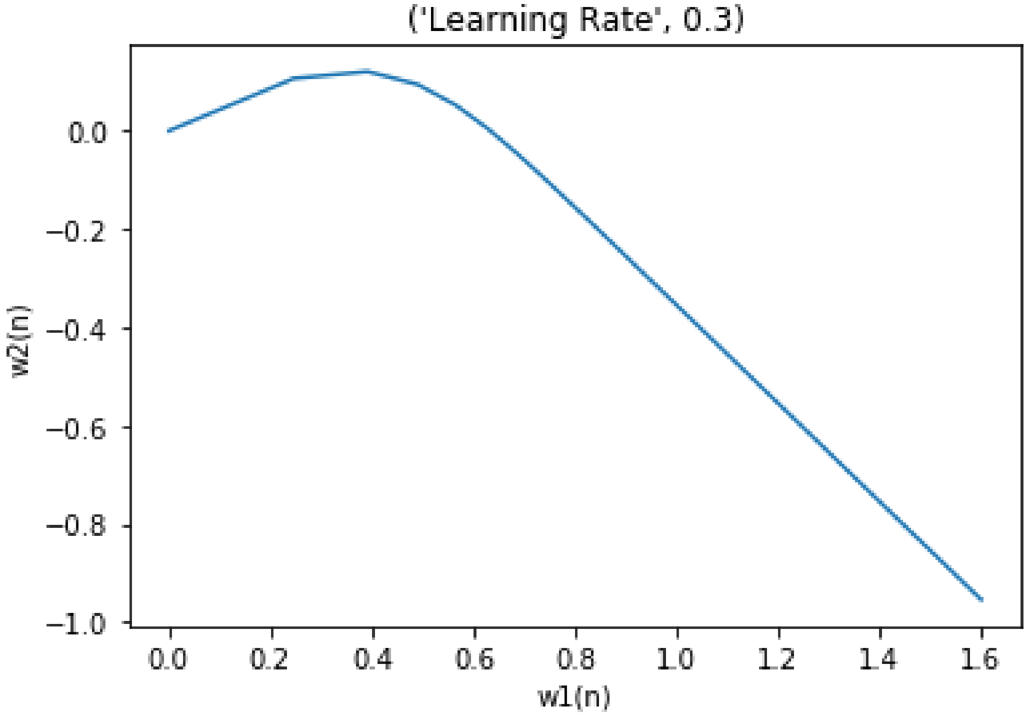

**FIGURE 3.2** Trajectory of the method of steepest descent in a two-dimensional space for two different values of learning-rate parameter: (a) $\eta = 0.3$, (b) $\eta = 1.0$. The coordinates $w_1$ and $w_2$ are elements of the weight vector **w**.

**Answer 3(b):**

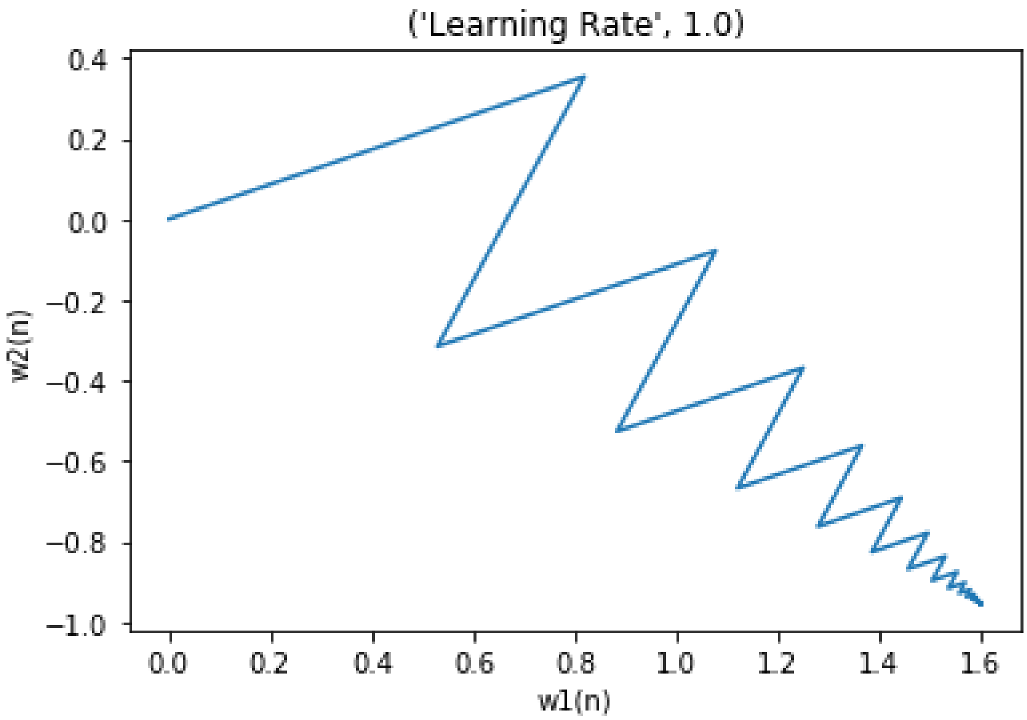
```
In [188]: #Lambda function for substitution of weight into symbolic jacobian matrix.
          delta_wt_func=sp.lambdify((w),symbolic_jacobian)

          def steepest_desc(weight, eta, max_iteration):
              found_optimum=False
              iteration=0
              wt_matrix=np.array(weight).T
```

```
        print "Learning Rate: ", eta
        while (found_optimum==False and iteration<max_iteration):
            iteration+=1
            delta_wt=delta_wt_func(weight)

            if(abs(delta_wt.all())<1e-5):
                """
                If absolute change in weight is less than 10^(-5),
                we consider we have reached the optimal solution
                """
                print "Reached optimum value"
                print "Absolute change in weight is less than 10^(-5), exiting loop."
                found_optimum=True
                break
            else:
                weight+=-eta*delta_wt.T
                wt_matrix=np.vstack((wt_matrix,weight.T))

        if iteration==max_iteration:
            print "Max iteration reached!"
        print "optimum values of weight, w*:",np.array(weight).T
        print "Total iterations required:",iteration
        print wt_matrix.shape
        plt.plot(wt_matrix[:,0],wt_matrix[:,1])
        plt.xlabel("w1(n)")
        plt.ylabel("w2(n)")
        plt.title(("Learning Rate",eta))
        plt.show()

    initial_weight = sp.Matrix([[0],[0]])
    print initial_weight.shape
    eta1 = 0.3
    eta2 = 1.0
    max_iteration =1000
    steepest_desc(initial_weight,eta1,max_iteration)
    steepest_desc(initial_weight,eta2,max_iteration)
```

```
(2, 1)
Learning Rate:  0.3
Reached optimum value
Absolute change in weight is less than 10^(-5), exiting loop.
optimum values of weight, w*: [[1.59897673946915 -0.954273186504682]]
Total iterations required: 181
(181, 2)
```




```
Learning Rate:  1.0
Reached optimum value
Absolute change in weight is less than 10^(-5), exiting loop.
optimum values of weight, w*: [[1.59899794732667 -0.954307093426509]]
Total iterations required: 55
(55, 2)
```

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### 3.5 Question 4 (Haykin Edition:2, Ex: 3.4):

The correlation matrix $R_x$ of the input vector $x(n)$ in the LMS algorithm is defined by

$$R_x = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

Define the range of values for the learning-rate parameter $\eta$ of the LMS algorithm for it to be convergent in the mean square.

**Answer 4:** LMS algorithm is convergent in mean square in the below mentioned region:

$$0 < \eta < \frac{2}{\lambda_{max}}$$

Where, $\lambda_{max}$ is the largest eigen value of the correlation matrix of the input vector.

```
In [191]: R_x=np.array([[1,0.5],[0.5,1]])

          eigval,eigvec=np.linalg.eig(R_x)
          max_eigvalue=eigval[np.argmax(eigval)]

          print "Ramge of values for learning rate: [0,",2./max_eigvalue,"]"
```

```
Ramge of values for learning rate: [0, 1.33333333333 ]
```

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## 3.6 Question 5 (Haykin Edition:2, Ex: 3.8):

The ensemble-averaged counterpart of the sum of error squares viewed as a cost function is the mean-square value of the error signal:

$$\begin{aligned} J(w) &= \frac{1}{2}E[e^2(n)] \\ &= \frac{1}{2}E[(d(n) - x^T(n)w)^2] \end{aligned}$$

### 3.6.1 5(a)

Assuming that the input vector $x(n)$ and the desired response $d(n)$ are drawn from a stationary environment, show that

$$J(w) = \frac{1}{2}\sigma_d^2 - r_{xd}^T w + \frac{1}{2}w^T R_x w$$

where,

$$\begin{aligned} \sigma_d^2 &= E[d^2(n)] \\ r_{xd} &== E[x(n)d(n)] \\ R_x &== E[x(n)x^T(n)] \end{aligned}$$

### 3.6.2 Answer:

Given:

$$\begin{aligned} J(w) &= \frac{1}{2}E[(d(n) - x^T(n)w)^2] \\ &= \frac{1}{2}E[d^2(n) - 2 \cdot d(n)x^T(n)w + (x^T(n)w)^2] \end{aligned}$$

Since expectation is a linear operator:

$$\begin{aligned} &= \frac{1}{2}E[d^2(n)] - \frac{1}{2} \cdot 2 \cdot E[d(n)x^T(n)w] + \frac{1}{2}E[(x^T(n)w)^2] \\ &= \frac{1}{2}E[d^2(n)] - E[d(n)x^T(n)w] + \frac{1}{2}E[(x^T(n)w)^T(x^T(n)w)] \\ &= \frac{1}{2}E[d^2(n)] - E[d(n)x^T(n)w] + \frac{1}{2}E[w^T x(n)x^T(n)w] \\ &= \frac{1}{2}E[d^2(n)] - E[d(n)x^T(n)]w + \frac{1}{2}w^T E[x(n)x^T(n)]w \end{aligned}$$

Substituting $\sigma_d^2, r_{xd}^T = E[d(n)x^T(n)]$ and $R_x$, we have:

$$J(w) = \frac{1}{2}\sigma_d^2 - r_{xd}^T w + \frac{1}{2}w^T R_x w$$

which is the required proof.

#### 3.6.3 5(b)

For this cost function, show that the gradient vector and the Hessian matrix of $J(w)$ are as follows, respectively:

$$g = -r_{xd} + R_x w$$
$$H = R_x$$

#### 3.6.4 Answer:

The gradient vector is obtained by partial differentiation of J(w) with respect to w:

$$\nabla J(w) = g = -r_{xd} + R_x w \tag{1}$$

The Hessian matrix is the partial derivative of gradient vector with respect to w:

$$H = \nabla g = R_x \tag{2}$$

#### 3.6.5 5(c)

In the *LMS/Newton algorithm*, the gradient vector g is replaced by its instantaneous value (*Widrow and Stearns, 1985*). Show that this algorithm, incorporating a learning rate parameter $\eta$, is described by

$$\hat{w}(n+1) = \hat{w}(n) + \eta R_x^{-1} x(n)(d(n) - x^T(n)w(n))$$

The inverse of the correlation matrix $R_x$, assumed to be positive definite, is calculated ahead of time.

#### 3.6.6 Answer:

From Newton's Method:

$$\Delta w = -H^{-1} g \tag{3}$$

For the given cost function, from 5(b), we know that Hessian matrix is:

$$H = R_x \tag{4}$$

We also know the instantaneous value of the gradient vector is given by:

$$\hat{g} = -x(n)(e(n)) \tag{5}$$
$$\hat{g} = -x(n)(d(n) - x^T(n)w(n)) \tag{6}$$

Therefore,

$$\Delta w = -R_x^{-1}(-x(n)(d(n) - x^T(n)w(n))) \tag{7}$$
$$\Delta w = R_x^{-1}x(n)(d(n) - x^T(n)w(n)) \tag{8}$$

Hence, after incorporating learning rate $\eta$,

$$\hat{w}(n+1) = \hat{w}(n) + \eta R_x^{-1}x(n)(d(n) - x^T(n)w(n))$$

which is the required proof.

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